MAT 531

Geometry/Topology II

Introduction to Smooth Manifolds

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Stony Brook University

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are all linear maps $\mathbb{V} \to \mathbb{R}$, for any fixed vectors $\mathsf{V}_1, \mathsf{V}_2, \ldots, \mathsf{V}_k \in \mathbb{V}$.

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Undergraduates are usually taught to mistake this for a function.

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Since any smooth n-manifold M is locally diffeomorphic to $\mathbb{R}^n$, any $\varphi \in \Omega^k(M)$ looks like this in local coordinates.

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