CONFORMAL DIMENSION OF THE ANTENNA SET

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Abstract. We show that the self-similar set known as the "antenna set" has the property that $\inf_f \dim(f(X)) = 1$, (where the infimum is over all quasiconformal mappings of the plane), but that this infimum is not attained by any quasiconformal map, indeed, is not attained for any quasisymmetric map into any metric space.

Given a compact metric space X we define its conformal dimension as

$$\mathrm{Cdim}(X) = \inf_f \dim(f(X)),$$

where the infimum is over all quasisymmetric maps of X into some metric space and "dim" denotes Hausdorff dimension. Recall that f is quasisymmetric if there is a homeomorphism η of $[0, \infty)$ to itself such that

$$|x-y| \le t|x-z| \quad \Rightarrow \quad |f(x)-f(y)| \le \eta(t)|f(x)-f(z)|.$$

For $f : \mathbb{R}^n \to \mathbb{R}^n$, $n \ge 2$, these are exactly the same as quasiconformal maps. This notion of conformal dimension was introduced by Pansu [8]. See also [3] and [9]. In this note we are interested in the question of whether the infimum above must be attained by some quasisymmetric image of X. It is known that for some sets (e.g., the usual Cantor middle thirds set in $\mathbb{R}$), the conformal dimension is zero, whereas any quasisymmetric image must have strictly positive dimension. Here we answer a question of Heinonen's (Section 19, [6]) by proving

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Theorem 1. *There is a compact, connected* $X \subset \mathbb{R}^2$ *so that for every quasisymmetric map* f *of* X *into a metric space,* $\dim(f(X)) > 1 = \mathrm{Cdim}(X)$.

In particular, this holds for all quasiconformal self-maps of the plane, which is already an interesting case. Our set is the self-similar set associated to the four conformal contractions

$$f_1(z) = \frac{1}{2}z, \qquad f_2(z) = \frac{1}{2}z + \frac{1}{2},$$
$$f_3^\alpha(z) = \alpha i z + \frac{1}{2}, \qquad f_4^\alpha(z) = -\alpha i z + \frac{1}{2} + \alpha i,$$

where $0 < \alpha < \frac{1}{2}$. For $E \subset \mathbb{R}^2$, let

$$F_\alpha(E) = f_1(E) \cup f_2(E) \cup f_2^\alpha(E) \cup f_4^\alpha(E),$$

and let X_α be the unique compact set such that $X = F_\alpha(X)$ (e.g., see Section 9.1 of [4] for the existence and uniqueness of this set). The set X_α is shown in Figure 1 for a few different values of α.

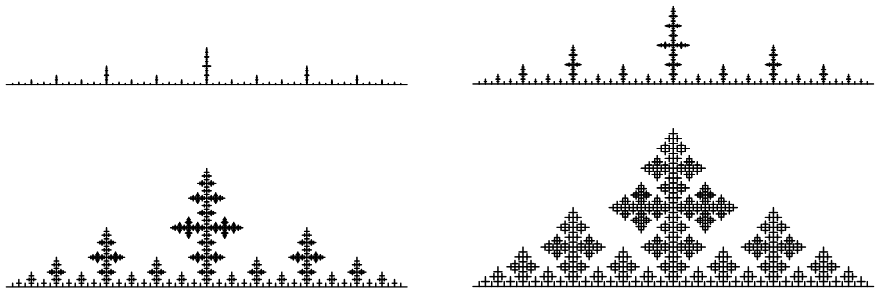

FIGURE 1. The antenna set for $\alpha = .1, .2, .3, .4$

Choose some γ with $\alpha < \gamma < \frac{1}{2}$ and consider the open triangle $T = T_\gamma$ with vertices $0, 1, \frac{1}{2} + i\gamma$. Then $F_\alpha(T)$ has four disjoint components and each is contained in T (see Figure 2). Hence the open set condition (page 118 of [4]) is satisfied and so X_α has Hausdorff dimension d where d is the unique solution to

$$2(\frac{1}{2})^d + 2(\alpha)^d = 1,$$

and will have positive, finite Hausdorff measure in this dimension (e..g., Theorem 9.3 of [4]). Note that $\mathrm{Cdim}(X_\alpha) \geq 1$ (since it is a continuum) and that $d \to 1$ as $\alpha \to 0$. Thus in order to show X_α has the desired properties, we need only verify the following two claims.

Lemma 2. *For any* $0 < \alpha, \beta < \frac{1}{2}$*, there is a quasiconformal* f *with* $f(X_\alpha) = X_\beta$*.*

Lemma 3. *For any* $0 < \alpha < \frac{1}{2}$ *and any quasisymmetric map* f*,* $\dim(f(X_\alpha)) > 1$*.*

Proof of Lemma 2: The proof of this is little more than a picture which describes how to build f. Choose a γ so that $\max(\alpha, \beta) < \gamma < \frac{1}{2}$ and as above let $T = T_\gamma$ be the triangle with vertices 0, 1 and $\frac{1}{2} + i\gamma$. Let $P_\alpha = T \setminus F_\alpha(T)$ and $P_\beta = T \setminus F_\beta(T)$. See Figure 2. T can clearly be mapped to itself by a mapping g which is the identity on the boundary, is linear and conformal on each component of $F_\alpha(T)$ and which is piecewise affine from P_α to P_β. Define f to be the identity outside T and define it in $T \setminus X_\alpha$ using

$$f(z) = f^\beta_{i_n} \circ \cdots \circ f^\beta_{i_1} \circ g \circ (f^\alpha_{i_1})^{-1} \circ \cdots \circ (f^\alpha_{i_n})^{-1}(z),$$

if $z \in f^\alpha_{i_n} \circ \cdots \circ f^\alpha_{i_n}(P_\alpha)$, where $i_1, \ldots, i_n \in \{1, 2, 3, 4\}$ and $f^\alpha_1 = f_1$, $f^\alpha_2 = f_2$. Then f is clearly quasiconformal on $\Omega = \mathbb{R}^2 \setminus X_\alpha$ and is a homeomorphism of the whole plane. Thus by Corollary 1 of [7], f is quasiconformal on the whole plane (since the complement of X_α is clearly a John domain). $\square$

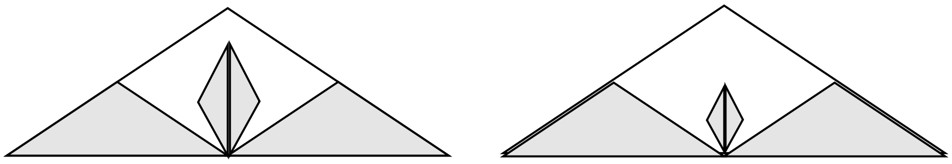

FIGURE 2. Building the map from X_α to X_β

Proof of Lemma 3: Since all the antenna sets are quasiconformally equivalent to one another, it is enough so prove the lemma for one of them, so we fix $\alpha = \frac{1}{4}$ and let $X = X_{1/4}$. Suppose $f : X \to Y$ is quasisymmetric and let η be the corresponding homeomorphism of $[0, \infty)$. In order to show that $\dim(f(X)) > 1$, we need only construct a positive measure ν on $f(X)$ such that

$$\nu(B(y, r)) \leq Cr^s, \quad \forall y \in Y, \forall r > 0, \tag{1}$$

for some $C < \infty$ and $s > 1$ (independent of y and r). Then the mass distribution principle shows that for any covering of $f(X)$ by balls $B(y_j, r_j)$, we have

$$0 < \nu(f(X)) \leq \sum_k \nu(B(y_j, r_j)) \leq C \sum r_j^s,$$

which implies $\dim(f(X)) \geq s > 1$.

Let $I = [0, 1] \subset X$. For any string $\sigma = (i_1, \ldots, i_n) \in \{1, 2, 3, 4\}^n$, let $\sigma^* = (i_1, \ldots, i_{n-1})$. Moreover, define

$$X_\sigma = f_{i_1} \circ \cdots \circ f_{i_n}(X),$$

$$I_\sigma = f_{i_1} \circ \cdots \circ f_{i_n}(I),$$

$$x_\sigma = f_{i_1} \circ \cdots \circ f_{i_n}(0), \quad y_\sigma = f_{i_1} \circ \cdots \circ f_{i_n}(1),$$

$$d_\sigma = |f(x_\sigma) - f(y_\sigma)|.$$

The first step is to show

Lemma 4. *There is an $s > 1$ (depending only on η) so that $\sum_{\sigma:\sigma^* = \tau} d_\sigma^s \geq d_\tau^s$.*

Proof. Since $x_\tau = x_{\tau 1}$ and $y_\tau = y_{\tau 2}$, the triangle inequality implies $d_{\tau 1} + d_{\tau 2} \geq d_\tau$ and by quasisymmetry $d_{\tau 3} \simeq d_{\tau 1}$. Thus

$$\sum_{\sigma:\sigma^* = \tau} d_\sigma \geq (1 + C_1) d_\tau,$$

for some $C_1 > 0$ depending only on η. Thus

$$\sum_{\sigma:\sigma^*=\tau} \frac{d_\sigma}{d_\tau} \geq 1 + C_1.$$

Since this is a finite sum, there is a $s > 1$, depending only on C_1 and the number of terms in the sum, such that

$$\sum_{\sigma:\sigma^*=\tau} (\frac{d_\sigma}{d_\tau})^s \geq 1,$$

which is the desired inequality. □

Fix $s > 1$ as in the previous lemma. To define a measure μ on X it suffices to define the measure of each X_σ. We let $\mu(X) = 1$ and in general

$$\mu(X_\sigma) = \mu(X_{\sigma^*}) \frac{d_\sigma^s}{\sum_{\tau:\tau^*=\sigma^*} d_\tau^s}.$$

Lemma 5. *There is a $C_2 < \infty$, depending only on η and* $\mathrm{diam}(Y)$ *such that the following holds. For every σ, we have $\mu(X_\sigma) \leq C_2 d_\sigma^s \leq C_3 \mathrm{diam}(f(X_\sigma))^s$.*

Proof. This is easy by induction. The estimate holds for X with $C_2 = \mathrm{diam}(Y)^{-1}$ since μ is a probability measure. In general,

$$\mu(X_\sigma) \quad = \quad \mu(X_{\sigma^*}) \frac{d_\sigma^s}{\sum_{\tau:\tau^*=\sigma^*} d_\tau^s} \leq C_2 d_{\sigma^*}^s \frac{d_\sigma^s}{d_{\sigma^*}^s} = C_2 d_\sigma^s,$$

as desired. The estimate $d_\sigma \simeq \mathrm{diam}(f(X_\sigma))$ follows from quasisymmetry. □

Let ν be the push forward of μ to Y via f, i.e., $\nu(E) = \mu(f^{-1}(E))$. Now we prove (1) for the measure ν. Let $B = B(y, r)$ be any ball in Y. We may assume $B \cap f(X) \neq \emptyset$ (or else the estimate is trivial). Furthermore, if B hits $f(X)$ then it is contained in a ball of radius $2r$ centered at a point of $f(X)$, so we may assume $y \in f(X)$. Let $x = f^{-1}(y)$ and $E = f^{-1}(B(y, r))$. By quasisymmetry, there are balls $B(x, r_1) \subset E \subset B(x, r_2)$, with the ratio r_2/r_1 bounded only depending on η.

Lemma 6. *There are constants $C_4, C_5 < \infty$ (depending only on η) so that E can be covered by $N \le C_4$ sets of the form $\{X_{\sigma_j}\}_1^N$, each of which has diameter $\le r_1$ and so that $d_{\sigma_j} \le C_5 d_{\sigma_k}$ for any choice of $j, k \in \{1, \dots, N\}$.*

If we can prove the lemma, then (1) follows because

$$\begin{aligned} \nu(B(y,r)) &= \mu(E) \\ &\le \sum_{j=1}^{N} \mu(X_{\sigma_j}) \\ &\le C_4 C_5 \mu(X_{\sigma_1}) \\ &\le C_3 C_4 C_5 \operatorname{diam}(f(X_{\sigma_1}))^s. \end{aligned}$$

If σ_1 is chosen so that $x \in X_{\sigma_1}$, then $\operatorname{diam}(X_{\sigma_1}) \le r_1$ implies $X_{\sigma_1} \subset B(x, r_1) \subset E$, and thus $\operatorname{diam}(f(X_{\sigma_1}) \le \operatorname{diam}(f(E)) = r$. This completes the proof of (1) except for the proof of the lemma which we now give.

Proof. We shall need to use that X is quasi-convex, i.e., that there is a $C_6 < \infty$ so that any two points $x, y \in X$ can be connected by a path in X of length at most $C_6|x - y|$. The proof of this is left to the reader.

Given E, let $\{X_{\sigma_j}\}$ be all the sets of the form X_σ with diameter $\le r_1$ and which are maximal with respect to inclusion. Because the sets are nested, such a maximal collection is disjoint in the sense that their convex hulls have disjoint interiors. Moreover, each such set has diameter $\ge r_1/4$ (otherwise its parent would be in the collection which contradicts maximality). Thus each of the disjoint hulls described above has area $\ge r_1^2/128$. Every such set which hits E is contained in $B(x, r_1 + r_2) \subset B(x, 2r_2)$ and so adding areas shows $N \le C_4 = 512\pi(r_2/r_1)^2$.

To prove the second claim, note that any two of our sets X_{σ_1}, X_{σ_2} which hit E are at most distance $2r_2$ apart and hence by the quasi-convexity of X can be joined by a path γ of length at most $2C_6 r_2$ in X. Assume γ is chosen as short as possible. Then it has the property that if it hits a set X_σ which does not contain the endpoints of γ then $\gamma \cap X_\sigma$ is either a single point or is I_σ (otherwise we can shorten γ). Thus X_{σ_1} and X_{σ_2} can be joined by a chain of intervals of the form $\{I_\sigma\}$ where each of these has diameter between $r_1/4$ and r_1 and the number of elements in the chain is at most the length of γ divided by $r_1/4$, and hence is bounded depending only on η. By quasisymmetry the f-images of any two adjacent intervals in the chain have comparable diameters depending only on η, and hence the same is true of X_{σ_1} and X_{σ_2}. This completes the proof of the lemma and hence the proof of Theorem 1. $\square$

Remark 1: If we are only interested in quasiconformal maps $f : \mathbb{R}^2 \to \mathbb{R}^2$ then the proof of Lemma 3 can be considerable shortened using a result from [2]. It is proven there that compact, connected sets which are uniformly wiggly (i.e., deviate from a line segment uniformly at all points and all scales; see [2] for the precise definition) must have dimension strictly larger than 1. Using the compactness of K-quasiconformal maps, it is easy to check that any quasiconformal image of the antenna set is uniformly wiggly and hence has dimension > 1. This proof does not apply directly to metric spaces since we don't know what "line segment" should mean in this setting. However, the proof given in this paper suggests the following possibility: Y is uniformly wiggly if there is a $\beta > 0$ so that for every $y \in Y$ and $r > 0$, and for any collection $\{y_0, \dots, y_n\} \subset B(y, r)$, we have

$$\sum |y_k - y_{k-1}| + \sup_{z \in B(y,r)} \inf_k |z - y_k| \geq (1+\beta)|y_0 - y_n|.$$

In Euclidean space this is equivalent to the definition from [2]. Does this imply $\dim(Y) > 1$ if Y is a compact, connected metric space?

Remark 2: The construction could be modified to give an example which is a Jordan arc. On the other hand, the infimum is attained for some arc (e..g., any quasi-arc). Thus topology alone cannot insure the infimum in the definition of conformal dimension is attained.

Remark 3: Given a set $E \subset \mathbb{R}^2$ we can define D_E to be the set of values of $\dim(f(E))$ as f ranges over all quasiconformal maps of the plane. This must be a single point or an interval since any quasiconformal map can be connected to the identity by a path of maps along which the dimension changes continuously. From [1] and [5], we know that the only possibilities are of the form $\{0\}$, $\{2\}$, $(0,2)$, $[\alpha, 2)$ and $(\alpha, 2)$ for $0 < \alpha < 2$. The first three are well known to occur, as is $[1,2)$. This paper shows that $(1,2)$ can also occur. The second author showed in [9] that intervals of the form $[\alpha, 2)$ can occur if $1 < \alpha < 2$ and conjectured $[\alpha, 2)$, $(\alpha, 2)$ never occur for $0 < \alpha < 1$. Do the intervals $(\alpha, 2)$ occur for $1 < \alpha < 2$?

Remark 4: Computing the conformal dimension seems difficult, even for some fairly "simple" fractal sets. For example, what is the conformal dimension of the Sierpinski carpet illustrated in Figure 3? Its dimension is $\log 8 / \log 3 \approx 1.89279$ and its conformal dimension is $\geq 1 + \log 2 / \log 3 \approx 1.63093$ (since it contains the middle thirds Cantor set cross an interval, for which the results of [9] imply $\mathrm{Cdim} = \dim$).

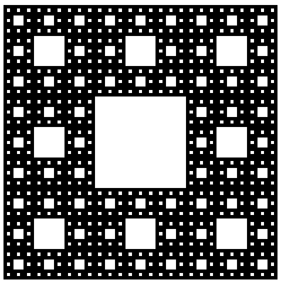

FIGURE 3. The Sierpinski carpet

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