ON THE SHAPES OF RATIONAL LEMNISCATES

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Abstract. A rational lemniscate is a level set of $|r|$ where $r : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is rational. We prove that any planar Euler graph can be approximated, in a strong sense, by a homeomorphic rational lemniscate. This generalizes Hilbert's lemniscate theorem; he proved that any Jordan curve can be approximated (in the same strong sense) by a polynomial lemniscate that is also a Jordan curve. As consequences, we obtain a sharp quantitative version of the classical Runge's theorem on rational approximation, and we give a new result on the approximation of planar continua by Julia sets of rational maps.

1. Introduction

1.1. Rational lemniscates and Euler graphs.

Definition 1.1. A *rational lemniscate* is a set of the form

$$L_r(c) := \{z \in \widehat{\mathbb{C}} : |r(z)| = c\},$$

where $0 < c < \infty$, r is a rational function and $\widehat{\mathbb{C}}$ is the Riemann sphere; in other words, a rational lemniscate is a level set of $|r|$. The constant c is often omitted from the notation, since by rescaling r we can always take $c = 1$, and for brevity we will write $L_r = L_r(1)$.

Definition 1.2. A *lemniscate graph* is a set $G \subset \widehat{\mathbb{C}}$ so that there is a finite set $V \subset G$ (called the *vertices* of G), so that:

(1) $G \setminus V$ has finitely many components (these are called the *edges* of G), each of which is either a (closed) Jordan curve, or else an (open) simple arc γ satisfying $\overline{\gamma} \setminus \gamma \subset V$.
(2) The degree of each vertex is even and at least four, where the *degree* of a vertex v is defined as the number of edges γ satisfying $v \in \overline{\gamma} \setminus \gamma$, and we count an edge γ twice if $\{v\} = \overline{\gamma} \setminus \gamma$.

See Figure 1. It is not hard to prove that every rational lemniscate is a lemniscate graph (see Proposition 2.4) and our main result will show that every lemniscate graph is homeomorphic to, and can be approximated by, a rational lemniscate. Before stating the precise result, we need a few more definitions.

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A lemniscate graph need not be a graph in the usual sense, since it can have Jordan curve components with no vertices (see Figure 1). However, if we add a vertex (of degree two) to each such curve component, we create an Euler graph in the usual sense (an Euler graph is one where every vertex has even degree; an Eulerian graph is a connected Euler graph). Thus as closed subsets of the plane (forgetting the vertex/edge structure), lemniscate graphs and Euler graphs are the same. In particular, the faces of the graph (that is, the connected components of the complement of the graph) are the same.

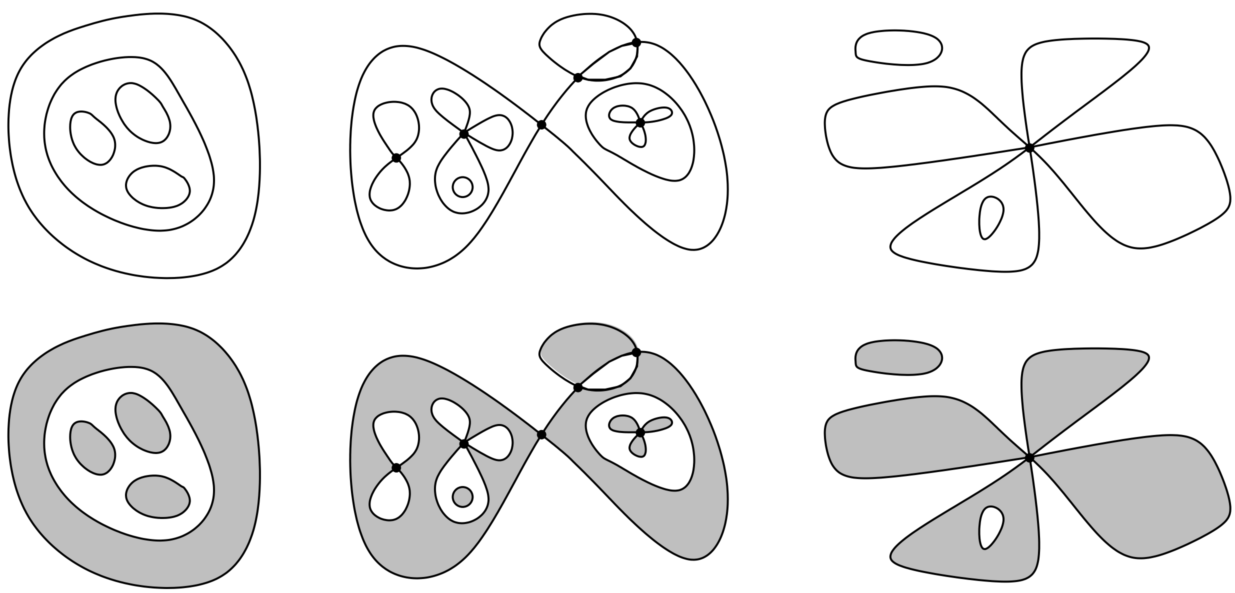

FIGURE 1. A lemniscate graph (top) and a 2-coloring of its faces (bottom). This graph has 14 connected components, 26 faces, and 7 vertices. Eight of the connected components are closed Jordan curves with no vertices.

Definition 1.3. Let G be a lemniscate graph. A 2-*coloring* of the faces of G assigns one of two colors to each face (we will use white and grey) so that any two faces sharing a common edge have different colors.

The fact that the faces of a planar Euler graph can be 2-colored is well known in graph theory. Moreover, there are exactly two such colorings, obtained from each other by swapping the colors. Similarly, the faces of a rational lemniscate have a natural 2-coloring where the components of $\{z \in \widehat{\mathbb{C}} : |r(z)| < 1\}$ are colored white, and the components of $\{z \in \widehat{\mathbb{C}} : |r(z)| > 1\}$ are colored grey (we can swap colors by replacing r by $1/r$). Clearly the poles of r must lie in the grey components. It turns out that this is the only restriction for the following type of approximation to hold.

Definition 1.4. Let $\varepsilon > 0$. Two sets E, $F \subset \widehat{\mathbb{C}}$ are said to be *ε-homeomorphic* if there exists a homeomorphism $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ satisfying $\phi(E) = F$ and $\sup_{z\in\widehat{\mathbb{C}}} d(\phi(z), z) < \varepsilon$, where $d(\cdot, \cdot)$ denotes the spherical metric on $\widehat{\mathbb{C}}$.

Theorem A. *Let G be a lemniscate graph, let $\varepsilon > 0$, fix a 2-coloring of the faces of G, and suppose that $P \subset \widehat{\mathbb{C}}$ contains exactly one point in each grey face of G. Then there exists a rational mapping $r : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ so that G and L_r are ε-homeomorphic and $r^{-1}(\infty) = P$.*

Remark 1.5. We might have called two sets E and F δ-homeomorphic if there is homeomorphism $\phi : E \to F$ so that $\sup_{z\in E} |\phi(z) - z| < \delta$, but this is weaker than Definition 1.4. However, we shall prove in Section 3 that if G is a lemniscate graph and $\phi : G \to G'$ is a δ-homeomorphism in this weaker sense, then it can be extended to an ε-homeomorphism of $\widehat{\mathbb{C}}$, assuming that δ is sufficiently small depending on G and ε (this is false for more general sets). Thus to prove Theorem A, it will suffice to verify that for every $\delta > 0$ there is a rational function r, so that G and L_r are δ-homeomorphic in the weaker sense.

1.2. **Hilbert's Lemniscate Theorem.**

Hilbert [Hil97] proved that any closed Jordan curve is ε-homeomorphic to a polynomial lemniscate. This is not how the result is usually stated, but it is an equivalent formulation, and it makes it easy to see that Theorem A generalizes Hilbert's result. More precisely, since a Jordan curve is a lemniscate graph (one edge, no vertices) with exactly two faces, we can choose the unbounded face to be colored grey, and place the pole in Theorem A at infinity. Thus the approximating rational lemniscate is actually a polynomial lemniscate, giving Hilbert's theorem. More generally, any finite collection of disjoint Jordan curves that are not nested (no curve separates another one from infinity) is ε-homeomorphic to a polynomial lemniscate, since we can color the bounded faces white and the unbounded face grey. This case is due to Walsh and Russell [WR34], and generalizing their result to arbitrary families of disjoint Jordan curves (i.e., allowing nesting) was the original motivation for the current paper. In addition to recovering the theorems of Hilbert and Walsh-Russell, Theorem A also gives the following new result about polynomial lemniscates.

Corollary 1.6. *If G is a lemniscate graph that is the boundary of its unbounded face, then G is ε-homeomorphic to a polynomial lemniscate for every $\varepsilon > 0$. If a lemniscate graph is not the boundary of its unbounded face, then it is not the image of a polynomial lemniscate under any homeomorphism of the plane.*

Both Hilbert's lemniscate theorem and Theorem A say that every topological version of some object (a Jordan curve or lemniscate graph) is ε-homeomorphic to an algebraic version of the same object.

1.3. Quantitative Approximation by Rational Functions.

Theorem B. *Let $K \subset \widehat{\mathbb{C}}$ be compact, let P contain exactly one point from each component of $\widehat{\mathbb{C}} \setminus K$, and suppose f is holomorphic in a neighborhood U of K. Then there exist constants $A, B \in (1, \infty)$ and a sequence of rational mappings R_n of degree $\leq n$ satisfying $R_n^{-1}(\infty) \subset P$ and such that*

$$\sup_{z \in K} |f(z) - R_n(z)| \leq \frac{A}{B^n} \text{ for all } n. \tag{1.1}$$

Replacing the geometric rate of convergence in (1.1) with $o(1)$ is exactly Runge's classical approximation theorem. When the neighborhood U in Theorem B is a disc, we can take R_n to be the degree n truncation of the Taylor series for f in U. We then see that Theorem B generalizes the well-known fact that the Taylor series of an analytic function f converges geometrically fast to f on compact subsets of the disc of convergence. For more general open sets U, we will first choose a rational map r whose lemniscate L_r separates K from ∂U, and then choose the maps R_n so that their derivatives, $R_n^{(k)}$, agree with the derivatives $f^{(k)}$ up to some order (depending on n) at the zeros of r. See Section 7 for details. To prove Theorem B, we will only need Theorem A in the case when the lemniscate graph has no vertices, i.e., it consists only of disjoint Jordan curves. We shall see that this case is much easier than the general case of graphs with vertices.

We will also show that the geometric decay in Theorem B is sharp for most functions: if there is a sequence of rational approximations that converge to f faster than geometrically along some subsequence of degrees, then f extends holomorphically to $\widehat{\mathbb{C}} \setminus P$. See Section 8 for the precise statement and proof.

In the case that $\widehat{\mathbb{C}} \setminus K$ is connected, $\infty \in \widehat{\mathbb{C}} \setminus K$, and $P = \{\infty\}$, Theorem B is exactly Theorem I of [WR34]. All other cases of Theorem B (namely, whenever $\widehat{\mathbb{C}} \setminus K$ is not connected) are new, to the best of our knowledge.

1.4. Approximation by Julia Sets.

If a rational map r has exactly two attracting cycles, and if the Fatou set $\mathcal{F}(r)$ is equal to the union of the two corresponding attracting basins, $\mathcal{A}_1$ and $\mathcal{A}_2$, then the sphere decomposes as (here $\sqcup$ denotes disjoint union)

$$\widehat{\mathbb{C}} = \mathcal{A}_1 \sqcup \mathcal{A}_2 \sqcup \mathcal{J}(r),$$

where $\mathcal{J}(r)$ is the Julia set of r, and $\partial\mathcal{A}_1 = \partial\mathcal{A}_2 = \mathcal{J}(r)$. See, for instance, Corollary 4.12 of [Mil06]. Note that each attracting basin is an open set, but need not be connected. Our next result implies that any disjoint pair of open sets sharing a common boundary can be approximated by a pair of attracting basins for some rational map.

Theorem C. *Let $\varepsilon > 0$ and A_1, $A_2 \subset \widehat{\mathbb{C}}$ be open, disjoint sets with common boundary J satisfying $\widehat{\mathbb{C}} = A_1 \sqcup A_2 \sqcup J$. Then there is a hyperbolic rational map r with two attracting* *basins, $\mathcal{A}_1$ and $\mathcal{A}_2$, sharing a common boundary $\mathcal{J}(r)$, satisfying $\widehat{\mathbb{C}} = \mathcal{A}_1 \sqcup \mathcal{A}_2 \sqcup \mathcal{J}(r)$, so that*

$$d_H(A_i, \mathcal{A}_i) < \varepsilon \text{ for } i = 1, 2 \text{ and } d_H(J, \mathcal{J}(r)) < \varepsilon, \tag{1.2}$$

where d_H denotes Hausdorff distance. Moreover, if A_1 has finitely many components, P contains one point from each component of A_1, and $p \in P$, then we may choose r so that $r(p) = p$, $r^{-1}(\infty) = P$ and $\mathcal{A}_1$ is the basin of attraction for p.

In the case when A_1 is connected, contains ∞ and $P = \{\infty\}$, this result is essentially the same as Theorem 1.2 of [LY19] by Lindsey and Younsi. They show that a compact set K not separating the plane can be approximated by the filled Julia set of a polynomial. Moreover, they show that such a polynomial approximation is possible only if the interior of K does not separate the plane; see Theorem 1.4 of [LY19]. Thus when A_1 is not connected in Theorem C, it is necessary to consider rational maps having finite poles.

Lindsey and Younsi give two proofs of their Theorem 1.2. The first argument does not seem to extend to the case that A_1 is disconnected, but their second proof is a short application of Hilbert's Lemniscate Theorem, and replacing Hilbert's result with our Theorem A yields a quick proof of Theorem C. See Section 9 for details. Like Theorem B, the proof of Theorem C only requires Theorem A in the easier case when the lemniscate graph has no vertices.

1.5. **Related Work.** Hilbert's interest in lemniscates seems to have arisen from a study of polynomial approximation [Hil97], where he used his lemniscate theorem to establish the special case of Theorem B in which K has simply-connected complement. Later, Walsh and Russell [WR34] also studied lemniscates in the course of their proof of Theorem B in the case that $K \subset \mathbb{C}$ has a connected complement.

Hilbert's theorem has been generalized to higher complex dimensions by Bloom, Levenberg and Lyubarskii [BLL08], and by Nivoche [Niv09]. Nagy and Totik consider placing a lemniscate between tangent curves in [NT05]. The rate of convergence in Hilbert's theorem has been studied by Andrievskii [And00], [And18], and Kosukhin [Kos05]. Results on generalized lemniscates for resolvents of operators are surveyed by Putinar in [Put05]. Lemniscate approximations via Runge's theorem are given in [Sta75] and [ANV22], and applied to various problems in functional analysis.

There has also been recent interest in rational lemniscates and Hilbert's theorem stemming from work of Ebenfelt, Khavinson, and Shapiro [EKS11], in which they propose coordinates on the space of Jordan curve polynomial lemniscates. By Hilbert's Lemniscate Theorem, this lemniscate space is dense in the larger space of smooth Jordan curves, and this larger space of smooth Jordan curves is the central object of study in the approaches of Kirillov [Kir87], Sharon and Mumford [SM06] and others to computer vision and pattern recognition. The results in [EKS11] led to a study of the conformal properties of lemniscates by Fortier Bourque and Younsi [FBY15], by Younsi [You16], and by Frolova, Khavinson and Vasil'ev [FKV18].

A well-known question of Erdős, Herzog, and Piranian [EHP58] asks what is the maximum length of a lemniscate of a monic polynomial: $z^n - 1$ is conjectured to be the extreme case. This problem is still open, but related results are given by Borwein in [Bor95], by the second author and Hayman in [EH99], and by Nazarov and Fryntov in [FN09]. Other recent work on lemniscates includes a study of the expected length of random rational lemniscates as considered by Lerario and Lundberg in [LL15], [LL16]. Random polynomial lemniscates are considered by Lundberg and Ramachandran [LR17], and by Epstein, Hanin and Lundberg [EHL20]. There has been work on the possible topologies of rational lemniscates, the tree structure of nested components, and the comparison of functions whose level sets are topologically equivalent, e.g., [CP91], [RY19], [Ric16], [RY17], [Ste86]. For a survey of recent results on lemniscates and level sets, see [Ric21].

The geometry of higher dimensional, real polynomial lemniscates is a very active field, e.g. the “polynomial ham sandwich theorem” of Stone and Tukey [ST42] gave rise to the “polynomial method” in discrete geometry and combinatorics, as described by Guth in [Gut13]. Recently, real variable polynomial lemniscates have been used to separate data points in the context of machine learning, e.g., [KLL$^+$] and [MP19].

We also mention again the recent work of Lindsey and Younsi [LY19] explaining the connection between lemniscates and the approximation of continua by polynomial Julia sets, a problem also studied by Lindsey in [Lin15], and by the first author and Pilgrim in [BP15]. Bok-Thaler [BT21] and Marti-Pete, Rempe and Waterman [MRW22] consider analogous problems in the setting of transcendental (i.e., non-polynomial) entire functions.

A slightly different, but related, problem is the study of pullbacks $r^{-1}(\Gamma)$ where Γ is a Jordan curve passing through the critical values of r (in our case, Γ is always a Euclidean circle, and need not contain all the critical values of r). Such pullbacks are called *nets*, and have been studied by the second author and Gabrielov [EG02], Thurston [Thu10], Koch and Lei [KL20], Tomasini [Tom15], and the third author [Laz23].

1.6. **Proof Sketch.** In Section 2, we show that any lemniscate graph is ε-homeomorphic to a graph with analytic edges making equal angles at each vertex. Thus Theorem A is reduced to this case, and this simplifies various arguments. In Section 3 we show that a homeomorphism moving points of a lemniscate graph G less than $\delta = \delta(\epsilon, G)$ can be extended to an ϵ-homeomorphism of the sphere; this fact further simplifies the proof of Theorem A.

In Section 4, we prove Theorem A in the case that G has no vertices (it is a union of disjoint Jordan curves). This case is much easier, but introduces several of the key ideas. Briefly, we consider Green's functions on the grey faces with poles at the points of P (one point per grey face), and note that G can be approximated by the level lines of a function u that is the sum of these Green's functions. This function u can be written as a convolution of the logarithmic kernel with the sum of harmonic measures for the grey faces, and negative point masses at the poles in P. The harmonic measures are then approximated by sums of point masses, and this leads to a rational function r so that $\log |r|$ approximates u away from G; thus L_r approximates the given level line of u, and hence it also approximates G.

The case when G has vertices is more difficult, and requires some new ideas. Given a lemniscate graph G with vertices, we claim that there is a graph H, without vertices, and a corresponding function u as above, so that a certain level set of u is ε-homeomorphic to G. The proof of this claim is postponed to Sections 10 and 11. Assuming the claim holds, we prove Theorem A in Sections 5 and 6: in Section 5 we place the poles of the rational functions as close to P as we wish, and in Section 6 we give a fixed point argument to position them exactly on P. Sections 7 and 9 are devoted to deducing Theorems B and C, and the sharpness of Theorem B is proven in Section 8.

As noted above, the proof of Theorem A is much shorter when G has no vertices, and this special case is sufficient for the proofs of Theorems B and C. The reader who is only interested in this case can skip the proof of Theorem 2.2, and the proofs in Sections 5, 6, 10 and 11; this will omit about half of the paper. We also note that only the first half of Section 11 is needed when all vertices of G have degree four.

1.7. **Notation.** We will generally use boldfaced symbols for vectors, e.g., $\boldsymbol{x} = (x_1, \ldots, x_n)$ or $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$. An open disk of radius r centered at z will be denoted by $D(z, r)$. When A and B are both quantities that depend on a common parameter, then we use the usual notation $A = O(B)$ to mean that the ratio A/B is bounded independent of the parameter. We write $A \simeq B$ if both $A = O(B)$ and $B = O(A)$. The notation $A = o(B)$ means $A/B \to 0$ as the parameter tends to infinity. We will use G to denote general lemniscate graphs and H to denote lemniscate graphs without vertices (finite unions of disjoint closed Jordan curves). A G with a subscript, such as G_W, will denote the Green's function for the domain W, and we define such functions to be zero off of W. In general, we use $A := B$ to mean that A is being defined in terms of B, and $A = B$ to show equality between two (already) defined quantities.

2. From topological graphs to analytic graphs

In the introduction, we defined lemniscate graphs so that the edges are just Jordan arcs; no smoothness is assumed. However, we will show every topological graph is ε-homeomorphic (for every $\varepsilon > 0$) to an analytic version of itself, that is, a graph with analytic edges. This allows us to reduce the proof of Theorem A to the case when the lemniscate graph has smooth edges, and this simplifies some of the arguments. We can think of this as a "stepping stone" to the main result of this paper, that every lemniscate graph is ε-homeomorphic to an algebraic version of itself, i.e., a rational lemniscate.

By an analytic edge e, we mean the image of the line segment $[0, 1]$ under a locally injective holomorphic map defined on some neighborhood of $[0, 1]$ (the map is conformal on a neighborhood of $[0, 1]$ if the endpoints of e are distinct). In particular, such an edge is a subset of a slightly longer analytic curve (the image of a longer segment $[-\varepsilon, 1+\varepsilon]$ under the same map), and hence it has a well defined direction at each endpoint, and it has uniformly bounded curvature.

Theorem 2.1. *Suppose that H is a closed Jordan curve. For any $\varepsilon > 0$, H is ε-homeomorphic to an analytic Jordan curve, and the homeomorphism may be taken to be the identity outside an ε-neighborhood of H.*

Theorem 2.2. *Suppose that G is a connected lemniscate graph. Then for any $\varepsilon > 0$, G is ε-homeomorphic to a lemniscate graph whose edges are analytic, and so that the edges meeting at any vertex form equal angles. The homeomorphism may be taken to be the identity outside an ε-neighborhood of G.*

Clearly, the first result is a special case of the second, but in the proof it is convenient to first deal with the case when there are no vertices (and Theorem 2.2 is not needed for the proofs of Theorems B and C).

For disconnected graphs, we can apply one of these results to each of the connected components. If ε is less than half the minimal distance between components, then the composition of these maps gives an ε-homeomorphism of the entire graph to an analytic version of itself (each map in the composition only moves points that all the other maps fix). Our proof gives a new edge e' that is disjoint from the edge e it replaces (except for the endpoints), and can place it inside either of the two faces of G bounded by e.

Proof of Theorem 2.1. Let $\phi : \mathbb{D} \to \Omega$ denote a Riemann map onto one of the complementary components of H. The map ϕ extends continuously to a map $\phi : \mathbb{T} \to H$, and this extension further extends to a homeomorphism $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by the Jordan-Schönflies theorem. Let ρ_δ be an increasing homeomorphism from $I_\delta = [1 - 2\delta, 1 + 2\delta]$ to itself so that $\rho_\delta(1) = 1 - \delta$. Since ρ_δ is increasing, it fixes both endpoints of I_δ, and we can extend ρ_δ to the complex plane by $\rho_\delta(z) = \rho_\delta(|z|)z/|z|$ if $1 - 2\delta < |z| < 1 + 2\delta$, and as the identity otherwise. We claim that the homeomorphism

$$\psi_\delta := \phi \circ \rho_\delta \circ \phi^{-1} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$$

satisfies the conclusions of the theorem if $\delta > 0$ is small enough. Indeed, we have $\psi_\delta(H) = \phi((1 - \delta)\mathbb{T})$ is an analytic Jordan curve, and

$$\operatorname{dist}(\psi_\delta(z), z) = \operatorname{dist}(\phi(\rho_\delta(\phi^{-1}(z))), z) \text{ for all } z \in \widehat{\mathbb{C}}, \tag{2.1}$$

where we use dist to denote spherical distance. Since ρ_δ tends to the identity as $\delta \searrow 0$ and ϕ is uniformly continuous on $\widehat{\mathbb{C}}$ (since ϕ is continuous on the compact set $\widehat{\mathbb{C}}$), we conclude that ψ_δ is an ε-homeomorphism for δ close enough to 0. $\square$

To prove Theorem 2.2, we first need to modify the graph near each vertex.

Lemma 2.3. *Let G' be a lemniscate graph and $\varepsilon > 0$. Then there exists an ε-homeomorphism of G' onto a lemniscate graph G having the property that in a neighborhood of each vertex of degree $2n$, the graph G is a union of $2n$ straight line segments each terminating at v,* *and making equal angles. The homeomorphism may be taken to be the identity outside an ε-neighborhood of G'.*

Proof. Let v be a vertex of G' of degree $2n$, and D a Jordan domain containing v so that G' intersects ∂D at exactly $2n$ points. We will define G by replacing G' in D with $2n$ Jordan arcs which start at the points $G' \cap \partial D$, and in a small ball around v consist of straight lines terminating at v and making equal angles. Doing this for each vertex v of G' defines the graph G. Suppose each D has diameter $< \varepsilon$. Define a homeomorphism as the identity outside of each D, and inside each D by first by mapping the Jordan arcs $G' \cap D$ onto the corresponding Jordan arcs in $G \cap D$, and then extending by the Jordan-Schönflies theorem throughout D (e.g., Corollary 12.15 in [Mar19]). The result is an ε-homeomorphism of $\widehat{\mathbb{C}}$ mapping G' to G. $\square$

Proof of Theorem 2.2. We may assume that G has the form given in Lemma 2.3, but that G is not a Jordan curve, since that case is covered by Theorem 2.1. Thus G has vertices and every vertex has even degree at least four. It suffices to replace each edge e of G by a new edge that is analytic (in the sense discussed earlier) and tangent to e at their common endpoints. There are two cases, depending on whether the endpoints of e are two distinct points (e is an arc) or a single point (e is a loop).

For any edge e in a lemniscate graph G, we claim that there is a Jordan curve $\gamma \subset G$ containing e. We can prove this by forming a directed path in G that starts with the edge e (choose either orientation). Since every vertex has even degree the path can be continued until it returns to the initial vertex of e. For any vertex v on this path, we erase the sub-path between the first and last visits to v. This gives a Jordan curve γ containing e. If e has distinct endpoints, then γ strictly contains e. If both ends of e are the same point, then we can take $\gamma = e$, but for the proof below, it will be more convenient to consider a different curve. If we remove e from G, we still have an Eulerian graph, so there is a Jordan curve γ' in G containing v but not e. Then $\gamma = e \cup \gamma'$ is a "figure 8" contained in G.

First assume the endpoints of e are two different points. Fix an endpoint v of e. The curve γ contains two line segments, s_1 and s_2, each with endpoint v. If these two segments lie on the same line, then their union is a line segment with v in its interior. In this case, we do nothing. If the two segments are not on the same line, then we modify γ by extending the segment s_1 a short distance past v, and then connect the end of this extended segment by a circular arc (centered at v) to an interior point of s_2. See Figure 2. This gives a new Jordan curve that contains a segment with midpoint v, and that equals γ outside a small neighborhood of v. Do this replacement, if necessary, at both endpoints of e, and call the new curve γ also.

Let Ω denote a component of $\widehat{\mathbb{C}} \setminus \gamma$ (it does not matter which component we pick). Let $\phi : \mathbb{D} \to \Omega$ denote a Riemann map. Since $\partial\Omega$ is locally connected, ϕ extends to a homeomorphism $\phi : \mathbb{T} \to \gamma$. We may choose ϕ so that the points -1 and 1 map to the two endpoints of e, and so that $e = \phi(\Gamma_0)$, where Γ_0 is the upper half-circle, $\Gamma_0 = \mathbb{T} \cap \mathbb{H}$. Because γ contains

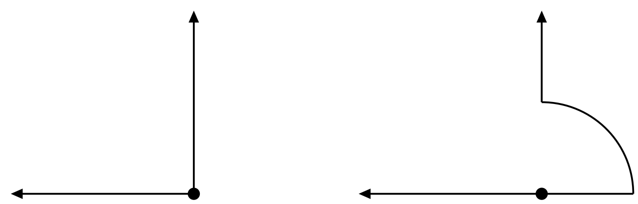

FIGURE 2. We modify γ near each vertex v so it contains a line segment centered at v.

line segments centered at each of the endpoints of e, there is some $\eta > 0$ so that ϕ extends analytically to a η-neighborhood around each of -1 and 1 by Schwarz reflection. Extend ϕ from $\mathbb{D} \cup D(-1, \eta) \cup D(1, \eta)$ to a homeomorphism $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by the Jordan-Schönflies Theorem.

For $|\delta|$ small, consider the rational map $R_\delta(z) = (z + \delta/z)/(1 + \delta)$. On the unit circle $1/z = \overline{z}$, so

$$R(x + iy) = (x + iy + \delta(x - iy))/(1 + \delta) = x + iy/\mu,$$

where $\mu = (1 + \delta)/(1 - \delta)$. Thus R maps the unit circle onto the ellipse

$$E_\delta = \{(x, y) \in \mathbb{R}^2 : x^2 + \mu^2 y^2 = 1\}.$$

When $\delta \in (0, 1)$, we have $\mu > 1$, which implies this ellipse is contained in the open unit disk, except at the points $-1, 1$ where it is tangent to the unit circle. Thus ϕ is analytic on a neighborhood of E_δ and $\phi(E_\delta)$ is an analytic closed curve that is tangent to e at both of its endpoints. We define the replacement edge to be $e' = \phi(\Gamma_\delta)$, where $\Gamma_\delta = E_\delta \cap \mathbb{H}$ is the upper half of E_δ.

Let G' be the lemniscate graph obtained from G by replacing the edge e by e'. We want to show these two graphs are ε-homeomorphic by a map that is the identity off a neighborhood of e (recall that edges are open arcs and do not contain their endpoints). As above, let $\Gamma_t = E_t \cap \mathbb{H} = R_t(\Gamma_0)$, and let U_δ be the union of Γ_t for $t \in I = [-2\delta, 2\delta]$. See Figure 3. For $z \in U_\delta$, define $t(z) = s$ if $z \in \Gamma_s$. Let $\rho : I \to I$ be a homeomorphism that fixes each endpoint and satisfies $\rho(0) = \delta$. Then $h_\delta(z) = R_{\rho(t(z))}(R^{-1}_{t(z)}(z))$ is a homeomorphism of U_δ that is the identity on ∂U_δ and maps Γ_0 to Γ_δ. It moves points in U_δ by at most $O(\delta)$. Set $W_\delta := \phi(U_\delta)$ and define $\psi_\delta(z) = \phi(h_\delta(\phi^{-1}(z)))$ if $z \in W_\delta$ and $\psi(z) = z$ elsewhere. For δ small enough, ψ_δ is a ε-homeomorphism of the sphere taking G to G', and it is the identity on every edge of G except e.

Next, suppose the endpoints of e are the same point v. This case can be reduced to the previous one. See Figures 4 and 5. As noted earlier, in this case e is part of a "figure 8" curve $\gamma = e \cup \gamma' \subset G$. The curve γ has three complementary components: one with boundary e, one with boundary γ', and a third component Ω with boundary γ. Choose points a and b in the first two, and let $\tau(z) = (z - a)/(z - b)$; this is a linear fractional transformation sending a to 0 and b to ∞. Define $f(z) = \tau^{-1}(\tau(z)^2)$ (we could write the formula explicitly, but we

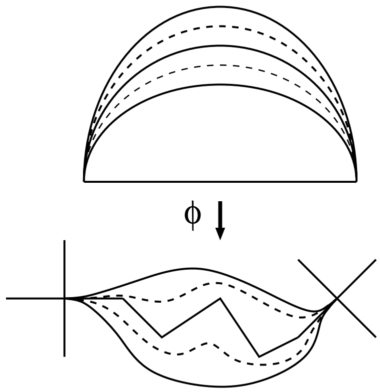

FIGURE 3. We define a neighborhood U of the upper half circle as a union of elliptical arcs, and define a homeomorphism of U to itself by shifting the arcs. This is the identity outside U. When conjugated by ϕ this becomes an ε-homeomorphism of G to G' taking e to e', that is the identity on $G \setminus e$.

don't need it). This is rational map of degree 2, and we can define a branch of f^{-1} on Ω by taking a branch of $z^{1/2}$ on $\tau(\Omega)$ (which is simply connected and omits both 0 and ∞). Then f^{-1} has two different values at v and it maps Ω to a Jordan region Ω', and v corresponds to two points on the boundary of Ω'. See Figure 4.

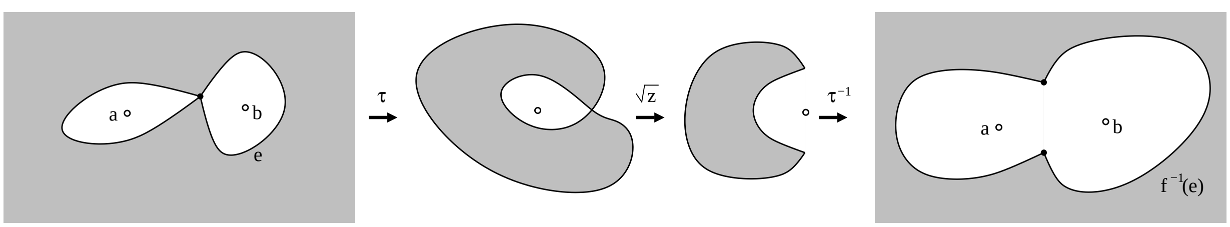

FIGURE 4. Using linear fractional transformations and a square root, we can map the non-Jordan face of γ to a Jordan domain. The inverse of this map is a rational map.

We repeat the earlier construction for Jordan faces given above to find an analytic curve σ that approximates $f^{-1}(e)$, and that is tangent to $f^{-1}(e)$ at its endpoints. Then apply the rational map f to σ and obtain an analytic curve e' approximating e. See Figure 5.

Thus we may proceed edge by edge, replacing edges which have not previously been replaced, and leaving previously altered edges fixed. Each graph we obtain is ε-homeomorphic to the previous one, and since at most one of these homeomorphisms moves any given point of $\widehat{\mathbb{C}}$, the composition is an ε-homeomorphism of $\widehat{\mathbb{C}}$, as desired. $\square$

The following result was promised in the introduction.

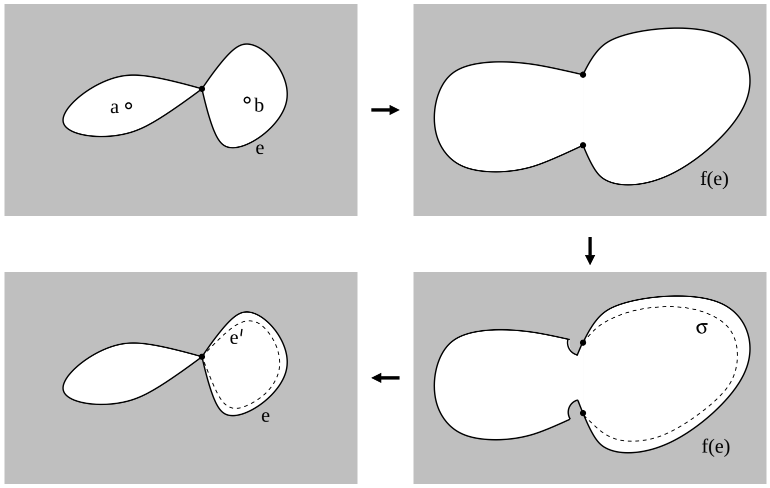

FIGURE 5. If both endpoints of e are the same, we can find a "figure 8" curve γ containing e and use a branch of f^{-1} to map the non-Jordan face of γ to a Jordan domain. Then the earlier construction gives an analytic approximation to $f(e)$ and applying the the rational map f gives the an analytic approximation to e.

Proposition 2.4. *Let L be a rational lemniscate. Then L is a lemniscate graph.*

Proof. We will show L satisfies Definition 1.2. We set $V := \{z \in L_r : r'(z) = 0\}$. Recall the local normal form for holomorphic maps (see for instance Theorem 1.59 in [Zak21]); namely that for any $z_0 \in \mathbb{C}$, up to holomorphic changes of coordinates in the domain and co-domain, r is locally given by $z \mapsto z^n$ near z_0, and $n = 1$ if and only if $r'(z) \neq 0$. Thus each component of $L_r \setminus V$ is a connected 1-manifold. The only connected 1-manifolds are (closed) Jordan curves or (open) simple arcs (see for instance [Mil97]). This shows that L satisfies condition (1) in Definition 1.2. The fact that L satisfies condition (2) in Definition 1.2 also follows from the local normal form for holomorphic mappings near critical points. $\square$

3. Extending ε-homeomorphisms

Next, we record a fact that we will use several times later in the paper.

Lemma 3.1. *Let G be a lemniscate graph and $\varepsilon > 0$. Then there exists $\delta > 0$ so that if $h : G \to G'$ is any homeomorphism satisfying $\sup_{z \in G} |h(z) - z| < \delta$, then h admits an ε-homeomorphic extension $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.*

This fact is probably well known, but we are not aware of a reference, so we give a proof for completeness. The main difficulty is not that the extension exists, but that it can be chosen to move points very little.

Definition 3.2. We will call a domain Ω a *lemniscate domain* iff Ω is simply connected and $\partial\Omega$ is a lemniscate graph.

Lemniscate domains include Jordan domains, but also include non-Jordan domains such as the unbounded complementary component of a figure-8 curve in the plane. Lemniscate graphs are locally connected, and so Carathéodory's Theorem implies that any Riemann map $\phi : \mathbb{D} \to \Omega$ onto a lemniscate domain Ω extends continuously to $\mathbb{T}$.

Lemma 3.3. *Let Ω be a lemniscate domain, $z_0 \in \Omega$ and set $\gamma := \partial\Omega$. Assume $(\delta_n)_{n=1}^{\infty}$ is a positive sequence converging to* 0*, and $h_n : \gamma \to h_n(\gamma)$ is a sequence of δ_n-homeomorphisms. Let Ω_n be the component of $\widehat{\mathbb{C}} \setminus h_n(\gamma)$ containing z_0. Denote by $\phi : \mathbb{D} \to \Omega$, $\phi_n : \mathbb{D} \to \Omega_n$ the Riemann maps normalized to map* 0 *to z_0 and have positive derivative at* 0*. Then $\phi_n \to \phi$ uniformly on $\overline{\mathbb{D}}$.*

Proof. It is straightforward to check that the domains Ω_n converge to Ω in the *Carathéodory kernel* sense (see Section 1.4 of [Pom92]), and so Carathéodory's convergence theorem implies that $\phi_n \to \phi$ uniformly on compact subsets of $\mathbb{D}$; in particular $\phi_n \to \phi$ pointwise on $\mathbb{D}$.

Thus, by Corollary 2.4 of [Pom92], in order to conclude that $\phi_n \to \phi$ uniformly on $\overline{\mathbb{D}}$, it suffices to check that the sets $\widehat{\mathbb{C}} \setminus \Omega_n$ are *uniformly locally connected* (see Section 2.2 of [Pom92]); that is, it suffices to check that for all $\varepsilon > 0$, there exists $\delta > 0$ (independent of n) so that any two points a, $b \in \widehat{\mathbb{C}} \setminus \Omega_n$ satisfying $|a - b| < \delta$ can be joined by a continuum $K \subset \widehat{\mathbb{C}} \setminus \Omega_n$ of diameter $< \varepsilon$.

Fix $\varepsilon > 0$. Since $\gamma = \partial\Omega$ is a lemniscate graph, γ is locally connected and hence there exists $\delta_\gamma > 0$ so that any two points a, $b \in \gamma$ satisfying $|a - b| < \delta_\gamma$ can be joined by a continuum $K \subset \gamma$ of diameter $< \varepsilon/2$. Set $\delta = \delta_\gamma/2$. Let $z, w \in h_n(\gamma)$. For n large enough so that $\delta_n < \delta_\gamma/4$, we have that $|z - w| < \delta$ implies that $a = h_n^{-1}(z)$, $b = h_n^{-1}(w) \in \gamma$ satisfy $|a - b| < \delta_\gamma$. Thus (by our choice of δ_γ) there is a continuum $K \subset \gamma$ of diameter $< \varepsilon/2$ joining a, b, and so $h_n(K) \subset h_n(\gamma)$ is a continuum of diameter $< \varepsilon/2 + 2\delta_n$ joining z, w. Since $\varepsilon/2 + 2\delta_n < \varepsilon$ for large n, we have demonstrated that the $(h_n(\gamma))_{n=1}^{\infty}$ are uniformly locally connected. The $(h_n(\gamma))_{n=1}^{\infty}$ being uniformly locally connected implies, in turn, that the $\widehat{\mathbb{C}} \setminus \Omega_n$ are uniformly locally connected (see Theorem 2.1 of [Pom92]). □

Corollary 3.4. *Let Ω be a lemniscate domain, $z_0 \in \Omega$, set $\gamma := \partial\Omega$ and let $\varepsilon > 0$. Then there exists a $\delta > 0$ so that if $h : \gamma \to h(\gamma)$ is a δ-homeomorphism, and Ω_h is the component of $\widehat{\mathbb{C}} \setminus h(\gamma)$ containing z_0, then the Riemann maps $\phi_h : \mathbb{D} \to \Omega_h$, $\phi : \mathbb{D} \to \Omega$ (normalized as in Lemma 3.3) satisfy* $\sup_{z\in\overline{\mathbb{D}}} |\phi(z) - \phi_h(z)| < \varepsilon$.

Proof. If there existed Ω, z_0, ε for which the corollary failed, then extracting sequences h_n, ϕ_n from the counterexamples arising from $\delta_n := 1/n$ would contradict Lemma 3.3. □

Lemma 3.5. *Let Ω be a lemniscate domain with $\gamma := \partial\Omega$, $\varepsilon > 0$ and U a neighborhood of γ. Then there exists $\delta > 0$ so that any δ-homeomorphism $h : \gamma \to h(\gamma)$ admits an ε-homeomorphic extension $h : \overline{\Omega} \to \overline{\Omega_h}$ satisfying $h(z) = z$ for $z \notin U$, where Ω_h denotes the component of $\widehat{\mathbb{C}} \setminus h(\gamma)$ containing z_0.*

Proof. Fix $z_0 \in \Omega$ and $\varepsilon > 0$. By the Riemann Mapping Theorem, there exists a conformal map $\phi : \mathbb{D} \to \Omega$ satisfying $\phi(0) = z_0$ and $\phi'(0) > 0$. For a homeomorphism $h : \gamma \to h(\gamma)$, let $\phi_h : \mathbb{D} \to \Omega_h$ denote a conformal map also normalized so that $\phi_h(0) = z_0$, $\phi_h'(0) > 0$.

Consider the following composition (for now defined only in a formal sense, not as a mapping):

$$f := \phi_h^{-1} \circ h \circ \phi. \tag{3.1}$$

The map ϕ_h is not injective on $\mathbb{T}$ whenever γ (and hence $h(\gamma)$) has at least one vertex; thus ϕ_h^{-1} may be multi-valued on $h(\gamma)$. We claim that the expression (3.1) nevertheless gives a well-defined homeomorphism $f : \mathbb{T} \to \mathbb{T}$ for δ small enough. Indeed, for δ small, we have that v is a vertex of γ having n accesses from z_0 (within Ω) if and only if $h(v)$ is a vertex of $h(\gamma)$ having n accesses from z_0 (within Ω_h). Moreover, the degrees of the vertices occur in the same order counterclockwise around γ, $h(\gamma)$ (as seen from z_0). Thus, for δ small, the mapping f is a homeomorphism off of the ϕ-images of the vertices of γ, and extends continuously to a homeomorphism $f : \mathbb{T} \to \mathbb{T}$.

For all $\eta > 0$, Corollary 3.4 implies that there exists a $\delta > 0$ so that if h is a δ-homeomorphism of γ, then

$$\sup_{z \in \overline{\mathbb{D}}} |\phi(z) - \phi_h(z)| < \eta. \tag{3.2}$$

Thus, for all $\eta > 0$, there exists $\delta > 0$ so that if h is a δ-homeomorphism of γ, then the homeomorphism

$$f := \phi_h^{-1} \circ h \circ \phi : \mathbb{T} \to \mathbb{T}$$

is an η-homeomorphism.

Let $r < 1$ satisfy that $\phi(r\mathbb{T}) \subset U$. We claim that $f : \mathbb{T} \to \mathbb{T}$ extends to a homeomorphism $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ satisfying $f(z) = \phi_h^{-1} \circ \phi(z)$ for $z \in r\mathbb{D}$. In order to verify this, we need to define f on the annulus $\{z : r < |z| < 1\}$. Since $\phi_h^{-1} \circ \phi$ is uniformly close to the identity, the Jordan curve $\phi_h^{-1} \circ \phi(r\mathbb{T})$ intersects any radial line $\{z : \arg(z) = \theta\}$ in at most one point. Thus, we may define a homeomorphism $\mathcal{R}$ of the annulus with boundary components $\mathbb{T}$, $\phi_h^{-1} \circ \phi(r\mathbb{T})$ onto the annulus $\{z : r < |r| < 1\}$ by specifying $\mathcal{R}$ preserves (set-wise) radial lines, $\mathcal{R}$ is the identity on $\mathbb{T}$, and $\mathcal{R}$ maps each point $\zeta \in \phi_h^{-1} \circ \phi(r\mathbb{T})$ to the point on $r\mathbb{T}$ having the same argument as ζ. We can interpolate between $f : \mathbb{T} \to \mathbb{T}$ and $\mathcal{R} \circ \phi_h^{-1} \circ \phi : r\mathbb{T} \to r\mathbb{T}$ to give a homeomorphism we call $g : \{z : r < |z| < 1\} \to \{z : r < |z| < 1\}$ by using the interpolation which is linear in logarithmic coordinates (i.e. when g is lifted by the exponential to give a self-map of a horizontal strip, this lift maps straight line segments to straight line segments). The homeomorphism g is close to the identity as long as h is close to the identity. Then $f := \mathcal{R}^{-1} \circ g$ gives the desired homeomorphic interpolation between $f : \mathbb{T} \to \mathbb{T}$ and $\phi_h^{-1} \circ \phi : r\mathbb{T} \to \phi_h^{-1} \circ \phi(r\mathbb{T})$, and f is close to the identity as long as δ is small.

Thus, for δ small enough (depending on ε, ϕ), if h is any δ-homeomorphism of γ, the map

$$\phi_h \circ f \circ \phi^{-1} : \overline{\Omega} \to \overline{\Omega_h}$$

is an ε-homeomorphic extension of $h : \gamma \to h(\gamma)$, which is the identity on $\phi(r\mathbb{D})$. By our choice of r, this means h is the identity outside of U. □

Proof of Lemma 3.1. For each component E of G, take a neighborhood U_E of E which is disjoint from all other components of G. Then by Lemma 3.5, there exists a $\delta > 0$ (depending on E, U_E, and ε) so that if $h|_E$ is a δ-homeomorphism, then $h|_E$ extends to an ε-homeomorphism of $\widehat{\mathbb{C}}$ which is the identity outside of U. Taking δ to be the smallest value that works for all components G, we see that the desired extension of any δ-homeomorphism h may be defined piecewise in a neighborhood of each component of G and the identity outside of these neighborhoods; this gives the desired ε-homeomorphic extension of h. □

4. Approximating Graphs Without Vertices

We are now ready to start the proof of our main result, Theorem A. The results in Section 2 show that it suffices to consider lemniscate graphs that have analytic edges, and form equal angles at each vertex. In this section, we prove Theorem A for lemniscate graphs with no vertices, so it suffices to assume H is a union of pairwise disjoint analytic Jordan curves. This special case is sufficient for the proofs of Theorems B and C (and its proof only uses Theorem 2.1, not Theorem 2.2).

We recall the convention that general lemniscate graphs (possibly with vertices) will be denoted by G, and lemniscate graphs without vertices will be denoted by H.

For any lemniscate graph G, the boundary of each face is a finite union of non-trivial continua, so each face is regular for the Dirichlet problem. Thus for each face B and each point $p \in B$, we can define the harmonic measure $\omega_p = \omega(\cdot, p, B)$ with base point p. This is a probability measure on ∂B that satisfies

$$u(p) = \int_{\partial B} f(\zeta) d\omega_p(\zeta),$$

where u is the harmonic extension of $f \in C(\partial B)$ to B (e.g., on the unit disk harmonic measure is given by the Poisson kernel). For a bounded face B of G, the Green's function for B with pole at p is defined as

$$G_B(z, p) = \int \log |\zeta - p| d\omega_z(\zeta) - \log |z - p|, \text{ for } z \in B, \tag{4.1}$$

and we set it to zero outside B, by convention. This is the unique harmonic function on $B \setminus \{p\}$ that vanishes on ∂B and has a logarithmic pole at p. For the basic properties of harmonic measure and Green's function see, e.g., Chapters II and III of [GM08].

Notation 4.1. Suppose that H is a lemniscate graph without vertices, and that $P \subset \widehat{\mathbb{C}}$ is a finite set that contains at least one point in each grey face of H. If B is a grey face of H, we set $P_B := P \cap B$. For a grey face B of H, we define the signed measure $\mu_B := \sum_{p \in P_B} (\omega_p - \delta_p)$ where (as above) ω_p is harmonic measure for B with base point p and and δ_p is a unit mass at p. Note that μ_B has total mass zero.

For lemniscate graphs H without vertices, the proof of Theorem A will only require considering sets of poles that have one element in each grey face of H. However, we will need to consider multiple poles per face when proving Theorem A for graphs with vertices (see Section 5), so we allow this possibility here. Throughout this section, we fix a lemniscate graph H without vertices, a 2-coloring of the faces of H, and a (non-empty) finite set of points $P_B \subset B$ for each grey face B of H. We will assume ∞ is contained in a white face of H. Set $P := \cup_B P_B$. For $p \in P$, let $B(p)$ denote the face of H containing p.

Notation 4.2. Given H and P, define a $[0, \infty]$-valued function

$$u(z) = u_{H,P}(z) := \sum_{p \in P} G_{B(p)}(z, p)$$

for $z \in \widehat{\mathbb{C}}$, where $G_{B(p)}(z, p)$ denotes the Green's function for $B(p)$ with pole at p. As above, we set $G_{B(p)}(z, p) = 0$ for $z \notin B(p)$.

The following two formulas are immediate from the definitions above.

Proposition 4.3. *For each grey face B of H, and all $z \in \widehat{\mathbb{C}}$,*

$$\sum_{p \in P_B} G_B(z, p) = \int_{\widehat{\mathbb{C}}} \log|z - \zeta| d\mu_B(\zeta). \tag{4.2}$$

For all $z \in \widehat{\mathbb{C}}$,

$$u_{H,P}(z) = \sum_B \int_{\widehat{\mathbb{C}}} \log|z - \zeta| d\mu_B(\zeta). \tag{4.3}$$

Next, we approximate harmonic measure by δ-masses at points $(\zeta_j^B)_{j=1}^m \in \partial B$.

Definition 4.4. First consider the case that ∂B consists of a single Jordan curve. We fix some large $m \in \mathbb{N}$ and a point $\zeta_1^B \in \partial B$, and for $j > 1$ we define points $\{\zeta_j^B\}_2^m$ so that the segment $I_j \subset \partial B$ from ζ_{j-1}^B to ζ_j^B oriented positively with respect to B, satisfies

$$\sum_{p \in P_B} \omega(I_j, p, B) = \frac{|P_B|}{m},$$

where $|P_B|$ denotes the number of elements in the set P_B.

When ∂B has more than one component, we follow a similar procedure, now placing on each component γ of ∂B either $\lfloor m \cdot \sum_{p \in P_B} \omega(\gamma, B, p) \rfloor$ points or $\lfloor m \cdot \sum_{p \in P_B} \omega(\gamma, B, p) \rfloor + 1$

points, so that each edge connecting two adjacent points on γ has measure $= |P_B|/m$, except, possibly, for one edge which has measure $< |P_B|/m$. Define $\{\omega_m^B\}$ by placing mass $|P_N|/m$ at each point constructed above, and define $\mu_m^B = \omega_m^B - \sum_{p\in P_B}\delta_p$.

We claim that the measures $\{\omega_m^B\}$ converge weak-$*$ to $\sum_{P\in P_B}\omega_p$. This follows because harmonic measure is non-atomic, i.e., single points always have harmonic measure zero (this is true for general domains in $\mathbb{R}^n$, $n \geq 2$, but we only need it for finitely connected domains in the plane). Because of this, the maximum size δ_m of the arcs I connecting adjacent points in Definition 4.4 tends to zero as m tends to infinity. Any continuous function g on the graph G is uniformly continuous and hence $|x-y| \leq \delta_m$ implies $|g(x)-g(y)| \leq \epsilon_m$ for some sequence ϵ_n tending to zero. Thus

$$\left| \int g \sum_{P\in P_B} d\omega_p - \int g d\omega_m^B \right| \leq \epsilon_m$$

for any continuous function g, which is the definition of weak convergence of measures.

For future reference, we note that this convergence also holds for $g(z) = \log|z-\zeta|$ when $\zeta \in G$ but $z \notin G$, since g is uniformly continuous outside any neighborhood of the pole. Moreover, if $z \in K$ for some compact set K disjoint from G, then we have a uniform modulus bound for the family $\{\log|z-\zeta|\}_{z\in K,\zeta\in G}$ depending only on $\mathrm{dist}(K,G)$. Thus

$$\int \log|z-\zeta| d\omega_m^B(\zeta) \to \int \log|z-\zeta| \sum_{P\in P_B} d\omega_p(\zeta),$$

uniformly on K.

Definition 4.5. Given the points $\{\zeta_j^B\}_{j=1}^m$ from Definition 4.4, define the rational function

$$r_m(z) := \frac{\prod_{j,B}(z-\zeta_j^B)^{|P_B|}}{\prod_{p\in P}(z-p)^m}, \tag{4.4}$$

where the product in the numerator is over all grey faces B and $1 \leq j \leq m$, and the product in the denominator is over $P = \cup_B P_B$. Set

$$u_m(z) := \frac{1}{m}\log|r_m(z)|. \tag{4.5}$$

Proposition 4.6. *For all $m \in \mathbb{N}$ and all $c \in \mathbb{R}$, we have $r_m^{-1}(e^{mc}\mathbb{T}) = u_m^{-1}(c)$.*

Proof. This is easy since $|r_m| = e^{cm}$ implies $u_m = \frac{1}{m}\log|r_m| = \frac{1}{m}\log e^{cm} = c$. $\square$

Theorem 4.7. *The sequence $(u_m)_{m=1}^\infty$ converges uniformly to $u_{H,P}$ on compact subsets of $\widehat{\mathbb{C}} \setminus (H \cup P)$.*

Proof. Recall from Notation 4.2 that $u_{H,P}$ denotes the sum over P of the Green's functions $G_{B(p)}(x,p)$. Also recall that we defined $\mu_m^B = \omega_m^B - \sum_{p\in P_B}\delta_p$; this is the discrete measure with mass -1 at each $p \in P_B$, and mass $|P_B|/m$ at each point ζ_j^B which lies on ∂B. A computation using this definition shows that

$$\begin{aligned} u_m(z) &= \sum_B \left(\sum_{p \in P_B} \log \left| \frac{1}{z-p} \right| + \sum_{j=1}^m \frac{|P_B|}{m} \log \left| z - \zeta_j^B \right| \right) \\ &= \sum_B \left(\int_{\widehat{\mathbb{C}}} \log |z - \zeta| d\mu_m^B(\zeta) \right). \end{aligned}$$

Our remarks following Definition 4.4 imply the measures $(\omega_m^B)_{m=1}^\infty$ converge weak-$*$ to the measure μ_B for each B. This fact and Equation (4.3) imply the theorem. $\square$

Proof of Theorem A for lemniscate graphs with no vertices. Theorem 2.1 says that for any $\varepsilon > 0$, H is ε-homeomorphic to a union of analytic closed curves, so without loss of generality, we may assume H has this form. Under this assumption, the function $u = u_{H,p}$ has an analytic extension across the boundary of each face, and these extensions have non-zero gradients on the boundary, since u is a sum of Green's functions, each of which have positive inward pointing normal derivative. Hence the gradient of u is non-zero in a neighborhood of the boundary of each face. (But note that the analytic extension of u across the boundary of a face does not equal u on the adjacent face; u is always non-negative, but the analytic extension becomes negative when we cross the boundary.)

Thus for $c > 0$ small enough, the level set $H_c = \{z : u(z) = c\}$ has n components that are Jordan curves approximating the n components of H. For such c, there is a homeomorphism from H_c to H given by the steepest descent curves of u (i.e., following the vector field $-\nabla u$), and we may assume maximum diameter of these connecting curves is as small as we wish, by taking c small enough. Thus by Corollary 3.1, the level set H_c and H are ε-homeomorphic if $c > 0$ is small enough.

For $0 < s < t$ let $H_{s,t} = \{z : s \leq u_{H,P}(z) \leq t\}$. As $m \nearrow \infty$, the functions u_m uniformly approximate u on the compact set $H_{c/4,4c}$. Hence for m large enough, the level set $H_c^m = \{z : u_m(z) = c\}$ lies inside $H_{c/2,2c}$. Suppose δ is the distance from $H_{c/2,2c}$ to the complement of $H_{c/4,4c}$. Since the functions u_m are harmonic, the uniform convergence of $u_m \to u$ on a δ-neighborhood of $H_{c/2,2c}$ implies the gradients of u_m converge uniformly to the gradient of u on $H_{c/2,2c}$. Hence for large enough m, we have

$$\sup |\nabla u_m - \nabla u| \leq \frac{1}{2} \inf |\nabla u|,$$

where both the supremum and infimum are take over the set $H_{c/2,2c}$. This inequality implies ∇u_m is never zero and is never perpendicular to ∇u anywhere on $H_{c/2,2c}$. Thus following the gradient line of u through a point $z \in H_c$ will reach a unique point of H_c^m before leaving $H_{c/2,2c}$. This defines a homeomorphism $H_c \to H_c^m$. Hence by Corollary 3.1, H_c and H_c^m are ε-homeomorphic if m is large enough, and hence H_c^m is 2ε-homeomorphic to H. $\square$

5. Graphs With Vertices: Approximate Pole Placement

In this section, we will show that any lemniscate graph is ε-homeomorphic to a rational lemniscate whose poles can be prescribed with error at most ε. The fixed point argument that proves we can place the poles exactly is given in the next section.

The proof for lemniscate graphs with no vertices (given in the previous section) is easier than the general case because we were free to choose any small enough value $c > 0$ to define a level line. All small enough choices gave level sets whose components were Jordan curves, and thus they automatically have the same topology as the components of H.

In the general case, the union of disjoint curves H is replaced by a lemniscate graph G that can have vertices, and to mimic this graph with level sets of a harmonic function u requires the level sets to run through critical values of u, of which there are only finitely many. Moreover, we need to use the same critical value in every face. Thus only very particular values of c will work, and only after we have made delicate adjustments to the shape of the lemniscate G. The description of these adjustments is delayed to Section 11, but we now state the needed result and finish the proof of Theorem A using it. Recall from Notation 4.2 that the function $u_{H,P}$ is a sum of Green's functions with poles in the set P inside the grey faces (with possibly more than one pole per face).

Theorem 5.1. *Let G be a lemniscate graph, and fix $\varepsilon > 0$. Then there exists a lemniscate graph H without vertices and $\delta > 0$ so that each grey face of G is contained in a grey face of H, and so that $H_\delta = u_{H,P}^{-1}(\delta)$ and G are ε-homeomorphic to each other.*

See Figure 6. In Section 11, H is constructed by modifying G in a small neighborhood of each vertex, and then taking a quasiconformal image of the result. Assuming Theorem 5.1 for now, we continue with the proof of Theorem A.

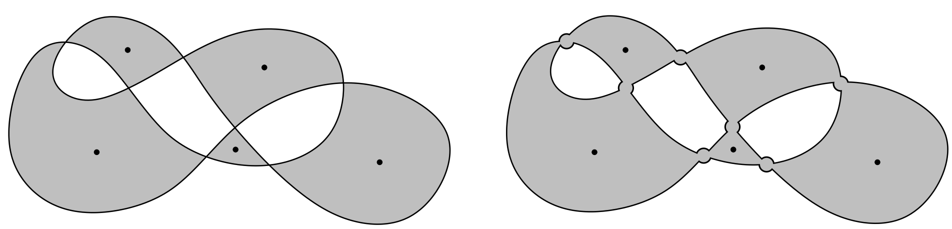

Figure 6. On the left is a 2-colored lemniscate graph G and on the right is a vertex-free graph H that approximates G. Note that the grey face of H contains every grey face of G. Here the grey face of H equals the grey faces of G, together with disks centered at the vertices of G. The actual construction involves other steps, and is explained in Section 11.

Notation 5.2. Throughout the remainder of this section, we will fix $\varepsilon > 0$ and a lemniscate graph G (perhaps with vertices). We also fix a 2-coloring of the faces of G, and a point p_B

in each grey face B of $\widehat{\mathbb{C}} \setminus G$. Set $P := \cup_B p_B$. Apply Theorem 5.1 to G and ε to obtain $\delta > 0$, H and H_δ. Set $u := u_{H,P}$. Since H has no vertices, the results and definitions of Section 4 apply to u. Note that several distinct grey faces of G may be contained in a single grey of H, so there may be several points in P contained in a single grey face of H. Let X denote the set of vertices of H_δ. For each $x \in X$ we denote by $D_x := D(x, r_x)$ a Euclidean disc centered at x of sufficiently small radius $r_x > 0$ so that $D(x, 2r_x) \cap (H \cup P) = \emptyset$, and the collection $\{D(x, 2r_x)\}_{x \in X}$ are pairwise disjoint.

Let $\{r_m\}_{m=1}^\infty$ be as in Definition 4.5, and let $(u_m)_{m=1}^\infty = (\frac{1}{m} \log |r_m|)_{m=1}^\infty$. By Theorem 4.7, $u_m \to u$ uniformly on compact subsets of $\widehat{\mathbb{C}} \setminus (H \cup P)$.

Proposition 5.3. *For each $m \in \mathbb{N}$ and $x \in X$, there exists a branch of the logarithm $\log(r_m)$ in D_x so that $h_m := (1/m) \log(r_m)$ converges uniformly to an analytic function h in D_x, as $m \to \infty$.*

Proof. Given a sequence of holomorphic functions $f_n = u_n + iv_n$, it is a general fact that that if the real parts u_n converge uniformly on a closed disk, then so do the imaginary parts v_n, assuming they converge at the center of the disk. This holds because the partial derivatives of u_n converge by the Cauchy estimates, and hence so do the partials of v_n by the Cauchy-Riemann equations. Thus to prove the proposition, it is enough to verify that $\{v_n\}$ converges at x.

For large m, the map r_m does not have any zeros in D_x since for $z \in D_x$,

$$(1/m) \log |r_m(z)| = u_m(z) \to u(z) > 0.$$

Thus for m large enough, a branch of $\log(r_m)$ exists in D_x. Next, for $z \in D_x$ we have

$$\operatorname{Im}\left(\frac{1}{m} \log(r_m(z))\right) = \frac{1}{m} \arg(r_m(z)) = \sum_B \sum_{p \in P_B} \sum_{j=1}^m \frac{1}{m} \arg\left(\frac{z - \zeta_j^B}{z - p}\right), \tag{5.1}$$

where the first sum $\sum_B$ is over all grey components B of $\widehat{\mathbb{C}} \setminus H$. As in the proof of Theorem 4.7, consider the discrete measure μ_m^B defined by having mass -1 at each $p \in P_B$, and mass $|P_B|/m$ at each of the points $(\zeta_j^B)_{j=1}^m$. Thus, the right-hand side of (5.1) can be rewritten as

$$\sum_B \int_{\widehat{\mathbb{C}}} \arg(z - \zeta)\, d\mu_m^B(\zeta). \tag{5.2}$$

Let $\mu_B = \sum_{p \in P_B} (\omega_p - \delta_p)$ be the measure from Notation 4.1, and recall the points ζ_j^B were chosen in Definition 4.4 so that $\mu_m^B \to \mu_B$ weak-$*$. Thus

$$\sum_B \int_{\widehat{\mathbb{C}}} \arg(z - \zeta)\, d\mu_m^B(\zeta) \to \sum_B \int_{\widehat{\mathbb{C}}} \arg(z - \zeta) d\mu_B(\zeta). \tag{5.3}$$

Combining (5.1)-(5.3), we see that the imaginary part of $(1/m) \log(r_m)$ converges uniformly in D_x, as desired. $\square$

For each $x \in X$ and large $m \in \mathbb{N}$, we define a set $X_m(x)$ as follows. Note that since $x \in X$ is a vertex of H_δ, it follows that x is a critical point of u of degree $\deg(x)/2-1$; here we are using $\deg(x)$ to denote the degree of x as a vertex of H_δ (see Definition 1.2). So, for instance, degree 4 vertices are simple (= degree 1) critical points of u. Since $(u_m)_{m=1}^\infty$ converges to u uniformly on compact subsets of $\widehat{\mathbb{C}} \setminus (H \cup P)$, we have that for each $x \in X$, there is a set $X_m(x)$ consisting of $\deg(x)/2-1$ many critical points (counted with multiplicity) of u_m, so that each $x_m \in X_m(x)$ converges to x as $m \to \infty$.

Definition 5.4. We set $h_m := (1/m)\log(r_m)$ and $h := \lim_{m\to\infty} h_m$ in D_x (see Proposition 5.3). As usual, Let $D(h(x), s)$ denote the Euclidean disc centered at $h(x)$ of radius $s > 0$. Fix s sufficiently small so that for all sufficiently large m,

(1) $D(h(x), s) \subset h(D_x)$,
(2) $h_m^{-1}(D(h(x), 2s)) \cap D_x$ is a Jordan domain containing $X_m(x)$, and
(3) $h_m : h_m^{-1}(D(h(x), 2s) \cap D_x) \to D(h(x), 2s)$ is a proper map.

By taking s smaller, if necessary, we can ensure (1)-(3) also hold when h replaces h_m (and X replacing $X_m(x)$ in (1)). For large m, let A_m^x be the topological annulus with outer boundary $h_m^{-1}(\partial D(h(x), 2s))$ and inner boundary $h^{-1}(\partial D(h(x), s))$. Note $A_m^x \subset D_x$, and that there will be corresponding annuli around each point of X.

We have used h to define the inner boundary of A_m^x and have used h_m to define its outer boundary. Our next proposition will show that A_m^x is "very close" to the annulus obtained when we use h to define both boundaries. The measure of closeness is given by quasiconformal maps. For the definitions and basic results about quasiconformal mappings see.e.g., [Ahl06] or [LV73]. The dilatation of a quasiregular map g is $\mu_g = g_{\overline{z}}/g_z$ and we set $k = \|\mu_f\|_\infty < 1$. The quasiconformal constant of g is $K = K(g) = (k+1)/(k-1) \geq 1$. Geometrically, $|\mu_g(z)|$ bounds the eccentricity of the ellipse that is the image of a circle centered at zero under the tangent map of g, at almost every point z. When $K(g)$ is close to 1, then g is close to holomorphic.

Proposition 5.5. *For all large m, there exists a quasiconformal mapping*
$$\psi_m : A_m^x \to h^{-1}(\{z : s < |z - h(x)| < 2s\})$$
so that $\psi_m(z) = z$ on the inner boundary of A_m^x, and $h \circ \psi_m(z) = h_m(z)$ on the outer boundary of A_m^x. Moreover, $K(\psi_m) \to 1$ as $m \nearrow \infty$.

Proof. Consider the sets A_m^x,
$$U := h^{-1}(\{z : s < |z - h(x)| < 2s\}),$$
and
$$A(s, 2s) := \{z : s < |z - h(x)| < 2s\}.$$
The sets U and $A(s, 2s)$ are topological annuli, and A_m^x is also a topological annulus for large enough m. Without loss of generality we may assume that $x = h(x) = 0$. Then the

exponential map is a covering from the vertical strip

$$H := \{z : \log(s) < \mathrm{Re}(z) < \log(2s)\}$$

onto the round annulus $A(s, 2s)$. Similarly, there are "curved", but $2\pi i$-periodic vertical strips H_A and H_U so that exp is a covering map from these domains to A_m^x and U, respectively. Although the boundary components of H_A and H_U need not be straight lines, the left boundary component of H_A coincides with the left boundary component of H_U (since the inner boundary of A_m^x coincides with the inner boundary of U).

Let R_A denote the right boundary component of H_A. By the lifting property,

$$h_m : h_m^{-1}(\partial D(h(x), 2s)) \to \partial D(h(x), 2s)$$

lifts to a periodic homeomorphism

$$\widehat{h}_m : R_A \to \{z : \mathrm{Re}(z) = \log(2s)\},$$

and $h : U \to A(s, 2s)$ lifts to a periodic homeomorphism $\widehat{h} : H_U \to H$. Since $h_m \to h$ as $m \to \infty$, the lifts may be chosen so that the map $\widehat{h}^{-1} \circ \widehat{h}_m$ (defined on R_A) converges to the identity as $m \to \infty$. Thus there exists a periodic quasiregular interpolation $f_m : H_A \to H_U$ so that f_m interpolates between $\widehat{h}^{-1} \circ \widehat{h}_m$ on R_A and the identity on the left boundary component of H_A, and satisfies $K(f_m) \to 1$ as $m \to \infty$. The map

$$\psi_m := \exp \circ f_m \circ \log : A_m^x \to U \tag{5.4}$$

is well-defined and satisfies the conclusions of the proposition. □

Remark 5.6. In Section 6, we will need a little extra information about ψ_m. In the construction above, we can take the quasiregular interpolating map f_m to be smooth and to equal the identity in a neighborhood of the left boundary of H_A, and to equal the analytic map $\widehat{h}^{-1} \circ \widehat{h}_m$ on the right boundary, R_A. This implies that the dilatation of ψ_m is continuous in A_m^x and vanishes near the boundary of A_m^x, and thus the dilatation extends to be uniformly continuous on the whole plane. In Section 6, we will replace the holomorphic function h by a parameterized family of holomorphic functions $h^{\boldsymbol{q}}$ that depend analytically on a vector $\boldsymbol{q}$ inside some open set in $\mathbb{C}^n$ (n is the number of poles in P). The annulus A_m^x is replaced by a family of annuli $A_m^{x;\boldsymbol{q}}$ that move analytically with q, and the map ψ_m becomes a parameterized family $\psi_m^{\boldsymbol{q}}$. In the proof of Theorem 6.5, we will use the fact that the uniform continuity of the dilatation of $\psi_m^{\boldsymbol{q}}$ implies that the dilatations of these maps move continuously in the supremum norm metric as functions of $\boldsymbol{q}$, i.e., that $\boldsymbol{q} \to \mu_{\psi_m^{\boldsymbol{q}}}$ is a continuous map from a neighborhood in $\mathbb{C}^n$ into the unit ball of $L^\infty(\mathbb{C})$.

Definition 5.7. We define a map $g_m : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ as follows.

$$g_m(z) := \begin{cases} r_m(z) & \text{for } z \text{ in the unbounded component of } \widehat{\mathbb{C}} \setminus \cup_x A_m^x, \\ \exp(m \cdot h(z)) & \text{for } z \text{ in a bounded component of } \widehat{\mathbb{C}} \setminus \cup_x A_m^x, \\ \exp(m \cdot h(\psi_m(z))) & \text{for } z \in A_m^x. \end{cases} \tag{5.5}$$

Proposition 5.8. *The map $g_m : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is quasiregular, satisfies $K(g_m) \to 1$ as $m \nearrow \infty$, and has a degree $deg(x)/2 - 1$ critical point at each $x \in X$ satisfying $|g_m(x)| = e^{m\delta}$.*

Proof. We claim that the definitions of g_m agree on both components of ∂A_m^x. For z on the inner boundary component of ∂A_m^x, this follows from Proposition 5.5 (since $\psi_m(z) = z$ on the inner boundary). For z on the outer boundary of ∂A_m^x, Proposition 5.5 says $h \circ \psi_m(z) = h_m(z)$, and so for such z we have

$$\exp(m \cdot h(\psi_m(z))) = \exp(m \cdot h_m(z)) := \exp(m\frac{1}{m}\log(r_m(z))) = r_m(z),$$

which says that the definitions of g_m agree on the outer boundary of ∂A_m^x.

The components of ∂A_m^x are analytic curves, and hence they removable for quasiregular mappings. (e.g., Theorem I.8.3 of [LV73]). Since g_m is holomorphic in $\widehat{\mathbb{C}} \backslash \cup_x A_m^x$, quasiregular in each A_m^x, and continuous across each ∂A_m^x, we can deduce that g_m is quasiregular on $\widehat{\mathbb{C}}$ with $K(g_m) = K(\psi_m)$, and $K(\psi_m) \to 1$ as $m \nearrow \infty$ by Proposition 5.5.

Recall that $\mathrm{Re}(h) = u$ by definition in Proposition 5.3. Since u has a degree $\deg(x)/2 - 1$ critical point at each $x \in X$, and since $g_m(z) = \exp(mh(z))$ in a neighborhood of each $x \in X$, the map h also has a degree $\deg(x)/2 - 1$ critical point at each $x \in X$. Moreover, by the definition of g_m, we have that for any $x \in X$,

$$|g_m(x)| = |\exp(mh(x))| = \exp\left(\mathrm{Re}\,[mh(x)]\right) = \exp\left(mu(x)\right) = \exp(m\delta). \qquad \square$$

Theorem 5.9. *Let $\varepsilon > 0$. Then, for all sufficiently large m, we have that $g_m^{-1}(e^{m\delta}\mathbb{T})$ is ε-homeomorphic to H_δ.*

Proof. The homeomorphism f of $g_m^{-1}(e^{m\delta}\mathbb{T})$ onto H_δ may be described as follows. Since $g_m(z) = \exp(mh(z))$ in a neighborhood of each $x \in X$ and $\mathrm{Re}(h) = u$, we have $|g_m(z)| = \exp\left(mu(z)\right)$ in a neighborhood of each $x \in X$. Thus the sets $g_m^{-1}(e^{m\delta}\mathbb{T})$ and H_δ in fact coincide in a neighborhood of each $x \in X$. Thus we may set $f(z) = z$ for z in such a neighborhood.

Moreover, x, $\tilde{x} \in X$ are connected by an edge in H_δ if and only if x, $\tilde{x}$ are connected by an edge in $g_m^{-1}(e^{m\delta}\mathbb{T})$. We claim that for any $\eta > 0$, if m is large enough, then f extends to a homeomorphism $f : g_m^{-1}(e^{m\delta}\mathbb{T}) \to H_\delta$ satisfying $d(f(z), z) < \eta$ for all $z \in g_m^{-1}(e^{m\delta}\mathbb{T})$. This map can be constructed by following the steepest gradient descent lines of $u_{H,P}$, just as in the proof at the end of Section 4; the details are the same, since H consists of disjoint Jordan curves. Using Theorem 2.2, this proves that for any $\varepsilon > 0$, H_δ and $g_m^{-1}(e^{m\delta}\mathbb{T})$ are ε-homeomorphic if m is large enough. $\square$

Theorem 5.10. *For every $m \in \mathbb{N}$ there exists a quasiconformal mapping ϕ_m so that $g_m \circ \phi_m^{-1} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is holomorphic (and hence a rational mapping), and moreover as $m \nearrow \infty$,*

$$\sup_{z \in \widehat{\mathbb{C}}} d(\phi_m(z), z) \to 0, \tag{5.6}$$

where, as usual, $d(\cdot, \cdot)$ denotes the spherical metric on $\widehat{\mathbb{C}}$.

Proof. The first statement follows from the Measurable Riemann Mapping theorem. The map ϕ_m is uniquely determined once we choose a normalization; we may choose to normalize ϕ_m so that $\phi_m(z) = z + O(1/|z|)$ as $z \to \infty$, or else to normalize ϕ_m so as to fix 3 distinct points in X (either normalization will work in what follows). With either choice, the relation (5.6) follows since $K(g_m) \to 1$ by Proposition 5.8. □

Definition 5.11. We set $R_m := g_m \circ \phi_m^{-1}$ where ϕ_m is as in Theorem 5.10, so that R_m is a rational mapping.

Proposition 5.12. *Let $\varepsilon > 0$. For all large m, we have that the rational lemniscate $R_m^{-1}(e^{mc}\mathbb{T})$ is ε-homeomorphic to H_δ.*

Proof. We already know that $g_m^{-1}(e^{mc}\mathbb{T})$ is $(\varepsilon/2)$-homeomorphic to H_δ if m is sufficiently large, and we know that the quasiconformal map ϕ_m is an $(\varepsilon/2)$-homeomorphism if m is large enough. Thus $R_m^{-1}(e^{mc}\mathbb{T}) = \phi_m(g_m^{-1}(e^{mc}\mathbb{T}))$ is ε-homeomorphic to H_δ. □

We have now proved (assuming Theorem 5.1) that any lemniscate graph is ε-homeomorphic to a rational lemniscate with one pole in each grey component, and that these poles may be specified with error at most ε. Indeed, three of the poles of R_m can be placed exactly by applying a linear fractional transformation. In the next section, we show that all the poles can be prescribed exactly.

6. Exact Placement of Poles

To place the poles exactly, we will apply a fixed point argument. More precisely, given a desired set of n poles P, and fixing a large integer m (that determines a discrete approximation to harmonic measure), we introduce a parameterized family $R_m^{\boldsymbol{q}}$ of rational functions, where $\boldsymbol{q}$ ranges over a neighborhood of P, considered as a point in $\mathbb{C}^n$. We will show that if m is sufficiently large, then there exists a value of $\boldsymbol{q}$ so that the poles of $R_m^{\boldsymbol{q}}$ are exactly P.

Throughout this section we will fix G, ε and P as in Theorem A. By applying a Möbius transformation, we may assume without loss of generality that the union of the grey components of $\widehat{\mathbb{C}} \setminus G$ is contained in $D(0, 1/2)$. It will be convenient to list the points in P as $\boldsymbol{p} := (p_1, ..., p_n) \in \mathbb{C}^n$, so that $\cup_j\{p_j\} = P$. Let H, δ, $u := u_{H,P}$ be as in Notation 5.2; namely H is a lemniscate graph without vertices whose grey faces properly contain the grey faces of G, and $H_\delta := u^{-1}(\delta)$ is ε-homeomorphic to G.

The index m will denote the parameter of r_m, the rational function defined in Equation (4.4). As the reader may recall, r_m was defined by cutting the boundary of each grey face of H into approximately m arcs of approximately equal harmonic measure. In this section, n (the number of points in P) will remain fixed, but we shall take m as large as is needed to make our arguments work.

Lemma 6.1. *Given a set of n distinct points $P = \{p_1, \ldots, p_n\} \subset D(0, 1/2)$, there exists $\rho > 0$ so that for each $\boldsymbol{q} = (q_1, ..., q_n) \in \prod_{j=1}^n D(p_j, \rho)$, there exists an injective, holomorphic mapping $\psi_{\boldsymbol{q}} : \mathbb{D} \to \psi_{\boldsymbol{q}}(\mathbb{D})$ satisfying:*

(1) $\psi_{\boldsymbol{q}}(q_j) = p_j$ *for* $1 \leq j \leq n$.
(2) $\psi_{\boldsymbol{p}}(z) = z$ *for all* $z \in \mathbb{D}$, *and*
(3) $\psi_{\boldsymbol{q}}$ *depends continuously on* $\boldsymbol{q}$.

Proof. This is stated without proof as Lemma 1 in [Roy54], but we will give a proof for the sake of completeness.

Set $\psi_{\boldsymbol{q}}(z) := z + f_{\boldsymbol{q}}(z)$ where $f_{\boldsymbol{q}}$ is the Lagrange interpolating polynomial for data $f_{\boldsymbol{q}}(q_j) = p_j - q_j$ for $1 \leq j \leq n$. More explicitly,

$$f_{\boldsymbol{q}}(z) = \sum_{j=1}^{n} (p_j - q_j) \prod_{k \neq j} \frac{z - q_k}{q_j - q_k}, \tag{6.1}$$

is a polynomial of degree at most $n-1$ that takes the n given values $\{p_j - q_j\}_1^n$ at the n specified distinct points $\{q_j\}_1^n$. It is easy to check from this definition that (1) and (2) hold, and that $\psi_{\boldsymbol{q}}$ depends continuously on $\boldsymbol{q}$ as long as the components of $\boldsymbol{q}$ are distinct. Let δ be half the minimal distance between distinct points of P. If $\rho < \delta/2$ and $|q_j - p_j| < \rho$ for all $j = 1, \ldots, n$, then the components of $\boldsymbol{q}$ are distinct and we even have $|q_j - q_k| \geq \delta$ for $j \neq k$.

It only remains to verify that ψ_q is injective. Our assumption that $\rho < \delta/2$ implies that for $z \in D(0, 3/2)$ each of the n products on the right side of (6.1) is bounded by $(2/\delta)^n$. By the Cauchy estimates, this implies $|f_{\boldsymbol{q}}'| = O(\rho \cdot n(2/\delta)^n)$ on $\mathbb{D}$. Let ρ be sufficiently small (depending on δ and n) so that $|f_{\boldsymbol{q}}'(z)| < 1$ for $z \in \mathbb{D}$. Then integration gives the inequality:

$$|\psi_{\boldsymbol{q}}(z_1) - \psi_{\boldsymbol{q}}(z_2)| = \left| \int_{z_1}^{z_2} (1 + f_{\boldsymbol{q}}') \right| \geq |z_1 - z_2| - \int_{z_1}^{z_2} |f_{\boldsymbol{q}}'| > 0, \tag{6.2}$$

for $z_1, z_2 \in \mathbb{D}$. This proves the injectivity of $\psi_{\boldsymbol{q}}$. □

We henceforth fix $\rho > 0$ as in the conclusion of Lemma 6.1.

Definition 6.2. For each $\boldsymbol{q} \in \prod_{j=1}^n D(p_j, \rho)$, we introduce parameterized versions $G_{\boldsymbol{q}}$, $H_{\boldsymbol{q}}$, $u_{\boldsymbol{q}}$ of the objects G, H, u as follows. Set $G_{\boldsymbol{q}} := \psi_{\boldsymbol{q}}^{-1}(G)$, $H_{\boldsymbol{q}} := \psi_{\boldsymbol{q}}^{-1}(H)$, $P_{\boldsymbol{q}} := \psi_{\boldsymbol{q}}^{-1}(P)$ and define $u_{\boldsymbol{q}}$ by setting $u_{\boldsymbol{q}}(z) := u_{H,P} \circ \psi_{\boldsymbol{q}}(z)$ for z in a grey face of $H_{\boldsymbol{q}}$, and $u_{\boldsymbol{q}}(z) := 0$ otherwise. As before, we set $u := u_{H,P}$ to simplify notation.

It will be useful to note that $G_{\boldsymbol{p}} = G$, $H_{\boldsymbol{p}} = H$, and $u_{\boldsymbol{p}} = u$ by Lemma 6.1(2). Moreover, $u_{\boldsymbol{q}}^{-1}(\delta) = \psi_{\boldsymbol{q}}^{-1}(u^{-1}(\delta)) = \psi_{\boldsymbol{q}}(G)$ is ε-homeomorphic to G for all $\boldsymbol{q} \in \prod_{j=1}^n D(p_j, \rho)$ after taking ρ smaller if need be. On the other hand, Notation 5.2 defines the function $u_{H_{\boldsymbol{q}}, \psi_{\boldsymbol{q}}(P)}$ for each $\boldsymbol{q} \in \prod_{j=1}^n D(p_j, \rho)$. In fact, we have the following.

Proposition 6.3. *For each* $\boldsymbol{q} \in \prod_{j=1}^n D(p_j, \rho)$, *we have* $u_{\boldsymbol{q}} = u_{H_{\boldsymbol{q}}, P_{\boldsymbol{q}}}$.

Proof. Both functions vanish except in the grey faces of $H_{\boldsymbol{q}}$, where they are both harmonic except for logarithmic poles at $P_{\boldsymbol{q}}$. The result follows from the maximum principle. □

Proposition 6.3 implies that the definitions of Section 5 apply to each $u_{\boldsymbol{q}}$ to produce parameterized families

$$\psi_m^{\boldsymbol{q}}, g_m^{\boldsymbol{q}}, \phi_m^{\boldsymbol{q}}, \text{ and } R_m^{\boldsymbol{q}} := g_m^{\boldsymbol{q}} \circ (\phi_m^{\boldsymbol{q}})^{-1} \tag{6.3}$$

so that

$$\psi_m^{\boldsymbol{p}} = \psi_m, g_m^{\boldsymbol{p}} = g_m, \phi_m^{\boldsymbol{p}} = \phi_m, \text{ and } R_m^{\boldsymbol{p}} = R_m. \tag{6.4}$$

Moreover, the results of Section 5 apply to the maps listed in (6.3) for each $\boldsymbol{q} \in \prod_{j=1}^n D(p_j, \rho)$. The important point is that all these functions depend continuously on the parameter $\boldsymbol{q}$.

Definition 6.4. For each $m \in \mathbb{N}$, define a mapping $\Psi_m : \prod_{j=1}^n D(p_j, \rho) \to \widehat{\mathbb{C}}^n$ by setting

$$\Psi_m(\boldsymbol{q}) := ((\phi_m^{\boldsymbol{q}})^{-1}(p_1), ..., (\phi_m^{\boldsymbol{q}})^{-1}(p_n)).$$

Note that

$$(g_m^{\boldsymbol{q}})^{-1}(\infty) = \boldsymbol{q}, \tag{6.5}$$

where we are abusing notation slightly in (6.5) by setting $(q_1, ..., q_n) = \{q_1, ..., q_n\}$. So if $\boldsymbol{q}$ is a fixed point of Ψ_m then $\Psi_m(\boldsymbol{q}) = \boldsymbol{q}$, or equivalently

$$(\phi_m^{\boldsymbol{q}})^{-1}(p_j) = q_j, \quad j = 1, \ldots, n,$$

or

$$p_j = \phi_m^{\boldsymbol{q}}(q_j), \quad j = 1, \ldots, n,$$

or

$$\boldsymbol{p} = \phi_m^{\boldsymbol{q}}((g_m^{\boldsymbol{q}})^{-1}(\infty)).$$

Therefore, if we can prove that Ψ_m has a fixed point $\boldsymbol{q}$, and we set $R_m = g_m^{\boldsymbol{q}} \circ (\phi_m^{\boldsymbol{q}})^{-1}$, then R_m is a rational function with poles equal to

$$R_m^{-1}(\infty) = \phi_m^{\boldsymbol{q}}((g_m^{\boldsymbol{q}})^{-1}(\infty)) = \boldsymbol{p}.$$

Assuming Ψ_m has a fixed point $\boldsymbol{q}$ if m is large enough (Theorem 6.5 below), and assuming Theorem 5.1 (proven in Section 11) we can complete the proof of our main result.

Proof of Theorem A. The map $R_m^{\boldsymbol{q}}$ with m and $\boldsymbol{q}$ chosen as above satisfies the conclusions of Theorem A. Indeed, we just showed that $(R_m^{\boldsymbol{q}})^{-1}(\infty) = P$ and the remaining conclusions of Theorem A were already verified in Section 5 (see Proposition 5.12). $\square$

Let us now prove that Ψ_m does, indeed, have a fixed point.

Theorem 6.5. *Let $\rho > 0$ be as given in Lemma 6.1. If $m \in \mathbb{N}$ is sufficiently large, then $\Psi_m : \prod_{j=1}^n D(p_j, \rho) \to \widehat{\mathbb{C}}^n$ has a fixed point.*

Proof. By Brouwer's fixed point theorem [Bro11], it suffices to verify that:

(1) $\Psi_m(\prod_{j=1}^n D(p_j, \rho)) \subset \prod_{j=1}^n D(p_j, \rho)$ for sufficiently large m,
(2) Ψ_m is continuous.

By (5.6), the image set $\Psi_m(\prod_{j=1}^n D(p_j,\rho))$ converges to the point $\boldsymbol{p}$ as $m \nearrow \infty$, so it is certainly contained inside $\prod_{j=1}^n D(p_j,\rho)$ for sufficiently large m.

To prove the continuity of Ψ_m, we write it as the composition $\Psi_m = \Psi_m^1 \circ \Psi_m^2$, where

$$\Psi_m^1 : \{\mu \in L^\infty(\widehat{\mathbb{C}}) : ||\mu|| < 1\} \to \widehat{\mathbb{C}}^n \text{ is defined by } \Psi_m^1(\mu) := (\phi_\mu^{-1}(p_1), ..., \phi_\mu^{-1}(p_n)).$$

and

$$\Psi_m^2 : \prod_{j=1}^n nD(p_j,\rho) \to \{\mu \in L^\infty(\widehat{\mathbb{C}}) : ||\mu|| < 1\} \text{ is defined by } \Psi_m^2(\boldsymbol{q}) := (g_m^{\boldsymbol{q}})_{\overline{z}}/(g_m^{\boldsymbol{q}})_z$$

Here ϕ_μ is the unique quasiconformal mapping (normalized so $\phi(z) = z + O(1/|z|)$ as $z \to \infty$) so that $(\phi_\mu)_{\overline{z}}/(\phi_\mu)_z = \mu$. The map Ψ_m^1 is continuous by a standard result saying that pointwise evaluation of quasiconformal maps depends continuously on the dilatation, e.g., Theorem I.7.5 of [CG93].

The continuity of Ψ_m^2 is the claim that the complex dilatation of $g_m^{\boldsymbol{q}}$ moves continuously in the supremum norm as a function of $\boldsymbol{q}$. By Definition 5.7, $g_m^{\boldsymbol{q}}$ is holomorphic except where it equals $g_m^{\boldsymbol{q}} = \exp \circ (m \cdot h^{\boldsymbol{q}}) \circ \psi_m^{\boldsymbol{q}}$. Since post-composing by holomorphic maps does not change the dilatation of a quasiregular map, the dilatation of $g_m^{\boldsymbol{q}}$ is the same as the dilatation of $\psi_m^{\boldsymbol{q}}$, and the latter dilatation moves continuously in L^∞ as a function of $\boldsymbol{q}$; this was explained in Remark 5.6. □

7. Proof of Theorem B: quantitative Runge's theorem

The proof of Theorem B roughly follows the proof of Theorem I in [WR34] (which assumes the set K is connected, and $P = \{\infty\} \subset \widehat{\mathbb{C}} \setminus K$), except that we replace their application of a weaker polynomial lemniscate result with our Theorem A. There are also some other non-trivial adjustments to handle the case of disconnected K.

Let K, P and f be as in the statement of Theorem B. Let U be the neighborhood of K in which f is holomorphic.

Lemma 7.1. *Given K, U and P as above, we can find $K' \supset K$, $U' \subset U$, and $P' \subset P$, so that both K' and U' are bounded by finitely many pairwise disjoint smooth Jordan curves, and that P' contains exactly one point in each connected component of $\widehat{\mathbb{C}} \setminus K'$.*

Proof. First replace U by a subset U' that still covers K and that is bounded by finitely many disjoint smooth curves. For example, take a union of sufficiently small grid squares and round the corners of the resulting polygon, as in Figure 7. Then U' and the connected components of $\widehat{\mathbb{C}} \setminus K$ cover $\widehat{\mathbb{C}}$. This is an open cover of a compact space, so U' and a finite subcollection $\{\Omega_k\}_1^n$ of the components of $\widehat{\mathbb{C}} \setminus K$ also cover $\widehat{\mathbb{C}}$. Let $P' \subset P$ be the n points contained in these finitely many open sets. Since K has positive distance from $\widehat{\mathbb{C}} \setminus U'$, we see that $\Omega_j \setminus U'$ is a compact subset of Ω_j. Replace each Ω_j by a connected subset Ω_j' that is bounded by finitely many smooth curves, and that contains the compact set $P \cup (\Omega_j \setminus U')$ (again we may take a union of small squares and round the remaining corners). Let $K' = \widehat{\mathbb{C}} \setminus \cup_{j=1}^{n} \Omega_j'$. Then $K \subset K' \subset U' \subset U$, and P' contains exactly one point in each complementary component of K'. Thus the claim is verified. $\square$

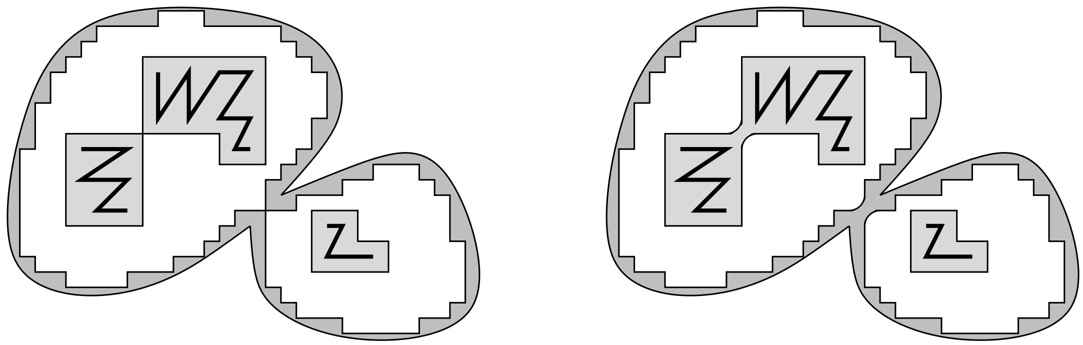

FIGURE 7. Replacing K and U by regions with smooth Jordan curve boundaries. If $\delta <\lll \operatorname{dist}(K, \partial U)$ we can take all squares from a δ-grid that lie inside U, and then round the corners of the resulting polygon to get a an open set $U' \subset U$ that covers K and has smooth boundaries. A similar construction gives a smooth set K' containing K.

It suffices to prove Theorem B for these new sets K', U', P'; this immediately implies the same result for the original sets K, U, P. To simplify notation, we henceforth refer to the new sets simply as K, U and P (dropping the prime notation).

Considering ∂K as a lemniscate graph, each of its faces either contains points of K or a single point of P, and we color these faces white and grey respectively. Consider the function $u_{\partial K,P}$ as in Notation 4.2. Recall that inside each grey face of ∂K, this is the Green's function for that face with pole at the corresponding point of P (there is one such point per grey face), and $u_{\partial K,P}$ is zero outside all the grey faces. If $S > 1$, set $C_S := \{z : u_{\partial K,P}(z) = \log S\}$. Then fix some $R > 1$ so that C_R separates K from ∂U. When R is close to one, C_R is a union of level lines of Green's functions for all the grey faces, and this union is as close to ∂K as we wish, so such a choice is possible.

Lemma 7.2. *For every $\rho \in (1, R)$, there exists a rational mapping r so that the lemniscates $L_r(1) = \{z : |r(z)| = 1\}$ and $L_r((R/\rho)^d) = \{z : |r(z)| = (R/\rho)^d\}$ both separate K from C_R, where $d = deg(r)$.*

Proof. Fix $\rho \in (1, R)$ and consider $C_\rho = \{z : u_{\partial K,P}(z) = \log \rho\}$. By Theorem A, there exists a rational mapping r with $r^{-1}(\infty) = P$ so that the lemniscate $L_r(1)$ separates K from C_ρ. Note that

$$u_{L_r,P}(z) = \frac{1}{d} \log |r(z)| \tag{7.1}$$

and

$$u_{C_\rho,P}(z) = u_{\partial K,P}(z) - \log(\rho) \tag{7.2}$$

for all z. Since L_r separates K from C_ρ, we have that $u_{C_\rho,P} - u_{L_r,P} < 0$ on C_ρ and hence by the maximum principle

$$u_{C_\rho,P} - u_{L_r,P} < 0 \text{ on } \{z : 0 < u_{C_\rho,P}(z) < \infty\}. \tag{7.3}$$

Now assume $z \in L_r((R/\rho)^d)$, i.e., assume that $|r(z)| = (R/\rho)^d$. Then by (7.1) we have that $u_{L_r,P}(z) = \log(R/\rho)$, and so by (7.3) we conclude $u_{C_\rho,P}(z) < \log(R/\rho)$. Thus (7.2) implies that for $z \in L_r((R/\rho)^d)$

$$u_{\partial K,P}(z) = u_{C_\rho,P}(z) + \log(\rho) < \log(R/\rho) + \log\rho = \log(R).$$

Hence the lemniscate $L_r((R/\rho)^d)$ separates K from C_R. Since we already arranged for the lemniscate $L_r(1)$ to separate K from C_ρ (and it separates K from C_R since $1 < \rho < R$), the proof of the lemma is finished. □

Recall that we have fixed $R > 1$ so that C_R separates K from ∂U. For the remainder of this section, we fix some $\rho \in (1, R)$ and fix a rational map r by applying Lemma 7.2 using this ρ. After applying a Möbius transformation, if necessary, we may assume that $\infty \notin K$ and $\infty \in P$. Thus if $r(z) = p(z)/q(z)$, where p and q are polynomials, then r has a pole at ∞ and hence $\deg(r) = \deg(p) > \deg(q)$.

We will refer to the *interior* of C_R as the union of the components of $\widehat{\mathbb{C}} \setminus C_R$ which do not contain a point of P. Equivalently, this is the union of components where $u_{\partial K,P} < R$, and hence these are the components that contain points of K. If we orient the individual curves in C_R so the interior regions (shown as white in Figure 8) are on the left, and note that the unbounded component is not in the interior of C_R, then any white point is surrounded by an odd number of nested boundary curves, with alternating orientations. The outermost curve is oriented in the counterclockwise direction, so the total winding number of C_R around any white point must equal one. Similarly, the winding number around any grey point is zero.

We can now define the rational maps that will prove Theorem B.

Definition 7.3. For $n \geq 1$, $d = \deg(r)$, and z in the interior of C_R, define

$$R_{dn-1}(z) := f(z) - \frac{1}{2\pi i} \int_{C_R} \frac{f(\zeta)}{\zeta - z} \left[\frac{r(z)}{r(\zeta)}\right]^n d\zeta. \tag{7.4}$$

Lemma 7.4. *For each n, R_{dn-1} is a rational map of degree $\leq dn - 1$ with $R_{dn-1}^{-1}(\infty) \subseteq P$.*

Proof. Assume z is interior to C_R. By the remarks above, C_R has winding number one around z. By the homology version of Cauchy's integral theorem (e.g., Theorem 2.41 of [Zak21]), we have that

$$f(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(\zeta)}{\zeta - z} d\zeta. \tag{7.5}$$

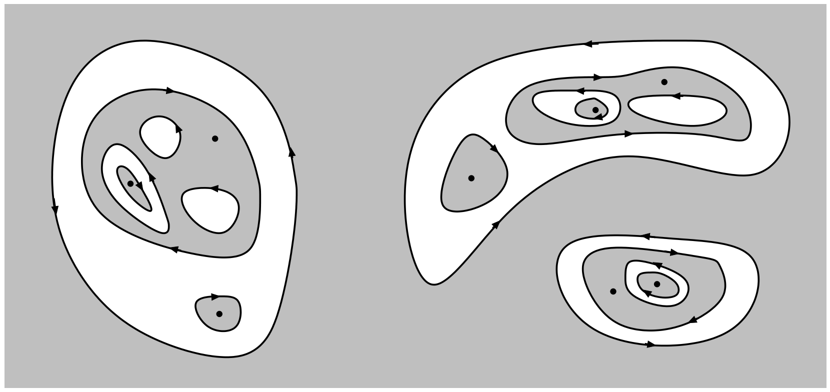

FIGURE 8. The white regions are the interior of C_R, and the gray regions each contain a point of P (black dots, plus ∞). Each component of C_R is oriented with the interior on the left, and the total winding number of C_R around any white point is one.

Thus, (7.4) and (7.5) combine to give

$$R_{dn-1}(z) = \frac{1}{2\pi i}\int_{C_R} f(\zeta)\left[\frac{1}{\zeta - z} - \frac{r^n(z)}{(\zeta - z)r^n(\zeta)}\right] d\zeta.$$

Recall that $r(z) = p(z)/q(z)$, where p and q are polynomials with $\deg(p) > \deg(q)$. Observe

$$\frac{1}{\zeta - z} - \frac{r^n(z)}{(\zeta - z)r^n(\zeta)} = \frac{1}{q^n(z)}\frac{p^n(\zeta)q^n(z) - p^n(z)q^n(\zeta)}{(\zeta - z)p^n(\zeta)},$$

and hence

$$R_{dn-1}(z) = \frac{1}{2\pi i}\frac{1}{q^n(z)}\int_{C_R} f(\zeta)\left[\frac{p^n(\zeta)q^n(z) - p^n(z)q^n(\zeta)}{(\zeta - z)p^n(\zeta)}\right] d\zeta. \tag{7.6}$$

Since $p^n(\zeta)q^n(z) - p^n(z)q^n(\zeta)$ vanishes when $\zeta = z$, we see that for fixed ζ,

$$\frac{p^n(\zeta)q^n(z) - p^n(z)q^n(\zeta)}{(\zeta - z)p^n(\zeta)}$$

is a polynomial in z of degree at most $dn - 1$. Therefore

$$\int_{C_R} f(\zeta)\left[\frac{p^n(\zeta)q^n(z) - p^n(z)q^n(\zeta)}{(\zeta - z)p^n(\zeta)}\right] d\zeta \tag{7.7}$$

is a linear combination of polynomials with degrees at most $dn-1$, and hence is a polynomial of degree at most $dn - 1$. From (7.6) and (7.7) we conclude that R_{dn-1} is a rational map, whose numerator has degree at most $dn - 1$, and whose denominator has degree $n\deg(q) < n\deg(p) \leq dn - 1$. Thus the degree of R_{dn-1} is $\leq dn - 1$. Moreover, from (7.6) and (7.7) we also see that the poles of R_{dn-1} are a subset of the poles of r (the zeros of q). $\square$

Proof of Theorem B. Let $z \in K$. From (7.4), we see that

$$|R_{dn-1}(z) - f(z)| \leq \frac{1}{2\pi} \int_{C_R} \frac{|f(\zeta)|}{|\zeta - z|} \left| \frac{r(z)}{r(\zeta)} \right|^n |d\zeta| \tag{7.8}$$

Note that

$$\int_{C_R} \frac{|d\zeta|}{|\zeta - z|} < \frac{\text{length}(C_R)}{\text{dist}(K, C_R)}. \tag{7.9}$$

Furthermore, Lemma 7.2 implies that for all $z \in K$ and $\zeta \in C_R$ we have

$$\left| \frac{r(z)}{r(\zeta)} \right|^n < \frac{1}{(R/\rho)^{dn}}. \tag{7.10}$$

Combining (7.8) with (7.9) and (7.10), we see that

$$|R_{dn-1}(z) - f(z)| \leq \frac{\sup_{w \in K} |f(w)| \cdot \text{length}(C_R)}{2\pi \cdot \text{dist}(K, C_R)} \cdot \frac{1}{(R/\rho)^{dn}}.$$

Thus, setting

$$A := \frac{\sup_{w \in K} |f(w)| \cdot \text{length}(C_R)}{2\pi \cdot \text{dist}(K, C_R) \cdot (R/\rho)} \quad \text{and} \quad B := (R/\rho),$$

we see that

$$|R_{dn-1}(z) - f(z)| \leq \frac{A}{B^{dn-1}} \tag{7.11}$$

as desired. Since R_{dn-1} is a rational map of degree $\leq dn-1$ with poles at P by Lemma 7.4, the inequality (7.11) proves Theorem B for the subsequence $(R_{dn-1})_{n=1}^{\infty}$. Setting

$$R_m := R_{dn-1} \text{ for } dn - 1 \leq m < d(n+1) - 1$$

and increasing A by a factor of B^d in (7.11), we readily deduce the desired estimate for all $m \in \mathbb{N}$. □

Remark 7.5. It is worth noting that in the special case when f is analytic on a disk U centered at p, we may take $r(z) = (z - p)$, and take $C_R \subset U$ to be a circle centered at p. Then the residue theorem shows that (7.6) computes the degree n truncation of the power series for f around p. Thus our calculation mimics the well known fact that these truncations converge geometrically fast to f on compact subsets of the disk of convergence.

The following fact is not needed for the proof of Theorem B, but it illustrates the structure of the approximants R_{dn-1} quite clearly, and explains how they generalize Taylor series approximations. We let $R^{(k)}$ denote the kth derivative of R.

Proposition 7.6. *The map R_{dn-1} is a rational map of degree $\leq dn-1$ satisfying $R_{dn-1}^{-1}(\infty) \subseteq P$ and*

$$\text{if } r(a) = 0 \text{ then } R_{dn-1}^{(k)}(a) = f^{(k)}(a) \text{ for each } 0 \leq k \leq n-1. \tag{7.12}$$

Moreover, if R is another rational map of degree $\leq dn-1$ with the same poles of the same order as R_{dn-1} so that R also satisfies (7.12), then $R \equiv R_{dn-1}$.

Proof. The fact that R_{dn-1} satisfies (7.12) follows from (7.4) and the quotient rule (it shows the first $n-1$ derivatives of the integral in (7.4) are all zero). The other conclusions about R_{dn-1} were proven in Lemma 7.4. If $R(z)$ is another rational map satisfying (7.12), then the difference $R_{dn-1}(z) - R(z)$ is a rational function of degree $\leq dn-1$ that has dn zeros counted with multiplicity (since (7.12) has dn equations). Hence $R_{dn-1}(z) - R(z)$ must be identically 0. □

8. Geometric decay in Theorem B is sharp

Theorem 8.1. *Let K, P and f be as in Theorem B, and assume f does not admit a holomorphic extension to $\widehat{\mathbb{C}} \setminus P$. Then there exists a constant $D \in (1, \infty)$ with the following property. For any $C \in (0, \infty)$ and any sequence of rational maps R_n of degree $\leq n$ satisfying $R_n^{-1}(\infty) \subseteq P$, we have that*

$$\sup_{z \in K} |f(z) - R_n(z)| > \frac{C}{D^n}$$

for all sufficiently large n.

We recall the following notation from the proof of Theorem B. Let U be the neighborhood of K in which f is holomorphic. After taking a superset of K and a subset of U, we may assume that each of U, K is bounded by finitely many, pairwise disjoint Jordan curves, and $U \cap P = \emptyset$. Consider ∂K as a lemniscate graph, with a face of ∂K colored grey if and only if the face contains a point in P. Consider the function $u_{\partial K,P}$ (in any face of ∂K containing a point in P we defined $u_{\partial K,P} =$ Green's function with pole at that point in P, and $u_{\partial K,P} = 0$ otherwise). Set $C_S := \{z : u_{\partial K,P}(z) = \log S\}$ for $S > 1$. As in the previous section, the *interior* of C_S (denoted $\mathrm{int}(C_S)$) refers to the union of the components of $\widehat{\mathbb{C}} \setminus C_S$ which do not contain a point of P. After applying a Möbius transformation, if need be, we may assume that $\infty \notin K$ and $\infty \in P$.

The proof of Theorem 8.1 relies on the following lemma.

Lemma 8.2. *Suppose Q is a rational map satisfying $Q^{-1}(\infty) \subseteq P$ and $\sup_{z\in K} |Q(z)| \leq M$ for some constant $M < \infty$. Let $p \in P$, let d denote the local degree of Q at the pole p, and let F denote the face of ∂K containing p. Then for any $S > 1$ we have $|Q(z)| \leq M \cdot S^d$ for $z \in \mathrm{int}(C_S) \cap F$.*

Proof. Set $u := u_{\partial K,P}$. We have that

$$\frac{\log |Q|}{d} - u \tag{8.1}$$

is subharmonic on the face F (including at the point p), and so by the maximum principle, (8.1) takes its maximum over F on ∂K. Recalling that $u|_K = 0$ and that $|Q(z)| \leq M$ for

$z \in K$ by assumption, we conclude that

$$\frac{\log|Q(z)|}{d} - u(z) \leq \frac{\log(M)}{d} \text{ for } z \in F.$$

In particular, we have that

$$\frac{\log|Q(z)|}{d} - \log(S) \leq \frac{\log(M)}{d}$$

for $z \in C_S \cap F$, and hence by the maximum principle we have that

$$\frac{\log|Q(z)|}{d} \leq \log(S) + \frac{\log(M)}{d} \tag{8.2}$$

for $z \in \mathrm{int}(C_S) \cap F$. Inequality (8.2) implies the lemma. □

Proof of Theorem 8.1. Since we have assumed that f does not admit a holomorphic extension to $\widehat{\mathbb{C}} \setminus P$, there exists some $S < \infty$ so that f does not admit a holomorphic extension to the interior of C_S. We set $D := S$. Now suppose there were a $C \in (0, \infty)$ and a sequence of rational maps R_n of degree $\leq n$ with poles only in P, so that

$$\sup_{z \in K} |f(z) - R_n(z)| \leq \frac{C}{D^n} \tag{8.3}$$

for some subsequence in n. It will simplify notation to assume that (8.3) holds for all n and not just along a subsequence $\{n_k\}_1^\infty$ (to prove the general case, just replace the subscript n by n_k in what follows).

We will show that the sequence $R_n(z)$ converges uniformly on compact subsets of the interior of C_D; this will be a contradiction since such a limit would necessarily constitute a holomorphic extension of f to the interior of C_D. Note that

$$|R_{n+1}(z) - R_n(z)| \leq |R_{n+1}(z) - f(z)| + |f(z) - R_n(z)| \leq \frac{C}{D^{n+1}} + \frac{C}{D^n}$$

for all $z \in K$ and $n \in \mathbb{N}$. Let F be a face of ∂K containing a point $p \in P$. The local degree of $R_{n+1} - R_n$ at p is $\leq n+1$. Thus Lemma 8.2 implies that for any $D_1 < D$, we have

$$|R_{n+1}(z) - R_n(z)| \leq D_1^{n+1} \left(\frac{C}{D^n} + \frac{C}{D^{n+1}} \right) = \left(\frac{D_1}{D} \right)^n C D_1 \left(1 + \frac{1}{D} \right)$$

for $z \in \mathrm{int}(C_{D_1}) \cap F$. As F was arbitrary, it follows that

$$|R_{n+1}(z) - R_n(z)| \leq \left(\frac{D_1}{D} \right)^n C D_1 \left(1 + \frac{1}{D} \right)$$

for $z \in \mathrm{int}(C_{D_1})$. The right-hand side is summable in n, and so the sequence $(R_n)_{n=1}^\infty$ is uniformly Cauchy and hence uniformly convergent on $\mathrm{int}(C_{D_1})$, as desired. □

9. Proof of Theorem C: approximation by Julia sets

Our proof of Theorem C follows the second of the two proofs that Lindsey and Younsi give for their Theorem 1.2 in [LY19], except that we replace their application of Hilbert's lemniscate theorem with an application of our Theorem A. We provide the details here for the convenience of the reader. As with Theorem B, we only need to use Theorem A in the easier case of lemniscate graphs without vertices.

Proof of Theorem C. For $E \subset \widehat{\mathbb{C}}$, denote the ε-neighborhood of E by

$$N_\varepsilon(E) := \{z \in \widehat{\mathbb{C}} : d(z, E) < \varepsilon\}.$$

As usual, d denotes the spherical metric. Note that for any $\varepsilon > 0$, the open set $N_\varepsilon(E)$ has finitely many connected components, even if E has infinitely many. This holds since each such component contains a distinct point of any $(\varepsilon/2)$-dense set in $\widehat{\mathbb{C}}$ and such a set can be finite.

After applying a Möbius transformation, we may assume without loss of generality that $p = \infty \in A_1$ and $0 \in A_2$. By Theorem A, there exists a lemniscate $L_r \subset N_\varepsilon(A_2)$ so that L_r consists of a union of pairwise disjoint Jordan curves, and L_r separates $\partial N_\varepsilon(A_2)$ from A_2. Since each component is a Jordan curve, we know that r has no critical points on L_r.

We may color white exactly those faces of L_r which contain at least one component of A_2, and arrange for $r(\infty) = \infty$. Furthermore, if A_1 has finitely many components, and P contains ∞ as well as one point from each bounded component of A_1, we may arrange (by taking ε smaller) for r to satisfy $r^{-1}(\infty) = P$.

Consider the map r^n, where r^n denotes the product r with itself n times, not the nth iteration of r. Since r has a pole at ∞, we know r^n has a super-attracting fixed point at ∞. Since $0 \in A_2$, we must have $|r| < 1$ on some disk $D(0, \delta)$, and so $|r|^n < \delta/2$ on $D(0, \delta)$ if n is large enough. This implies r_n maps $D(0, \delta)$ into $D(0, \delta/2)$ and hence r^n has an attracting fixed point in $D(0, \delta) \subset A_2$.

Denote the corresponding basins of attraction by $\mathcal{A}_n(\infty)$ and $\mathcal{A}_n(0)$, where we emphasize the dependence on n. Since A_2 is compactly contained in $r^{-1}(\mathbb{D})$ (the union of the white faces), we have that $A_2 \subset \mathcal{A}_n(0)$ for large n. Similarly, any compact subset of $r^{-1}(\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}})$ is contained in $\mathcal{A}_n(\infty)$ for large enough n. As a consequence we have $\mathcal{A}_n(0) \subset N_\varepsilon(A_2)$ for large n. This proves

$$d_H(A_2, \mathcal{A}_n(0)) < \varepsilon,$$

and the other two inequalities in Theorem C follow by similar considerations. Thus, r^n and $\mathcal{A}_1 := \mathcal{A}_n(\infty)$ and $\mathcal{A}_2 := \mathcal{A}_n(0)$ satisfy the conclusions of the theorem for all large enough n.

We note the fact that $\mathcal{F}(r^n) = \mathcal{A}_1 \sqcup \mathcal{A}_2$ for large n follows since all critical values of r^n lie in $\mathcal{A}_1 \sqcup \mathcal{A}_2$ for large enough n (since r has no critical points on $L_r := \{|r| = 1\}$, all the critical values of r^n have very large or very small absolute value if n is large). This observation uses some standard but deep facts: by Sullivan's "no wandering domains" theorem (e.g., [Sul85], Theorem IV.1.3 of [CG93] or Theorem F.1 of [Mil06]), every Fatou component is eventually periodic, and it is known that every periodic Fatou component has an associated critical orbit in the domain or that accumulates on the boundary of the component (e.g. Theorems 8.6 and 11.17, Corollary 10.11 and Lemma 15.7 of [Mil06]). In our case, every critical orbit is attracted to fixed points in $\mathcal{A}_1$ or $\mathcal{A}_2$, so there can be no other Fatou components except those that eventually land on one of these two. □

10. A Topological Lemma

Recall that the proof of our main result (Theorem A) relied on Theorem 5.1, but that Theorem 5.1 has not yet been proven in general (only in the case when there are no vertices). Section 11 is devoted to establishing this result. In this section, we give a topological result that will be used in Section 11.

The n-dimensional simplex Δ_n is the convex hull of the standard unit vectors $\{e_k\}_1^{n+1} \subset \mathbb{R}^{n+1}$; e_k is 1 in the kth coordinate and zero elsewhere. These points are called the vertices of the simplex. A face of Δ_n is a convex hull of some non-empty subset of its vertices. A facet is a face of dimension $n-1$ (the convex combination of all but one vertex). Every face is the intersection of facets that contain it, so a continuous map $f : \Delta_n \to \Delta_n$ that maps each facet into itself must also map each face into itself. Every point $x \in \Delta_n$ is in the interior of some face, where interior means that x does not lie in any strictly lower dimensional face (the vertices are interior points of themselves).

The following is Lemma 2.1 of [JR76], but was probably known much earlier.

Theorem 10.1. *Suppose $f : \Delta_n \to \Delta_n$ is a continuous map that sends each facet of Δ_n into itself. Then f is surjective.*

Proof. For completeness, we recall the proof from [JR76]. Since n is fixed, we write $\Delta := \Delta_n$, to simplify notation. Since $f(\Delta)$ is compact and the interior of Δ is dense in Δ, it suffices to show that every interior point is in $f(\Delta)$. By way of contradiction, suppose p is an interior point that is not in the image, and let g be the radial projection of $\Delta \setminus \{p\}$ onto $\partial\Delta$. This map is continuous and fixes each point of the boundary. Let $h : \Delta \to \Delta$ be the linear map extending the cyclic permutation

$$(x_1, x_2, \dots, x_n) \to (x_2, \dots, x_n, x_1)$$

of the vertices. Then $\phi := h \circ g \circ f : \Delta \to \partial\Delta$ is clearly continuous.

We will show ϕ has no fixed point, contradicting Brouwer's theorem [Bro11] (that a continuous self-map of a compact, convex set in $\mathbb{R}^n$ must have a fixed point). This proves that $p \in f(\Delta)$, and hence that f is surjective. Since $\phi(\Delta) \subset \partial\Delta$, any fixed point x must be in $\partial\Delta$, and hence in the interior of some face F_1. By assumption $f(x)$ is also in F_1, and so $g(f(x)) = f(x) \in F_1$, since g is the identity on ∂Q. But h maps each face of Δ to a distinct face (every subset of vertices is sent to a different subset by the cyclic permutation). Hence h maps the interior of F_1 to the interior of some different face F_2, and hence $\phi(x) = h(f(x)) \neq x$. Thus ϕ has no fixed points. □

By projecting e_{n+1} to zero, the simplex $\Delta_n \subset \mathbb{R}^{n+1}$ can identified with $X_n \subset \mathbb{R}^n$, the convex hull of $\{0\}$ and the n unit vectors in $\mathbb{R}^n$. Note that $\Delta_{n-1} \subset X_n$ is the unique facet of X_n that does not contain the origin. Let us denote projection from $\mathbb{R}^n$ onto the jth coordinate by $\pi_j(x_1, ..., x_n) := x_j$ for $1 \le j \le n$, and denote the vector $(x_1..., x_n) \in \mathbb{R}^n$ by $\boldsymbol{x}$.

Theorem 10.2. *Let $F : X_n \to [0, \infty)^n$ be a continuous mapping satisfying*

$$F_j(\boldsymbol{x}) = 0 \text{ iff } x_j = 0. \tag{10.1}$$

Then $\varepsilon X_n \subset F(X_n)$ for some $\varepsilon > 0$.

Proof. The condition in the theorem says that each facet of X_n, except for Δ_{n-1}, is mapped into the hyperplane containing that facet. Because our assumptions imply $F^{-1}(0) = 0$, $F(\Delta_{n-1})$ is a compact set that does not contain zero, and so it has a positive distance ε from the origin. Let R denote the radial retraction of $[0, \infty)^n$ onto $\epsilon \cdot X_n$, i.e.,

$$R(\boldsymbol{x}) := \varepsilon \cdot \frac{(x_1, \dots, x_n)}{x_1 + \cdots + x_n}.$$

if $x_1 + \cdots + x_n \ge \varepsilon$, and $R(\boldsymbol{x}) = \boldsymbol{x}$ if $x_1 + \cdots + x_n \le \varepsilon$. Then $\boldsymbol{x} \to (R \circ F(\boldsymbol{x}))/\varepsilon$ is a continuous map of the simplex X_n into itself, and every facet of X_n maps into itself. Hence $(R \circ F)/\varepsilon$ is surjective by Theorem 10.1, or equivalently, $\varepsilon X_n = R(F(X_n))$. Since R maps $[0, \infty)^n \setminus \varepsilon X_n$ onto $\varepsilon \Delta_{n-1} \subset \partial(\varepsilon X_n)$, an interior point of εX_n can't be in $R(F(X_n))$ unless it is in $F(X_n)$. Thus $F(X_n)$ contains the interior of εX_n. Since $F(X_n)$ is compact, it must contain all of εX_n, as desired. $\square$

Corollary 10.3. *Let $F : [0, 1]^n \to [0, \infty)^n$ be a continuous mapping satisfying*

$$F_j(\boldsymbol{x}) = 0 \text{ iff } x_j = 0. \tag{10.2}$$

Then there exists $\delta > 0$ so that $\{\boldsymbol{y} \in \mathbb{R}^n : y_1 = ... = y_n \text{ and } 0 \le y_1 < \delta\} \subset F([0, 1]^n)$.

Proof. Restrict F to $X_n \subset [0, 1]^n$ and apply Theorem 10.2. If $\delta < \varepsilon/n$, then the diagonal segment is inside $\varepsilon X_n \subset F(X_n) \subset F([0, 1]^n)$, and so the corollary follows. $\square$

11. Proof of Theorem 5.1: harmonic level sets

In this section, we prove Theorem 5.1. Recall that if H is a colored lemniscate graph with no vertices, and if P is a finite set of points from the grey faces, then the function $u_{H,P}$ was defined in Notation 4.2 as the sum of Green's functions for faces of H with the poles in the set P, i.e.,

$$u_{H,P}(z) := \sum_{p \in P} G_{B(p)}(z, p),$$

where $B(p)$ denotes the face of H containing p (we define the Green's function of a domain to be zero off that domain). In this section, we will say that two positive functions f, g are relatively close if $|f - g|/g$ is close to zero. In other words, f/g is uniformly close to 1.

For the convenience of the reader, we re-state the desired result.

Theorem 11.1. *Let G be a 2-colored lemniscate graph, $\varepsilon > 0$, and P a set of points consisting of one point in each grey face of G. Then, for all $\delta > 0$ sufficiently small, there exists a lemniscate graph H without vertices so that each grey face of G is contained in a grey face of H, and so that $u_{H,P}^{-1}(\delta)$ and G are ε-homeomorphic.*

The proof breaks into several steps:

(1) Modify G near each vertex v to consist of straight segments meeting at v and making equals angles there.
(2) We further modify G so that the Green's function for each grey face of G takes the same value at every point of a certain finite set (d points are chosen a fixed distance from every vertex of degree $2d$).
(3) Define a vertex-free lemniscate graph H by adding disks around each vertex v of G. See Figure 10.
(4) Define a harmonic function w_v on the faces of H near each vertex v, so that w_v has a single critical point of order $d-1$ at v, where $2d$ is the degree of v as a vertex of G.
(5) Use a partition of unity to combine w_v and $u_{G,P}$ in an annulus around each vertex v. We make this new function harmonic using the measurable Riemann mapping theorem, but this correction causes the poles to move slightly.
(6) Use a fixed point argument to show that the poles can be placed precisely on P. This gives the desired harmonic function with poles in P, and with a critical level set that is ε-homeomorphic to G.

Step 1: modifying G to be straight near its vertices. By Theorem 2.2 we can assume that G has analytic edges that form equal angles at each vertex. It is easy to modify such a graph G so that its edges are smooth arcs and are straight line segments near each vertex. In other words, we may assume there is a $\rho_0 > 0$ so that for every vertex v of G, $G \cap D(v, 2\rho_0)$ consists of line segments making equal angles at v. Thus ρ_0 represents a scale below which G looks like line segments around each vertex. Note that ρ_0 is the same value at every vertex. We also assume that ρ_0 is so small that no point of P lies inside any of the disks $D(v, 2\rho_0)$.

A sector S with vertex $v \in \mathbb{C}$, radius $r \in (0, \infty]$ and angle $\theta \in (0, 2\pi]$ is a set of the form

$$S := v + \{z : 0 < |z| < r, |\arg z - \theta_0| \le \theta/2\}$$

for some $\theta_0 \in [0, 2\pi)$ and a truncated sector is a set of the form

$$v + \{z : s < |z| < r, |\arg z - \theta_0| \le \theta/2\}$$

for some $0 < s < r < \infty$.

By our choice of G and ρ_0, for each grey face B of G and vertex $v \in \partial B$, $B \cap D(v, \rho_0)$ is a union of sectors of radius ρ_0. If v has degree $2d$ in G, then each sector at v has angle π/d. Each such sector can be conformally mapped to the half-disk $W := \mathbb{D} \cap \mathbb{H} = \{z : |z| < 1, \mathrm{Im}(z) > 0\}$ by a power map $\tau(z) := a(z-v)^d$, for some $a \in \mathbb{C} \setminus \{0\}$. Then $U(z) := u_{G,P} \circ \tau^{-1}(z)$ is harmonic on the half-disk W and it vanishes on $I := [-1,1]$, so U extends harmonically across I by the Schwarz reflection principle. Thus

$$U(z) = U(x+iy) = by + O(y^2) + O(xy) = b \cdot \mathrm{Re}(z) + O(|z| \cdot \mathrm{Re}(z)),$$

as z approaches zero, where $b > 0$ is the normal derivative of U at zero. The normal derivative is positive since U positive on W, for if $\frac{\partial U}{\partial n}(0) \le 0$, then U would take negative values somewhere in W. This implies that $u_{G,P}$ is asymptotically equal to $c \cdot \mathrm{Re}((\lambda(z-v))^d)$ as $z \to v$ through the sector, for some $|\lambda| = 1$ and $c > 0$. More precisely, we have

$$c_S := \lim_{z \to v, z \in S} \frac{u_{G,P}(z)}{\mathrm{Re}(\lambda(z-v)^d)} \in (0, \infty). \tag{11.1}$$

Step 2: modifying G to equalize the Green's function. In this step we use the topological fact recorded in Corollary 10.3. For the remainder of this section we use a plain G to denote a lemniscate graph and a G with a subscript, e.g., G_W, to denote Green's function for a domain W.

We will choose $\eta, \rho > 0$ so that $\eta \ll \rho \ll \rho_0$. Given a grey face B of G, let v be a vertex on the boundary of B. Let $2d$ be the degree of v. Then $\partial D(v, \eta) \cap B$ consists of d arcs each of angle measure π/d. We call the center of each such arc a "base point" b. See Figure 9. We claim that we can modify G to obtain a new graph G' so that $u_{G',P}(b)$ has the same positive value at every base point b.

Figure 9 shows the case where the vertex v has degree eight, so $d = 4$ and $B \cap D(v, \rho_0)$ consists of four sectors. Let S be one of these sectors. We modify its boundary inside the annulus $\{\rho/2 \le |z| \le \rho\}$. Let γ be one of the radial sides of the sector S. We modify γ as illustrated in Figure 9. The part of γ between the circles $\{|z| = \rho\}$ and $\{|z| = \rho/2\}$ is replaced by a new arc consisting of three parts: one subarc on each of these two circles, and a radial segment joining them. The circular arcs lie inside S and both have angle measure $\theta \in (0, \pi/2d)$. The second radial segment in ∂S is modified symmetrically. Doing this for every sector associated to every vertex of G gives a new lemniscate graph G' that has the same vertices as G.

Let $\{S_k\}_1^m$ be a listing of all the sectors of radius ρ_0 associated to G, and let $S_k' \subset S_k$ be the subregion obtained after the modification. This list is over all sectors of all grey faces of G; since half the sectors at any vertex v are grey, the number of sectors is $\frac{1}{2}\sum_v \deg(v)$. Let θ_k be the angular width of the channel in S_k'. We let v_k denote the vertex of S_k; observe that we can have $v_k = v_j$ even if $j \ne k$ (different sectors can share a vertex). Also, $S_k' \cap D(v, \rho/4)$ is still a sector and it contains exactly one base point, which we denote b_k.

As we continuously change the value of θ_k, the value of $u_{G',P}(b_k)$ changes continuously, and it tends to zero as θ_k tends to zero (this is intuitively clear, and it is an immediate consequence of the Ahlfors distortion theorem, e.g., Theorem IV.6.2 of [GM08]). By Corollary 10.3 there is a choice of $\boldsymbol{\theta} = (\theta_1, \dots, \theta_m)$ so that $u_{G',P}$ takes the same value at every base point.

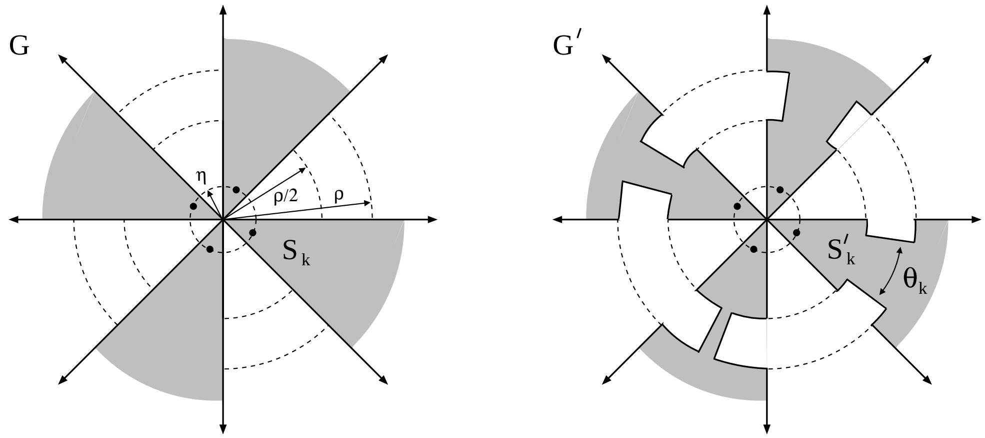

FIGURE 9. By "pinching" the sectors that lead to a vertex v, we can decrease the harmonic measure near v. Hence we can make Green's function as small as we wish at the base points (black dots), and make the values at all the base points as close to each other as we wish.

Step 3: constructing the vertex-free approximation. To simplify notation we refer to G' as just G; we will have no further need to refer to the previous versions of the graph.

We want to modify G inside even smaller neighborhoods of each vertex v to obtain a vertex free graph H. For each v we will choose a value $\delta_v > 0$, remove the segments $G \cap D(v, \delta_v)$, and add the d arcs of $\partial D(v, \delta_v)$ that are outside the grey faces of G. This construction is illustrated in Figure 10. The new lemniscate graph H has no vertices (we just removed all of them) and H equals G outside the δ_0-disks around each vertex of G. We shall see later that δ_v is the same for vertices with the same degree, although this is not crucial to the argument. More important is that we will be able to choose every δ_v to be as small as we wish, say all smaller than some $\delta_0 \ll \eta$.

Remark 11.2. Figure 10 also shows that each grey face of H is the union of some grey faces of G, together with disks centered at all the vertices on the boundaries of these faces. If G is connected, then so is the closure of it grey faces, and hence H has only one grey face (although H itself may be disconnected). Similarly, if G is disconnected, but has no multiply connected white faces, then the closure of its grey faces is still connected, and hence H has a single grey face in this case too.

Remark 11.3. The construction in the following Steps 4-6 is not needed if all the vertices of G have degree four. In that case, one can show that if the δ_v's are all small enough, then $u_{H,P}$ must have a simple critical point near each of the vertices v of G. Then Corollary 10.3 lets us choose the δ_v's so that these critical values are all the same. One can then prove that the level set of $u_{H,P}$ passing through these critical points is ε-homeomorphic to G.

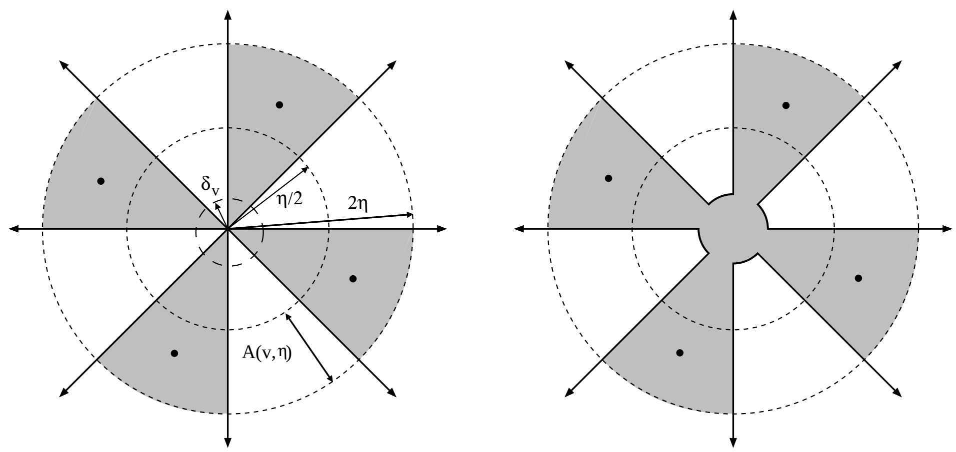

FIGURE 10. We modify G near each vertex v by removing radial segments and adding circular arcs, so that d different sectors are now joined by a disk centered at v. We let $A(v, \eta) := \{z : \eta/2 < |z - v| < 2\eta\}$.

However, a more complicated argument seems necessary for higher degree vertices. Although it is not too hard to show that $u_{H,P}$ has $d-1$ critical points (counted with multiplicities) near each vertex v of degree $2d$, it is not obvious whether these points form a single critical point of order $d-1$; probably they do not. Instead, we will construct a function that has most of the necessary properties: it is zero on H, has logarithmic poles at P, has critical points of the correct orders at the vertices of G, and it is "nearly harmonic", i.e., it fails to be harmonic only in the annuli $A(v, \eta) := \{z : \eta/2 < |z - v| < 2\eta\}$. Using the measurable Riemann mapping theorem, we will obtain a quasiconformal map $\psi : \mathbb{C} \to \mathbb{C}$ that is close to the identity, and so that pre-composing our nearly harmonic function with ψ gives a function V that is harmonic on the faces of $H' := \psi^{-1}(H)$ and has its poles at $P' := \psi^{-1}(P)$. The final step will be to show the poles can be placed exactly at P.

Step 4: adding high degree critical points. To start the construction described in the previous paragraph, let $\Omega_d := D(0, 1) \cup \{\mathrm{Im}(z^d) < 0\}$. This is an infinite domain that is symmetric under rotation by $2\pi/d$. (It looks like the right side of Figure 10: a union of the unit disk and d evenly spaced infinite sectors of angle π/d.) Let f_d be the holomorphic map from Ω_d to the right half-plane that is given by the following composition of four maps (see Figure 11):

(1) the power map z^d sending Ω_d to $W_1 := \{z : \mathrm{Im}(z) < 0\} \cup \{z : |z| < 1\}$,
(2) the linear fractional map $(1 + z)/(1 - z)$ sending W_1 to $W_2 := \{-\pi < \arg z < \pi/2\}$,
(3) the power map $e^{i\pi/6} z^{2/3}$ sending W_2 to the right half-plane,
(4) the linear fractional map $(4/3)/(z + i)$ (this preserves the right half-plane).

The pole of the final map is chosen so that $f_d(\infty) = \infty$. The last three maps define a conformal map from a perturbed version of the lower half-plane to the right half-plane that fixes ∞, and it is easy to check that this map is asymptotically linear near infinity, and the "4/3" is chosen so that it is asymptotic to iz. Thus f_d is asymptotic to z^d near infinity (for $z \in \Omega_d$).

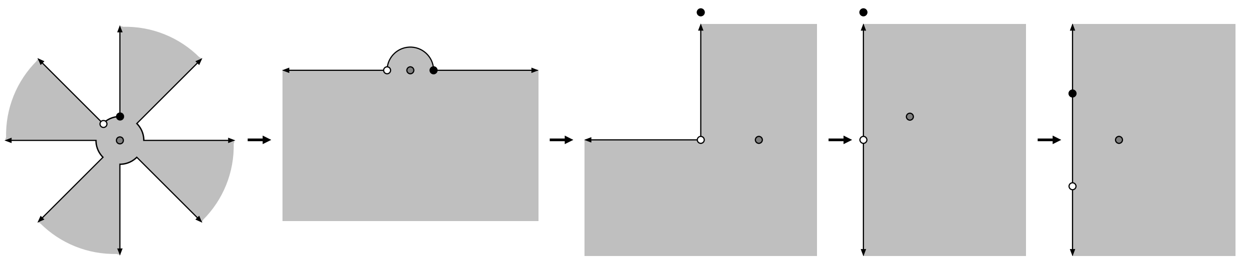

FIGURE 11. The domain Ω_d can be mapped to a half-plane by a composition f_d of explicit maps: the power map z^d, the linear fractional transformation $(1+z)/(1-z)$, the power map $e^{i\pi/6}z^{2/3}$, and another linear fractional transformation $(4/3)/(1+z)$ (chosen so that $f_d(\infty) = \infty$).

Set $w_d(z) := \mathrm{Re}(f_d(z))$. This is a positive harmonic function on Ω_d that is zero on $\partial\Omega_d$, and is asymptotic to $\mathrm{Re}(z^d)$ as z tends to infinity in Ω_d. More precisely,

$$\lim_{z\to\infty, z\in\Omega_d} \frac{w_d(z)}{\mathrm{Re}((z-v)^d)} = 1. \tag{11.2}$$

It is easy to check that w_d has a critical point of order $d-1$ at the origin and no other critical points. Figure 12 shows contour plots of w_d for $d = 2, 3, 4, 5, 6, 10$.

Suppose B is a grey face of H. For each vertex v of G that is in B, we define a function by translating, rotating and rescaling w_d, i.e., on $B \cap D(v, 2\eta)$ we set

$$w_v(z) := c_v \cdot w_d(\lambda_v(z-v)/\delta_v) \tag{11.3}$$

where $|\lambda_v| = 1$ is chosen to rotate the sectors of B at v to match the "arms" of Ω_d, and the constant c_v is chosen so that $w_v(b) = u_{G,P}(b)$ at each of the base points b surrounding v (recall $u_{G,P}$ has the same value at each of these points).

Then w_v is a harmonic function that has a critical point of the correct degree located at the correct point. However, this function is only defined near each vertex, not on entire faces of H. On the other hand, the function $u(G,P)$ is not harmonic on $B \cap D(v, \delta_v) = D(v, \delta_v)$ for each vertex v (inside this disk, $u_{G,P}$ is positive and harmonic on half of the sectors touching v and vanishes on the other sectors). Thus neither w_v nor $u_{G,P}$ can be $u_{H,P}$, but we will construct $u_{H',P}$ for some $H' \approx H$ by combining these two functions with a partition of unity, and then applying the measurable Riemann mapping theorem.

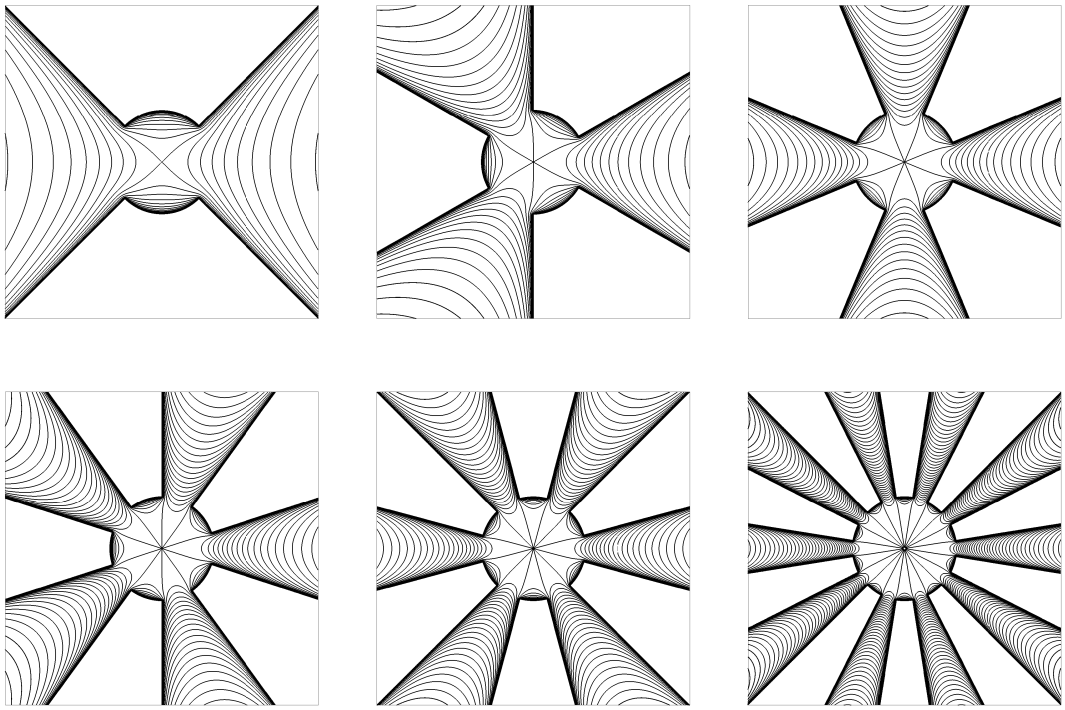

FIGURE 12. Contour plots of the function w_d when $d = 2, 3, 4, 5, 6$ and 10. These were computed as $w_d = \mathrm{Re}(f_d(z))$, where f_d is the holomorphic map shown in Figure 11 sending Ω_d to the right half-plane. Since f_d' only has a zero at 0, the only critical point of w_d is at the origin.

Step 5: merging two harmonic functions. By our earlier remarks, (11.1) holds if $\eta \ll \rho$ and (11.2) holds if $\delta_v \ll \eta$. If both these conditions hold then $u_{G,P}(z)$ and $w_v(z)$ are both relatively close to multiples of the same function in $B \cap A(v, \eta)$, and hence are relatively close to each other. In other words, $u_{G,P}(z)/w_v(z)$ is uniformly close to a constant on $B \cap A(v, \eta)$. Previously, we had multiplied w_v by a constant to make it agree with $u_{G,P}$ at the basepoints around each vertex, so we must have $u_{G,P}(z)/w_v(z) \approx 1$. That is, for any $\varepsilon > 0$ we can ensure

$$1 - \varepsilon \leq \frac{u_{G,P}(z)}{w_v(z)} \leq 1 + \varepsilon$$

on $B \cap A(v, \eta)$ by choosing ρ, η and δ_0 all sufficiently small (and each sufficiently smaller than the previous one).

Note that when we decrease δ_v by a factor of t, the constant c_v in (11.3) has to decrease by a factor of approximately t^d, in order to maintain the equality $w_v(b) = u_{G,P}(b)$. Thus as

δ_v tends to zero, so does c_v, and to make the values of $w_v(v)$ the same for every v, we must take c_v to have the same value for all vertices with the same degree. Since each point v is a critical point of w_v, this implies that all the critical values are the same.

Suppose $S = S_k$ is one of the sectors we are considering, and suppose f_1 is a holomorphic function in $W := S \cap \{\eta/3 < |z - v| < 3\eta\}$ that has real part w_v; this truncated sector is simply connected, so w_v has a harmonic conjugate on W. Similarly suppose f_2 is holomorphic on W with real part $u_{G,P}$. By Schwarz reflection, both these functions extend analytically across both radial sides of W and define a holomorphic function on the union W' of W and its reflections. Now let $\Omega := S \cap \{\eta/2 < |z - v| < 2\eta\} \subset W$. This is compactly contained in W' and $\text{dist}(\partial W', \Omega) \simeq \eta$. See Figure 13.

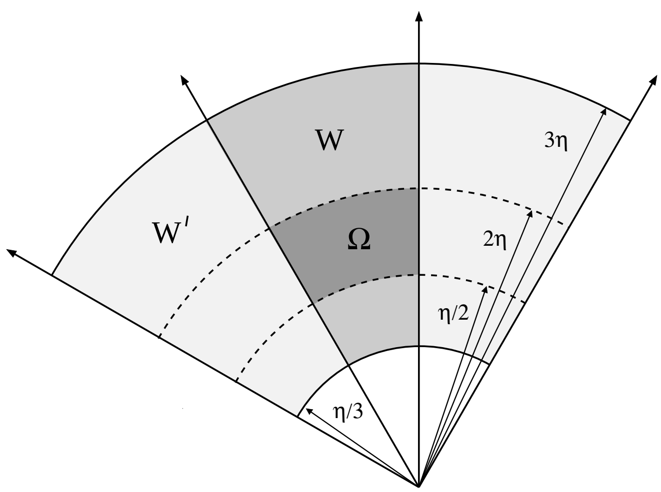

FIGURE 13. The functions w_v and $u_{G,P}$ are harmonic on different parts of the grey faces of H, but they are both defined on the intersection of these faces with the annuli $A(v, \eta)$ around each vertex. On Ω, a connected component of this intersection, we combine them using a partition of unity. Since both functions extend analytically to a neighborhood W' of $\overline{\Omega}$, and are very close there, the Cauchy estimates imply their partial derivatives are close in Ω, and thus the merged function is "nearly" harmonic.

To simplify notation, assume $v = 0$. By our remarks above, both the functions f_1, f_2 are very close to a multiple of z^d, and hence we can write

$$f_1(z) = cz^d(1 + \varepsilon_1(z)), \quad f_2(z) = cz^d(1 + \varepsilon_2(z)),$$

where ε_1 and ε_2 are holomorphic functions on W' that are both as small as we wish, say $|\varepsilon_1|, |\varepsilon_2| < \varepsilon$. By the Cauchy estimates, $|\varepsilon_1'|, |\varepsilon_2'| = O(\varepsilon/\eta)$ on Ω. Take a smooth, decreasing function φ on $[0, \infty) \to [0, 1]$ so that $\varphi = 1$ on $[0, \eta/2]$ and $\varphi = 0$ on $[2\eta, \infty)$ and extend it to the plane by $\varphi(z) := \varphi(|z|)$. We can choose φ so that its partial derivatives are bounded by $O(\eta^{-1})$. Define

$$\begin{aligned} F(z) &= \varphi(z)f_1(z) + (1-\varphi(z))f_2(z) \\ &= \varphi(z)cz^d(1+\varepsilon_1(z)) + (1-\varphi(z))cz^d(1+\varepsilon_2(z)) \\ &= c\cdot z^d[1+\varepsilon_2(z)+\varphi(z)(\varepsilon_1(z)-\varepsilon_2(z))]. \end{aligned}$$

Since $|z| \simeq \eta$ on $A(0,\eta)$ we have

$$\partial_{\overline{z}}F(z) = cz^d\left(\partial_{\overline{z}}\varphi(z)(\varepsilon_1(z)-\varepsilon_2(z))\right) = cz^dO(\eta^{-1}\varepsilon) = cz^{d-1}O(\varepsilon)$$

and

$$\begin{aligned} \partial_z F(z) &= c\cdot d\cdot z^{d-1}[1+\varepsilon_2(z)+\varphi(z)(\varepsilon_1(z)-\varepsilon_2(z))] \\ &\qquad +cz^d[\partial_z\varepsilon_2(z)+\partial_z\varphi(z)(\varepsilon_1(z)-\varepsilon_2(z))+\varphi(z)(\partial_z\varepsilon_1(z)-\partial_z\varepsilon_2(z)) \\ &= c\cdot d\cdot z^{d-1}[1+O(\varepsilon)+O(|z|(\eta^{-1}\varepsilon))] \\ &= c\cdot d\cdot z^{d-1}[1+O(\varepsilon)], \end{aligned}$$

so

$$\frac{\partial_{\overline{z}}F}{\partial_z F}(z) = \frac{c\cdot z^{d-1}O(\varepsilon)}{c\cdot dz^{d-1}[1+O(\varepsilon)]} = O(\varepsilon).$$

Next we use the measurable Riemann mapping theorem to find a quasiconformal map ψ of the plane to itself whose dilatation is $F_{\overline{z}}/F_z$ on Ω and zero elsewhere. Then ψ is conformal off Ω and $F\circ\psi^{-1}$ is holomorphic on Ω. The dilatation of ψ is bounded $O(\varepsilon)$, which is as small as we wish. Thus we can take ψ to be a ε-homeomorphism for any $\varepsilon > 0$ that we want.

Now define

$$V = \begin{cases} \mathrm{Re}(F\circ\psi^{-1}), & \text{on } \Omega, \\ w_v\circ\psi^{-1}, & \text{on } B\cap\cup_v D(v,\eta/2), \\ u_{G,P}\circ\psi^{-1}, & \text{on } B\setminus\cup_v D(v,3\eta). \end{cases}$$

Then V is positive and harmonic on the grey faces of $H' := \psi(H)$ with logarithmic poles at $P' := \psi(P)$, and it vanishes on H' and on the white faces of H' (V is harmonic in the regions of the form Ω because the quasiconformal map ψ was chosen to make this happen; elsewhere ψ is conformal so V harmonic since $\mathrm{Re}(F)$ is harmonic.) Thus $V = u_{H',P'}$. Also, V has critical points at the images under ψ of the vertices of G, and the critical values of V at these points are not changed by pre-composition with ψ^{-1}. Thus V has the same critical values as $\mathrm{Re}(F)$, and so the degree of the critical level set (as a graph) at its vertex $\psi(v)$ equals the degree of v in G. Using this and Lemma 3.1, it is easy to prove that the level set is ε-homeomorphic to G.

Step 6: placing the poles precisely. The one problem that remains is that the new poles $P' = \psi(P)$ are not the same as P. However, this can be dealt with as follows. Suppose P has n points and we think of this as a vector in $\mathbb{C}^n$. Let $B(P,\sigma)$ be a small ball of radius

$\sigma > 0$ around P in $\mathbb{C}^n$, and for $Q \in B(P, \sigma)$ we repeat the construction above, obtaining a harmonic function V_Q that has logarithmic poles at $Q' := \psi_Q(Q)$. In the construction, when we move P continuously, the value of $u_{G,P}$ at each base point changes continuously, so our choices of the channel angles $(\theta_1, \ldots, \theta_m)$ also change continuously. Thus the values of $u_{G,P}$ and its partial derivatives on each $A(v, \eta)$ change continuously with P, which implies the same for F (w_d and φ do not change), and hence so does our solution of the Beltrami equation ψ (properly normalized). Thus the dependence of Q' on Q is continuous, and we can make $|Q - Q'|$ as small as we wish, say $< \sigma/2$, by choosing ε to be small enough. Then the following lemma shows that there exists a Q so that $Q' = P$.

Lemma 11.4. *Let $\mathbb{B}^n$ denote the unit ball in $\mathbb{C}^n$ and suppose that $F : \mathbb{B}^n \to \mathbb{C}^n$ is a continuous map with the property that $|F(\boldsymbol{x}) - \boldsymbol{x}| < 1/2$. Then $0 \in F(\mathbb{B}^n)$.*

Proof. Suppose $0 \notin F(\mathbb{B}^n)$. Choose a continuous, increasing function $\phi : [0, 1] \to [0, 1]$ so that $\phi(t) = 0$ if $t \in [0, 1/2]$ and $\phi(1) = 1$. Then $G(\boldsymbol{x}) := (1 - \phi(|\boldsymbol{x}|))F(\boldsymbol{x}) + \phi(|\boldsymbol{x}|)\boldsymbol{x}$ is continuous on $\overline{\mathbb{B}^n}$, equals F on $\frac{1}{2}\mathbb{B}^n$, and is the identity on $\partial\mathbb{B}^n$. If $|\boldsymbol{x}| \leq 1/2$, then $G(\boldsymbol{x}) = F(\boldsymbol{x}) \neq 0$. If $|\boldsymbol{x}| > 1/2$, then the ball $B(\boldsymbol{x}, 1/2)$ does not contain 0, but it does contain both $\boldsymbol{x}$ and $F(\boldsymbol{x})$, and hence it contains $G(\boldsymbol{x})$, which is on the line segment from $\boldsymbol{x}$ to $F(\boldsymbol{x})$. Therefore G is never zero on $\mathbb{B}^n$. Taking $R(\boldsymbol{x}) := \boldsymbol{x}/|\boldsymbol{x}|$ to be the radial projection of $\mathbb{C}^n \setminus \{0\}$ onto $\partial\mathbb{B}^n$, we see that $R \circ G : \overline{\mathbb{B}^n} \to \partial\mathbb{B}^n$ is continuous and equals the identity on $\partial\mathbb{B}^n$, i.e., it is a retraction of $\mathbb{B}^n$ onto $\partial\mathbb{B}^n$. But following such a retraction by the antipodal map $\boldsymbol{x} \to -\boldsymbol{x}$ on $\partial\mathbb{B}^n$ gives a continuous map of the closed ball into itself with no fixed point, contradicting Brouwer's theorem. Therefore we must have $0 \in F(\mathbb{B}^n)$. □

This completes the proof of Theorem 11.1, and hence the proof of Theorem A.

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