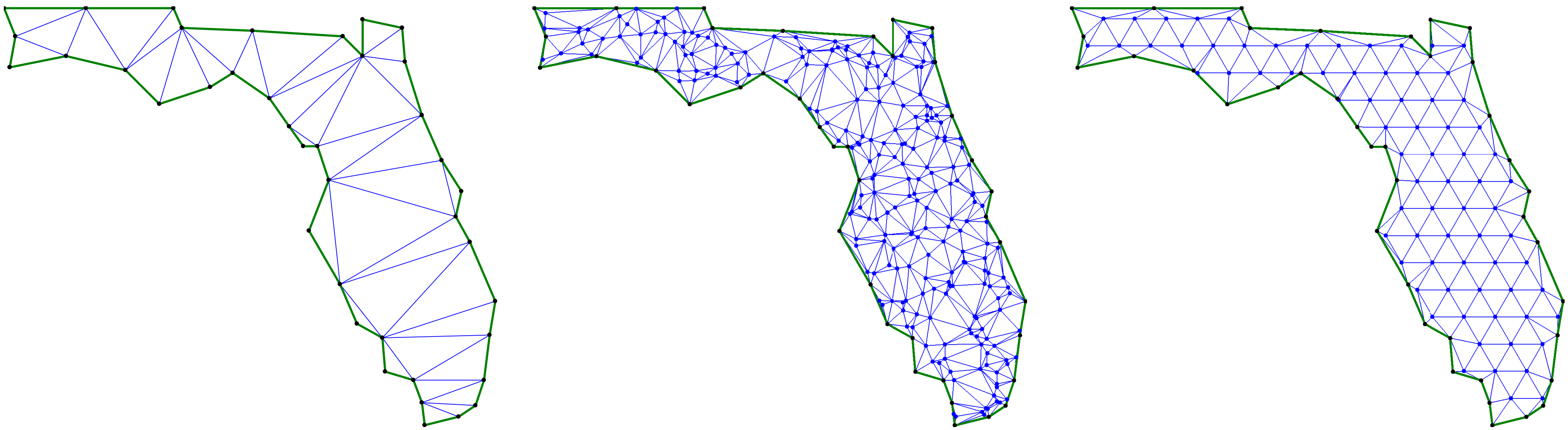
A Brief History of Triangulations

Christopher Bishop, Stony Brook University

R. Kent Nagle Lecture, USF, April 9, 2026



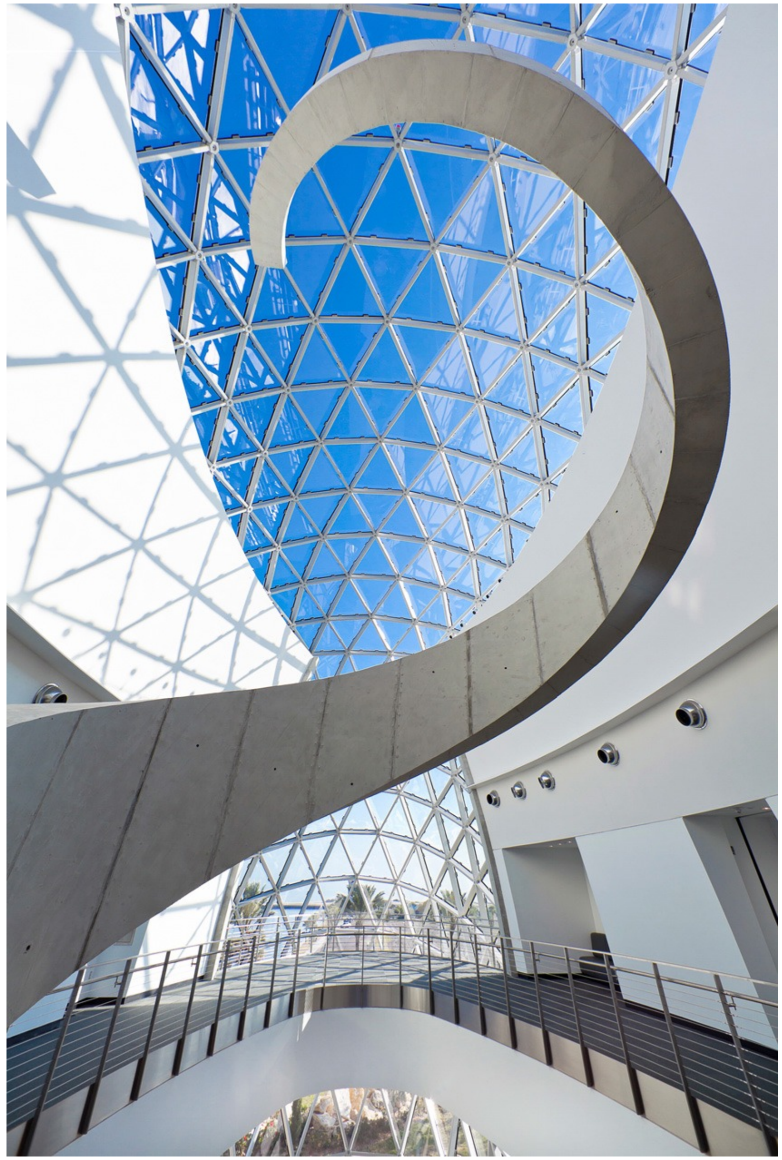
slides posted at `www.math.stonybrook.edu/~bishop/lectures`

Egyptian triangles

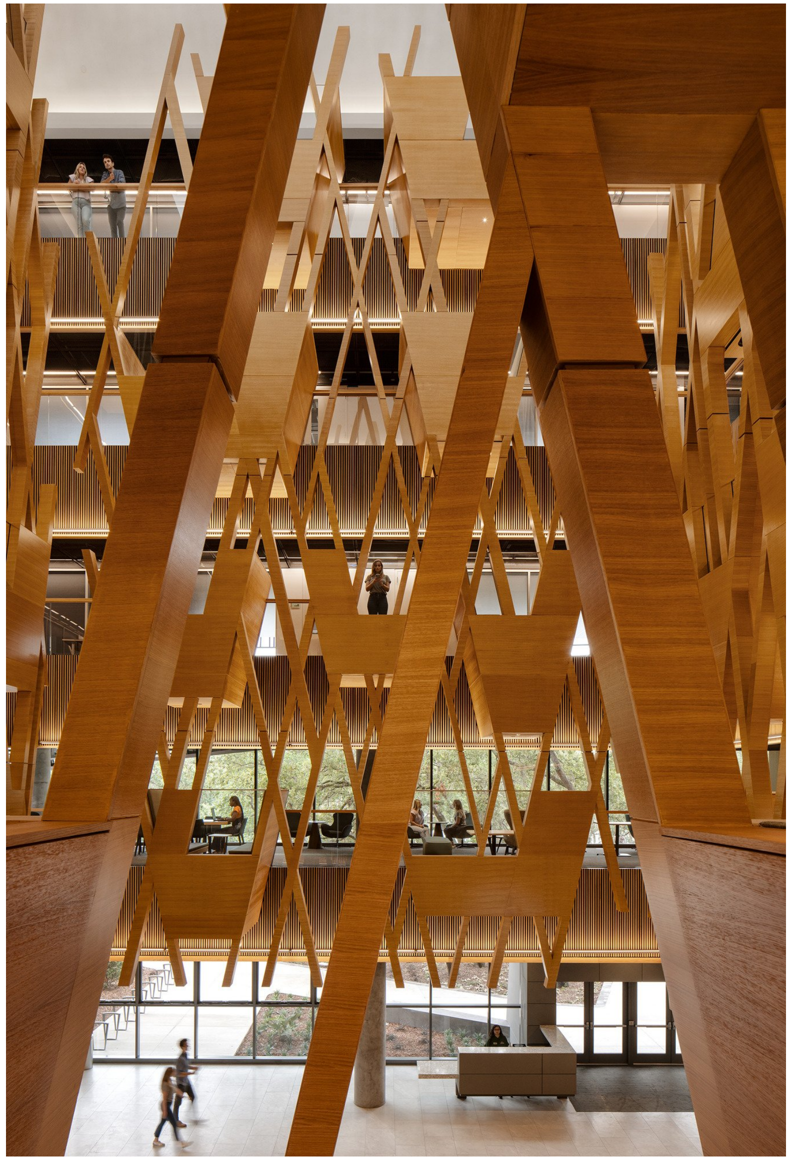
French triangles

British triangles

Floridian triangles

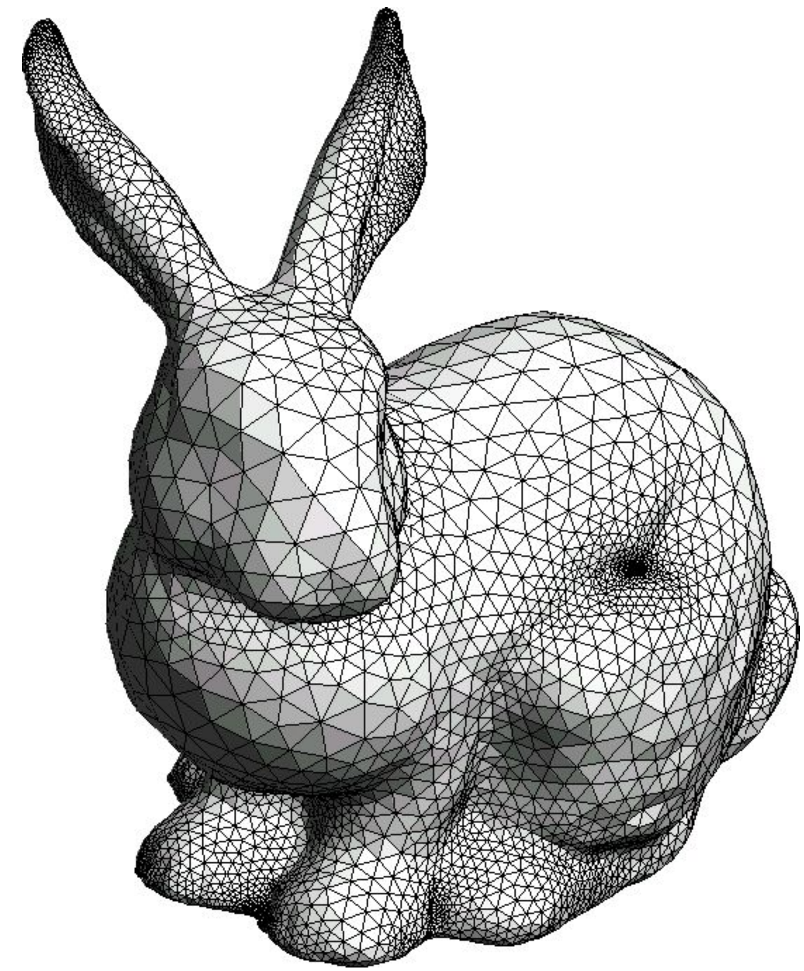


St Petersburg triangles

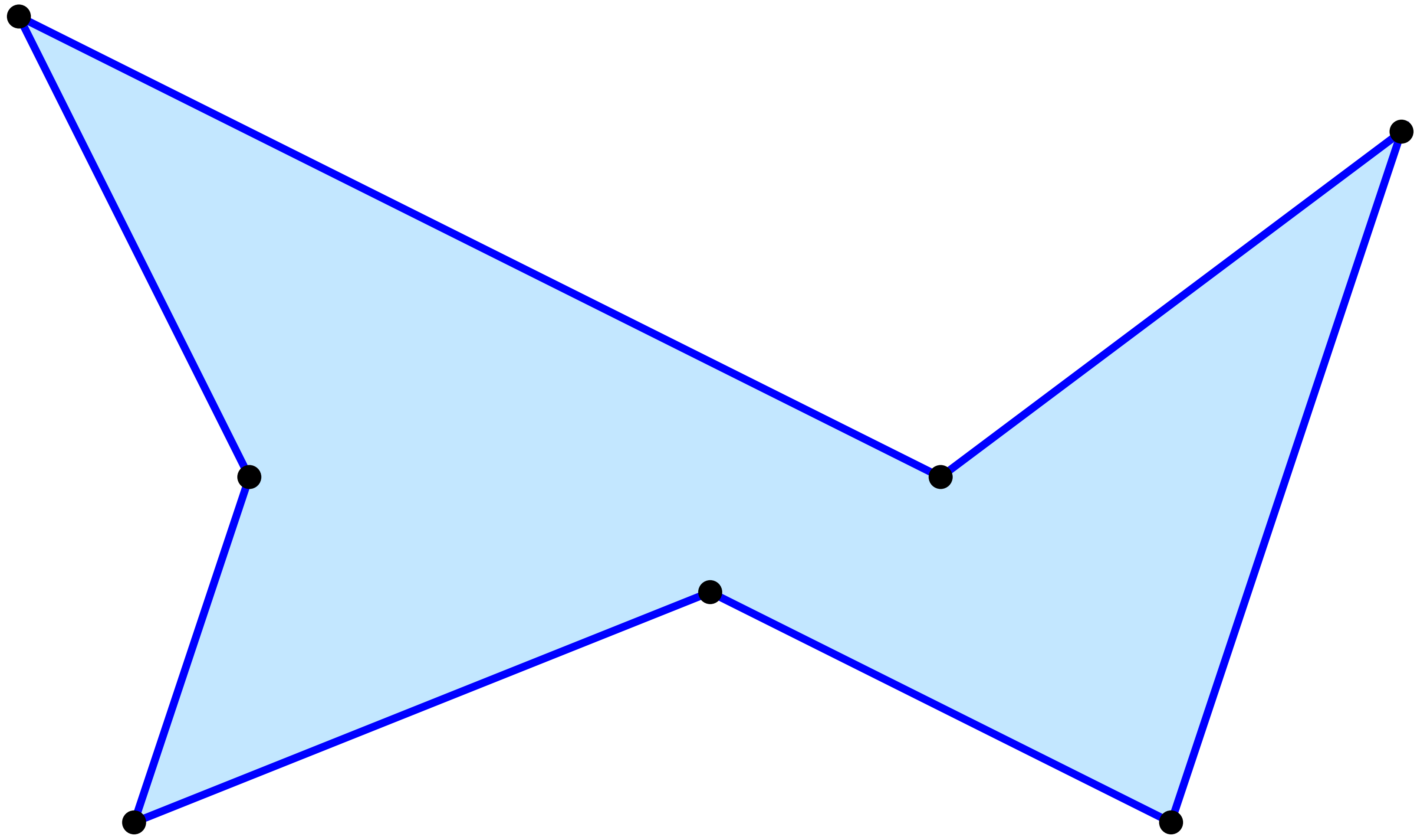


USF triangles

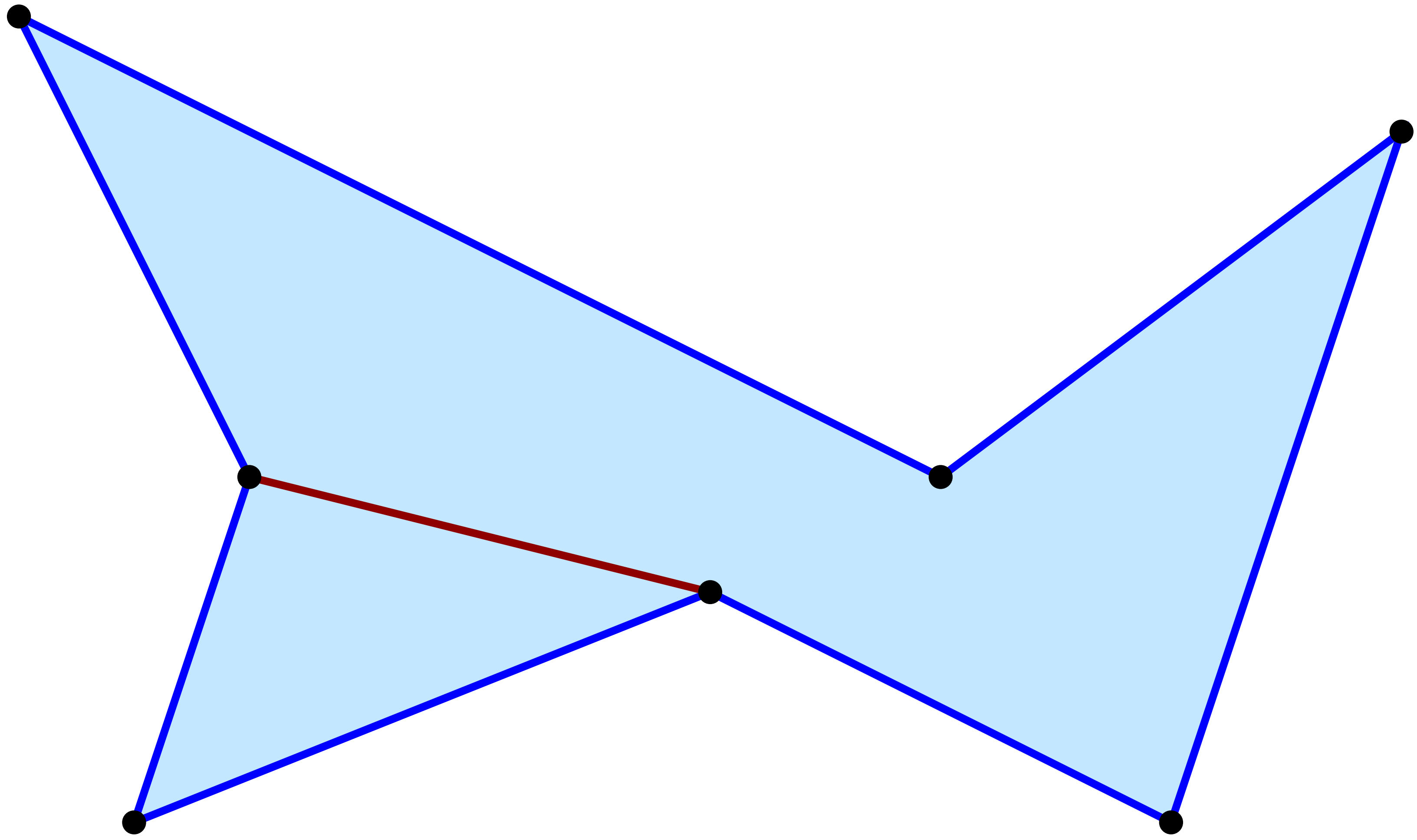
Triangles in engineering



Triangles in computer graphics
(The Stanford Bunny)



Triangulation of a polygon



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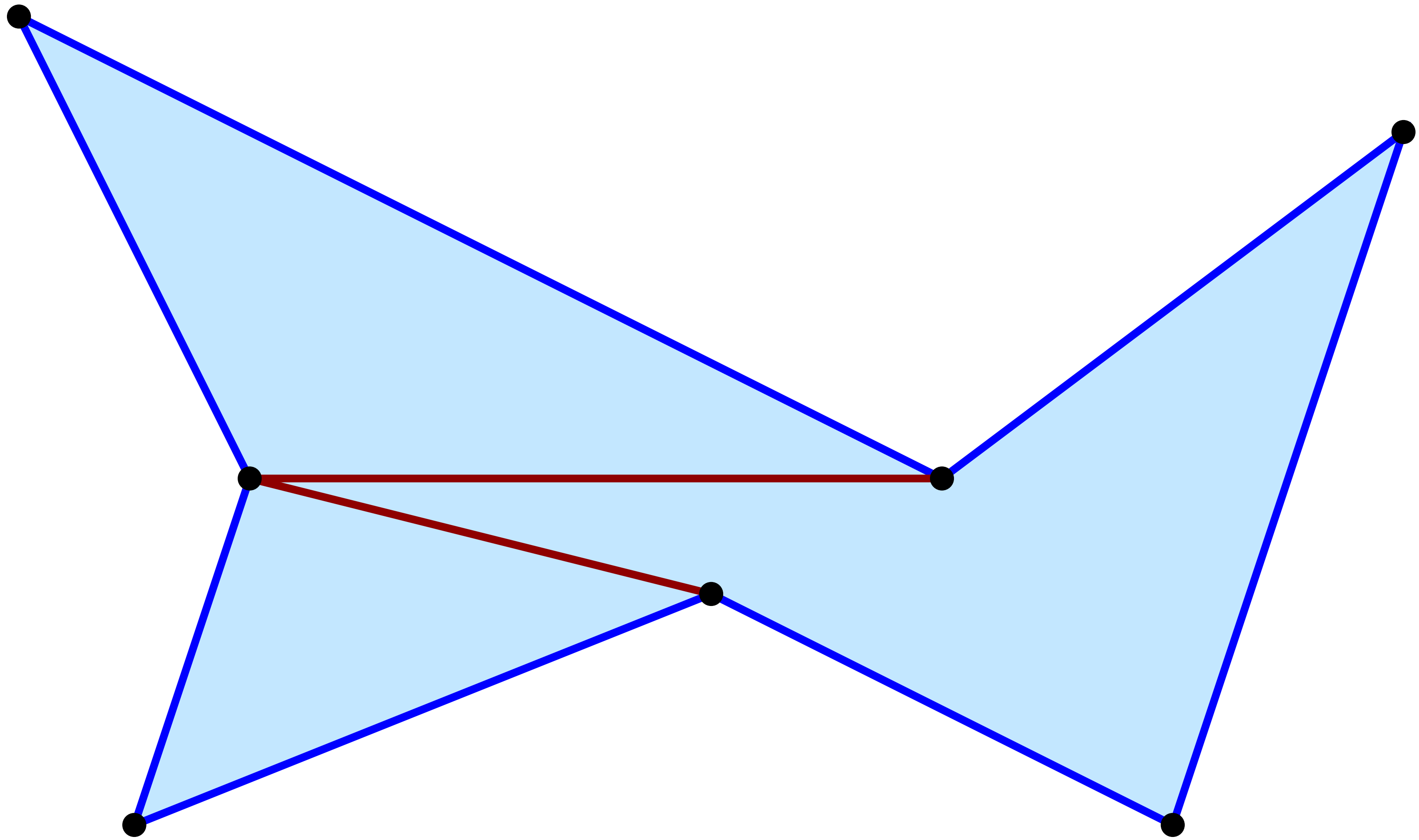

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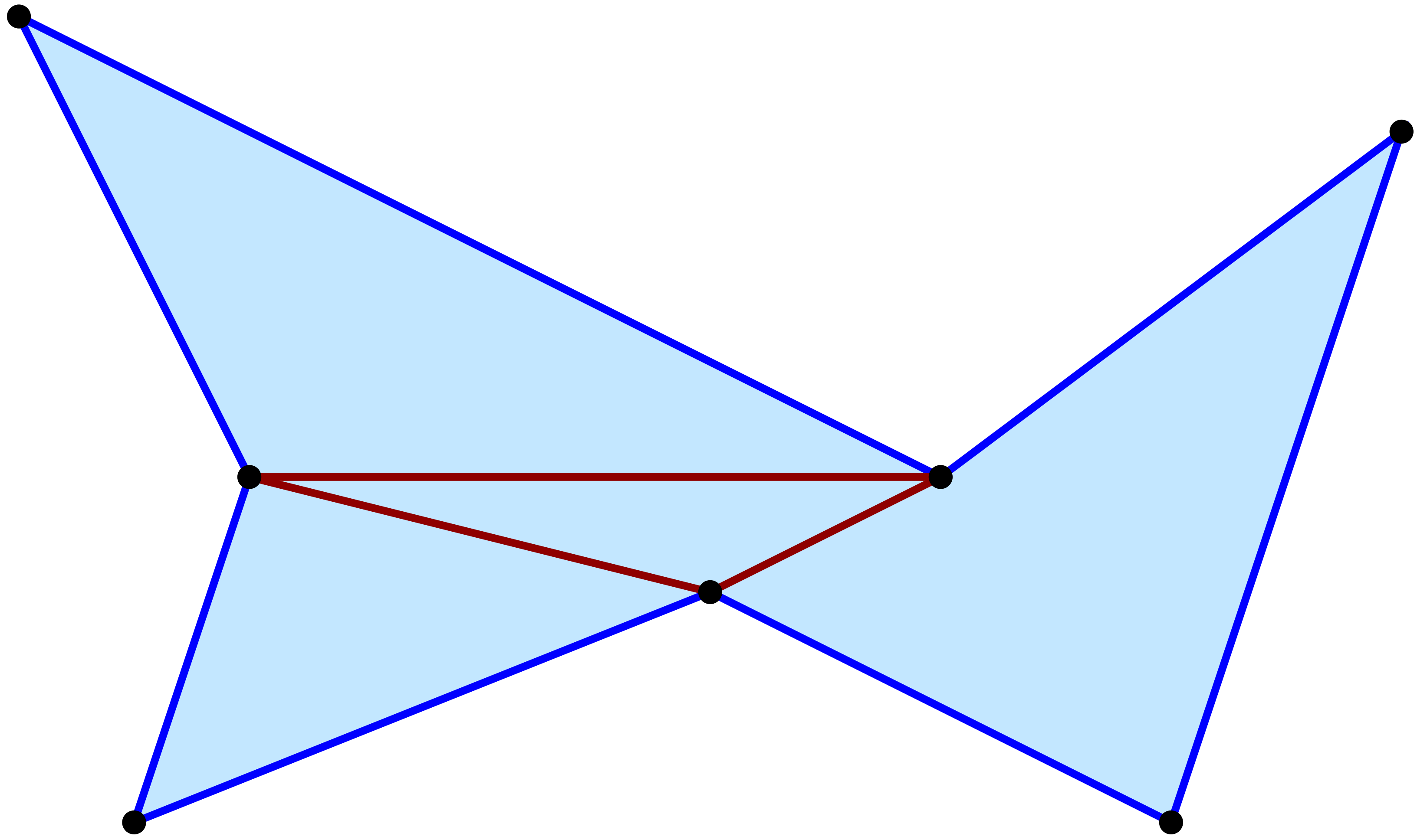

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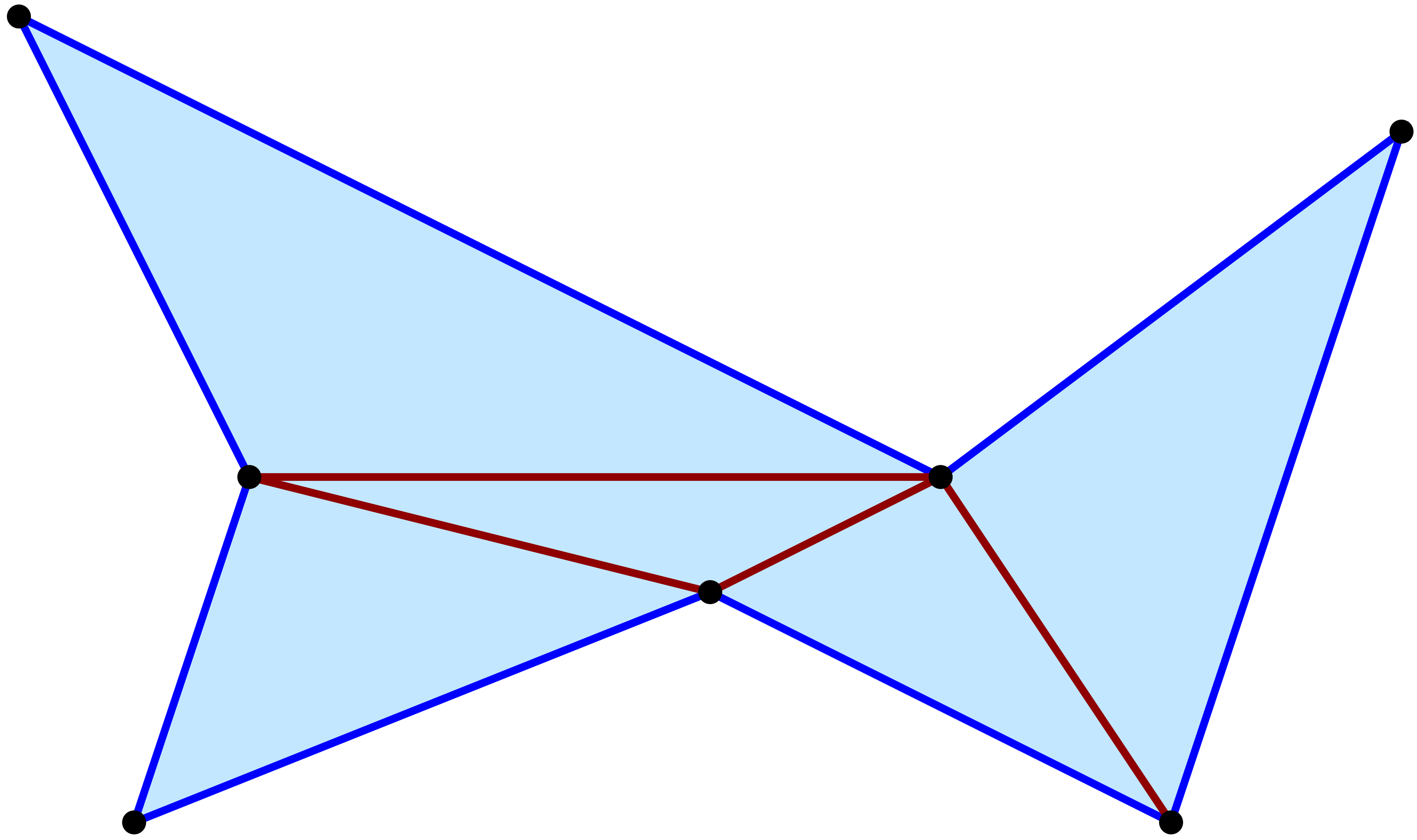
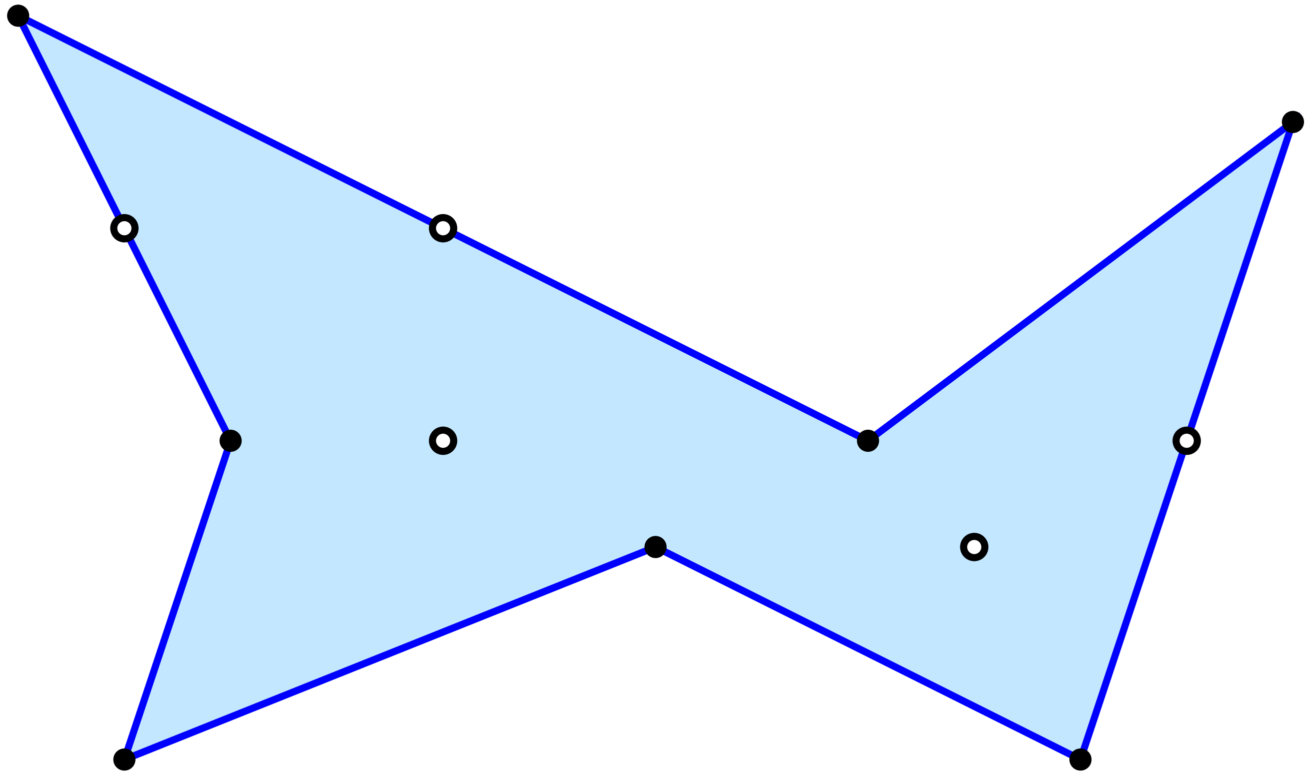


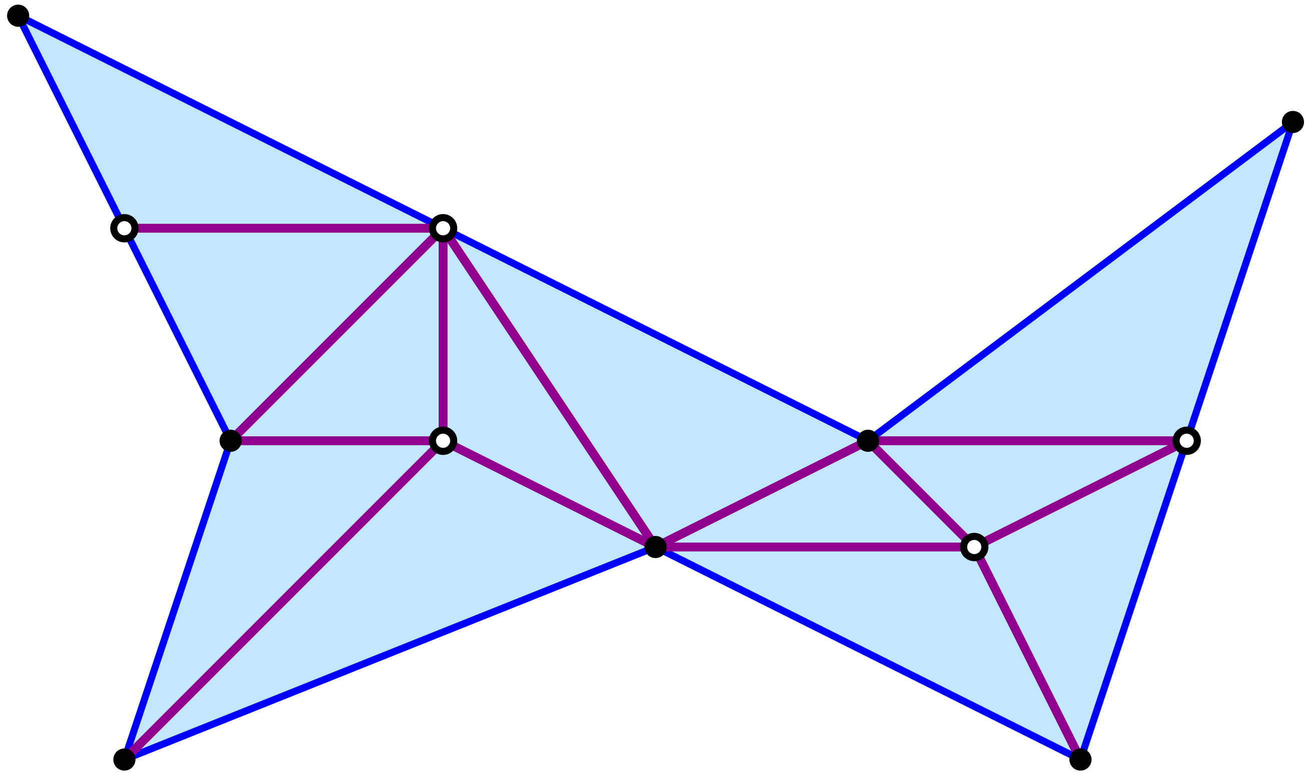
Triangulation of a polygon
using only the vertices of the polygon

There are three types of triangulations:

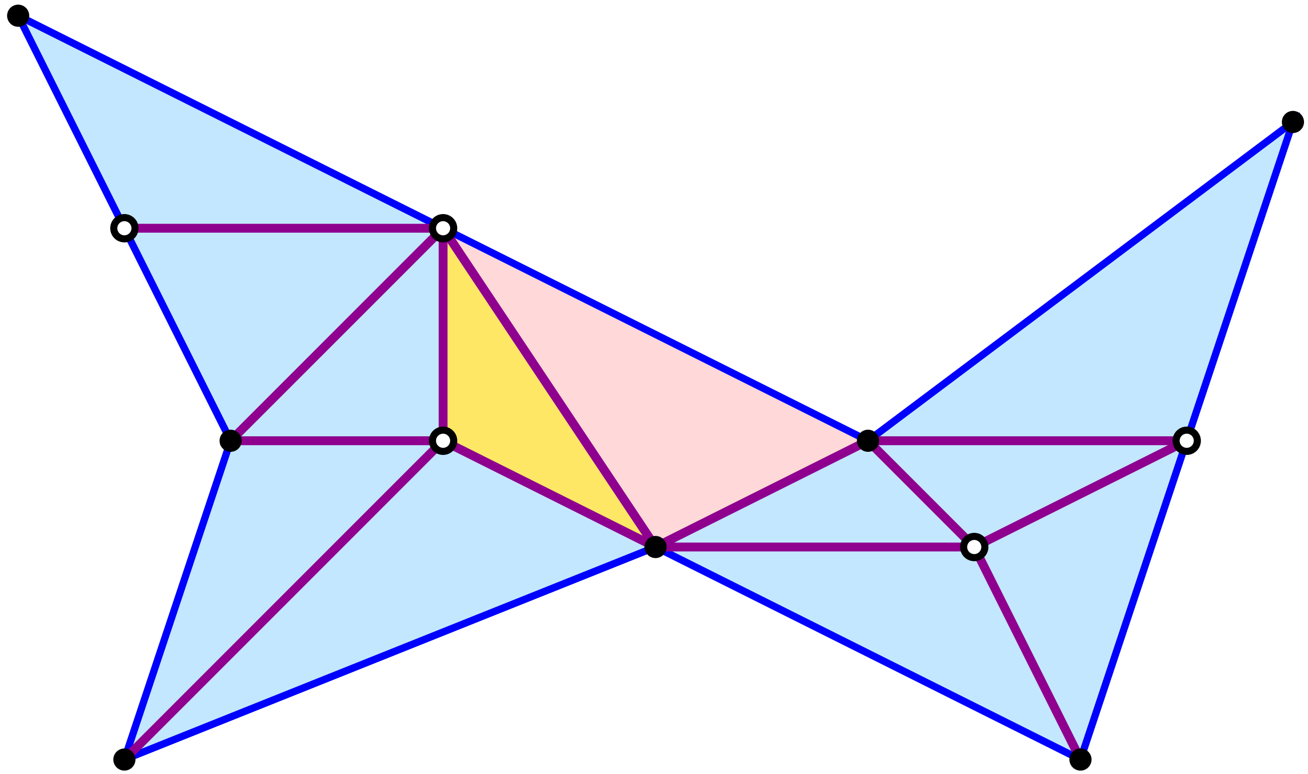
- Triangulation (no extra vertices)
- Steiner triangulation (extra vertices allowed, edges match)
- Triangular dissections (extra vertices allowed, edges need not match)



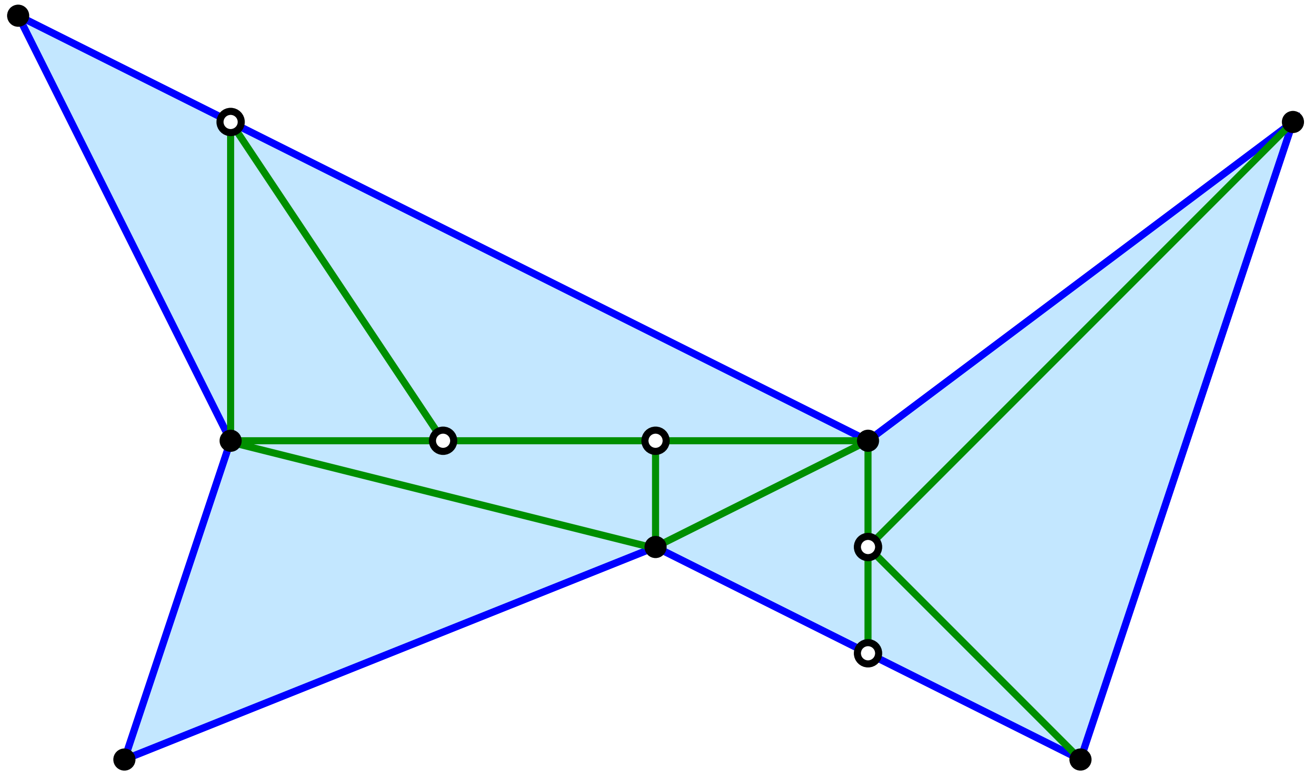
We can add extra points, called Steiner points



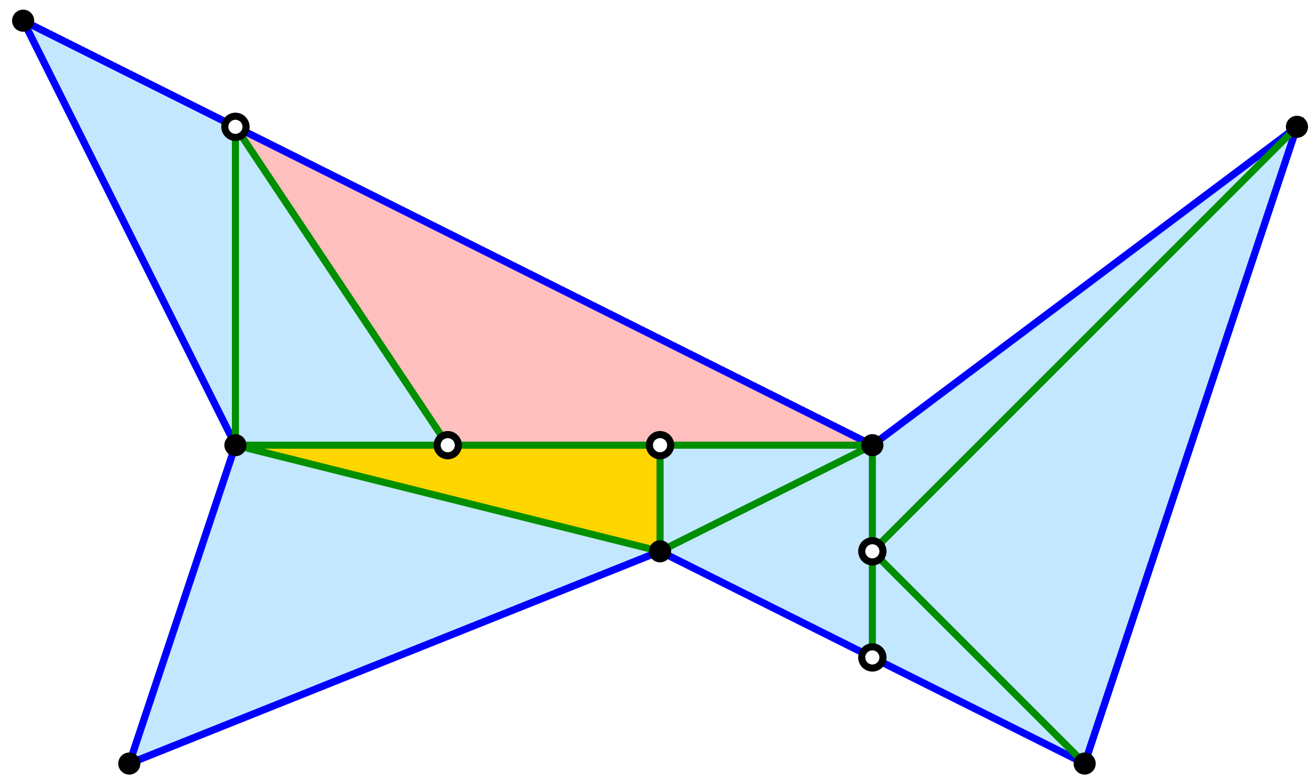
Using these, we get a Steiner triangulation.



Simplex condition: adjacent triangles have matching edges.
Most applications use this condition.



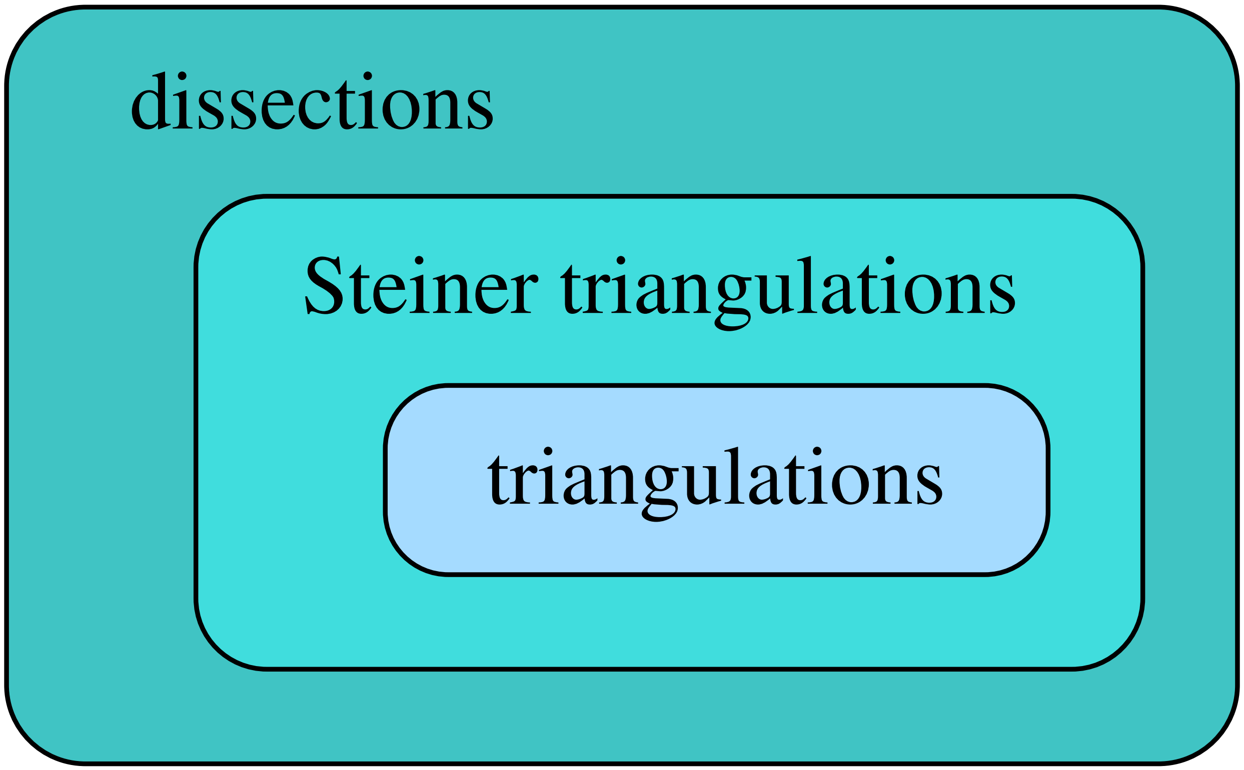
A dissection of a polygon, edges can mis-match

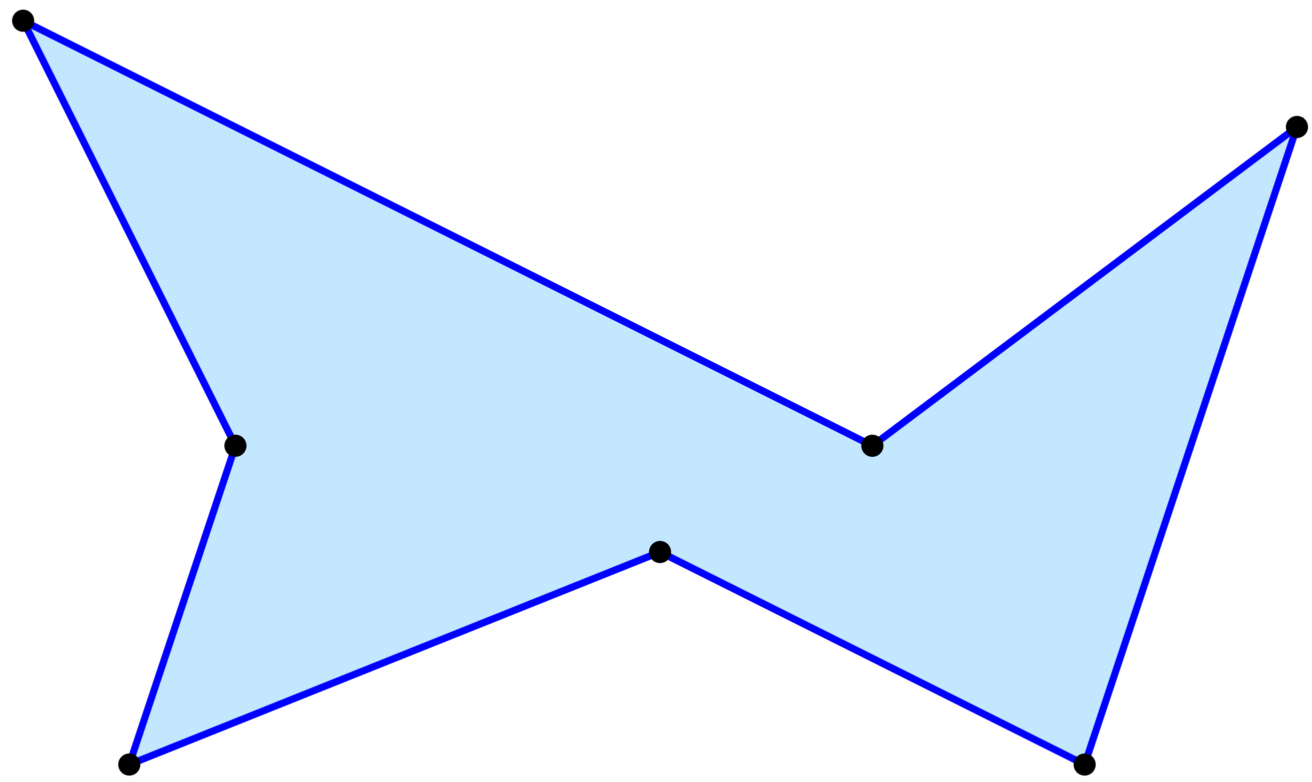


In a **triangulation**, triangles meet only at vertices or full edges.

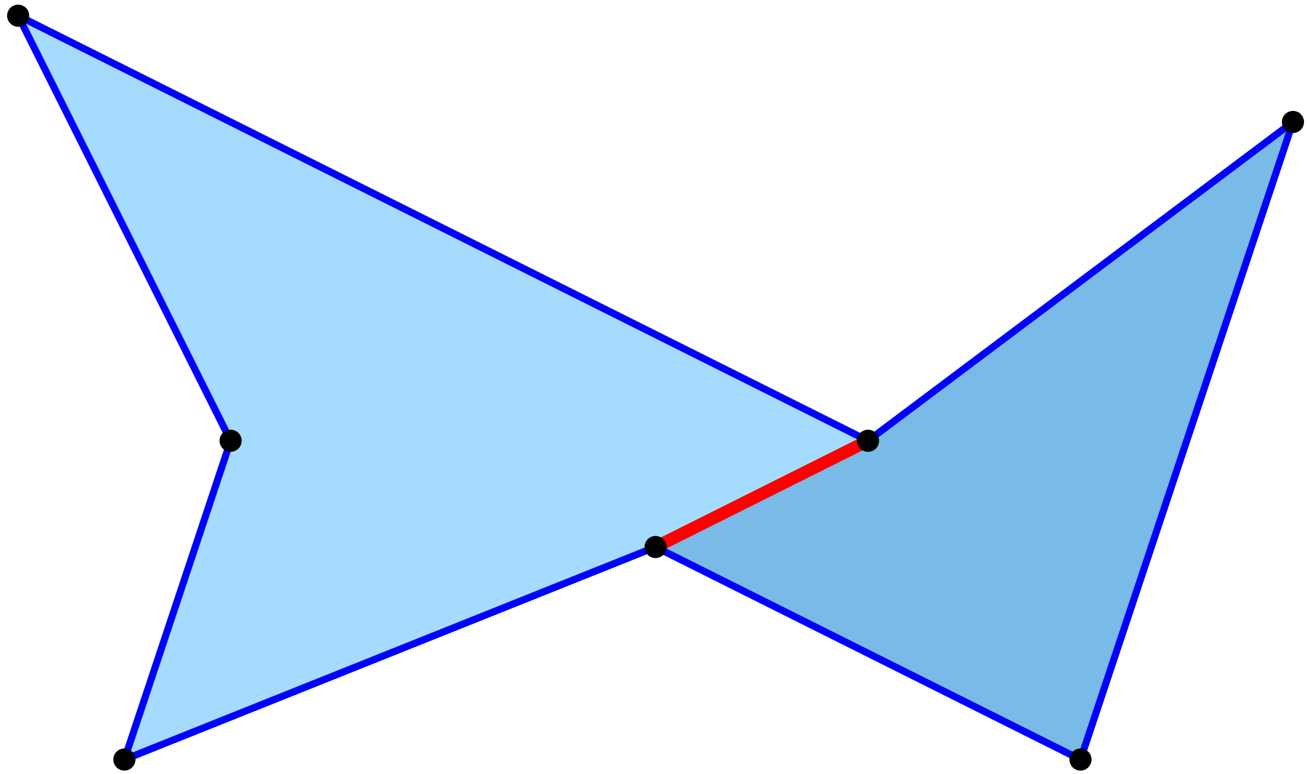
In a **dissection**, triangles can meet along partial edges.

Triangular dissection at Tampa airport

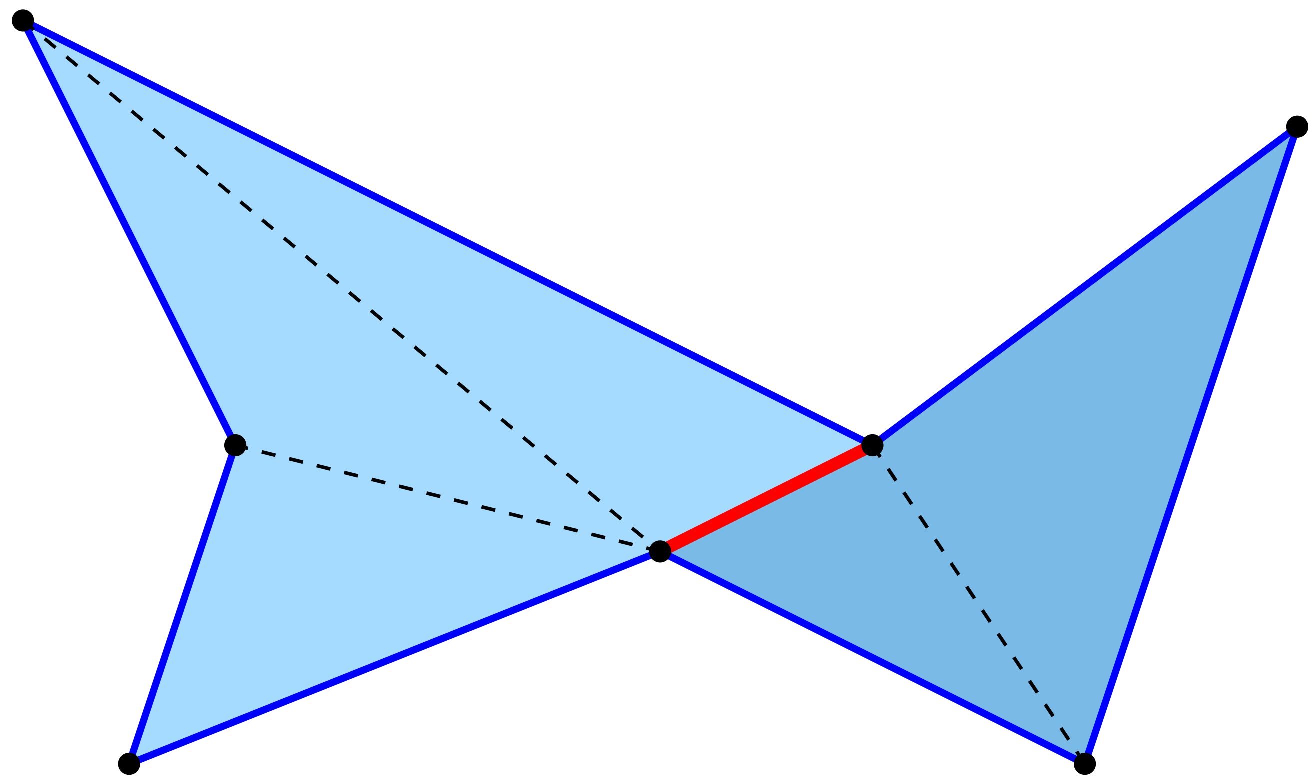

dissections
Steiner triangulations
triangulations



Does every polygon have a triangulation? (no extra vertices)



Yes. Can prove by induction on number of sides N.
$N = 3$ is obvious. For $N > 3$, it suffices to find one diagonal.
Diagonal = segment connecting two vertices **inside** the polygon.



A diagonal cuts polygon into two pieces each with fewer sides.
Each piece can be triangulated by the induction hypothesis.

Proof that a diagonal always exists:

Claim: every polygon has an interior angle $< 180°$.

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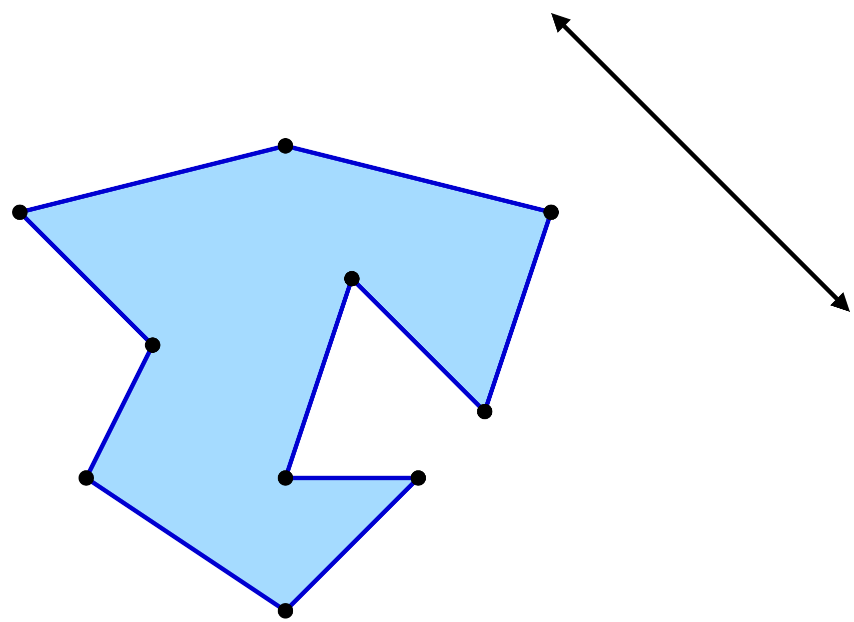
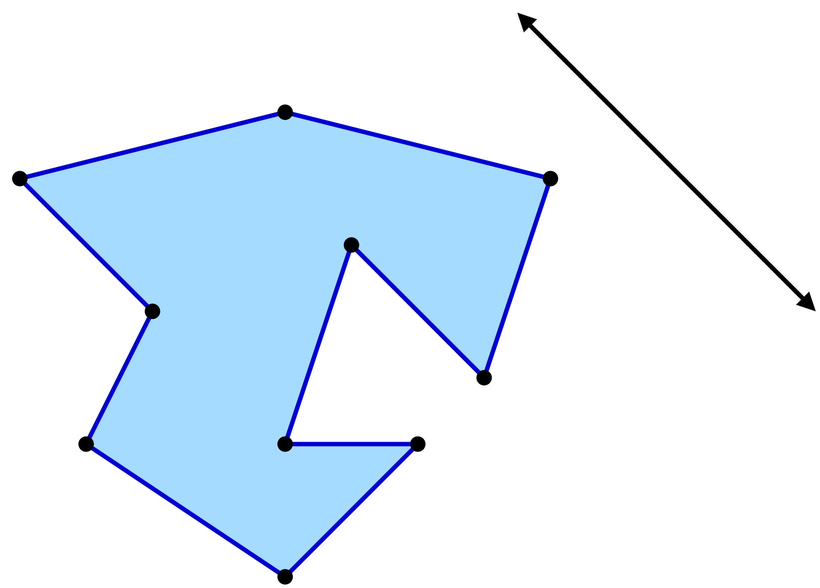


Choose line outside P and not parallel to any side.

Proof that a diagonal always exists:

Claim: every polygon has an interior angle $< 180°$.



Choose line outside P and not parallel to any side.
Slide the line until it first hits P.

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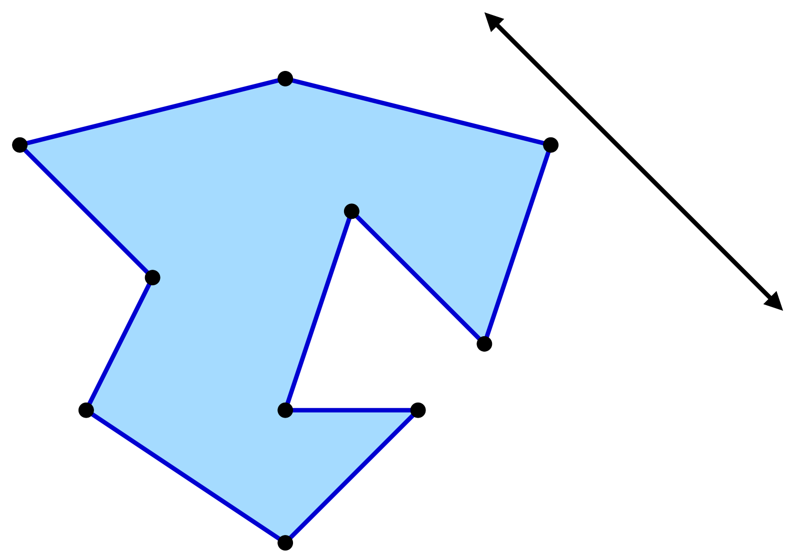

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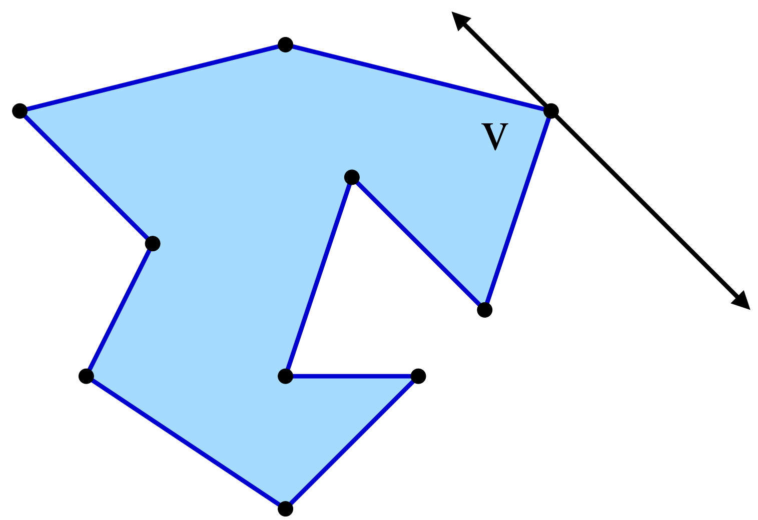
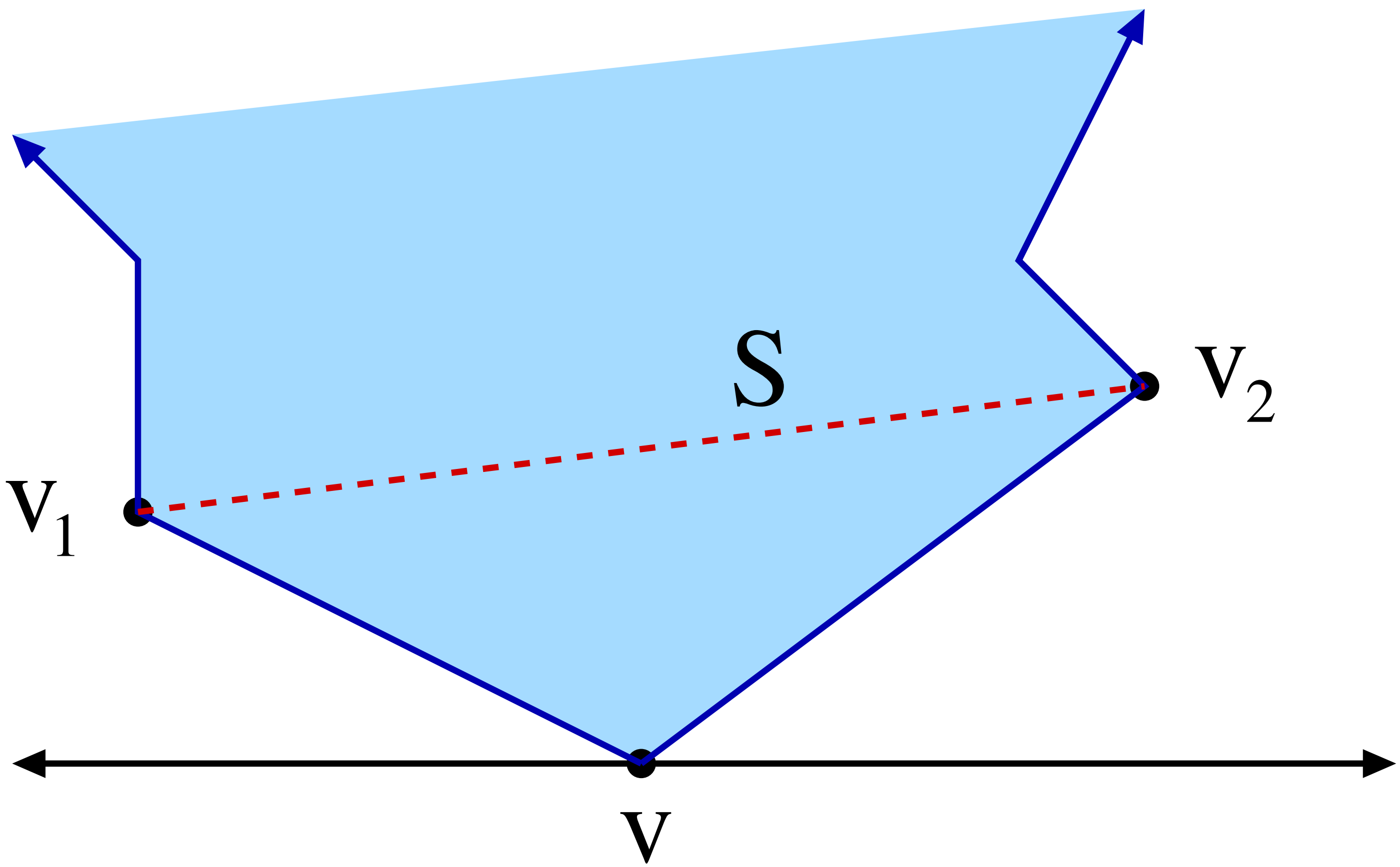


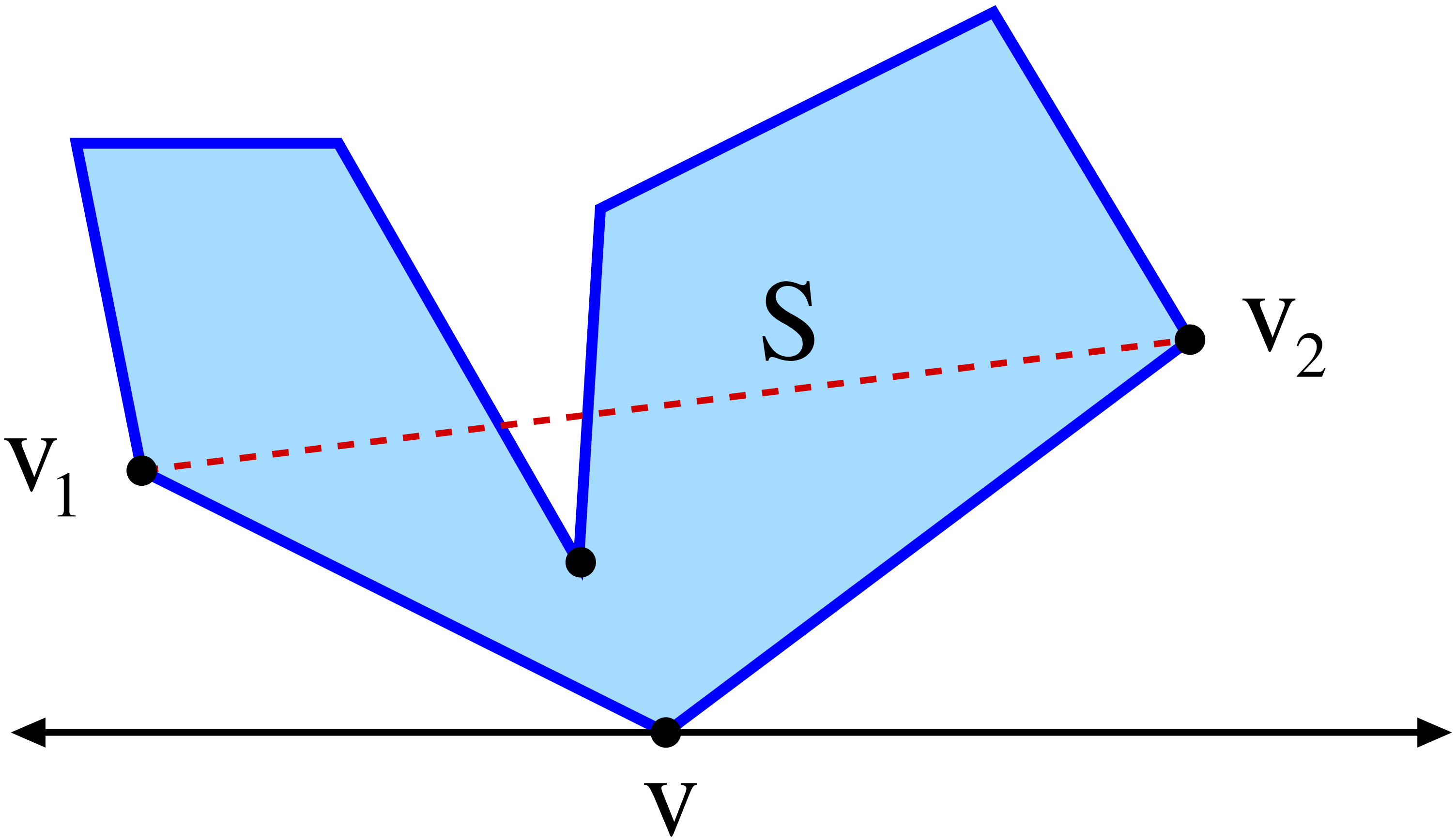
Choose line outside P and not parallel to any side.
Slide the line until it first hits P.
Intersection must contain a vertex v with angle $< 180°$.

Proof that a diagonal always exists, Case 1:



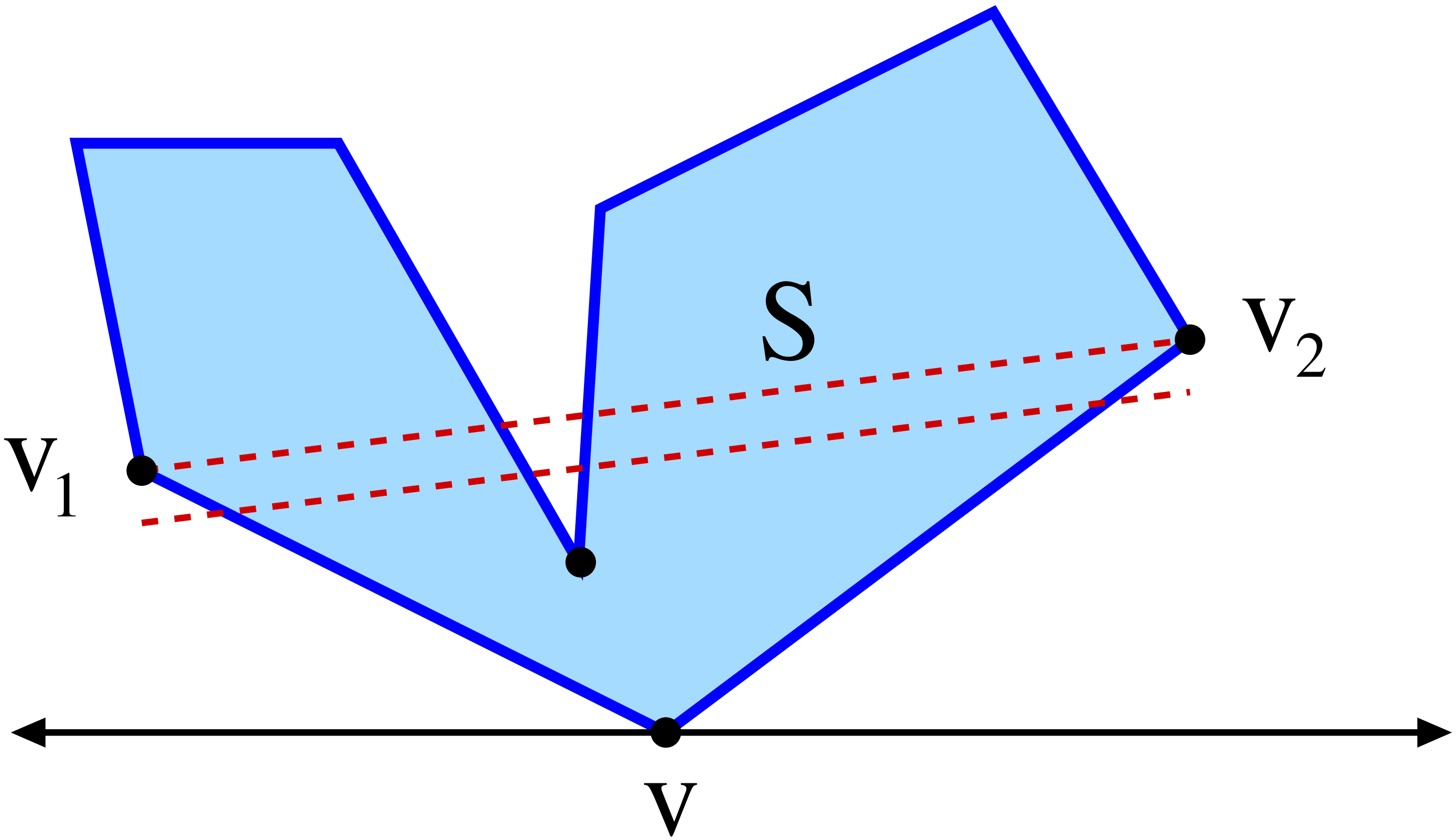
Connect vertices v_1, v_2 adjacent to v by a segment S.
If this segment S inside P, we are done.

Proof that a diagonal always exists, Case 2:



Otherwise, segment $[v_1, v_2]$ hits other edges.

Proof that a diagonal always exists, Case 2:



Move S towards v, parallel to itself,
until last time it hits an edge not adjacent to v.

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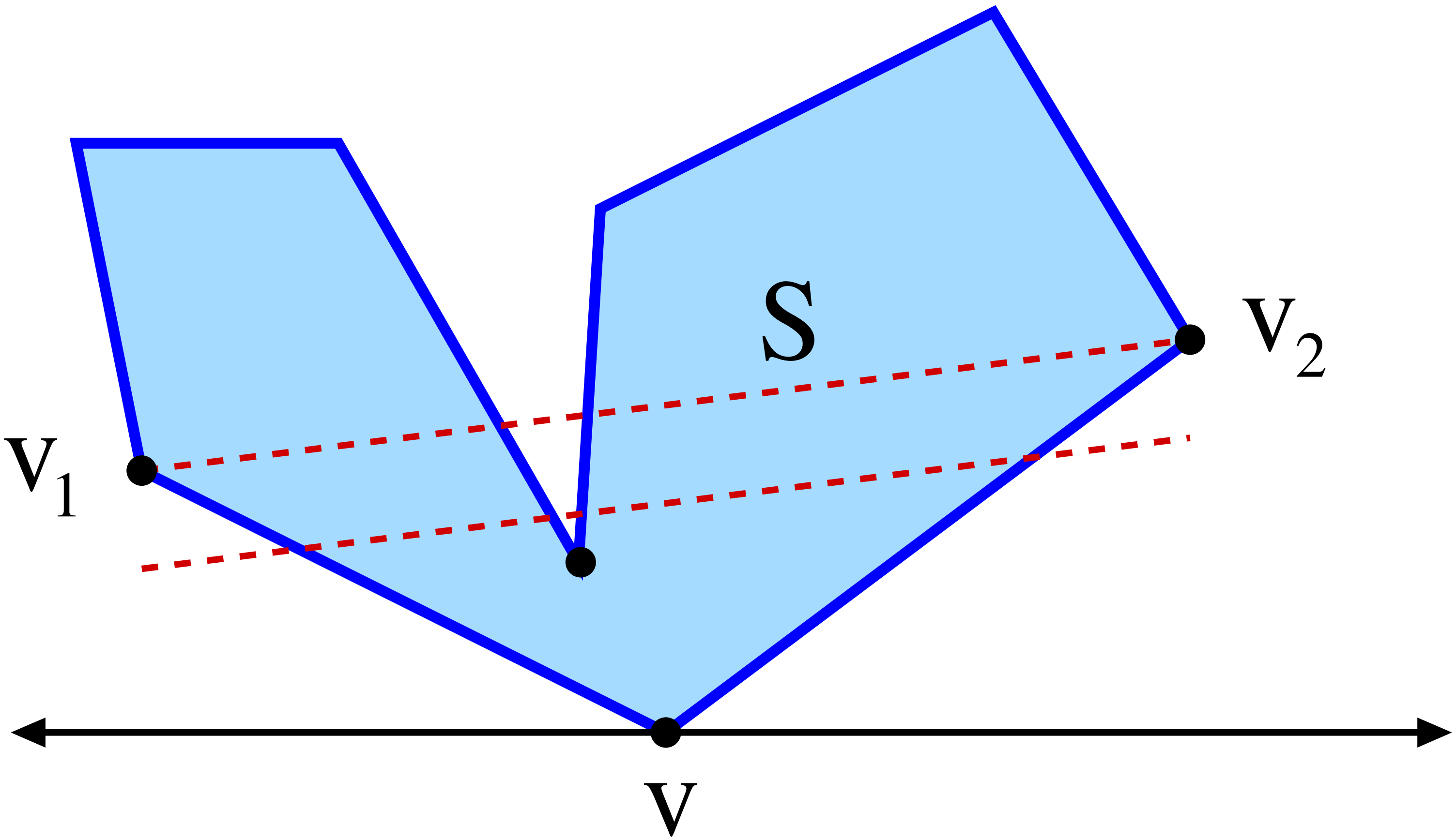

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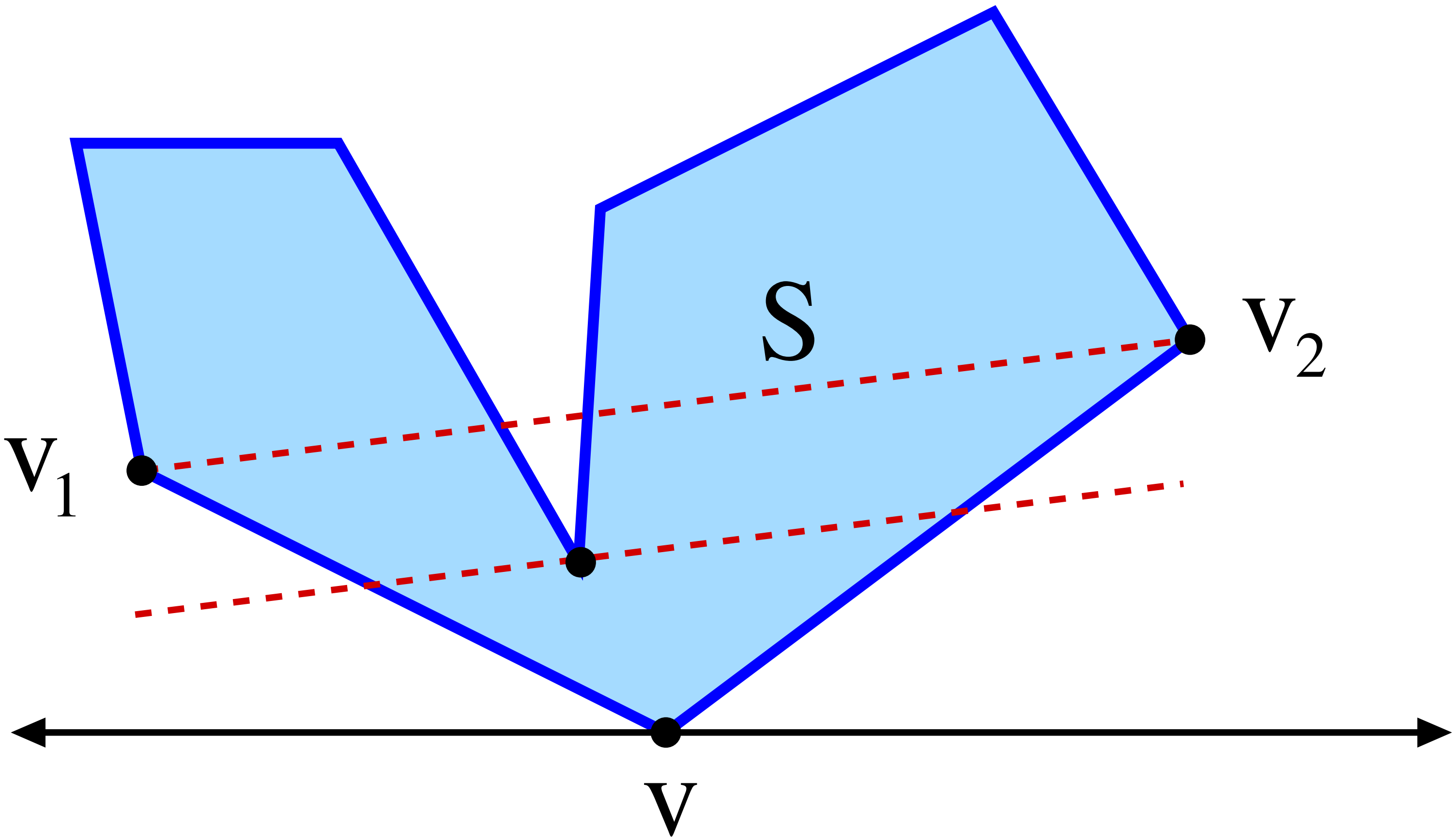

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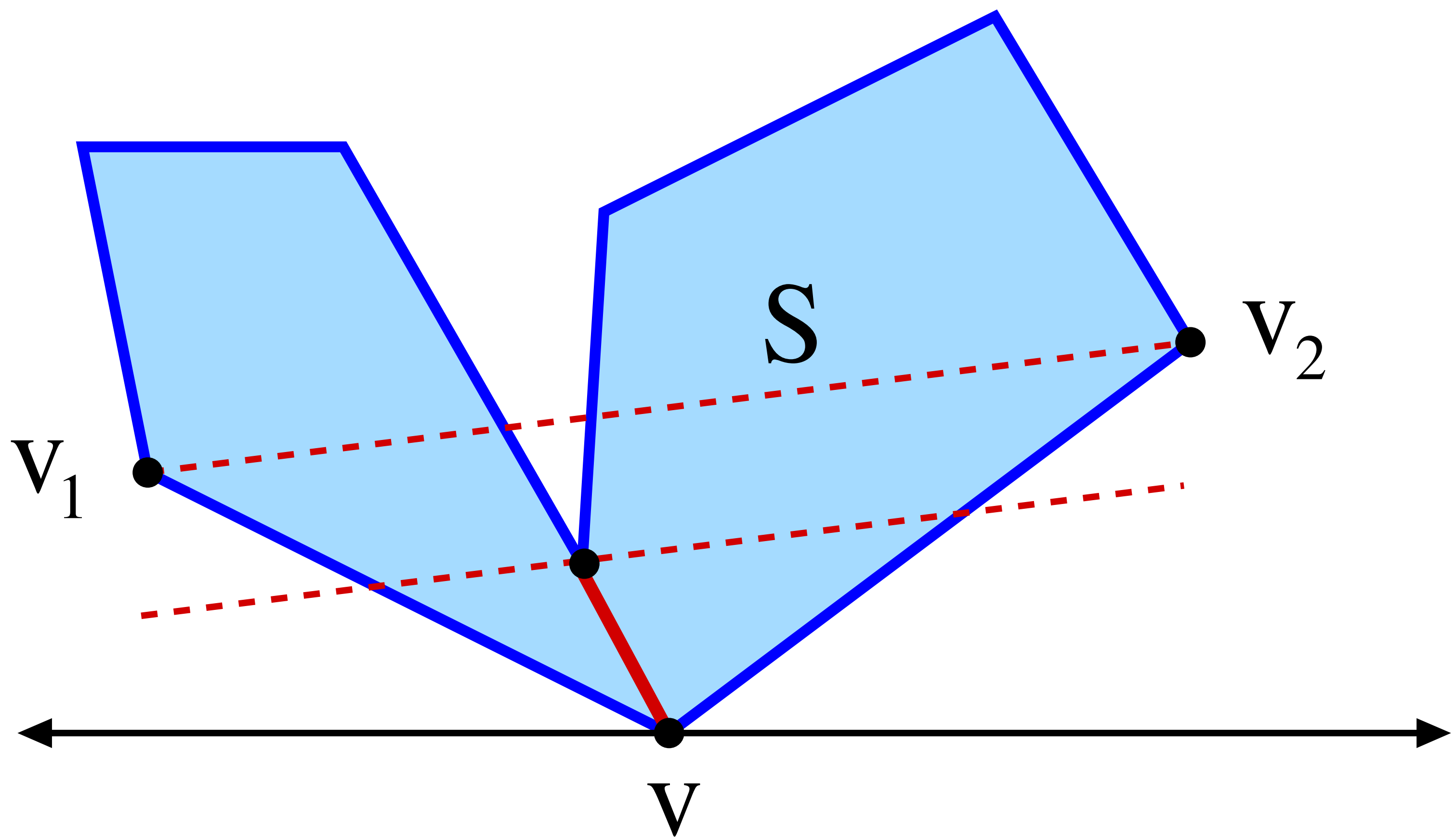
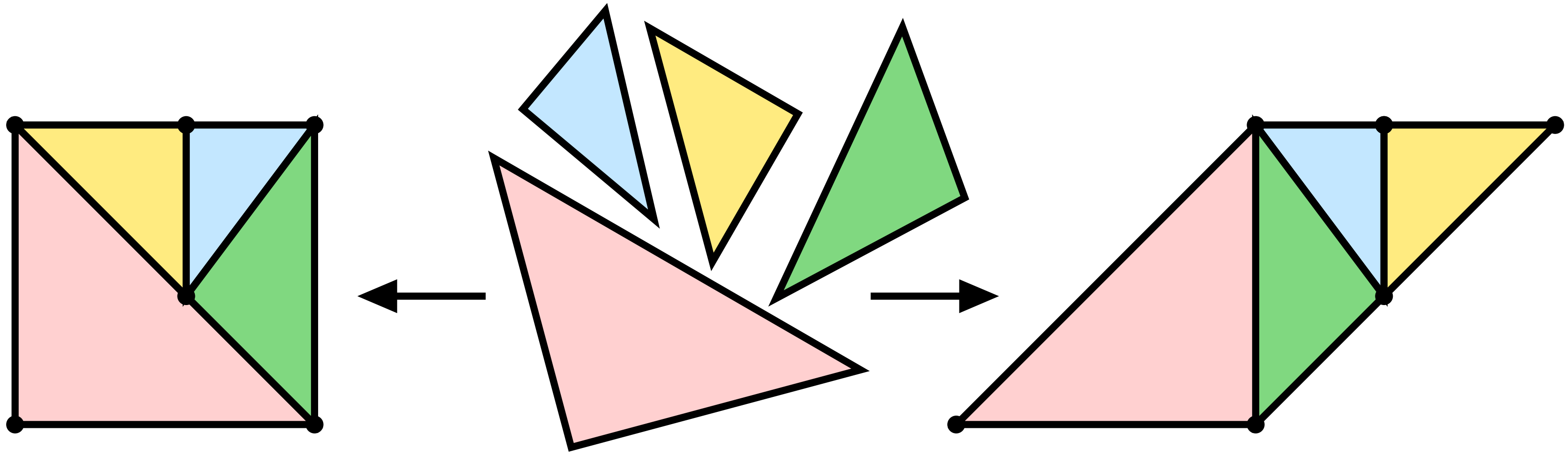


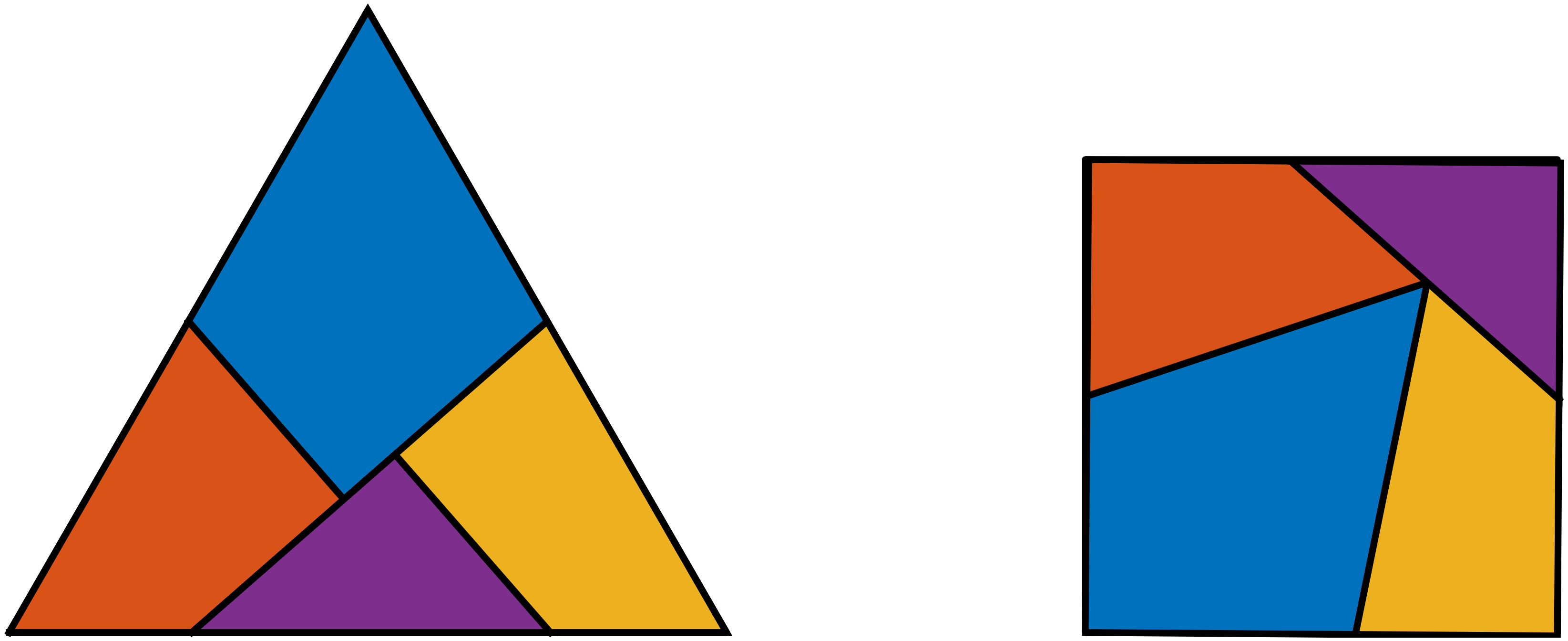
Then S contains a vertex w that can be connected to v. QED



Two polygons cut into same set of triangles must have same area.

Wallace–Bolyai–Gerwien Theorem (early 1800's):

Any two polygons with the same area have dissections into similar sets of sub-polygons (up to rotations and translations).



An equi-dissection of an equilateral triangle and a square.

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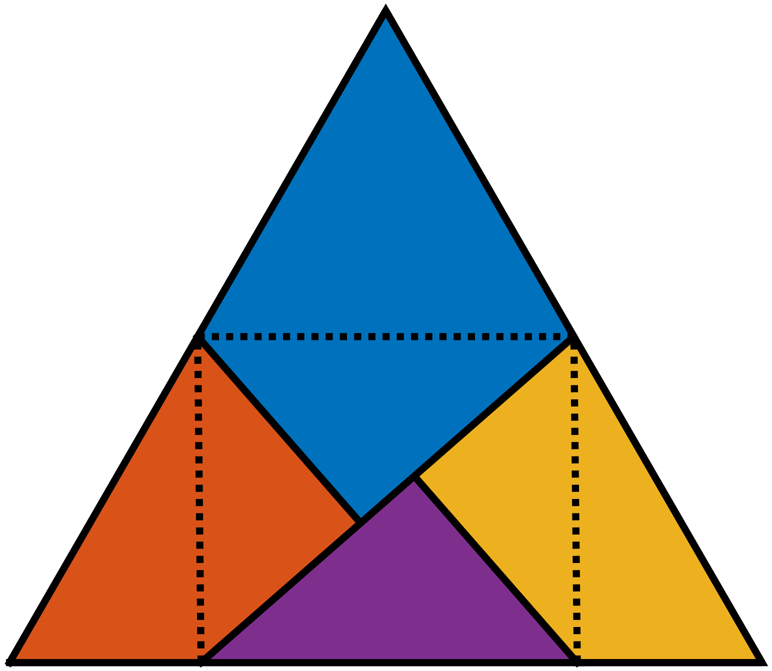
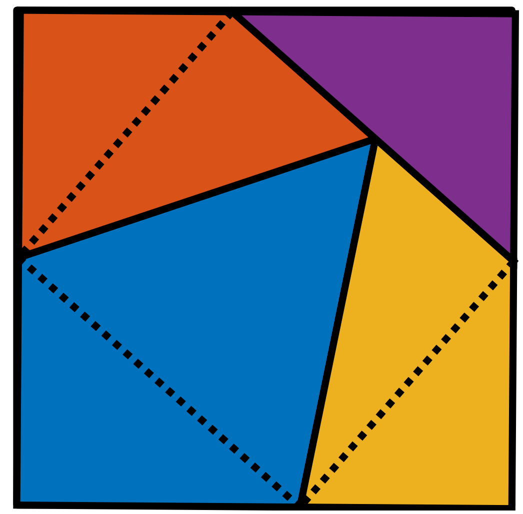




An equi-dissection of an equilateral triangle and a square.

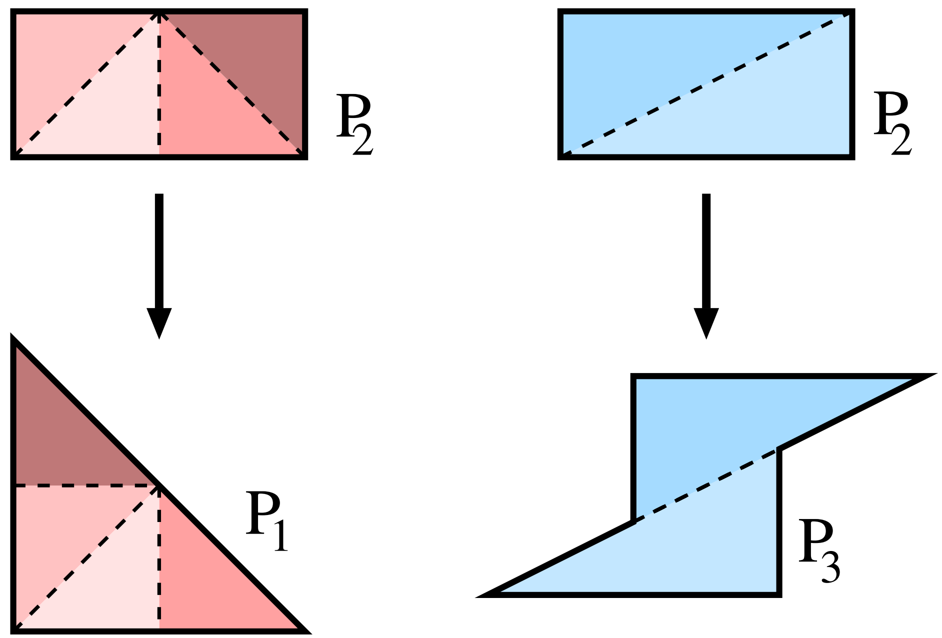
Who proved the WBG theorem?

Not clear. Contradictory information given in:

I. Stewart, *From here to infinity*, The Clarendon Press, 1996.

E.N. Giovannini, David Hilbert and the foundations of the theory of plane area, *Arch. Hist. Exact Sci.*, vol 75, 2021, pages 649–698.

Lemma: If P_1 and P_2 have an equi-dissection, and if P_2 and P_3 have an equi-dissection, then so do P_1 and P_3. (Equi-dissection is transitive.)



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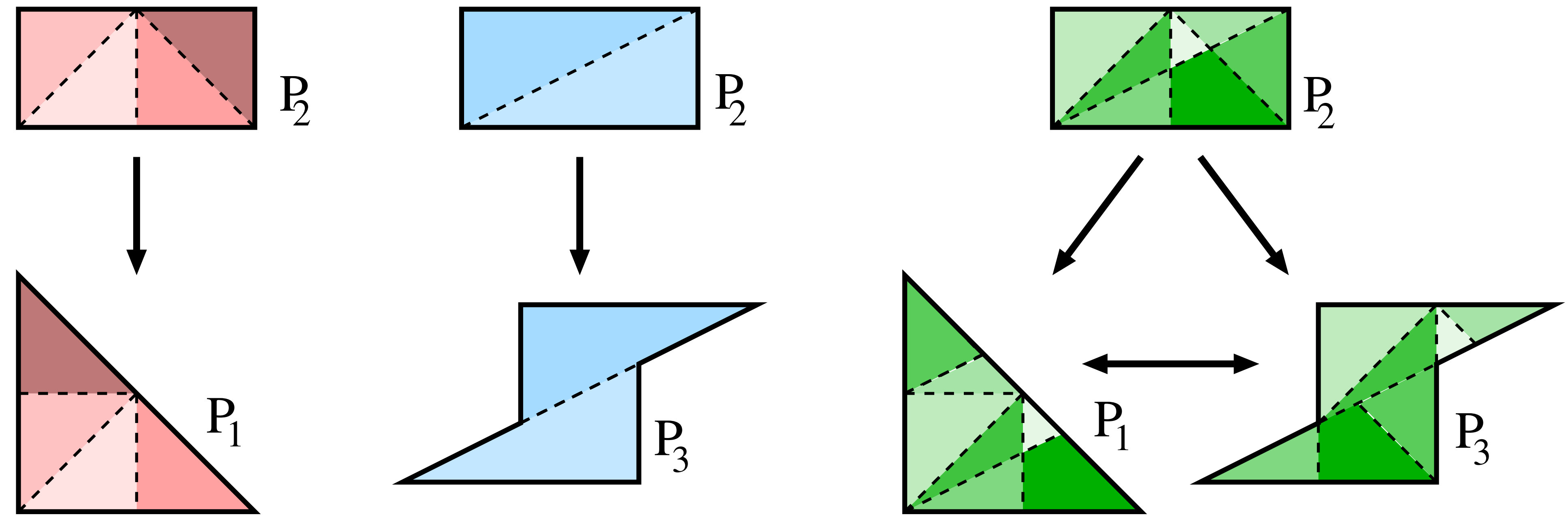
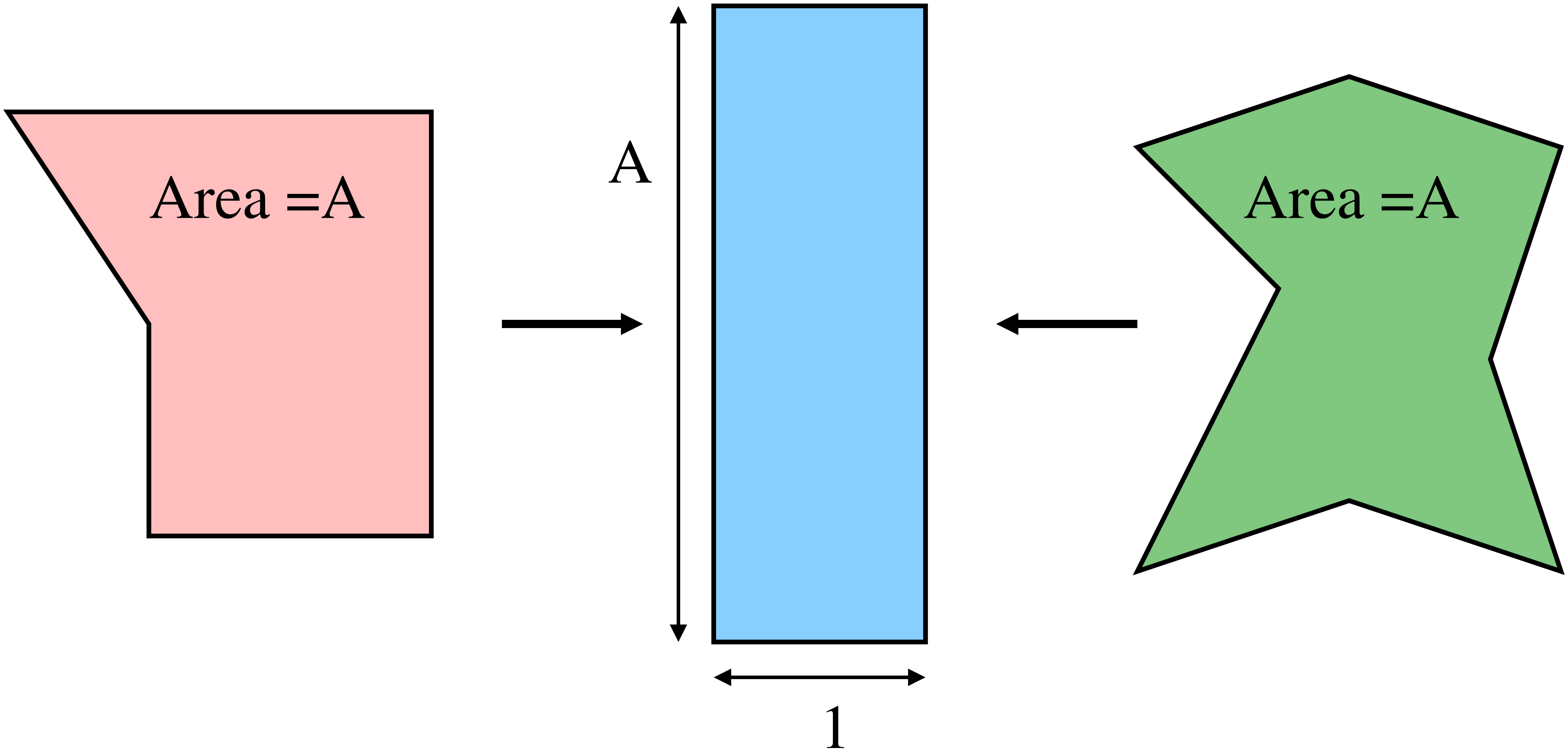


Proof: Intersect the two dissections of P_2. Smaller pieces can be rearranged into either P_1 or P_3.

Proof of WBG theorem.

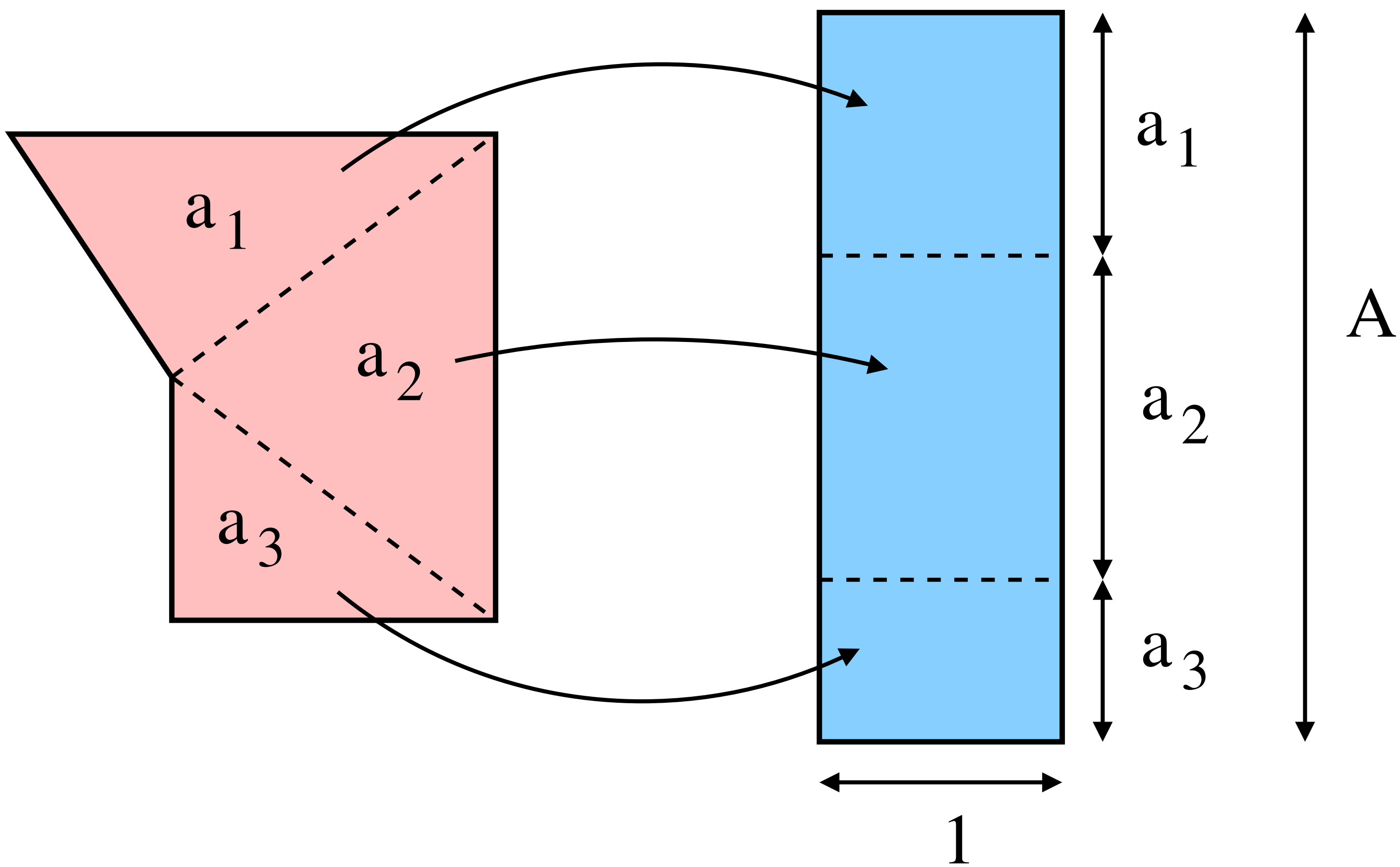
Enough to show a polygon of area A is equivalent to a $1 \times A$ rectangle.



Proof of WBG theorem.

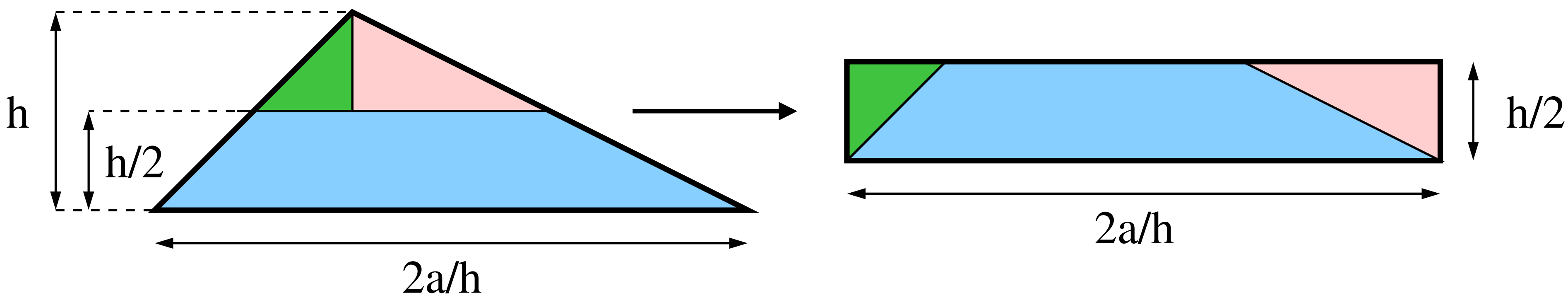
It is enough to show a triangle of area a is equivalent to a $1 \times a$ rectangle.

Triangulate P, convert each triangle to a rectangle and stack them.



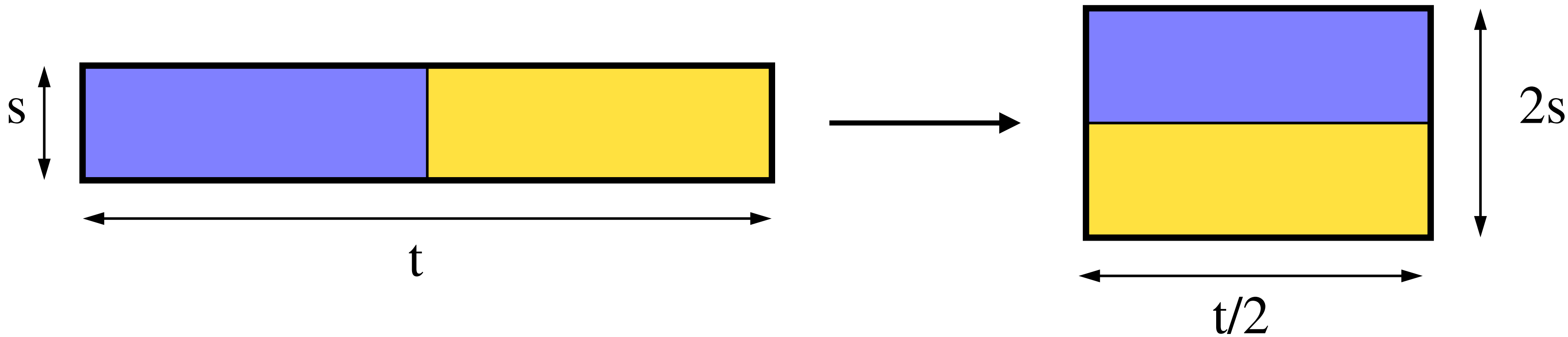
Proof of WBG theorem.

Claim 1: every triangle is dissection equivalent to some rectangle:



Proof of WBG theorem.

Claim 2: a $t \times s$ rectangle is equivalent to a $(t/2) \times (2s)$ rectangle.



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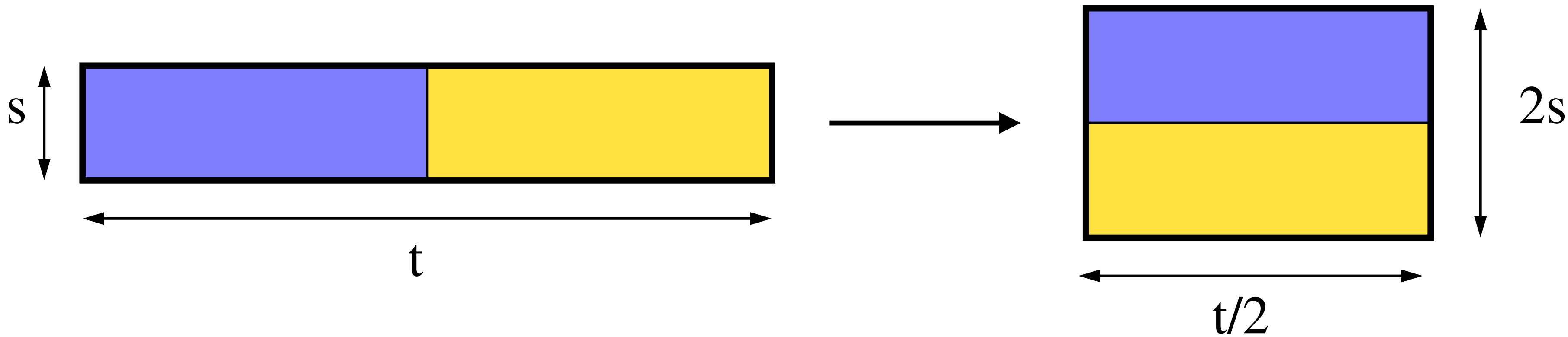


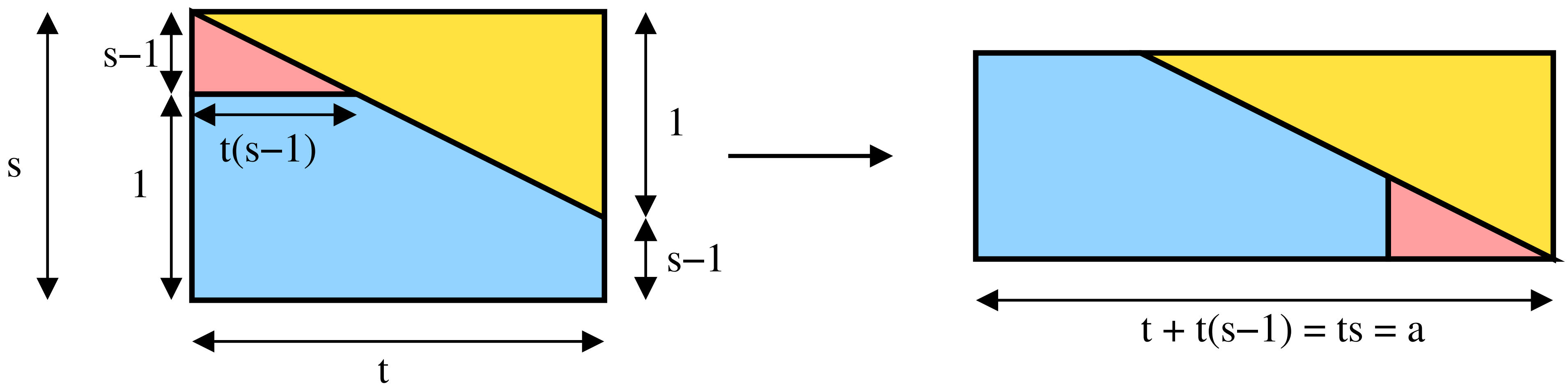
Claim 3: a rectangle of area a is equivalent to a rectangle with side lengths $s \in [1, 2)$ and $t \in (a/2, a]$.

Keep doubling (or halving) sides until this is true.

Proof of WBG theorem.

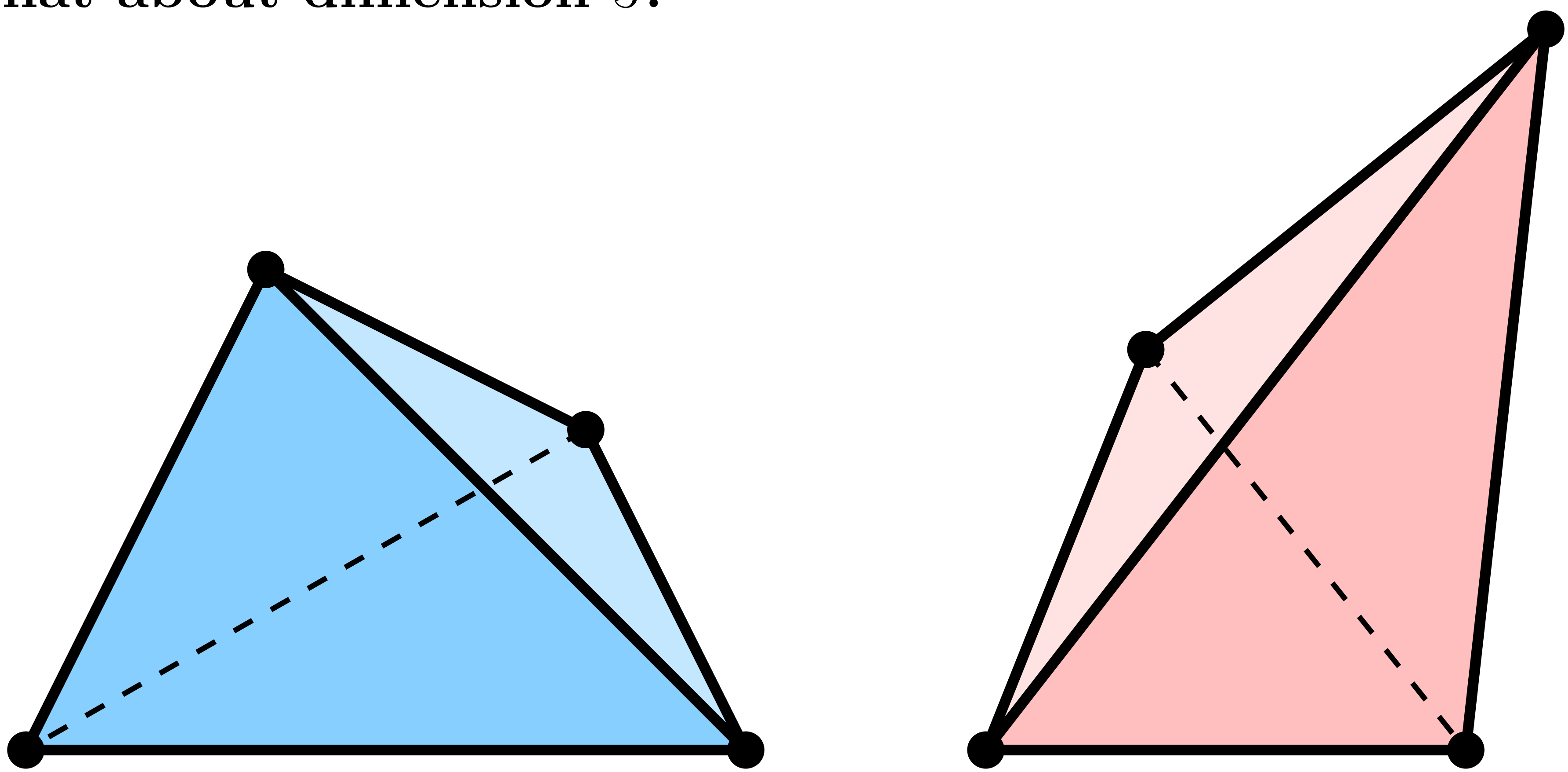
Claim 4: a rectangle of area a is equivalent to a $1 \times a$ rectangle.

By previous steps we may assume $t \times s$ with $1 \leq s < 2$.



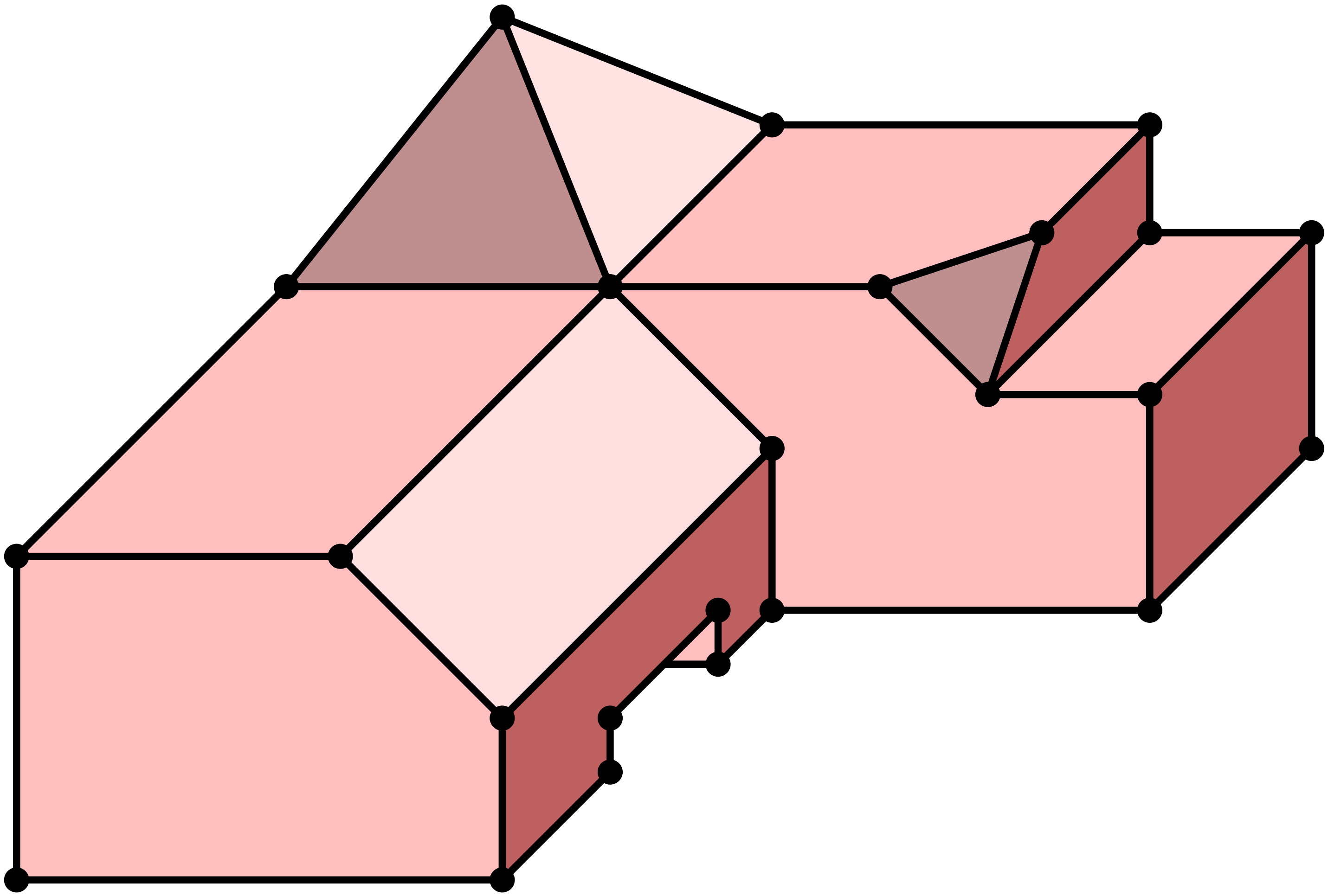
This completes the proof of the WBG theorem.

What about dimension 3?



3D version of a triangle is a tetrahedron = four faces, all triangles

What about dimension 3?



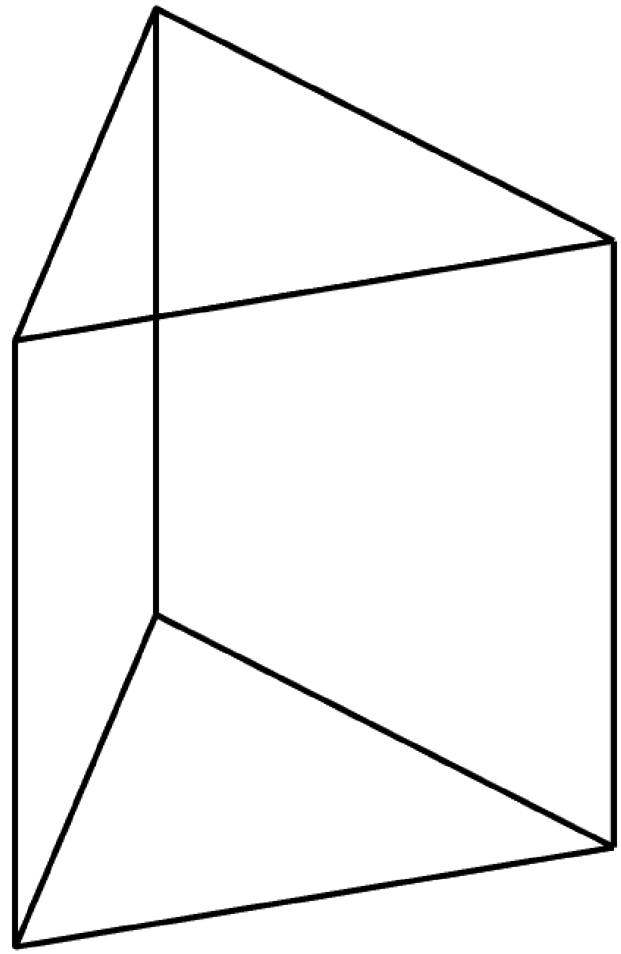
3D version of a polygon is a polyhedron.

So far we have proven

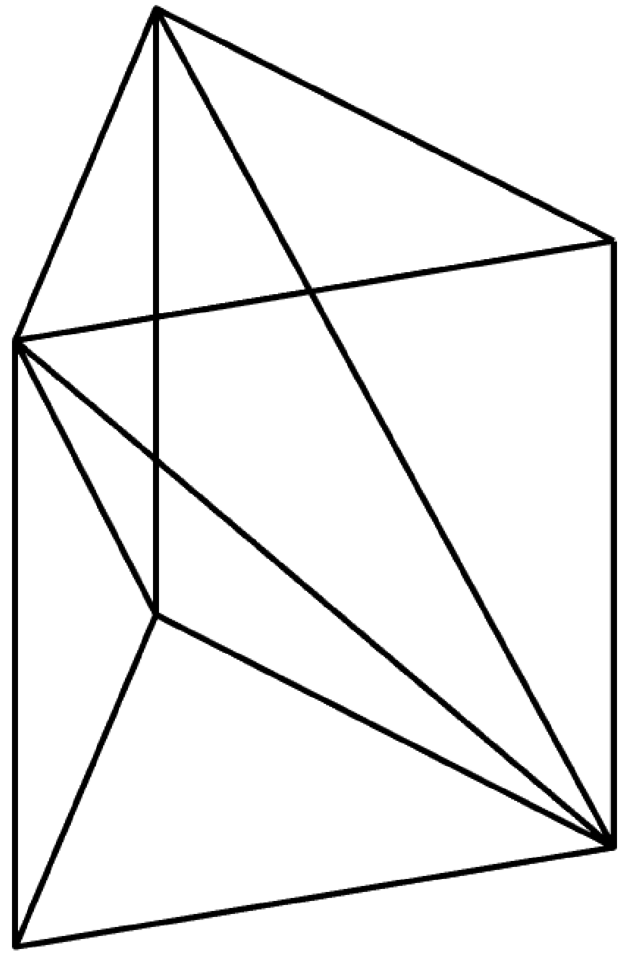
- Every 2D polygon has a triangulation with no extra vertices.
- The Wallace-Bolyai-Gerwien theorem.

Are these true in 3 dimensions?

- Is every polyhedron a union of tetrahedron using no extra vertices?
- Can two equal volume polyhedra be cut into the same set of tetrahedra?



Some polyhedra can be cut into tetrahedra.
A triangular prism can be cut into three tetrahedra.



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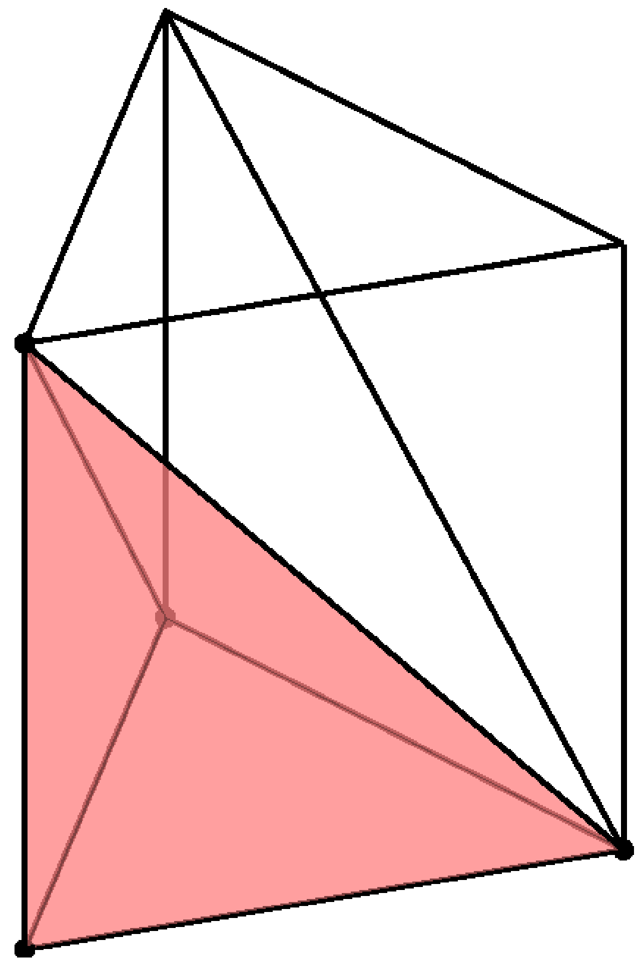

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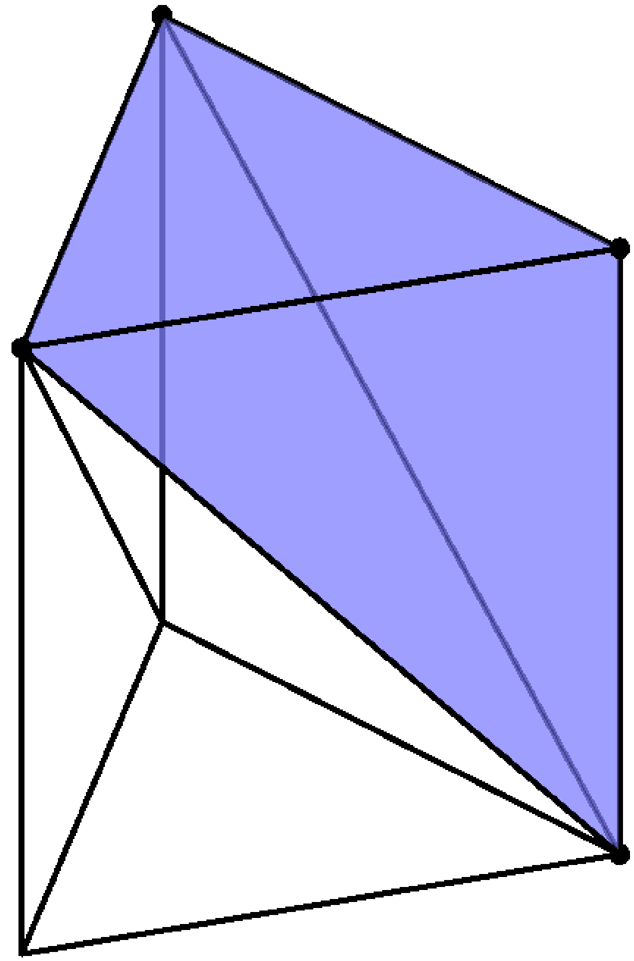

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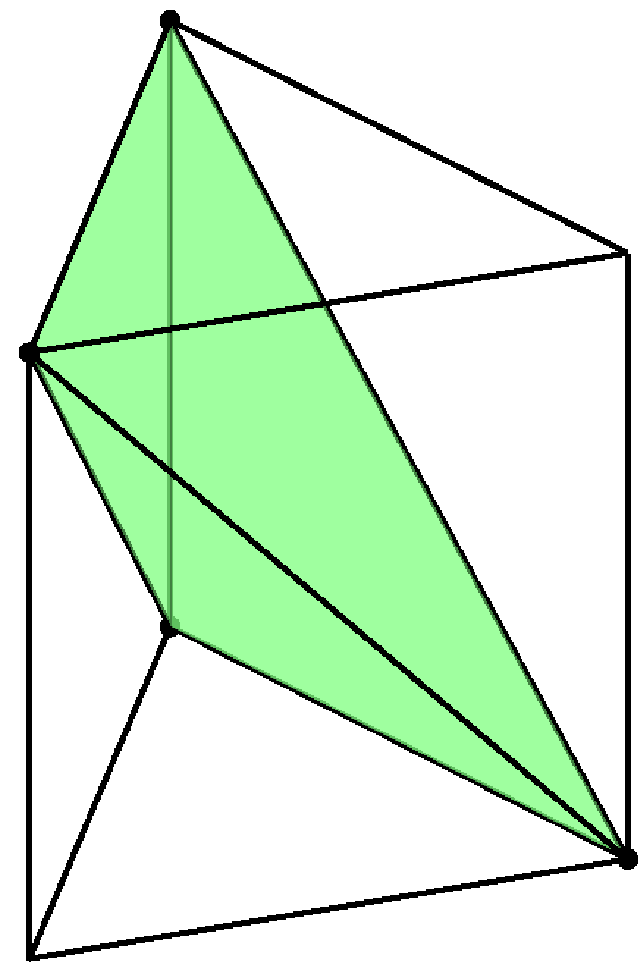
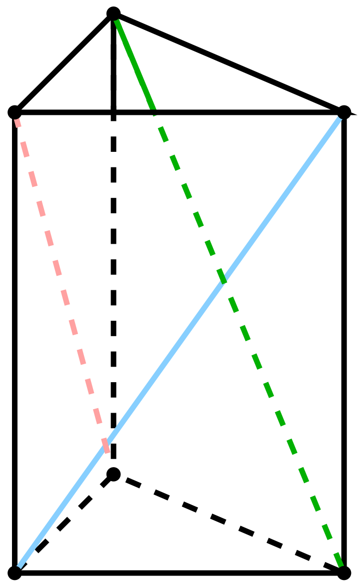


Some polyhedra can be cut into tetrahedra.
A triangular prism can be cut into three tetrahedra.

3D counterexample due to Erich Schönhardt in 1928.



The Schönhardt polyhedron does not contain any tetrahedra defined by four of it vertices.

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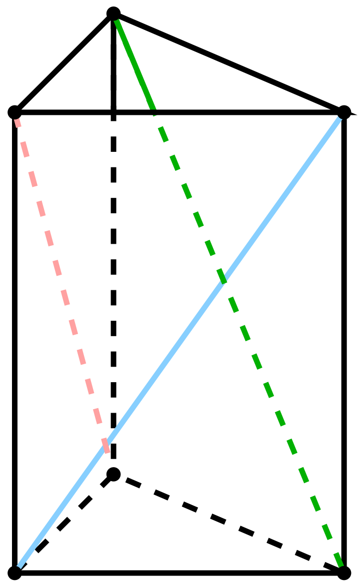
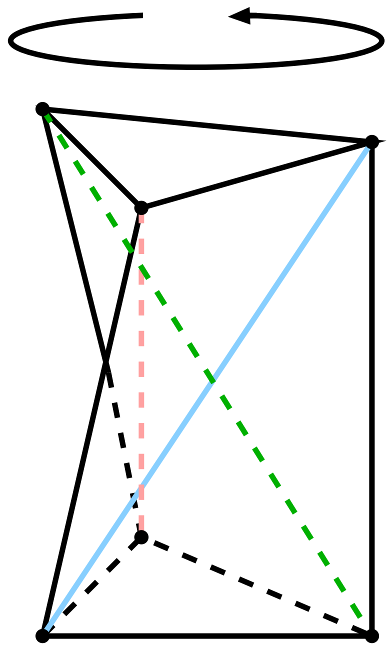
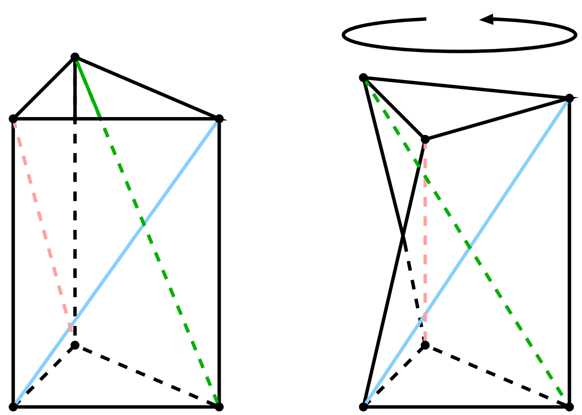


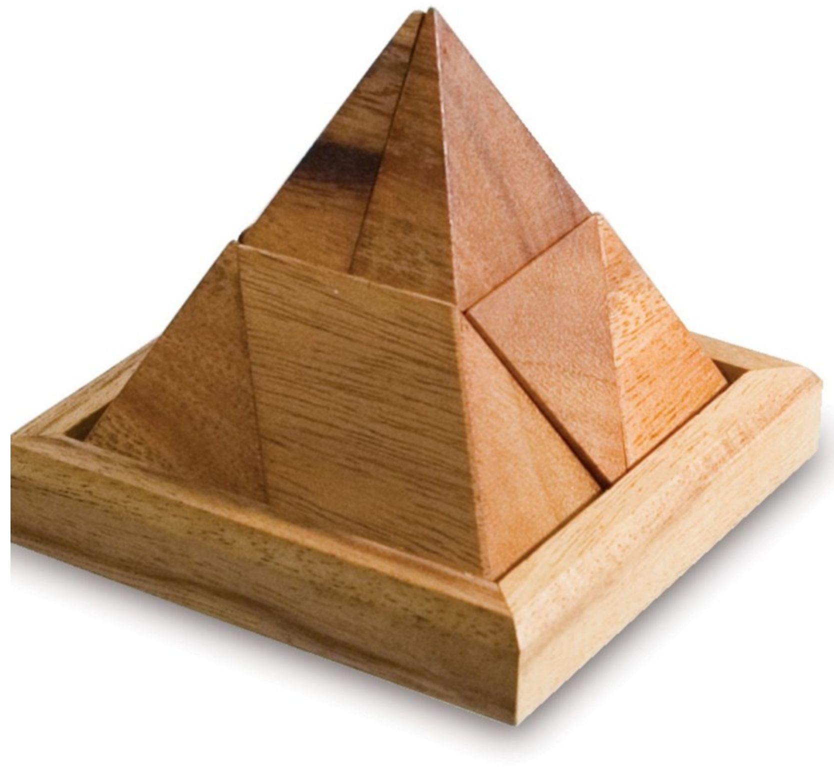


The Schönhardt polyhedron does not contain
any tetrahedra defined by four of it vertices.

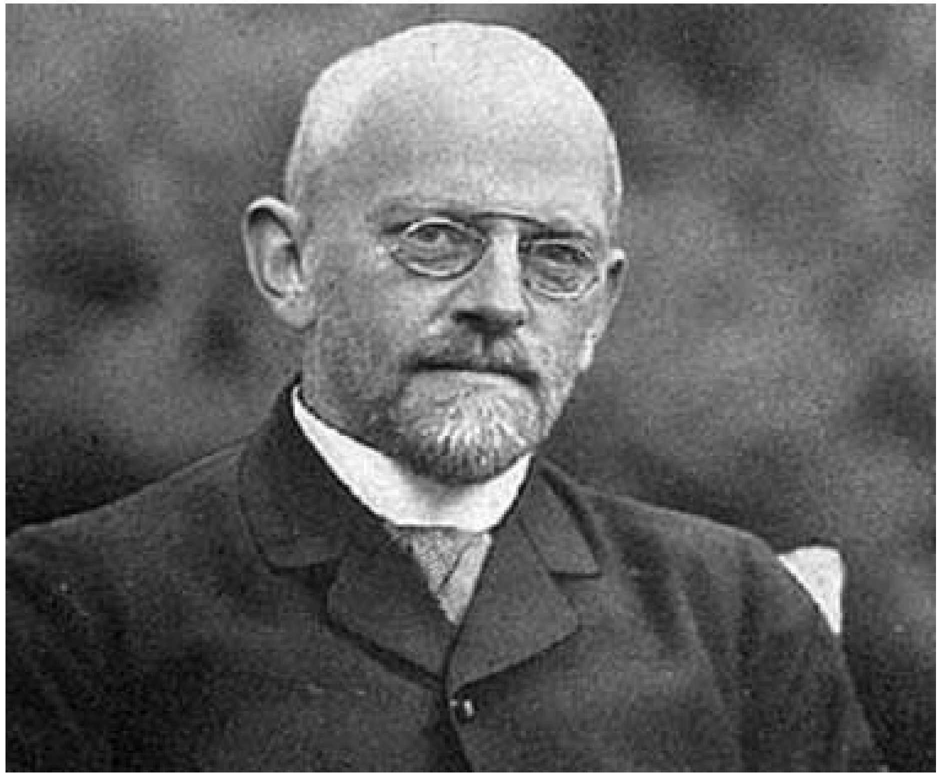
Proof:

- Let P be the Schönhardt polyhedron.
- Suppose T is a tetrahedron using 4 of the 6 vertices of P.
- T is inside P iff none of its vertices can "see" another one.
- Every vertex v on top side of P, can see one vertex on bottom side.
- If T has 3 vertices on top, the vertex on bottom sees one of these. $\Rightarrow\Leftarrow$
- If T has 2 vertices on top, only one on bottom is "invisible" $\Rightarrow\Leftarrow$





Is the WBG Theorem true in 3 dimensions?
Can we cut a cube into tetrahedra and re-assemble as a pyramid?

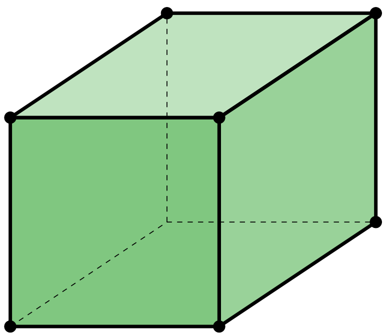


This is “**Hilbert’s third problem**”. Ten problems given at the 1900 International Congress of Mathematicians; 23 later appeared in print.

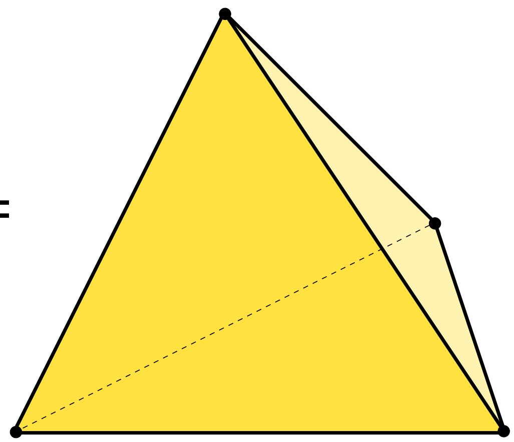
The Dehn Invariant is a “number” associated to each polyhedron.

Dehn proved that dissection equivalent polyhedra have same value.

But unit cube and regular tetrahedron have different Dehn invariants.



$\neq$



Max Dehn

In 1965 Jean-Pierre Sydler proved converse:

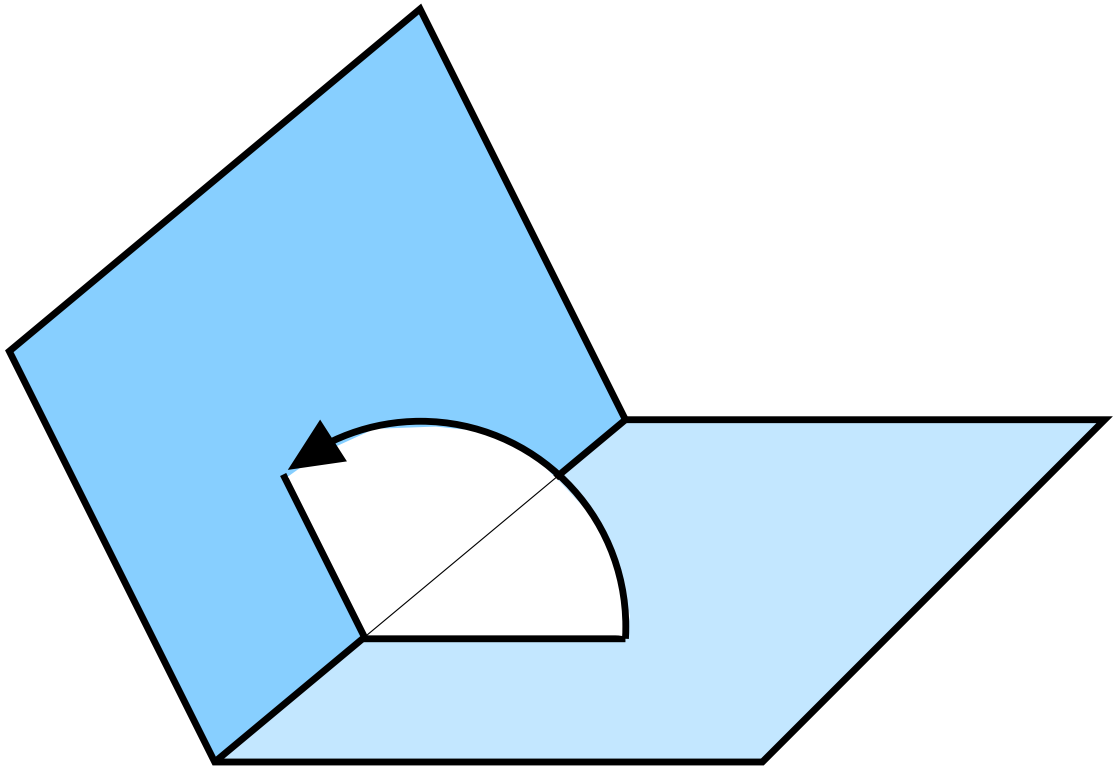
Theorem: Two polyhedra are dissection equivalent iff equal same Dehn invariants.

Easier algebraic proof given in 1990 by Johan Dupont and Han Sah.

Proof is related to topology and the homology the 3-D rotation group.

Defining the Dehn invariant:

- “Angles” means dihedral angles between faces. One per edge.



Defining the Dehn invariant:

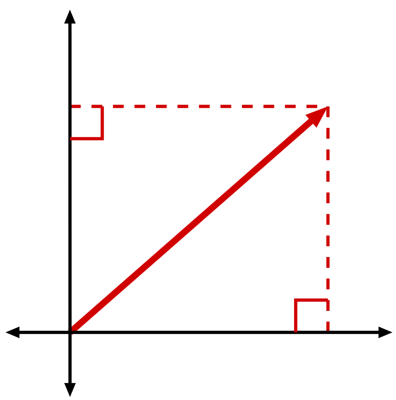
• The reals form an (infinite dimensional) vector space over the rationals.

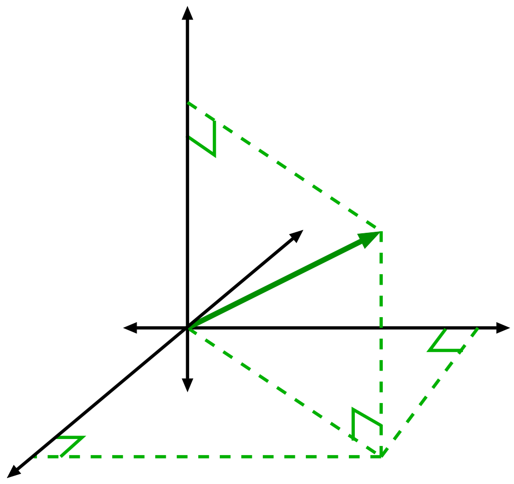
Vector space = set with addition and multiplication by some scalar field.

For example: $\log 120 = 3\log 2 + \log 3 + \log 5$.

Exericse: $\{\log p\}$ are independent over rationals for p prime.

Research problem: is e a rational multiple of π?





Defining the Dehn invariant:

Given a polyhedron:

(1) Find a basis for the dihedral angles over the rationals. Add π.

(2) Write each dihedral angle as a sum of basis elements, i.e., a vector.

(3) Multiply each vector by corresponding edge length.

(4) Dehn Invariant = sum vectors over all edges and drop the π coordinate.

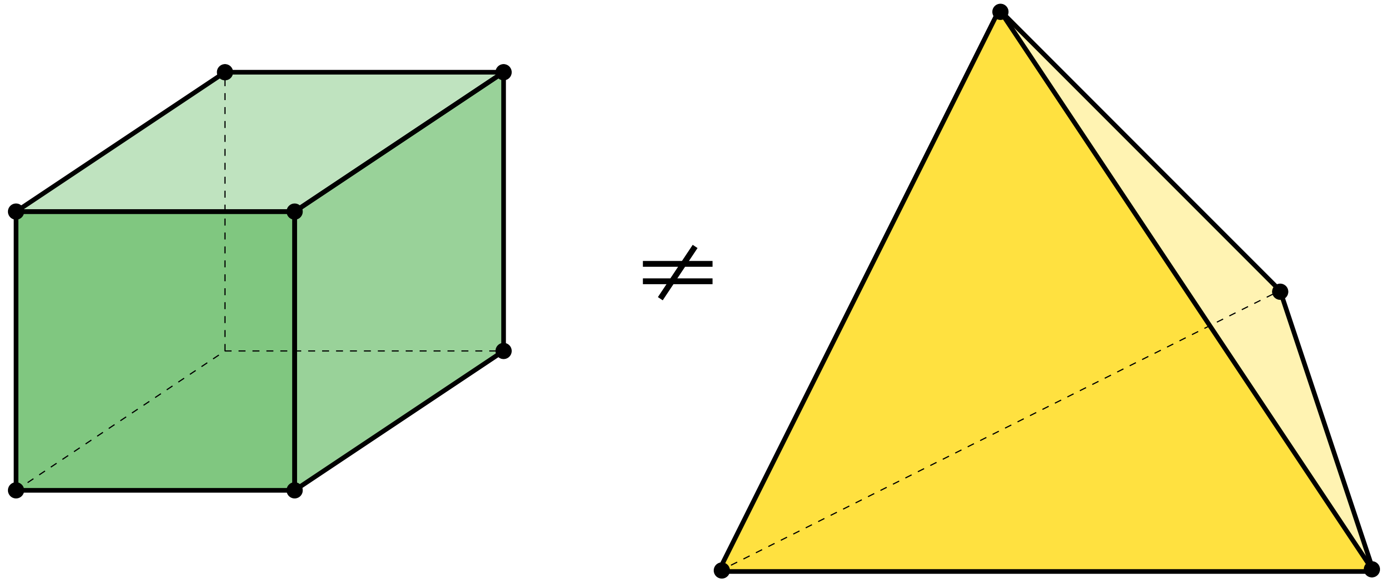
Platonic solids have only one angle, so Dehn invariant is 1-dimensional.

Enough to check if dihedral angles are rational multiples of each other.

Dihedral angles: Cube $= \pi/2$, Regular Tetrahedron $= \arccos(1/3)$.

Fact: $\arccos(1/3)$ is not a rational multiple of π.

$\Rightarrow$ Cube and Tetrahedron and not dissection equivalent.



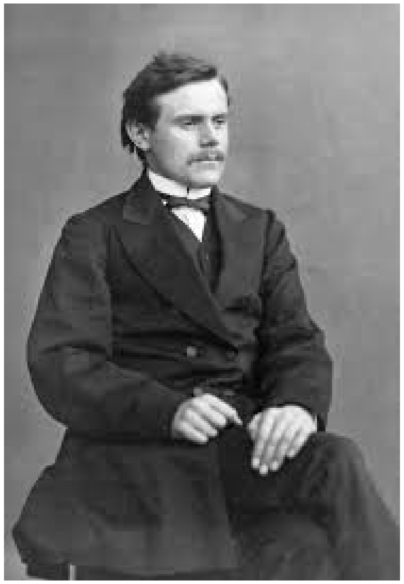
$$\frac{\arccos(1/3)}{\pi} =$$

$$
\begin{aligned}
0.\ &3918265520306072701708555592224309113166286494394 00\\
&28728793829802200140867744224687021182617577451307 9\\
&52678273049545797823752623500338545526659941597292 0\\
&90579705090502501993335698129955261034483258984876 3\\
&53909685738717008648924701920801818935311629606629 3\\
&19041723144886747469900680232768563277798404277743 1\\
&65957708490506779879792090618821421778117897274056 1\\
&61091627577866209458898862048432163043892901819390 0\\
&35529660145457982049693839042773662499741280604880 0\\
&795743074277018712424377133609510363005 15\ldots
\end{aligned}
$$

Doesn't look rational, but proof given at end of slides.

Surprising historical note:

Hilbert's problem had been solved even before he stated it.



Władyslaw Kretkowski

Ludwig Birkenmajer

In 1882, when a Polish nobleman Władysław Kretkowski offered a prize of 500 French francs (about $10,000 today) for its solution.

Awarded by the Polish Academy of Arts and Sciences to a 28-year-old math teacher, Ludwig Birkenmajer, later a professor at Jagiellonian University.

Equidecomposability of polyhedra: a solution of Hilbert's third problem in Kraków before ICM 1900" by D. and K. Ciesielski, *Mathematical Intelligencer* vol 40, 2018.

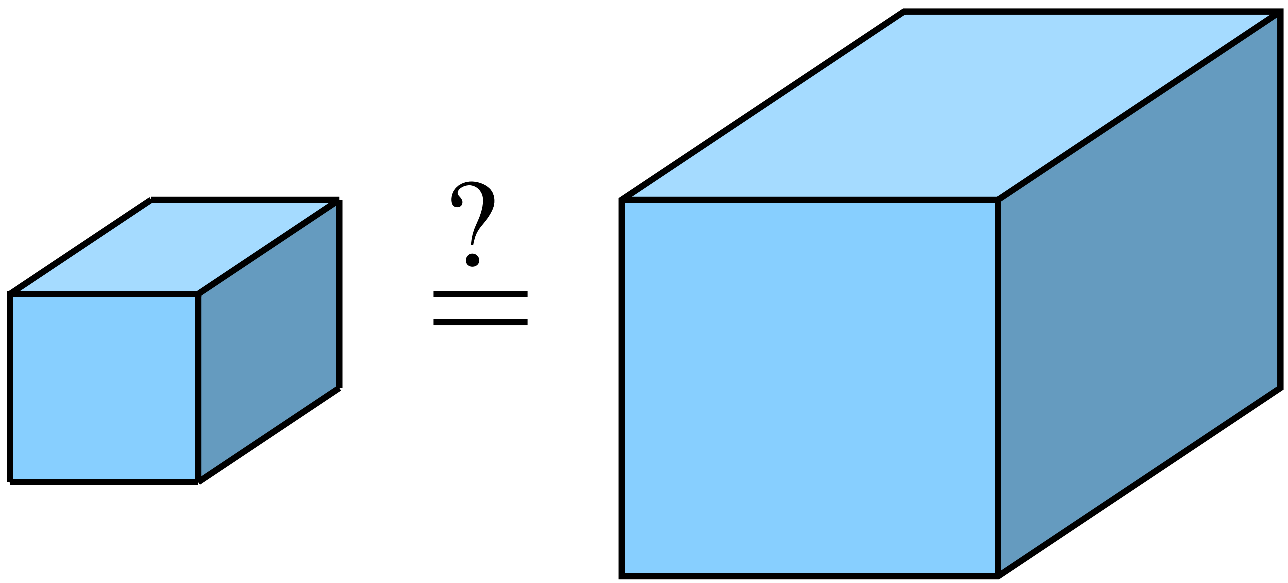
Side remark:

Equal volume polyhedra can't always be equi-partitioned into tetrahedra.

What if we replace tetrahedra by more general sets?

Can two polyhedra of equal volume be decomposed into finite collections of sets that are the same, up to rotation and translation?

Banach-Tarski paradox (1926): Given any two polyhedra in 3-space, the first can be decomposed into finitely many disjoint subsets, that can be rotated and translated to give the second as their disjoint union.



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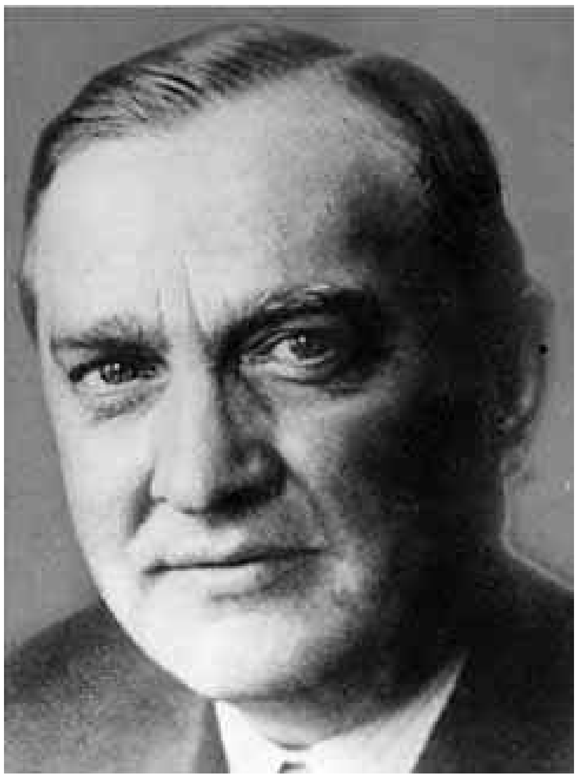
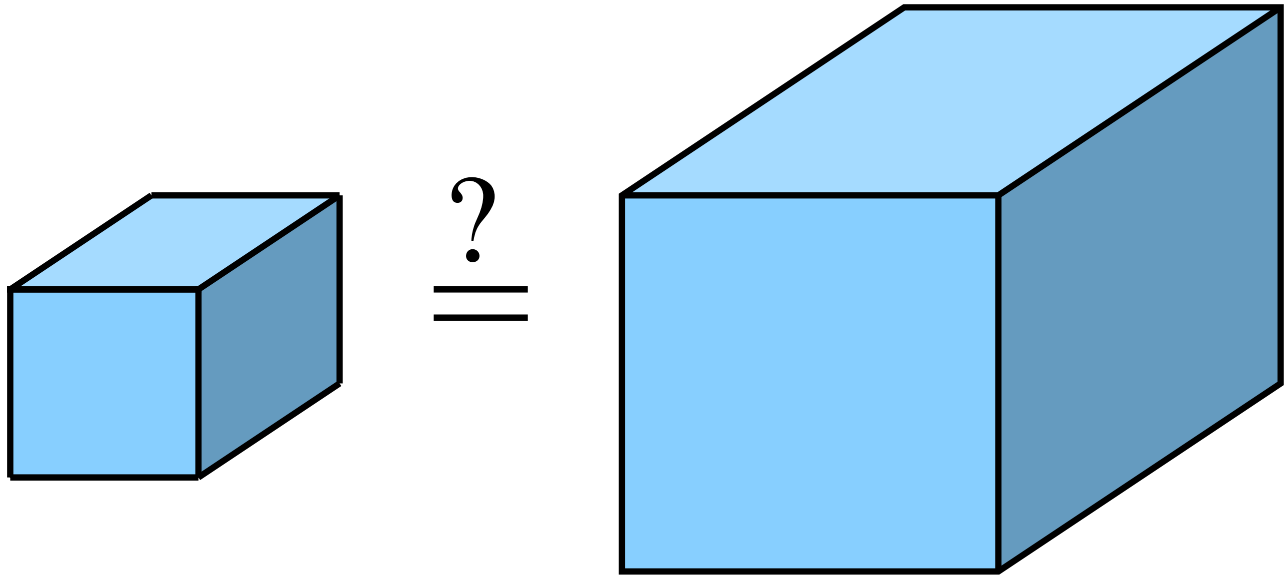


Stephen Banach

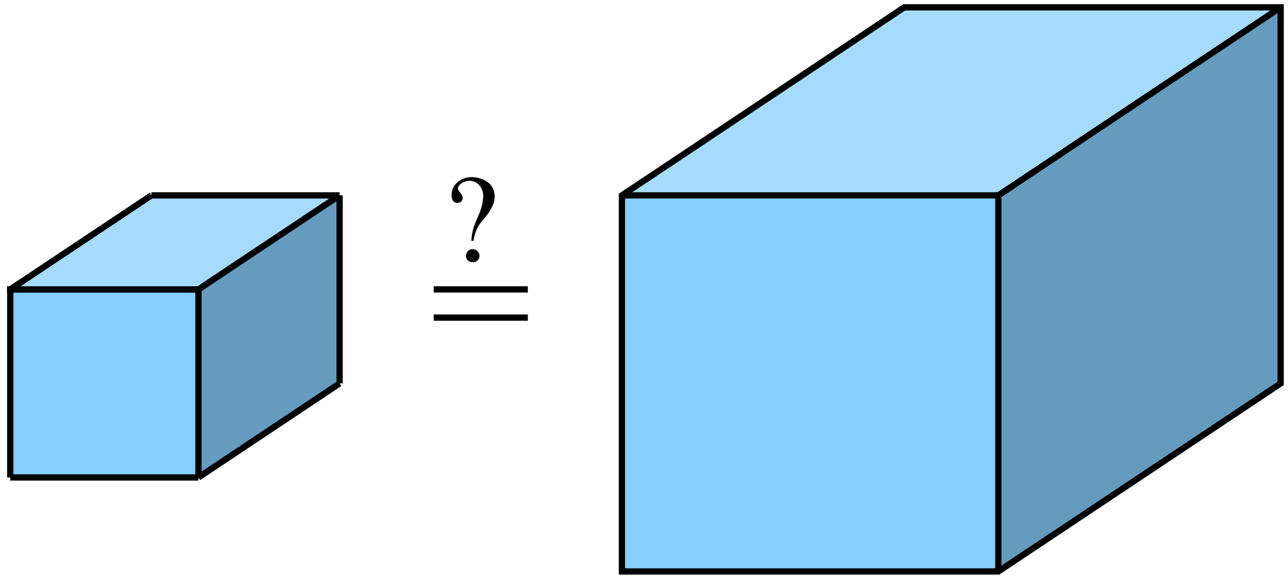
Alfred Tarski



This violates our intuition that

(1) volume does not change under translations or rotations,

(2) the volume of a disjoint union should be the sum of the volumes.

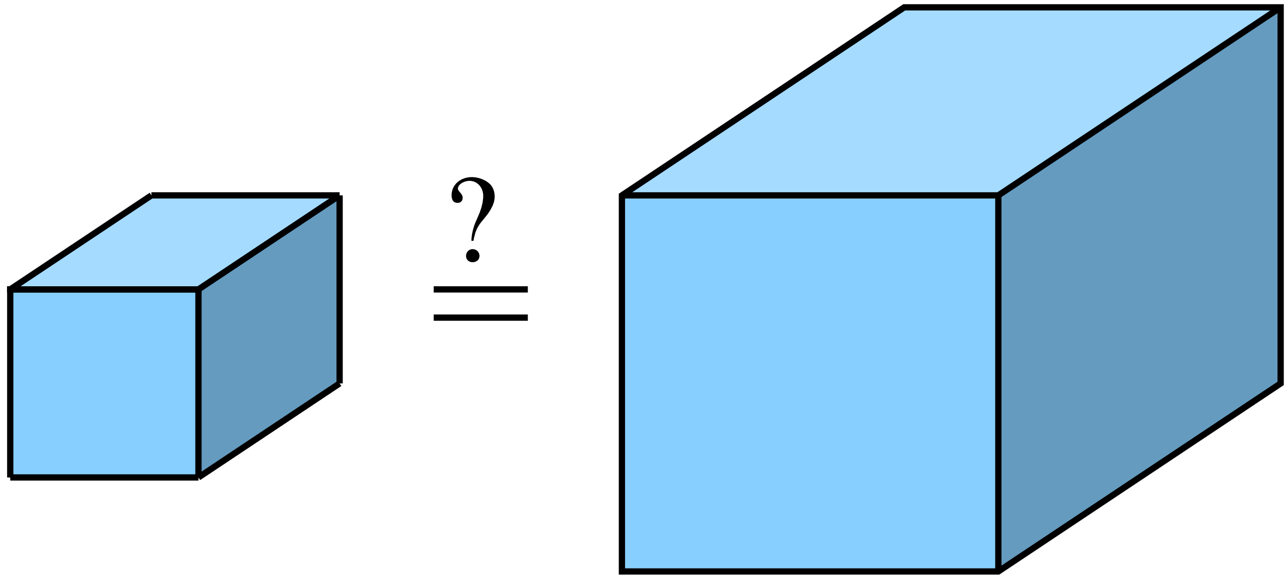


This violates our intuition that

(1) volume does not change under translations or rotations,

(2) the volume of a disjoint union should be the sum of the volumes.

We can't define 3D-volume so it has both properties for all sets.

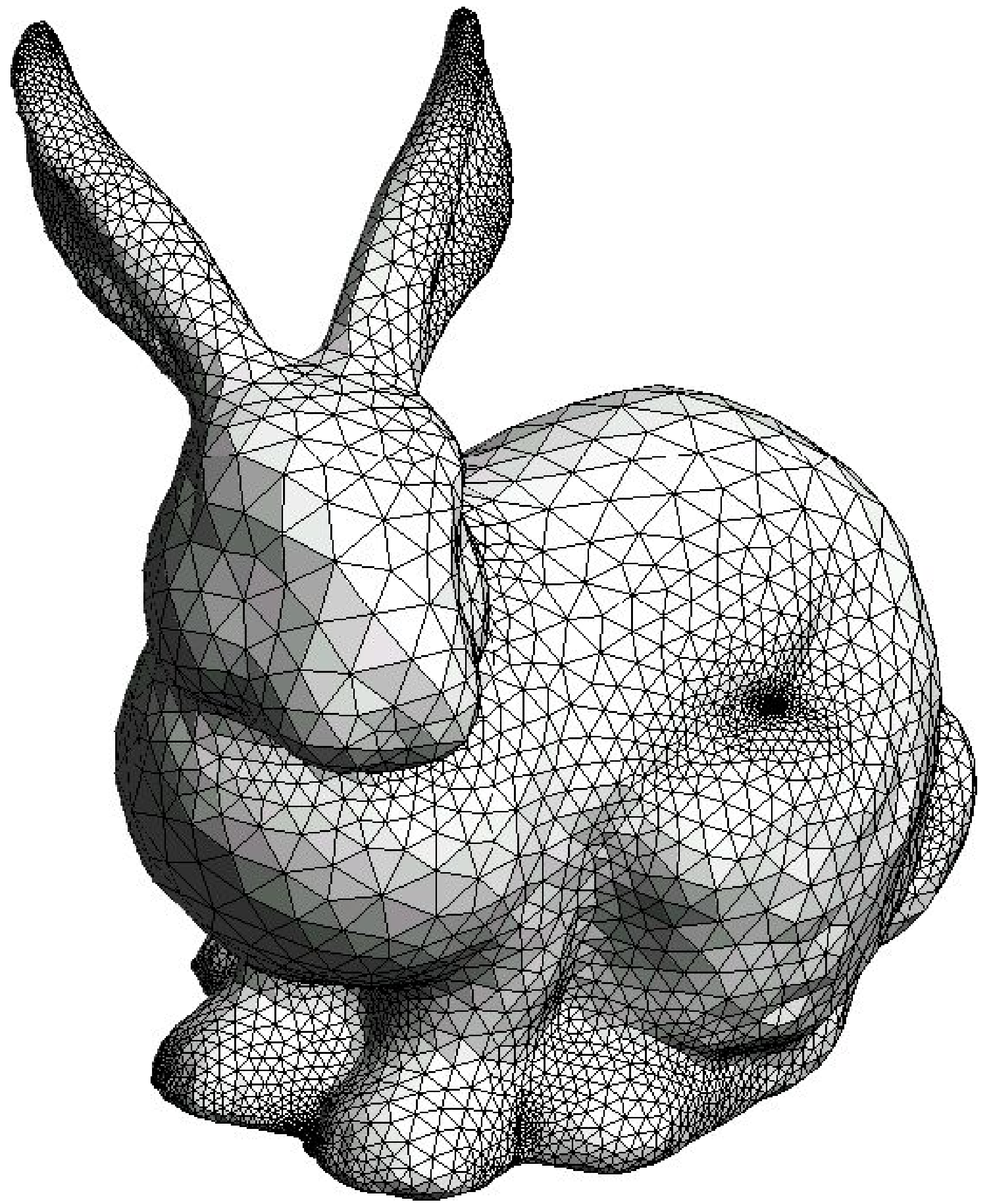


This is why a field called “Measure Theory” exists.

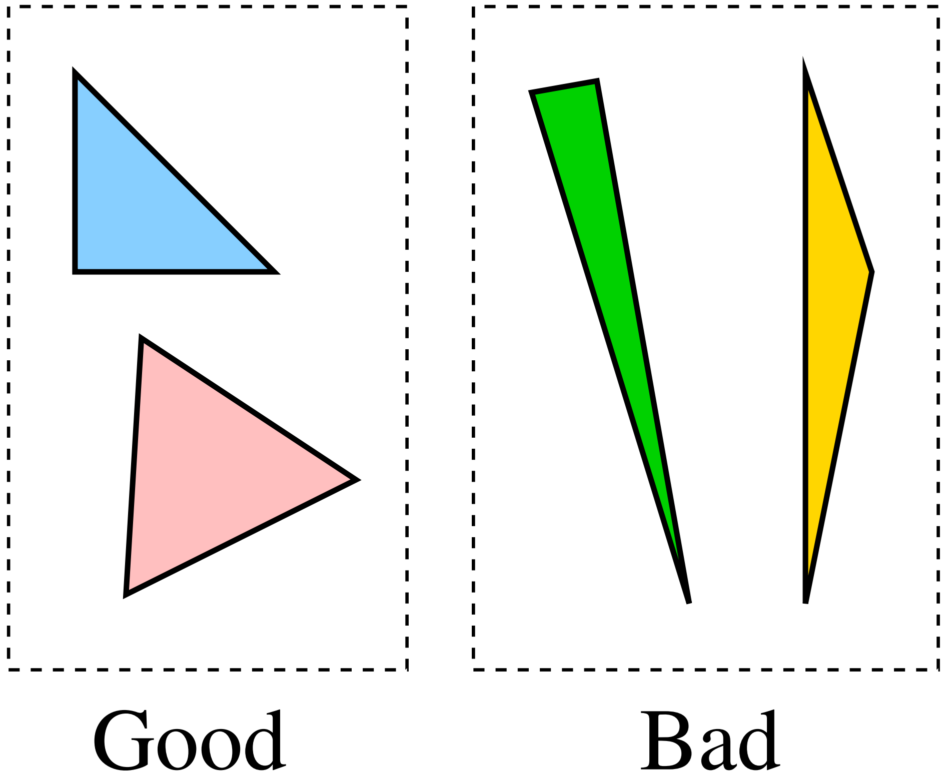
Volume can only be defined for special “measurable sets”.

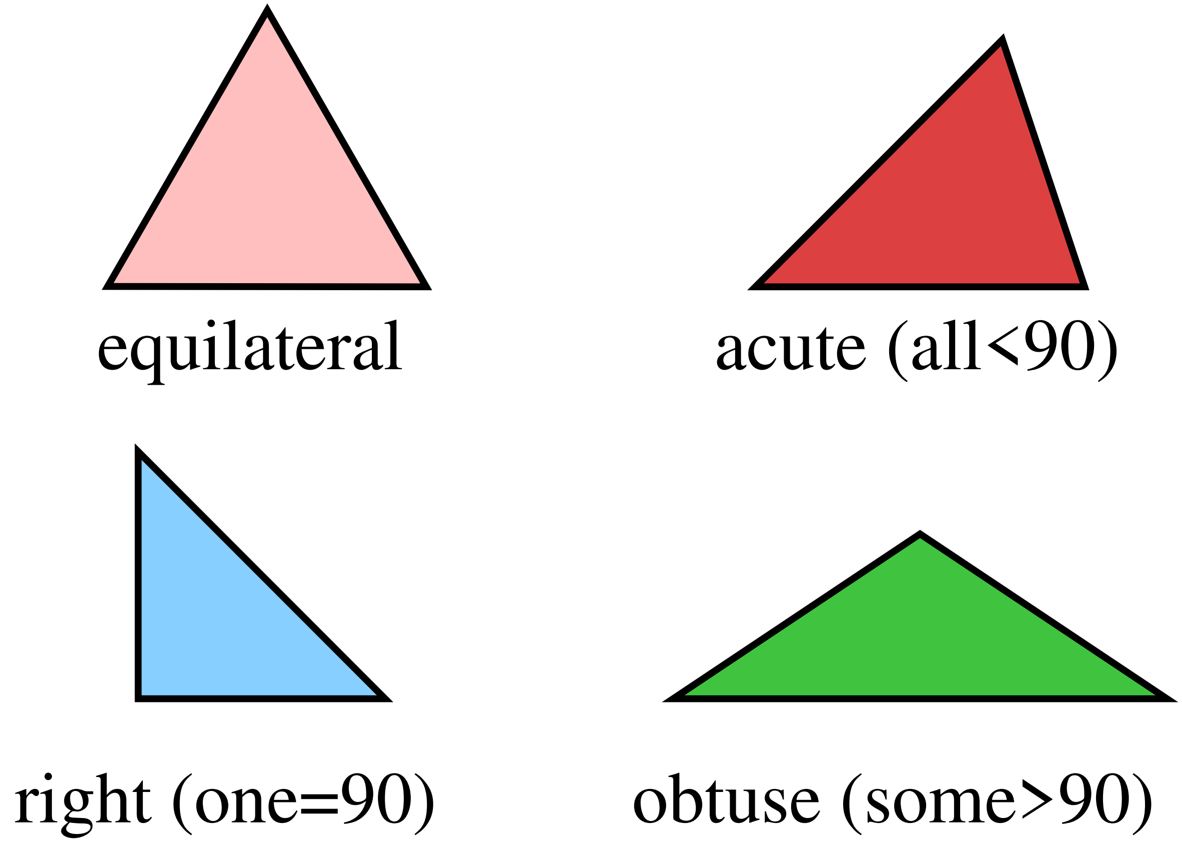
“Measurable” includes every set you can explicitly describe.

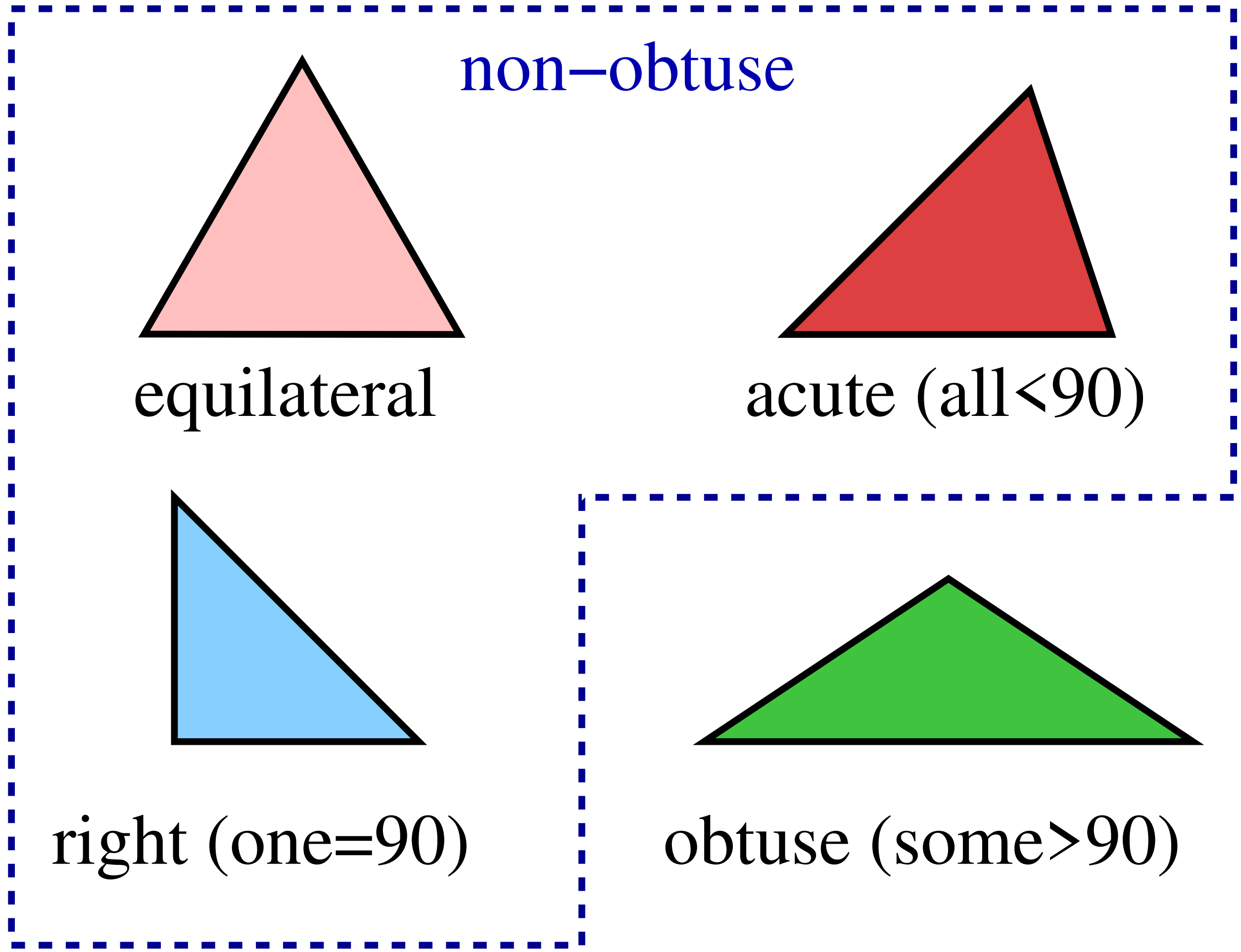
Non-measurable sets require the Axiom of Choice to construct.



For many applications, it is important to have triangulations with "nice" triangles. ("nice" = angles not too close to 0 or 180.)

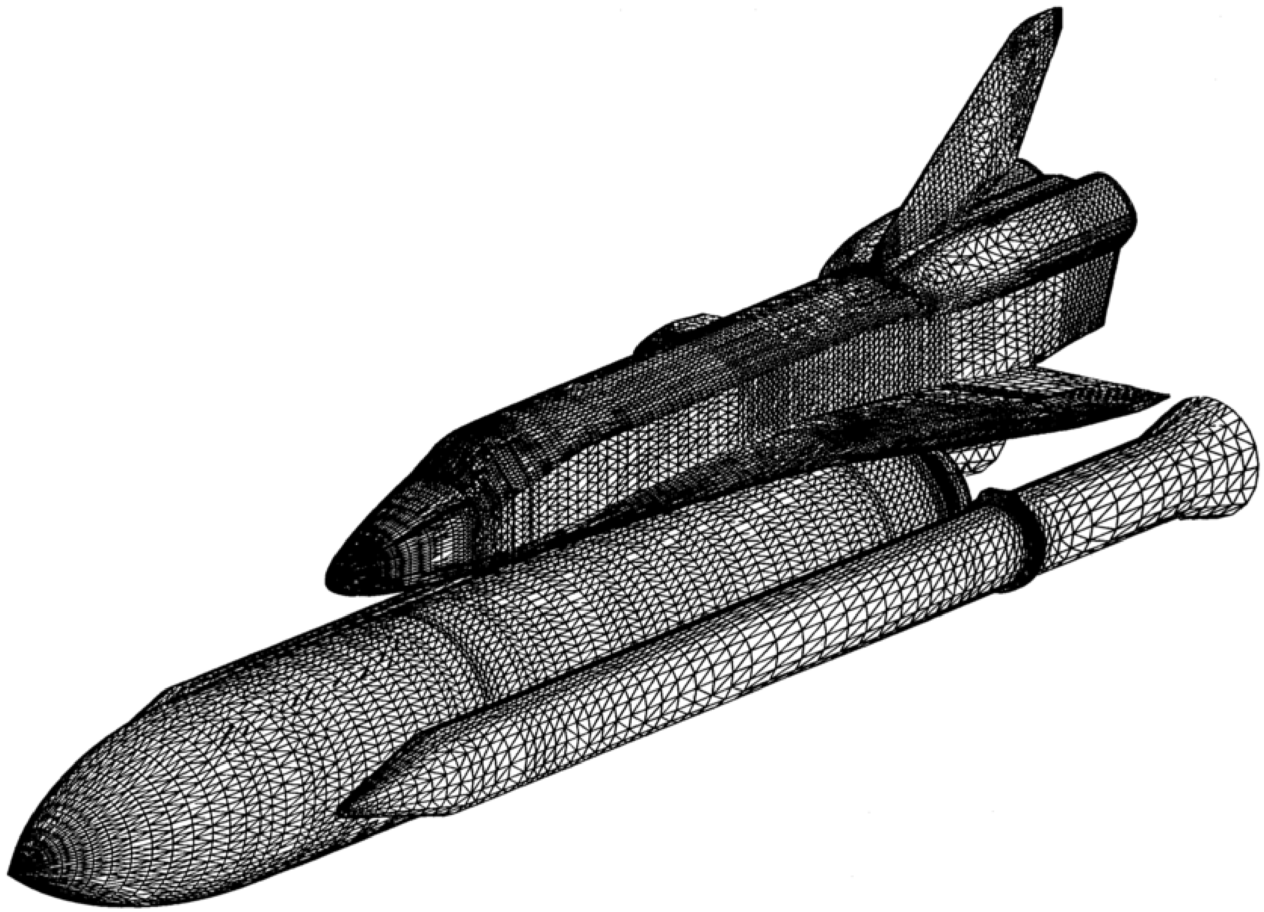



equilateral
acute (all<90)
right (one=90)
obtuse (some>90)


non–obtuse
equilateral
acute (all<90)
right (one=90)
obtuse (some>90)

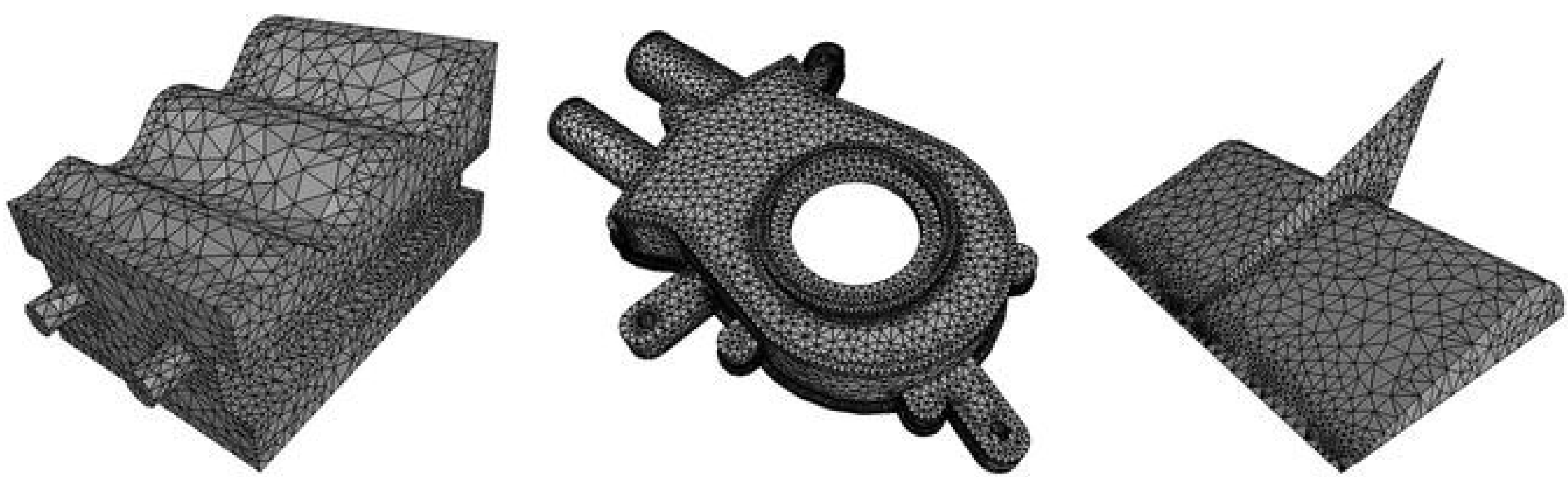
Acute triangles are often better for computer graphics and engineering.

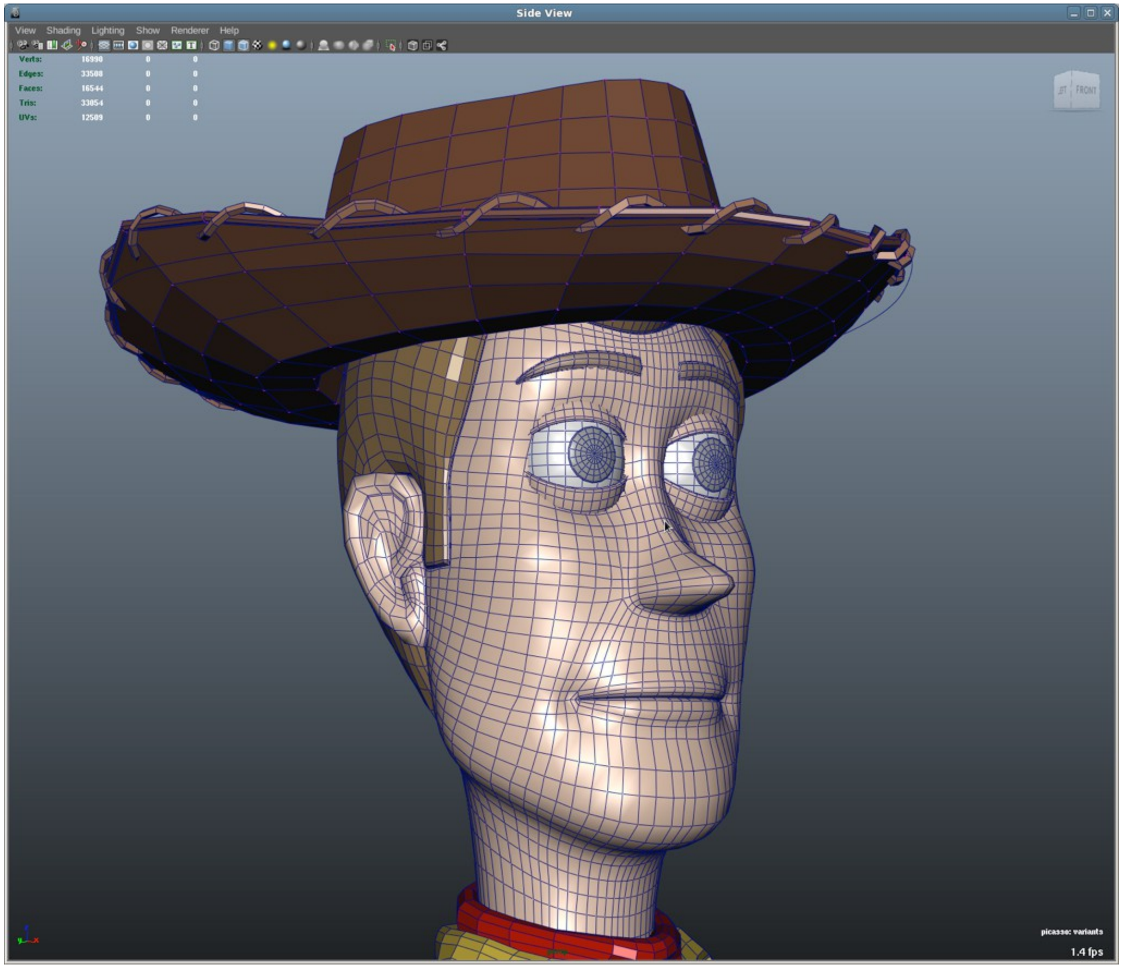
For example, in finite elements methods for differential equations, the systems arising from acute triangulations are often easier and faster to solve.



"An essential element of the numerical solution of partial differential equations (PDEs) on general regions is the construction of a grid (mesh) on which to represent the equations in finite form . . . it can take orders of magnitude more man-hours to construct the grid than it does to perform and analyze the PDE solution on the grid."

Quoted from the introduction *Delaunay Mesh Generation*, by Cheng, Dey and Shewchuk, Chapman & Hall, 2013.



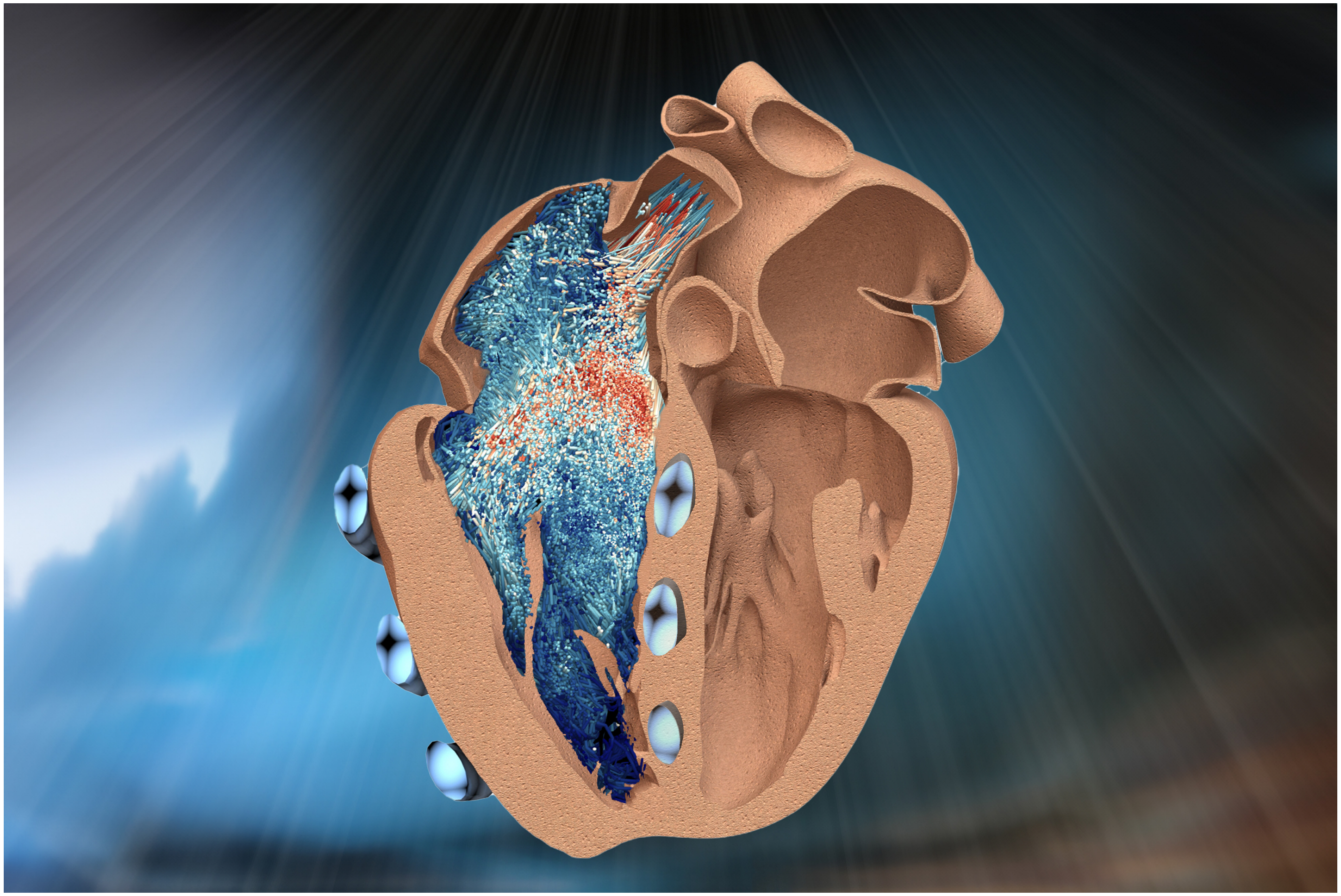


The same text points out that motion pictures are now the most economically significant consumers of high quality meshing algorithms.

Applications of triangular meshes (according to Google AI):

- **Real-time Rendering & Computer Graphics:** video games, virtual reality (VR), and augmented reality (AR).
- **3D Modeling and Animation:** Sculpting, 3D printing.
- **Scientific Visualization & Simulation:** Finite Element Analysis for mechanical engineering, physical sciences, and continuum mechanics.
- **Computer Vision & Reconstruction:** Meshes are generated from 3D point clouds, for 3D terrain modeling and facial recognition.
- **Surface Segmentation & Smoothing:** They allow for edge detection and adaptive surface smoothing.
- **Physical Simulations:** In game development meshes are used to computeollisions and physical interactions.
- **Digital Fabrication:** They are used in manufacturing processes to create objects from CAD models.
- **Global Circulation:** ocean and atmospheric modeling in geoscience.

3D meshes used in mapping applications



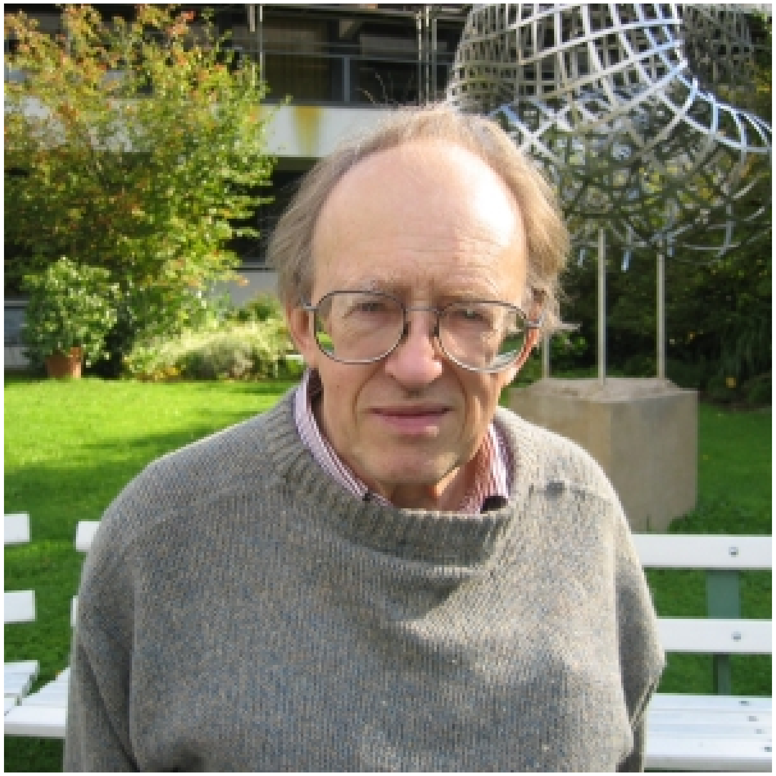
Medical applications (MIT 3D model of heart)

We want to mesh surfaces using acute triangulations.

Do acute triangulations always exist?

If so, how many acute triangles are needed?

Burago-Zalgaller (1960): all polyhedral surfaces have acute triangulations.

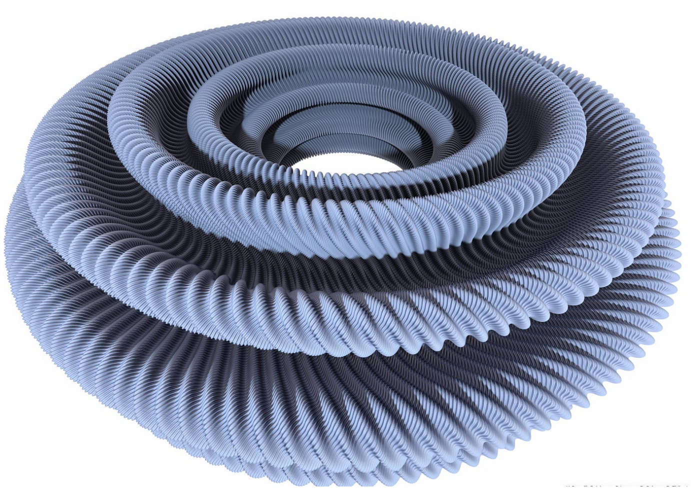


Yuri Burago (1936 -)

Victor Zalgaller (1920 - 2020)

However, their result was a lemma buried inside a Russian language paper on topology and went unnoticed in the West. First reference I know to Burago-Zalgaller in CS papers is in 2004.

They were proving a polyhedral version of the Nash Embedding Theorem.



Nash embedding

John Nash

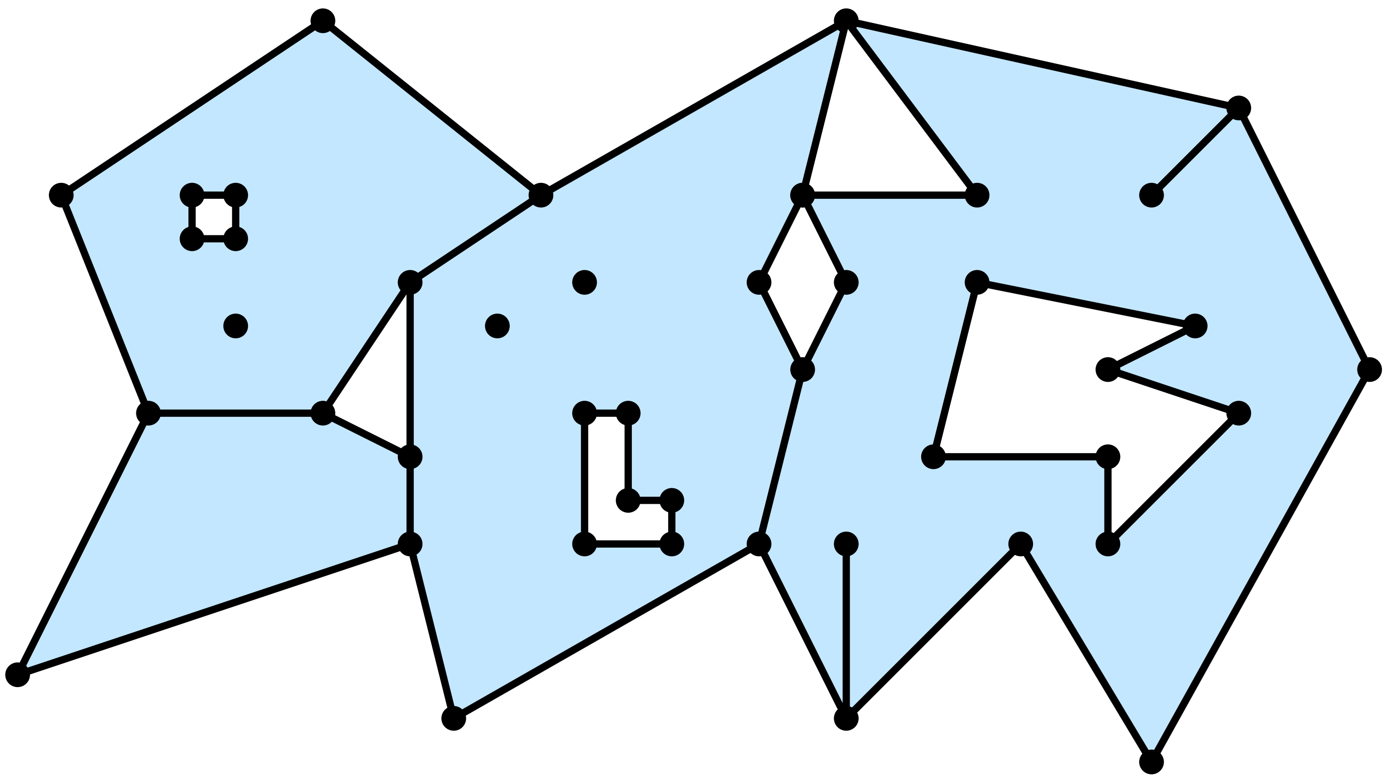
not John Nash

Computational complexity = number of triangles needed.

• Baker-Grosse-Rafferty (1987) proved every n-gon has a non-obtuse triangulation (weaker than the Burago-Zalgaller theorem).

• Bern-Mitchell-Ruppert (1995): Cn triangles suffice for simple polygons.

• Maehara (2002): any nonobtuse triangulation, can be subdivided to give an acute triangulation $\Rightarrow$ Burago-Zalgaller + linear complexity.



Non-simple polyon = PSLG = Planar Straight Line Graph

- Bishop (2016): $Cn^{2.5}$ acute triangles suffice for PSLGs.

Some examples need $\simeq n^2$. Sharp power remains unknown.

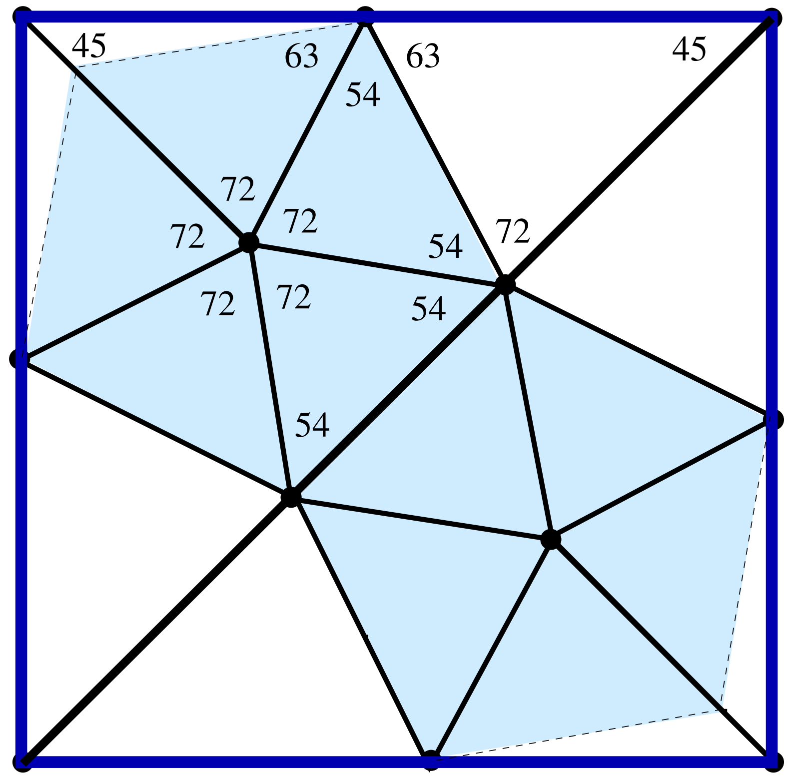
Defn: A θ-triangulation is one where every angle is $\leq \theta$.

Burago-Zalgaller: any polygon has a θ-triangulation for some $\theta < 90°$.

How does θ depends on the polygon?

From here on, "triangulation" means "Steiner triangulation".

For example, a square has a 72°-triangulation, and this is optimal.



Optimality proven using Euler's formula $F - E + V = 1$ where

$F =$ number of faces, $E =$ number edges, $V =$ number of vertices.

Proof that 72° is optimal, at end of slides.

For general polygons, we have (Bishop, DCG 2026):

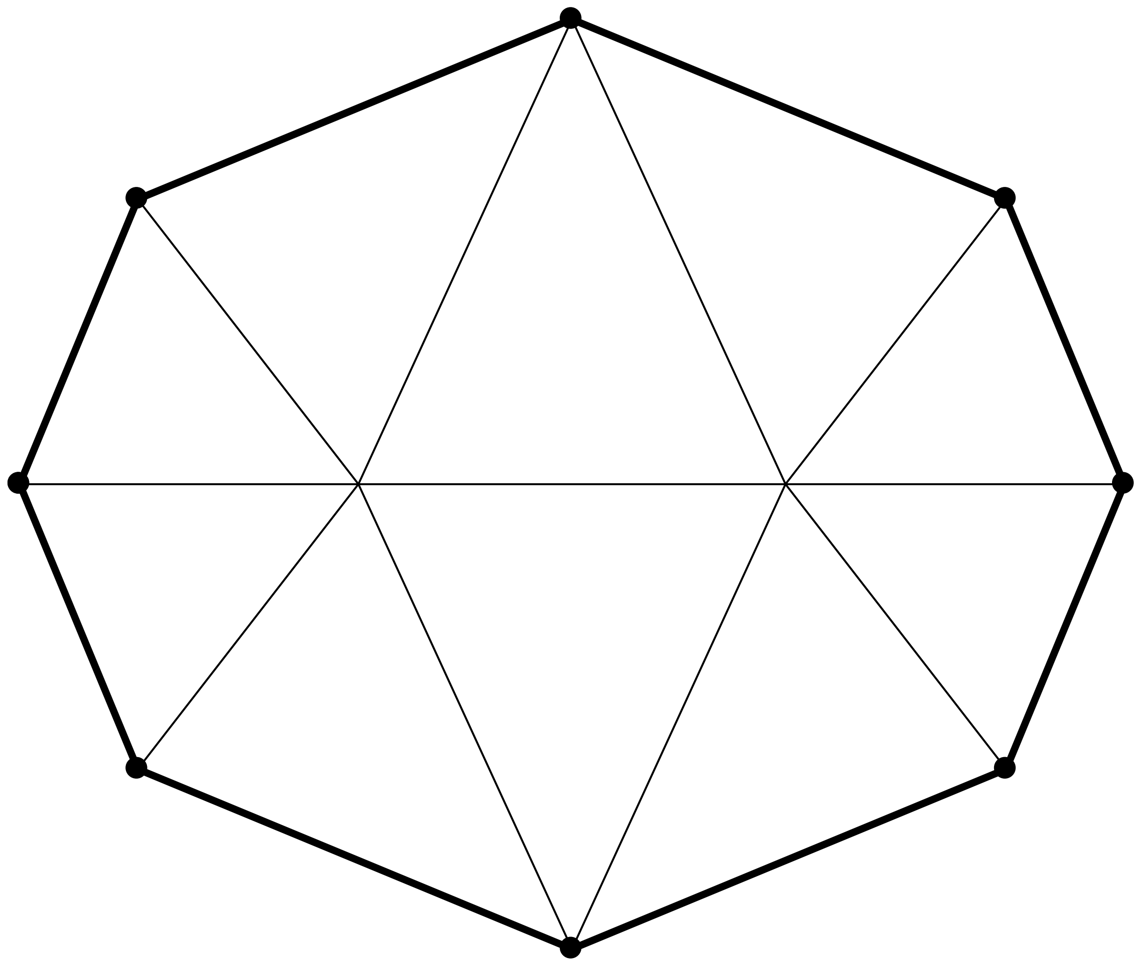
Theorem: The optimal angle bound for triangulating an n-gon can be computed in time $O(n)$.

Theorem: The optimal angle bound only depends on the set of angles, not their order or the edge lengths.

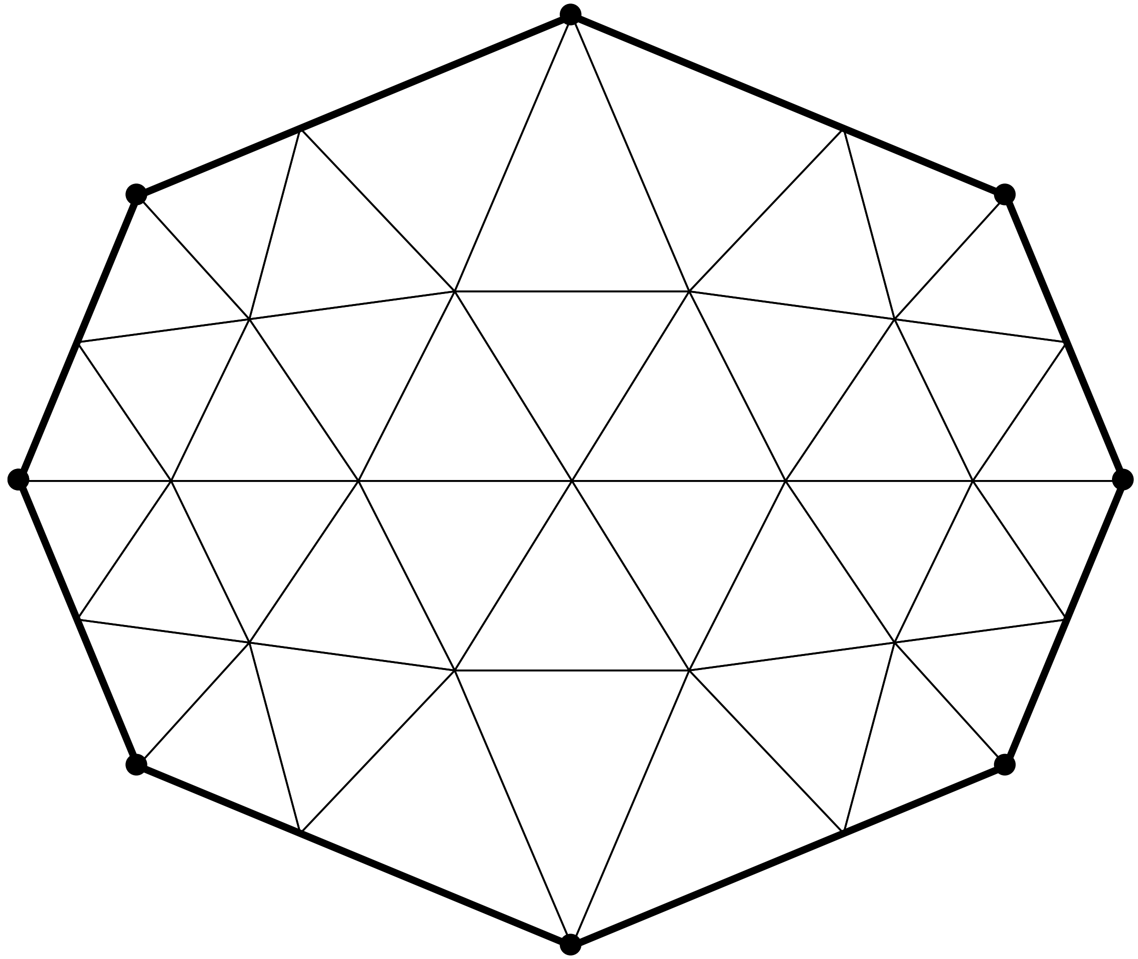
Theorem: If P has minimal angle $\geq 36^\circ$ then it has a 72°-triangulation.

Theorem: The optimal angle bound for triangulations is the same as for triangular dissections.

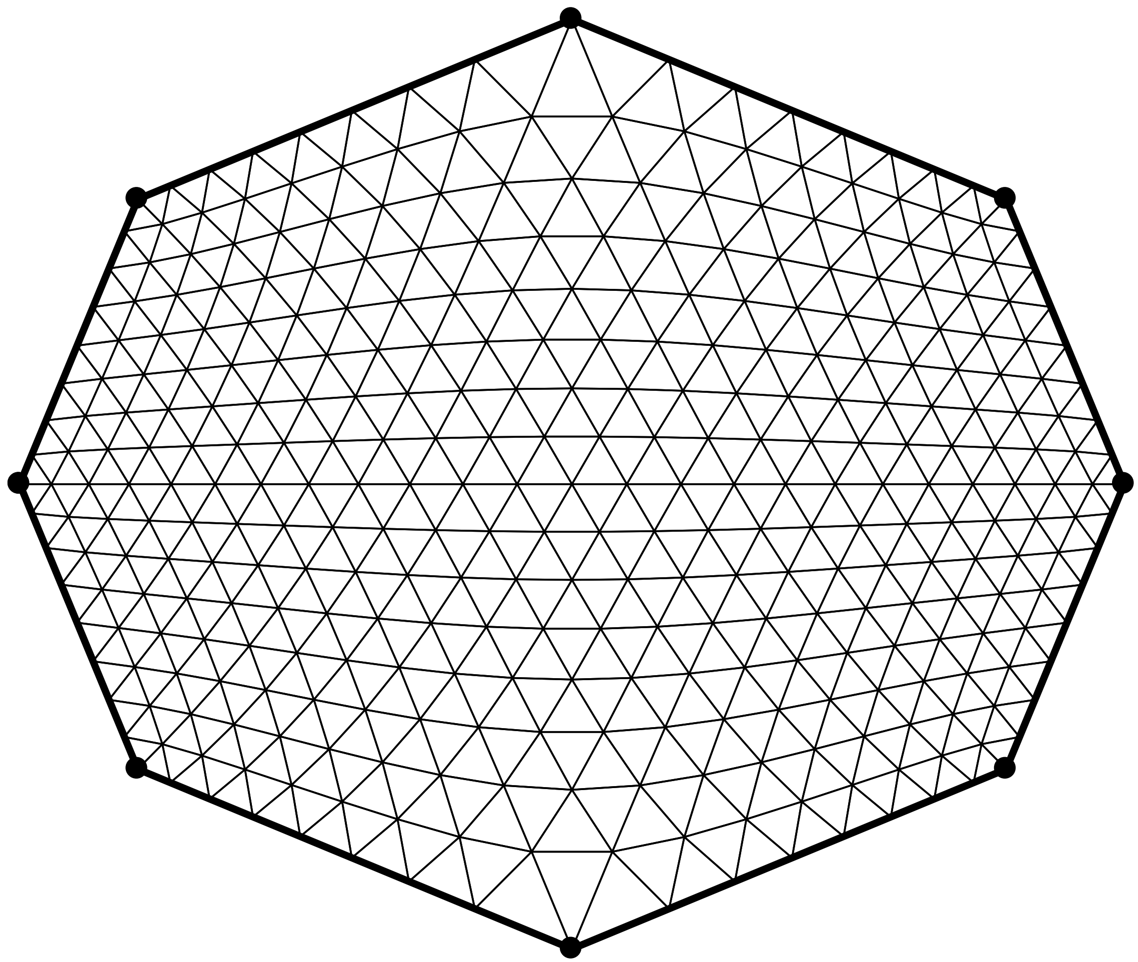
Theorem: An optimal triangulation "usually" exists.



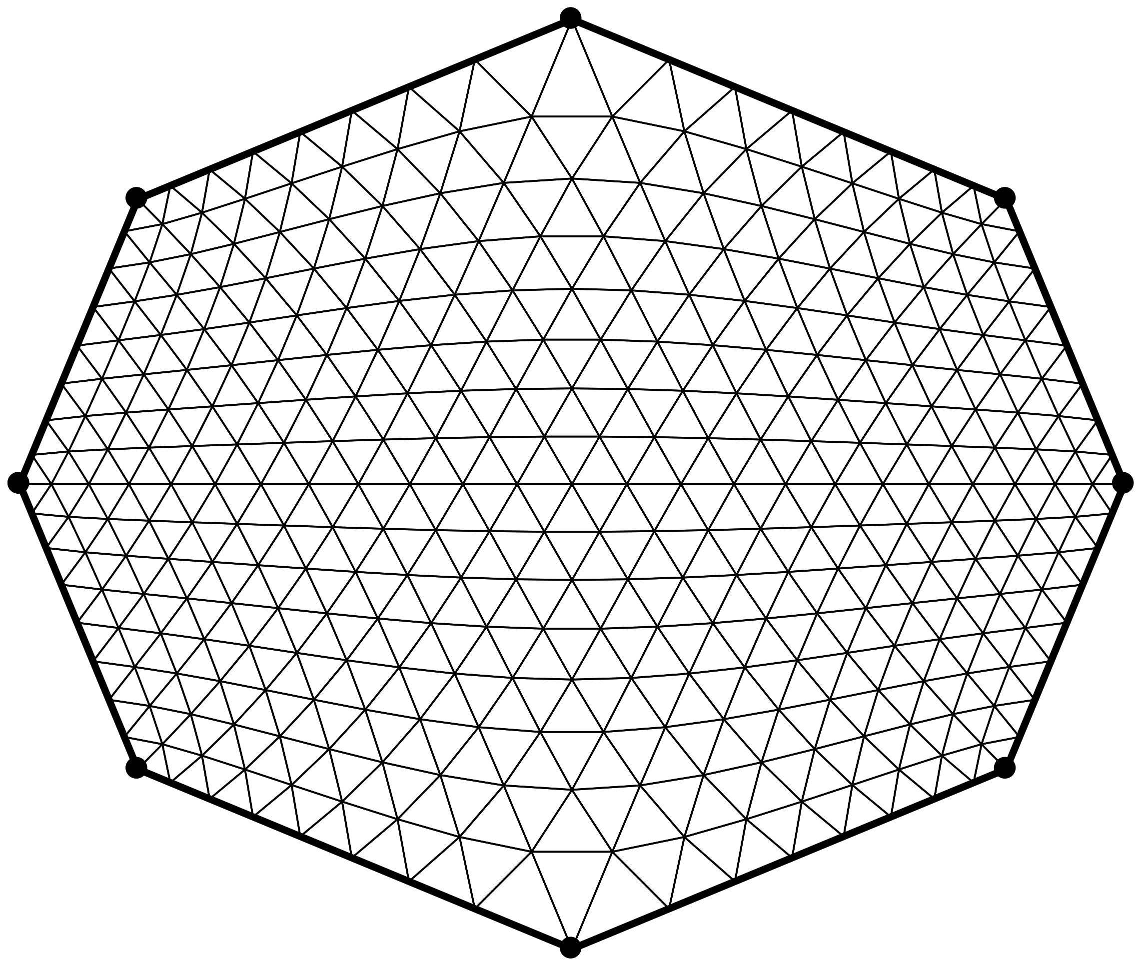
Maximum angle ≈ 74.7482.



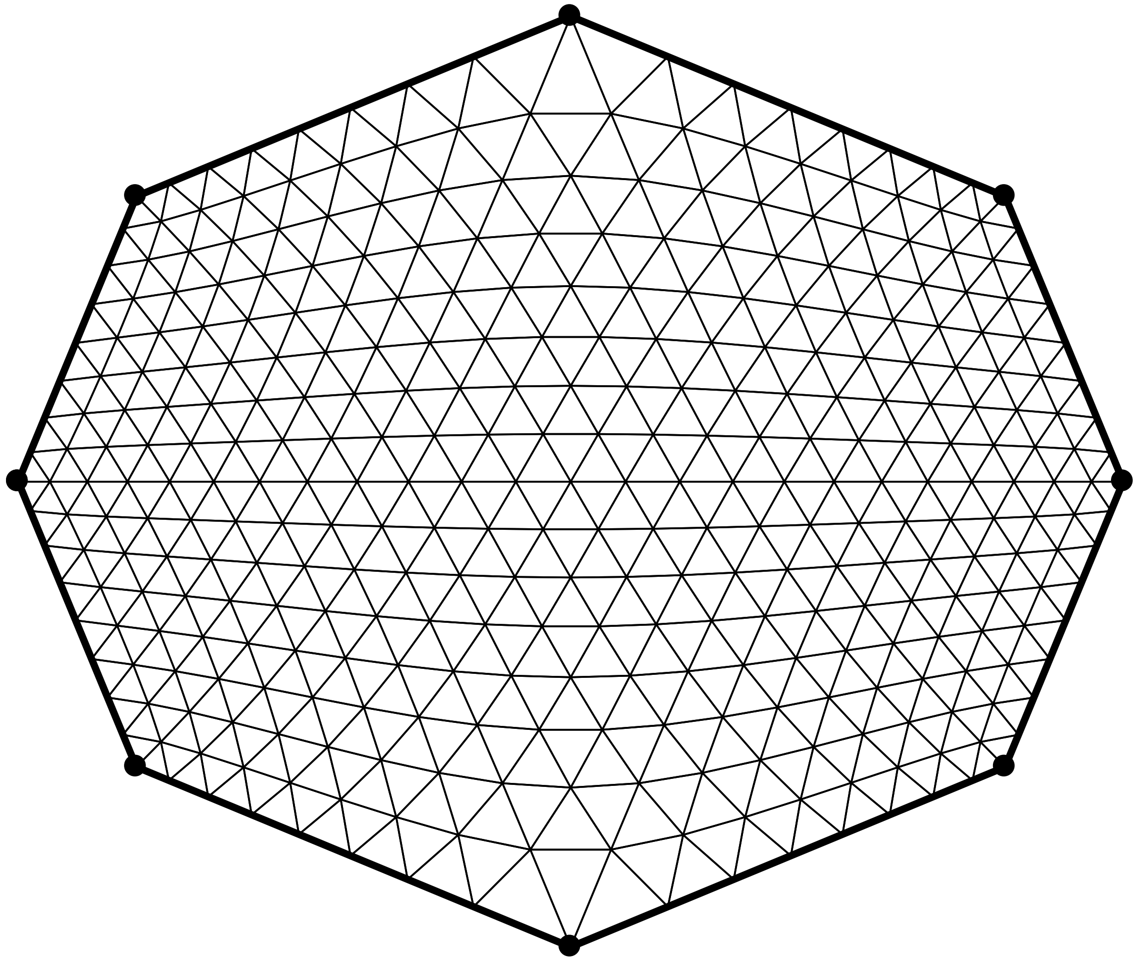
Maximum angle ≈ 70.3590.



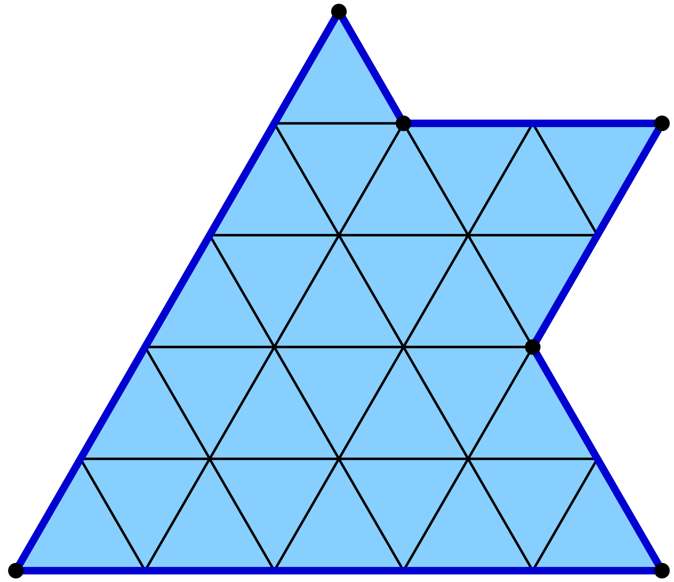
Maximum angle ≈ 67.8690.



Optimum bound is $= 67.5°$. Can it be attained?



Optimum bound is $= 67.5^\circ$. Can it be attained?
Yes in this case, but not for all polygons.



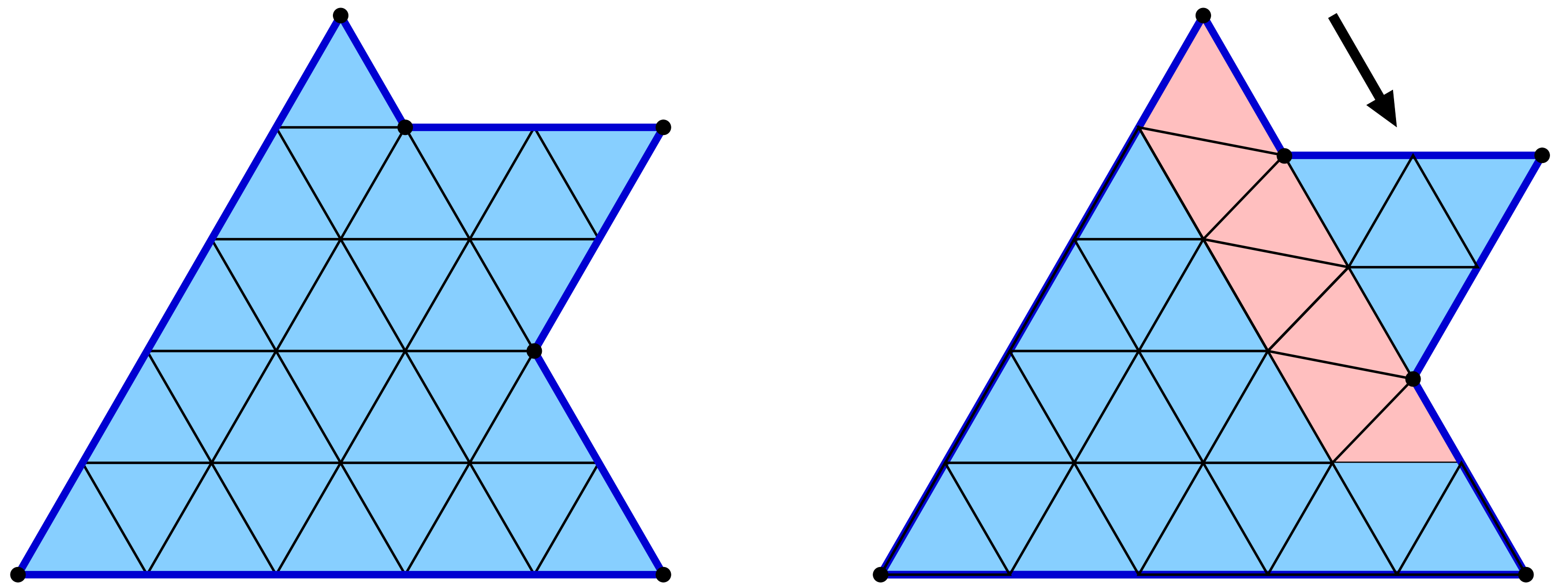
If a polygon P has an equilateral triangulation, then:

(1) all angles of P are integer multiples of 60° (P is a 60°-polygon) .

(2) all triangles are same size.

$\Rightarrow$ all edges of P are multiples of this length

$\Rightarrow$ length ratios of any two edges is rational



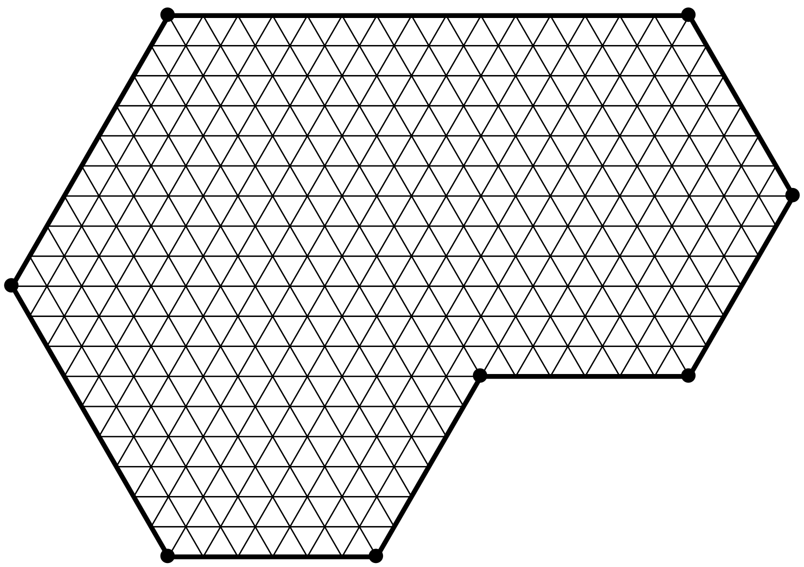
Lemma: every 60°-polygon has a θ-triangulation for every $\theta > 60^\circ$.

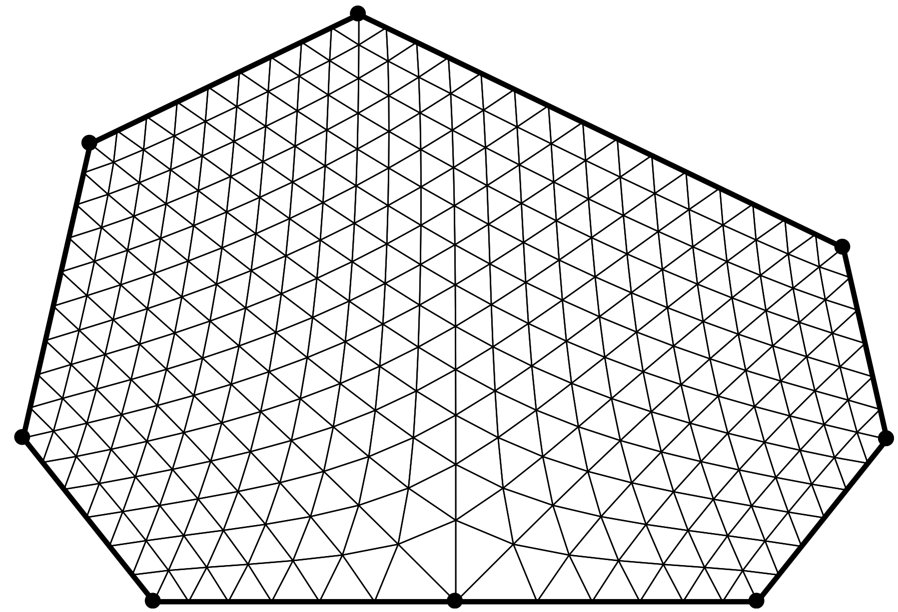
But if some edge ratio is irrational no equilateral triangulation exists.

Conclusion: for such polygons no optimal triangulation exists.

Thm: An optimum triangulation exists for every other polygon.

Very, very rough idea behind proofs:





- Given P "approximate" it by a 60°-polygon P'.
- Use conformal map $P' \to P$ to transfer equilateral triangulation.
- Prove worst angle distortion is near vertices.

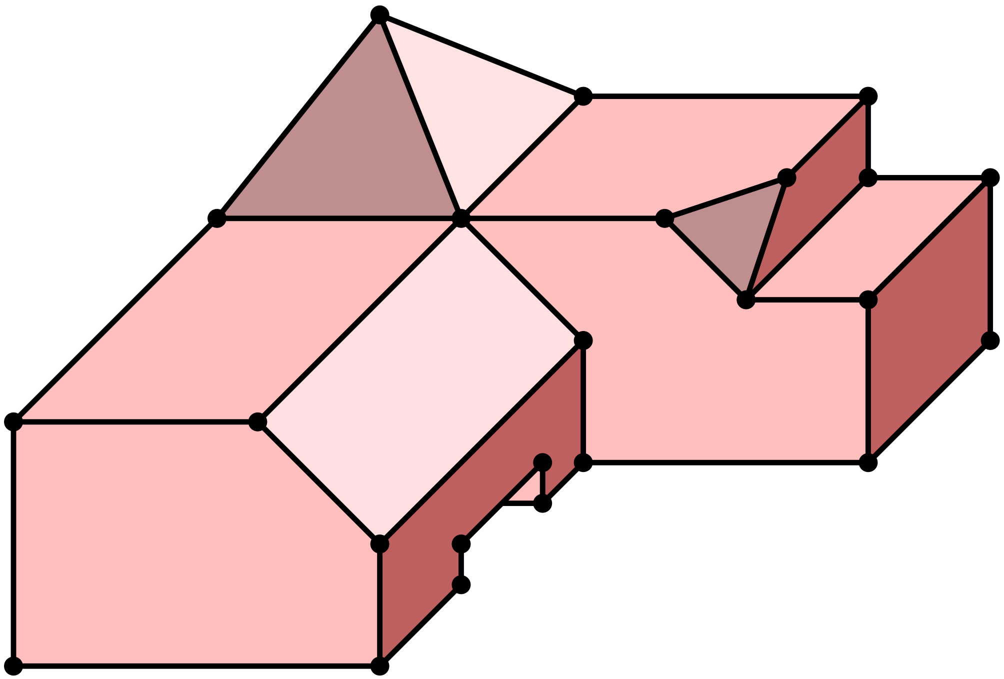
Choosing the correct P' is the hard part.

The Future (open problems)

1. Acute Triangulation in Dimension 3:

Question: Does a polyhedron with n vertices have a polynomial sized acute tetrahedralization?

(dihedral angles $< 90^\circ$, number of elements less than a power of n.)



This is pretty hard even for the unit cube.

The “smallest” known acute triangulation of a cube uses **1370** tetrahedra.

Found by VanderZee, Hirani, Zharnitsky and Damrong in 2010, using a computer search. A more conceptual construction with **2715** tetrahedra was given in 2012 by Kopczyński, Pak and Przytycki.

446 E. VanderZee et al. / Computational Geometry 43 (2010) 445–452

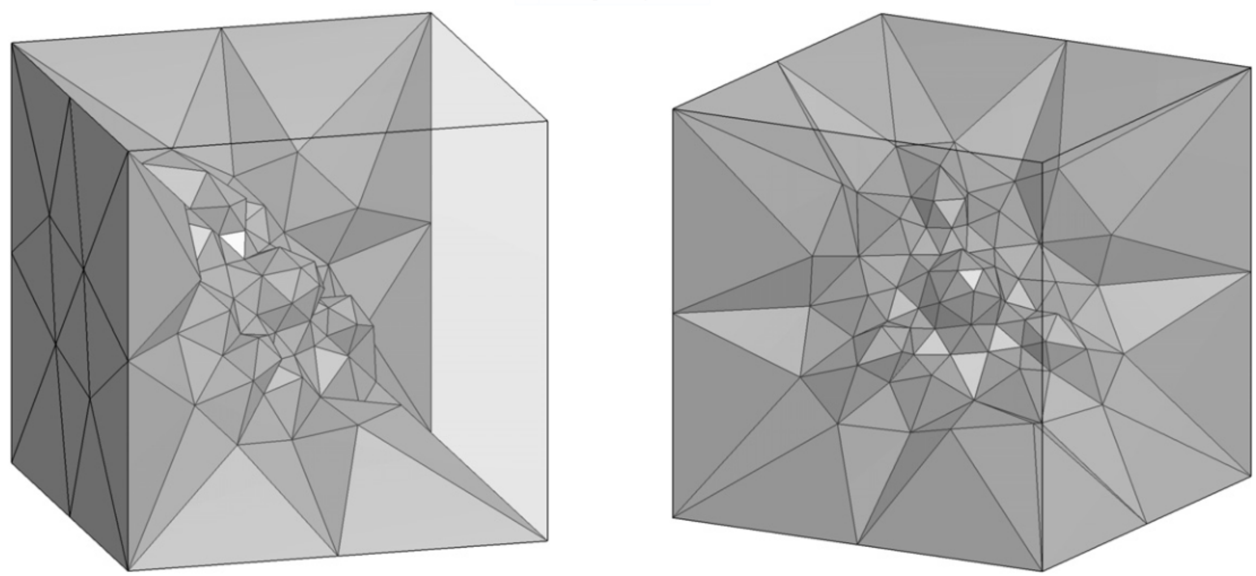


Fig. 1. Two views of a cutaway section of the first-known acute triangulation of the cube. The view at right is a 45° rotation about the z-axis from the view at left. On the left a 14-triangle triangulation of one of the square faces of the cube is visible. This 14-triangle triangulation of the square is used on each face of the cube.

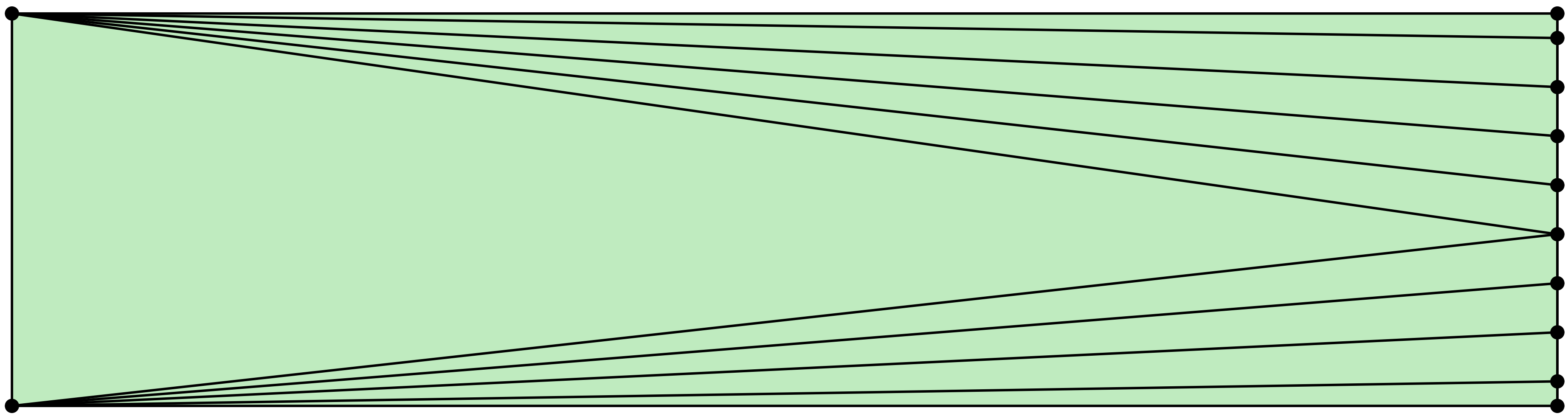
2. Minimal Weight Triangulations

Find triangulation that minimizes total edge length.

$1 \times R$ rectangle with n extra points on right side.

2. Minimal Weight Triangulations

Find triangulation that minimizes total edge length.



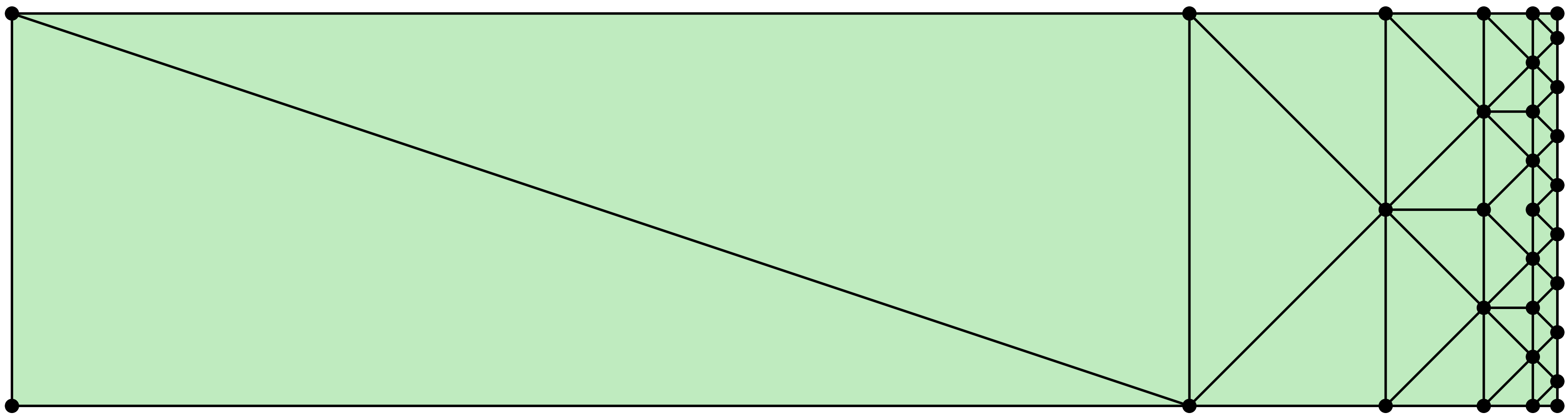
$1 \times R$ rectangle with n extra points on right side.

Without Steiner points, all triangulations length $> R \cdot n$.

For general polygon finding (non-Steiner) optimum is NP hard.

2. Minimal Weight Triangulations

Find triangulation that minimizes total edge length.

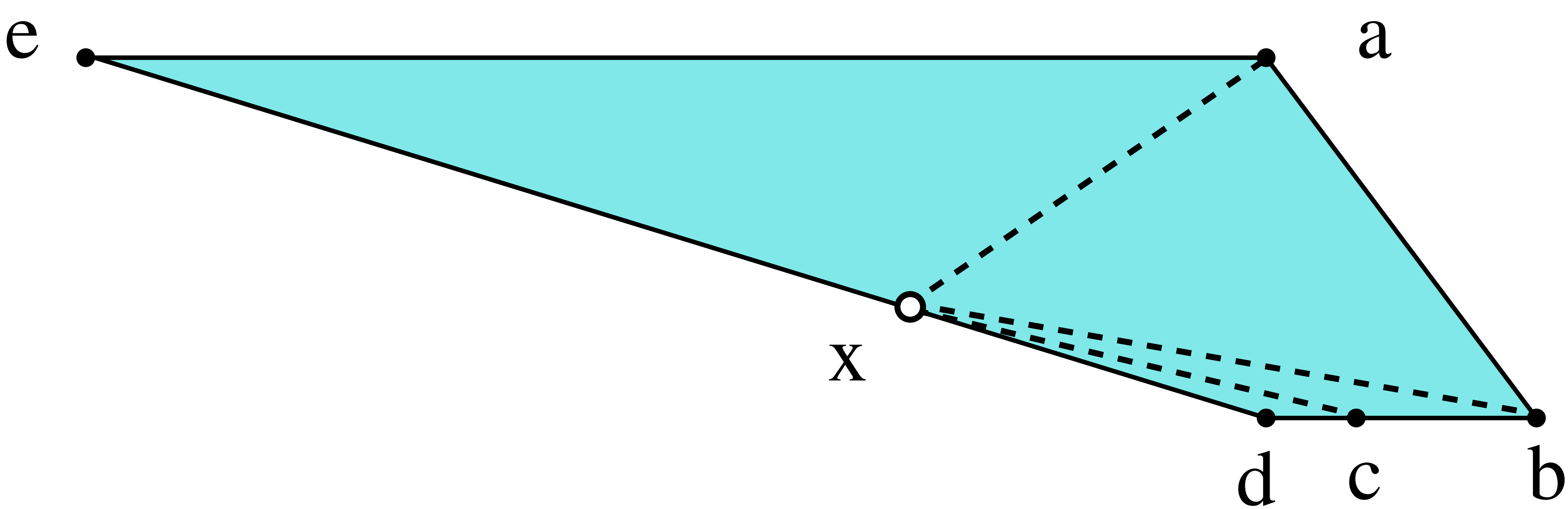


$1 \times R$ rectangle with n extra points on right side.

With Steiner points, length $\leq R + O(\log n)$ is possible.

What is optimal? Does an optimal triangulation exist?

2. Minimal Weight Triangulations



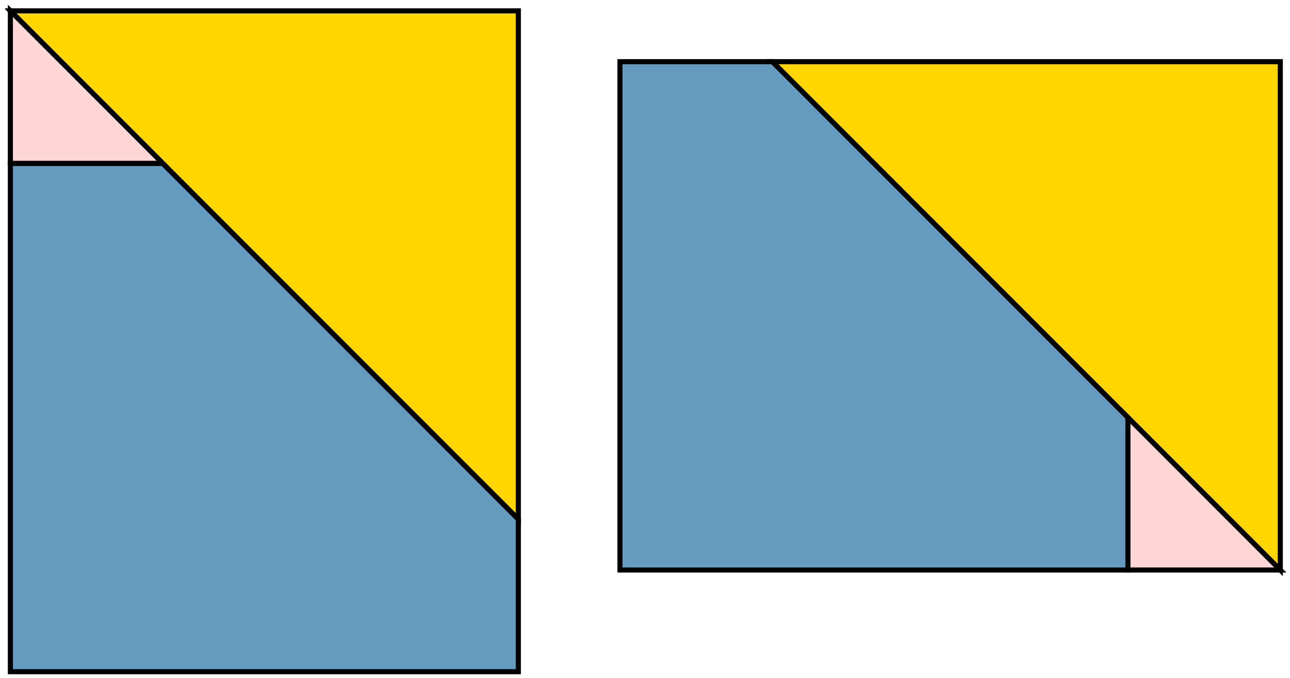
As x approached d, total length tends to infimum.

A minimal weight Steiner triangulation (MWST) does not exist.

But there are three co-linear vertices (b, c, d).

Question: Does MWST exist if no three vertices are co-linear?

3. A stronger Wallace–Bolyai–Gerwien Theorem:

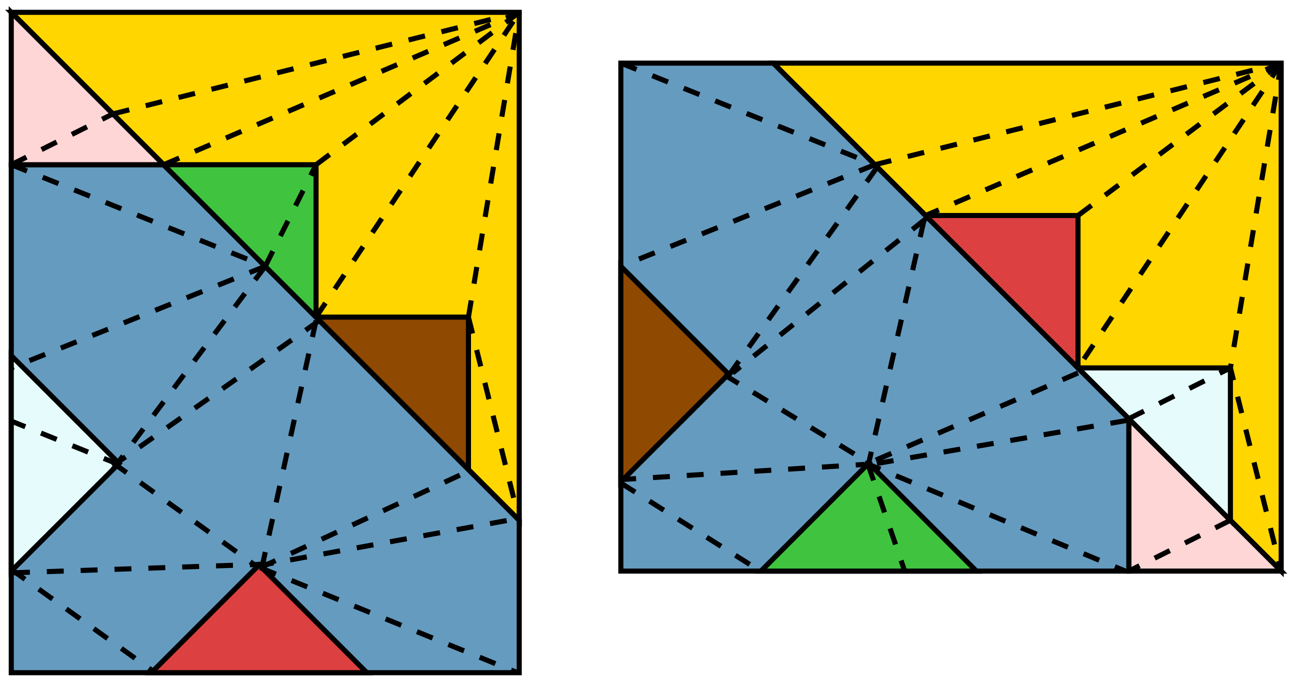


WBG ⇒ equal area polygons can be dissected using same triangles.

Can "dissection" be improved to "triangulation" (with Steiner points)?

In other words, all triangles meet along full edges.

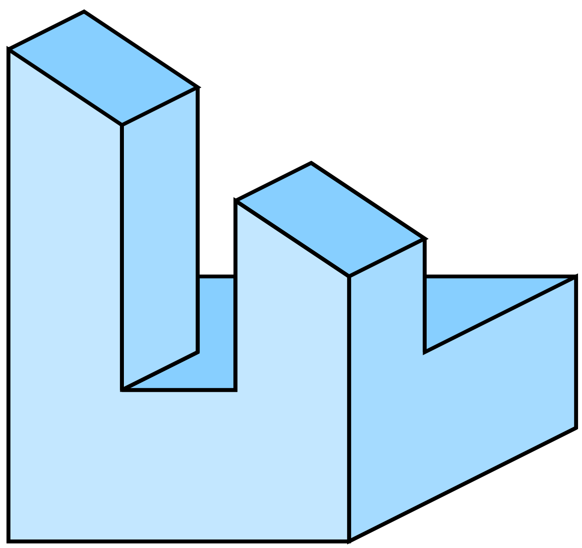
3. A stronger Wallace–Bolyai–Gerwien Theorem:



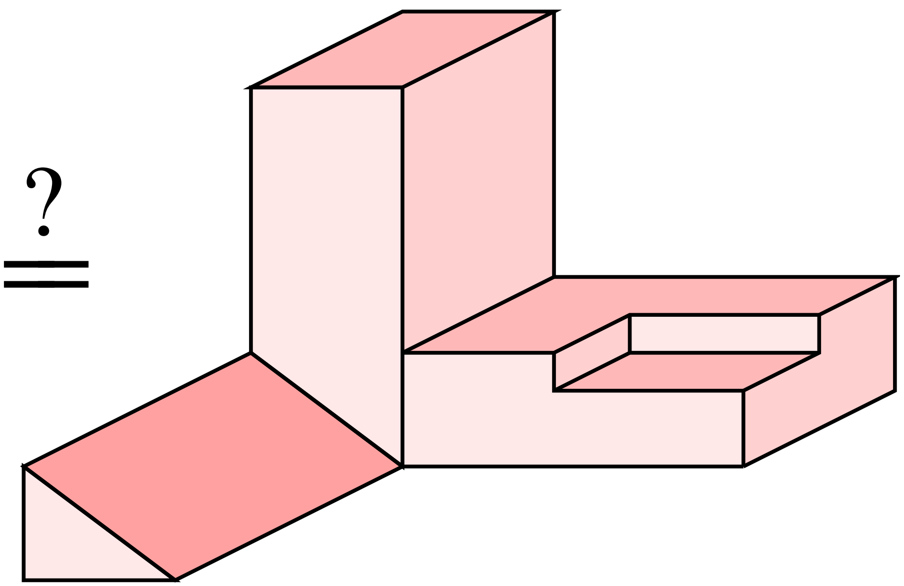
Thm (B, Amer. Math. Monthly 2026): Any two equal area polygons can be triangulated using same set of triangles.

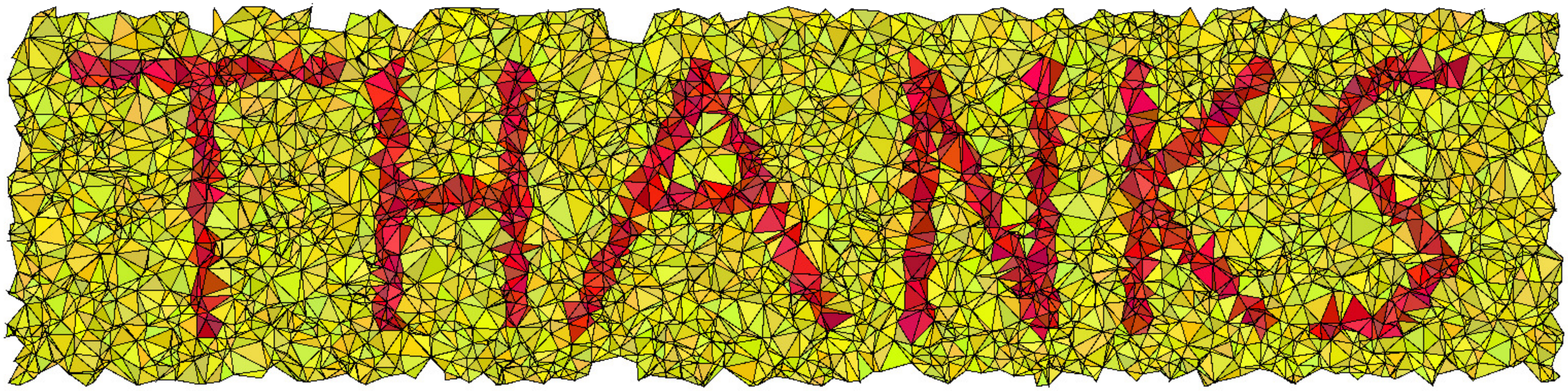
Proof uses ideas from dynamical systems. Boundaries of the polygons play a very important role. What is there is no boundary?

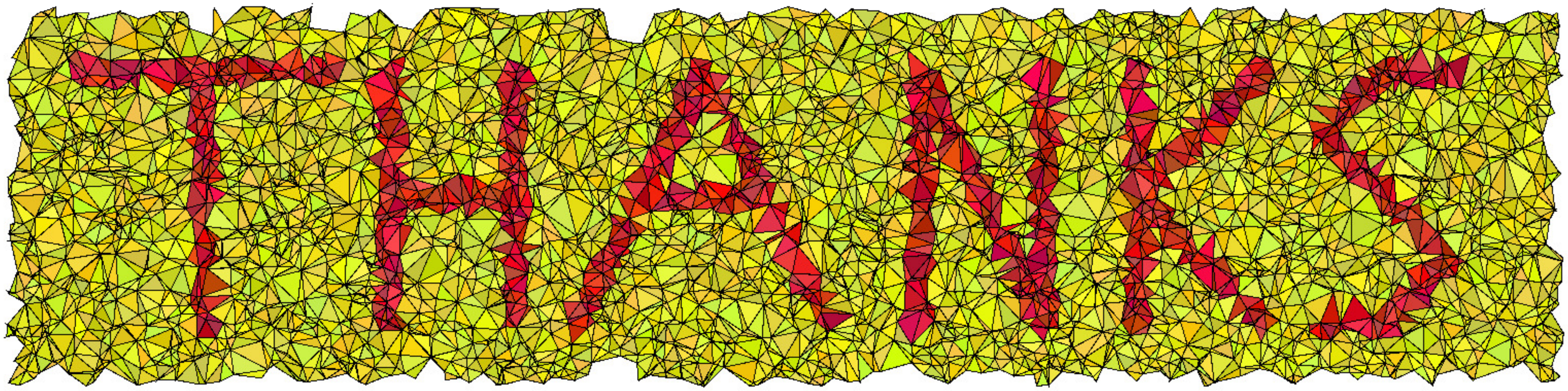
Question: Can two closed polyhedral surfaces with equal area be triangulated (with Steiner points) using the same set of triangles?



$\stackrel{?}{=}$




THANKS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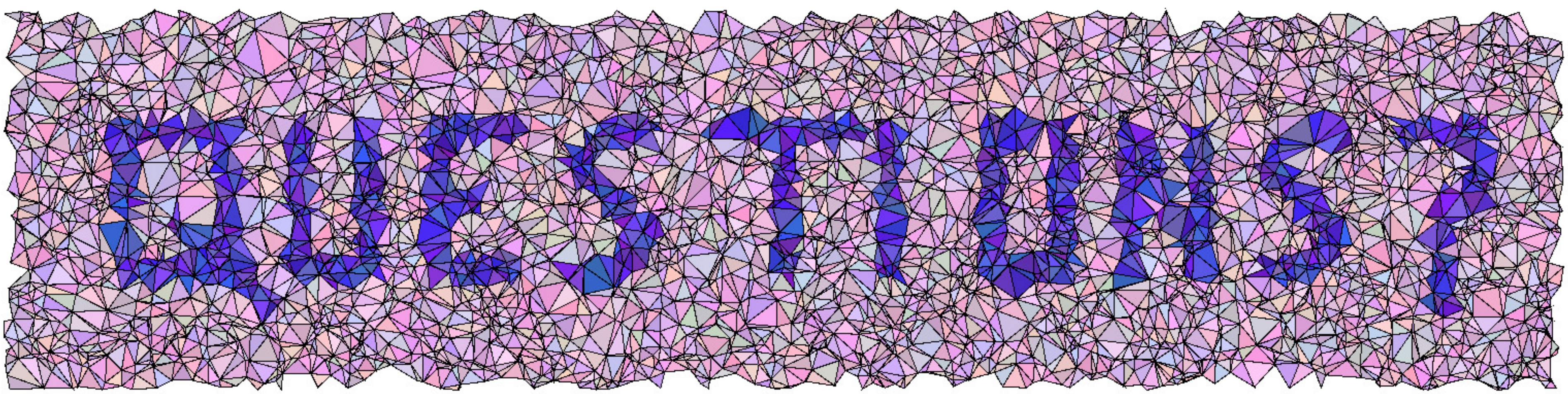
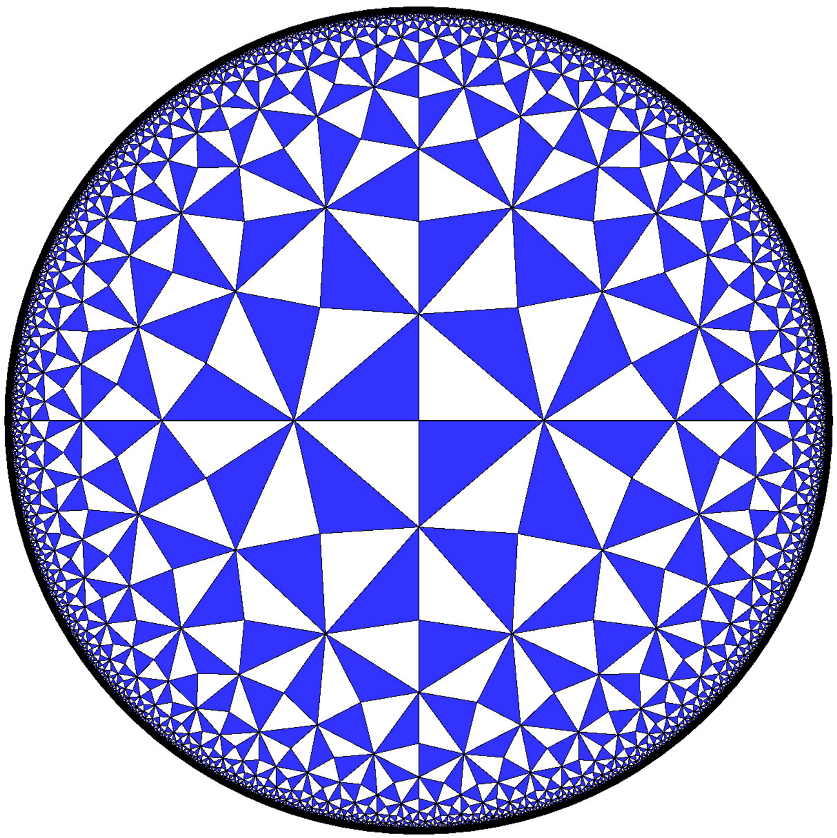
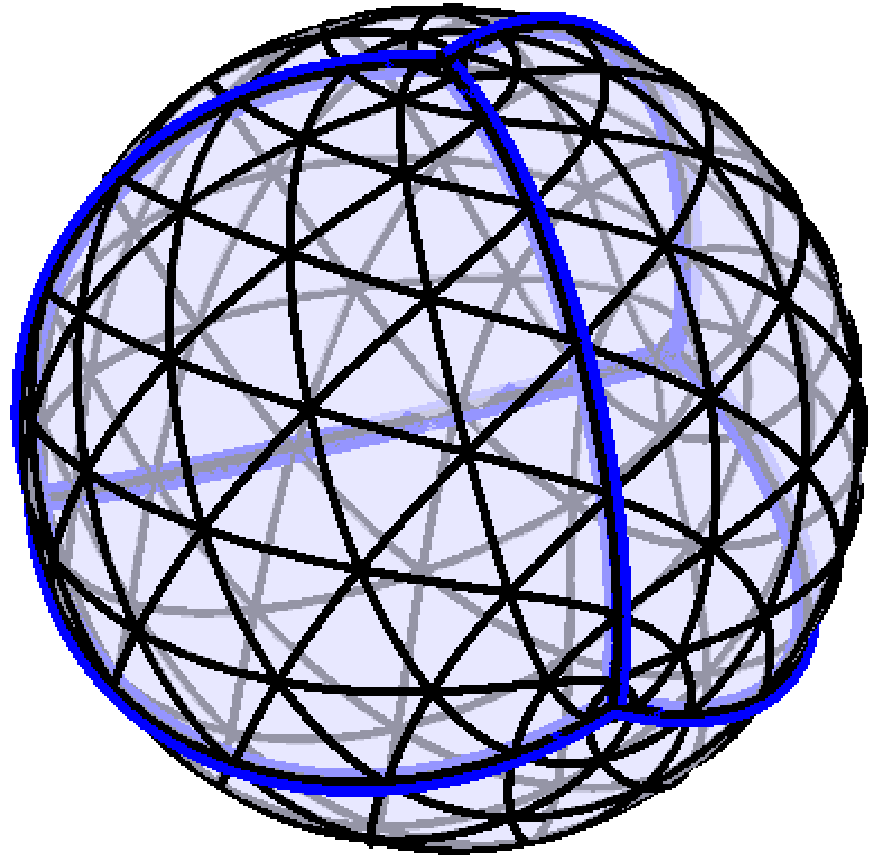

QUESTIONS?

Hyperbolic equilateral triangulations:



Triangulation of disk



Triangulation of sphere

Exercise: Prove that $\theta = \arccos(1/3)$ is not a rational multiple of π.

Solution: We give a proof by contradiction. Suppose $\theta = m\pi/n$. Then $\sin(n\theta) = \sin(m\pi) = 0$, and hence $\cos(n\theta) = \pm 1$. Let $z = \cos\theta + i\sin\theta$. Then $z^n = \cos(n\theta) + i\sin(n\theta) = \pm 1$, so z is an algebraic integer (the root of an equation with integer coefficients, $z^n \pm 1$).

Similarly, $(1/z)^n = \pm 1$, so $1/z$ is also an algebraic integer.

Thus $z + 1/z = 2\cos(\theta) = 2/3$ is an algebraic integer, which is impossible since the only real rational numbers that are algebraic integers are the regular integers. $\square$

Exercise: square has no triangulation with all angles $< 72^\circ$.

Given a triangulation of square S, and $k = 1, 2, 3, \ldots,$ define:

- r_k = number of square's corners that are in k triangles,
- q_k = number of interior triangulation vertices in k triangles,
- s_k = number of non-corner boundary vertices in k triangles.

We have the relations

- $F - E + V = 1$ (Euler's formula),
- $V = \sum_k (q_k + r_k + s_k)$ (every vertex is one of the three types)
- $3F = \sum_k k(q_k + r_k + s_k)$ (triangle corners counted two ways),
- $2E = 3F - \sum_k (r_k + s_k)$ (triangle sides counted two ways.)

Combining these four equations and eliminating variables gives

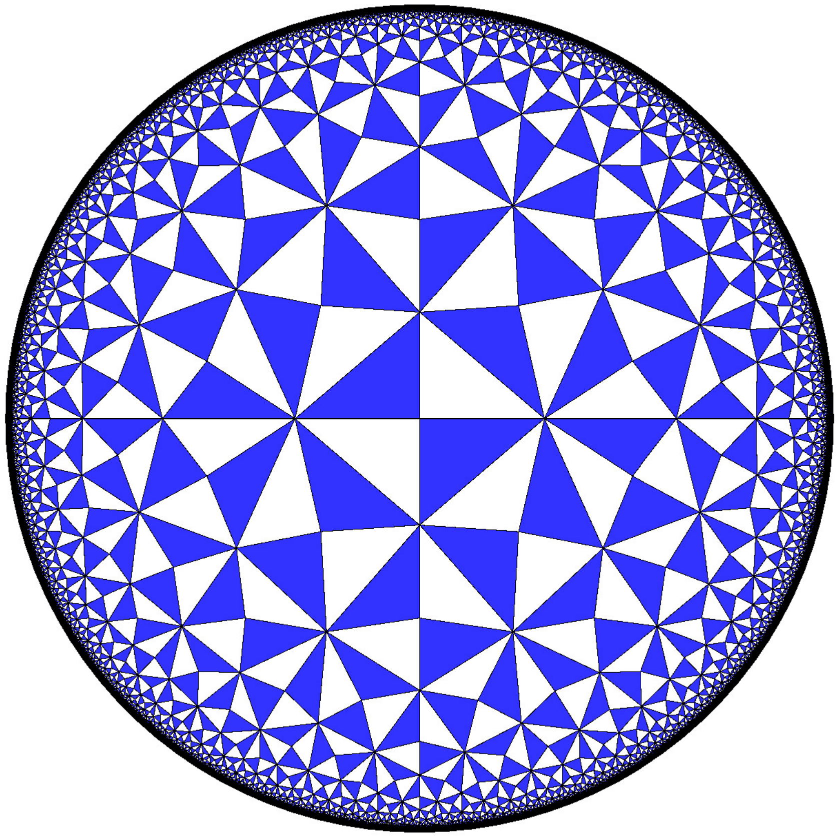
$$\sum_k (6-k)q_k + \sum_k (3-k)r_k + \sum_k (3-k)s_k = 6,$$

If every angle is $< 72^\circ$ then every interior vertex is on ≥ 6 triangles and every non-corner boundary vertex is on ≥ 3 triangles. Thus

$$2r_1 + r_2 \geq \sum_k (3-k)r_k \geq 6.$$

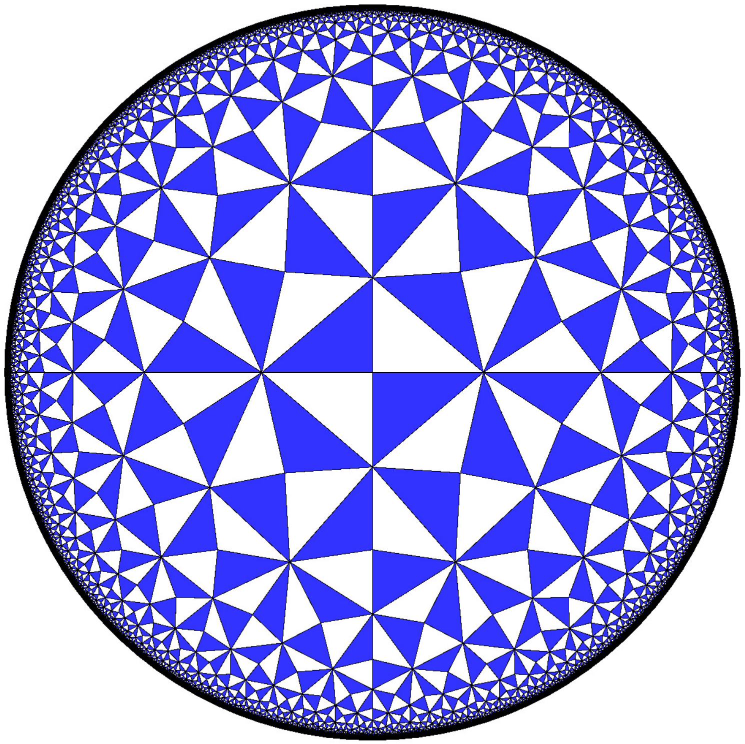
Since $r_2 \leq 4$ (the square has four corners), we have $r_1 > 0$. Thus a 90° angle occurs at some corner, a contradiction. $\square$

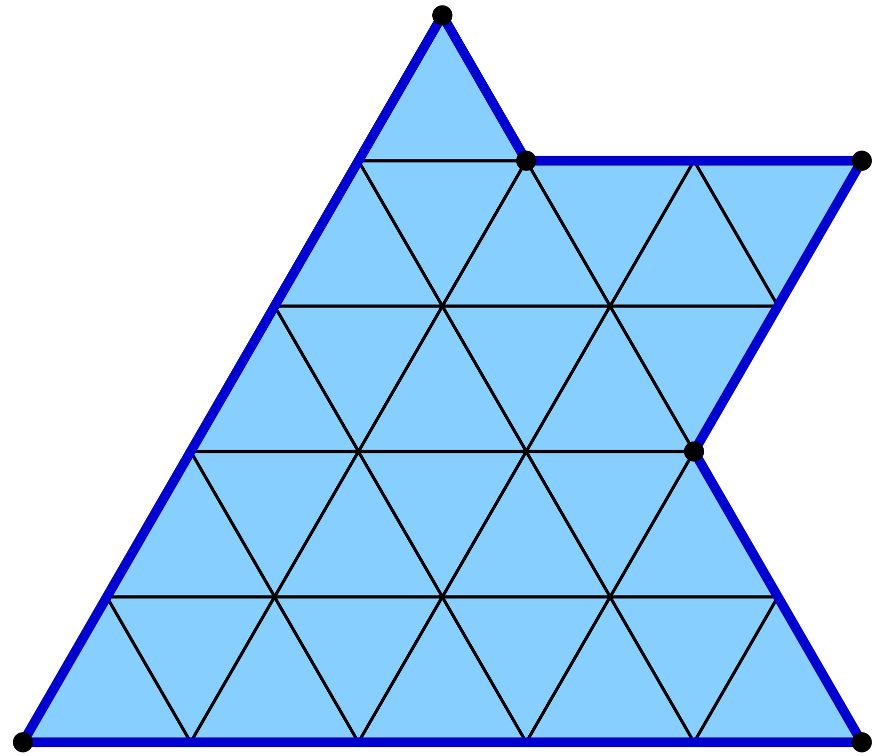
Hyperbolic triangulations:



Equilateral triangulation of disk in hyperbolic metric.

Hyperbolic triangulations:

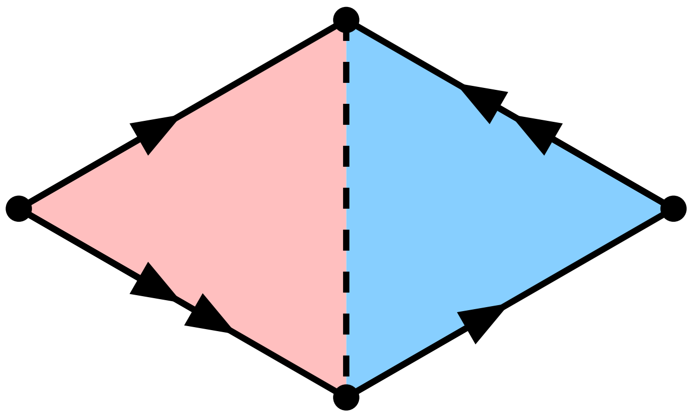


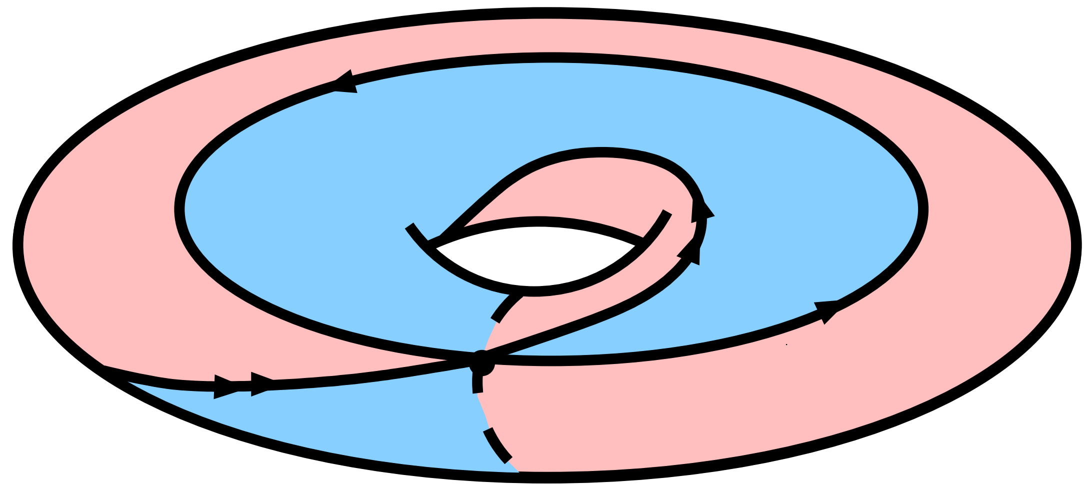


We say a triangulation is equilateral if any two triangles sharing an edge are reflections of each other across that edge.

This works in either geometry (if we know how to define reflections).

Hyperbolic triangulations:



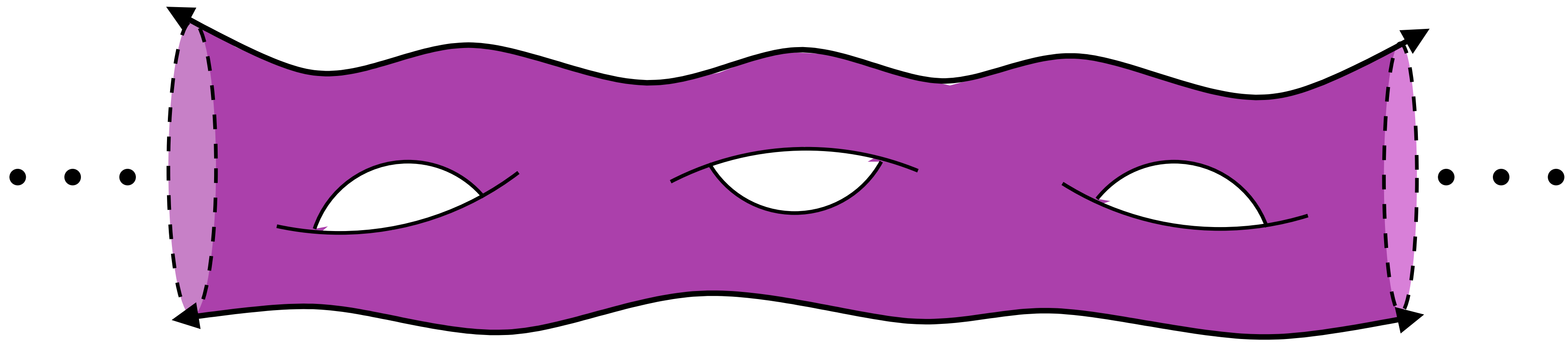


Some compact surfaces have an equilateral triangulation.

Most do not; only finitely many ways to glue together n triangles.

Compact surfaces with equilateral triangles have been intensely studied: related to Grothendieck's theory of *dessins d'enfants*, graph theory, probability, algebraic number theory, Galois theory, ...

Hyperbolic triangulations:

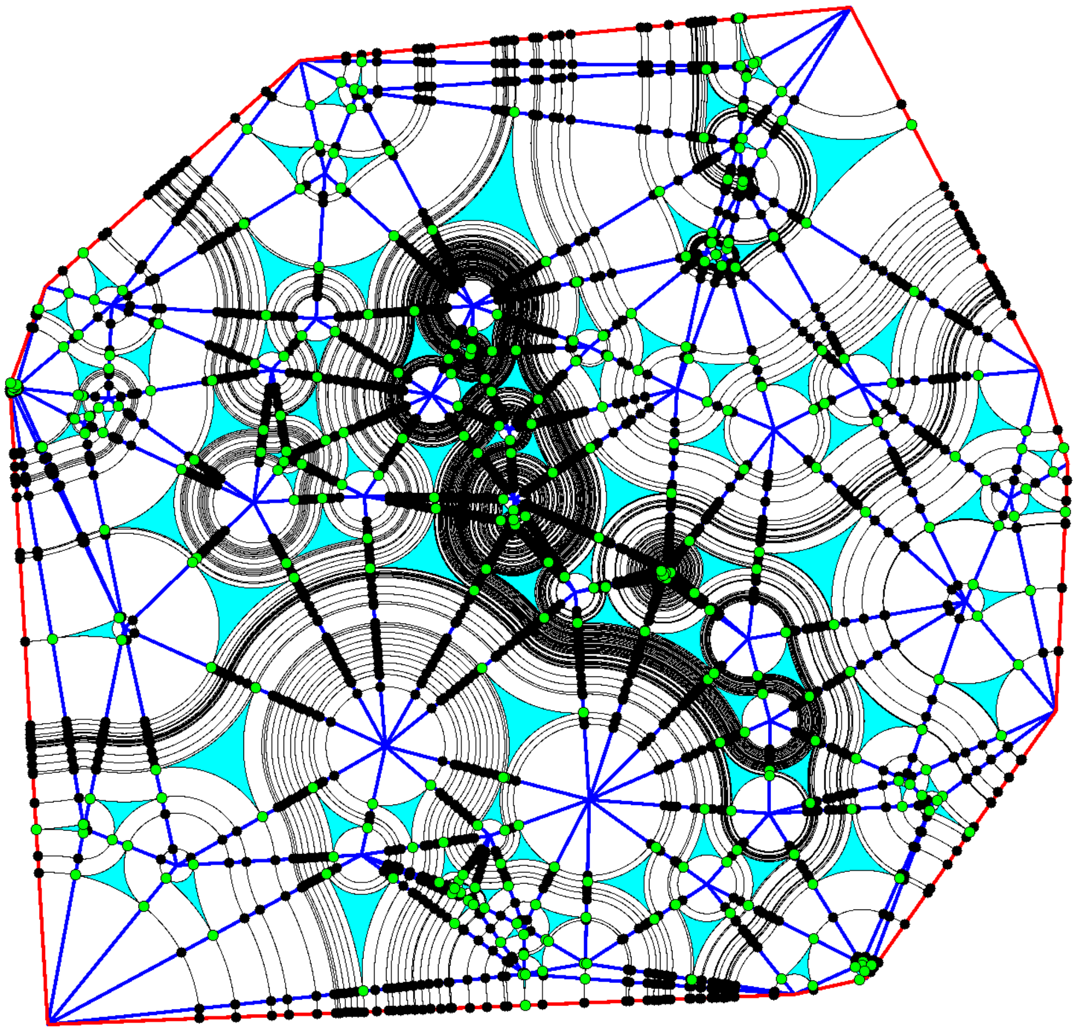


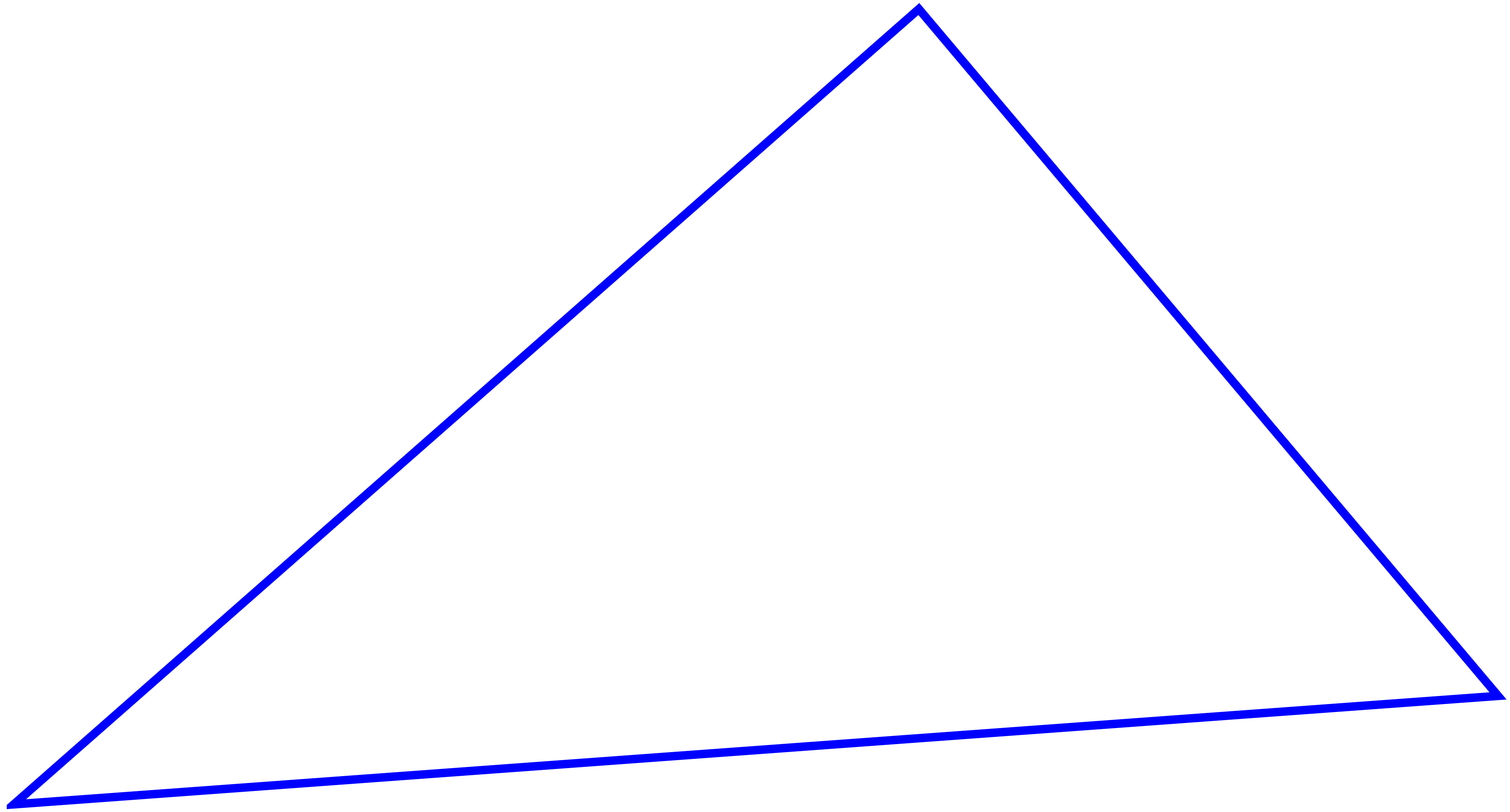
Thm (Bishop-Rempe, 2026): Every non-compact hyperbolic surface has an equilateral triangulation.

Proof combines differential equations and planar acute triangulation.

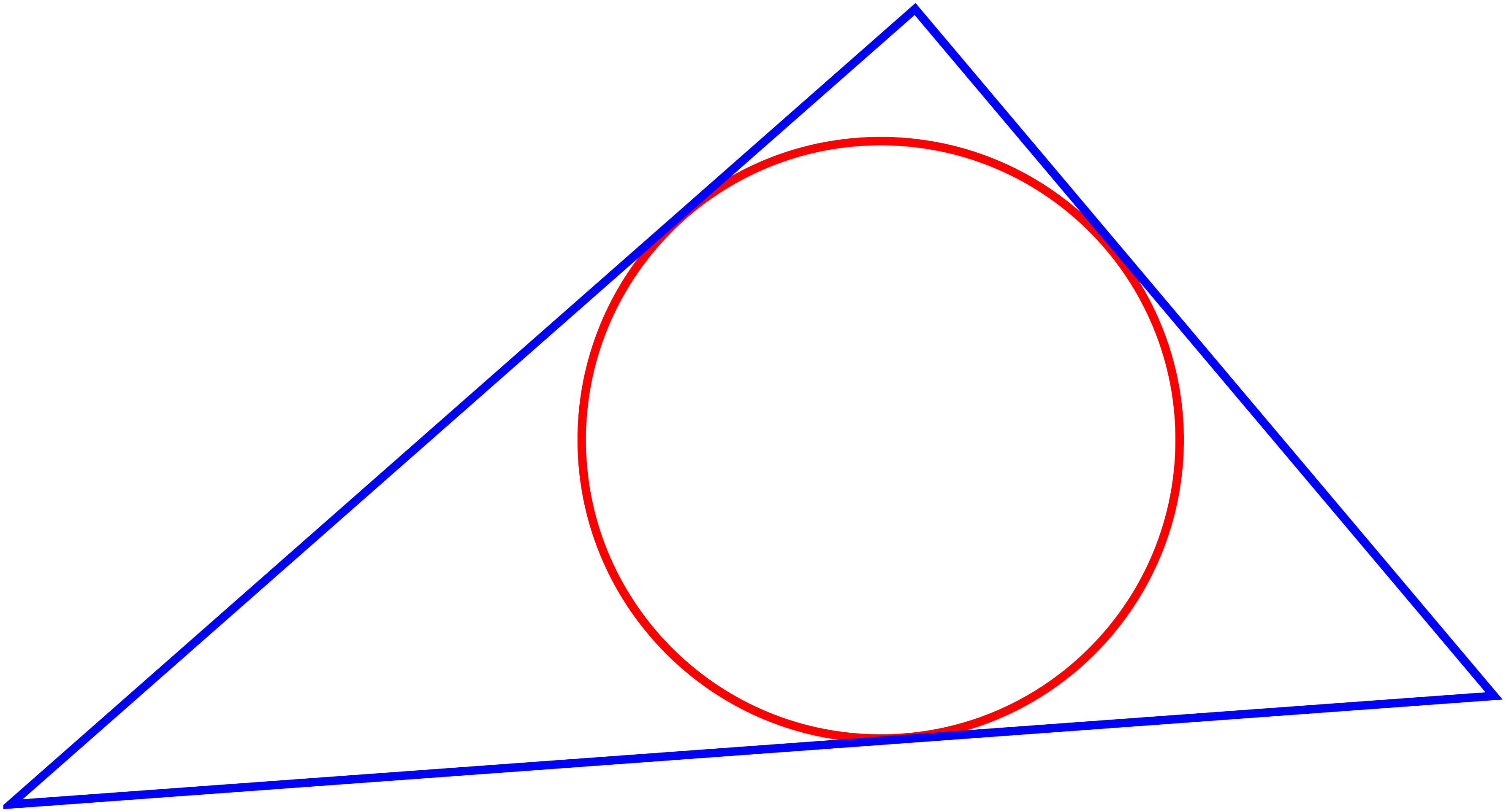
Many questions about hyperbolic triangulations remain open.

TRIANGULATION FLOWS

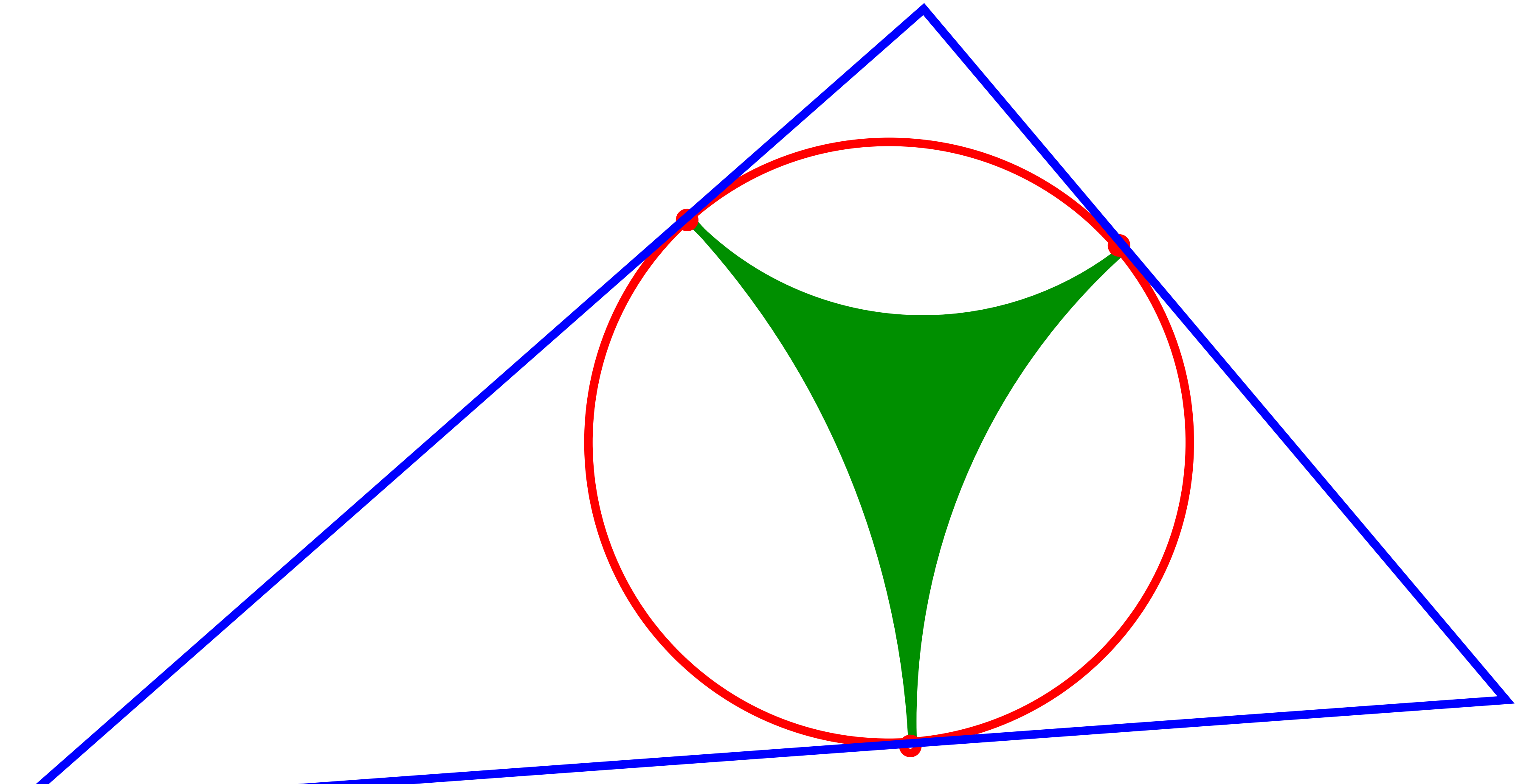




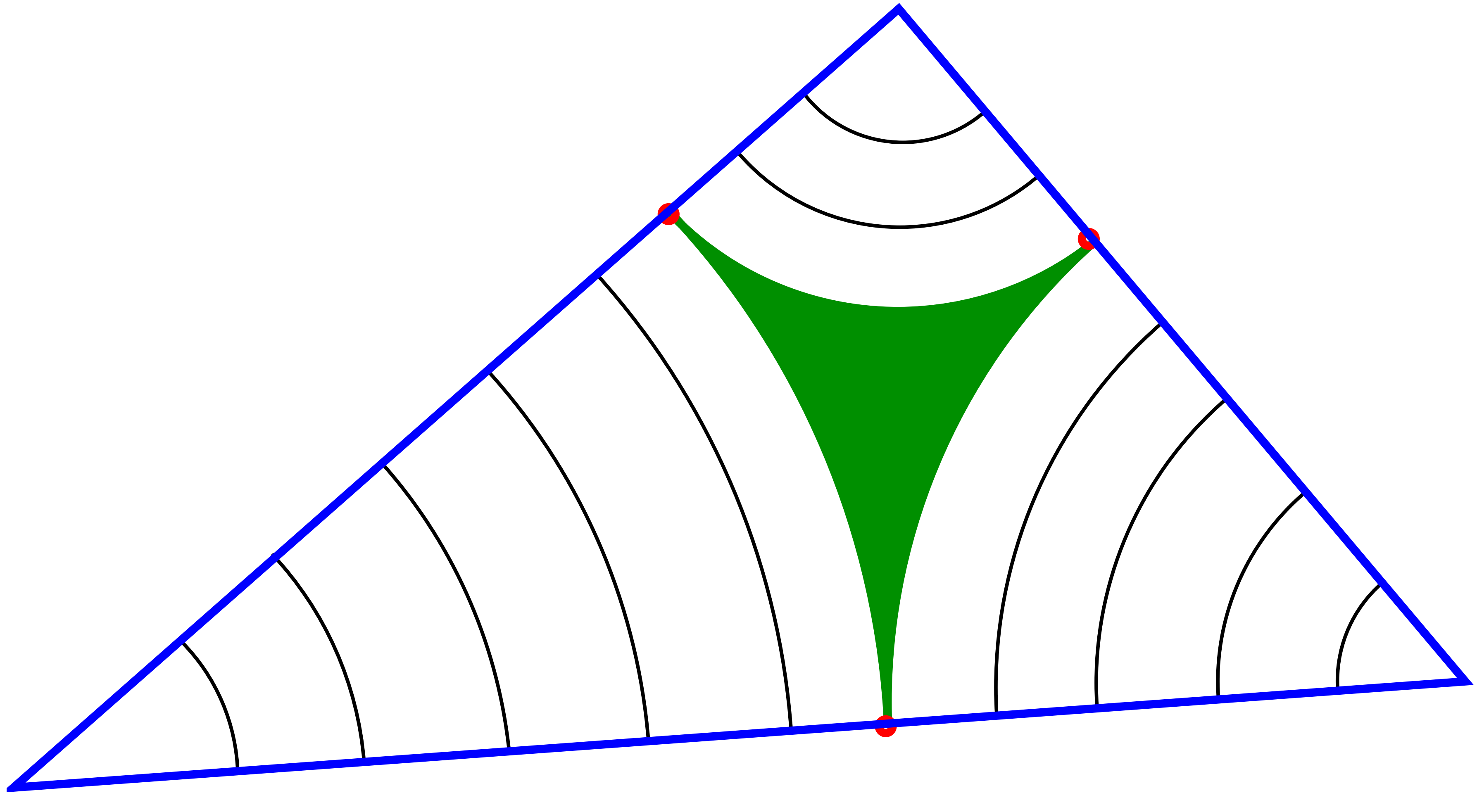
A triangle.



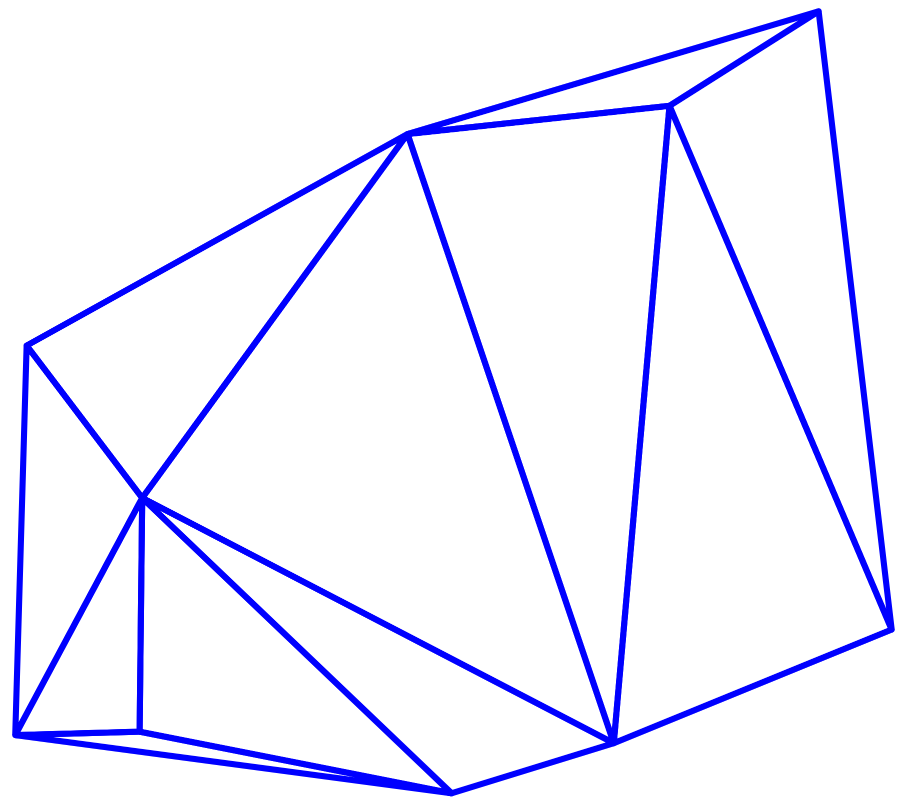
Its in-circle.



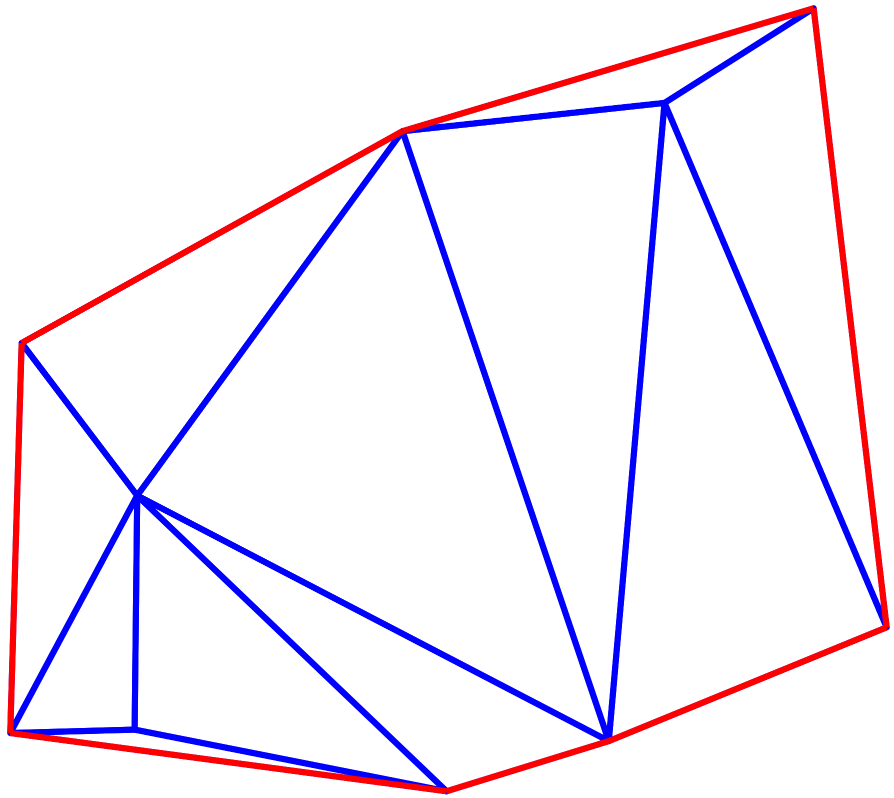
The central region and three sectors (thin version).



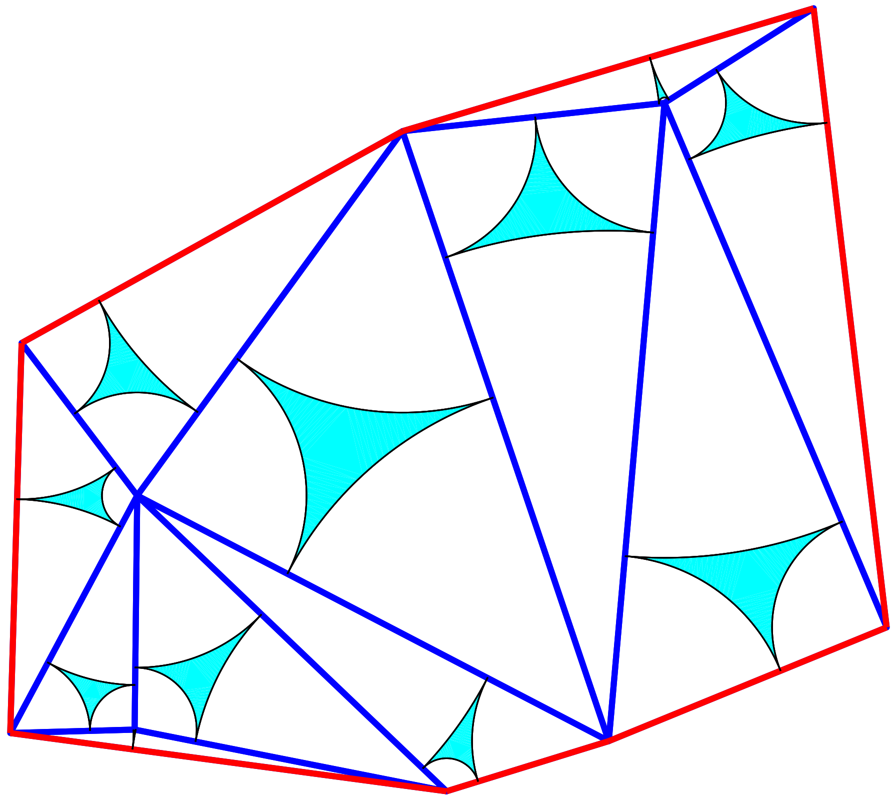
The three sectors are foliated by circular arcs.
Defines flow on a triangulation that stops at boundary or cusp point.



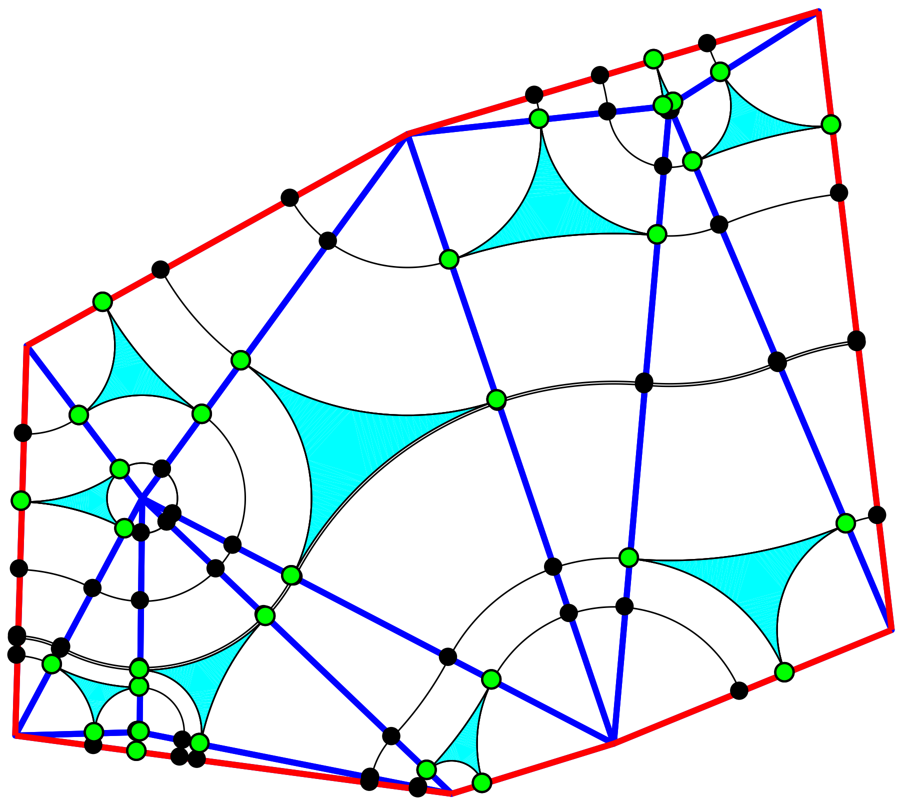
Delaunay triangulation of 10 random points,



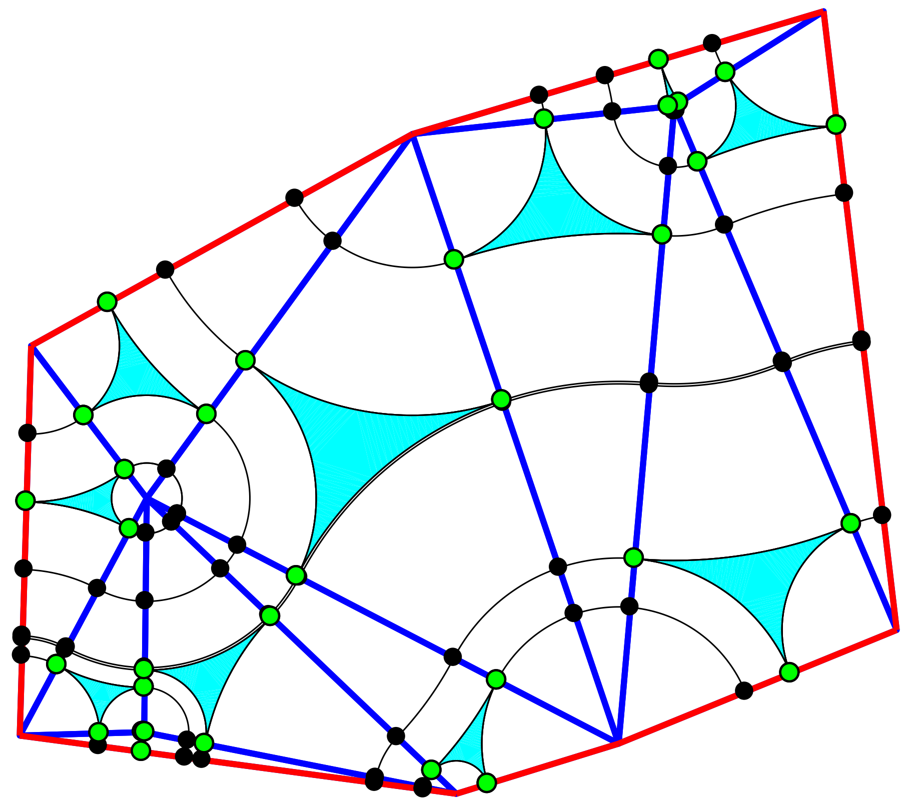
The boundary of the triangulation



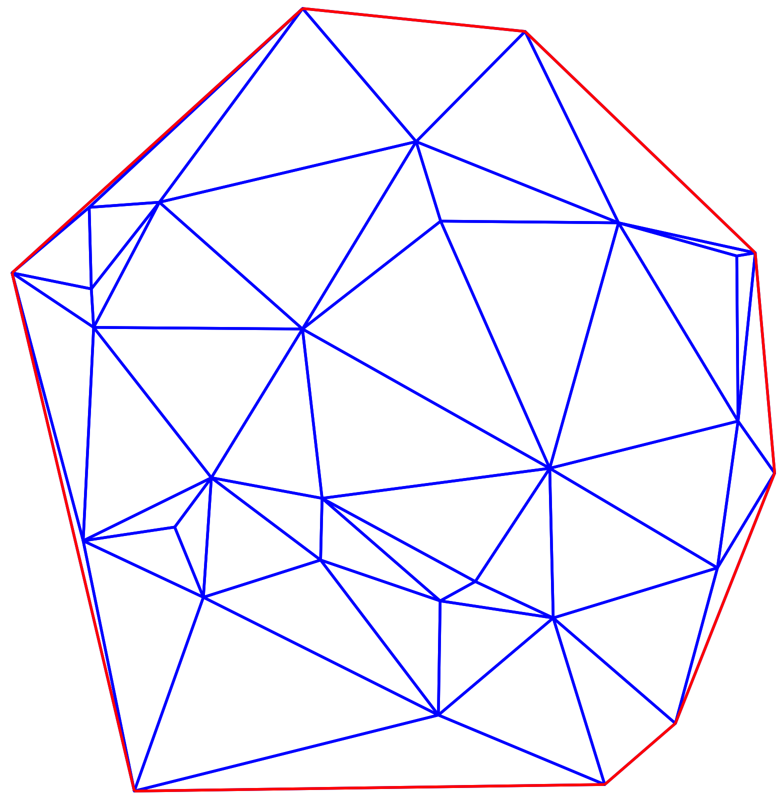
The central regions.



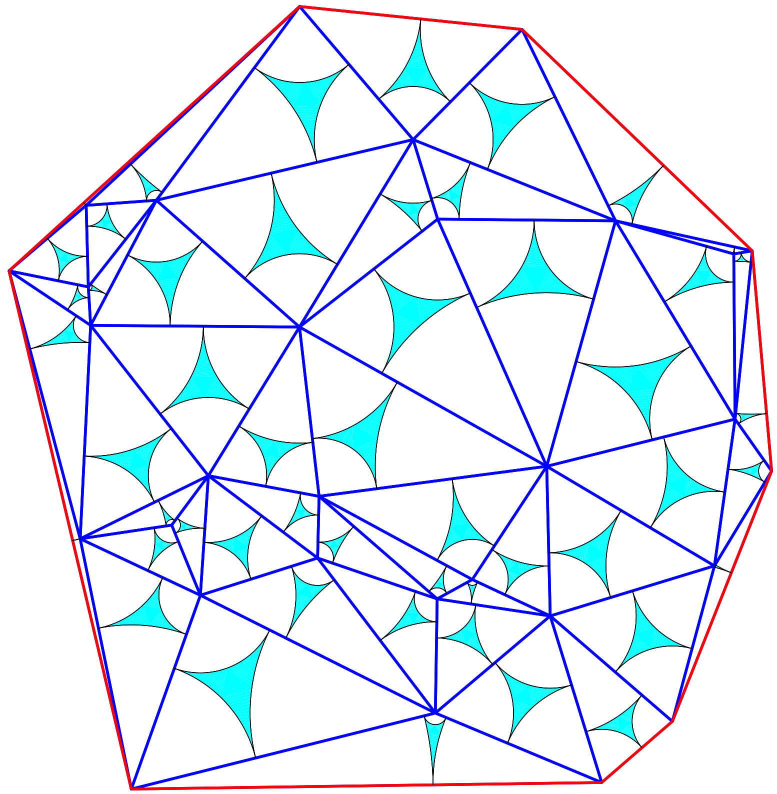
Propagation lines starting at all cusp points.



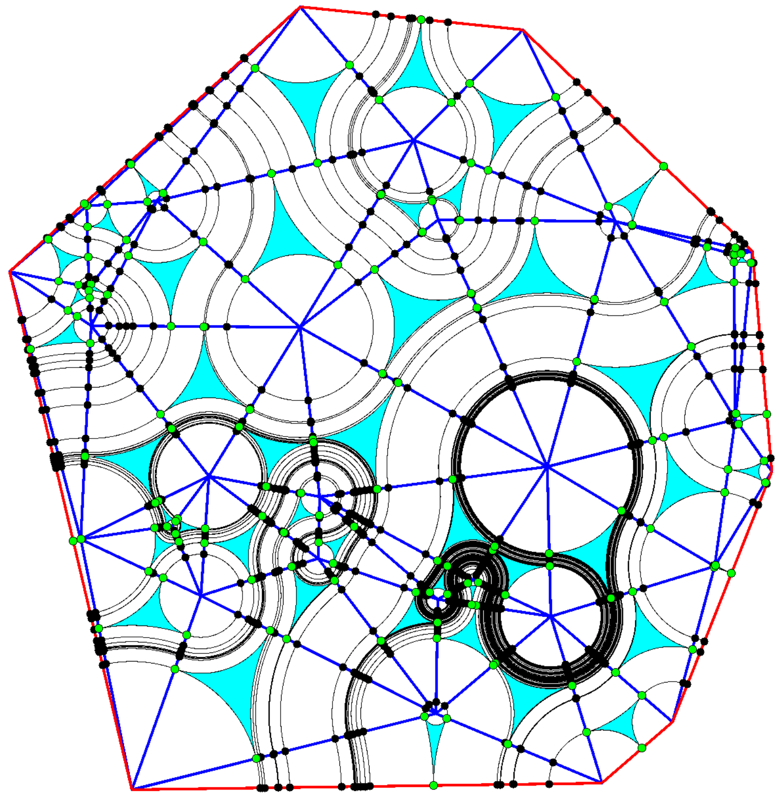
Propagation lines identify boundary points; induces tree.
Discontinuous, but piecewise length preserving.



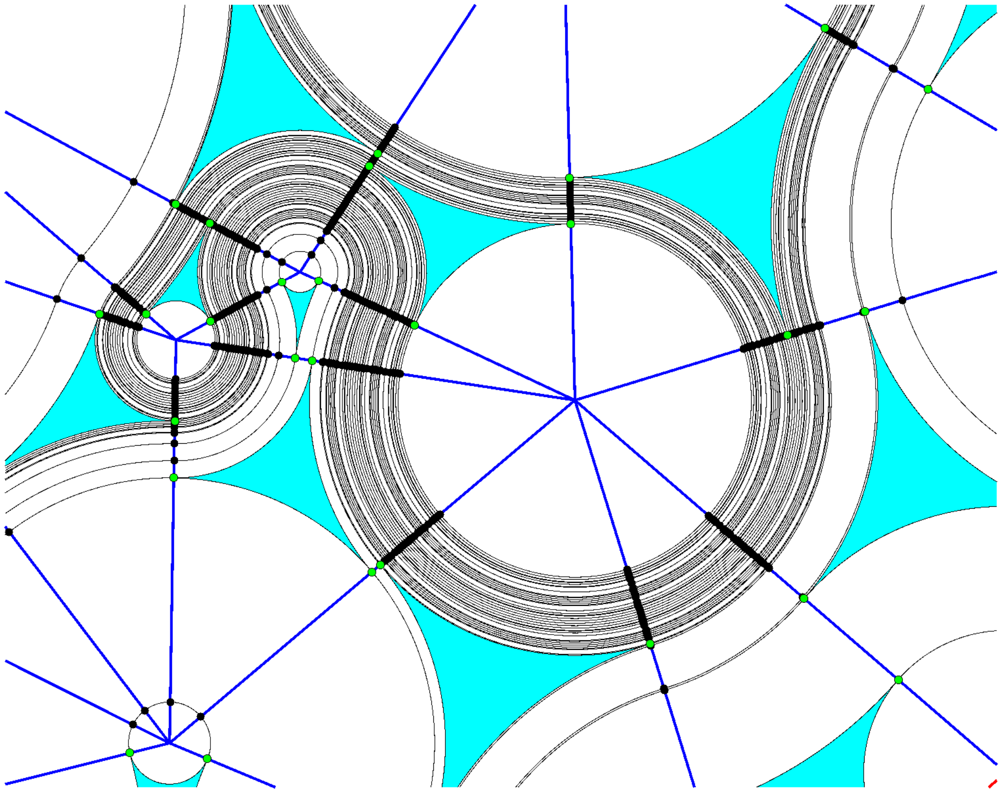
Delaunay triangulation of 30 random points in disk.



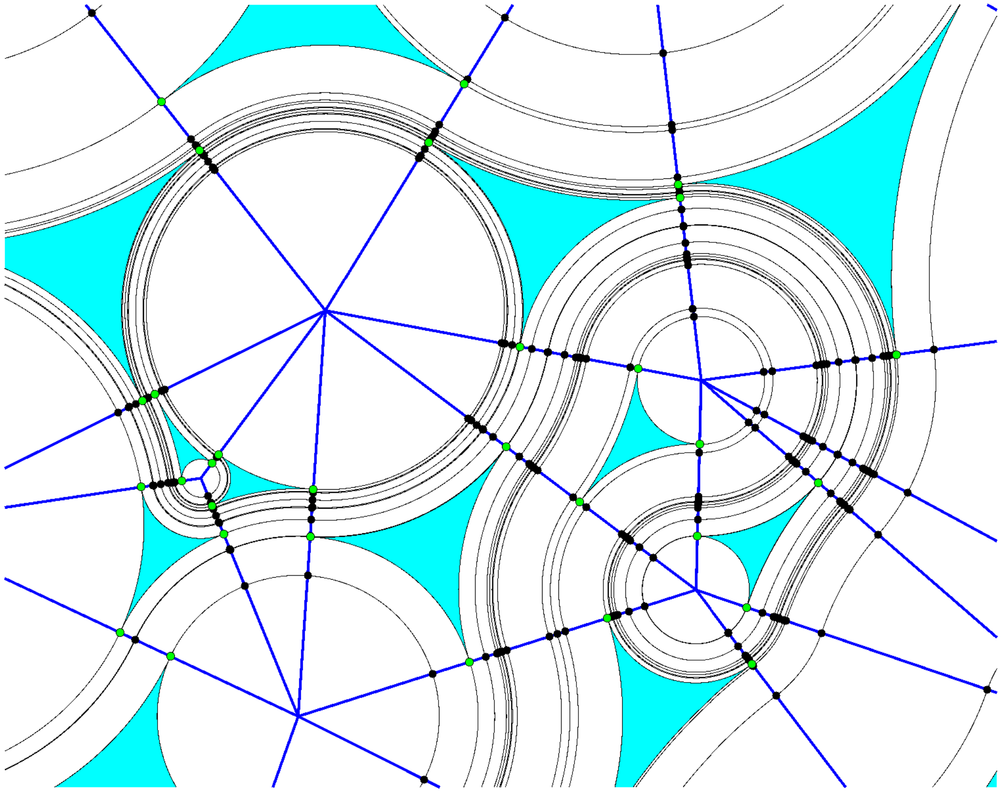
The central regions.



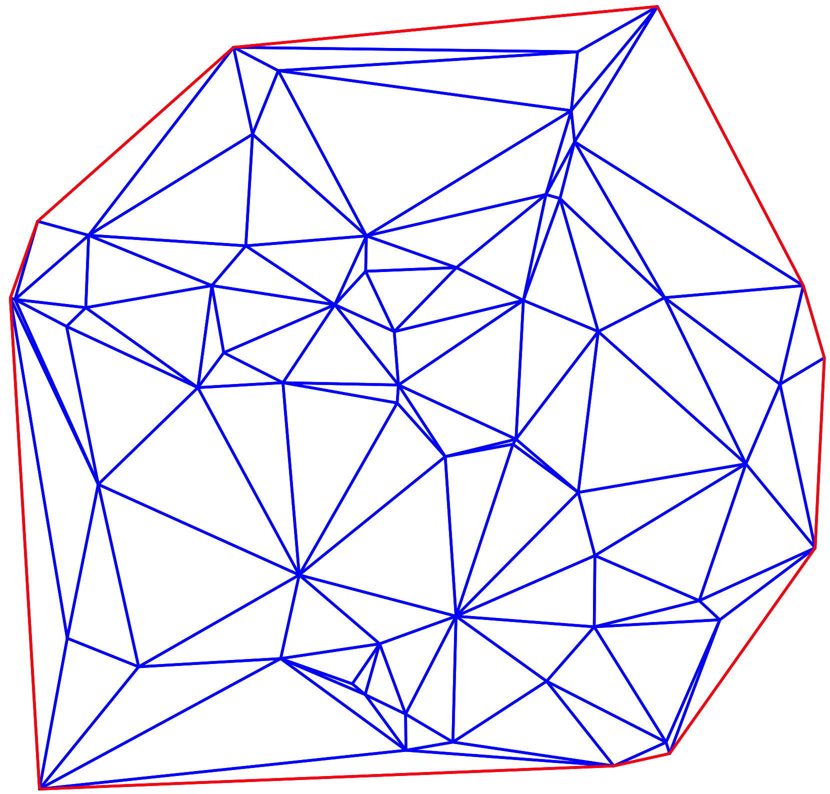
Propagation lines starting at all cusp points.



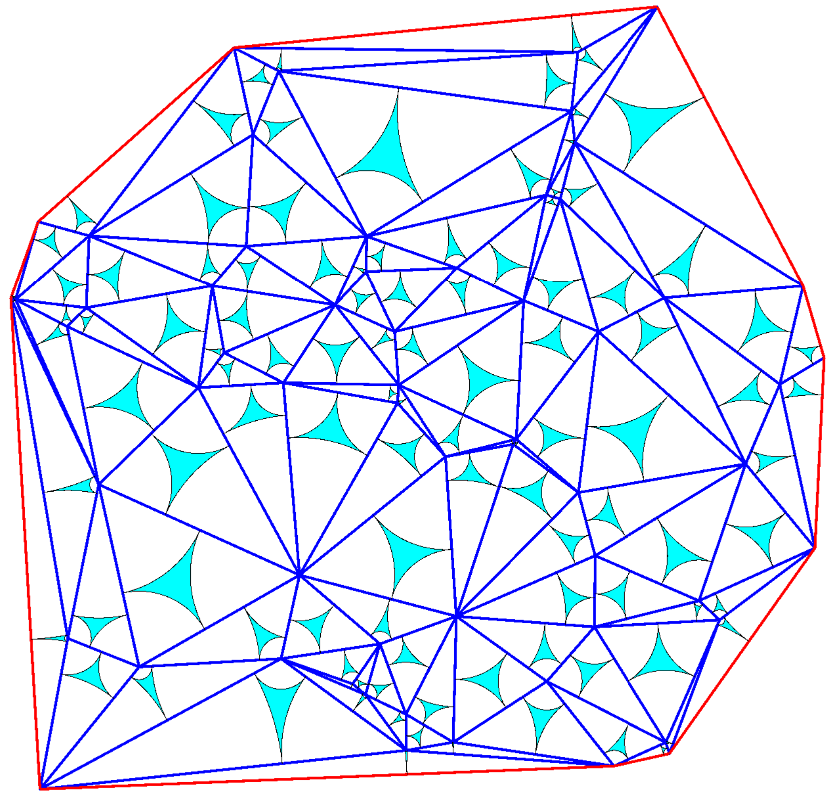
Enlargement 1.



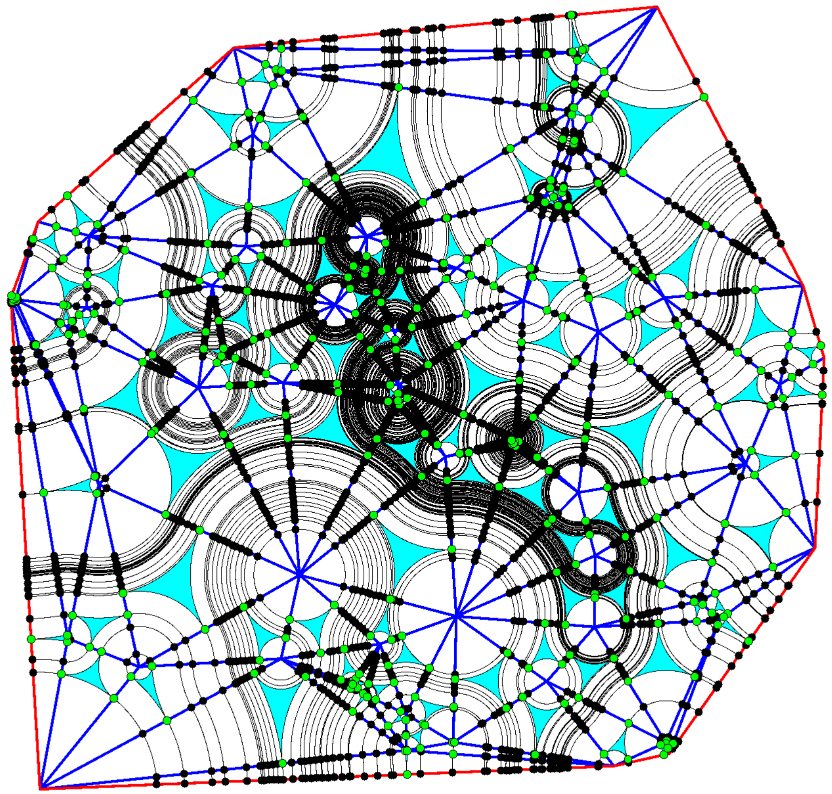
Enlargement 2.



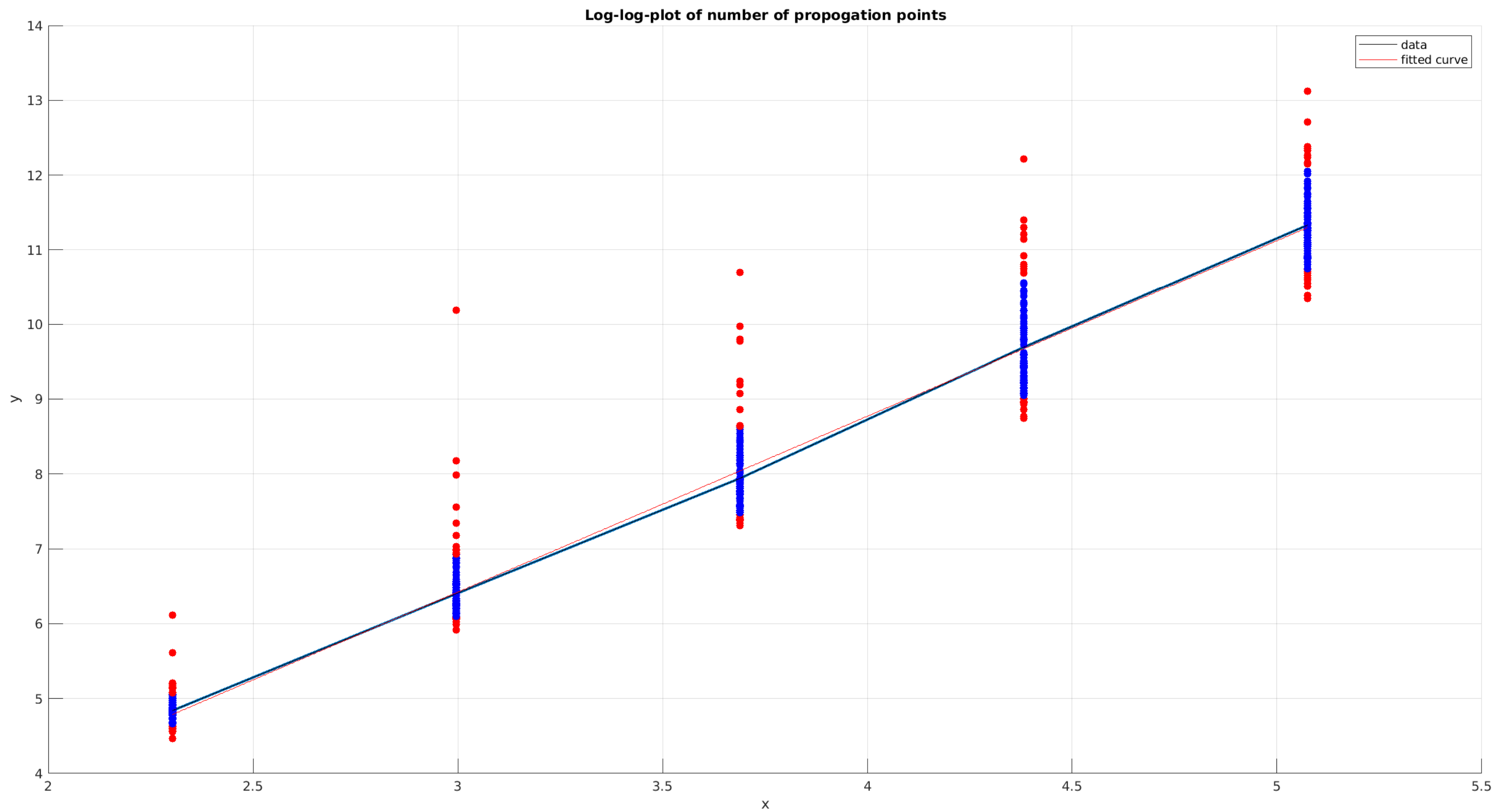
60 points



The central regions.



Propagation lines starting at all cusp points.



Log-log plot of number points created versus n.
Slope ≈ 2.5