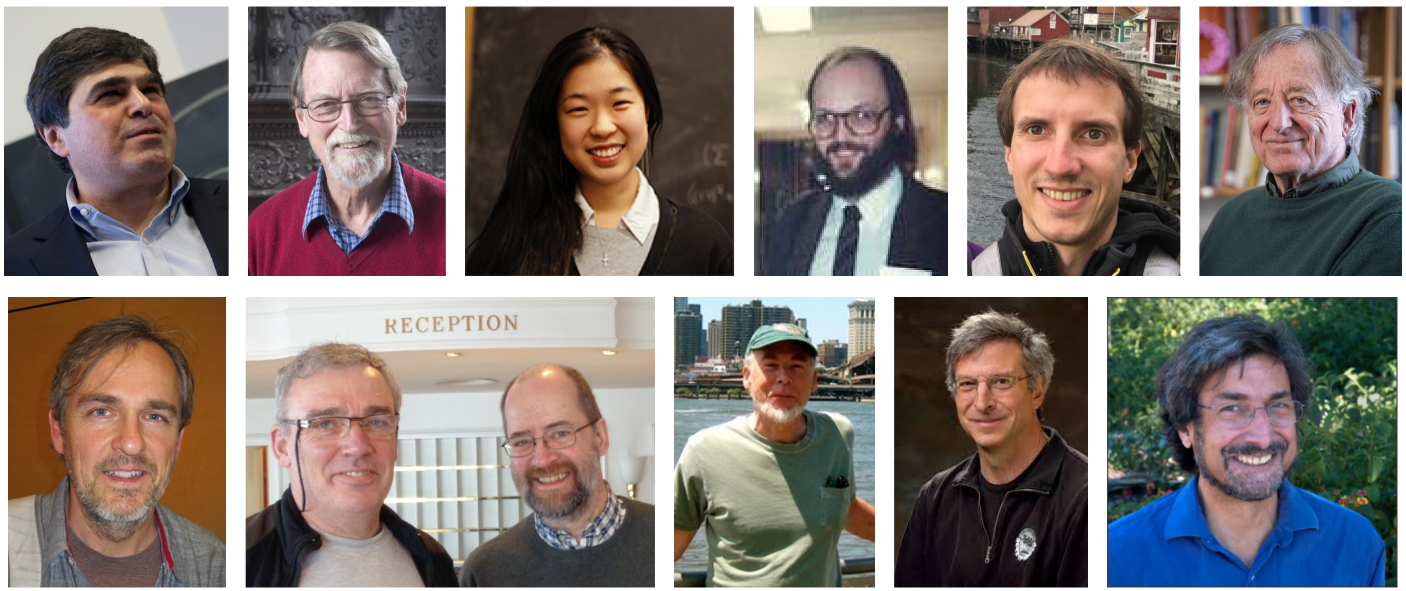
# WEIL-PETERSSON CURVES, TRAVELING SALESMAN THEOREMS AND MINIMAL SURFACES


## Christopher Bishop, Stony Brook


## Complex Faces — Venice, Italy, April 7-11, 2025



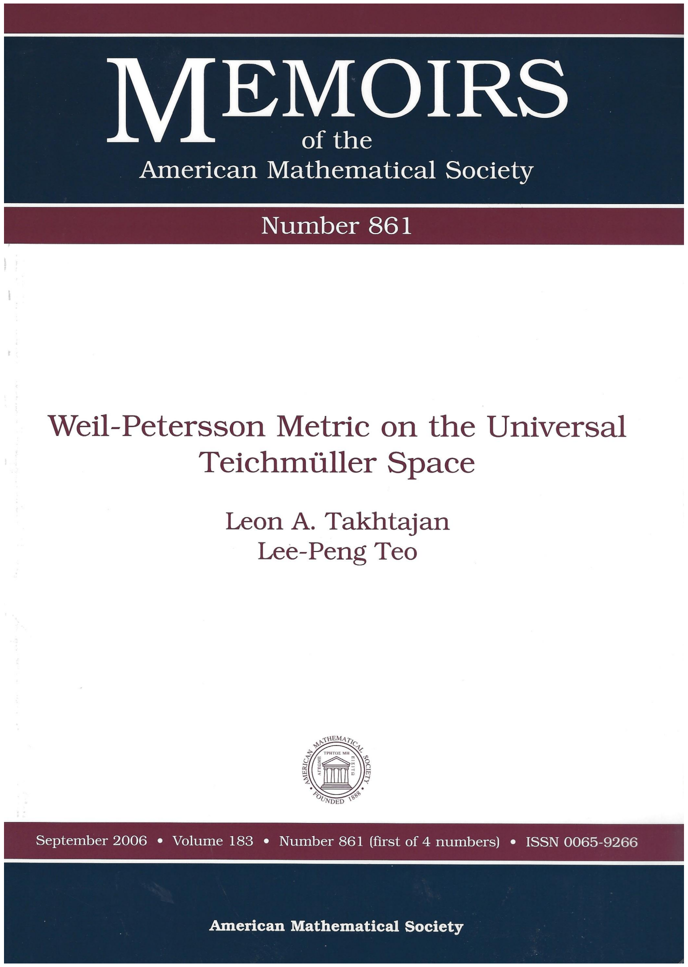
- Problem: put a metric on closed loops appropriate for string theory.

- $T(1)$ = universal Teichmüller space = quasicircles

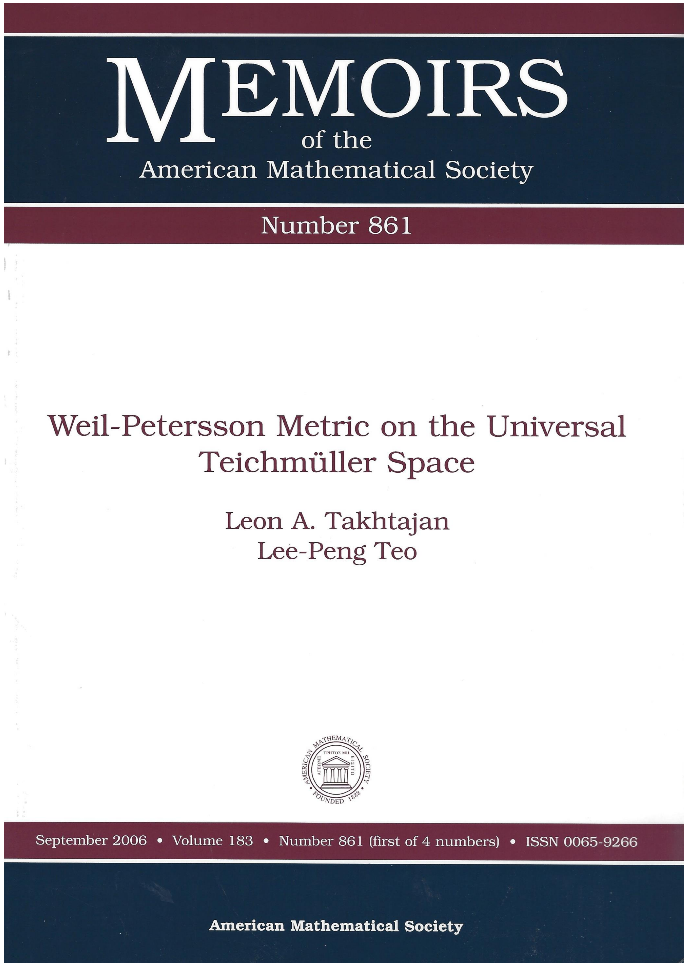
- Usual Teichmüller metric based on $L^\infty$ (supremum norm).

  $\Rightarrow T(1)$ is Banach manifold, not Hilbert manifold.

  $\Rightarrow$ not so good for physics or computations.

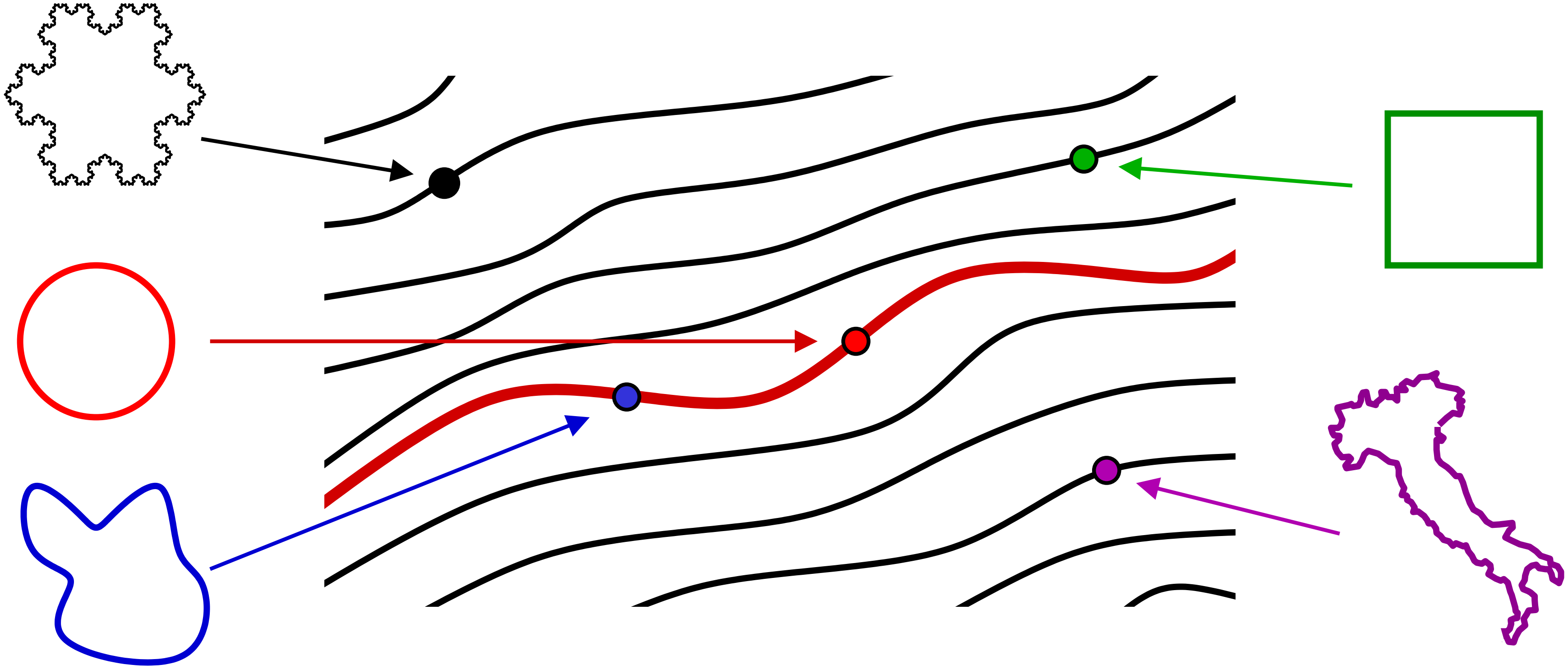



"In this memoir, we prove that the universal Teichmüller space $T(1)$ carries a new structure of a complex Hilbert manifold and show that the connected component of the identity of $T(1)$, the Hilbert submanifold $T_0(1)$, is a topological group. ..."




"Weil-Petersson class boundary parameterizations provide the correct analytic setting for conformal field theory." — Radnell, Schippers and Staubach, 2017

## Cartoon of universal Teichmüller space



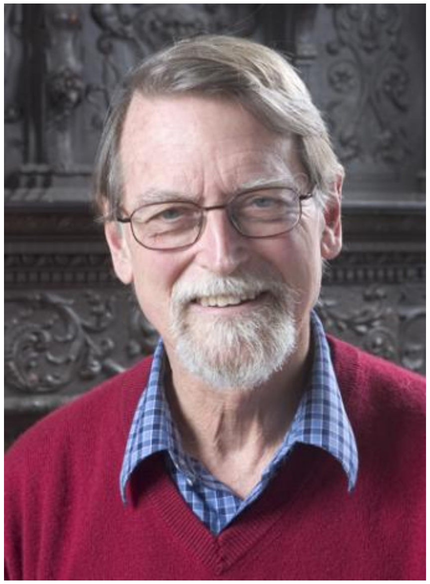
Takhtajan and Teo make $T(1)$ a (disconnected) Hilbert manifold.

$T_0(1) =$ Weil-Petersson class

$=$ connected component containing the circle

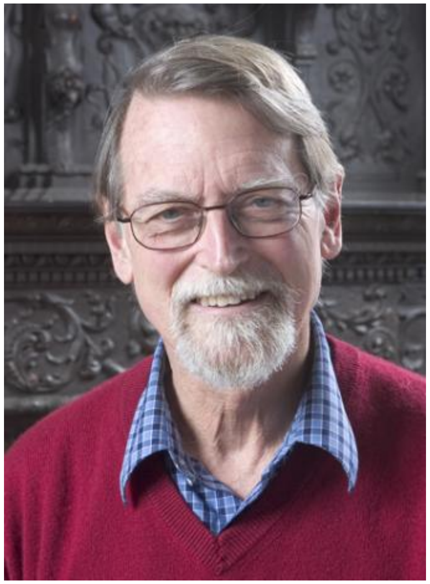
$=$ closure of smooth curves

$= \infty$-dim Kähler-Einstein manifold.



In Dec 2017 email David Mumford asked me which non-smooth curves are in WP? Motivated by computer vision and pattern recognition.

"Riemannian geometries on spaces of plane curves, Michor and Mumford, *JEMS*, 2006.



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Jan 2019 IPAM workshop: Analysis and Geometry of Random Sets.

Lecture by Yilin Wang:

"Loewner energy via Brownian loop measure and action functional analogs of SLE/GFF couplings"

So the Weil-Petersson class (undefined so far) is linked to:

- String theory
- Kähler-Einstein manifolds
- Teichmüller theory
- Pattern recognition
- Brownian loops, SLE, Gaussian free fields, ...

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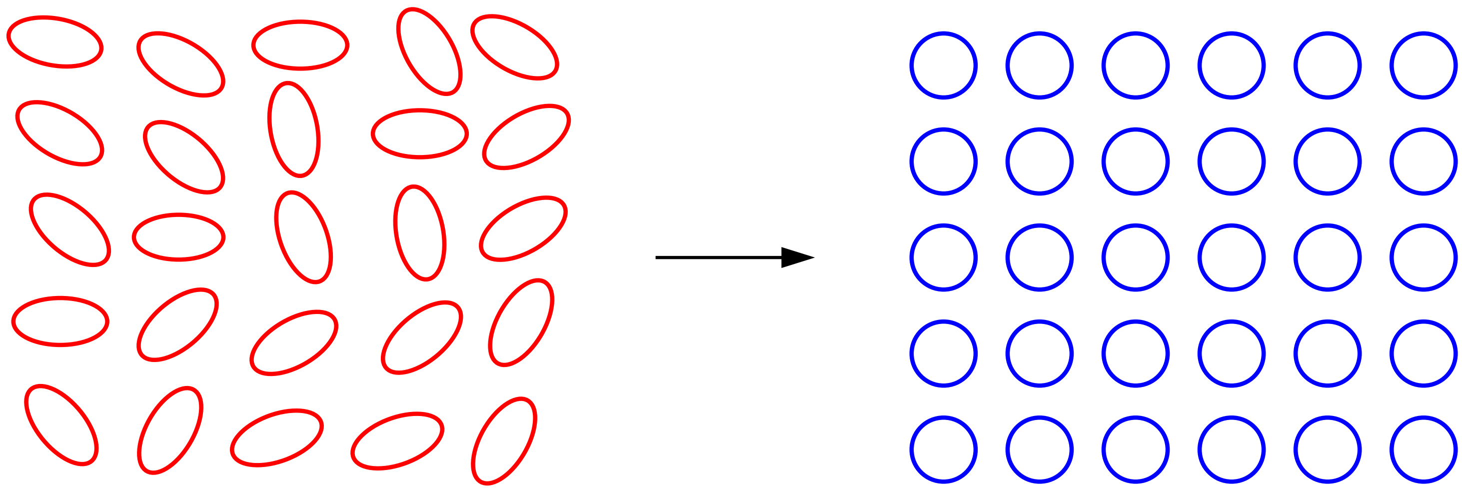
In today's talk I will discuss further connections to:

- Geometric function theory
- Sobolev spaces
- Knot theory
- The traveling salesman theorem
- Convex hulls in hyperbolic space
- Minimal surfaces
- Renormalized area

We start with a quick review of **quasiconformal** maps.

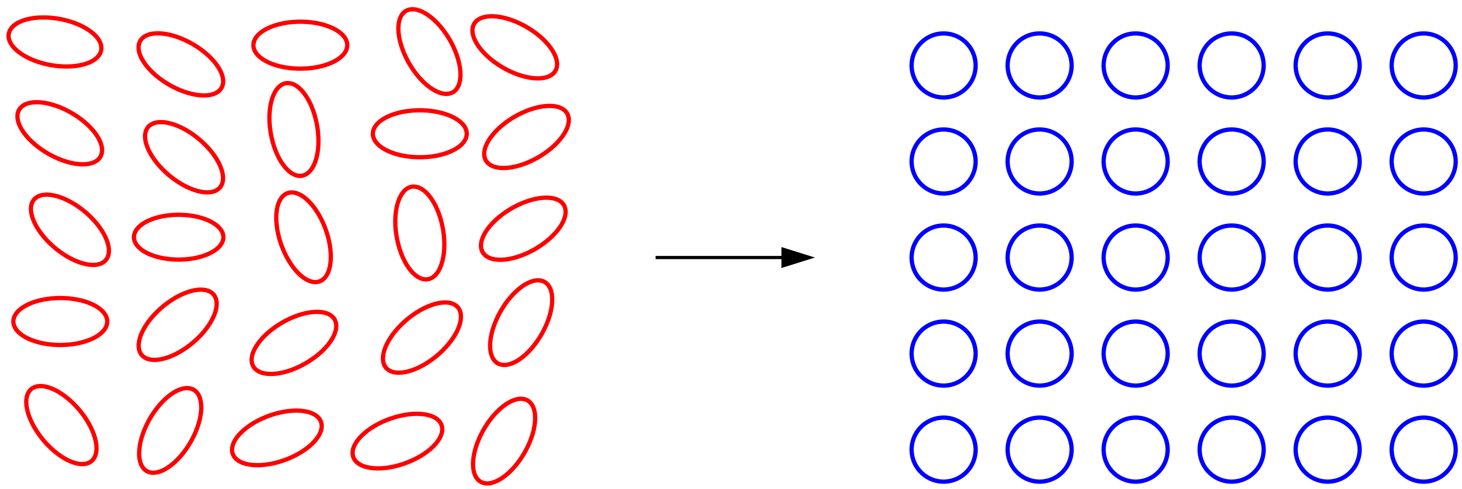
Diffeomorphisms send infinitesimal ellipses to circles.



Eccentricity = ratio of major to minor axis of ellipse.

$K$-quasiconformal = ellipses have eccentricity $\leq K$ almost everywhere

Diffeomorphisms send infinitesimal ellipses to circles.



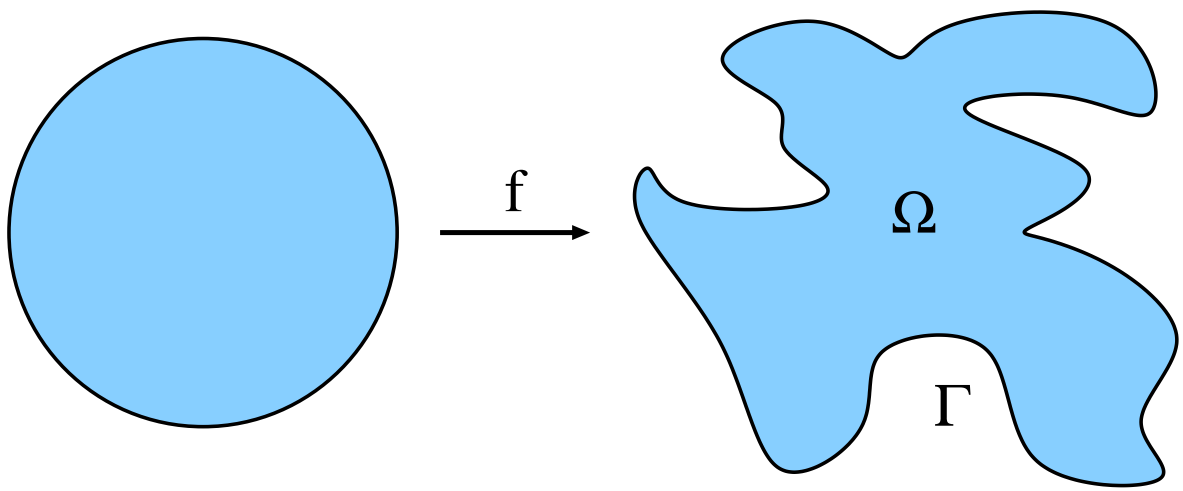
Ellipses determined by dilatation $\mu = f_{\overline{z}}/f_z$ with $f_{\overline{z}}, f_z = \frac{1}{2}(f_x \pm i f_y)$.

$$|\mu| = \frac{K-1}{K+1} < 1, \qquad \arg(\mu) \text{ gives major axis.}$$

$f$ is QC $\Leftrightarrow \|\mu\|_\infty < 1$. $\quad$ $f$ is conformal = 1-1 holomorphic $\Leftrightarrow \mu \equiv 0$.

A **quasicircle** is the image of circle under a quasiconformal map of $\mathbb{R}^2$.

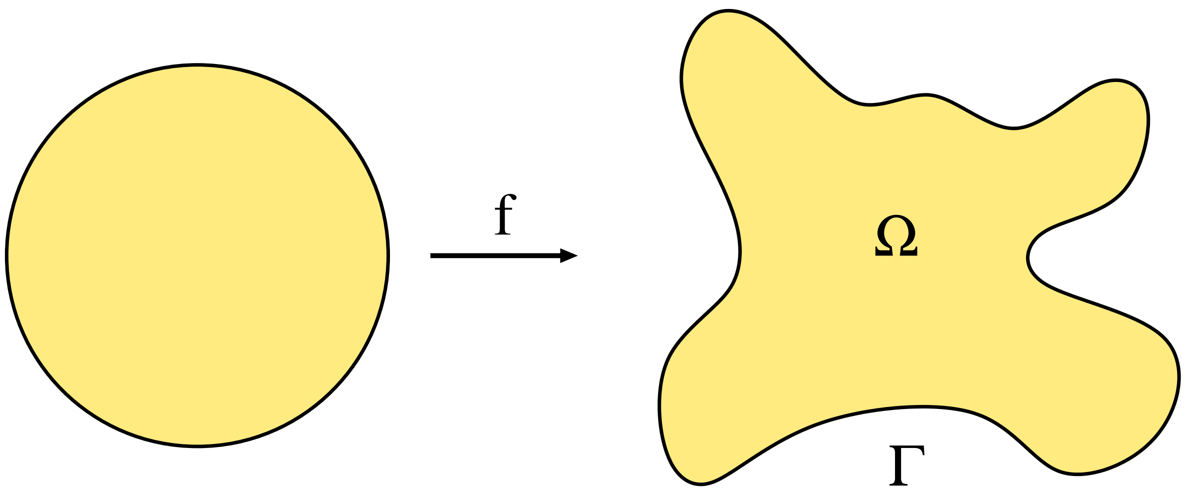
$T(1)$ = Universal Teichmüller space = quasicircles modulo similarities.




All smooth closed curves are quasicircles.

**Defn:** $\Gamma = f(\mathbb{T})$ is **Weil-Petersson** if $\mu \in L^2(dA_\rho)$.

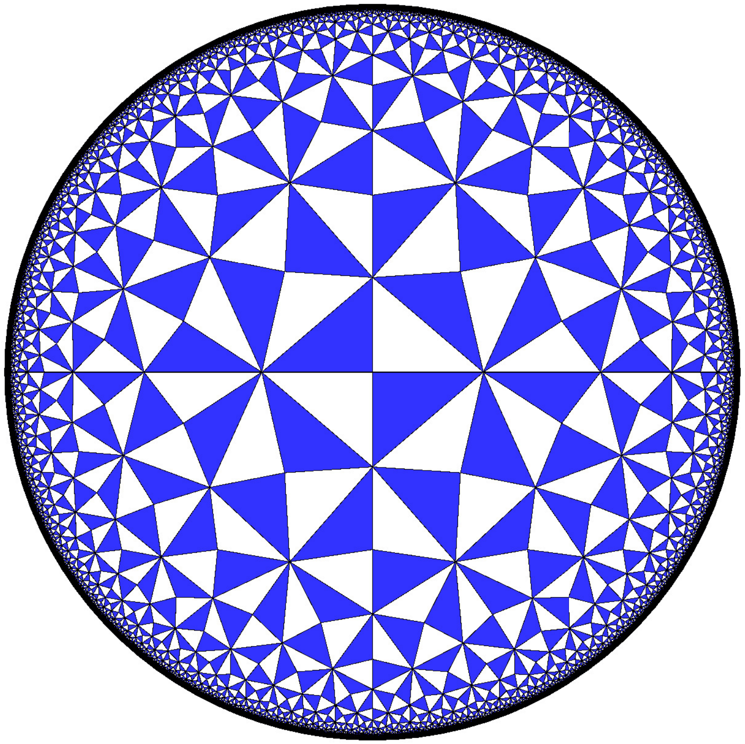
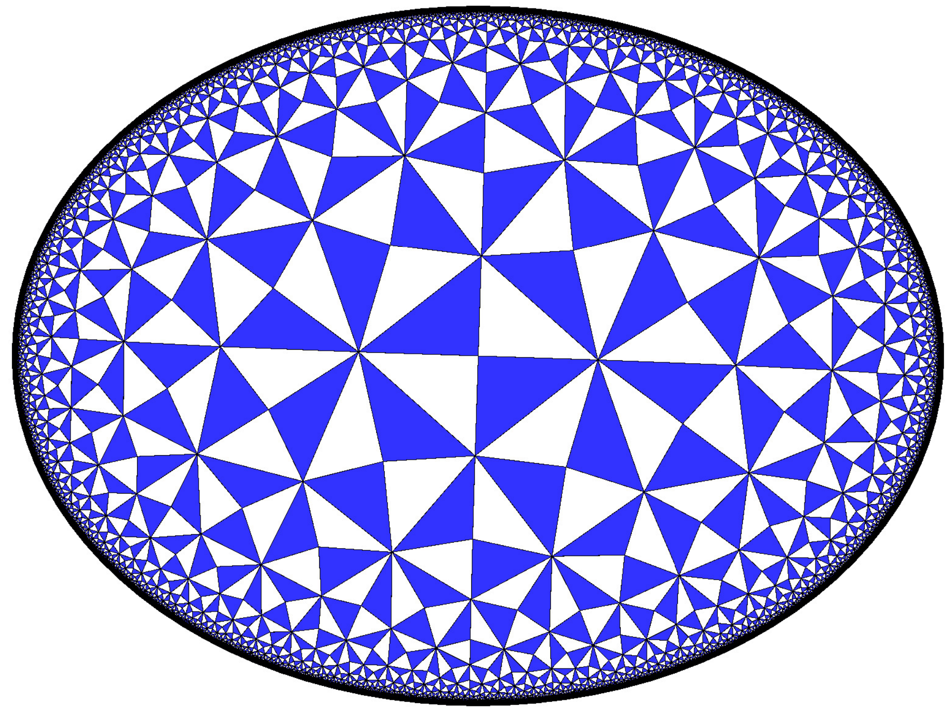
Here $dA_\rho = \frac{dxdy}{(1-|z|^2)^2}$ = hyperbolic area on $\mathbb{C} \setminus \mathbb{T}$.




**Informally: WP is to $L^2$, as QC is to $L^\infty$.**

All smooth curves are Weil-Petersson.

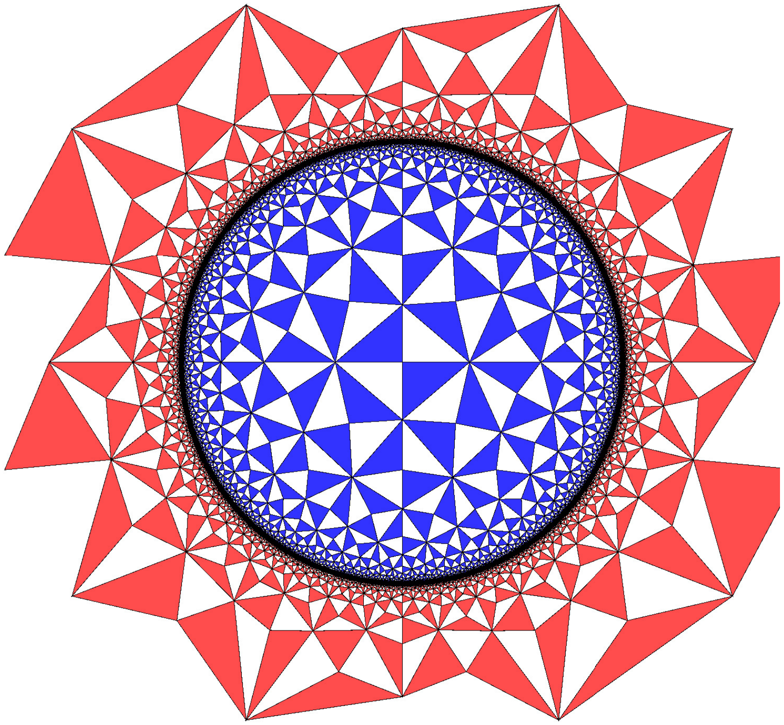
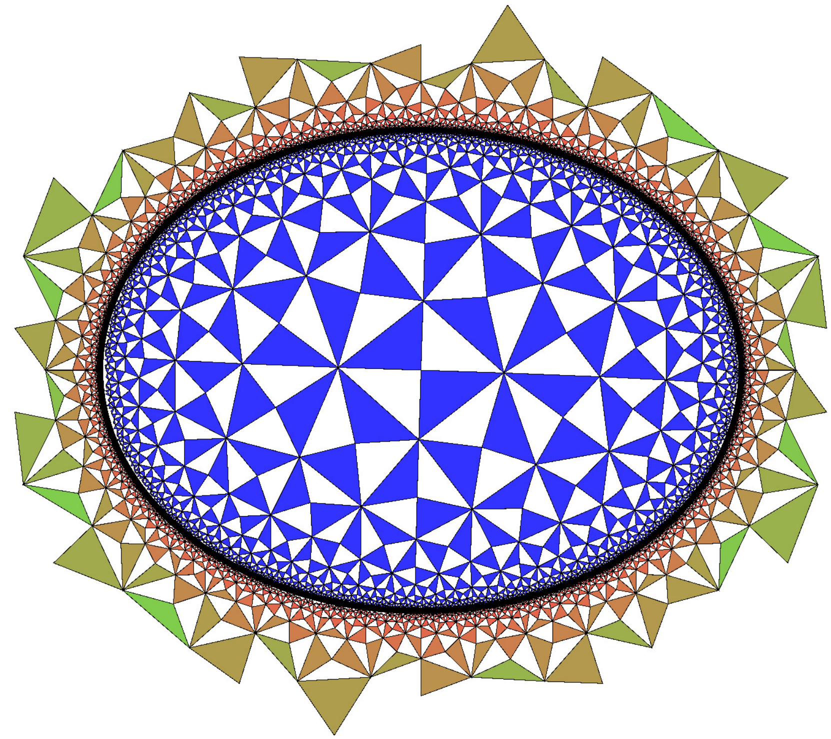
The Weil-Petersson class is Möbius invariant.

Riemann Mapping Thm: any Jordan domain is conformal image of $\mathbb{D}$.

Liouville's Theorem $\Rightarrow$ any conformal map $\mathbb{C} \to \mathbb{C}$ is linear.

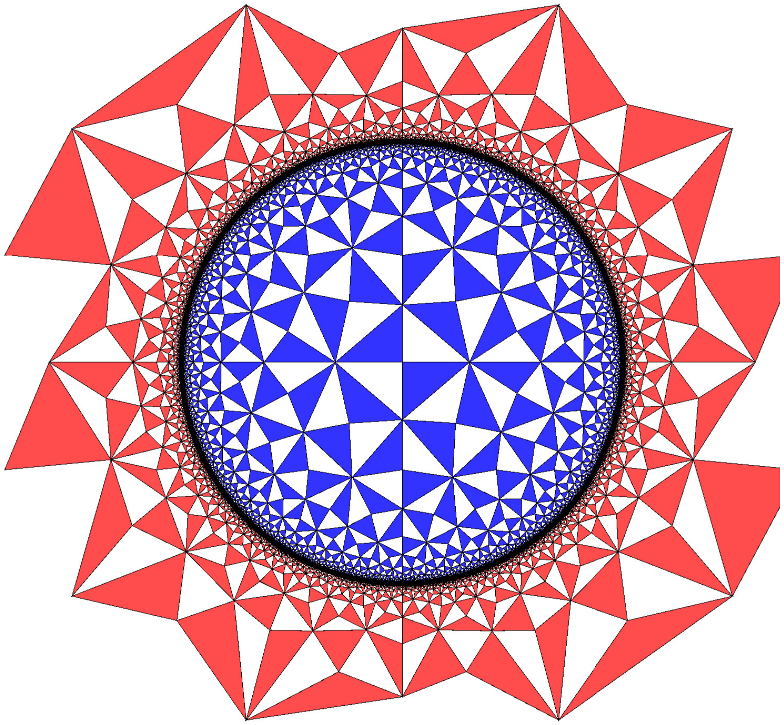
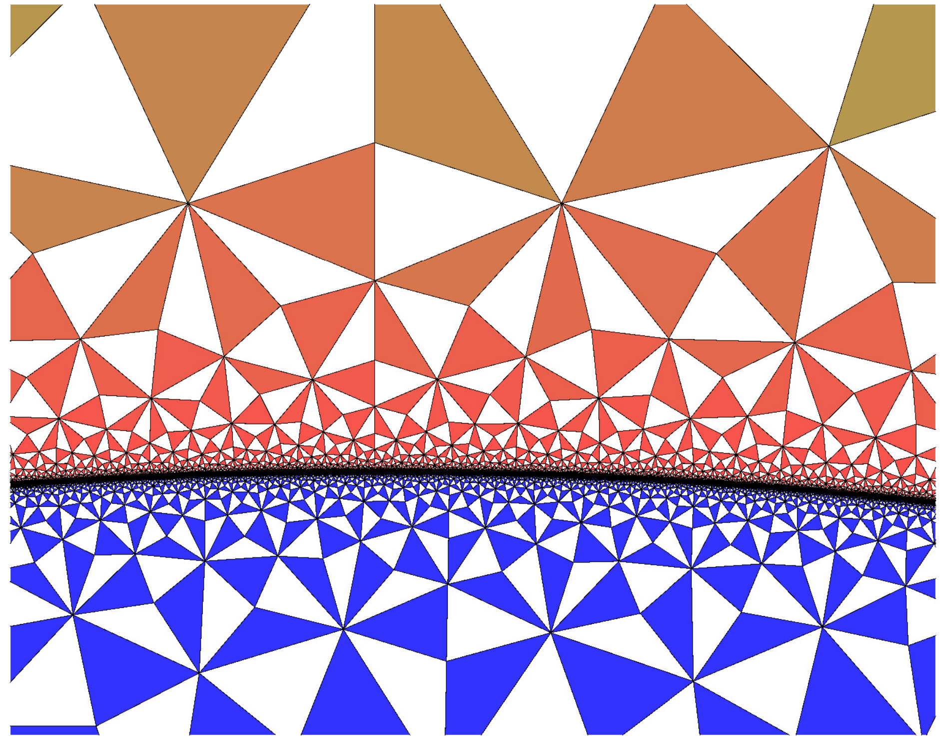
$\Rightarrow$ map above can't be extended to be conformal in whole plane.

Color distortion = angle distortion

Teichmüller metric = maximum dilatation

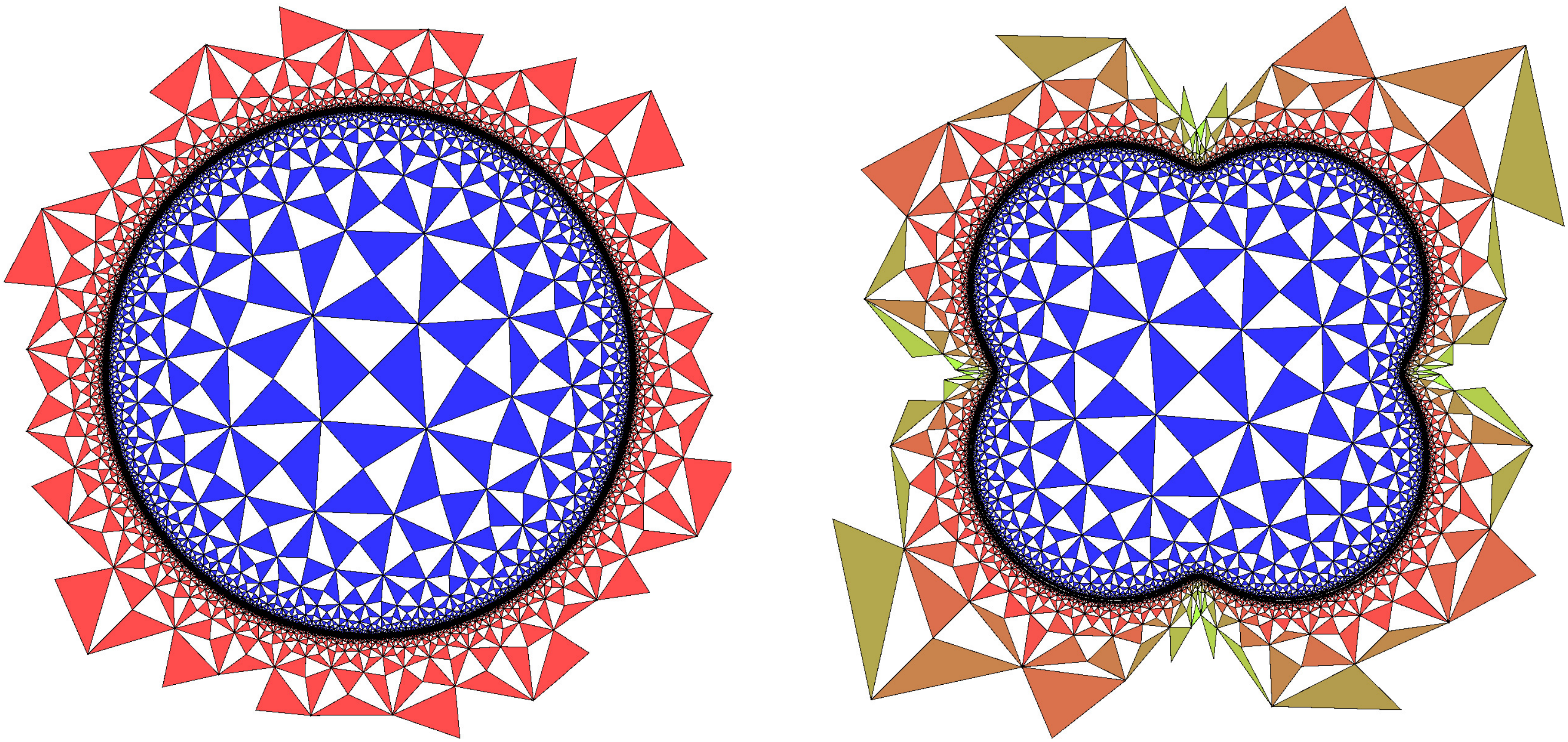
Weil-Petersson metric = $\sum(\text{dilatation})^2$

Distortion decreases near boundary (for smooth domains)

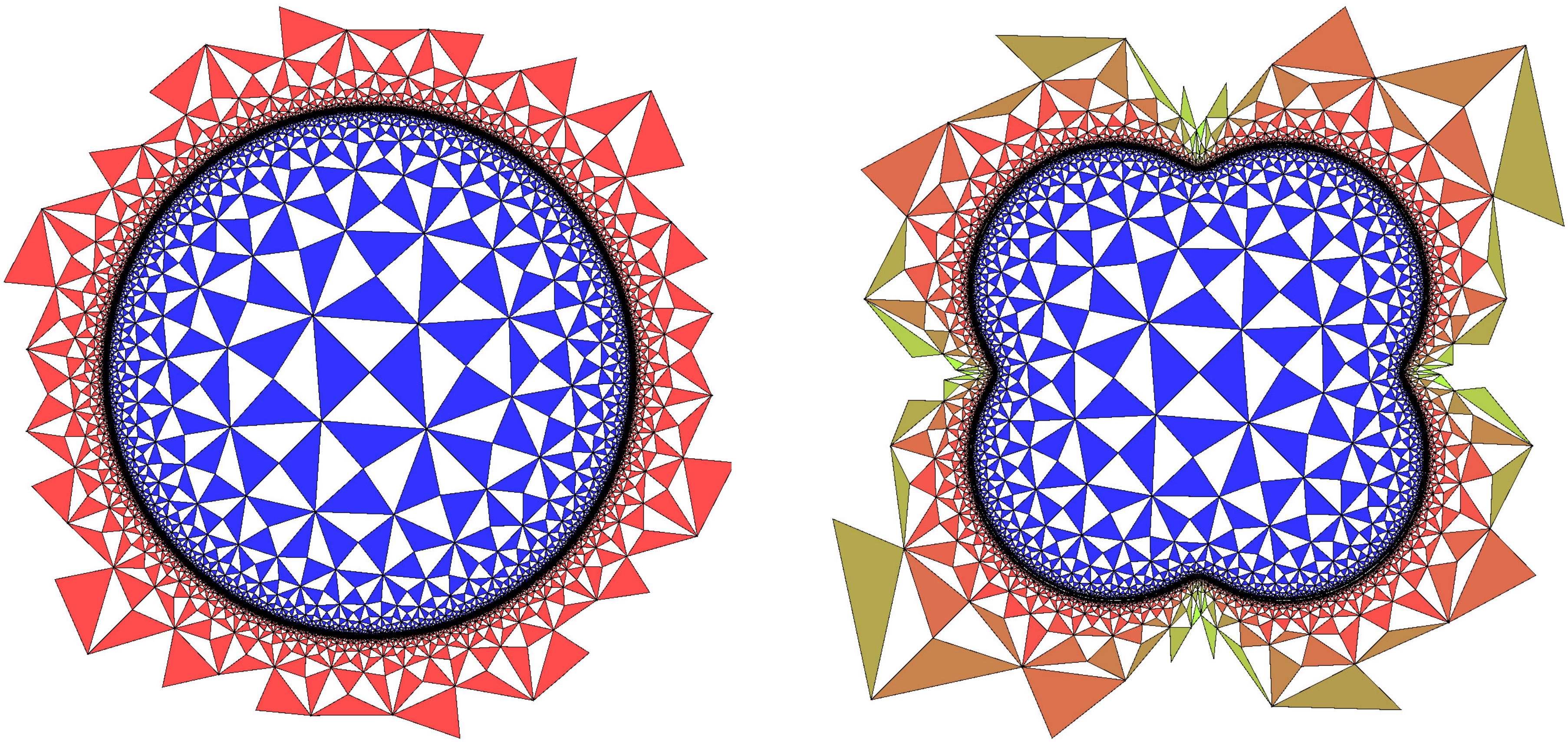
Teichmüller metric = maximum dilatation

Weil-Petersson metric = $\sum(\text{dilatation})^2$



Triangles are (approximately) unit hyperbolic size.

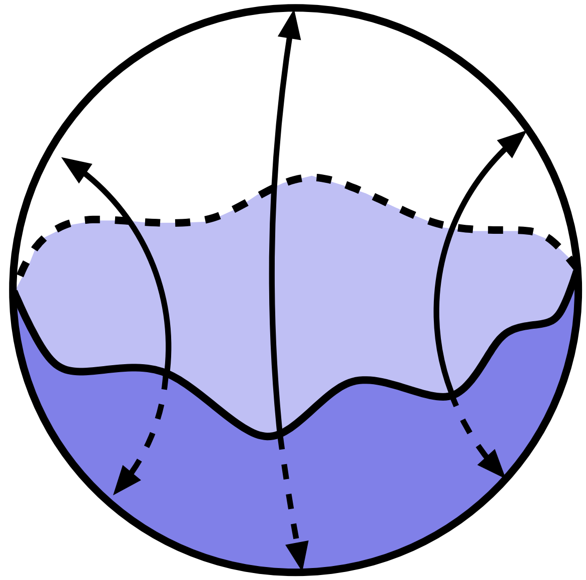
$$\int |\mu|^2 \frac{dxdy}{(1-|z|^2)^2} \quad \approx \quad \sum (\text{distortions})^2.$$



Circle reflection: $R : z \to 1/\overline{z}$. Triangles reflect across circle.

$f \circ R \circ f^{-1}$ is biLipschitz reflection over $\Gamma$, $L^2$ dilatation.

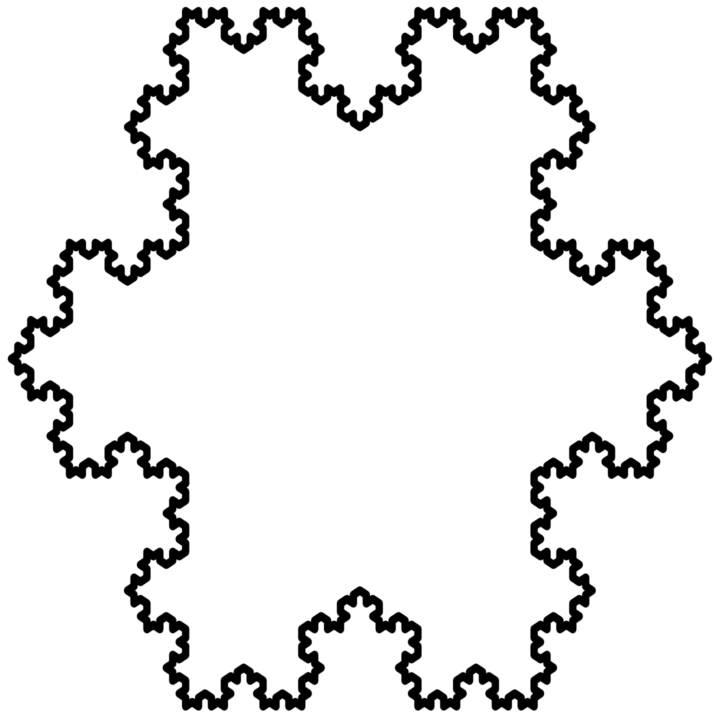
**Theorem:** $\Gamma = f(\mathbb{T})$ is **Weil-Petersson** if $\Gamma$ is pointwise fixed for biLipschitz involution of $S^2$ with $\mu \in L^2$ for hyperbolic area on $\mathbb{S}^2 \setminus \Gamma$.



We will use this later. Generalizes to dimensions $d \geq 4$.

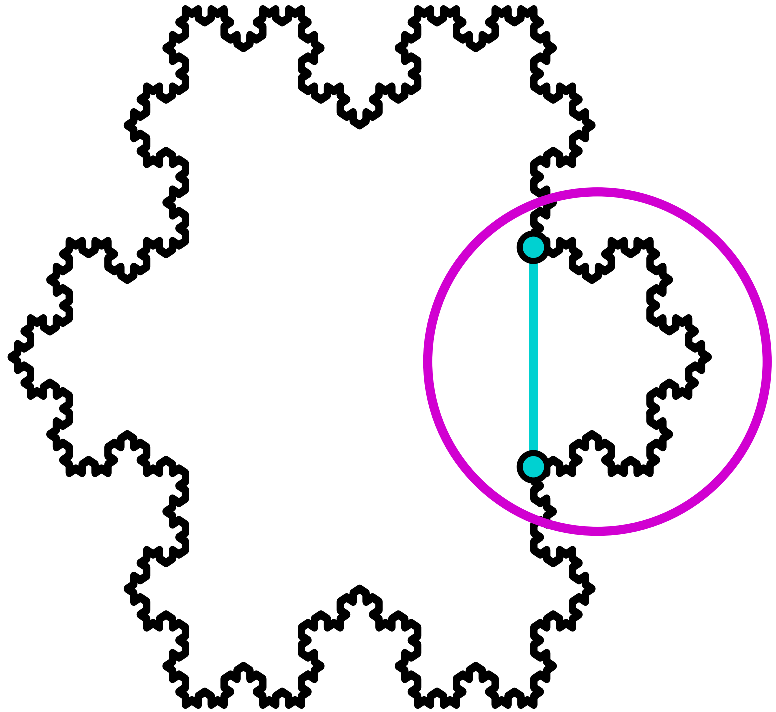
(Smith conjecture implies such a curve in unknotted in dimension 3.)

Smooth curves are quasicircles, but some quasicircles are not smooth.



Quasicircles have a simple geometric characterization due to Ahlfors.

Smooth curves are quasicircles, but some quasicircles are not smooth.



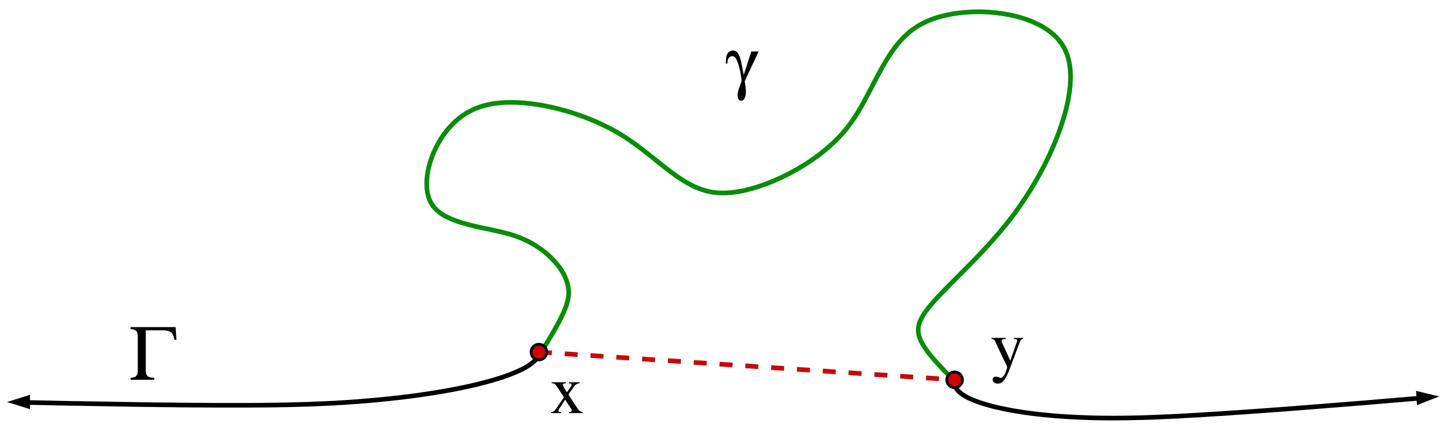
$\Gamma$ is a quasicircle iff $\quad \mathrm{diam}(\gamma) = O(\mathrm{crd}(\gamma)) \quad$ for all $\gamma \subset \Gamma$.

$\mathrm{crd}(\gamma) = |x - y|$, $x, y$, the endpoints of $\gamma$.

Weil-Petersson curves **are never** fractal.

WP-curves are rectifiable (= finite length), in fact, are chord-arc.

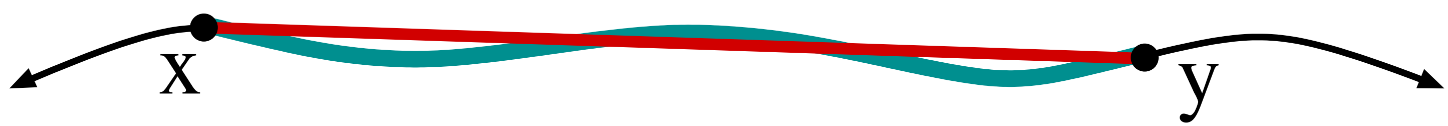
$\Gamma$ is a **chord-arc** iff $\ell(\gamma) = O(|x-y|)$ for all $\gamma \subset \Gamma$.



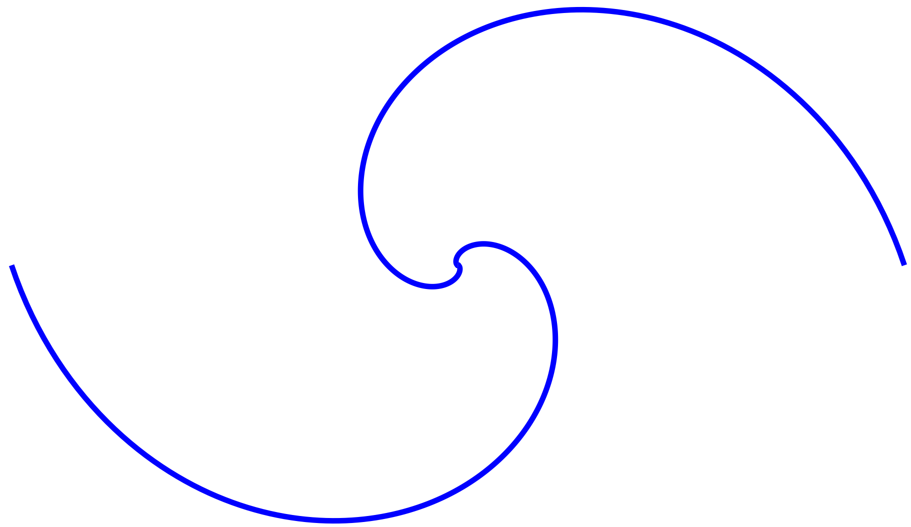
**Even stronger:** Weil-Petersson $\Rightarrow$ Asymptotically smooth

Asymptotically smooth means that $\gamma \subset \Gamma$, $\ell(\gamma) \to 0$ implies

$$\frac{\ell(\gamma)}{|x-y|} \to 1, \quad \text{or equivalently,} \quad \frac{\ell(\gamma) - |x-y|}{|x-y|} \to 0.$$

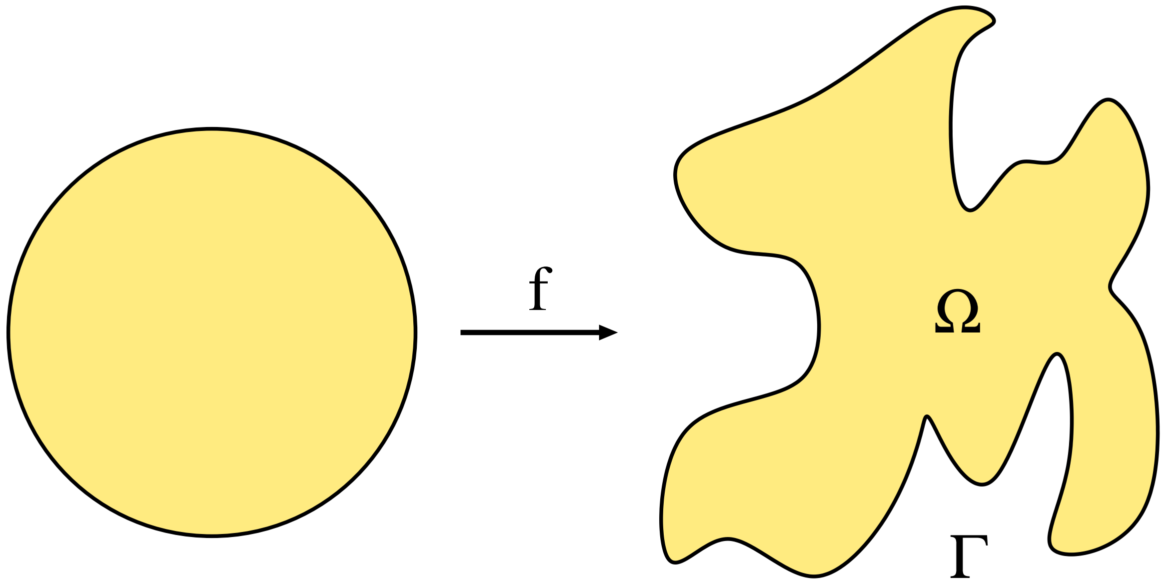



Weil-Petersson curves are almost $C^1$ (but not quite).



Weil-Petersson curves need not be $C^1$.

$z(t) = \exp(-t + i \log t)$, infinite spiral.



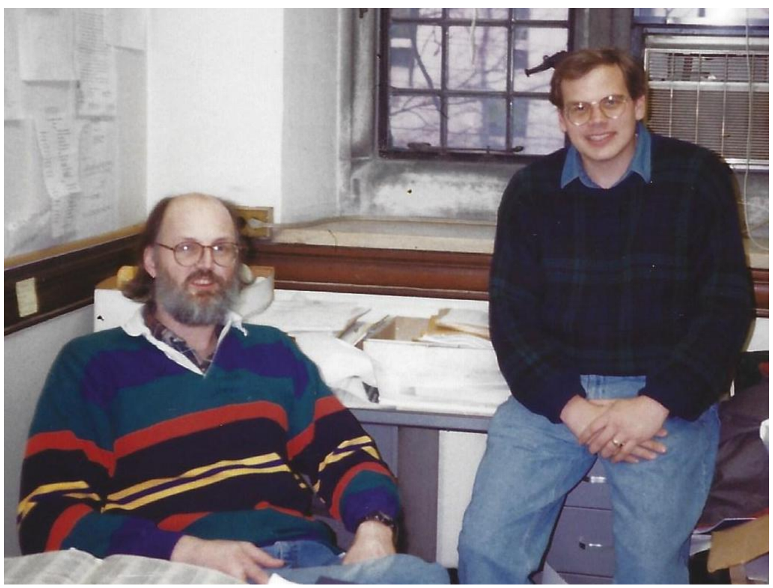

Suppose $X$ is a space of holomorphic functions on $\mathbb{D}$,

e.g., $L^p$, VMO, BMO, Hardy spaces, Bergman, Bloch, Sobolev, ....

**Problem:** Characterize $\Gamma = f(\mathbb{T})$ so that $\log f' \in X$.

Kari Astala and Michel Zinsmeister invented "BMO-Teichmüller theory" where $\log f' \in \mathrm{BMO}$ (1990's).

(BMO = Bounded Mean Oscillation, recall $L^\infty \subset \mathrm{BMO} \subset L^p$, $p < \infty$)

Peter Jones and I characterized curves with $\log f' \in \mathrm{BMO}$.

(roughly speaking, $\Gamma$ has "good" approximations by chord-arc curves)

In their memoir, Takhtajan and Teo prove:

**Theorem:** $\Gamma$ is Weil-Petersson iff $u = \log f' \in W^{1,2}(\mathbb{D})$.

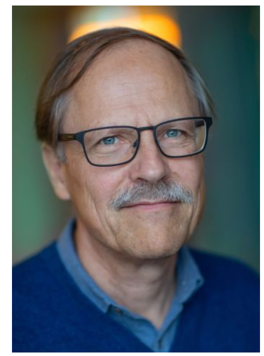
$W^{1,2}(\mathbb{D}) = \{u : |\nabla u| \in L^2(dxdy)\}$ = one derivative in $L^2$

Hence $\Gamma$ is WP iff $\int_{\mathbb{D}} |(\log f')'|^2 dxdy < \infty$.

I learned this in the IPAM lecture of Yilin Wang.

She and Rohde proved $\log f' \in W^{1,2}$ iff $\Gamma$ has finite **Loewner energy**.

Her work connects WP to large deviations of Schramm-Loewner evolutions (small $\kappa$) and the Brownian loop soup of Lawler and Werner.

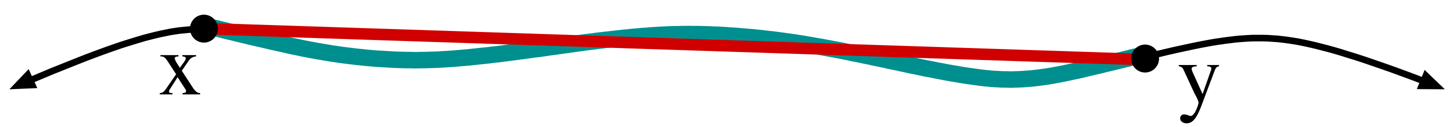
Wang and Viklund give other connections of WP curves to SLE (large $\kappa$).

Johansson and Viklund have connected WP curves to Coulomb gas.

Previous work on BMO suggests $\log f' \in W^{1,2}$ is same as:

**Jones Conjecture (B, 2020):** $\Gamma$ is Weil-Petersson iff

$$\int_\Gamma \int_\Gamma \frac{\ell(x,y) - |x-y|}{|x-y|^3} ds dt < \infty$$




$\ell(x,y)$ = arclength distance between $x, y$ along curve

Stronger version of "asymptotically smooth"

Does not look like an $L^2$ condition.

For a chord-arc curve, $|x-y| \le \ell(x,y) \le C|x-y|$, so

$$\begin{aligned}
\frac{\ell(x,y)-|x-y|}{|x-y|^3} &\simeq \frac{\ell(x,y)-|x-y|}{|x-y|^2\cdot\ell(x,y)}\cdot\frac{\ell(x,y)+|x-y|}{\ell(x,y)} \\
&= \frac{\ell(x,y)^2-|x-y|^2}{|x-y|^2\ \ell(x,y)^2} \\
&= \frac{1}{|x-y|^2}-\frac{1}{\ell(x,y)^2}
\end{aligned}$$

This looks more like $L^2$.

The **Möbius energy** of a curve $\Gamma \in \mathbb{R}^n$ is

$$\text{Möb}(\Gamma) = \int_\Gamma \int_\Gamma \left( \frac{1}{|x-y|^2} - \frac{1}{\ell(x,y)^2} \right) ds dt.$$

**Theorem:** $\Gamma$ is WP iff $\text{Möb}(\Gamma) < \infty$.

Möbius energy is Möbius invariant (hence the name).

The **Möbius energy** of a curve $\Gamma \in \mathbb{R}^n$ is

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Möbius energy is Hadamard renormalization of divergent integral

$$\int_\Gamma \int_\Gamma \frac{dsdt}{|x-y|^2}$$

= energy to place arclength charge on $\Gamma$ under inverse-cube force.

(= electrostatic repulsion in 4 dimensions.)

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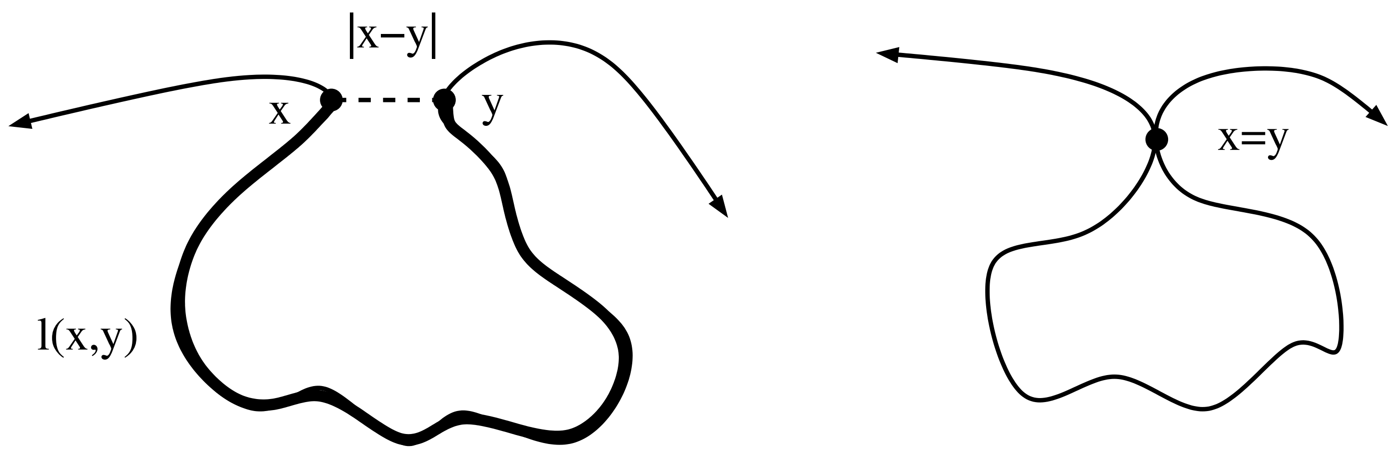


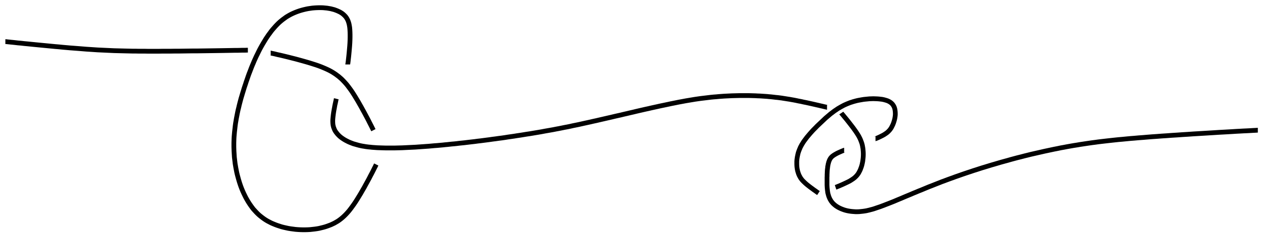

Möbius energy blows up if curve self-intersects.

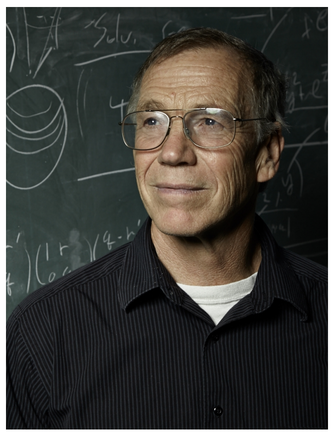
$\Rightarrow$ deforming $\Gamma$ to lower energy doesn't change topology.

$\Rightarrow$ minimizing should give canonical representation of a knot.

Möbius energy is one of several “knot energies” due to Jun O’Hara.

Studied by Freedman, He and Wang in 1990’s.





**Theorem (Blatt, 2012):** $\text{Möb}(\Gamma) < \infty$ iff arclength parameterization is $H^{3/2}$.

$H^{3/2}$ = Sobolev space = $\frac{3}{2}$-derivative in $L^2$.

**Cor:** $\Gamma$ is WP iff arclength parameterization is in $H^{3/2}$.

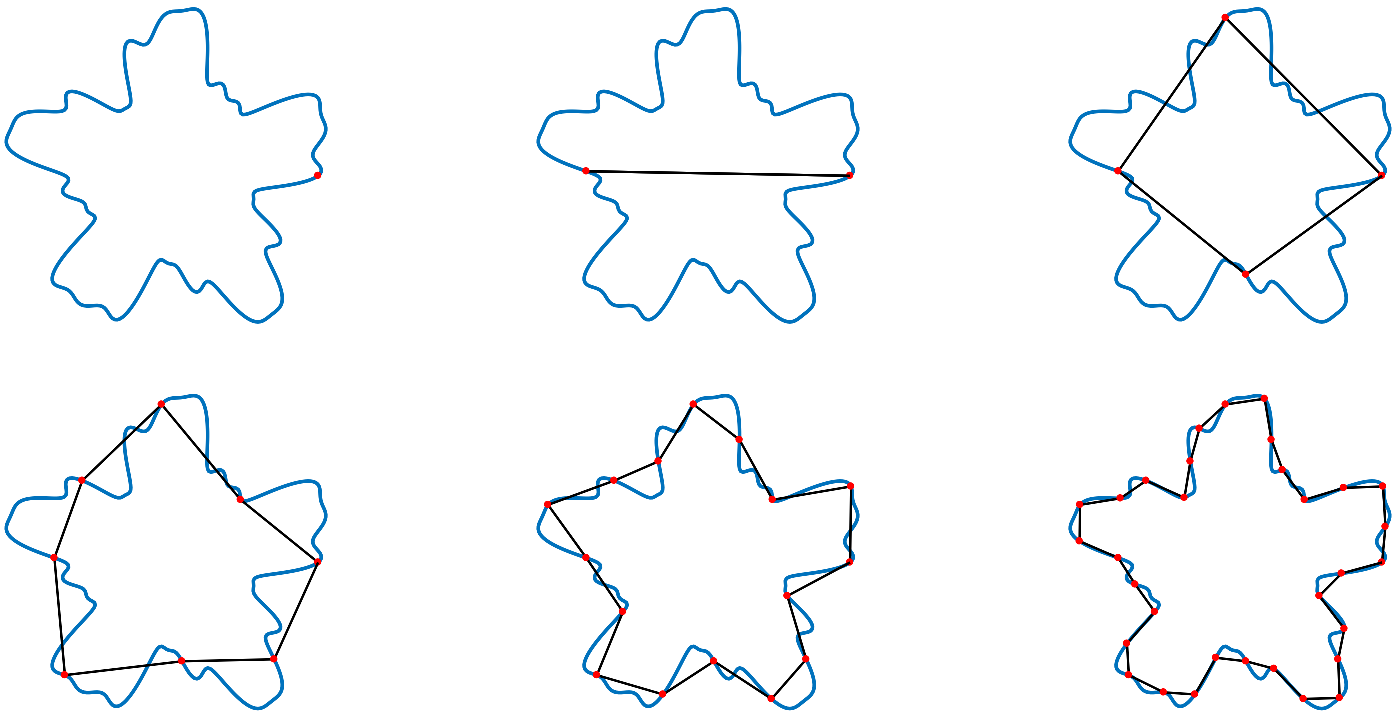
For $s > 3/2$, is known that $H^s \Rightarrow C^1$, so WP curves are "almost" $C^1$.

Quasiconformal maps and $H^{3/2}$ and are pretty sophisticated.

How can you describe WP curves to a calculus student?

**Dyadic decomposition.**

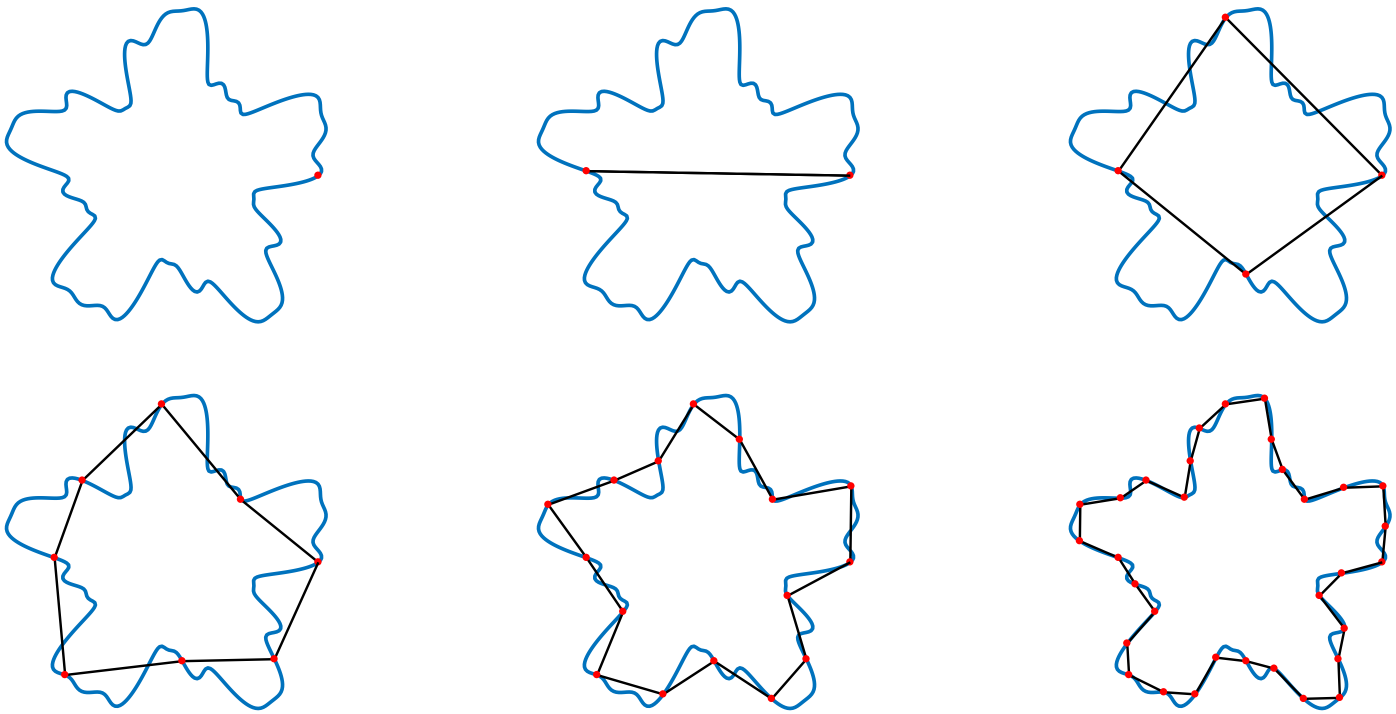
- Divide $\Gamma$ into nested families of $2^n$ equal length arcs.
- Inscribe a polygon $\Gamma_n$ at these points.
- Clearly $\ell(\Gamma_n) \nearrow \ell(\Gamma)$. How fast?



**Theorem:** $\Gamma$ is Weil-Petersson if and only if

$$\sum_{n=1}^{\infty} 2^n \left[\ell(\Gamma) - \ell(\Gamma_n)\right] < \infty$$

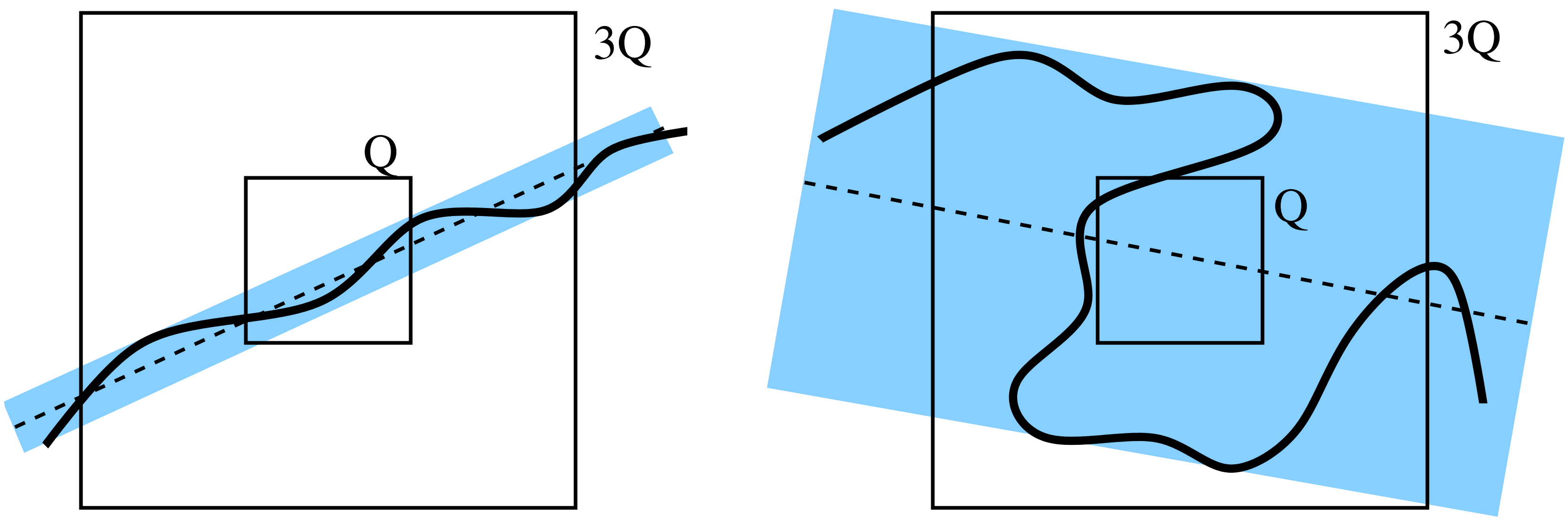
with a bound that is independent of the dyadic family.



**Peter Jones's $\beta$-numbers:**

$$\beta_\Gamma(Q) = \inf_L \sup_{z \in 3Q \cap \Gamma} \frac{\text{dist}(z, L)}{\text{diam}(Q)}$$

Infimum is over lines $L$ that hit $3Q$.

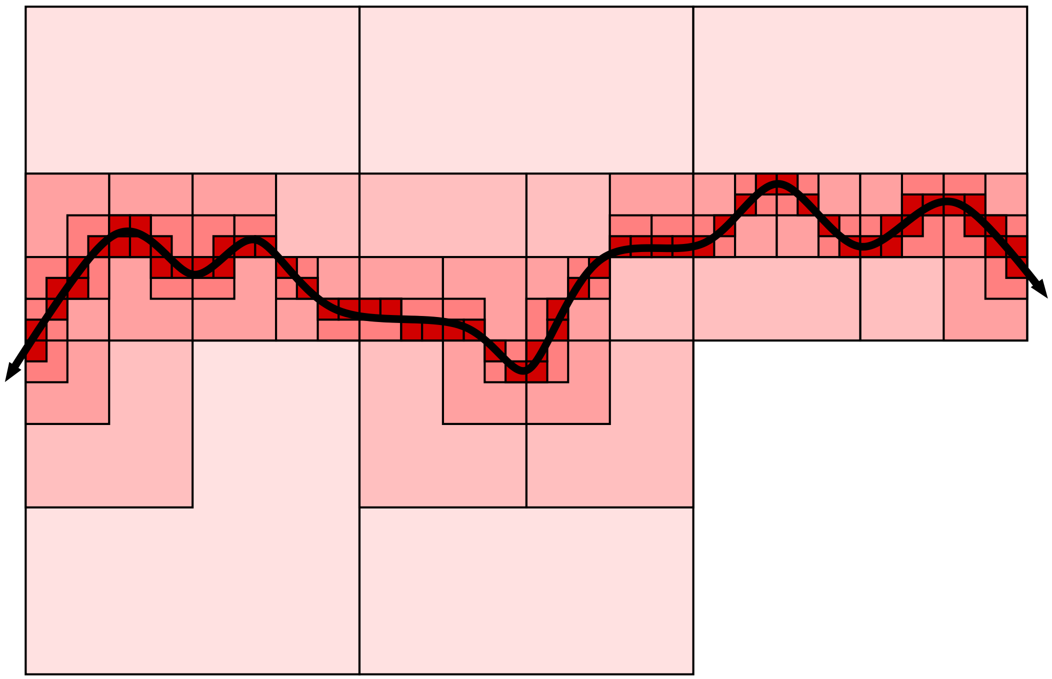



$\beta \approx 0$ $\beta \approx 1$

Jones invented the $\beta$-numbers for his traveling salesman theorem:

$$\ell(\Gamma) \simeq \operatorname{diam}(\Gamma) + \sum_{Q} \beta_{\Gamma}(Q)^2 \operatorname{diam}(Q),$$

where the sum is over all dyadic cubes $Q$ in $\mathbb{R}^n$ hitting $\Gamma$.



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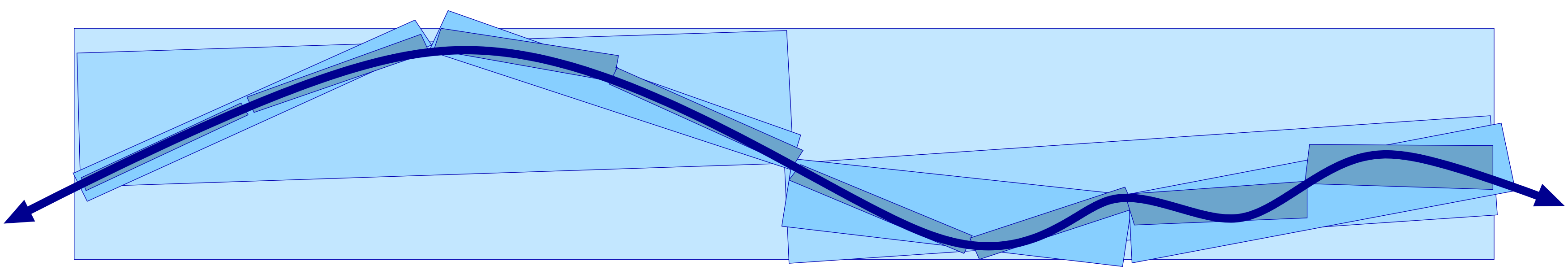
**Theorem:** $\Gamma$ is Weil-Petersson iff $\sum_Q \beta_\Gamma(Q)^2 < \infty$.

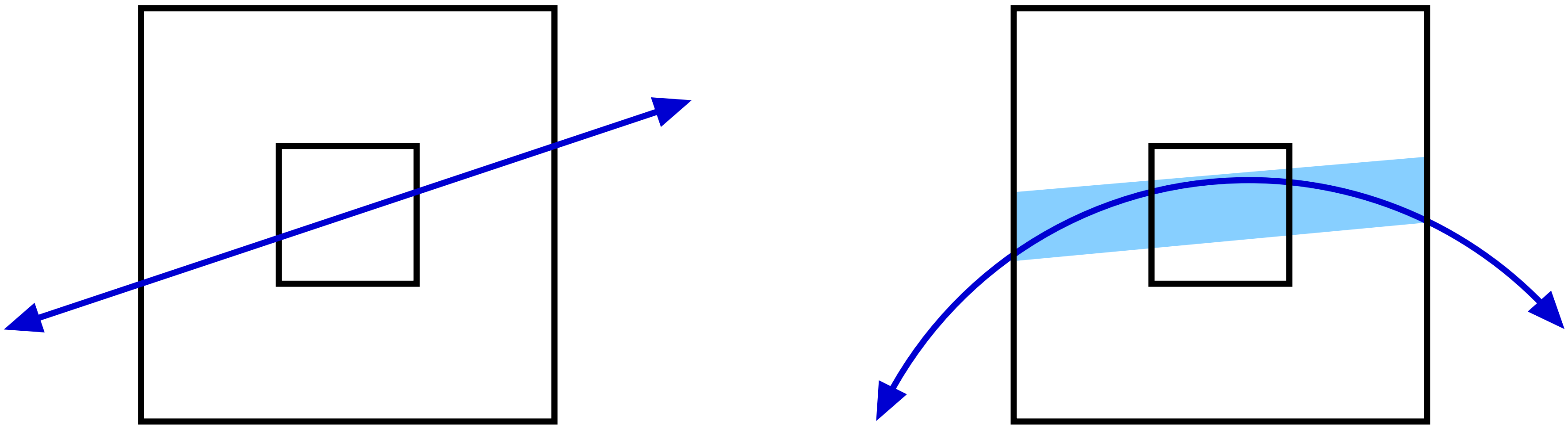


WP = "curvature in $L^2$, summed over all positions and scales".

= "rectifiable in scale invariant way".

The Weil-Petersson class is Möbius invariant.

$\beta$-numbers are not: lines ($\beta = 0$) can map to circles ($\beta > 0$).

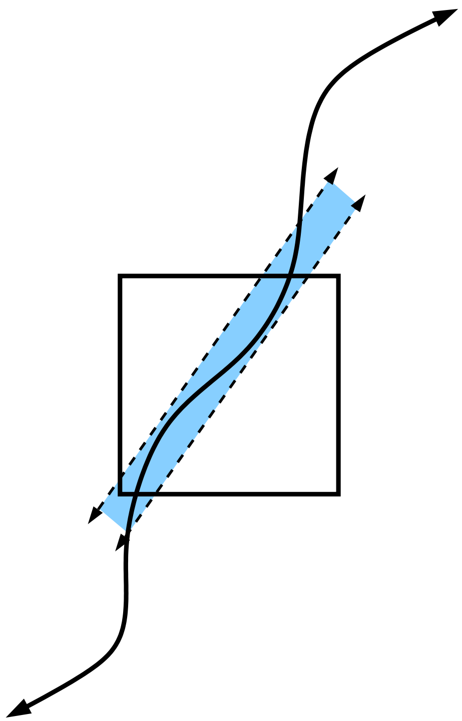


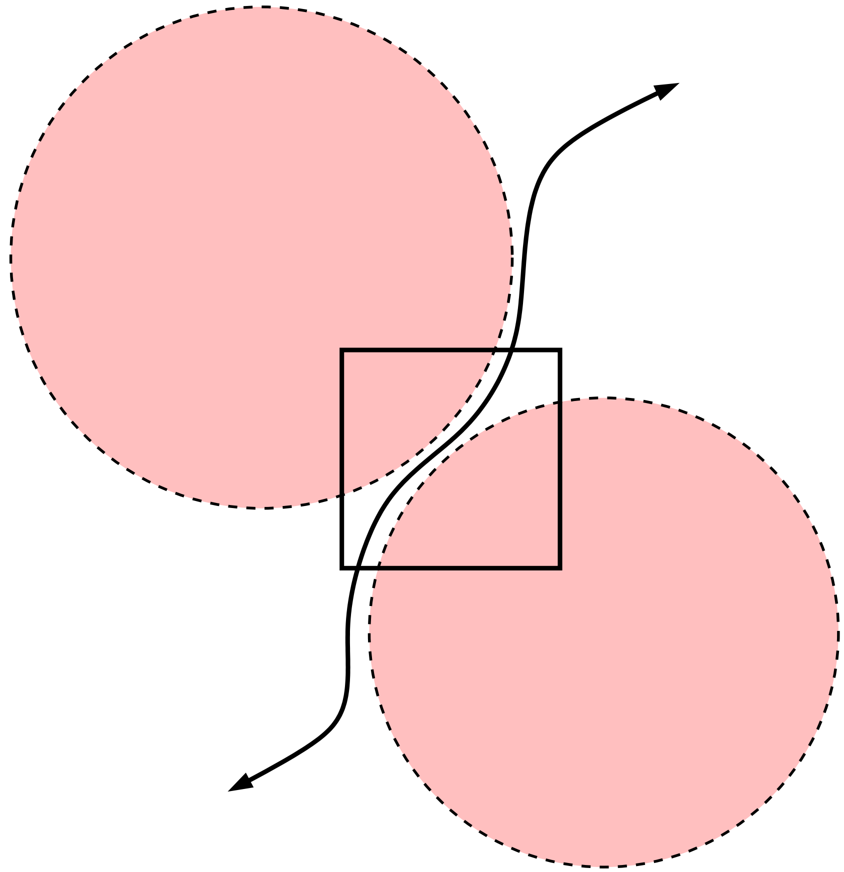
What is a Möbius invariant version of the $\beta$-numbers?

Möbius = linear fractional = $\frac{az+b}{cz+d}$ = conformal self-maps of sphere

Möbius transformations preserve lines/circles.

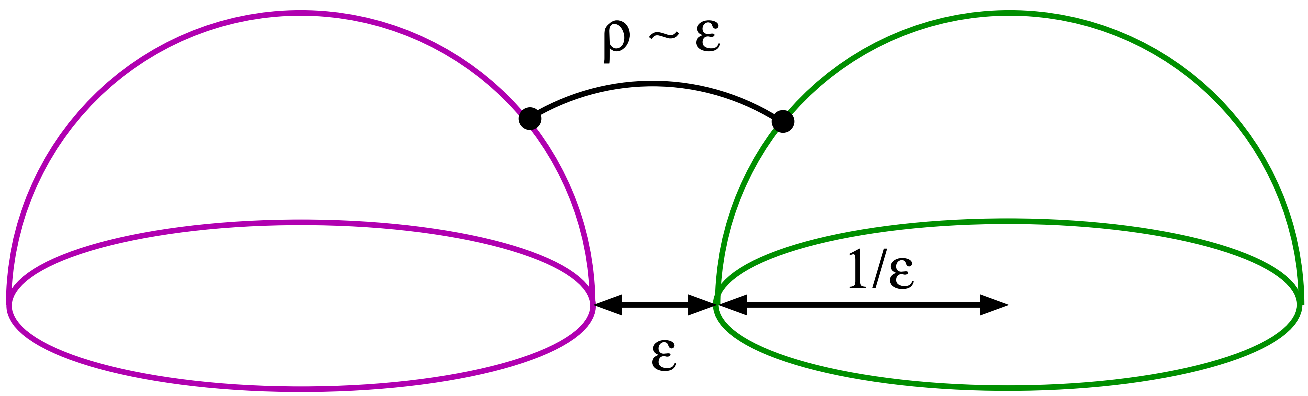
$\beta$-numbers trap curve between lines. Trap curve between disks instead.





Each disk is the base of a hemisphere in the upper half-space $\mathbb{H}^3 = \mathbb{R}^3_+$.

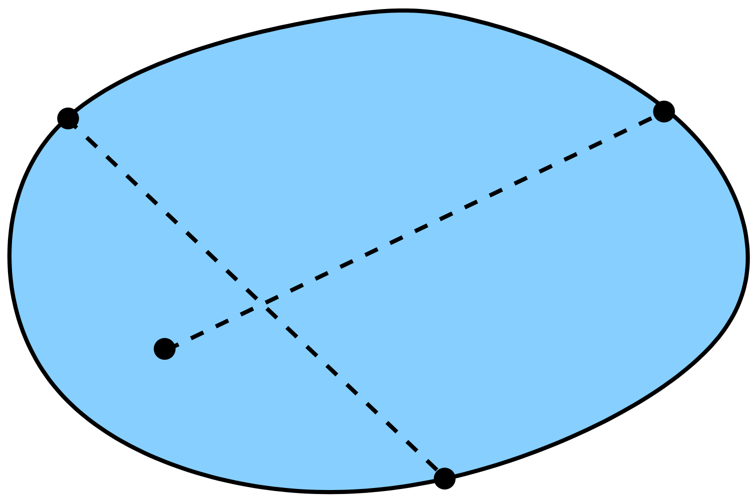
These Euclidean hemispheres are hyperbolic half-spaces.



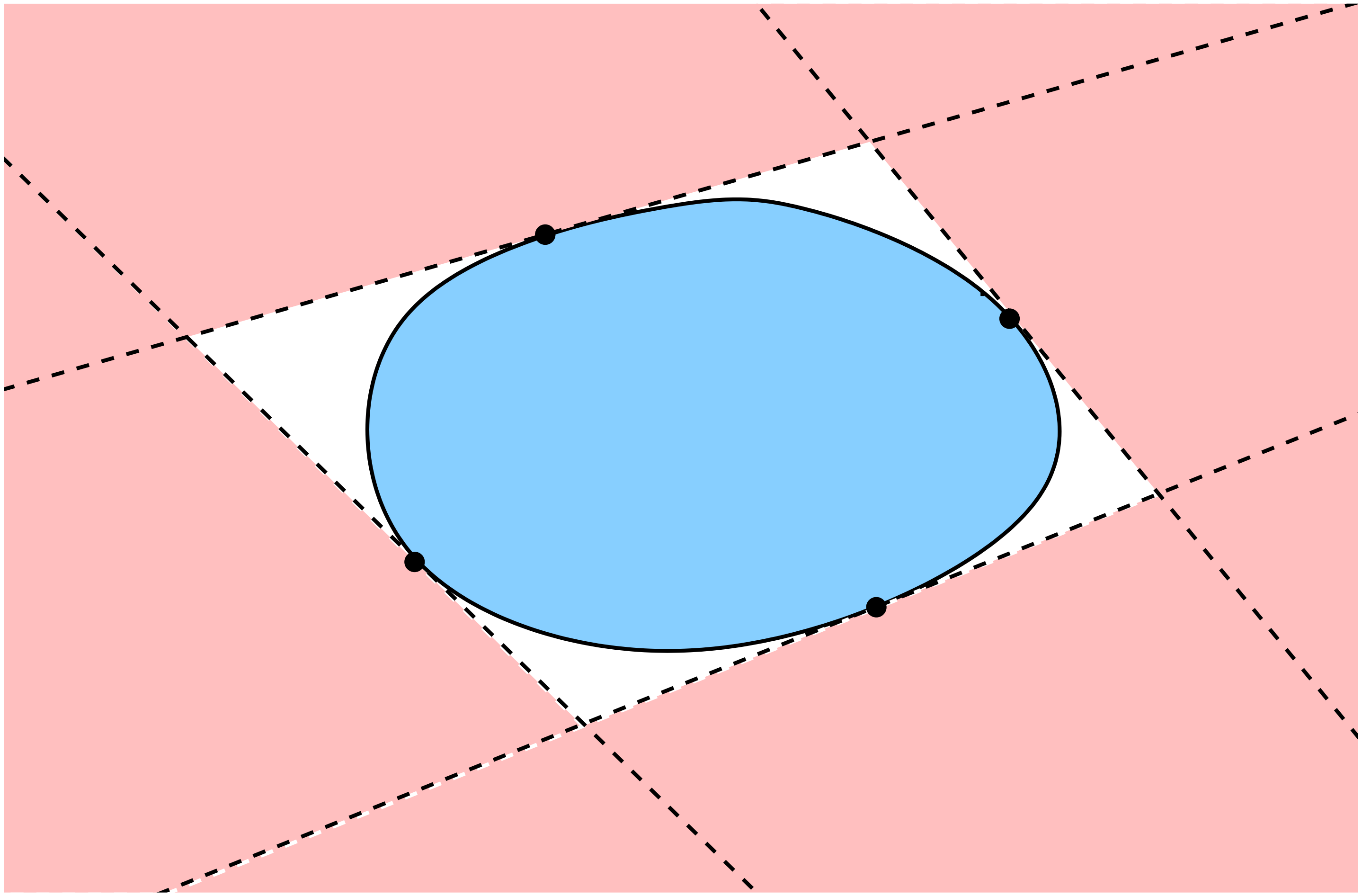

For $(1/\epsilon)$-disks that are $\epsilon$ apart, the hyperbolic distance is $\approx \epsilon$.

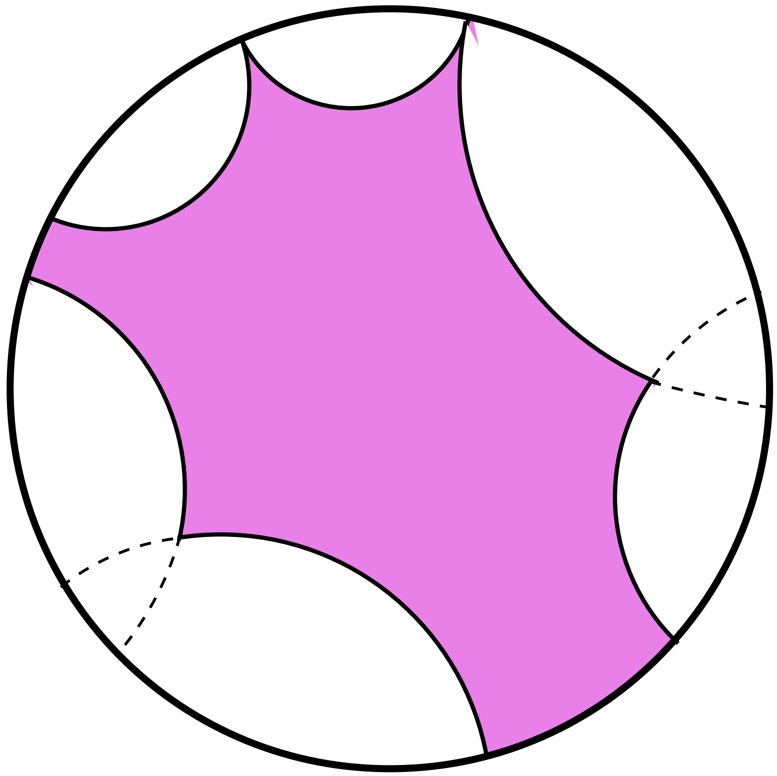
Möbius transformation of plane extends to isometry of upper half-space.

Usual definition of convex: contains geodesic between any two points.



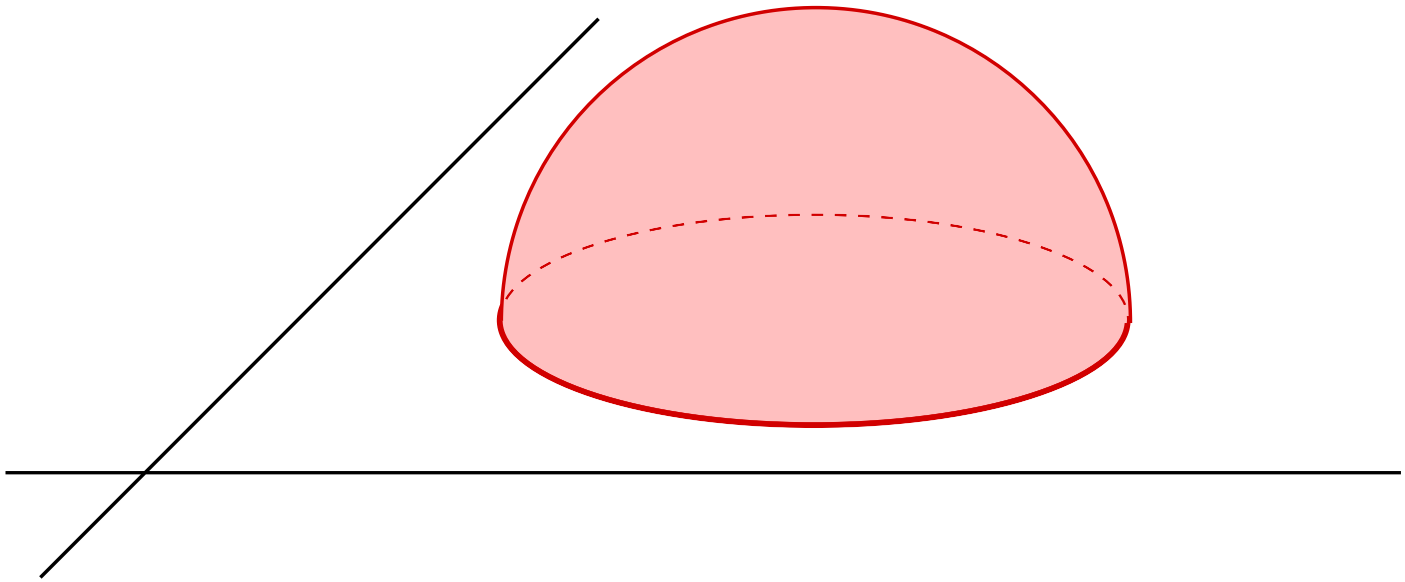
More useful for us: complement is a union of half-spaces.





Convex set in hyperbolic disk

Complement = union of half-spaces



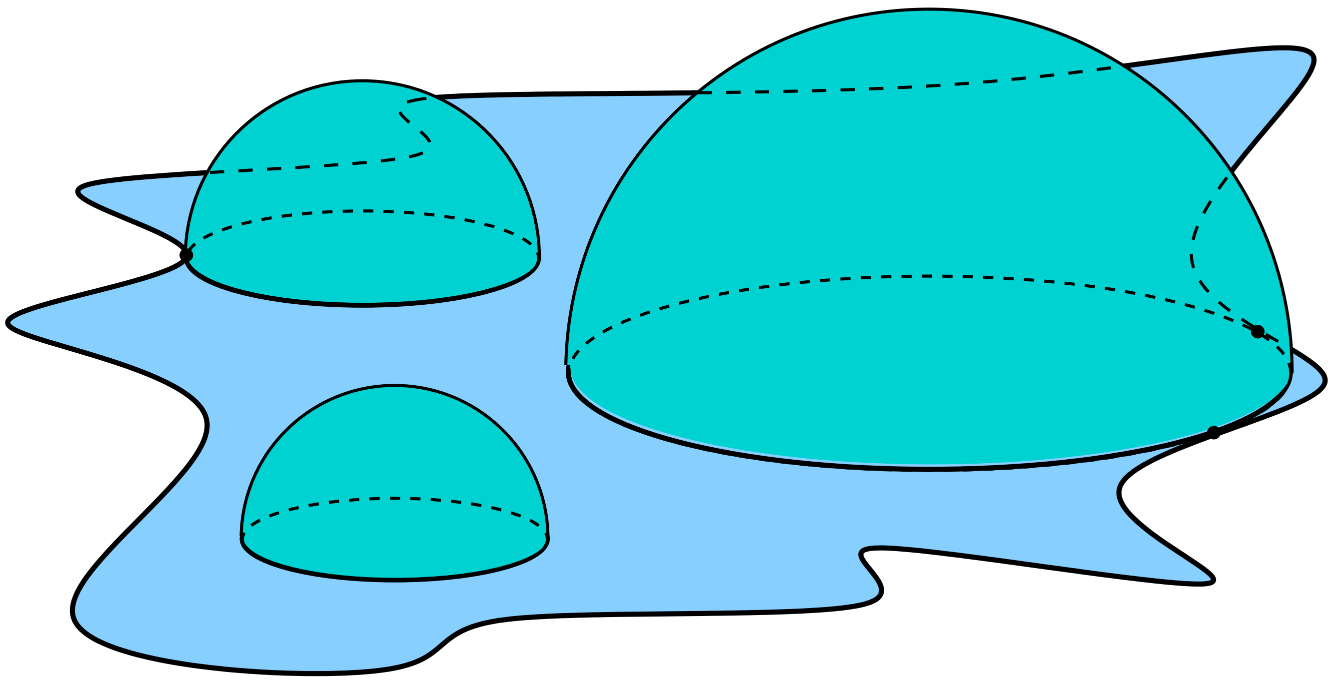
In $\mathbb{R}^3_+$, a hyperbolic half-space = hemisphere.

$\mathrm{CH}(\Gamma)$ = complement of all open half-spaces that miss $\Gamma$.

In general, $\mathrm{CH}(\Gamma)$ has non-empty interior and 2 boundary components.

A hyperbolic half-space missing CH(Γ) has boundary disk missing Γ.

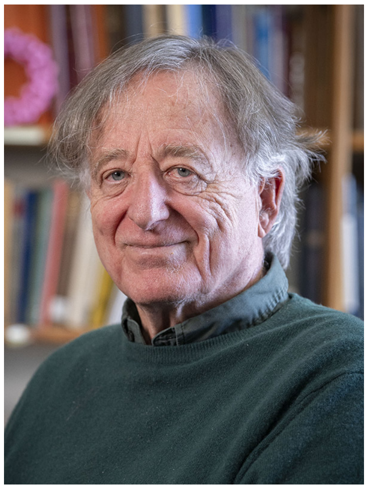
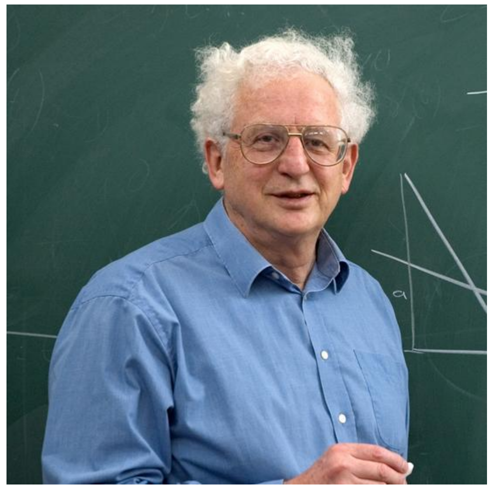
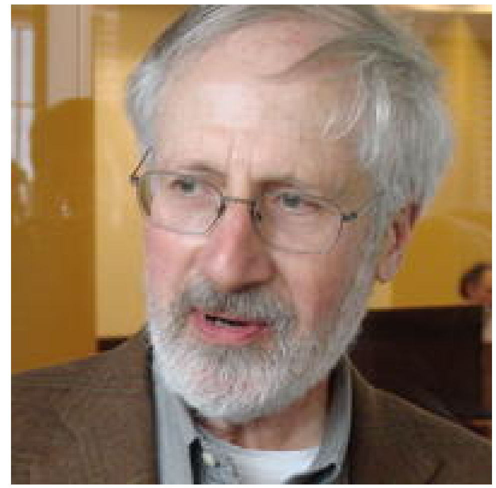
The Dome(Ω) is union of hemispheres with base disk inside Ω.



Region above dome is intersection of half-spaces, hence convex.

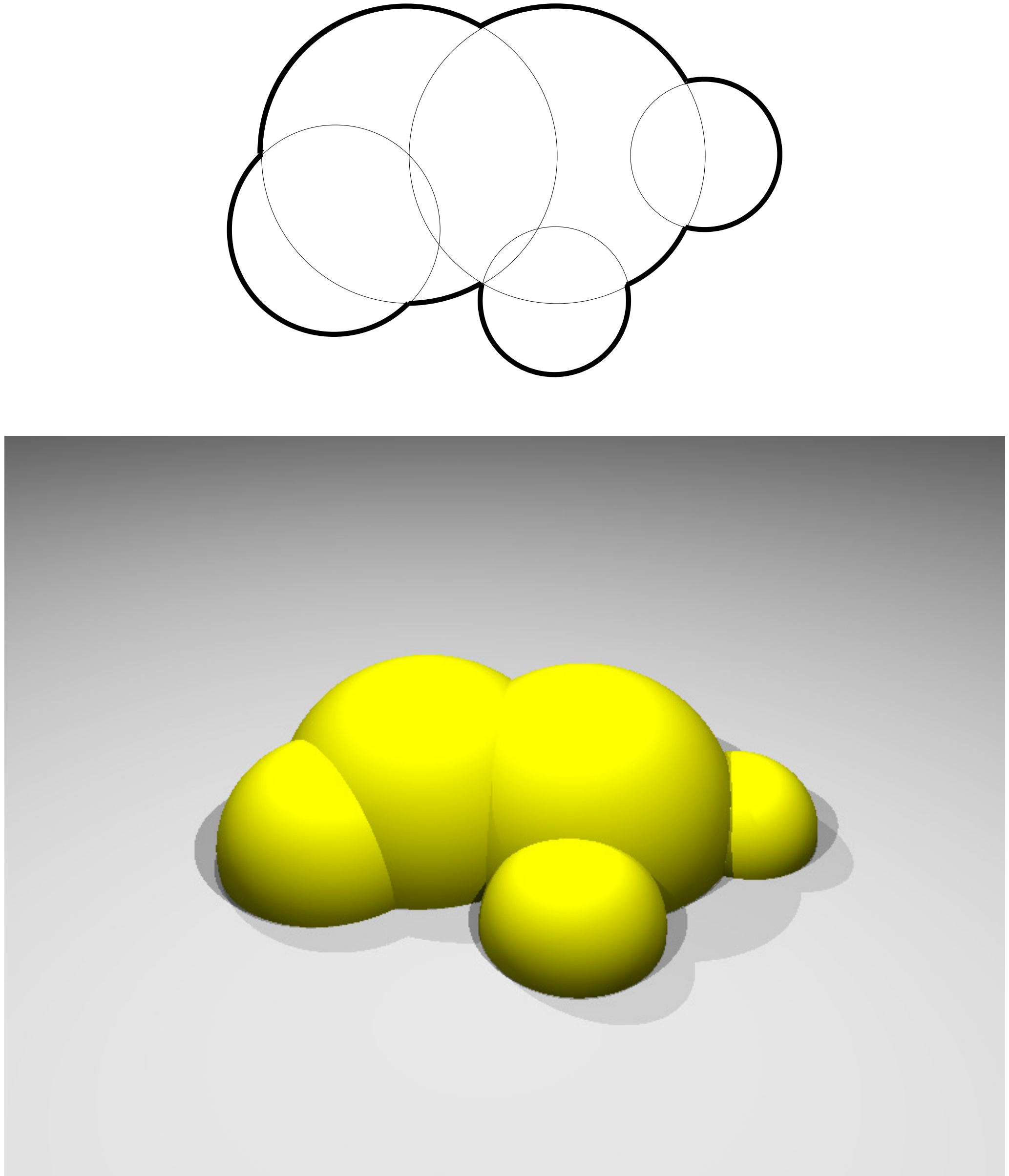
CH(Γ) is region between domes for "inside" and "outside" of Γ.

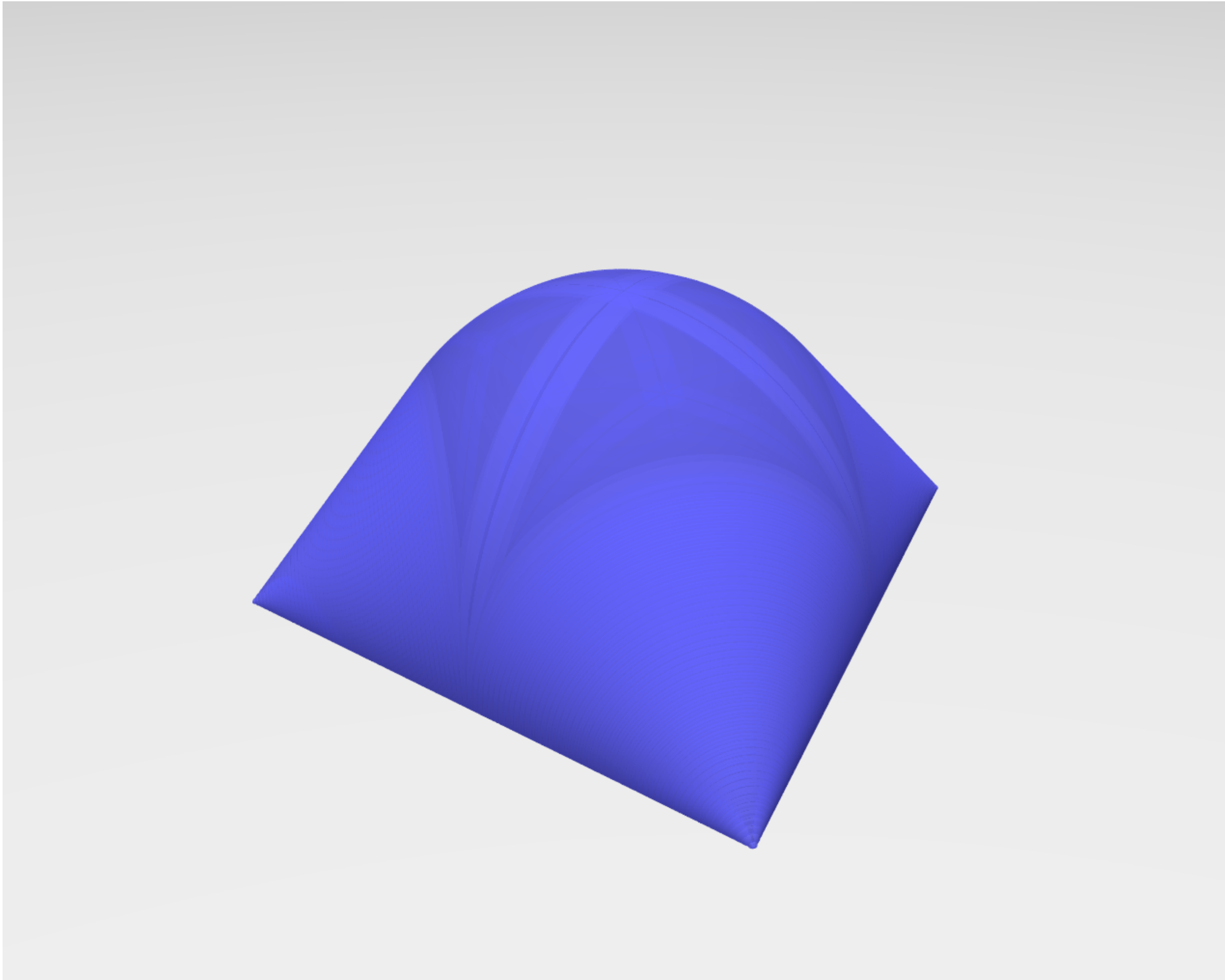
Domes arise from work of Dennis Sullivan, David Epstein and Al Marden.

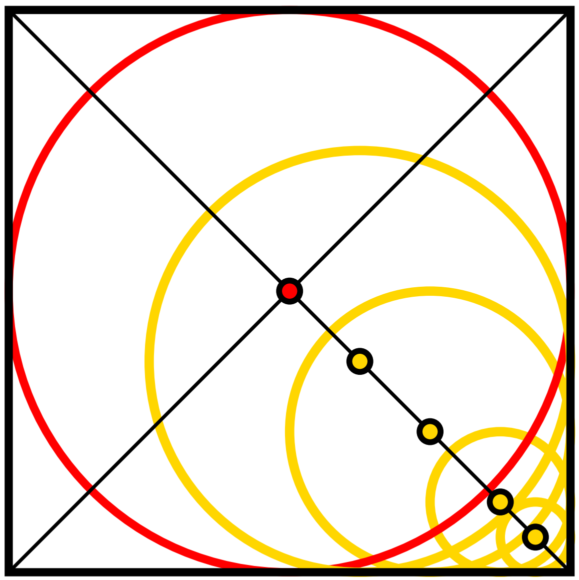
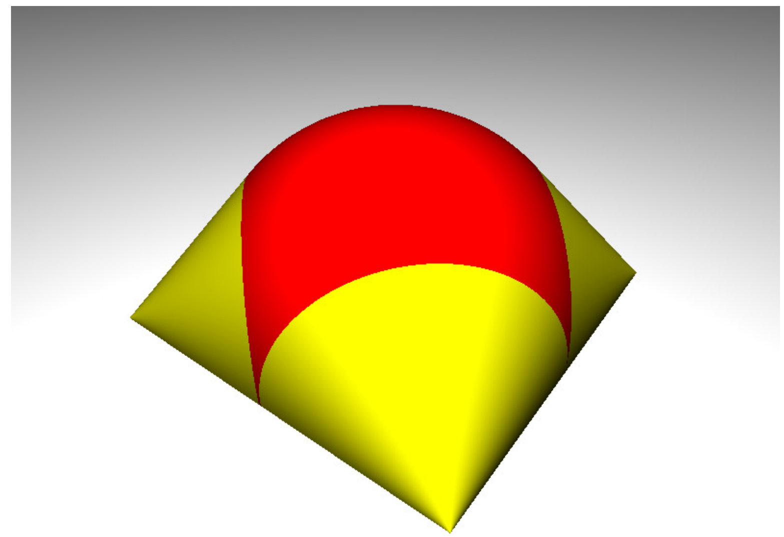




Many applications to Kleinian and Fuchsian groups.

Also (surprisingly) to conformal mapping and optimal meshing.



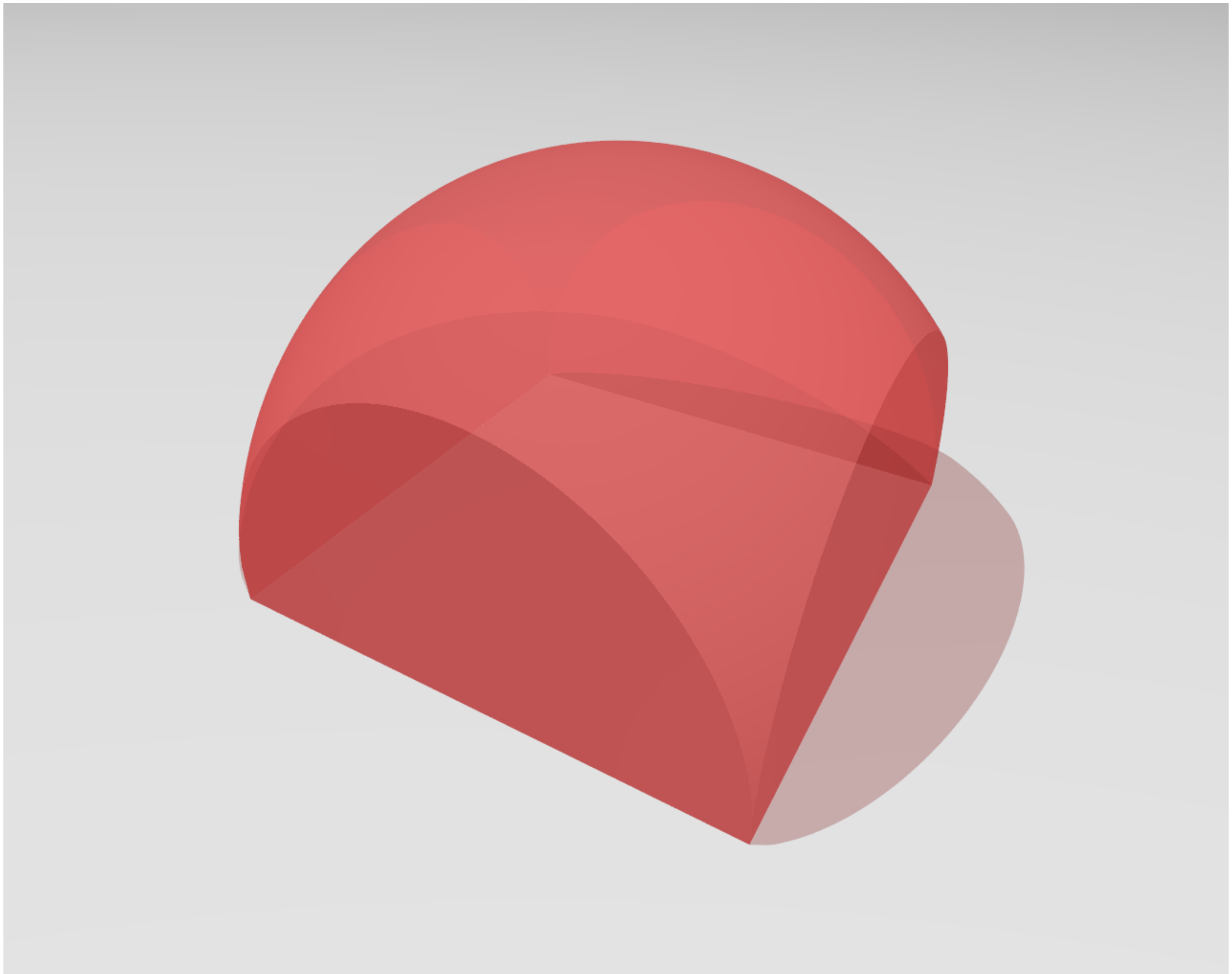


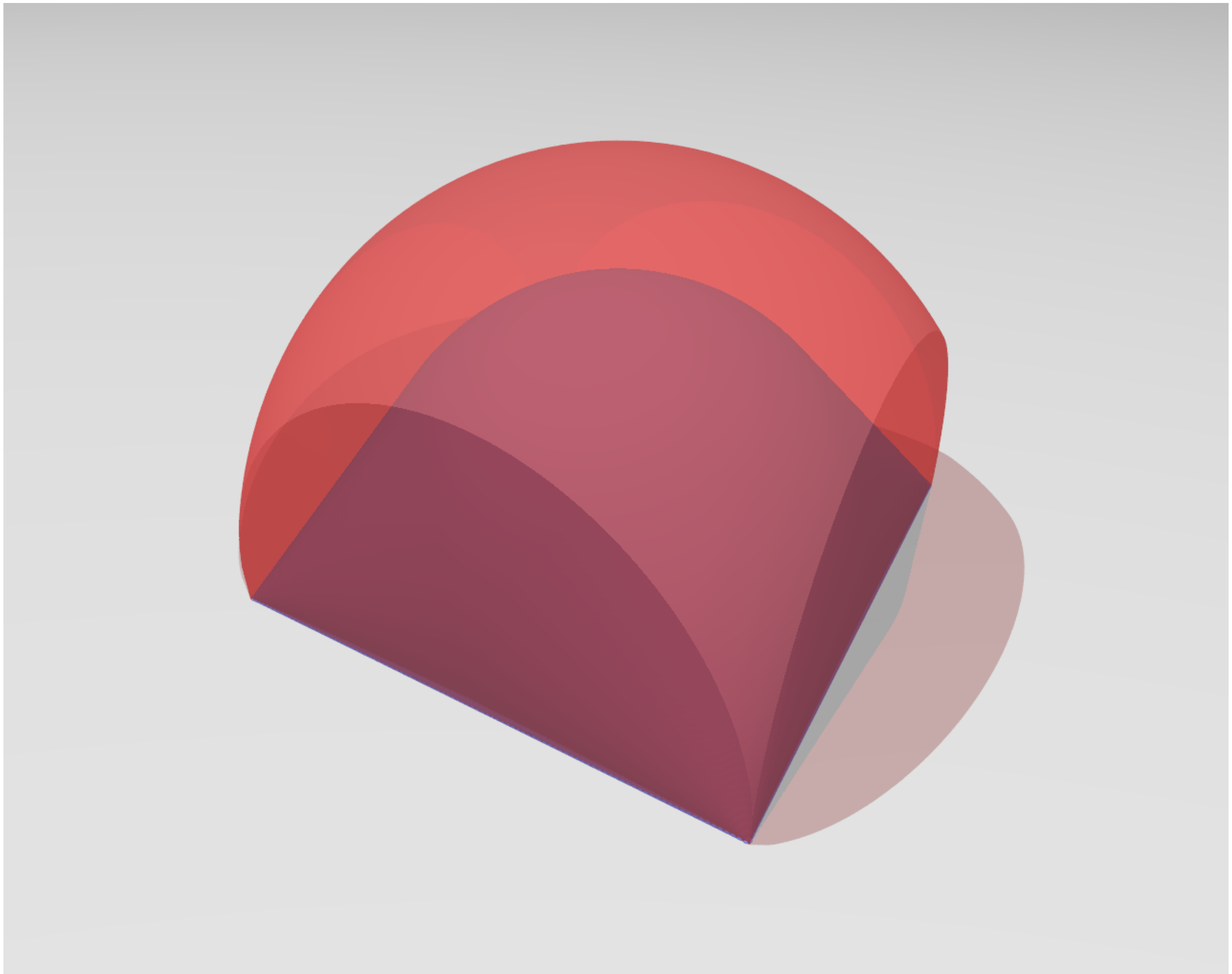
 

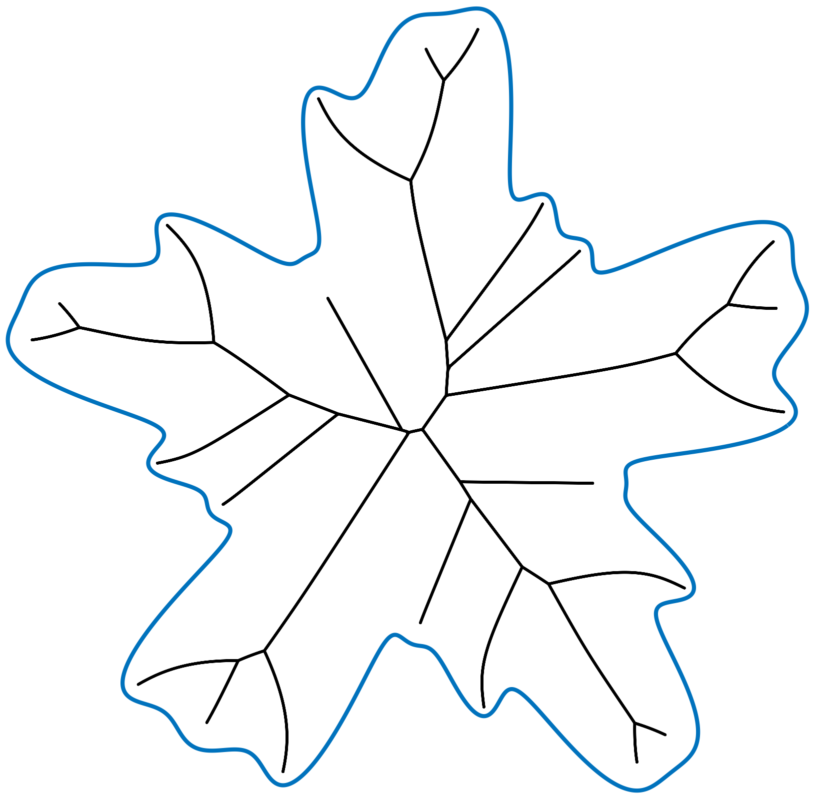
The medial axis of square.

= points equidistant from at least two boundary points.

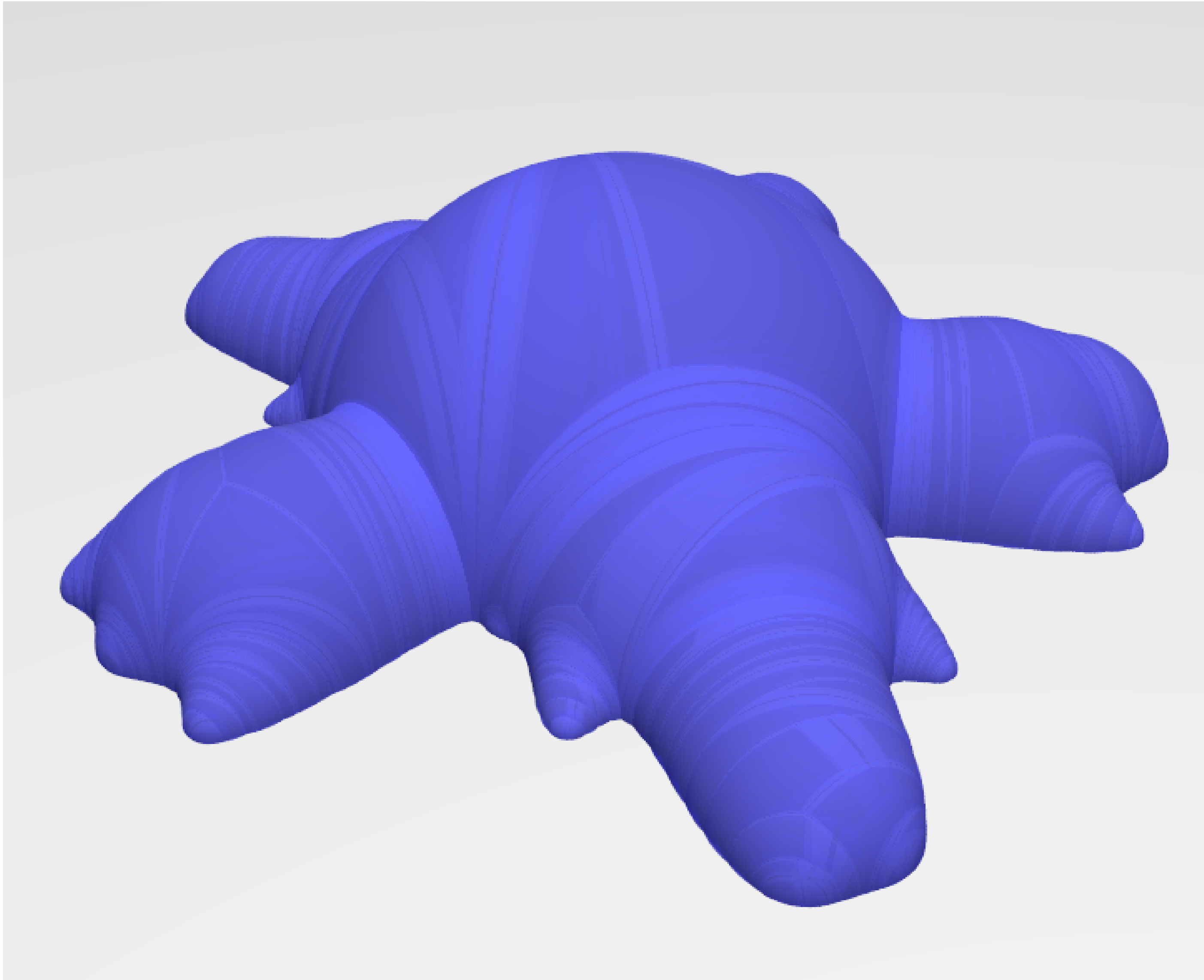
Corresponding hemispheres give the dome.

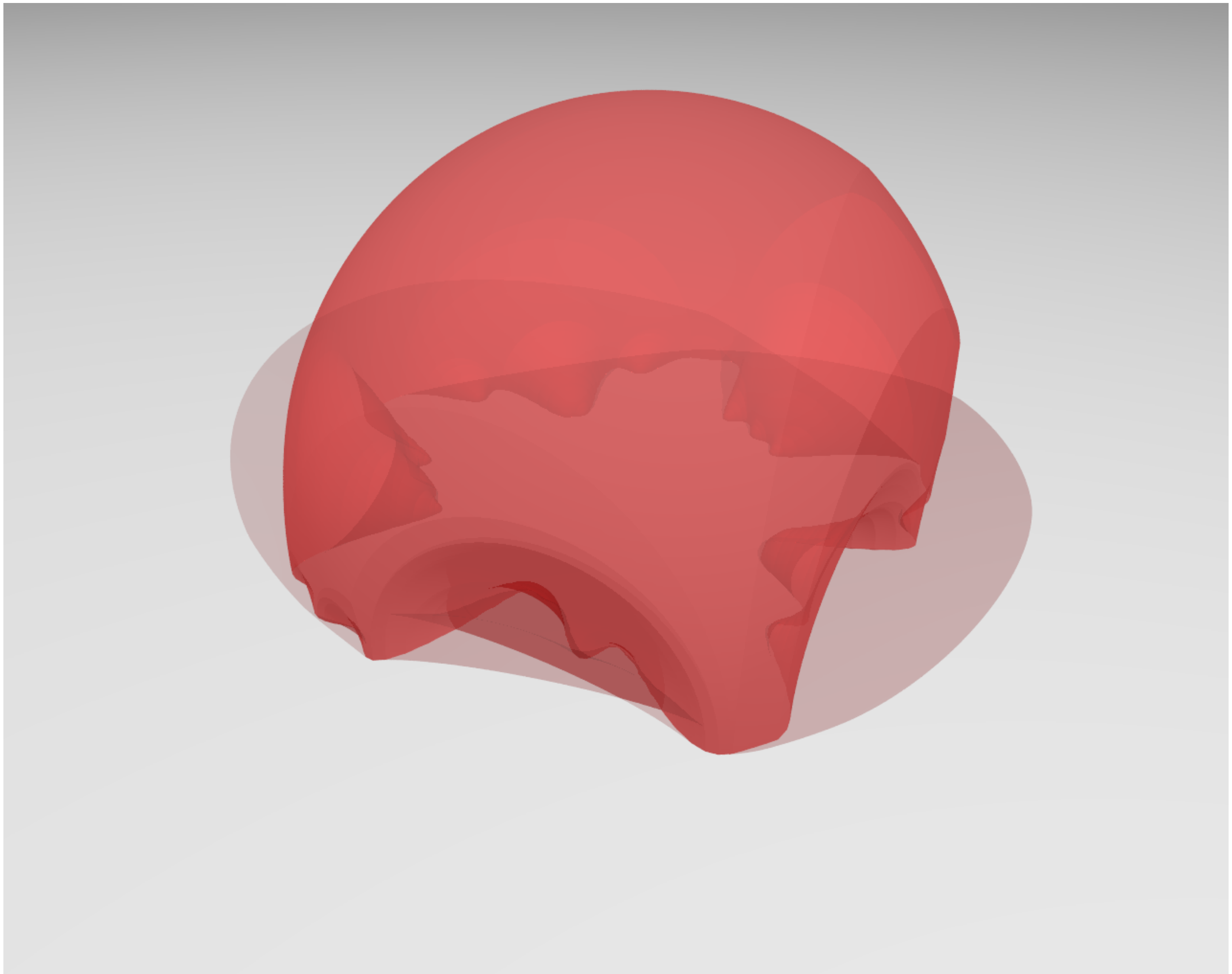


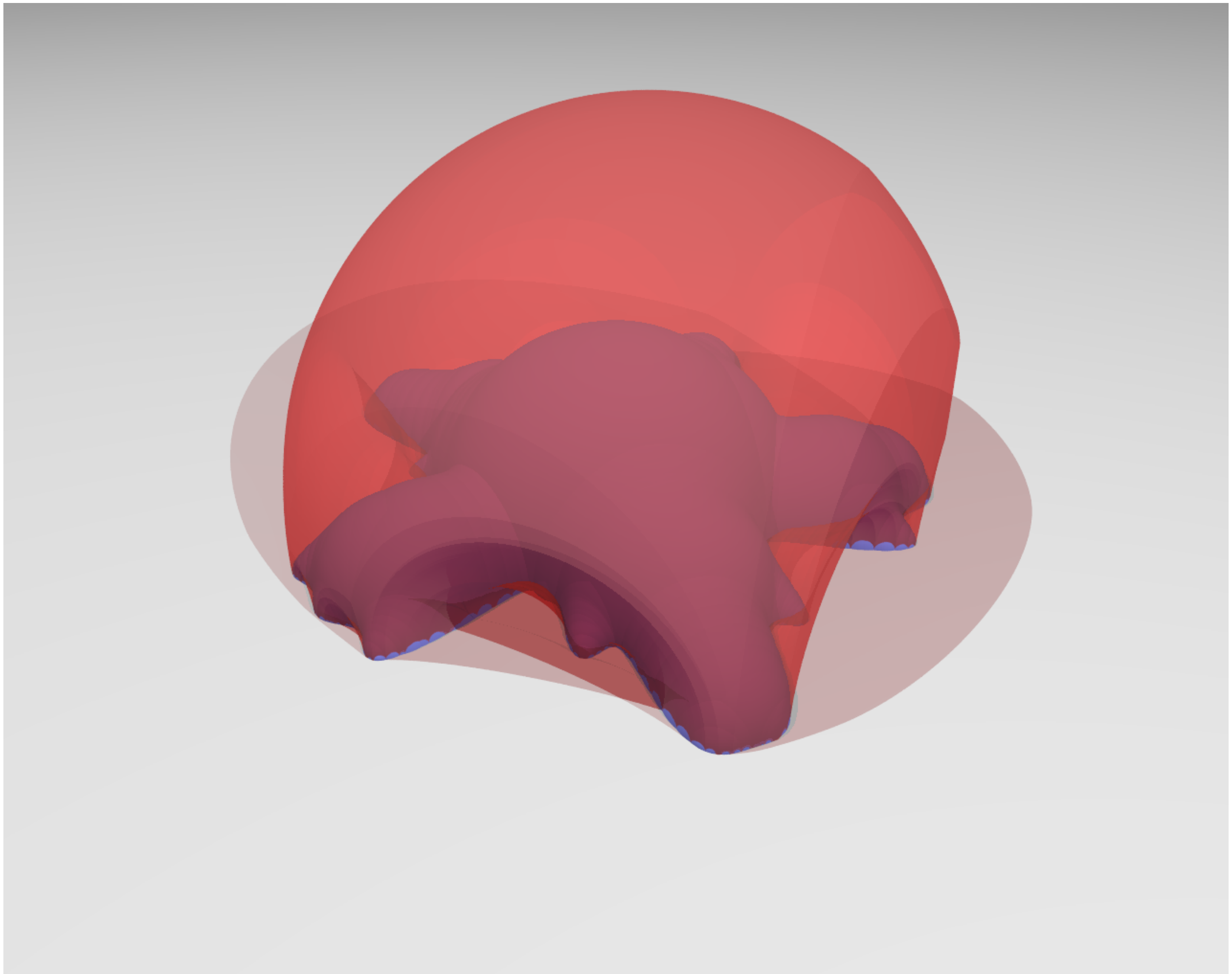




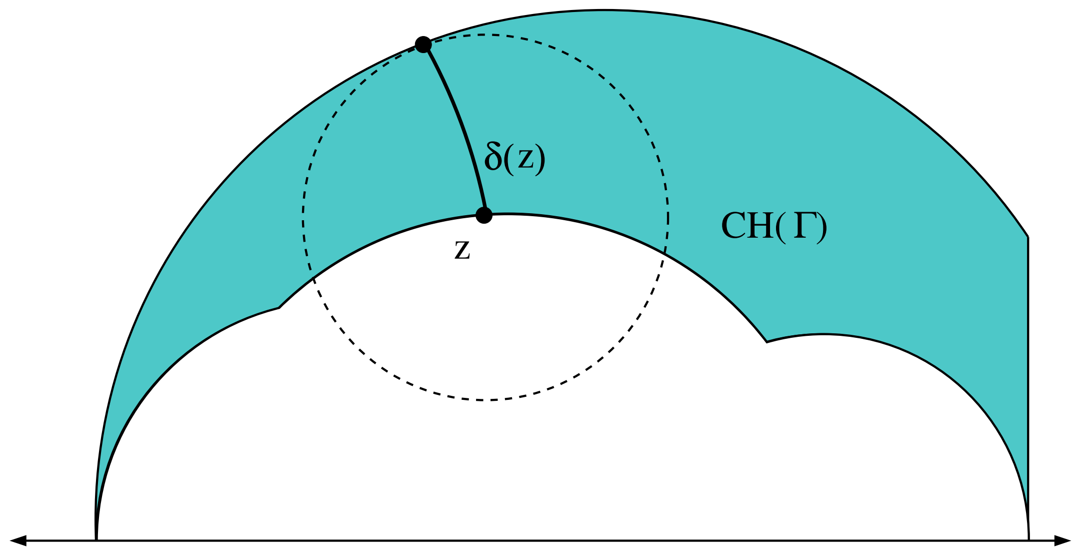
The medial axis. Equidistant from at least two boundary points.
Corresponding hemispheres give the dome.
Well studied in computational geometry. Fast to compute.







Let $\delta(z)$ be the maximum distance from $z$ to the components of $\partial \mathrm{CH}(\Gamma)$.




**Theorem:** $\Gamma$ is Weil-Petersson implies $\int_{\partial \mathrm{CH}(\Gamma)} \delta^2(z) dA_\rho < \infty$.

$\delta =$ "conformally invariant $\beta$"

$\partial \mathrm{CH}(\Gamma)$ has two components. Nicer to have single surface with $\partial S = \Gamma$.

Let $S$ be a surface in $\mathbb{H}^3$ that has asymptotic boundary $\Gamma$.

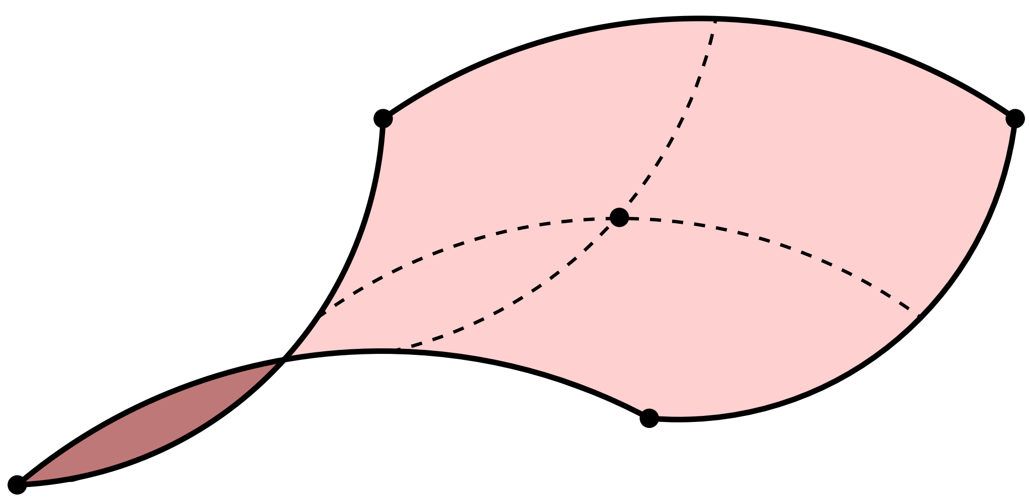
$K(z) =$ Gauss curvature of $S$ at $z$.

$\kappa_1, \kappa_2 =$ principle curvatures.

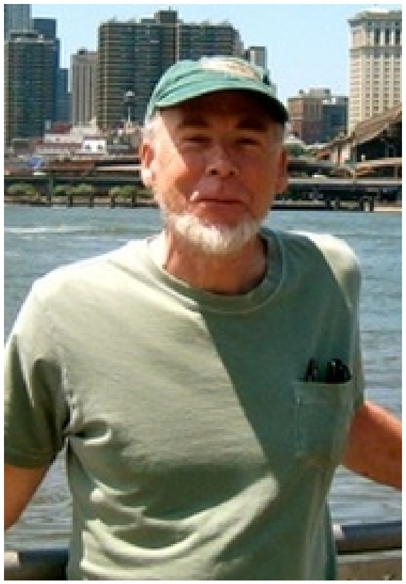
Gauss equation: $K(z) = -1 + \kappa_1(z)\kappa_2(z)$.

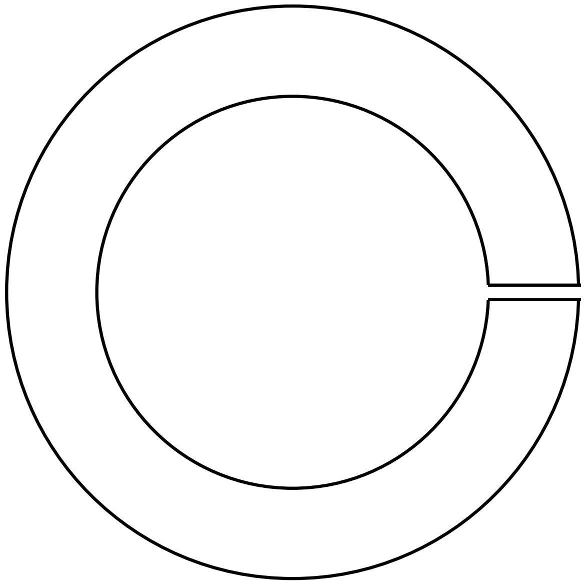
$S$ is a **minimal surface** if $\kappa_1 = -\kappa_2$ (the mean curvature is zero).

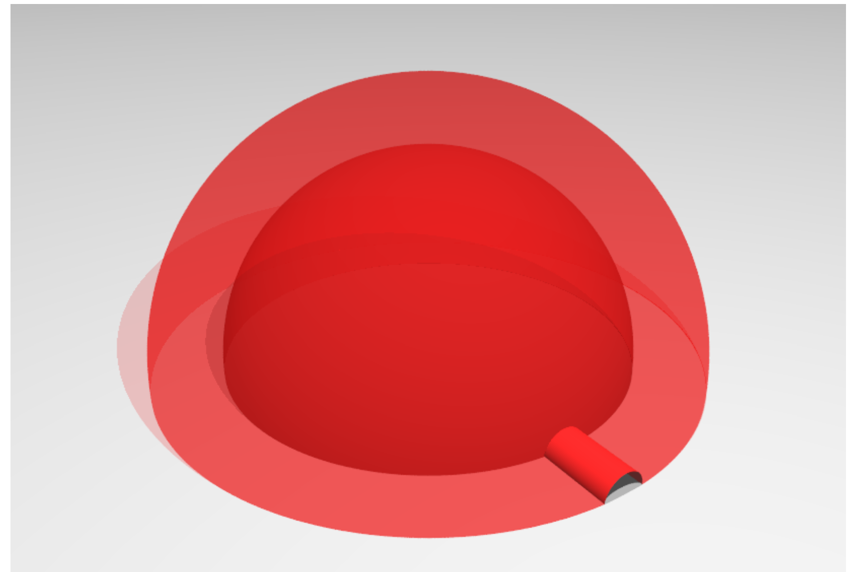
In that case, $K(z) = -1 - \kappa^2(z) \leq -1$.



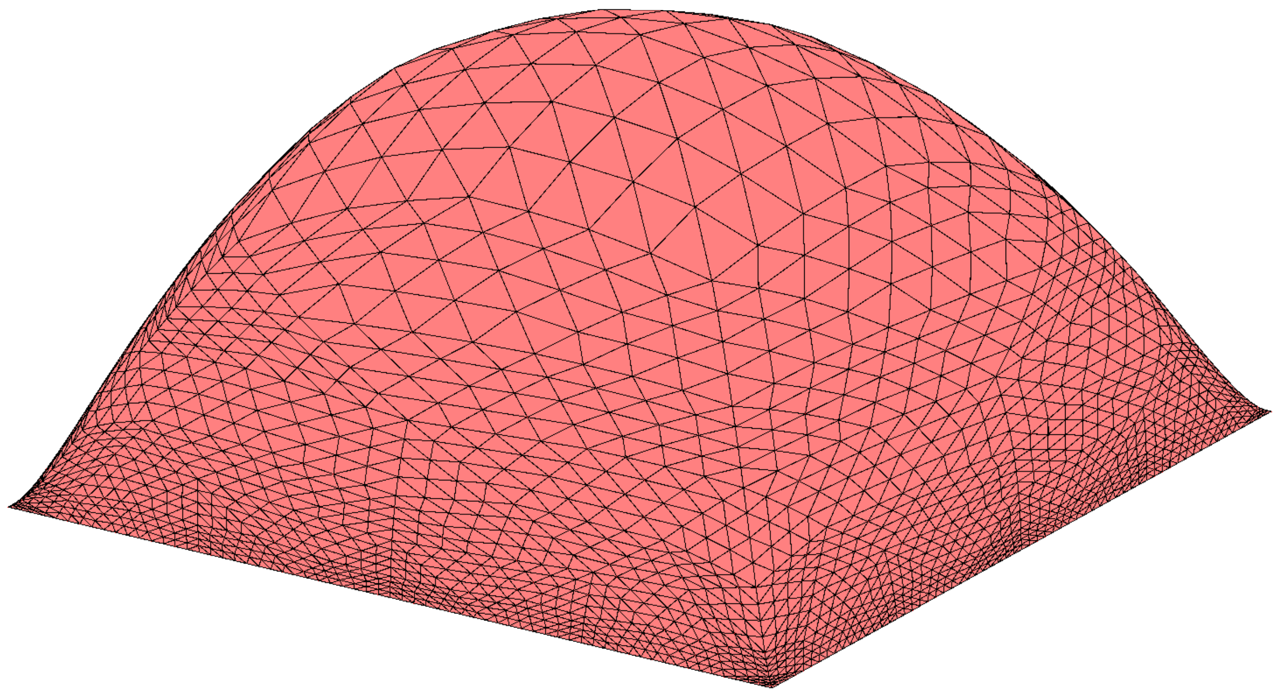
**Theorem (Anderson, 1983):** Every closed Jordan curve $\Gamma \subset \mathbb{R}^2$ bounds a minimal disk $S \subset \mathrm{CH}(\Gamma) \subset \mathbb{H}^3$.



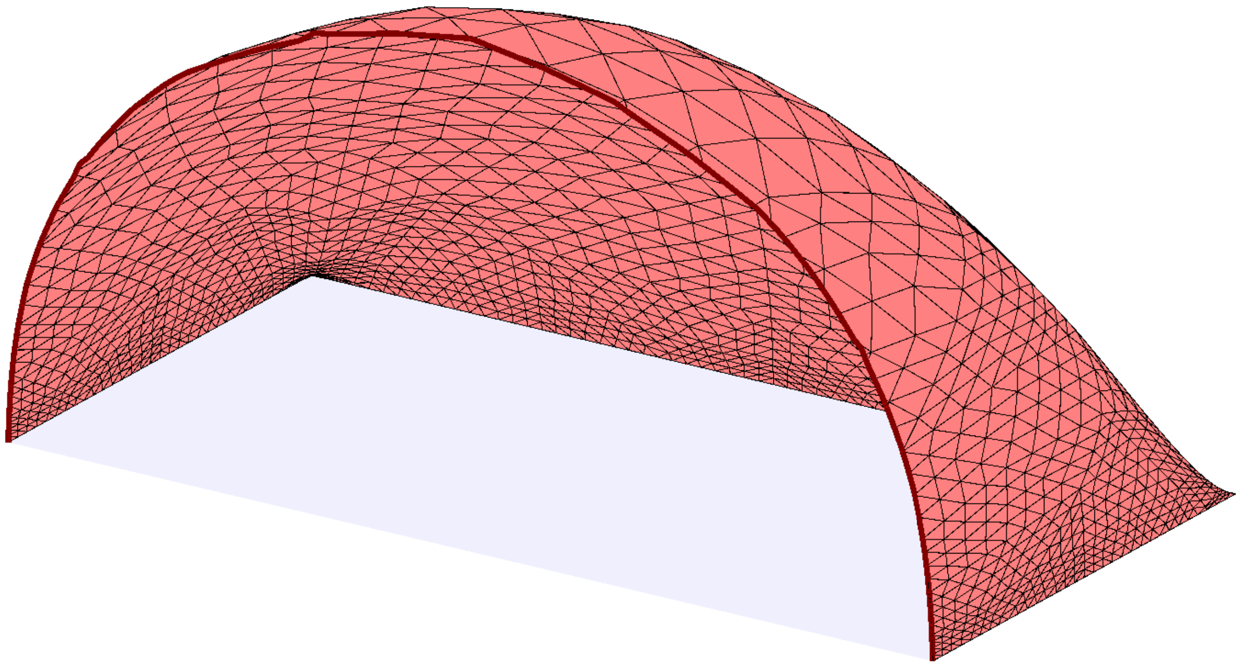




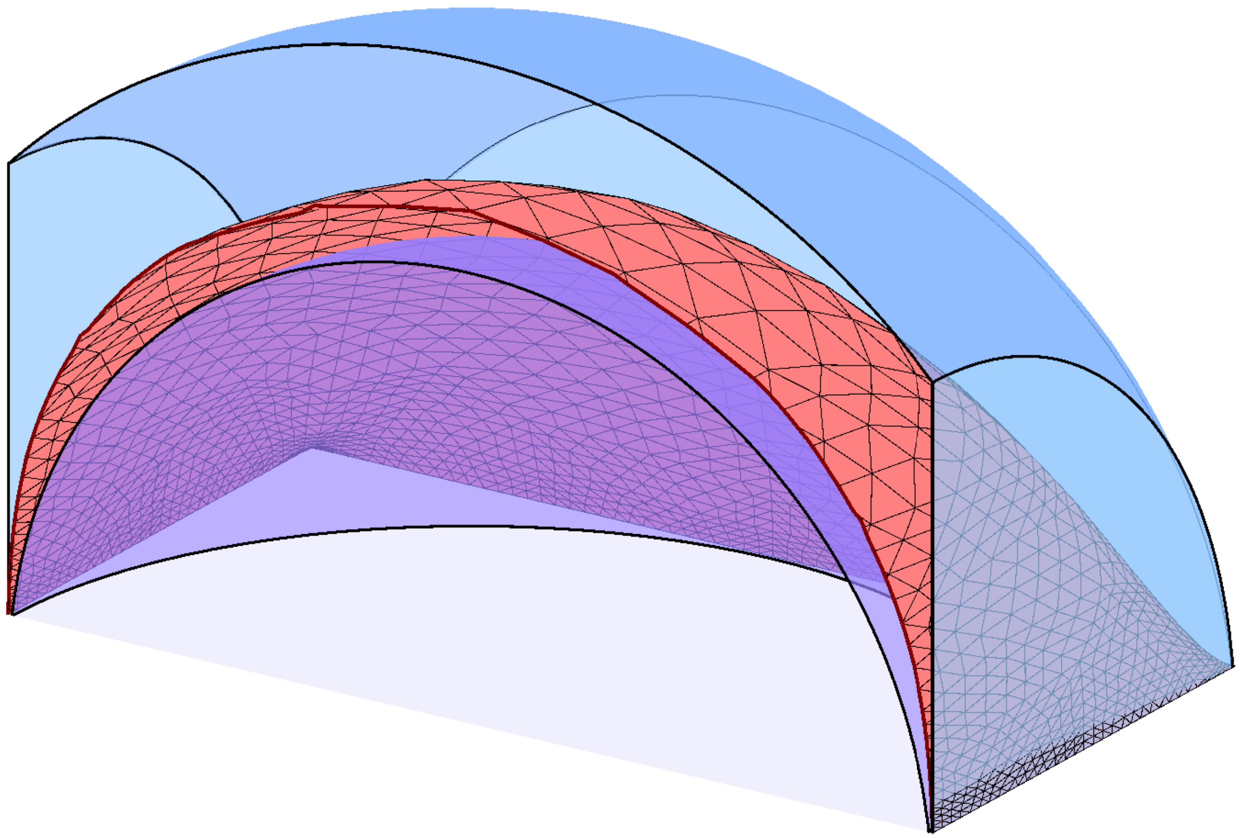
Minimal surface with boundary $\Gamma$ is contained in convex hull of $\Gamma$.



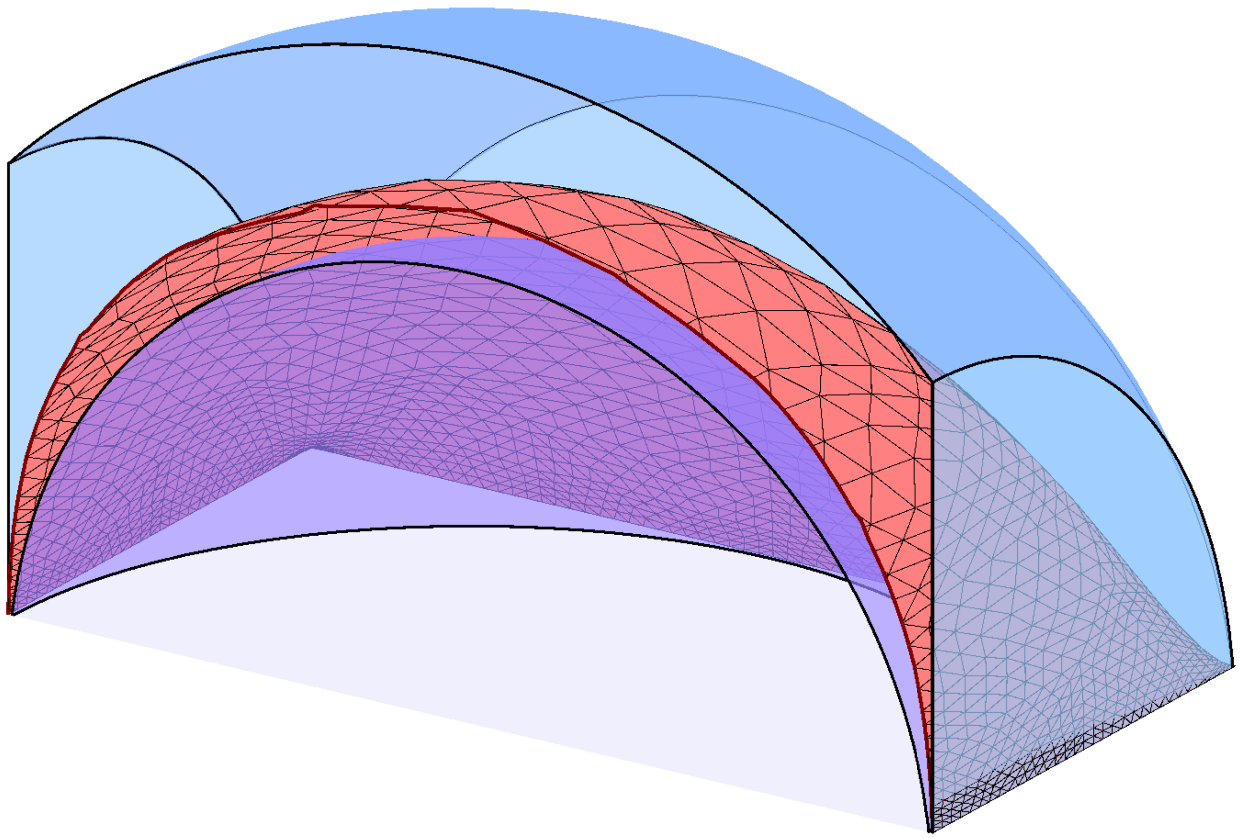
Hyperbolic minimal surface with boundary curve a square
Drawn with `Surface Evolver` by Kenneth Brakke



The surface cut in half.

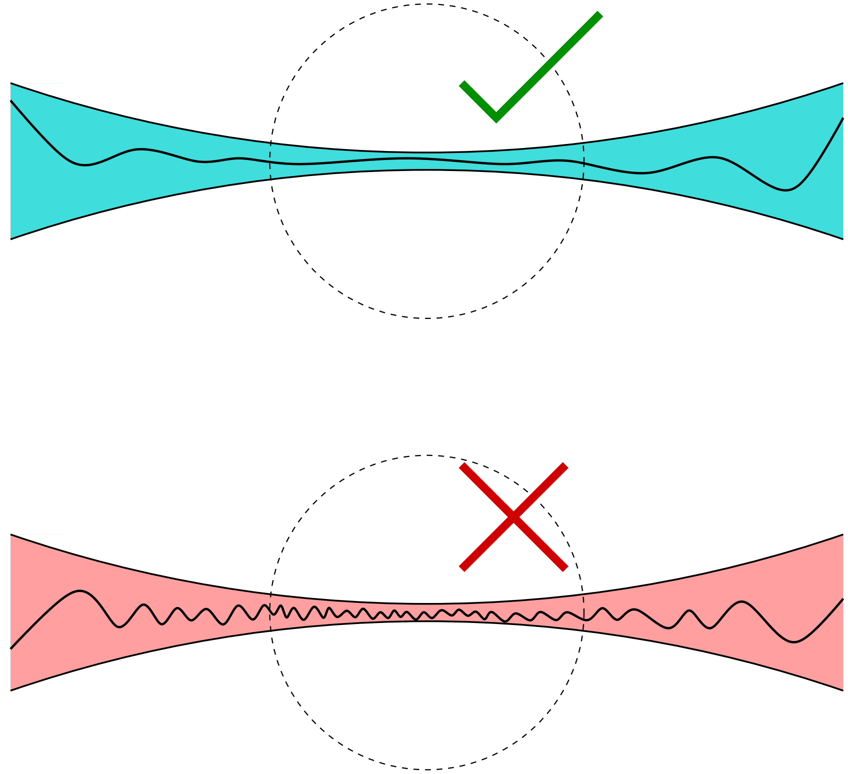


Minimal surface compared to convex hull boundaries.



Minimal surface should be “flat” if convex hull is “thin”.
“flat” = sectional curvatures are small.

**Theorem (Seppi, 2016):** Principle curvatures satisfies $\kappa(z) = O(\delta(z))$.



$$u(z) = \sinh(\mathrm{dist}(z, P)) \text{ satisfies } \Delta_S u - 2u = 0.$$

Use Schauder estimate $\|\nabla^2 u\|_\infty \leq C\|u\|_\infty = O(\delta)$.

Seppi's estimate + "$\int \delta^2 < \infty$" $\Rightarrow$

**Theorem:** If $\Gamma$ is WP then it bounds a minimal disk with

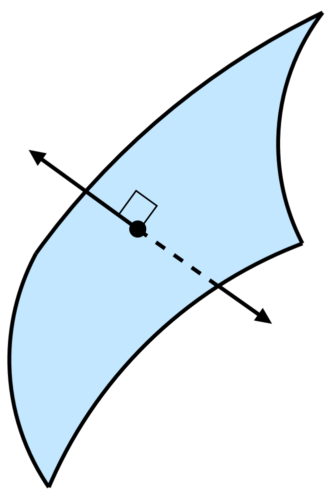
$$\int_S |K+1| dA_\rho = \int_S \kappa^2(z) dA_\rho < \infty.$$

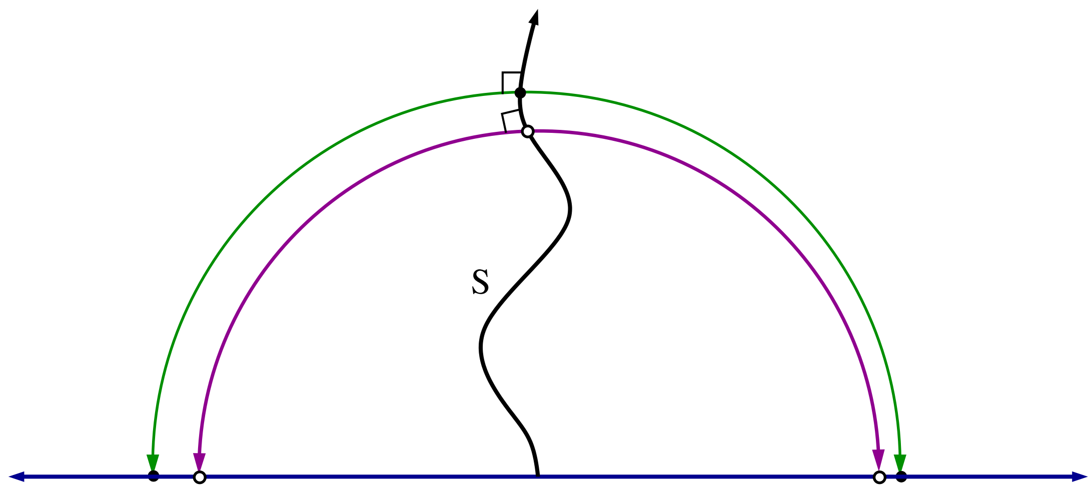
We say such a surface has **finite total curvature**.

**Cor:** Boundary of surface of finite total curvature need not be $C^1$.

**Gauss map:** follow normal geodesic from surface $S$ to $\mathbb{R}^2 = \partial\mathbb{H}^3$.

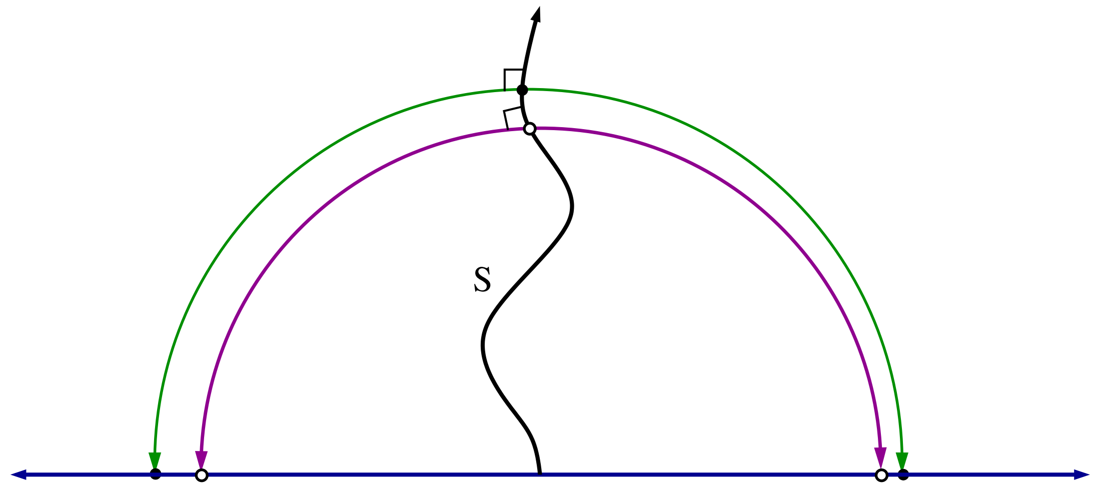
Two directions. Defines reflection across $\Gamma$.





**Gauss map:** follow normal geodesic from surface $S$ to $\mathbb{R}^2 = \partial\mathbb{H}^3$.

Two directions. Defines reflection across $\Gamma$.




**Theorem (C. Epstein, 1986):** If $|\kappa_1|, |\kappa_2| < 1$, then the Gauss maps define a quasiconformal reflection across $\Gamma$. Moreover, if $S$ has finite total curvature, then $\int_{\mathbb{C}\setminus\Gamma} |\mu|^2 d\mathrm{A}_\rho < \infty$.

$\Rightarrow$ $\Gamma$ is fixed by a QC involution with $\mu \in L^2(dA_\rho) \Rightarrow$ Weil-Petersson.

Weil-Petersson ($\mu$ in $L^2(dA_\rho)$)

$\Rightarrow \log f'$ in $W^{1,2}$

$\Rightarrow$ finite Möbius energy

$\Rightarrow$ parameterization in $H^{3/2}$

$\Rightarrow$ inscribed polygons

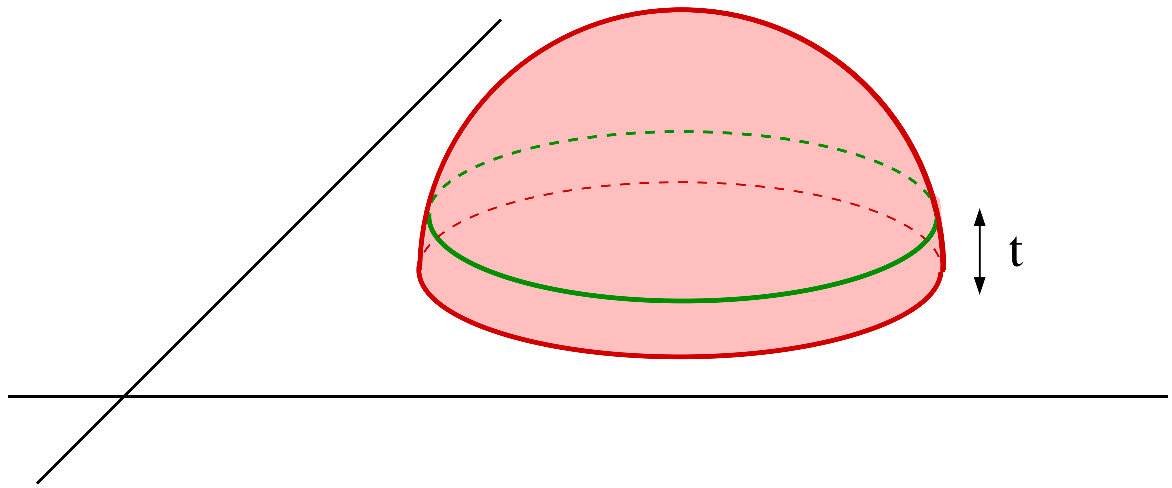
$\Rightarrow \sum \beta^2 < \infty$

$\Rightarrow \int_S \delta^2 dA_\rho < \infty$

$\Rightarrow \int_S \kappa^2 dA_\rho < \infty$

$\Rightarrow$ fixed by "nice" involution of $S^2$

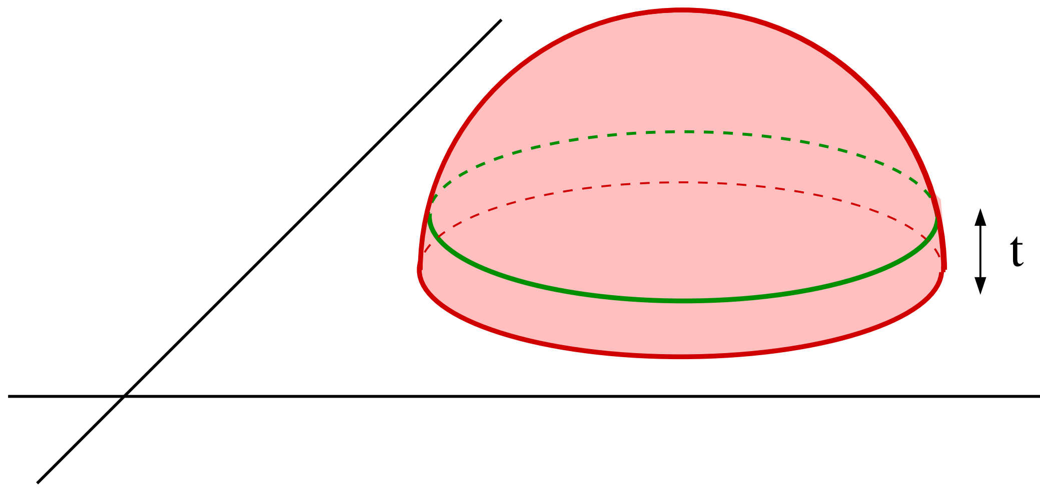
$\Rightarrow$ Weil-Petersson




Truncate $S \subset \mathbb{R}^3_+$ at a fixed height above the boundary, i.e.,

$$S_t = S \cap \{(x, y, s) \in \mathbb{R}^3_+ : s > t\},$$

Boundary length $\ell(\partial S_t)$ and interior area $A(S_t)$ both grow to $\infty$.



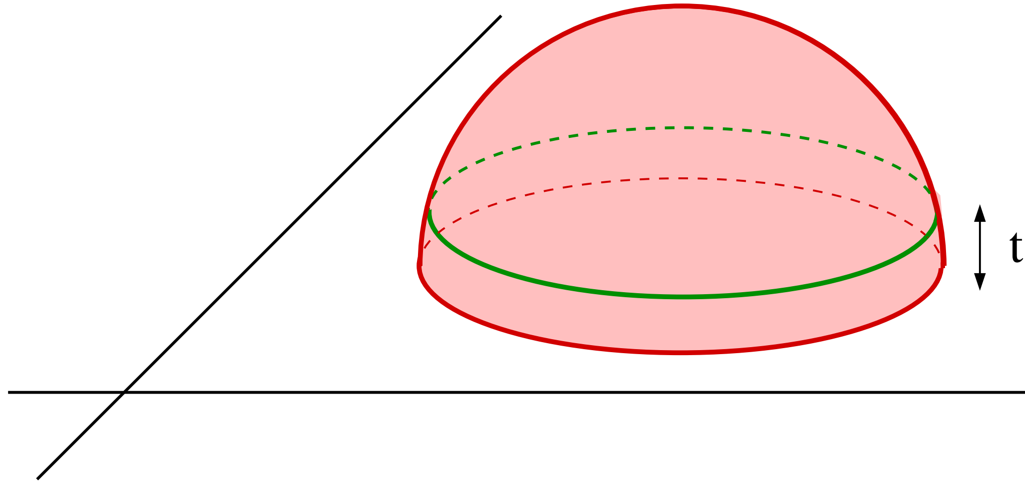

Truncate $S \subset \mathbb{R}^3_+$ at a fixed height above the boundary, i.e.,

$$S_t = S \cap \{(x, y, s) \in \mathbb{R}^3_+ : s > t\},$$

**Isoperimetric inequality:** if $K(z) \leq -1$, then

$$\ell(\partial S_t) \geq A(S_t) + 4\pi\chi(S_t).$$

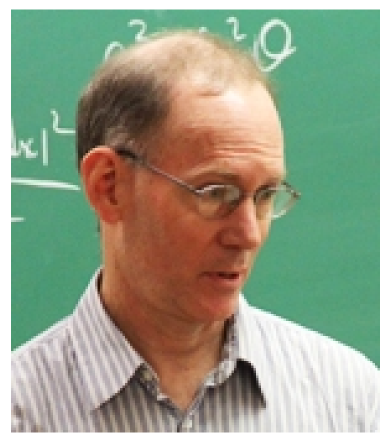
Does the gap $\ell(\partial S_t) - A(S_t)$ stay bounded or grow to $\infty$?




**Renormalized area**: $\mathcal{A}_R(S) = \lim_{t \searrow 0} \left[ \mathrm{A}_\rho(S_t) - \ell_\rho(\partial S_t) \right].$

Graham and Witten proved well defined.

Related to quantum entanglement,
AdS/CFT correspondence.



**Theorem:** For any closed curve $\Gamma \subset \mathbb{R}^2$ and for any minimal surface $S \subset \mathbb{R}^3_+$ with finite Euler characteristic and asymptotic boundary $\Gamma$,

$$\mathcal{A}_R(S) = -2\pi\chi(S) - \int_S \kappa^2(z) d\mathrm{A}_\rho.$$

In particular, $\mathcal{A}_R(S) < \infty$ iff $\Gamma$ is Weil-Petersson.

- One direction uses Seppi's estimate and Gauss-Bonnet theorem.
- Converse uses isoperimetric inequalities on negatively curved surfaces.

**Theorem:** For any closed curve $\Gamma \subset \mathbb{R}^2$ and for any minimal surface $S \subset \mathbb{R}^3_+$ with finite Euler characteristic and asymptotic boundary $\Gamma$,

$$\mathcal{A}_R(S) = -2\pi\chi(S) - \int_S \kappa^2(z)d\mathrm{A}_\rho.$$

Formula is due to Alexakis and Mazzeo (2010) when $\Gamma$ is $C^{3,\alpha}$.

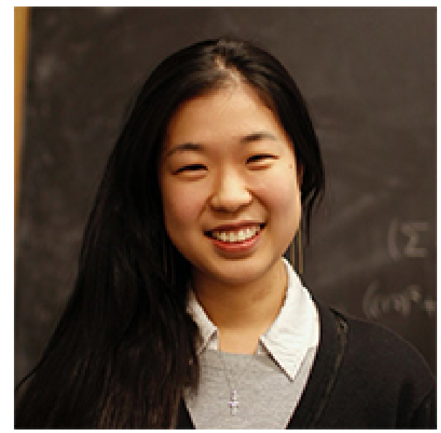
Their work valid in Poincaré-Einstein manifolds.

Is there a "holographic principle": 2-D quantity = 3-D quantity?

April 2023 preprint of Bridgeman-Bromberg-Pallete-Wang relates Loewner energy of $\Gamma$ to volume between Epstein-Poincaré surfaces associated to $\Gamma$.

EP surfaces are similar to convex hull boundaries, but defined using horoballs instead of hemispheres.



They give explicit formula for $C^{5,\alpha}$ curves. True in general?

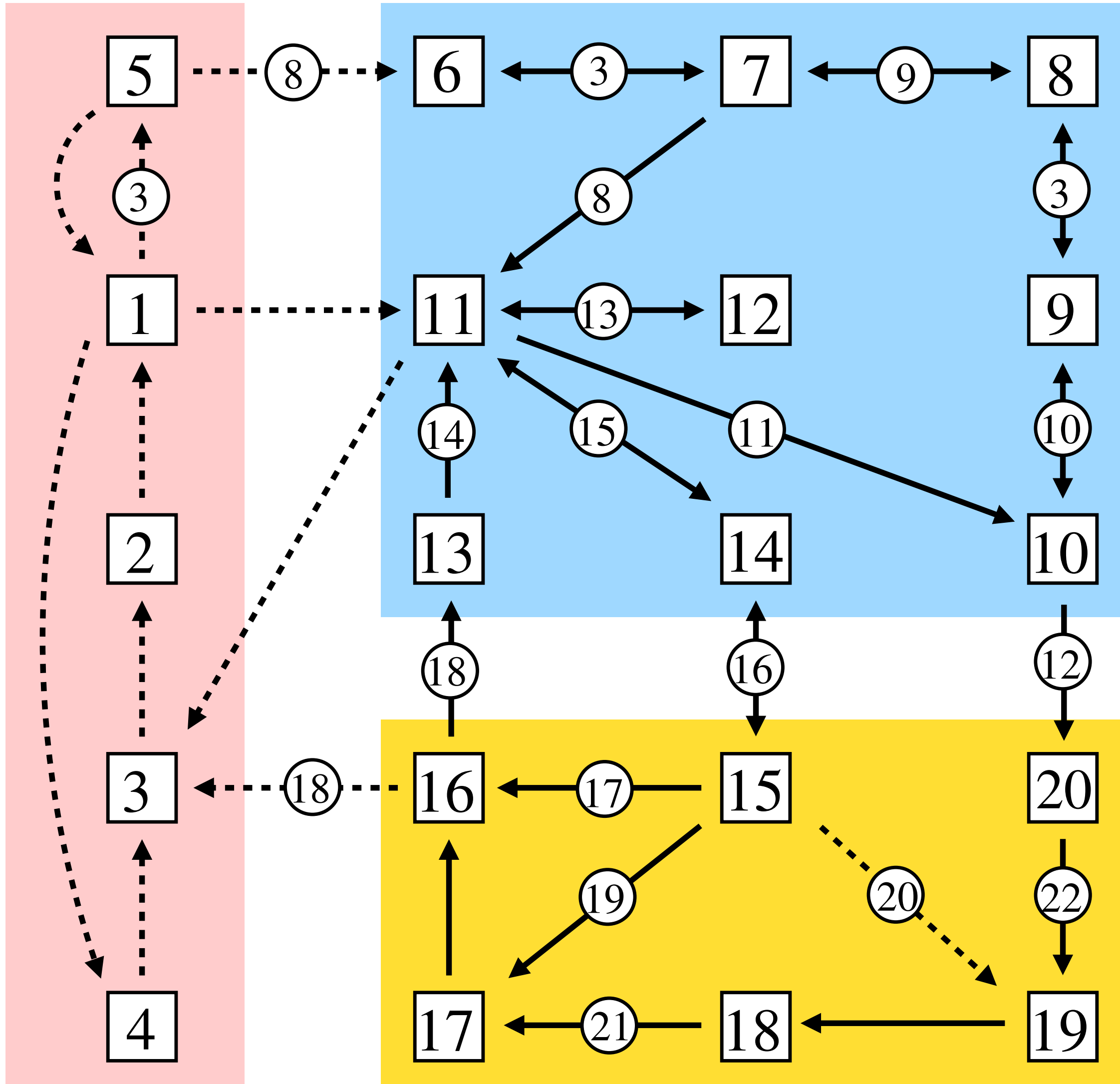
| Definition | Description |
|---|---|
| 1 | $\log f'$ in Dirichlet class |
| 2 | Schwarzian derivative |
| 3 | QC dilatation in $L^2$ |
| 4 | conformal welding midpoints |
| 5 | $\exp(i \log f')$ in $H^{1/2}$ |
| 6 | arclength parameterization in $H^{3/2}$ |
| 7 | tangents in $H^{1/2}$ |
| 8 | finite Möbius energy |
| 9 | Jones conjecture |
| 10 | good polygonal approximations |
| 11 | $\beta^2$-sum is finite |
| 12 | Menger curvature |
| 13 | biLipschitz involutions |
| 14 | between disjoint disks |
| 15 | thickness of convex hull |
| 16 | finite total curvature surface |
| 17 | minimal surface of finite curvature |
| 18 | additive isoperimetric bound |
| 19 | finite renormalized area |
| 20 | dyadic cylinder |

Weil-Petersson curves

André Weil

Hans Petersson

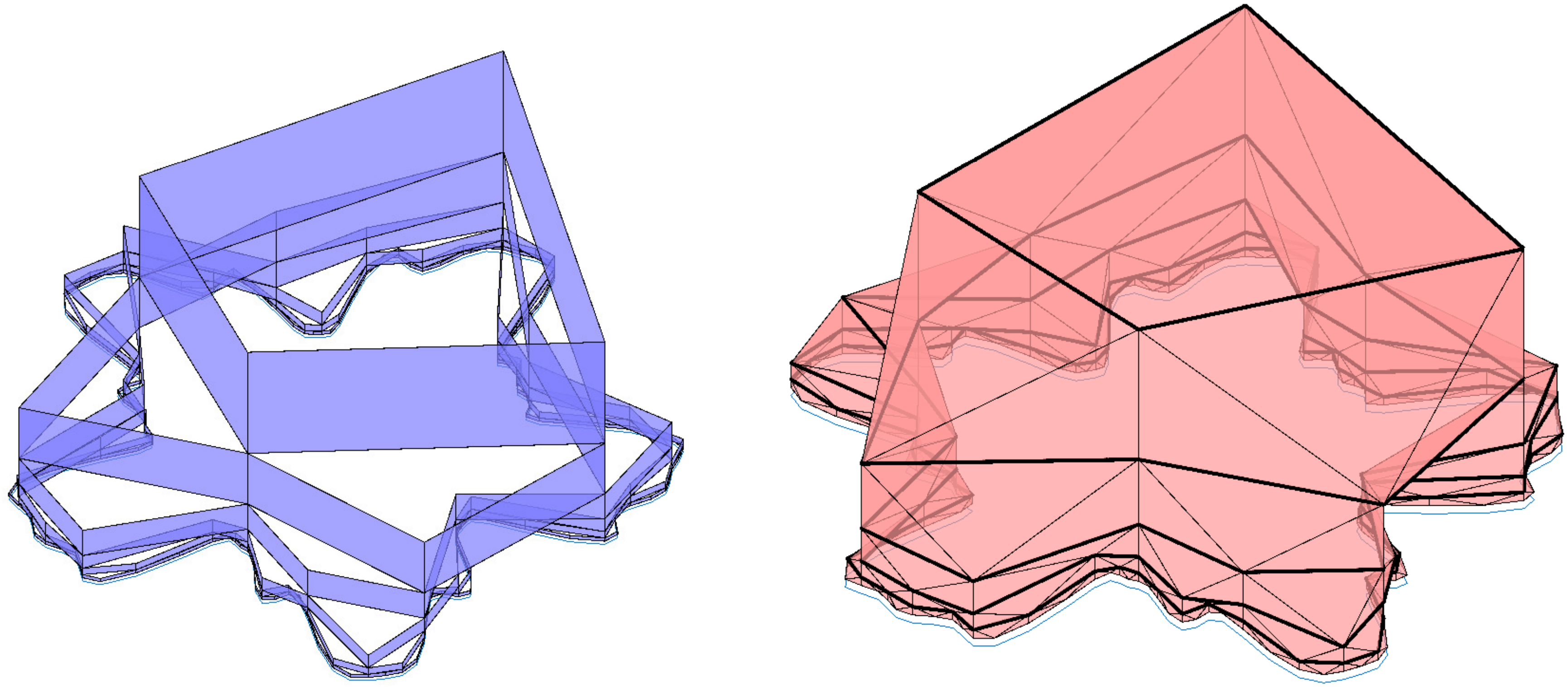
# THE THEOREM




# THANKS FOR LISTENING. QUESTIONS?

Congratulations to Nick for many wonderful results!

A Complex Faces Game



The dyadic dome.
Polyhedral approximation to minimal surface.
Intermediary between Euclidean and hyperbolic regimes.

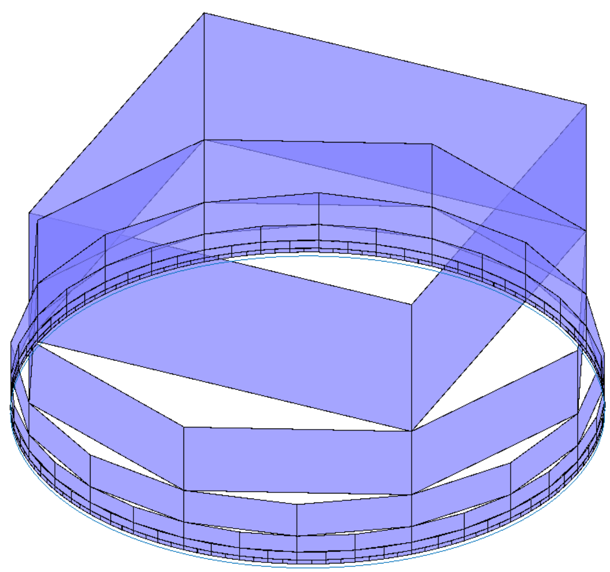
An idea connecting Euclidean and hyperbolic results.

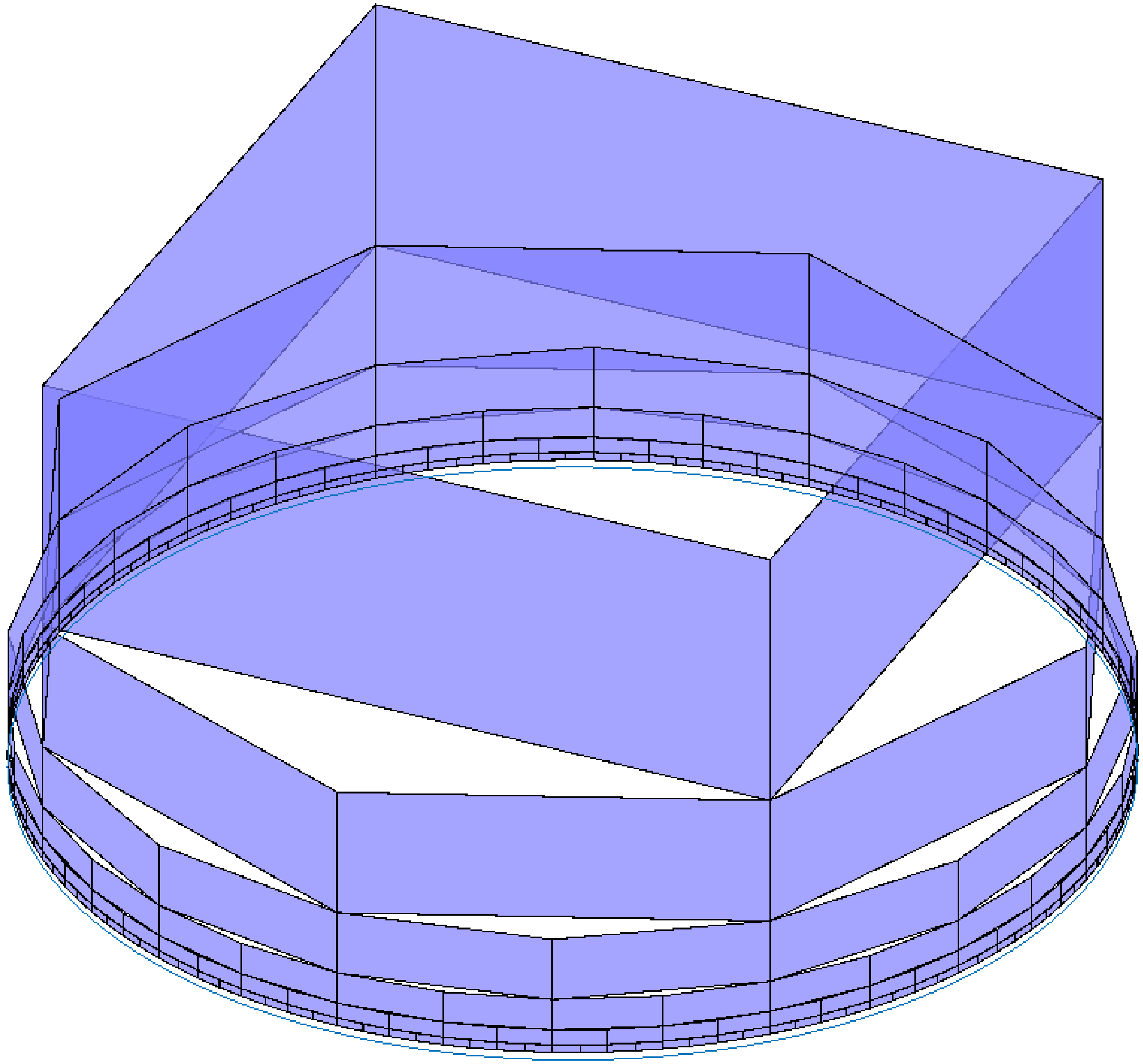
Define a dyadic cylinder in the upper half-space:

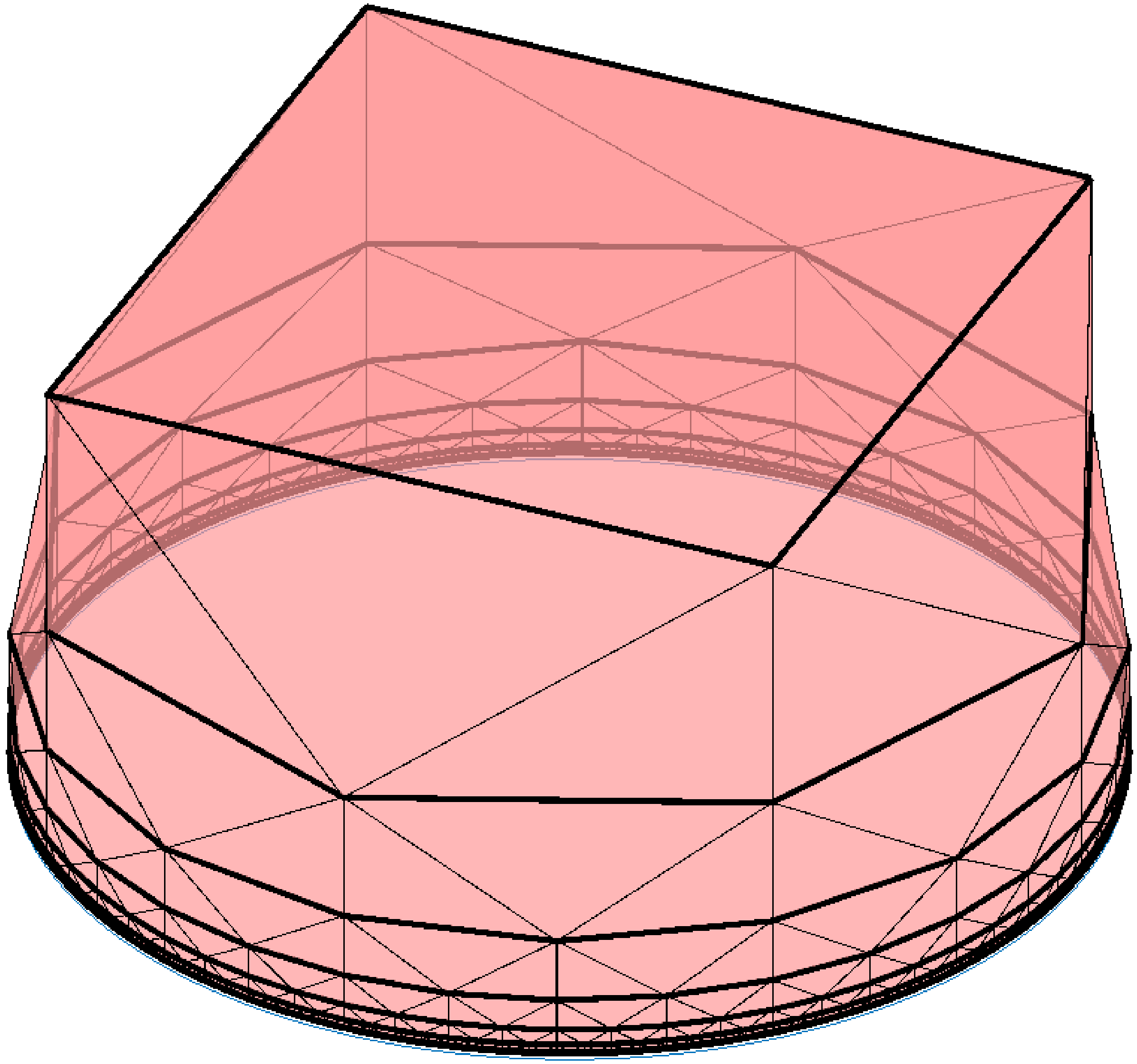
$$X = \bigcup_{n=1}^{\infty} \Gamma_n \times [2^{-n}, 2^{-n+1}),$$

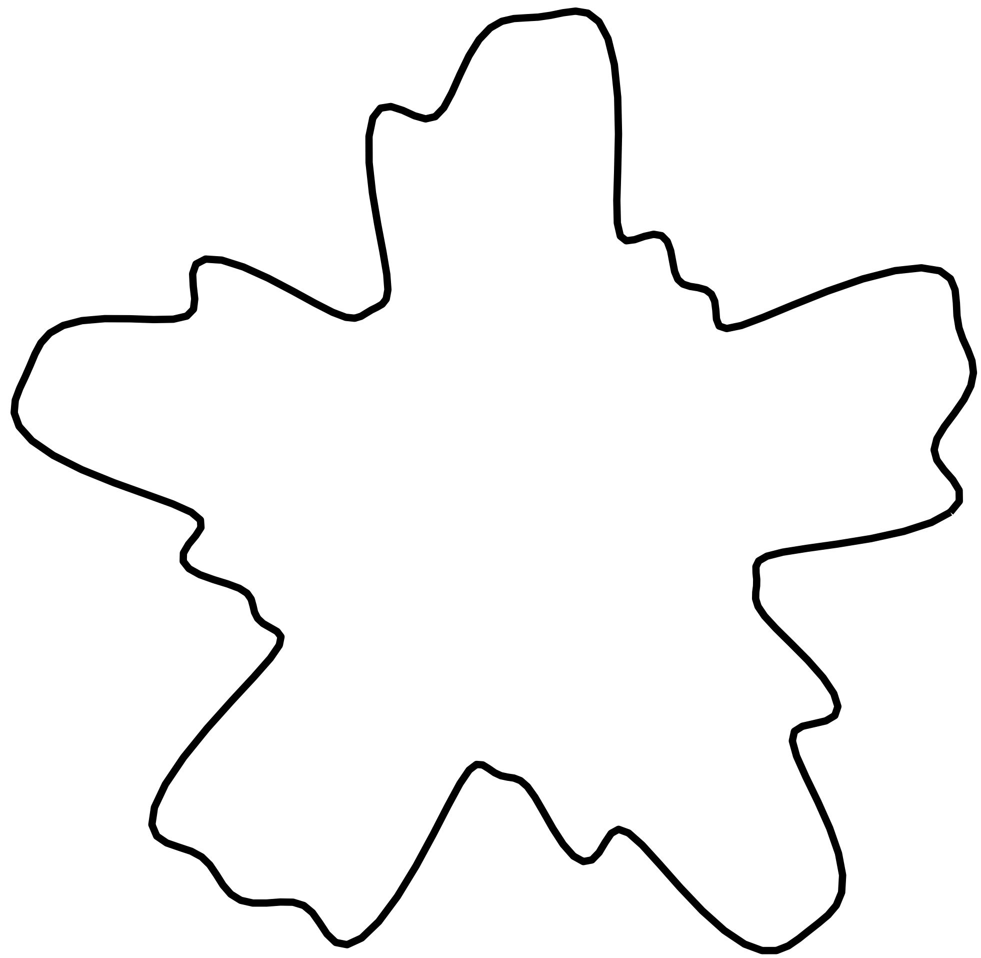
where $\{\Gamma_n\}$ are inscribed dyadic polygons in $\Gamma$.

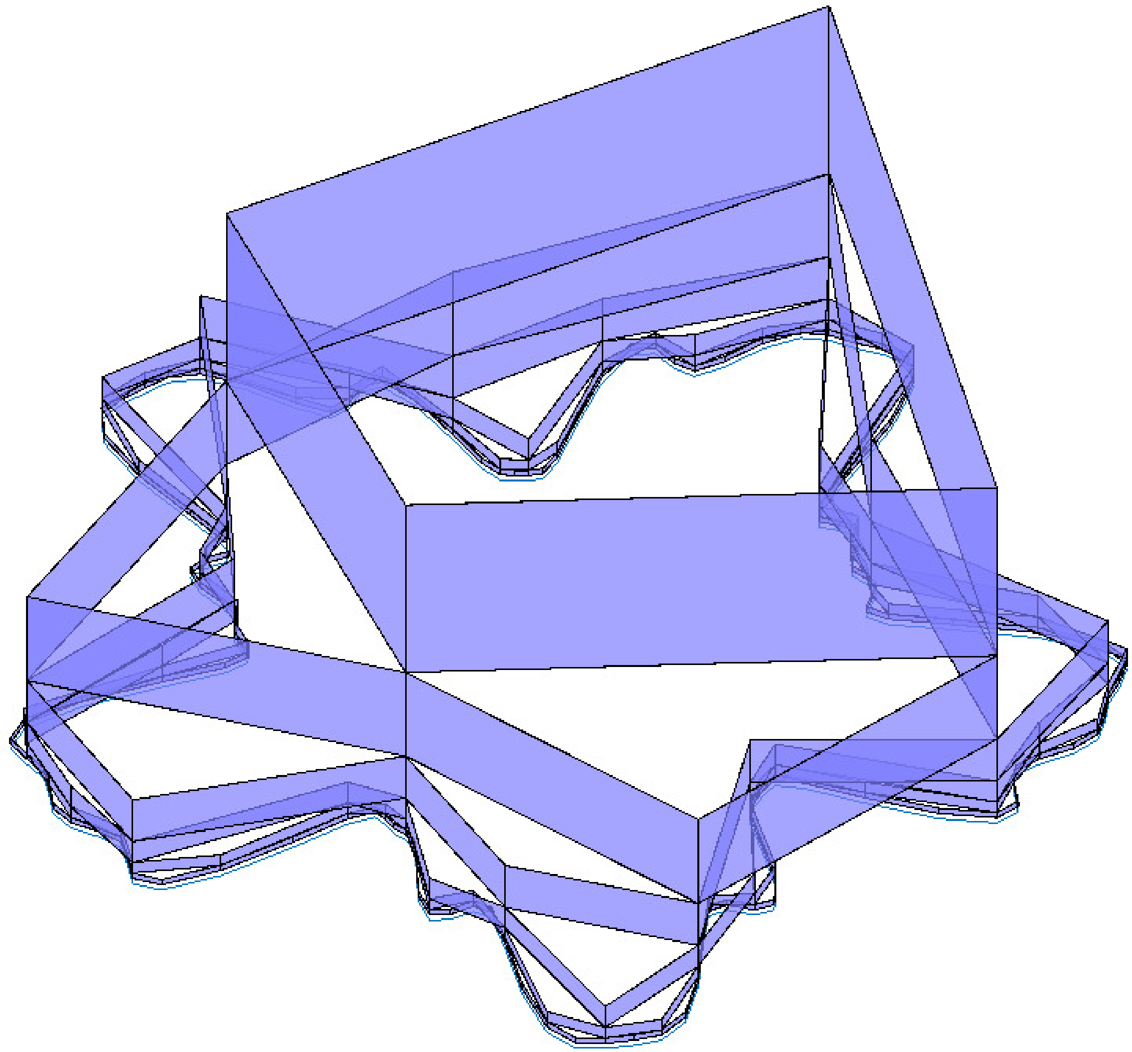
Discrete analog of minimal surface with boundary $\Gamma$.

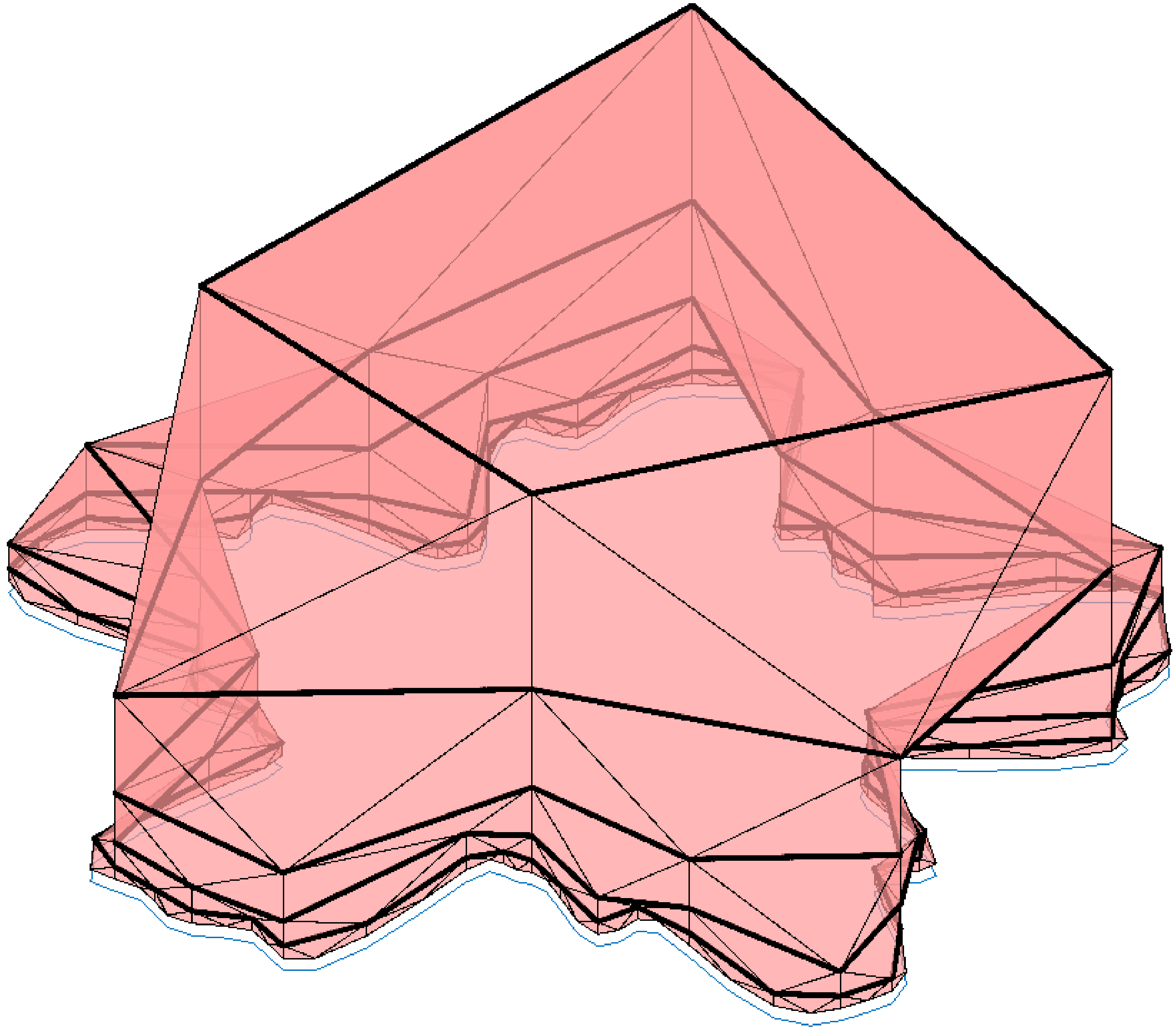












Our earlier estimate

$$\sum_n 2^n(\ell(\Gamma) - \ell(\Gamma_n)) < \infty$$

is equivalent to the dyadic cylinder having finite renormalized area.

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$$\sum_n 2^n(\ell(\Gamma) - \ell(\Gamma_n)) < \infty$$

is equivalent to the dyadic cylinder having finite renormalized area.

Obvious "normal projection" from the dyadic cylinder to minimal surface, distorts length and area each by a bounded additive error.

We can deduce finite renormalized area for the minimal surface from the same result for the dyadic cylinder.