PACIFIC JOURNAL OF MATHEMATICS
Vol. 99, No. 2, 1982

BMO FROM DYADIC BMO

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We give new proofs of four decomposition theorems for functions of bounded mean oscillation by first obtaining each theorem in the easier dyadic case and then averaging the results of the dyadic decomposition over translations in R_m.

1. Introduction. Let φ be a locally integrable real function on R^m, let Q be a bounded cube in R^m, with sides parallel to the axes, and let $|Q|$ be the Lebesgue measure of Q. Then

$$\varphi_Q = \frac{1}{|Q|}\int_Q \varphi dx$$

is the average of φ over Q. We say φ has *bounded mean oscillation*, $\varphi \in \mathrm{BMO}$, if

$$\|\varphi\| = \sup_Q \frac{1}{|Q|}\int_Q |\varphi - \varphi_Q| dx < \infty .$$

A *dyadic cube* is a cube of the special form

$$Q = \{k_j 2^{-n} < x_j < (k_j + 1)2^{-n};\ 1 \leqq j \leqq m\}$$

where n and k_j, $1 \leqq j \leqq m$, are integers, and φ has *bounded dyadic mean oscillation*, $\varphi \in \mathrm{BMO}_d$, if

$$\|\varphi\|_d = \sup_{Q \text{ dyadic}} \frac{1}{|Q|}\int_Q |\varphi - \varphi_Q| dx < \infty .$$

Then clearly $\mathrm{BMO} \subset \mathrm{BMO}_d$ with $\|\varphi\|_d \leqq \|\varphi\|$, but BMO and BMO_d are not the same space; the function $\log|x_j|\chi_{\{x_j>0\}}$ is in BMO_d but not in BMO. In analysis BMO is more important than BMO_d because BMO is translation invariant, but BMO_d is not. On the other hand, BMO_d is very much the easier space to work with because dyadic cubes are nested (if two open daydic cubes intersect then one of them is contained in the other). For example, for BMO the original proofs [1], [6], [8], [11] of the four theorems stated below were rather technical, while for BMO_d the analogous results are comparatively trivial. In this paper we derive the four theorems from their dyadic counterparts.

Here is the idea. Let $T_\alpha\varphi(x) = \varphi(x - \alpha)$. Then

$$\varphi(x) = \lim_{N\to\infty} \frac{1}{(2N)^m}\int_{|\alpha_j| \leqq N} T_\alpha \varphi(x + \alpha) d\alpha .$$

Each of the theorems amounts to showing $\varphi \in \mathrm{BMO}$ has the form $\varphi = F_1 + F_2$ where F_1 and F_2 are BMO functions satisfying certain additional growth conditions. By the BMO_d result we have

$$T_\alpha \varphi = F_1^{(\alpha)} + F_2^{(\alpha)}$$

where $F_1^{(\alpha)}, F_2^{(\alpha)} \in \mathrm{BMO}_d$ satisfy the extra growth conditions on dyadic cubes. To prove each theorem we show the averages

$$F_j(x) = \lim_{N\to\infty} \frac{1}{(2N)^m} \int_{|\alpha_j| \leq N} F_j^{(\alpha)}(x+\alpha)d\alpha$$

are in BMO and have the correct growth. The method yields this general result.

THEOREM. *Suppose that* $\alpha \to \varphi^{(\alpha)}$ *is a measurable mapping from* $\boldsymbol{R}^m$ *to* BMO_d *such that all* $\varphi^{(\alpha)}(x)$ *have support a fixed dyadic cube, such that* $\|\varphi^{(\alpha)}\|_d \leqq 1$ *and such that*

$$\int \varphi^{(\alpha)}(x)dx = 0 .$$

Then

$$\varphi_N(x) = \frac{1}{(2N)^m} \int_{|\alpha_j| \leq N} \varphi^{(\alpha)}(x+\alpha)d\alpha$$

is in BMO *and* $\|\varphi_N\| \leqq C$.

By duality, this theorem implies Davis's result connecting H^1 and H^1_{dyadic} on the unit circle. The proof of theorem is implicit in the arguments below. In §4 we show

$$\varphi_N = g + \sum_{n=1}^{\infty} f_n$$

where $g \in L^\infty$ and where $f_n(x)$ satisfies the Lipschitz condition (3.3) and the thinness condition (4.2). From these $\|\varphi_N\| \leqq C$ follows easily. This general result is not explicitly used in the proofs of Theorem 1 to 4.

Let $\ell(Q)$ denote the sidelength of the cube Q. A *Carleson measure* is a signed measure on the upper half space $\boldsymbol{R}_+^{m+1} = \boldsymbol{R}^m \times (0, \infty)$ such that for some constant $N(\sigma)$,

$$|\sigma|(Q \times (0, \ell(Q)]) \leqq N(\sigma)|Q|$$

for all cubes $Q \subset \boldsymbol{R}^m$. Here $|\sigma|$ is the total variation of σ. Let $K(x)$ be a positive function for which

$$K(x) = O((1+|x|)^{-m-1}) \tag{1.1}$$

and

$$\int K(x)dx = 1 .$$

Write $\check{K}_y(x) = y^{-m}K(x/y)$, $y > 0$.

THEOREM 1 (Carleson [1]). *If $\varphi \in \mathrm{BMO}$ has compact support, then there is $g \in L^\infty$ and there is a Carleson measure σ such that*

$$\varphi(x) = g(x) + \int_{\boldsymbol{R}_+^{m+1}} K_y(x - t)d\sigma(t, y) , \tag{1.2}$$

where

$$\| g \|_\infty \leqq C \| \varphi \|$$

and

$$N(\sigma) \leqq C \| \varphi \| ,$$

where the constant C depends only on $K(x)$.

Theorem 1 implies Fefferman's Theorem [5] that $H^1(\boldsymbol{R}^m)$ has dual space BMO. Under the additional hypotheses

$$|\nabla K(x)| = O((1 + |x|)^{-m-1}) ,$$

the converse of Theorem 1 is true (and not difficult). It then follows that $H^1(\boldsymbol{R}^m) = \{f \in L^1 : f_K^* \in L^1\}$ where f_K^* is the maximal function $\sup_{|t-x|<y} |f * K^y(t)|$. See [5].

By the theorem of John and Nirenberg [7], $\varphi \in \mathrm{BMO}$ if and only if there is $A > 0$ such that

$$\sup_Q \frac{1}{|Q|} \int_Q e^{A|\varphi - \varphi_Q|} dx < \infty . \tag{1.3}$$

In fact, (1.3) holds with $A = c\| \varphi \|^{-1}$, c depending only on the dimension. Set

$$A(\varphi) = \sup\{A: (1.3) \text{ holds}\} .$$

THEOREM 2 ([6]). *There are constants $c_1(m)$ and $c_2(m)$ such that if $\varphi \in \mathrm{BMO}$ then*

$$\frac{c_1(m)}{A(\varphi)} \leqq \inf_{g \in L^\infty} \| \varphi - g \| \leqq \frac{c_2(m)}{A(\varphi)} .$$

The left inequality is immediate since $A(\varphi - g) \geqq c\| \varphi - g \|^{-1}$, $g \in L^\infty$. We prove the other inequality.

Let $w(x) > 0$ be a locally integrable function on $\boldsymbol{R}^m$, and let $1 < p < \infty$. We say $w \in A_p$ if

$$\| w \|_{A_p} = \sup_Q \left(\frac{1}{|Q|}\int_Q w dx\right)\left(\frac{1}{|Q|}\int_Q \left(\frac{1}{w}\right)^{1/(p-1)} dx\right)^{p-1} < \infty \,;$$

The Riesz transforms and the Hardy-Littlewood maximal functions are bounded on $L^p(wdx)$ if and only if $w \in A_p$ [2]. As $p \to 1$ the limiting form of A_p is

$$\| w \|_{A_1} = \sup_Q \left(\frac{1}{|Q|}\int_Q w dx\right)\left(\left\|\frac{1}{w}\right\|_{L^\infty(Q)}\right),$$

and we say $w \in A_1$ if $\| w \|_{A_1} < \infty$.

THEOREM 3 ([8]). *If $1 < p < \infty$, then $w \in A_p$ if and only if*

$$w = w_1(w_2)^{1-p} \tag{1.4}$$

where $w_1, w_2 \in A_1$.

Hölder's inequality shows that (1.4) is sufficient. Obtaining the factorization (1.4) for $w \in A_p$ is more difficult.

Theorem 2 is a simple consequence of Theorem 3. Indeed, let $\varphi \in \mathrm{BMO}$ and take $A(\varphi)/2 < A < A(\varphi)$. Write $w = e^{A\varphi}$. Then for any Q,

$$\begin{aligned}\left(\frac{1}{|Q|}\int_Q w dx\right)\left(\frac{1}{|Q|}\int_Q \frac{1}{w} dx\right) &= \left(\frac{1}{|Q|}\int_Q e^{A(\varphi-\varphi_Q)} dx\right)\left(\frac{1}{|Q|}\int_Q e^{-A(\varphi-\varphi_Q)} dx\right) \\ &\leqq \left(\frac{1}{|Q|}\int_Q e^{A|\varphi-\varphi_Q|} dx\right)^2,\end{aligned}$$

so that $w \in A_2$. By Theorem 3,

$$A\varphi = \log w = F_1 - F_2$$

where $e^{F_1}, e^{F_2} \in A_1$. From A_1 it follows easily that

$$e^{F_j} \leqq M(e^{F_j}) \leqq c e^{F_j}$$

almost everywhere, where $M(f)$ denotes the Hardy-Littlewood maximal function of f. Coifman and Rochberg [3] have shown $\| \log M(f) \|_{\mathrm{BMO}} \leqq C(m)$ whenever $f \in L^1_{\mathrm{loc}}$. Consequently

$$\begin{aligned} F_j &= \log M(e^{F_j}) + \log(e^{F_j}/M(e^{F_j})) \\ &= \psi_j + g_j \end{aligned}$$

where $g_j \in L^\infty$ and $\| \psi_j \| \leqq C(m)$. Hence

$$\varphi = \frac{g_1 - g_2}{A} + \frac{\psi_1 - \psi_2}{A} = g + \psi$$

with $g \in L^\infty$ and $\|\psi\| \leqq 2C(m)/A$.

The above reasoning also explains why Theorem 3 is a theorem about BMO. See [8] for further application of Theorem 3.

THEOREM 4 (Uchiyama [11]). *Let* $\lambda > 0$ *and let* $E_1, E_2, \cdots, E_N$ *be measurable subsets of* $\boldsymbol{R}^m$ *such that*

$$\operatorname*{Min}_{1 \leqq i \leqq N} \frac{|Q \cap E_i|}{|Q|} \leqq 2^{-2m\lambda} \tag{1.5}$$

for every cube Q. *Then there exists functions* $f_1(x), f_2(x), \cdots, f_N(x)$ *such that almost everywhere*

$$f_i(x) = 0\ , \quad x \in E_i\ . \tag{1.6}$$

$$0 \leqq f_i(x) \leqq 1\ . \tag{1.7}$$

$$\sum_{i=1}^{N} f_i(x) = 1 \tag{1.8}$$

and such that

$$\|f_i\| \leqq C(m, N)/\lambda\ , \quad 1 \leqq i \leqq N\ . \tag{1.9}$$

The converse (with $\|f_i\| \leqq C'(m, N)/\lambda$) of this theorem is not difficult. Theorem 4 for $N = 2$ is roughly equivalent to Theorem 2. For $N > 2$ it has interesting applications to function theory. See [8] and [11].

In §2 we prove the dyadic versions of Theorem 1 and Theorem 3. Although the arguments are well known (see [13] and [8]), they are included for completeness and because some of their by-products will be needed later. Theorem 3 is proved in §3 and Theorem 1 is proved in §4. In §5 we discuss Theorem 4 and its dyadic analogue.

We would like to acknowledge our indebtedness to Davis [4], who showed on the circle that $T_\alpha f \in H^1_{\mathrm{dyadic}}$ for almost every α if $f \in H^1$, and to Varopoulos [12], who proved Theorem 2 by adapting the argument of the dyadic case to Brownian motion.

2. Two dyadic theorems.

THEOREM 2.1. *Let* $\varphi \in \mathrm{BMO}_d$ *and let* Q_0 *be a fixed dyadic cube. Then there exists a sequence* $\{Q_k\}$ *of dyadic cubes* $Q_k \subset Q_0$, *and a sequence* $\{a_k\}$ *of real numbers such that*

$$\sum_{Q_k \subset Q} |a_k|\,|Q_k| \leqq C\|\varphi\|_d\,|Q|\,, \tag{2.1}$$

for all dyadic cubes Q, and there exists $g \in L^\infty$,

$$\|g\|_\infty \leqq 2\|\varphi\|_d\,,$$

such that

$$\varphi(x) - \varphi_{Q_0} = g(x) + \sum a_k \chi_{Q_k}(x) \tag{2.2}$$

almost everywhere on Q_0. The constant C depends only on the dimension.

To understand why Theorem 2.1 is the dyadic formulation of Theorem 1, replace $\boldsymbol{R}_+^{m+1}$ by its discrete subset $\mathscr{D} = \{p_Q = (c(Q), \ell(Q)),$ Q dyadic$\}$ where $c(Q) \in \boldsymbol{R}^m$ is the center of Q and $\ell(Q)$ is the sidelength of Q. The correspondence between p_Q and $K_Q(x) = \chi_Q(x)/_{|Q|}$ resembles the correspondence between $(t, y) \in \boldsymbol{R}_+^{m+1}$ and $K_y(x - t)$. Let σ be the measure on $\mathscr{D}$ having mass $a_k|Q|$ at p_{Q_k}. Then (2.1) says that

$$\begin{aligned} |\sigma|(Q \times (0, \ell(Q)] &= \sum_{Q_k \subset Q} |a_k|\,|Q_k| \\ &\leqq C\|\varphi\|_d|Q| \end{aligned}$$

and σ can be viewed as a dyadic Carleson measure. Since

$$\int K_Q(x)d\sigma(p_Q) = \sum a_k \chi_{Q_k}(x)\,,$$

(2.2) is now the dyadic version of (1.2).

Proof. We suppose $\varphi_{Q_0} = 0$. Fix $\lambda = 2\|\varphi\|_d$ and set

$$G_1 = \{Q_k \subset Q_0\colon Q_k \text{ dyadic},\ |\varphi_{Q_k}| > \lambda, \quad \text{and} \quad Q_k \text{ maximal}\}\,.$$

Because $Q_k \in G_1$ is maximal, we have

$$|\varphi_{Q_k}| \leqq \lambda + 2^m\|\varphi\|_d \leqq 2^{m+1}\|\varphi\|_d\,. \tag{2.3}$$

Indeed, if Q_k^* is that dyadic cube with $Q_k^* \supset Q_k$ and $|Q_k^*| = 2^m|Q_k|$, then

$$|\varphi_{Q_k} - \varphi_{Q_k^*}| \leqq \frac{1}{|Q_k|}\int_{Q_k^*}|\varphi - \varphi_{Q_k^*}|dx \leqq 2^m\|\varphi\|_d$$

and $|\varphi_{Q_k^*}| \leqq \lambda$ as Q_k is maximal. The Q_k in G_1 are pairwise disjoint, because they are maximal, so that

$$\sum_{G_1}|Q_k| \geqq \frac{1}{\lambda}\sum_{G_1}\left|\int_{Q_k}\varphi dx\right| \leqq \frac{1}{\lambda}\int_{Q_0}|\varphi|dx \leqq \frac{\|\varphi\|_d|Q_0|}{\lambda} \leqq |Q_0|/2\,. \tag{2.4}$$

Write $a_k = \varphi_{Q_k}$, $Q_k \in G_1$. Then we have

$$\varphi(x) = g_1(x) + \sum_{G_1} a_k \chi_{Q_k}(x) + \varphi_1(x) ,$$

where $g_1 = \varphi\chi_{E_1}$, $E_1 = Q_0 \setminus \cup \{Q_k \colon Q_k \in G_1\}$, satisfies $|g_1| \leqq \lambda$ by Lebesgue's theorem on differentiating the integral, where $|a_k| \leqq 2^{m+1}\|\varphi\|_d$ by (2.3), and where

$$\varphi_1 = \sum_{G_1} (\varphi(x) - \varphi_{Q_k})\chi_{Q_k}(x) .$$

Now because $\varphi \in \mathrm{BMO}_d$, $\varphi_1\chi_{Q_k} = (\varphi - \varphi_{Q_k})\chi_{Q_k}$ has the same behavior on Q_k that φ has an Q_0, and we can repeat the construction with each $\varphi_1\chi_{Q_k}$, and continue by induction. At stage n we have a family G_{n-1} of disjoint dyadic cubes and $\varphi_{n-1} = \sum_{G_{n-1}} (\varphi(x) - \varphi_{Q_j})\chi_{Q_j}(x)$. For each $Q_j \in G_{n-1}$ we set

$$G_1(Q_j) = \{Q_k \subset Q_j \colon Q_k \quad \text{dyadic}, |\varphi_{Q_k} - \varphi_{Q_j}| > \lambda, Q_k \quad \text{maximal}\}$$

and $G_n = \cup \{G_1(Q_j) \colon Q_1 \in G_{n-1}\}$. Then

$$\varphi_{n-1}(x) = g_n(x) + \sum_{G_n} a_k \chi_{Q_k}(x) + \varphi_n(x) ,$$

where $g_n = \varphi_{n-1}\chi_{E_n}$, $E_n = \bigcup_{G_{n-1}} Q_j \setminus \bigcup_{G_n} Q_k$, satisfies $|g_n| \leqq \lambda$ and where $a_k = \varphi_{Q_k} - \varphi_{Q_j}$, $Q_k \subset Q_j \in G_{n-1}$, satisfies

$$|a_k| \leqq 2^{m+1}\|\varphi\|_d \tag{2.5}$$

by the proof of (2.3). Moreover, the proof of (2.4) now gives

$$\sum_{\substack{Q_k \subset Q_j \\ Q_k \in G_n}} |Q_k| \leqq |Q_j|/2 \tag{2.6}$$

for all $Q_j \in G_{n-1}$. Consequently $\varphi_n(x) \to 0$ almost everywhere, because φ_n has support $\bigcup_{G_n} Q_k$ and this set has measure $\leqq 2^{-n}|Q_0|$. Summing, we obtain

$$\varphi(x) = \sum_{n=1}^{\infty} g_n(x) + \sum_{n=1}^{\infty} \sum_{G_n} a_k \chi_{Q_k}(x) .$$

Since $|g_n| \leqq \lambda$ and the g_n have pairwise disjoint supports, $g = \sum g_n$ satisfies $\|g\|_\infty \leqq \lambda = 2\|\varphi\|_d$ and we have the representation (2.2).

To prove (2.1) fix a dyadic cube and set $G_1(Q) = \{Q_j \in \cup G_n \colon Q_j \subset Q, Q_j \text{ maximal}\}$. The Q_j in $G_1(Q)$ are disjoint and

$$\sum_{Q_k \subset Q} |a_k|\,|Q_k| = \sum_{Q_j \in G_1(Q)} \sum_{Q \subseteqq Q_j} |a_k|\,|Q_k| .$$

Hence by (2.5), (2.6) and induction,

$$\sum_{Q_k \subset Q} |a_k|\,|Q_k| \leqq 2^{m+1}\|\varphi\|_d \sum_{Q_j \in G_1(Q)} \sum_{\substack{Q_k \subseteqq Q_j \\ Q_k \in \cup G_n}} |Q_k|$$
$$\leqq 2^{m+1}\|\varphi\|_d \sum_{Q_j \in G_1(Q)} 2|Q_j|$$
$$\leqq 2^{m+2}\|\varphi\|_d |Q| .$$

Theorem 2.1 is proved.

Notice that when applied to the translates $T_\alpha\varphi$, $\varphi \in \mathrm{BMO}$, the costruction above produces functions $g^{(\alpha)}(x)$ and coefficients $a_k^{(\alpha)}$ which vary measurably in α.

Now let $w \geqq 0$ and set $\varphi = \log w$. Then $w \in A_p$, $1 < p < \infty$, if and only if

$$\sup_Q \left(\frac{1}{|Q|}\int_Q e^{\varphi-\varphi_Q}dx\right)\left(\frac{1}{|Q|}\int e^{-(\varphi-\varphi_Q)/(p-1)}dx\right)^{p-1} < \infty .$$

By Jensen's inequality each factor is at least 1, and hence $w \in A_p$ if and only if

$$\sup_Q \frac{1}{|Q|}\int_Q e^{\varphi-\varphi_Q}dx < \infty \tag{2.7}$$

and

$$\sup_Q \frac{1}{|Q|}\int_Q e^{-(\varphi-\varphi_Q)/(p-1)}dx < \infty . \tag{2.8}$$

For the dyadic form of Theorem 3, the suprema in (2.7) and (2.8) are taken over dyadic subcubes of Q_0 only.

THEOREM 2.2. *Let $\varphi(x)$ be a real function on a dyadic cube Q_0 and let $1 < p < \infty$. Assume*

$$\sup_{\substack{Q\subset Q_0 \\ Q \text{ dyadic}}} \frac{1}{|Q|}\int_Q e^{\varphi-\varphi_Q}dx < \infty , \tag{2.7d}$$

and

$$\sup_{\substack{Q\subset Q_0 \\ Q \text{ dyadic}}} \frac{1}{|Q|}\int_Q e^{-(\varphi-\varphi_Q)/(p-1)}dx < \infty . \tag{2.8d}$$

Then

$$\varphi - \varphi_Q = g + F - G ,$$

where $g \in L^\infty$, $\|g\|_\infty \leqq C_1$, where

$$\sup_{\substack{Q\subset Q_0 \\ Q \text{ dyadic}}} \left\{\left(\frac{1}{|Q|}\int_Q e^F dx\right)\|e^{-F}\|_{L^\infty(Q)}\right\} < C_2 \tag{2.9}$$

and where

$$\sup_{\substack{Q\subset Q_0\\ Q \text{ dyadic}}} \left\{\left(\frac{1}{|Q|}\int_Q e^{G/(p-1)}dx\right)\|e^{-G/(p-1)}\|_{L^\infty(Q)}\right\} < C_3 . \tag{2.10}$$

The constants C_1, C_2, C_3 depend only on m and the bounds in (2.7d) *and* (2.8d).

Thus if $w = e^{\varphi}$ satisfies the dyadic A_p condition (i.e., if 2.7d) and (2.8d), then $w = w_1(w_2)^{1-p}$ where $w_1 = e^{\varphi_{Q_0}+g+f}$ and $w_2 = e^{G/(p-1)}$ satisfy A_1 on dyadic subcubes of Q_0.

Proof. The construction is the same as in the proof of Theorem 2.1. By (2.7d) and (2.8d) and by Jensen's inequality, $\varphi\chi_{Q_0} \in \mathrm{BMO}_d$. Fix $\lambda > 2\|\varphi\|_d$ to be determined later and set $G_1 = \{Q_k \subset Q_0\colon Q_k$ dyadic, $|\varphi_{Q_k} - \varphi_{Q_0}| > \lambda$, Q_k maximal$\}$ and by induction

$$G_n = \bigcup_{Q_j\in G_{n-1}} \{Q_k \subset Q_j\colon Q_k \text{ dyadic}, |\varphi_{Q_k} - \varphi_{Q_j}| > \lambda, Q_k \text{ maximal}\} .$$

For $Q_k \in G_n$, $Q_k \subset Q_j \in G_{n-1}$, set $a_k = (\varphi_{Q_k} - \varphi_{Q_j})$. The proof of (2.3) gives

$$\lambda < |a_j| < \lambda + 2^m\|\varphi\|_d . \tag{2.11}$$

As in the proof of Theorem 2.1, we have

$$\varphi = \varphi_{Q_0} + g + \sum_{n=1}^{\infty}\sum_{G_n} a_k\chi_{Q_k}(x) ,$$

where $\|g\|_\infty \leqq \lambda$. Write

$$F = \sum_{a_k>0} a_k\chi_{Q_k} , \tag{2.12}$$

$$G = -\sum_{a_k<0} a_k\chi_{Q_k} . \tag{2.13}$$

Then $\varphi = \varphi_{Q_0} + g + F - G$.

To prove (2.9) and (2.10) we recall that there is $\varepsilon > 0$, depending only on the bounds in (2.7d) and (2.8d), such that

$$\sup_{\substack{Q\subset Q_0\\ Q \text{ dyadic}}} \frac{1}{|Q|}\int_Q e^{(1+\varepsilon)(\varphi-\varphi_Q)}dx < \infty , \tag{2.14}$$

and

$$\sup_{\substack{Q\subset Q_0\\ Q \text{ dyadic}}} \frac{1}{|Q|}\int_Q e^{-(1+\varepsilon)(\varphi-\varphi_Q)/(p-1)}dx < \infty . \tag{2.15}$$

See [3] or [10].

We prove (2.9). Fix $Q_j \in \cup G_n$ with $a_j > 0$ and set

$$G_1^+(Q_j) = \{Q_k \in \cup G_n\colon Q_k \subsetneqq Q_j,\ a_k > 0,\ Q_k \text{ maximal}\}\,,$$

and by induction $G_{n+1}^+(Q) = \cup\{G_1^+(Q_k)\colon Q_k \in G_n^+(Q_j)\}$. The critical inequality for the proof of (2.9) is

$$\sum_{G_n^+(Q_j)} \frac{|Q_k|}{|Q_j|} \leqq (2C)^n e^{-n(1+\varepsilon)\lambda}\,, \tag{2.16}$$

where C is the sumpremum in (2.14). By induction we need only obtain (2.16) for $n = 1$. There are two case.

Case 1. $Q_j \in G_n$ and $Q_k \in G_{n+1}$. Then $\varphi_{Q_k} - \varphi_{Q_j} > \lambda$, so by Jensen's inequality

$$e^{(1+\varepsilon)\lambda} \leqq \frac{1}{|Q_k|}\int_{Q_k} e^{(1+\varepsilon)(\varphi-\varphi_{Q_j})}dx\,.$$

Since the Q_k are disjoint this gives

$$\begin{aligned}\sum_{\text{Case 1}} \frac{|Q_k|}{|Q_j|} &\leqq e^{-(1+\varepsilon)\lambda} \sum_{\text{Case 1}} \frac{1}{|Q_j|}\int_Q e^{(1+\varepsilon)(\varphi-\varphi_{Q_j})}dx \\ &\leqq e^{-(1+\varepsilon)\lambda}\frac{1}{|Q_j|}\int_Q e^{(1+\varepsilon)(\varphi-\varphi_{Q_j})}dx \\ &\leqq Ce^{-(1+\varepsilon)\lambda}\,.\end{aligned}$$

Case 2. $Q_j \in G_n$ and $Q_k \in G_{n+p}$, $p \geqq 2$. Then if $Q_\ell \in G_{n+r}$, $1 \leqq r \leqq p-1$, and if $Q_k \subset Q_\ell$, it must be that $a_\ell < 0$. Let $D_1 = \{Q_\ell \in G_{n+1}\colon Q_\ell \subset Q_j,\ a_\ell < 0\}$ and by induction $D_r = \{Q_\ell \in G_{n+r}\colon \exists Q_m \in D_{r-1},\ Q_\ell \subset Q_m,\ a_\ell < 0\}$. Then as in the proof of Case 1, $|\bigcup D_1| \leqq Ce^{-(1+\varepsilon)\lambda/p-1}|Q_j| \leqq 1/2|Q_j|$ if λ is large enough. Induction then shows $|\bigcup D_r| \leqq 2^{-r}|Q_j|$. For $Q_\ell \in D_r$, $r \geqq 1$, let $U(Q_\ell) = \{Q_k \in G_{n+r+1}\colon Q_k \subset Q_\ell,\ a_k > 0\}$. By Case 1, $|\bigcup U_\ell)| \leqq C|Q_\ell e^{-(1+\varepsilon)\lambda}$. Consequently,

$$\begin{aligned}\sum_{\text{Case 2}} \frac{|Q_k|}{|Q_j|} &= \sum_{r=1}^{\infty} \frac{1}{|Q_j|} \sum_{Q_\ell \in D_r} |\textstyle\bigcup U(Q_\ell)| \\ &\leqq Ce^{-(1+\varepsilon)\lambda} \sum_{r=1}^{\infty} \frac{1}{|Q_j|} \sum_{Q_\ell \in D_r} |Q_\ell| \\ &\leqq Ce^{-(1+\varepsilon)\lambda} \sum_{r=1}^{\infty} 2^{-r}\,,\end{aligned}$$

because the cubes $Q_\ell \in D_r$ are disjoint for each r. Summing the two cases gives (2.16) for $n = 1$.

Now fix a dyadic cube $Q \subset Q_0$ and set

$$F_1 = \sum_{\substack{Q_k \subsetneqq Q \\ a_k > 0}} a_k \chi_{Q_k}\,, \quad F_2 = \sum_{\substack{Q_k \supset Q \\ a_k > 0}} a_k \chi_{Q_k}\,.$$

On Q, $F = F_1 + F_2$, F_2 is constant, and $F_1 \geqq 0$. Hence

$$\left(\frac{1}{|Q|}\int_Q e^F dx\right)\|e^{-(F_1+F_2)}\|_{L^\infty(Q)} = \left(\frac{1}{|Q|}\int_Q e^{F_1+F_2}dx\right)\|e^{-(F_1+F_2}\|_{L^\infty(Q)}$$
$$\leqq \frac{1}{|Q|}\int_Q e^{F_1}dx ,$$

and it suffices to establish

$$\frac{1}{|Q|}\int_Q e^{F_1}dx \leqq C_2 . \tag{2.17}$$

If $\{Q_j\}$ denotes the set of maximal cubes $Q_j \subset Q$ having $a_j > 0$, then $\sum|Q_j| \leqq |Q|$ and

$$\frac{1}{|Q|}\int_Q (e^{F_1}-1)dx \leqq \sum \frac{|Q_j|}{|Q|}\left(\frac{1}{|Q_j|}\int_{Q_j} e^{F_1}dx\right) \leqq \sup_j\left(\frac{1}{|Q_j|}\int_{Q_j} e^{F_1}dx\right) .$$

Now by (2.11),

$$\{x \in Q : F_1(x) > (n+1)(\lambda + 2^m\|\varphi\|_d)\} \subset \bigcup_{G_n^+(Q_j)} Q_k ,$$

so that by (2.16),

$$\frac{1}{|Q_j|}\int_{Q_j} e^{F_1}dx \leqq \sum_{n=0}^{\infty} e^{(n+1)(\lambda+2^m\|\varphi\|_d)}(2C)^n e^{-n(1+\varepsilon)\lambda} .$$

If $\lambda > \lambda(\varepsilon, C, \|\varphi\|_d)$ the series sums and we obtain (2.17) and therefore (2.9).

The proof of (2.11) is the same except that (2.15) is used in place of (2.14).

3. The proof of Theorem 3. Let $w = e^\varphi \in A_p$ and let S_N be the cube $\{|x_i| \leqq 2^N, 1 \leqq i \leqq m\}$.

LEMMA 3.1. *There exist $g_N(x)$, $F_N(x)$ and $G_N(x)$, $x \in S_N$, such that $\|g_N\|^\infty \leqq C_1$, and*

$$\sup_{Q\subset S_N}\left(\frac{1}{|Q|}\int_Q e^{F_N}dx\right)\|e^{-F_N}\|_{L^\infty(Q)} \leqq C_2 , \tag{3.1}$$

$$\sup_{Q\subset S_N}\left(\frac{1}{|Q|}\int_Q e^{G_N/p-1}dx\right)\|e^{-G_N/p-1}\|_{L^\infty(Q)} \leqq C_3 , \tag{3.2}$$

and such that

$$\varphi(x) - \varphi_{S_N} = g_N(x) + F_N(x) - G_N(x) , \quad x \in S_N .$$

The constants C_1, C_2, C_3 do not depend on N.

We first show how Lemma 3.1 easily implies Theorem 3. We suppose $\varphi_{S_0} = 0$. By Lemma 3.1,

$$\varphi = R_N + (F_N - (F_N)_{S_0}) - (G_N - (G_N)_{S_0})$$
$$= R_N + \tilde{F}_N - \tilde{G}_N , \quad x \in S_N ,$$

where $R_N = \varphi_{S_N} + g_N + (F_N)_{S_0} - (G_N)_{S_0}$ satisfies $\|R_N\|_\infty \leqq 2C_1$ since $|\varphi_{S_N} + (F_N)_{S_0} - (G_N)_{S_0}| = |\varphi_{S_0} - (g_N)_{S_0}| \leqq C_1$. For $N > M$, (3.1) and (3.2) give

$$\frac{1}{|S_M|}\int_{S_M} |\tilde{F}_N - (\tilde{F}_N)_{S_M}|^2 dx \leqq C$$

$$\frac{1}{|S_M|}\int_{S_M} |\tilde{G}_N - (\tilde{G}_N)_{S_M}|^2 dx \leqq C\,,$$

and hence as $(\tilde{F}_N)_{S_0} = (\tilde{G}_N)_{S_0} = 0$, $|(\tilde{F}_N)_{S_M}| \leqq C_M$, $|(\tilde{G}_N)_{S_M}| \leqq C_M$. Consequently $\{(\tilde{F}_N\colon N \geqq M\}$ and $\{\tilde{G}_N\colon N \geqq M\}$ are bounded in $L^2(S_M)$. Choose $N_j \to \infty$ so that $\tilde{F}_{N_j} \to F$, $\tilde{G}_{N_j} \to G$ weakly in $L^2(S_M)$ for all M and so that $R_{N_j} \to g$ weak-star in L^∞. Then

$$\varphi = g + F - G$$

with $\|g\|_\infty \leqq C_1$. For any cube Q there is a sequence of finite convex combinations

$$F^{(n)} = \sum_j t_{j,n} F_{N_{j,n}}\,, \quad t_{j,n} \geqq 0\,, \quad \sum t_{j,n} = 1\,,$$

converging to F almost everywhere on $S_M \subset Q$. Then by Fatou's Lemma and Hölder's inequality

$$\left(\frac{1}{|Q|}\int_Q e^F dx\right)\| e^{-F}\|_{L^\infty(Q)} \leqq \lim_{n\to\infty} \prod \left(\frac{1}{|Q|}\int e^{F_{N_{j,n}}} dx\right)^{t_{j,n}} \prod \| e^{-F_{N_{j,n}}}\|^{t_{n,j}} \leqq C_2\,,$$

and hence $w_1 = e^{g+F} \in A_1$. Using (3.2) we see $w_2 = e^{G/p-1} \in A_1$ in the same way.

Proof of Lemma 3.1. We assume $\varphi_{S_N} = 0$. For $\alpha \in S_N$ we use Theorem 2.2 on $T_\alpha\varphi(x) = \varphi(x - \alpha)$ with $Q_0 = S_{N+1}$ (which we pretend is a dyadic cube) to obtain

$$T_\alpha\varphi = g^{(\alpha)} + F^{(\alpha)} - G^{(\alpha)}$$

where F_α and G_α satisfy (2.9) and (2.10) respectively and where $\|g\|_\infty \leqq C_1$ (since $\varphi \in \mathrm{BMO}$ and $\varphi_{S_N} = 0$, $\sup_{\alpha\in S_N}(T_\alpha\varphi)_{S_{N+1}}$ is bounded). Almost everywhere on S_N,

$$\begin{aligned}\varphi(x) &= \frac{1}{|S_N|}\int_{S_N} T_{-\alpha}(T_\alpha\varphi)(x)d\alpha \\ &= \frac{1}{|S_N|}\int_{S_N} T_{-\alpha}(g^{(\alpha)})(x)d\alpha + \frac{1}{|S_N|}\int_{S_N} T_{-\alpha}(F^{(\alpha)})(x)d\alpha \\ &\quad + \frac{1}{|S_N|}\int_{S_N} T_{-\alpha}(G^{(\alpha)})(x)d\alpha = g(x) + F(x) - G(x)\,.\end{aligned}$$

Clearly $\|g\|_\infty \leqq C_1$. By (2.12) there are $a_k^{(\alpha)} > 0$ such that

$$F^{(\alpha)}(x) = \sum_{n=0}^{\infty} \sum_{\ell(Q_k)=2^{-n}\ell(S_N)} a_k^{(\alpha)} \chi_{Q_k}(x)$$
$$= \sum_{n=0}^{\infty} f_n^{(\alpha)}(x)$$

and by (2.13), $G^{(\alpha)}(x)$ has a similar representation. Write

$$f_n(x) = \frac{1}{|S_N|} \int_{S_N} (T_{-\alpha} f_n^{(\alpha)})(x) d\alpha$$

so that $F = \sum_{n=0}^{\infty} f_n$.

Lemma 3.2. *If* $\sup_i |x_i - y_i| \leqq 2^{-n}\ell(S_N)$ *then*

$$|f_n(x) - f_n(y)| \leqq \frac{C_4 2^n}{\ell(S_N)} |x - y| , \tag{3.3}$$

with C_4 *independent of* n.

Proof. By (2.11), $|a^{(\alpha)}| \leqq C$, and hence

$$|f_n(x) - f_n(y)| \leqq \frac{C}{|S_N|} \int_{S_N} \sum_{\ell(Q_k)=2^{-n}\ell(S_N)} |\chi_{Q_k}(x + \alpha) - \chi_{Q_k}(y + \alpha)| d\alpha .$$

The integrand is twice the characteristic function of $\{\alpha \in S_N: x + \alpha$ and $y + \alpha$ fall in different Q_k, $\ell(Q_k) = 2^{-n}\ell(S_N)\}$, and this set has probability not exceeding

$$\sum_{i=1}^{m} \frac{|x_i - y_i|}{2^{-n}\ell(S_N)}$$

Returning to the proof of Lemma 3.1, we fix $Q \subset S_N$ with $2^{-k}\ell(S_N) < \ell(Q) \leqq 2^{-k+1}\ell(S_N)$. Then

$$F(x) = \sum_{n>k} f_n(x) + \sum_{n \leqq k} f_n(x) = F_1(x) + F_2(x)$$
$$= \frac{1}{|S_N|} \int_{S_N} F_1^{(\alpha)}(x + \alpha) d\alpha + \frac{1}{|S_N|} \int_{S_N} F_2^{(\alpha)}(x + \alpha) d\alpha .$$

By Lemma 3.2,

$$\sup_Q F_2(x) - \inf_Q F_2(x) \leqq \frac{C}{\ell(S_N)} \sum_{n=0}^{k} 2^n \ell(Q) \leqq C .$$

Hence as $F_1 \geqq 0$,

$$\left(\frac{1}{|Q|} \int_Q e^{F_1 + F_2} dx\right) \| e^{-(F_1+F_2)} \|_{L^\infty(Q)} \leqq C\left(\frac{1}{|Q|} \int_Q e^{F_1} dx\right) .$$

But by Jensen's inequality,

$$\frac{1}{|Q|}\int_Q e^{F_1(x)}dx = \frac{1}{|Q|}\int_Q \exp\Big(\frac{1}{|S_N|}\int_{S_N} F_1^{(\alpha)}(x+\alpha)d\alpha\Big)dx$$
$$\leqq \frac{1}{|S_N|}\int_{S_N}\Big(\frac{1}{|Q|}\int_Q e^{F_1^{(\alpha)}(x+\alpha)}dx\Big)d\alpha \leqq C_2 .$$

by (2.17). The proof of (3.2) is the same.

4. The proof of Theorem 1. We suppose $\varphi \in \mathrm{BMO}$ has support $S_0 = \{|x_i| \leqq 1, 1 \leqq i \leqq m\}$ and $\int \varphi dx = 0$. For each $\alpha \in S_0$ we have, by Theorem 2.1,

$$T_\alpha\varphi(x) = \varphi(x-\alpha) = g^{(\alpha)}(x) + \sum_{Q_k\subset Q_0} a_k^{(\alpha)}\chi_{Q_k}(x)$$

where $Q_0 = \{|x_i| \leqq 2, 1 \leqq i \leqq m\}$, where $\|g^{(\alpha)}\|_\infty \leqq C\|\varphi\|$, and where

$$\sum_{Q_k\subset Q}|a_k^{(\alpha)}|\,|Q_k| \leqq C\|\varphi\|\,|Q| . \tag{4.1}$$

Write

$$f_n^{(\alpha)}(x) = \sum_{\ell(Q)=2^{-n}} a_k^{(\alpha)}\chi_{Q_k}(x) ,$$

so that $T_\alpha\varphi(x) = g^{(\alpha)}(x) + \sum_{n=0}^\infty f_n^{(\alpha)}(x)$. Then as before

$$\varphi(x) = \frac{1}{|S_0|}\int_{S_0} g^{(\alpha)}(x+\alpha)d\alpha + \sum_{n=0}^\infty \frac{1}{|S_0|}\int_{S_0} f_n^{(\alpha)}(x+\alpha)d\alpha$$
$$= g(x) + \sum_{n=0}^\infty f_n(x)$$

where $\|g\|_\infty \leqq C\|\varphi\|$. For any cube Q we have

$$\frac{1}{|Q|}\int_Q \sum_{2^{-n}\leqq\ell(Q)} |f_n(x)|dx \leqq \sup_{\alpha\in S_0}\Big(\frac{1}{|Q|}\int_Q \sum_{2^{-n}<\ell(Q)} |f_n^{(\alpha)}(x)|dx\Big)$$
$$\leqq \sup_{\alpha\in S_0}\Big(\frac{1}{|Q|}\sum_{Q_k\subset\tilde{Q}}|a_k^{(\alpha)}|\,|Q_k|\Big) \leqq C\|\varphi\| , \tag{4.2}$$

where $\tilde{Q}$ is concentric with Q and $\ell(\tilde{Q}) = 3\ell(Q)$. Thus for any $\delta > 0$, $d\sigma = \sum f_n(x)d\sigma_n$, where $d\sigma_n$ is surface measure on $\mathbf{R}^m \times \{y = \delta 2^{-n}\}$, is a Carleson measure and $N(\sigma) \leqq C\delta^{-m}\|\varphi\|$, and

$$\int K_y(x-z)d\sigma(z, y) = \sum_{n=0}^\infty f_n * K_{\delta 2^{-n}}(x)$$
$$= \sum h_n(x) .$$

We will show that when δ is small,

$$\|\sum(f_n - h_n)\| \leqq \frac{1}{2}\|\varphi\| . \tag{4.3}$$

With an iteration, that will prove Theorem 1.

To prove (4.3) fix a cube Q and a point $x_0 \in Q$. We have

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |\sum (f_n(x) - h_n(x) - (f_n - h_n)_Q)|\,dx \\
&\leqq 2 \sum_{2^{-n} \geqq A\ell(Q)} \frac{1}{|Q|}\int_Q |(f_n - h_n)(x) - (f_n - h_n)(x_0)|\,dx \\
&\quad + 2 \sum_{\ell(Q) \leqq 2^{-n} < A\ell(Q)} \frac{1}{|Q|}\int_Q |f_n(x) - h_n(x)|\,dx \\
&\quad + 2 \sum \frac{1}{|Q|}\int_Q |f_n(x) - h_n(x)|\,dx \\
&= 2\sum_1 + 2\sum_2 + 2\sum_3 ,
\end{aligned}$$

where $A \geqq 2$ is a constant to be determined.

To estimate $\sum_1$, recall that

$$|f_n(x) - f_n(y)| \leqq C2^n \|\varphi\|\, |x - y| \tag{4.4}$$

by the proof of Lemma 3.2. The convolution $h_n = K_{\delta 2^{-n}} * f_n$ has the same continuity as f_n, since $\int K dx = 1$, and we have

$$\begin{aligned}
\sum_1 \leqq \sum_{2^{-n} > A\ell(Q)} \frac{1}{|Q|}\int_Q 2C2^n \|\varphi\|\, |x - x_0|\,dx \leqq C'\|\varphi\|\ell(Q) \sum_{2^n \leqq (A\ell(Q))^{-1}} 2^n \\
\leqq C'\|\varphi\|/A .
\end{aligned}$$

Hence $2\sum_1 \leqq \|\varphi\|/6$ if A is large.

To estimate $\sum_2$, note that by (4.4) and the bound $\|f_n\|_\infty \leqq C\|\varphi\|$ (because $|a_k^{(\alpha)}| \leqq C\|\varphi\|$), we have

$$\|f_n - f_n * K_{\delta 2^{-n}}\|_\infty \leqq \varepsilon \|\varphi\|$$

if δ is small, independent of n. Therefore

$$2\sum_2 \leqq 2\varepsilon\|\varphi\| \sum_{\ell(Q) \leqq 2^{-n} \leqq A\ell(Q)} \leqq C\varepsilon \|\varphi\| \log A \leqq \|\varphi\|/6$$

if $\varepsilon \log A$ is small.

Finally, we have

$$\sum_3 \leqq \sup_\alpha \frac{1}{|Q|}\int_Q \sum_{2^{-n} < \ell(Q)} |f_n^{(\alpha)} - f_n^{(\alpha)} * K_{\delta 2^{-n}}^{(\alpha)}|\,dx$$

by the definition of f_n. After a translation it is enough to consider $\alpha = 0$. Let $Q^{(0)} = Q$ and pave $\boldsymbol{R}^m$ with cubes $Q^{(j)}$ congruent to $\tilde{Q}$. Then

$$\sum_3 \leqq \sum_j \sum_{Q_k \subset Q^{(j)}} |a_k^{(0)}|\,|Q_k| \frac{1}{|Q|}\int_Q \left| \frac{\chi_{Q_k}(x)}{|Q_k|} - \frac{\chi_{Q_k} * K_{\delta\ell(Q_k)}(x)}{|Q_k|} \right| dx$$

By a change of scale,

$$\frac{1}{|Q|}\int_{R^m}\left|\frac{\chi_{Q_k}(x)}{|Q_k|}-\frac{\chi_{Q_k}*K_{\delta\ell(Q_k)}(x)}{|Q_k|}\right|dx$$

does not depend on $\ell(Q_k)$. Thus for $\varepsilon > 0$ we can choose δ so that

$$\frac{1}{|Q|}\int_{Q}\left|\frac{\chi_{Q_k}(x)}{|Q_k|}-\frac{\chi_{Q_k}*K_{\delta\ell(Q_k)}}{|Q_k|}\right|dx < \varepsilon/|Q|\ .$$

Moreover, if $Q_k \subset Q^{(j)}$, $Q^{(j)} \neq Q^{(0)}$, then $\chi_{Q_k}(x) = 0$ on Q and by (1.1),

$$\begin{aligned}\frac{1}{|Q|}\int_{Q}\frac{\chi_{Q_k}*K_{\delta\ell(Q_k)}(x)}{|Q_k|}\,dx &\leqq \sup_{\substack{x\in Q\\ x\in Q_k}} K_{\delta\ell(Q_k)}(x-t) \leqq \frac{C\delta\ell(Q_k)}{(\mathrm{dist}(Q_k, Q))^{m+1}}\\ &\leqq \frac{C\delta\ell(Q)}{(\mathrm{dist}(Q^{(j)}, Q))^{m+1}}\end{aligned}$$

since $\mathrm{dist}(Q_k, Q) \geqq \mathrm{dist}(Q^{(j)}, Q)$. Therefore

$$\begin{aligned}\sum_3 \leqq{}& \varepsilon \sum_{Q_k\subset Q^{(0)}} \frac{|a_k^{(0)}|\,|Q_k|}{|Q|}\\ &+ C\delta\ell(Q)\sum_{j\neq 0}\frac{1}{(\mathrm{dist}(Q^{(j)}, Q))^{m+1}}\sum_{Q_k\subset Q^{(j)}}|a_k^{(0)}|\,|Q_k|\ ,\end{aligned}$$

and by (4.1),

$$\begin{aligned}2\sum_3 &\leqq C\varepsilon\|\varphi\| + C\delta\|\varphi\|\ell(Q)\sum_{j\neq 0}\frac{|Q^{(j)}|}{(\mathrm{dist}(Q^{(j)}, Q))^{m+1}}\\ &\leqq C\varepsilon\|\varphi\| + C\delta\|\varphi\|\ell(Q)\int_{R^m/Q^{(0)}}\frac{dx}{|x-x_0|^{m+1}}\\ &\leqq C(\varepsilon+\delta)\|\varphi\| \leqq \|\varphi\|/6\end{aligned}$$

if ε and δ are small.

5. The proof of Theorem 4. We begin with the dyadic form of the theorem, which is also due to Uchiyama.

THEOREM 5.1. *Let $\lambda > 0$, let Q_0 be a dyadic cube in $\boldsymbol{R}^m$, and let $E_1, E_2, \cdots, E_N$ be measurable subsets of Q_0 such that*

$$\min_{1\leqq i\leqq N}\frac{|Q\cap E_i|}{|Q|} \leqq 2^{-2m\lambda} \tag{5.1}$$

for all dyadic $Q \subset Q_0$. Then there exist $f_1(x), f_2(x), \cdots, f_N(x)$ such that almost everywhere on Q_0,

$$f_i(x) = 0,\ x\in E_i\ , \tag{5.2}$$

(5.3) $$0 \leqq f_i(x) \leqq 1 \ ;$$

(5.4) $$\sum_{i=1}^{N} f_0(x) = 1 \ ;$$

and

(5.5) $$\sup_{\substack{Q \subset Q_0 \\ Q \text{ dyadic}}} \frac{1}{|Q|} \int_Q |f_i - (f_i)_Q| \, dx \leqq C_1(m, N)/\lambda \ .$$

Proof. By (5.1), $|\cap E_i| = 0$ and the bounded solutions $f_i(x) = (1 - \chi_{E_i}(x))/\sum_j(1 - \chi_{E_j}(x))$ satisfy (5.5) if λ is not large. Thus we assume $\lambda > N$.

We shall inductively choose families G_n of dyadic cubes $Q_k \subset Q_0$ and functions $\psi_i^{(n)}$, $1 \leqq i \leqq N$ such that

$$\psi_i^{(n)} = \psi_i^{(n-1)} + \sum_{Q_k \in G_n} a_{i,k} \chi_{Q_k}(x) \ ,$$

(5.6) $$0 \leqq \psi_i^{(n)} \leqq \lambda \ ,$$

(5.7) $$\sum_i \psi_i^{(n)} = \lambda \ ,$$

(5.8) $$(\psi_i^{(n)})_{Q_k} \leqq \operatorname{Max}\left(0, \ -N + \frac{1}{m} \log_2\left(\frac{|Q_k|}{|Q_k \cap E_i|}\right)\right)$$

if $Q_k \in G_n$, and

(5.9) $$|a_{i,k}| \leqq N^2 - 1, \ Q_k \in G_n, \ n \geqq 1 \ .$$

For each dyadic cube $Q \subset Q_0$, (5.1) ensures there exists an index $i(Q)$, $1 \leqq i(Q) \leqq N$, such that

(5.10) $$2\lambda \leqq \frac{1}{m} \log_2\left(\frac{|Q_k|}{|Q \cap E_{i(Q)}|}\right)$$

To start the induction take $G_0 = \{G_0\}$, and $\psi_i^{(0)}(x) = a_{i,0}\chi_{Q_0}(x)$, where

$$a_{i,0} = \begin{cases} 0 \ , & i \neq i(Q_0) \\ \lambda \ , & i = i(Q_0) \ . \end{cases}$$

Then (5.6) and (5.7) are trivial and (5.8) follows from (5.10) and our choice $\lambda > N$. At $n = 0$, (5.9) is not required.

Let G_n be the set of maximal dyadic cubes satisfying $Q_k \subset Q_j \in G_{n-1}$ and

(5.11) $$(\psi_i^{(n-1)})_{Q_k} > \frac{1}{m} \log_2\left(\frac{|Q_k|}{|Q_k \cap E_i|}\right) ,$$

for some i, $1 \leqq i \leqq N$. Define

$$a_{i,k} = \begin{cases} -\text{Min}(N+1, (\psi_i^{(n-1)})_{Q_k}, & i \neq i(Q_k) \\ -\sum_{j \neq i(Q_k)} a_{j,k} & , \quad i = i(Q_k)\,. \end{cases}$$

Then by definition $\psi_i^{(n)} = \psi_i^{(n-1)} + \sum_{Q_k \in G_n} a_{i,k}\chi_{Q_k}$ clearly satisfies $\psi_i^{(n)} \geqq 0$ and $\sum_i \psi_i^{(n)} = \lambda$. Thus (5.6) and (5.7) hold. Since $|a_{i,k}| \leqq N+1$ for $i \neq i(Q_k)$, and since $|a_{i(Q_k),k}| \leqq (N-1)(N+1)$, (5.9) holds.

We now verify inequality (5.8). If $i = i(Q_k)$, then by (5.6) and (5.10),

$$(\psi_i^{(n)})_{Q_k} \leqq \lambda \leqq -N + \frac{1}{m}\log_2\left(\frac{|Q_k|}{|Q_k \cap E_i|}\right).$$

Suppose $Q_k \in G_n$ and $i \neq i(Q_k)$. If $Q_k^* \supset Q_k$ is that dyadic cube with $|Q_k^*| = 2^m|Q_k|$, then $(\psi_i^{(n-1)})_{Q_k^*} = (\psi_i^{(n-1)})_{Q_k}$ and

$$\log_2\left(\frac{|Q_k^*|}{|Q_k^* \cap E_i|}\right) \leqq m + \log_2\left(\frac{|Q_k|}{|Q_k \cap E_i|}\right).$$

Since Q_k is maximal, (5.11) fails for Q_k^*, and so we have

$$1 = \frac{1}{m}\log_2\left(\frac{|Q_k|}{|Q_k \cap E_i|}\right) \geqq (\psi_i^{(n-1)})_{Q_k}\,. \tag{5.12}$$

If $a_{i,k} = -(\psi_i^{(n-1)})_{Q_k}$, then $(\psi_i^{(n)})_{Q_k} = 0$ and (5.9) is clear. If $a_{i,k} = -(N+1)$, then (5.9) follows from (5.12). Thus the induction is completed.

We thank J. Michael Wilson for this argument.

To obtain convergence and ultimately (5.5) we observe that if $Q_j \in G_{n-1}$, then

$$\sum_{\substack{Q_k \in G_n^{(i)} \\ Q_k \subset Q_j}} |Q_k| \leqq 2^{-mN}|Q_j|\,. \tag{5.13}$$

Indeed, if the left side of (5.13) is nonzero, we have $(\psi_i^{(n-1)})_{Q_j} > 0$, and then (5.11) and (5.8) yield

$$\begin{aligned}\sum_{\substack{Q_k \in G_n^{(i)} \\ Q_k \subset Q_j}} |Q_k| &\leqq 2^{m(\psi_i^{(n-1)})_{Q_j}} \sum_{\substack{Q_k \in G_n^{(i)} \\ Q_k \subset Q_j}} |Q_k \cap E_i| \\ &\leqq 2^{m(\psi_i^{(n-1)})_{Q_j}}|Q_j \cap E_i| \\ &\leqq 2^{-mN}|Q_j|\,.\end{aligned}$$

Since $N2^{-mN} < 1$, (5.9), (5.13) and induction show $\sum \|\psi_i^{(n)} - \psi_i^{(n-1)}\|_1 < \infty$, so that

$$\psi_i(x) = \lim_n \psi_i^{(n)}(x)$$

exists almost everywhere. Moreover, if $Q \subset Q_0$ is a dyadic cube, then by (5.9) and (5.13)

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |\psi_i - (\psi_i)_Q|\,dx \\
(5.19)\qquad &\leqq \frac{2}{|Q|}\int_Q \sum_{Q_k \subsetneqq Q} |a_{i,k}|\chi_{Q_k}(x)dx \\
&\leqq 2(N^2-1)\sum_{\substack{Q_k \subsetneqq Q \\ Q_k \in \cup G_n}} |Q_k|/|Q| \leqq 2(N^2-1)\sum_{\ell=1}^{\infty}(N2^{-mN})^{\ell} \\
&= C_1(m, N)\ .
\end{aligned}$$

Write $f_i = \psi_i/\lambda$. Then (5.5) follows from (5.14) and (5.6) and (5.7) give (5.3) and (5.4).

To conclude the proof we establish (5.2). Almost every point $x \in Q_0$ lies in a unique dyadic cube $Q_k(x)$, $|Q_k| = 2^{-mk}$, $k = 0, 1, 2, \cdots$. For almost such x, $Q_k(x) \in \cup G_n$ for only finitely k, because by (5.13), $\sum_n |\cup\{Q_k\colon Q_k \in G_n\}| < \infty$. Hence for almost every x there exist $k_x < \infty$ and $n_x < 8$ such that for $k > k_x$ and $n > n_x$

$$Q_k(x) \notin G_n$$

and

$$\psi_i(x) = \psi_i^{(n)}(x)\ .$$

So by the definition of G_n,

$$\psi_i(x) = (\psi_i^{(n-1)})_{Q_k(x)} \leqq \frac{1}{m}\log_2\left(\frac{|Q_k(x)|}{|Q_k(x) \cap E_i|}\right)$$

$k > k_x$, $n > n_x$, almost all x. On the other hand,

$$\log_2\left(\frac{|Q_k(x)|}{|Q_k(x) \cap E_i|}\right) \longrightarrow 0 \qquad (k \longrightarrow \infty)\ ,$$

almost everywhere on E_i. Therefore $f_i(x) = \psi_i(x) = 0$ almost everywhere on E_i.

Proof of Theorem 4. The argument is much like the proof of Theorem 3. Let S_M be the cube $\{x\colon |x_i| \leqq 2^M\}$. It is enough to produce $f_{1,M}(x), \cdots, f_{N,M}(x)$ which satisfy (1.6), (1.7) and (1.8) for $x \in S_M$ and also

$$(5.15)\qquad \sup_{Q \subset S_M} \frac{1}{|Q|}\int_Q |f_{i,M} - (f_{i,M})_Q|\,dx \leqq C(m, N)/\lambda\ ,$$

by then taking $f_i(x)$ an L^∞ weak-star limit of $\{f_{1,M}(x)\}_{M=1}^{\infty}$.

So fix S_M. For $\alpha \in S_M$ we set $E_i^{(\alpha)} = \{x + \alpha : x \in E_i \cap S_M\} \subset S_{M+1}$. With $Q_0 = S_{M+1}$, (5.1) holds for $E_1^{(\alpha)}, \cdots, E_N^{(\alpha)}$, and Theorem 5.1 gives us $f_1^{(\alpha)}(x), \cdots, f_N^{(\alpha)}(x)$ satisfying (5.2), (5.3), (5.4) and (5.5) on S_{M+1}. Define, as before,

$$f_{i,M}(x) = \frac{1}{|S_M|}\int f_i^{(\alpha)}(x+\alpha)d\alpha\ , \quad x \in S_M\ .$$

Then (1.6), (1.7) and (1.8) hold on S_M. To prove (5.15) write

$$\begin{aligned} f_i^{(\alpha)}(x) &= \frac{1}{\lambda}\sum_{n=0}^{\infty}\sum_{\ell(Q_k)=2^{-n}\ell(S_{M+1})} a_{i\,k}^{(\alpha)}\chi_{Q_k}(x) \\ &= \sum_{n=0}^{\infty} f_{i,n}^{(\alpha)}(x) \end{aligned}$$

and

$$\begin{aligned} f_{i,M}(x) &= \sum_{n=0}^{\infty}\frac{1}{|S_M|}\int f_{i,n}^{(\alpha)}(x+\alpha)d\alpha \\ &= \sum_{n=0}^{\infty} f_{i,n}(x)\ . \end{aligned}$$

If $Q \subset S_M$ and $2^{-k}\ell(S_{M+1}) < \ell(Q) \leqq 2^{-k+1}\ell(S_{M+1})$, then

$$\begin{aligned} \frac{1}{|Q|}\int_Q |f_{i,M}(x) - (f_{i,M})_Q|\,dx &\leqq \sum_{n<k}\frac{1}{|Q|}\int_Q |f_{i,n}(x) - (f_{i,n})_Q|\,dx \\ &\quad + 2\sum_{n \geqq k}\frac{1}{|Q|}\int_Q |f_{i,n}(x)|\,dx = \sum_1 + 2\sum_2\ . \end{aligned}$$

By the proof of Lemma 3.2

$$|f_{i,n}(x) - f_{i,n}(y)| \leqq \frac{C(N)2^n}{\lambda\ell(S_{M+1})}|x-y|\ ,$$

so that $\sum_1 \leqq C(N)/\lambda$, and by (5.13) and (5.9),

$$\begin{aligned} \sum_2 &\leqq \sup_{\alpha \in S_M}\sum_{Q_k \subset Q+\alpha} |a_{i\,k}^{(\alpha)}|\,|Q| \\ &\leqq \frac{C(N^2-1)}{\lambda}|Q|\ . \end{aligned}$$

Hence (5.15) holds and Theorem 4 is proved.

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Received November 19, 1980. The first author was partially supported by NSF Grant MCS80-02955 and the second by NSF Grant MCS810-2631.

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