The hyperbolic Gauss map and quasiconformal reflections.

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The hyperbolic Gauss map and quasiconformal reflections

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§ 0. Preliminary discussion

Since the results of this paper lie at the intersection of two disparate fields, hyperbolic geometry and univalent function theory, we begin with a short exposition of the problems and a discussion of our method of solution. By hyperbolic space we mean the complete, simply connected manifold with constant, sectional curvature -1; by a univalent function we mean a one to one conformal map defined over some domain in S^2.

Let ψ be a holomorphic function defined in the unit disk, D_1. Suppose $\psi'(z) \neq 0$ anywhere in D_1 or more geometrically that ψ is locally one to one. It is reasonable to inquire after conditions which imply that ψ is globally one to one. Beginning with Nehari, many authors have studied answers to this question phrased in terms of differential inequalities satisfied by ψ, see for instance [Ahl 1], [Ahl-We], [Be], [Be-Po], [Ge-Po]. Nehari's condition states: ψ is univalent in D_1 provided

$$\sup_{|\xi|<1} (1-|\xi|^2)^2 \, |\mathcal{S}_\psi(\xi)| \leqq 2. \tag{0. 1}$$

Here $\mathcal{S}_\psi(\xi) = \left(\frac{\psi''}{\psi'}\right)' - \frac{1}{2}\left(\frac{\psi''}{\psi'}\right)^2$ is the Schwarzian derivative of ψ. Another well known condition is that of Duren, Shields, Shapiro and Becker: ψ is univalent in D_1 if

$$\sup_{|\xi|<1} (1-|\xi|^2) \left|\xi \frac{\psi''}{\psi'}(\xi)\right| \leqq 1. \tag{0. 2}$$

The proofs of these results were of three main types:

1) Nehari's original proof used estimates on second order differential equations and a classical connection between second order linear equations and Schwarzian derivatives, see [Po].

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2) Proofs involving Löwner chains, see [Be] and [Be-Po].

3) A topological argument via quasiconformal extensions, see [Ahl-We].

Our argument is closest to the third type which was introduced by Ahlfors and Weill. This argument goes along the following lines:

First suppose that ψ is holomorphic in a neighborhood of $\bar{D}_1$ and that

$$\sup_{|\xi|<1} (1-|\xi|^2)^2 \, |\mathcal{S}_\psi(\xi)| \leqq 2\,k < 2.$$

Using the aforementioned connection between Schwarzian derivatives and second order equations, ψ was extended as a quasiconformal, locally one to one, map from S^2 to S^2. It is an elementary fact from topology that the extended map is globally one to one. The general case is treated by a limiting argument.

The significance of the existence of a quasiconformal extension for functions satisfying (0. 1) strictly was only recognized later, see [Ahl 2]. Using a method that involved the Löwner equation Becker and Pommerenke proved analogous results for conditions similar to (0. 2).

In this paper we prove generalizations of these results via a method similar in spirit to that of Ahlfors and Weill. We show, more or less directly, that the image of ψ is simply connected. Since ψ is holomorphic it is proper and therefore by degree theory ψ must be one to one. Without insisting on technicalities the differential inequality ψ must satisfy is:

$$\text{(0. 3)} \qquad \sup_{|\xi|<1} \left| \frac{(1-|\xi|^2)^2 \left[g_{\xi\xi} - g_\xi^2 - \frac{1}{2}\mathcal{S}_\psi(\xi) \right] - 2\,\xi(1-|\xi|^2)\,g_\xi}{1+(1-|\xi|^2)^2\, g_{\xi\bar{\xi}}} \right| \leqq 1$$

For g a real valued function in $C^5(D_1)$, see Theorem 7. 2. If the inequality is satisfied strictly then ψ has a quasiconformal extension to S^2.

In the last twenty years many connections have been established between hyperbolic geometry and conformal geometry on S^2. Fundamentally, they are connected because S^2 is the geometric boundary of $\mathbb{H}^3$ and the isometries of $\mathbb{H}^3$ extend to be conformal automorphisms of S^2. The link between the two geometries, which we exploit, is via hyperbolic analogues of the Gauss map and support function of a surface immersed in $\mathbb{R}^3$.

Recall that the Gauss map, G associates to a point, p on a surface, $\Sigma \hookrightarrow \mathbb{R}^3$ an oriented unit normal vector to Σ at p. This vector is thought of as a point on the unit sphere in $\mathbb{R}^3$. The support function, ϱ assigns to a point $p \in \Sigma$ the affine parameter of the plane tangent to Σ at p. Where the Gauss map is locally invertible, the support function can be thought of as a function defined on the Gauss image. That is, at each point $\theta = G(p)$ we assign a real number, $\varrho(\theta)$ such that the plane:

$$\{x \in \mathbb{R}^3 : X \cdot \theta = \varrho(\theta)\}$$

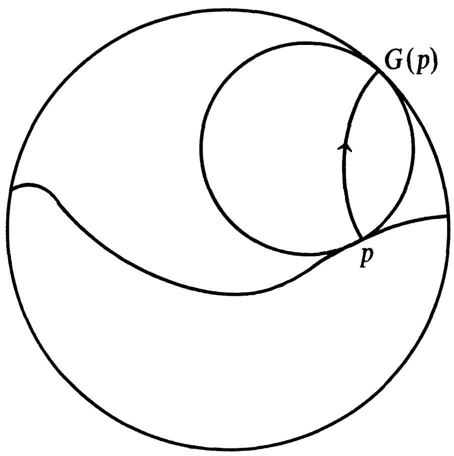
is tangent to Σ at p. The surface can be reconstructed by forming the envelope of these planes for $\theta \in G(\Sigma)$.

If we model $\mathbb{H}^3$ as the interior of the unit ball in $\mathbb{R}^3$ with the metric:

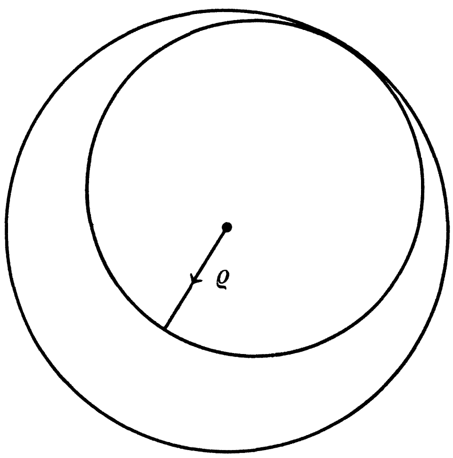
$$ds^2 = \frac{4(dx^2 + dy^2 + dz^2)}{(1 - r^2)^2}$$

then geodesics are circular arcs normal to S^2. Thus each geodesic defines a unique pair of points on S^2 and vice versa.

We define the Gauss map for a surface $\Sigma \hookrightarrow \mathbb{H}^3$ by first choosing an orientation and then following the geodesic normal to Σ at p to the sphere infinity in the given direction:

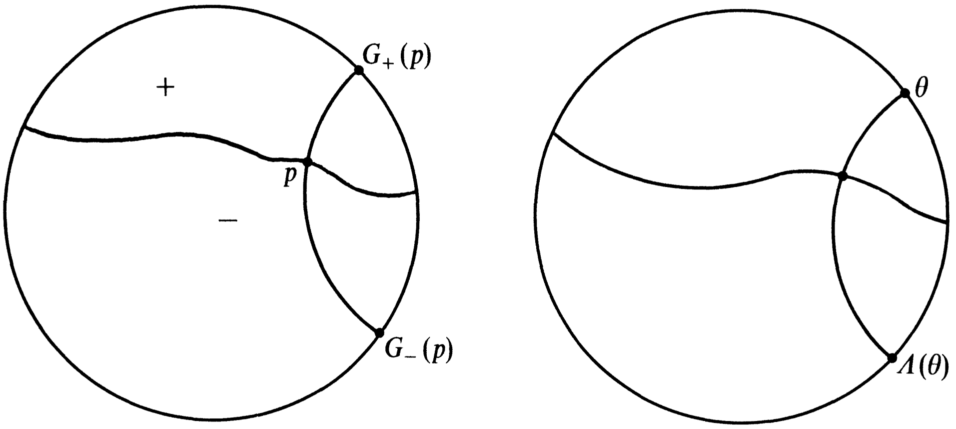


To define a support function we assign to each point $p \in \Sigma$ the "affine" parameter of the horosphere tangent to Σ at p on the side given by the orientation. In the ball model a horosphere is a Euclidean sphere internally tangent to the unit sphere. The "affine" parameter is the hyperbolic distance from a fixed origin, $\mathcal{O}$ to the nearest point on the horosphere.



In Euclidean space the Gauss map and support function merely change sign if we reverse the orientation of the surface; in hyperbolic space the differences are more significant so we usually speak of two Gauss maps, G_+ and G_-. Near a point where G_+ is invertible we can define a map Λ from S^2 to S^2 by:

$$\Lambda(\theta) = G_- \circ G_+^{-1}(\theta).$$



The dilatation of this map is expressed in terms of the principal curvatures (k_1, k_2) of Σ by:

(0. 4) $$K(A;\theta)=\max_{+,-}\left\{\left|\frac{1\mp k_1(p)}{1\pm k_1(p)}\cdot\frac{1\pm k_2(p)}{1\mp k_2(p)}\right|^{\frac{1}{2}}\right\},$$

where $\theta = G_+(p)$. From (0. 4) it is evident that surfaces whose principal curvatures satisfy

(0. 5) $$|k_i|<1 \quad i=1,2$$

play an important role in the construction of quasiconformal reflections.

In the first several sections we prove that a complete surface whose principal curvatures satisfy (0. 5) is properly embedded, homeomorphic to a disk and its Gauss maps are one to one onto disjoint regions of S^2, see Theorem 2. 2 and Corollary 4. 5.

To return to the proof of the univalence criterion we think of ψ as a conformal map from D_1 to $S^2=\partial \mathbb{H}^3$. Define a support function locally on $\psi(D_1)$ via:

$$\varrho(\theta)=\log\frac{|\phi'(\theta)|}{(1-|\phi|^2)}+g\circ\phi(\theta);$$

ϕ is a local inverse to ψ. We use this support function to piece together a complete surface, Σ_ψ. In virtue of (0. 3) we can show that the principal curvatures of Σ_ψ satisfy (0. 5) and therefore Σ_ψ is homeomorphic to a disk. We have constructed a disk in $\mathbb{H}^3$ whose homeomorphic, Gauss image is $\psi(D_1)$. By the topological result mentioned above ψ must be globally one to one. A limiting argument similar to that in [Ahl 1] completes the proof. If ψ satisfies (0. 3) strictly then the map A defined by Σ_ψ is a quasiconformal reflection with fixed curve $\psi(\partial D_1)$. Thus we obtain a quasiconformal extension of ψ to S^2, but a posteriori.

This approach unifies the different univalence criteria and construction of quasiconformal extensions. By recasting these questions as problems in hyperbolic geometry it gives a new way to think about such problems and brings to bear the considerable techniques of modern Riemannian geometry.

§ 1. Introduction

In [Ep 1] and [Ep 2] we introduced a hyperbolic analogue for the Gauss map and applied it to study several problems in differential geometry and function theory. In this paper we continue along those lines exploring a connection between surfaces in hyperbolic 3-space and quasiconformal reflections of S^2. As in the previous papers we will work in the Poincarè ball model of $\mathbb{H}^3$.

The surfaces which engender quasiconformal reflections are those whose principal curvatures, (k_1, k_2) satisfy:

$$(1.1)\qquad |k_i| < 1 \quad i = 1, 2.$$

In [Ep 1] we demonstrated that such a surface is necessarily imbedded and diffeomorphic to a disk; the proof presented there also shows that the imbedding is proper. In §§ 2—4 we will prove that both Gauss maps of such a surface are diffeomorphisms onto disjoint open subsets of S^2, and the asymptotic boundary of Σ, $\partial_\infty \Sigma$ is a Jordan curve.

Let N denote a globally defined unit normal field on the surface Σ. Recall that $\psi^t(p, X)$ is the constant speed geodesic with initial point p and initial velocity $X \in T_p \mathbb{H}^3$. The forward and backward Gauss maps of Σ are defined by:

$$(1.2)\qquad G_\pm(p) = \lim_{t \to \pm\infty} \psi^t(p, N).$$

The limit is taken in the Euclidean topology of $\mathbb{B}^3$. For surfaces which satisfy (1. 1) $G_\pm$ are diffeomorphisms onto disjoint domains $\Omega_\pm \subset S^2$. We can invert $G_\pm$ and then define:

$$(1.3)\qquad \Lambda = G_- \circ G_+^{-1}.$$

This map is a diffeomorphism from Ω_+ onto Ω_-. The closure of Ω_+ and Ω_- meet along $\partial_\infty \Sigma$. In § 5 we show that Λ is quasiconformal whenever Σ satisfies (1. 1). The dilatation at a point $G_+(p)$ is:

(1. 4)

$$K(\Lambda, G_+(p)) = \max \left\{ \left| \frac{1-k_1(p)}{1+k_1(p)} \cdot \frac{1+k_2(p)}{1-k_2(p)} \right|^{\frac{1}{2}}, \left| \frac{1+k_1(p)}{1-k_1(p)} \cdot \frac{1-k_2(p)}{1+k_2(p)} \right|^{\frac{1}{2}} \right\}.$$

As the Gauss maps are diffeomorphisms we can invert each of them by representing Σ as an envelope of horospheres:

$$\Sigma = \text{Outer envelope of } \{H(\theta, \varrho_\pm(\theta)) : \theta \in \Omega_\pm\}.$$

The support functions, $\varrho_\pm(\theta)$ are defined in $\Omega_\pm$ and have one less derivative than Σ. In § 6 we derive a formula for the Beltrami coefficient of Λ in terms of $\varrho_\pm$.

Using the relations from §§ 5—6 we obtain a sufficient condition for a conformal map defined on the disk to be univalent and to have a quasiconformal extension to the plane. This condition generalizes one given by Ahlfors in [Ahl 1]. It includes, as special cases, the classical condition on the Schwarzian derivative:

$$|\mathcal{S}_\psi(\zeta)\,(1-|\zeta|^2)^2| \leqq 2\,k < 2,$$

the condition of Duren, Shields, Shapiro and Becker for the unit disk

$$\left|\bar{\zeta}\,\frac{\psi''}{\psi'}(\zeta)\,(1-|\zeta|^2)\right| \leqq k < 1,$$

and for the right half plane

$$2\,\mathrm{Re}\,\zeta\left|\frac{\psi''}{\psi'}(\zeta)\right| \leqq k < 1.$$

Many of the geometric results in this paper are true in any number of dimensions. For ease of exposition we restrict ourselves to three dimensions.

Notation. The notation is identical to that introduced in [Ep 1] and [Ep 2] with a few additions:

$\mathbb{H}^3$ — Hyperbolic 3-space as modelled in the unit ball, $\mathbb{B}^3$ with the metric

$$ds^2 = \frac{4(dx^2+dy^2+dz^2)}{(1-r^2)^2}.$$

$\mathcal{O}$ — The point (0, 0, 0,), considered as the origin in $\mathbb{H}^3$.

$\partial\mathbb{H}^3$ — The ideal boundary of $\mathbb{H}^3$, usually identified with the unit sphere $S^2 = \partial\mathbb{B}^3$.

$d_H(p, q)$ — The hyperbolic distance between the points p and q.

$d_E(p, q)$ — The Euclidean distance between the points p and q.

$H(\theta, \varrho)$ — The horosphere tangent to S^2 at θ which satisfies:

$$|\varrho| = \inf_{q\in H(\theta,\varrho)} d_H(q, \mathcal{O}),$$

$\varrho > 0$ if $\mathcal{O}$ is in the exterior of $H(\theta, \varrho)$ and negative otherwise. ϱ is called the "horospheric radius" of $H(\theta, \varrho)$.

$B(\theta, \varrho)$ — The open ball bounded by $H(\theta, \varrho)$.

$[p, \theta]$ — The horospheric radius of the horosphere tangent to $\theta \in S^2$ passing through $p \in \mathbb{H}^3$; if $p \in H(\theta, \varrho)$ then

$$[p, \theta] = \varrho.$$

γ_{pq} — The hyperbolic geodesic connecting p to q.

$r\omega$ — Euclidean polar coordinates in $\mathbb{B}^3$ with respect to $\mathcal{O}$.

$\sigma\omega$ — Hyperbolic geodesic polar coordinates with respect to $\mathcal{O}$.

$b(\theta, r)$ — The disk in S^2 about θ of radius r in the spherical metric.

D — The gradient operator on S^2 with respect to the round metric.

S — Stereographic projection from S^2 to $\mathbb{C}$:

$$S^{-1}(z) = \frac{(2\,\mathrm{Re}\, z, 2\,\mathrm{Im}\, z, |z|^2 - 1)}{1 + |z|^2}.$$

D_r — The disk of radius r in $\mathbb{C}$ about $z = 0$.

Σ_t — The parallel surface at distance t to Σ.

$T_p^1 \Sigma$ — The space of unit tangent vectors to Σ at p.

$\partial_\infty \Sigma$ — The asymptotic boundary of Σ on $\partial \mathbb{B}^3 = \bar{\Sigma} \cap S^2$.

$G_\pm$ — The forward and backward Gauss maps.

Λ — The composition: $\Lambda = G_- \circ G_+^{-1}$.

(k_1, k_2) — The principal curvatures of $\Sigma \hookrightarrow \mathbb{H}^3$.

$\mathcal{S}_\psi(z)$ — The Schwarzian derivative of ψ:

$$\mathcal{S}_\psi(z) = \left(\frac{\psi''}{\psi'}\right)' - \frac{1}{2}\left(\frac{\psi''}{\psi'}\right)^2.$$

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§ 2. The Gauss maps

In this and the next section we will explore the differential geometry of smoothly imbedded surfaces in $\mathbb{H}^3$ whose principal curvatures satisfy:

$$(1.\,1) \qquad |k_i| < 1 \quad i = 1, 2.$$

Section 2 deals with the Gauss maps of such a surface. We will prove that the Gauss maps are diffeomorphisms onto disjoint open sets.

In §§ 2—4 the surface Σ will be normalized so that it passes through $\mathcal{O}$. We will use (r, ω) to denote Euclidean polar coordinates with respect to $\mathcal{O}$. If V is a tangent vector in $T_{\mathcal{O}} \Sigma$ then $\exp(t V)$ will denote the constant speed geodesic in Σ with initial point $\mathcal{O}$ and direction V. At times we will use the notation $(r_V(t), \omega_V(t))$ to denote $\exp(t V)$ in terms of the (r, ω)-coordinates, that is

$$\exp(t V) = r_V(t)\, \omega_V(t).$$

It follows from Schur's Theorem for $\mathbb{H}^n$ (see § 4) that the function

$$F(V, t) = \operatorname{ch} \{\log [(1 + r_V(t))(1 - r_V(t))^{-1}]\}$$

satisfies $F(V, t) \geqq 1 + \frac{t^2}{2}$ provided the principal curvatures satisfy $|k| \leqq 1$. Thus we see that Σ is necessarily a properly imbedded disk. By the Jordan Brower Theorem Σ separates $\mathbb{H}^3$ into two components, D_+ and D_-. Let $\bar{\Omega}_+$ and $\bar{\Omega}_-$ denote the asymptotic boundaries of D_+ and D_- respectively; these are closed subsets of S^2.

Lemma 2. 1. *The intersection of $\bar{\Omega}_+$ with $\bar{\Omega}_-$ lies in $\partial_\infty \Sigma$.*

Proof of Lemma 2. 1. Let $\theta \in \bar{\Omega}_+ \cap \bar{\Omega}_-$; there are two sequences $\{p_n^+\}$ and $\{p_n^-\}$ contained in D_+ and D_- respectively such that

$$\lim_{n \to \infty} p_n^\pm = \theta.$$

Let γ_n denote the geodesic from p_n^+ to p_n^-. As Σ separates D_+ from D_- $\gamma_n \cap \Sigma$ is not empty. Let q_n be a point in this intersection. Since $\{p_n^+\}$ and $\{p_n^-\}$ converge to the same point on $\partial \mathbb{H}^3$ the Euclidean diameter of γ_n tends to zero. From this it is immediate that:

$$\lim_{n \to \infty} q_n = \theta$$

as well. □

Let $\tilde{\Omega}_\pm = \bar{\Omega}_\pm \setminus \partial_\infty \Sigma \cap \bar{\Omega}_\pm$. As $S^2 = \bar{\Omega}_+ \cup \bar{\Omega}_- \cup \partial_\infty \Sigma$ and $\partial_\infty \Sigma$ is a closed set, it follows from Lemma 2. 1 that $\tilde{\Omega}_\pm$ are open sets as each is the union of connected components of the open set $S^2 \setminus \partial_\infty \Sigma$. We will show they are the images of $G_\pm$.

The proofs in this section make use of the following, well known geometric construction for a horosphere:

Let $S(p, N, r)$ be the hyperbolic sphere of radius r, passing through the point p, with inner normal vector N at p. The limit (in the Euclidean topology of $\mathbb{B}^3$) of $S(p, N, r)$, as r tends to infinity is the horosphere, $H(\theta, \varrho)$, where

$$\theta = \lim_{t \to \infty} \psi^t(p, N)$$

and $\varrho = [p, \theta]$. Recall $[p, \theta]$ is the real number for which

$$p \in H(\theta, [p, \theta]);$$

this number is called the horospheric radius of p relative to θ.

Through a series of lemmas we will prove the following theorem:

Theorem 2. 2. *If Σ is a complete, C^5 imbedding of D_1 into $\mathbb{H}^3$ whose principal curvatures, (k_1, k_2) satisfy*

$$\sup_{p\in\Sigma} \max\{|k_1(p)|, |k_2|(p)|\} < 1, \tag{2. 1}$$

then the Gauss maps $G_\pm$ are diffeomorphisms onto the disjoint open regions $\tilde{\Omega}_\pm$.

Remark. In the sequel $B(\theta, \varrho)$ will denote the open ball bounded by $H(\theta, \varrho)$.

The following lemma forms the core of the proof of the theorem:

Lemma 2. 3. *If $H(\theta, \varrho(\theta))$ is tangent to Σ then $\Sigma \cap B(\theta, \varrho(\theta))$ is empty provided the principal curvatures satisfy*:

$$\sup_{p\in\Sigma} \max\{|k_1(p)|, |k_2(p)|\} \leqq 1. \tag{2. 2}$$

Proof of Lemma 2. 3. Let N_p denote a unit normal vector to Σ at p, and $S(p, N_p, r)$ the family of spheres defined above. Each sphere in this family is tangent to Σ at p and clearly:

$$H(\theta, \varrho(\theta)) = \lim_{r\to\infty} S(p, N_p, r).$$

In virtue of the smoothness of Σ and the estimate on the principal curvatures, (2. 2) it is easy to see that

$$S(p, N_p, r) \cap \Sigma = \{p\}$$

for sufficiently small r. As the principal curvatures of a sphere of finite radius are larger than 1 there is, for each r, a neighborhood of p, U_r such that:

$$S(p, N_p, r) \cap U_r = \{p\}. \tag{2. 3}$$

If the lemma were false then, for some finite r, $S(p, N_p, r) \cap \Sigma$ would contain another point q. By choosing the smallest radius for which this pertains we obtain a second point of tangency between $S(p, N_p, r)$ and Σ. Let r_0 denote this radius, q the second point of tangency, and N_q the normal vector at q pointing into $S(p, N_p, r_0)$. It is clear that:

$$\psi^{r_0}(p, N_p) = Q = \psi^{r_0}(q, N_q). \tag{2. 4}$$

From formulae (2. 3) and (2. 4) it follows that the surface, Σ_{r_0}, parallel to Σ at distance r_0 has a self intersection at Q. However as Σ satisfies (2. 2) so does Σ_{r_0} and thus according to Theorem 3. 4 of [Ep 1], Σ_{r_0} can have no self intersections. The contradiction proves the lemma. □

Lemma 2. 4. *The Gauss maps are globally one to one if Σ is a C^3-imbedding of D_1 satisfying* (2. 1).

Proof. Let $\varrho_\pm(\theta)$ denote the horospheric radius of the horosphere tangent to Σ at p and to $\partial \mathbb{B}^3$ at $\theta = G_\pm(p)$. From Lemma 2. 3 it follows that $\varrho_\pm(\theta)$ are single valued functions on $G_\pm(\Sigma)$ whether or not $G_\pm$ are one to one. For suppose

$$G_\pm(p) = G_\pm(q) = \theta$$

but that the horosphere tangent to Σ at p is not the horosphere tangent to Σ at q; denote these by H_q and H_p respectively. As H_q and H_p are tangent to $\partial_\infty \mathbb{H}^3$ at the same point either $H_p \subsetneqq H_q$ or $H_q \subsetneqq H_p$. This contradicts Lemma 2. 3; thus we see that $\varrho_\pm(\theta)$ are single valued functions.

If $H(\theta, \varrho)$ is tangent to Σ then the order of contact at points of tangency is exactly one, for the second fundamental form of Σ strictly dominates that of $H(\theta, \varrho_\pm)$. We can therefore conclude that the intersection $H(\theta, \varrho_\pm(\theta)) \cap \Sigma$ consists of isolated points.

From Lemma 1. 4 of [Ep 2] it follows that $\varrho_\pm(\theta)$ are at least C^2-functions. In the derivation of the formula for the envelope of a C^1-family of horospheres,

$$\{H(\theta, \varrho(\theta)) \colon \theta \in U\}$$

we observed that the equation defining the outer envelope always has a unique solution (cf. § 2 of [Ep 1]). All that this derivation required was that the point of contact between $H(\theta, \varrho(\theta))$ and the envelope Σ be a differentiable function of θ.

The families of horospheres in question are given parametrically by:

$$R_\pm(\theta, \alpha) = \tfrac{1}{2}(1 + r_\pm(\theta))\, X(\theta) + \tfrac{1}{2}(1 - r_\pm(\theta))\, Y(\alpha),$$
$$r_\pm(\alpha) = (e^{\varrho_\pm(\theta)} - 1)\,(e^{\varrho_\pm(\theta)} + 1)^{-1}.$$

Here $X(\theta)$ and $Y(\alpha)$ are parametric representations of the unit sphere, with $X(\theta)$ the point of tangency between $H(\theta, \varrho)$ and $\partial \mathbb{H}^3$.

In § 5 we will show that dG_+ (resp. dG_-) is invertible provided $k_i \neq 1$ (resp. $k_i \neq -1$), $i = 1, 2$. Thus the Gauss map can be locally inverted to give parametrizations of the surface by its Gauss image. Let $F(\theta)$ denote one such determination of an inverse. By Lemma 1. 4 of [Ep 2], $F(\theta)$ is a C^2-function of θ. Thus we can write $Y(\alpha_\pm(\theta))$ as:

$$Y(\alpha_\pm(\theta)) = 2(1 - r_\pm(\theta))^{-1}\,(F(\theta) - \tfrac{1}{2}(1 + r_\pm(\theta))\, X(\theta)).$$

The right hand side is a twice differentiable S^2-valued function of θ. We can smoothly invert the coordinate representation $Y(\alpha)$ of S^2 to obtain:

$$\alpha_\pm(\theta) = Y^{-1}\left[2(1 - r_\pm(\theta))^{-1}\left(F(\theta) - \left(\frac{1 + r_\pm(\theta)}{2}\right) X(\theta)\right)\right].$$

From this formula it is evident that $\alpha_\pm(\theta)$ is a C^2-function of θ for any determination of an inverse of $G_\pm$. By the aforementioned uniqueness result the point of contact between $H(\theta, \varrho_\pm(\theta))$ and Σ must be unique. From this the lemma is immediate. □

Remark. The previous lemma is in effect a monodromy theorem for envelopes: if the generating function for a smooth envelope is single valued and the points of contact are isolated, then they are unique.

Lemma 2. 5. *If Σ is a complete C^2 imbedded surface satisfying* (2. 1) *then the images of the Gauss maps are disjoint from $\partial_\infty \Sigma$.*

Proof of Lemma 2. 5. As the argument for G_+ is identical to that for G_-, we present the argument only for G_+. Let $\theta_0 = G_+(p)$. In the previous proof we observed that G_+ is an open mapping and thus the image of G_+ contains an open disk, $b(\theta_0, r)$ about θ_0. For θ in $b(\theta_0, r)$ let $\varrho_+(\theta)$ be the horospheric radius such that $H(\theta, \varrho_+(\theta))$ is tangent to Σ. Evidently $\varrho_+(\theta)$ is a bounded function in $b(\theta_0, r)$. From Lemma 2. 3 it follows that Σ is disjoint from each of the balls $B(\theta, \varrho_+(\theta))$ and thus disjoint from the set V, defined by:

$$V = \bigcup_{\theta \in b(\theta_0, r)} B(\theta, \varrho_+(\theta)).$$

From this it is clear that no point in $b\left(\theta_0, \frac{r}{2}\right)$ lies in the asymptotic boundary of Σ. □

The following lemma completes the proof of the theorem:

Lemma 2. 6. *For any properly immersed surface Σ, every point in $\partial \mathbb{H}^3 \setminus \partial_\infty \Sigma$ is in the image of one of the Gauss maps.*

Proof of Lemma 2. 6. If θ lies in $\partial \mathbb{H}^3 \setminus \partial_\infty \Sigma$, then there exists an R such that

$$H(\theta, \varrho) \cap \Sigma = \emptyset$$

if $\varrho > R$. The horospheres tangent to $\partial \mathbb{H}^3$ at θ foliate $\mathbb{H}^3$ and thus there is a first ϱ, call it ϱ_0, such that the set

$$A = H(\theta, \varrho_0) \cap \Sigma$$

is not empty. The points in A are points of tangency between $H(\theta, \varrho_0)$ and Σ and therefore θ is either in the forward or backward Gauss map of some point in A. □

Now we assemble the proof of the Theorem:

Proof of Theorem 2. 2. From Lemma 2. 6 it follows that every point in $\tilde{\Omega}_+ \cup \tilde{\Omega}_-$ is in the image of either G_+ or G_-. From Lemma 2. 5 and the proof of Lemma 2. 3 it follows that the image of G_+ (resp. G_-) is contained in $\tilde{\Omega}_+$ (resp. $\tilde{\Omega}_-$). From Lemma 2. 4 it follows that G_+ (resp. G_-) is a diffeomorphism. This completes the proof of the theorem. □

In the proof of the theorem we also showed

Proposition 2. 7. *The boundary of* $\tilde{\Omega}_+$ *(resp.* $\tilde{\Omega}_-$*) is contained in* $\partial_\infty \Sigma$.

Remark. As $\tilde{\Omega}_+$ (resp. $\tilde{\Omega}_-$) is the interior of $\bar{\Omega}_+$ (resp. $\bar{\Omega}_-$) we will henceforth denote this set by Ω_+ (resp. Ω_-).

We close this section with a result on $\partial_\infty \Sigma$:

Proposition 2. 8. *If* Σ *is a complete imbedded surface satisfying* (2. 1) *then* $\partial_\infty \Sigma$ *has empty interior.*

Proof of Proposition 2. 8. Let θ_0 be a point in $\partial_\infty \Sigma$ and let $\{p_n\}$ be a sequence of points in Σ converging to θ_0. Set $\theta_n^\pm = G_\pm(p_n)$. From Lemma 2. 5 it follows that $\{\theta_n^\pm\}$ lies in $\partial \mathbb{H}^3 \setminus \partial_\infty \Sigma$. Let γ_n be the geodesic joining θ_n^+ to θ_n^-; note that p_n is a point on this geodesic. It is convenient, for this argument, to represent hyperbolic 3-space as the upper half space, $\{(x, y): x \in \mathbb{R}^2, y > 0\}$. Let θ_0 correspond to (0, 0, 0) and let (x_n, y_n) denote the coordinates of the points p_n. Because (x_n, y_n) tends to (0, 0, 0) as n tends to infinity it is evident that

$$d_n = \min\{d_E(\theta_n^+, (0,0,0)), d_E(\theta_n^-, (0,0,0))\}$$

(d_E: Euclidean distance in $\mathbb{R}^3$) must tend to zero as n tends to infinity. But this implies that we can extract a subsequence $\{\theta_{n_j}\}$ from $\{\theta_n^+, \theta_n^-\}$ which converges to θ_0. As θ_0 was arbitrary and $\{\theta_{n_j}\} \subset S^2 \setminus \partial_\infty \Sigma$ the proposition follows. □

Corollary 2. 9. *The asymptotic boundary of* Σ *is given by*:

$$\partial_\infty \Sigma = \partial \Omega_+ \cup \partial \Omega_- .$$

§ 3. The asymptotic boundary

In this section we will prove that under an additional hypothesis $\partial_\infty \Sigma$ is actually a Jordan curve. The additional topological hypothesis we make is that $\partial_\infty \Sigma$ is locally connected; in § 4 we will show this is always true. For the argument we will use the representations of Σ as envelopes of horospheres. Let $\varrho_+(\theta)$ and $\varrho_-(\theta)$ denote the support functions defined on Ω_+ and Ω_- respectively. In the proof of Lemma 2.4 we showed that these functions are single valued on their respective domains and have one less derivative than Σ.

As G_+ is invertible we can map Ω_+ to Ω_- by the mapping

$$\Lambda = G_- \circ G_+^{-1}.$$

Using Λ we can express ϱ_- in terms of ϱ_+:

Theorem 3. 1. *The support functions are related by*:

$$\varrho_-(\Lambda(\theta)) = \log(1 + |D\varrho_+(\theta)|^2) - \varrho_+(\theta). \tag{3.1}$$

Proof of Theorem 3. 1. In the notation for the horospheric radius introduced in § 2, the support functions are expressed by

$$\varrho_\pm(\theta) = [G_\pm^{-1}(\theta), \theta] \quad \text{for} \quad \theta \in \Omega_\pm.$$

Denote the point on S^2 antipodal to θ by θ^*. If $\Lambda(\theta) = \theta^*$ then it is clear that:

$$\varrho_-(\Lambda(\theta)) = -\varrho_+(\theta).$$

As is well known the horospheric radius satisfies the following transformation law under the action of a hyperbolic isometry, g:

$$[gp, g\theta] = [p, \theta] + [g \cdot \mathcal{O}, g \cdot \theta], \tag{3.2}$$

(see [L-P]).

Suppose that g is a parabolic transformation fixing θ and carrying $\Lambda(\theta)$ to θ^*, from formula (3. 2) we see that for the point $p = G_+^{-1}(\theta)$:

$$[p, \Lambda(\theta)] = [gp, g\Lambda(\theta)] - [g\mathcal{O}, g\Lambda(\theta)] = [gp, \theta^*] - [g\mathcal{O}, \theta^*]. \tag{3.3}$$

The point gp is on the geodesic joining θ to θ^* and therefore

$$[gp, \theta^*] = -[gp, \theta] = -[p, \theta].$$

From this and (3. 3) we obtain:

$$[p, \Lambda(\theta)] = -[p, \theta] - [g\mathcal{O}, \theta^*] = -\varrho_+(\theta) - [g\mathcal{O}, \theta^*]. \tag{3.4}$$

The problem is now reduced to calculating $[g\mathcal{O}, \theta^*]$ for a parabolic transformation g which fixes θ and carries $\Lambda(\theta)$ to θ^*. This is a two dimensional problem as θ, $\Lambda(\theta)$, $\mathcal{O}$ and the intersection point $H(\theta, \varrho_+(\theta)) \cap H(\theta, \varrho_-(\theta))$ lie in a hyperbolic plane, $\mathscr{H}$. We can assume without loss of generality, that $\mathscr{H} = \mathbb{H}^3 \cap \{z = 0\}$, $\theta = (1, 0)$ and $\Lambda(\theta) = e^{i\varphi}$. The parabolic subgroup of $SU(1, 1)$ that fixes θ is:

$$g_t = \begin{pmatrix} 1 + it & -it \\ it & 1 - it \end{pmatrix}, \quad t \in \mathbb{R}.$$

We need to solve equation (3. 5) for t:

$$g_t e^{i\phi} = -1. \tag{3. 5}$$

An elementary calculation shows that:

$$e^{i\phi} = \frac{2it-1}{2it+1}.$$

Thus

$$g_t \mathcal{O} = \frac{t(t-i)}{t^2+1}$$

and therefore:

$$[g_t \mathcal{O}, \theta^*] = -\log(1+4t^2) = \log\left(\frac{1-\cos\phi}{2}\right). \tag{3. 6}$$

To complete the calculation we need to determine the cosine of the angle $\sphericalangle\, \Lambda(\theta)\, \mathcal{O}\, \theta$ in the plane $\mathscr{H}$. Formula (2. 4) of [Ep 1] gives the parametric representation for Σ_t in terms of its Gauss image as:

$$R_{\varrho+t}(\theta) = \frac{|D\varrho(\theta)|^2 + (e^{2(\varrho+t)}-1)}{|D\varrho(\theta)|^2 + (e^{\varrho+t}+1)^2} X(\theta) + \frac{2\, D\varrho(\theta)}{|D\varrho(\theta)|^2 + (e^{\varrho+t}+1)^2}. \tag{3. 7}$$

Inserting ϱ_+ for ϱ in (3. 7) and allowing t to tend to infinity we arrive at a representation for Λ:

$$\Lambda\colon X(\theta) \mapsto \frac{|D\varrho_+(\theta)|^2-1}{|D\varrho_+(\theta)|^2+1} X(\theta) + \frac{2\, D\varrho_+(\theta)}{|D\varrho_+(\varrho)|^2+1}. \tag{3. 8}$$

To compute the cosine we use the fact that $X(\theta)$ and $\Lambda(X(\theta))$ are unit vectors to conclude that:

$$\cos\phi = X(\theta)\cdot\Lambda(\theta) = \frac{|D\varrho_+(\theta)|^2-1}{|D\varrho_+(\theta)|^2+1} \tag{3. 9}$$

where $X\cdot\Lambda$ is the Euclidean inner product in $\mathbb{R}^3$. To obtain (3. 9) we have used the fact that $D\varrho_+(\theta)$ is a vector tangent to S^2 at $X(\theta)$ and thus $D\varrho_+(\theta)\cdot X(\theta)=0$.

Putting the evaluation of $\cos\phi$ into (3. 6) we obtain

$$\log\left(\frac{1-\cos\phi}{2}\right) = -\log(|D\varrho_+(\theta)|^2+1). \tag{3. 10}$$

By combining (3. 4) and (3. 10) one completes the proof of (3. 1). □

The formula for the action of Λ on S^2 is worth noting as a separate proposition:

Proposition 3. 2. *If ϱ_+ is the support function of the smooth surface $\Sigma(\varrho_+)$ with Gauss maps G_+ and G_- then the map $\Lambda = G_- \circ G_+^{-1}$ is*

$$\Lambda: X(\theta) \mapsto \frac{|D\varrho_+(\theta)|^2 - 1}{|D\varrho_+(\theta)|^2 + 1} X(\theta) + \frac{2\,D\varrho_+(\theta)}{|D\varrho_+(\theta)|^2 + 1}. \tag{3. 11}$$

Remark. In this paper as in [Ep 1] we will frequently use $X(\theta)$ to denote a point on S^2 which we will often identify with θ itself. Though this is somewhat imprecise it should cause no confusion.

In Proposition 2. 8 we showed that $\partial_\infty \Sigma$ has empty interior. As

$$S^2 = \Omega_+ \amalg \partial_\infty \Sigma \amalg \Omega_-$$

it follows that the points of $\partial_\infty \Sigma$ are divided into three disjoint sets:

$$\partial_\infty \Sigma = A_+ \amalg B \amalg A_-.$$

A point θ is in A_+ (resp. A_-) if, for sufficiently small r, $b(\theta, r) \cap \Omega_-$ (resp. $b(\theta, r) \cap \Omega_+$) is empty. The points not in $A_+ \cup A_-$ are in B. We will show that $A_+ \cup A_-$ is empty when Σ satisfies (2. 1). The proof is effected by a series of lemmas; the argument for A_+ is identical to that for A_- so we present the argument only for A_+.

Lemma 3. 3. *If $\theta_0 \in A_+$ then there is a sequence of points $\{\theta_n\}$ in Ω_+ tending to θ_0 such that*

$$\lim_{n\to\infty} \varrho_+(\theta_n) = \infty.$$

Proof of Lemma 3. 3. Suppose the Lemma is false, then for some $r > 0$

$$b(\theta_0, r) \cap \Omega_- = \emptyset$$

and

$$\sup_{\theta \in b(\theta_0, r) \cap \Omega_+} \varrho_+(\theta) = M < \infty.$$

From Lemma 2. 3 it follows that Σ is disjoint from the open set V given by:

$$V = \bigcup_{\theta \in b(\theta_0, r) \cap \Omega_+} B(\theta, M).$$

The set $\partial_\infty \Sigma$ has empty interior; therefore $\Omega_+ \cap b(\theta_0, r)$ is dense in $b(\theta_0, r)$. From this it is evident hat

$$V = \bigcup_{\theta \in b(\Omega_0, r)} B(\theta, M)$$

as well. It is clear that if $\Sigma \cap V$ is empty, then θ_0 cannot be in the asymptotic boundary of Σ. The contradiction proves the Lemma. □

By combining Lemma 3. 3 with formula 3. 1 we can now prove:

Proposition 3. 4. *The set A_+ is empty.*

Proof of Proposition 3. 4. If the proposition is false then there exists a point $\theta_0 \in \partial_\infty \Sigma$ and an $r > 0$ such that

$$\Omega_- \cap b(\theta_0, r) = \emptyset.$$

Let $\{\theta_n\}$ be a sequence of points in A_+, whose existence follows from Lemma 3. 3, such that

$$\{\theta_n\} \subset \Omega_+$$

and

$$\lim_{n\to\infty} \varrho_+(\theta_n) = \infty. \tag{3.12}$$

As D_+ is open and disjoint from D_- and furthermore $H(\theta, \varrho_-(\theta))$ lies in D_- for every $\theta \in \Omega_-$ it is easy to see that $\varrho_-(\theta)$ is bounded from below. Thus it follows from formula (3. 1) that:

$$\lim_{n\to\infty} |D\varrho_+(\theta_n)| = \infty$$

as well. From formula (3. 12) we easily derive that

$$d_E(\Lambda(\theta), \theta) = 2(1 + |D\varrho_+(\theta)|^2)^{-\frac{1}{2}}. \tag{3. 13}$$

From (3. 12) and (3. 13) we conclude that

$$\lim_{n\to\infty} d_E(\theta_n, \Lambda(\theta_n)) = 0.$$

For sufficiently large n, $\Lambda(\theta_n)$ must lie in $b(\theta_0, r)$. However $\Lambda(\theta_n)$ lies in Ω_- and thus

$$b(\theta_0, r) \cap \Omega_- \neq \emptyset.$$

As $\Omega_- \cap \Omega_+ = \emptyset$ this contradiction completes the proof of Proposition 3. 4. □

Altogether we've now proved:

Theorem 3. 5. *If Σ is a complete, embedded surface satisfying* (2. 1) *then*

$$\partial\Omega_+ = \partial\Omega_- = \partial_\infty \Sigma.$$

If we make stronger topological assumptions we can show that $\partial_\infty \Sigma$ is a Jordan curve:

Theorem 3. 6. *If $\partial_\infty \Sigma$ is locally connected and satisfies the hypotheses of the previous theorem then $\partial_\infty \Sigma$ is a Jordan curve.*

Remark. The topological conditions $\partial\Omega_+ = \partial\Omega_- = \partial_\infty \Sigma$ do not suffice to show that $\partial_\infty \Sigma$ is a Jordan curve. A simple modification of the well known "dragon's teeth" example gives a set A which is a common boundary of two complementary disks but is not locally connected and therefore not a Jordan curve (see Figure 1).

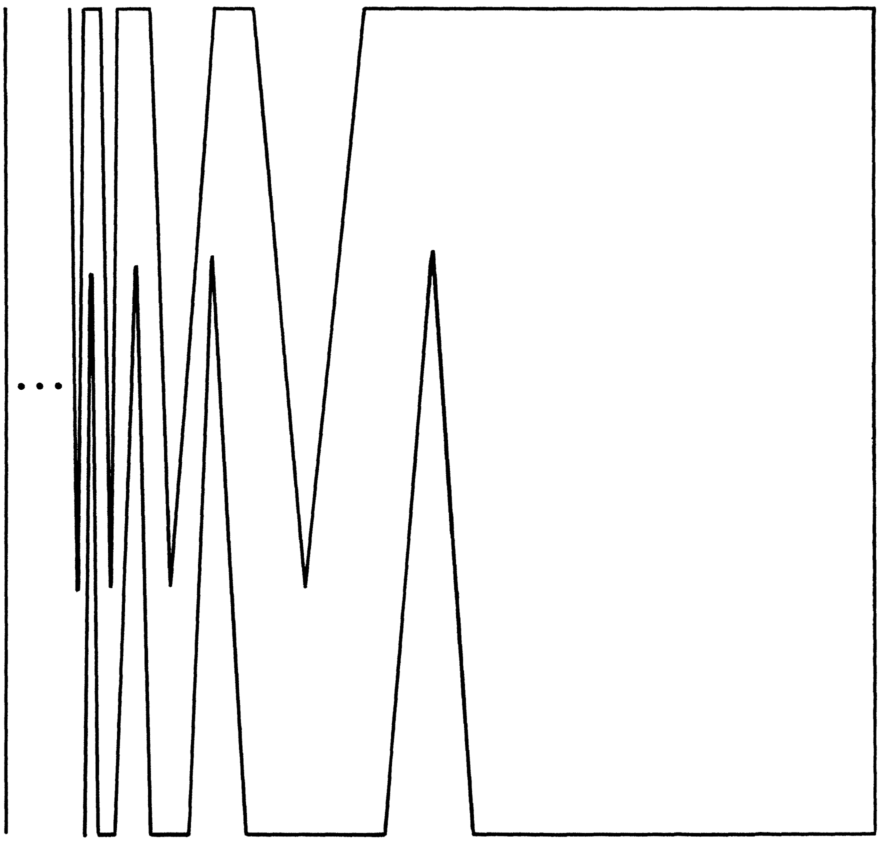

Figure 1

Proof of Theorem 3. 6. In this argument we use some well known facts about the boundaries of simply connected regions in S^2. These results can be found in Chapter 9 of [Po]. The set $\partial\Omega_+ = \partial\Omega_- = \partial_\infty\Sigma$ can be divided into two disjoint subsets A and B. A point, θ is in A if $\partial_\infty\Sigma\setminus\{\theta\}$ has at least two components and in B otherwise. As $\partial_\infty\Sigma$ is locally connected it is a Jordan curve if and only if A is empty. The points in A are called cut points.

Suppose $\theta_0 \in \partial_\infty\Sigma$ is a cut point. Let $\{\gamma_n\}$ be the connected components of $\partial_\infty\Sigma\setminus\{\theta_0\}$. At least one of the arcs $\gamma_n \cup \{\theta_0\}$ contains a simple closed curve for otherwise $S^2\setminus\partial_\infty\Sigma$ would have a single component. On the other hand only one of the arcs $\gamma_n \cup \{\theta_0\}$ can contain a simple closed curve for the complement of $\partial_\infty\Sigma$ has exactly two components.

Let $\gamma_1 \cup \{\theta\}$ be the arc which contains the simple closed curve. Each of the other arcs lies in one of the components of $S^2\setminus\gamma_1 \cup \{\theta_0\}$. Evidently the points on these arcs are only boundary points of one of the sets, Ω_+ or Ω_-. This contradicts Theorem 3. 5 and shows that A is empty. □

Proposition 3. 7. *If Σ is a complete embedded surface satisfying* (2. 1) *then* $|D\varrho_+(\theta)|$ *tends to infinity as* θ *tends to* $\partial\Omega_+$.

Proof of Proposition 3. 7. Formula (3. 1) states

$$\log(1+|D\varrho_+(\theta)|^2)=\varrho_+(\theta)+\varrho_-(\Lambda(\theta)).$$

To prove the proposition it suffices to show that

$$\lim_{n\to\infty}\varrho_+(\theta_n)+\varrho_-(\Lambda(\theta_n))=\infty \tag{3. 14}$$

for any sequence $\{\theta_n\}$ tending to $\partial\Omega_+$. If $\lim\limits_{n\to\infty}\varrho_+(\theta_n)=+\infty$ or $\lim\limits_{n\to\infty}\varrho_-(\Lambda(\theta_n))=\infty$ then (3. 14) follows for both ϱ_+ and ϱ_- are bounded below.

Suppose $\{\theta_n\}$ is a sequence tending to $\partial\Omega_+$ such that

$$|\varrho_+(\theta_n)|<M.$$

There are points $\{p_n\}$ on Σ such that

$$G_+(p_n)=\theta_n.$$

It is clear that p_n tends to $\partial_\infty\Sigma$ as n tends to infinity. Otherwise we could extract subsequences $\{p_{n_k}\}$ and $\{\theta_{n_k}\}$ converging to $p^*\in\Sigma$ and $\theta^*\in\partial\Omega_+$. As $\partial\Omega_+=\partial_\infty\Sigma$ it follows that $G_+(p^*)\in\partial_\infty\Sigma$. This contradicts Lemma 2. 5.

Let $H_n^\pm=H(G_\pm(p_n),\varrho_\pm(G_\pm(p_n)))$. Evidently

$$H_n^+\cap H_n^-=p_n.$$

If $|\varrho_+(\theta_n)|<M$ then H_n^- is a sphere tangent to the unit sphere and a sphere, tangent to the unit sphere, of radius at least M' (for some $M'>0$). The point of tangency tends to the unit sphere and therefore the Euclidean radius of H_n^- must tend to zero. From this it is immediate that

$$\lim_{n\to\infty}\varrho_-(\Lambda(\theta_n))=\infty. \qquad \square$$

§ 4. The asymptotic boundary continued

In this section we will show that $\partial_\infty\Sigma$ is a locally connected set by realizing it as the image of a continuous map. It then follows from the results in the previous section that $\partial_\infty\Sigma$ is a Jordan curve. The argument uses a hyperbolic analogue of Schur's Theorem proved in [Ep 4]:

Theorem 4. 1. *Let $c(t)$ be a smooth arclength parametrized curve in $\mathbb{H}^3$ with curvature $K(t)$. If $|K(t)|\leqq 1$ then*

$$\cosh[d(c(0),c(t))]\geqq 1+\frac{t^2}{2}. \tag{4. 1}$$

Proof of Theorem 4. 1. The analogue of Schur's theorem states:

Theorem A [Ep 4]. Let $c_1(t)$ be an arclength parametrized curve in $\mathbb{H}^3$ with curvature $K_1(t)$ and $c_2(t)$ a planar curve in $\mathbb{H}^2$ with curvature $K_2(t)$, finally let $\tilde{c}_1(t)$ be the planar curve with curvature $|K_1(t)|$. Suppose that all curves have length L and that $\tilde{c}_1(t)$ along with the chord from $\tilde{c}_1(0)$ to $\tilde{c}_1(L)$ and $c_2(t)$ along with the chord from $c_2(0)$ to $c_2(L)$ bound convex regions and that:

$$|K_1(t)| \leqq K_2(t)$$

then

$$d(c_1(0), c_1(L)) \geqq d(c_2(0), c_2(L)).$$

We can apply this to prove Theorem 4. 1 because of the following fact:

If $c(t)$ is a complete curve in $\mathbb{H}^2$ with curvature $K(t)$ satisfying $0 \leqq K \leqq 1$ then

1) $\lim_{t \to \pm\infty} c(t)$ *exist,*

2) $\{c(t): t \in (-\infty, \infty)\}$ *bounds a convex region.*

The fact that $\{c(t)\}$ is properly embedded follows easily from the fact, proved in [Ep 1], that

$$\operatorname{ch} d(c(0), c(t))$$

is a convex function. We will prove the fact under the simplifying assumptions that

1) $K(t) > 0$,

2) $K(t) \equiv 1$ for $|t| > T+1$.

From the second assumption it is evident that

$$\lim_{t \to \pm\infty} c(t)$$

exist and as $\{c(t)\}$ is a Jordan arc separating $\mathbb{H}^2$ into two simply connected regions. Let D denote the "convex" component. For each p on ∂D there is a neighborhood N_p on ∂D such that if $q \in N_p$ then γ_{pq} has in $\bar{D}$. To prove that D is convex it clearly suffices to prove that $N_p = \partial D$ for every $p \in \partial D$. By the strict local convexity it is clear that $N_p \neq \emptyset$; it is also clear that N_p is a closed set. We will be done if we can show that N_p is open. Suppose not. Then we can find a sequence $\{q_n\} \subset N_p^c$ such that

$$\lim_{n \to \infty} q_n = q \in N_p.$$

On each geodesic γ_{pq} there is an arc $\gamma_{r_n s_n} \subset D^c$ such that $\{r_n, s_n\} \subset \partial D$. We can extract a subsequence such that r_n converges to r and s_n converges to s. Either $r = s$ or not; if not then the arc $\gamma_{rs} \in \bar{D} \cap \bar{D}^c = \partial D$, but $K > 0$ and therefore this is not possible. If $r = s$ then $d(r_n, s_n)$ tends to zero and so we can find arbitrarily nearby points (r_n, s_n) such that $\gamma_{r_n s_n} \subset D^c$. For p in a compact subset of ∂D, $\operatorname{diam}(N_p)$ is uniformly bounded from below which contradicts the assertion that $\gamma_{r_n s_n} \subset D^c$ as n tends to infinity. Thus $N_p = \partial D$ for every p.

We remove the hypothesis that $K>0$ by choosing a function $\varphi \in C_0^\infty(\mathbb{R})$ such that

1) $\varphi \geqq 0$,

2) $0<K+\varphi \leqq 1$.

This implies that

2′) $0<K+\varepsilon\varphi \leqq 1, \quad \varepsilon \in (0, 1]$.

Let $\{c^\varepsilon(t)\colon t \in (-\infty, \infty)\}$ denote the curve with curvature $K+\varepsilon\varphi$ normalized so that

$$c^\varepsilon(0)=c(0) \quad \text{and} \quad \dot{c}^\varepsilon(0)=\dot{c}(0).$$

Standard estimates for ordinary differential equations imply that $\{c^\varepsilon(t)\}$ tends locally uniformly to $\{c(t)\}$. If p, q are a pair of point in $\operatorname{int}(D)$ then p, q are in $\operatorname{int}(D^\varepsilon)$ for sufficiently small ε. Thus $\gamma_{pq} \subset D^\varepsilon$ for small enough ε and so also in $\bar{D}$. From this it follows that D is a convex set.

We can therefore apply Theorem A to compare chordal distances on $\{c(t)\}$ to a horocycle, which is a curve of constant curvature 1. The estimate is:

$$\operatorname{ch}[d(c(0), c(t))] \geqq 1+\frac{t^2}{2}.$$

To complete the proof of the fact we need to remove the hypothesis that $|K(t)|=1$ for $t>T+1$. This is easily done by considering curves $c^T(t)$ with continuous curvature given by:

$$K^T(t)=\begin{cases} K(t) & |t| \leqq T, \\ 1 & |t| \geqq T+1 \end{cases}$$

and such that $c^T(0)=c(0)$; $\dot{c}^T(0)=\dot{c}(0)$. From the uniqueness theorem for ordinary differential equations it follows that:

$$c^T(t)=c(t) \quad |t|<T$$

thus we obtain

$$\operatorname{ch}[d(c(0), c(t))]=\operatorname{ch}[d(c^T(0), c^T(t))] \geqq 1+\frac{t^2}{2}, \quad |t| \leqq T.$$

As T is arbitrary we obtain the estimate for all t. The proof that $\lim_{t\to\pm\infty} c(t)$ exist follows from this estimate as in the proof of Proposition 4. 2. Theorem 4. 1 follows easily for the hypothesis that $|K_1(t)| \leqq 1$ implies that the planar curve $\tilde{c}_1(t)$ with curvature $|K_1(t)|$ bounds a convex region and thus we can apply Theorem A to conclude that

$$\operatorname{ch}[d(c_1(0), c_1(t))] \geqq 1+\frac{t^2}{2}. \qquad \square$$

Let $c(t)$ be an arclength parametrized geodesic in Σ. If the principal curvatures of Σ satisfy

$$|k_i| \leqq 1 \quad i=1, 2,$$

then the curvature of $c(t)$ as a curve in $\mathbb{H}^3$ is at most 1. This follows because

$$\nabla_{\dot{c}} \dot{c} = k N$$

where N is the normal to Σ, hence

$$|K| = |\langle \nabla_{\dot{c}} \dot{c}, N \rangle| = |\pi(\dot{c}, \dot{c})| \leqq 1.$$

We will represent $c(t)$ in hyperbolic polar coordinates:

$$c(t) = \sigma(t)\, \omega(t).$$

The radius, $\sigma(t)$ is related to the Euclidean radius by:

$$\sigma(t) = \log\left(\frac{1+r(t)}{1-r(t)}\right).$$

As the curve is parametrized by arclength we have:

$$|\dot{\sigma}|^2 + sh^2 \sigma\, |\dot{\omega}|^2 = 1. \tag{4. 3}$$

As usual we normalize Σ so that it passes through $\mathcal{O}$. Let V_θ denote an isometric map from the unit circle in $\mathbb{C}$ to the unit tangent space to Σ at $\mathcal{O}$. Let

$$\sigma_{V_\theta}(t)\, \omega_{V_\theta}(t) = \exp_{\mathcal{O}}(t\, V_\theta)$$

($\exp_{\mathcal{O}}$ is the exponential map $T_{\mathcal{O}} \Sigma \to \Sigma$).

We will show that $\omega_{V_\theta}(t)$ tends to a limit as t tends to infinity and limit depends continuously on V_θ:

From Theorem 4. 1 and (4. 3) it follows that for $t > T_0$

$$|\dot{\omega}_{V_\theta}(t)| \leqq \frac{C}{t^2} \tag{4. 4}$$

uniformly in V_θ for some fixed C. Clearly:

$$\begin{aligned} |\omega_{V_\theta}(T_2) - \omega_{V_\theta}(T_1)| &= \left| \int_{T_1}^{T_2} \dot{\omega}_{V_\theta}(s)\, ds \right| \\ &\leqq \int_{T_1}^{T_2} |\dot{\omega}_{V_\theta}(s)|\, ds \\ &\leqq C\left(\frac{1}{T_1} - \frac{1}{T_2}\right). \end{aligned} \tag{4. 5}$$

From (4. 5) it is clear that $\omega_{V_\theta}(t)$ tends to a limit. Define

$$\Phi(V_\theta) = \lim_{t \to \infty} \omega_{V_\theta}(t).$$

Proposition 4. 2. *If Σ is a complete embedded surface in $\mathbb{H}^3$ such that*

$$|k_i| \leqq 1 \quad i=1,2$$

then Φ is a continuous map.

Remark. Φ is thought of as a map from the unit circle in $T_{\mathcal{O}}\Sigma$ to the unit sphere in $\mathbb{R}^3$.

Proof of Proposition 4. 2. The proof follows easily from (4. 5):

$$|\Phi(V_{\theta_1}) - \Phi(V_{\theta_2})| = |\omega_{V_{\theta_1}}(T) - \omega_{V_{\theta_2}}(T) + \int_T^\infty \dot{\omega}_{V_{\theta_1}}(s) - \dot{\omega}_{V_{\theta_2}}(s)\, ds|$$

$$\leqq |\omega_{V_{\theta_1}}(T) - \omega_{V_{\theta_2}}(T)| + \frac{2C}{T}.$$

Given $\varepsilon > 0$ we choose T so large that $\frac{2C}{T} < \frac{\varepsilon}{2}$, then we choose $\delta > 0$ so that

$$|\omega_{V_{\theta_1}}(T) - \omega_{V_{\theta_2}}(T)| < \frac{\varepsilon}{2}$$

if

$$|V_{\theta_1} - V_{\theta_2}| < \delta.$$

If $|V_{\theta_1} - V_{\theta_2}| < \delta$ then $|\Phi(V_{\theta_1}) - \Phi(V_{\theta_2})| < \varepsilon$. □

Corollary 4. 3. *If Σ satisfies $|k_i| \leqq 1$ then Σ is the image of the unit disk by a mapping which is continuous in the closure of the disk.*

Proof. Let

$$i(s e^{i\theta}) = r_{V_\theta}\left(\log\left(\frac{1+s}{1-s}\right)\right) \omega_{V_\theta}\left(\log\left(\frac{1+s}{1-s}\right)\right).$$

The continuity in the closure follows easily from (4. 1), (4. 3) and Proposition 4. 2. □

Corollary 4. 4. *The asymptotic boundary of Σ is the image of Φ.*

Proof. Let θ be a point in $\partial_\infty \Sigma$ and let $\{p_n\}$ be a sequence in Σ converging θ. We can represent each point p_n as:

$$p_n = \exp_{\theta}(t_n V_n).$$

The sequence $\{t_n\}$ tends to infinity, whereas $\{V_n\}$ has a convergent subsequence; call it $\{V_n\}$ as well. Let V^* be the limit of $\{V_n\}$. Because Φ is continuous:

$$\lim_{n\to\infty} |\Phi(V^*) - \Phi(V_n)| = 0.$$

As $r(p_n)$ tends to one:

$$\lim_{n\to\infty} |\Phi(V_n) - p_n| = 0.$$

The conclusion follows by applying the triangle inequality. □

Corollary 4. 5. *If Σ is embedded in $\mathbb{H}^3$ and*

$$|k_i| < 1 \quad i = 1, 2$$

then $\partial_\infty \Sigma$ is a Jordan curve.

Proof. This follows immediately from Theorem 3. 6 as the previous corollary and Proposition 4. 2 imply that $\partial_\infty \Sigma$ is locally connected. □

Corollary 4. 6. *If Σ is an embedded surface in $\mathbb{H}^3$ with*

$$|k_i| < 1 \quad i = 1, 2$$

then Λ extends continuously to $\partial \Omega_+$ as the identity.

Proof. From Proposition 3. 7 it follows that there is an exhaustion of Ω_+ by relatively compact subsets $\{\Omega_n : n = 1, 2, \ldots\}$ such that

$$|D\varrho_+(\theta)| \geqq n \text{ for } \theta \text{ in } \Omega_+ \setminus \Omega_n. \tag{4. 6}$$

From formula (3. 8) for Λ we see that:

$$\begin{aligned} |\Lambda(\theta) - \Lambda(\phi)| = {} & |X(\theta) - X(\phi)| \\ & + 2\left|\frac{X(\theta)}{|D\varrho_+(\theta)|^2 + 1} - \frac{X(\phi)}{|D\varrho_+(\phi)|^2 + 1}\right| \\ & + 2\left|\frac{D\varrho_+(\theta)}{|D\varrho_+(\theta)|^2 + 1} - \frac{D\varrho_+(\theta)}{|D\varrho_+(\theta)|^2 + 1}\right|. \end{aligned} \tag{4. 7}$$

From (4. 6) and (4. 7) it is clear that if we extend Λ to $\partial \Omega_+$ as the identity then Λ is continuous in $\bar{\Omega}_+$. □

Remarks. 1) If the curvatures satisfy a stronger estimate

$$|k_i| \leqq \beta < 1 \quad i = 1, 2,$$

then one can show that Φ is $\left(\frac{1-\beta^2}{1+\beta^2}\right)^{\frac{1}{2}}$ Hölder continuous. The proof uses Schur's theorem and the Toponogov triangle comparison theorem.

2) If $|k_i| = 1$ on a sufficiently large set then $\partial_\infty \Sigma$ may fail to be a Jordan curve.

§ 5. The dilatation

The distortion of the map Λ is intimately connected to the curvature properties of the surface which defines it. In this section we will derive a formula for the dilatation of Λ in terms of the principal curvature of Σ. As a byproduct of this analysis we will also find a formula for the Jacobian determinant of Λ.

Recall that the dilatation of a smooth map between Riemannian manifolds:

$$\varphi\colon (M, g) \to (N, h),$$

is defined at a point $p \in M$ by:

$$K(\varphi, p) = \left[\frac{\max\limits_{X \in T_pM} [(\varphi^* h)(X)/g(X)]}{\min\limits_{X \in T_pM} [(\varphi^* h)(X)/g(X)]} \right]^{\frac{1}{2}}.$$

This quantity is invariant under composition with conformal maps (see [L-V]).

To compute the dilatation of Λ we normalize Σ so that the point $p = G_+^{-1}(\theta)$ is the origin in $\mathbb{H}^3$ and the unit normal vector at p is $(0, 0, 1)$. Effecting this normalization will lead to a conjugation of Λ by an isometry of $\mathbb{H}^3$. These maps act conformally on $\partial \mathbb{B}^3$ and thus the dilatation is unchanged.

Let $\{X_1, X_2\}$ be an orthonormal frame of principal directions at p, corresponding to the principal curvatures k_1 and k_2, respectively. Let $\{X_1(t), X_2(t)\}$ be the image of this frame under the linearization of the parallel flow, $p \mapsto \psi^t(p, N)$. According to equation (3. 7) of [Ep 1]

$$(5.1) \qquad X_i(t) = \frac{1}{2}[(1-k_i)e^t + (1+k_i)e^{-t}]\,\tilde{X}_i(t).$$

Here $\tilde{X}_i(t)$ is the hyperbolic parallel translation of $X_i(0)$. The hyperbolic geodesic $\psi^t(p, N_p)$ is also a Euclidean geodesic and the Euclidean and hyperbolic metrics agree at $\mathcal{O}$; thus we can express $\tilde{X}_i(t)$ in terms of the Euclidean parallel translate of $X_i(0)$, $\bar{X}_i(t)$ by:

$$(5.2) \qquad \tilde{X}_i(t) = \left(\operatorname{ch}\left(\frac{t}{2}\right)\right)^{-2} \bar{X}_i(t).$$

Putting together (5. 1), (5. 2) and the fact that $\bar{X}_i(t) = X_i(0)$ we see that:

$$(5.3) \qquad \lim_{t \to \pm\infty} X_i(t) = 2(1 \mp k_i)\,X_i(0).$$

If $k_i \neq 1$ we can use (5. 3) to compute the linearized action of Λ in the basis of $X_i(0)$:

$$(5.4) \qquad d\Lambda|_\theta(X_i) = \left(\frac{1+k_i}{1-k_i}\right) X_i.$$

As $\{X_1, X_2\}$ is an orthonormal frame at θ we can compute the dilatation of Λ at θ:

$$(5.5) \quad K(\Lambda, \theta) = \max\left\{ \left|\frac{1+k_1(p)}{1-k_1(p)} \cdot \frac{1-k_2(p)}{1+k_2(p)}\right|^{\frac{1}{2}}, \left|\frac{1-k_1(p)}{1+k_1(p)} \cdot \frac{1+k_2(p)}{1-k_2(p)}\right|^{\frac{1}{2}} \right\}.$$

In this normalization $\theta=(0,0,1)$ and $A(\theta)=(0,0,-1)$. We have tacitly identified all the tangent spaces along the z-axis by using $\{X_1(0), X_2(0)\}$ as a basis for these tangent spaces. It is important to note that this basis gives one orientation for the tangent space to S^2 at $(0,0,1)$ and the opposite orientation at $(0,0,-1)$. If

$$(1-k_1)(1-k_2)(1+k_1)(1+k_2)>0$$

then A reverses orientation and it preserves orientation otherwise.

As remarked above the normalization of Σ does not effect the dilatation so we've essentially proved:

Proposition 5. 1. *If Σ is a C^2 surface in $\mathbb{H}^3$ such that $k_i \neq 1$, $i=1,2$ then the forward Gauss map, G_+ is locally invertible and the dilatation of the composition is given by:*

$$K(A, G_+(p))=\max\left\{\left|\frac{1+k_1(p)}{1-k_1(p)}\cdot\frac{1-k_2(p)}{1+k_2(p)}\right|^{\frac{1}{2}}, \left|\frac{1-k_1(p)}{1+k_1(p)}\cdot\frac{1+k_2(p)}{1-k_2(p)}\right|^{\frac{1}{2}}\right\}.$$

Proof of Proposition 5. 1. We've proven everything except the local invertibility of G_+. This follows easily from the inverse function theorem. For the hypothesis $k_i \neq 1$ and (5. 3) imply that dG_+ is invertible. □

For surfaces defined as envelopes the restriction $k_i \neq 1$ is not important.

Proposition 5. 2. *Let ϱ be a C^3 function on a domain $U \subset S^2$. Suppose the envelope of $\{H(\theta, \varrho(\theta)) : \theta \in U\}$ given by:*

$$(5.6) \qquad R_\varrho(\theta)=\frac{|D\varrho(\theta)|^2+(e^{2\varrho}-1)}{|D\varrho(\theta)|^2+(e^\varrho+1)^2}X(\theta)+\frac{2D\varrho(\theta)}{|D\varrho(\theta)|^2+(e^\varrho+1)^2}$$

is an immersion, then neither principal curvature of $\Sigma(\varrho)$ ever attains the value 1.

Remark. If ϱ is a C^3 function then for a given θ the map $\theta \mapsto R_{\varrho+t}(\theta)$ is an immersion at every value of t with at most two exceptions; this result is proved in § 5 of [Ep 1]. The condition $k_i \neq 1$ is invariant under the parallel flow.

Proof of Proposition 5. 2. Let Σ_t denote the parallel surface at distance t from Σ. In light of the results in § 5 of [Ep 1] it follows that each $\theta \in U$ has an open neighborhood V such that the map

$$\Psi : U \times (-\varepsilon, \varepsilon) \to \mathbb{H}^3$$

defined by:

$$\Psi(\theta, t)=R_{\varrho+t}(\theta)$$

is a homeomorphism for some choice of $\varepsilon>0$ as the differential $d\Psi$ is an isomorphism at $(\theta_0, 0)$. If G_+ denotes the forward Gauss map from the surface Σ then:

$$(5.7) \qquad G_+ \circ \Psi(\theta, 0)=\theta.$$

The linearization of Ψ carries ∂_t to the unit normal vector to $\Sigma(\varrho)$ thus (5. 7) and the inverse function theorem imply that dG_+ must have rank two everywhere on $\Sigma(\varrho)$. If $k_i(p)=1$ for some i and some $p\in\Sigma(\varrho)$ then (5. 3) implies that:

$$dG_+|_p X_i(p)=0.$$

This contradiction proves the proposition. □

Remarks. 1) By combining the calculation leading up to Proposition 5. 1 and that used in the proof of Theorem 3. 1 one can easily show that the action of $d\Lambda$ on the orthonormal basis $\{X_1, X_2\}$ of $T_\theta S^2$ arising from the principal directions of Σ is given by:

$$d\Lambda(X_i)=\frac{e^{2\varrho_+(\theta)}}{(1+|D\varrho_+(\theta)|^2)}\left[\frac{1+k_i(G_+^{-1}(\theta))}{1-k_i(G_+^{-1}(\theta))}\right]Y_i.$$

Here $\{Y_1, Y_2\}$ is an orthonormal frame for the tangent space to S^2 at $\Lambda(\theta)$. The factor by which Λ distorts the spherical area element can be calculated from this:

$$J=\frac{e^{4\varrho_+}}{(1+|D\varrho_+|^2)}\cdot\left(\frac{1+k_1}{1-k_1}\cdot\frac{1+k_2}{1-k_2}\right).$$

2) The calculation also substantiates the claim made in an earlier section that in compact regions of Σ the Jacobian determinant of $dG_\pm$, $J_\pm$ satisfies:

$$c_1\,|(1\mp k_1)(1\mp k_2)|\leqq J_\pm\leqq c_2\,|(1\mp k_1)(1\mp k_2)|,$$

for some positive constants c_1 and c_2.

As an easy corollary of Proposition 5. 1 and Corollary 4. 6 we have:

Corollary 5. 3. *If Σ is a complete surface in $\mathbb{H}^3$ with*

$$|k_i|<1 \quad i=1, 2,$$

and

$$\sup_{p\in\Sigma} K(\Lambda, G_+(p))<K$$

then Λ is a K-quasiconformal reflection from Ω_+ to Ω_- fixing $\partial_\infty\Sigma$.

Corollary 5. 4. *If Σ is as in the previous corollary then $\partial_\infty\Sigma$ is a quasicircle.*

Proof. Follows immediately from the previous corollary and the theorem of Ahlfors characterizing quasicircles as fixed points of quasiconformal reflections [Ahl 2].

§ 6. The Beltrami coefficient

The Beltrami equation

(6. 1) $$v_z=\mu v_{\bar z}$$

is a powerful method for studying quasiconformal maps. This method requires a complex parameter and so it is convenient to work with the map

$$w=S\circ\Lambda\circ S^{-1}$$

instead of Λ; S is stereographic projection from S^2 to $\mathbb{C}$:

$$S^{-1}(z)=(2\,\mathrm{Re}\,z, 2\,\mathrm{Im}\,z, |z|^2-1)(1+|z|^2)^{-1}.$$

We work with equation (6. 1) instead of the more familiar

$$v_{\bar{z}} = \mu v_z$$

because our maps generally reverse orientation.

In this section we will derive a formula for

$$\mu = \frac{w_z}{w_{\bar{z}}}$$

in terms of the support function, ϱ_+. This formula is an algebraic consequence of formula (3. 11) for Λ:

$$\text{(3. 11)} \qquad \Lambda\colon X(\theta) \mapsto \frac{|D\varrho_+|^2 - 1}{|D\varrho_+|^2 + 1} X(\theta) + \frac{2\,D\varrho_+}{|D\varrho_+|^2 + 1}.$$

The operator, D is defined by

$$\text{(6. 2)} \qquad \langle Df, Y \rangle = df(Y).$$

Here and elsewhere in this section $\cdot$ denotes the Euclidean inner product; $\langle , \rangle$ is the induced inner product in TS^2, Y is a vector field on S^2; f is a differentiable function. The complex coordinate vector fields induced by stereographic coordinates are:

$$\text{(6. 3)} \qquad \begin{aligned} \partial_z &= (1 - \bar{z}^2, \ -i(1 + \bar{z}^2), 2\,\bar{z})\,(1 + |z|^2)^{-2} \\ \partial_{\bar{z}} &= (1 - z^2, i(1 + z^2), 2\,z)\,(1 + |z|^2)^{-2}. \end{aligned}$$

From (3. 11) we easily derive:

$$\text{(6. 4)} \qquad D\varrho = \frac{\Lambda - (\Lambda \cdot X)\,X}{1 - \Lambda \cdot X}.$$

From the definition of w it follows that

$$\text{(6. 5)} \qquad \Lambda \cdot S^{-1}(z) = (2\,\mathrm{Re}\,w(z), 2\,\mathrm{Im}\,w(z), |w(z)|^2 - 1)\,(|w(z)|^2 + 1)^{-1}.$$

From (6. 2) and (6. 4) we see that

$$\text{(6. 6)} \qquad \begin{aligned} d\varrho(\partial_z) &= \frac{\Lambda \cdot \partial_z}{1 - \Lambda \cdot X}, \\ d\varrho(\partial_{\bar{z}}) &= \frac{\Lambda \cdot \partial_{\bar{z}}}{1 - \Lambda \cdot X} \end{aligned}$$

as $X \cdot \partial_z = X \cdot \partial_{\bar{z}} = 0$.

From (6. 5) and (6. 6) we derive the formula for $d\varrho$ in the stereographic coordinate system; it is:

$$\text{(6. 7)} \qquad d\varrho = 2\,\mathrm{Re}\left(\frac{1 + w\bar{z}}{w - z} \cdot \frac{dz}{1 + |z|^2}\right).$$

Since

$$d\varrho = \varrho_z dz + \varrho_{\bar z} d\bar z$$

it follows from (6. 7) that

$$\text{(6. 8)} \quad \begin{aligned} &\text{a)} \quad \varrho_z = \frac{1 + w\bar z}{(w - z)(1 + |z|^2)}, \\ &\text{b)} \quad w = \frac{z(1 + |z|^2)\,\varrho_z + 1}{(1 + |z|^2)\,\varrho_z - z}. \end{aligned}$$

Formula (6. 8) leads to:

Theorem 6. 1. *The Beltrami coefficients μ of w is given by:*

$$\text{(6. 9)} \qquad \mu = \frac{(1 + |z|^2)^2\,(\varrho_{zz} - \varrho_z^2) + 2\,\bar z \varrho_z(| + |z|^2)}{(1 + |z|^2)^2\,\varrho_{z\bar z} - 1}.$$

Proof of Theorem 6. 1. We differentiate (6. 8) a) with respect to z and $\bar z$ to obtain:

$$\text{(6. 10)} \quad \begin{aligned} &\text{a)} \quad \varrho_{zz} = -w_z(w - z)^{-2} + \left(\frac{1 + w\bar z}{w - z}\right)\frac{(w - z)^{-1} - (1 + |z|^2)^{-1}}{1 + |z|^2}, \\ &\text{b)} \quad \varrho_{z\bar z} = -w_{\bar z}(w - z)^{-2} + (1 + |z|^2)^{-2}. \end{aligned}$$

Substituting from (6. 8) into (6. 10) a) and using the equation $w_z = \mu w_{\bar z}$ we get:

$$\text{(6. 11)} \qquad \varrho_{zz} = \varrho_z^2 - \frac{-2\,\bar z \varrho_z}{1 + |z|^2} - \mu w_{\bar z}(w - z)^{-2}.$$

Solving for w_z in (6. 10) b) and then substituting this expression into (6. 11) we obtain (6. 9). □

The formula for μ becomes neater if we eliminate the factors arising from the spherical metric. To that end we define:

$$f = \varrho - \log(1 + |z|^2).$$

In terms of f, formula (6. 9) becomes:

$$\text{(6. 12)} \qquad \mu = \frac{f_{zz} - f_z^2}{f_{z\bar z}}$$

and (6. 8) b) becomes:

$$\text{(6. 13)} \qquad w = z + \frac{1}{f_z}.$$

In the next section we apply these formulae to find sufficient conditions for a conformal map defined on the unit disk to be univalent and have a quasiconformal extension to $\mathbb{C}$.

§ 7. Sufficient conditions for univalence and quasiconformal extension

Let Ω be a simply connected domain on S^2 and $\hat{\Omega}$ its stereographic image. Let ϕ be a conformal map from $\hat{\Omega}$ to the unit disk and let ψ be the inverse of ϕ. If h is sufficiently smooth then

$$\varrho \circ S^{-1}(z) = \log\left(\frac{|\phi_z|}{1-|\phi|^2}\right) + h(z) + \log(1+|z|^2) \tag{7.1}$$

will generate a surface Σ in $\mathbb{H}^3$ whose asymptotic boundary is $\partial\Omega$. As before $G_\pm$ will be the Gauss maps of Σ and

$$\Lambda = G_- \circ G_+^{-1}, \quad w = S \circ \Lambda \circ S^{-1}.$$

First we need a formula for $\mu = \frac{w_z}{w_{\bar{z}}}$ incorporating the special form of ϱ:

Lemma 7.1. *The Beltrami coefficient of w is*

$$\mu = \frac{\dfrac{(1-|\phi|^2)^2}{|\phi_z|^2}\left(\dfrac{1}{2}\mathscr{S}_\phi - \dfrac{\phi_{zz}h_z}{\phi_z} + h_{zz} - h_z^2\right) - \dfrac{2\bar{\phi}(1-|\phi|^2)\,h_z}{\bar{\phi}_z}}{1 + \dfrac{(1-|\phi|^2)^2\,h_{z\bar{z}}}{|\phi_z|^2}}. \tag{7.2}$$

The pullback of μ to the unit disk via ψ is:

$$\nu = \mu \circ \psi\,\frac{\psi_\zeta}{\bar{\psi}_\zeta} = \frac{(1-|\zeta|^2)^2\left(g_{\zeta\zeta} - g_\zeta^2 - \frac{1}{2}\mathscr{S}_\psi\right) - 2\bar{\zeta}(1-|\zeta|^2)\,g_\zeta}{1+(1-|\zeta|^2)^2\,g_{\zeta\bar{\zeta}}}; \tag{7.3}$$

here $g(\zeta) = h \circ \psi(\zeta)$ and $\mathscr{S}_f$ is the Schwarzian derivative:

$$\mathscr{S}_f(z) = \left(\frac{f''}{f'}\right)'(z) - \frac{1}{2}\left(\frac{f''}{f'}\right)^2(z).$$

Proof of Lemma 7.1. To prove (7.2) one simply applies formula (6.12) with

$$f = \log\left(\frac{|\phi_z|}{(1-|\phi|^2)}\right) + h.$$

For the convenience of the reader we include the important steps in the calculation:

$$f_z = \frac{\frac{1}{2}\phi_{zz}}{\phi_z} + \frac{\phi_z \bar{\phi}}{1-|\phi|^2} + h_z$$

$$f_{zz} = \frac{1}{2}\left(\frac{\phi_{zz}}{\phi_z}\right)_z + \frac{\phi_{zz}\bar{\phi}(1-|\phi|^2) + \phi_z^2 \bar{\phi}^2}{(1-|\phi|^2)^2} + h_{zz}$$

and

$$f_{z\bar{z}} = \frac{|\phi_z|^2}{(1-|\phi|^2)^2} + h_{z\bar{z}}.$$

Inserting these expressions into (6. 12) we obtain:

$$\mu = \frac{\frac{1}{2}\mathscr{S}_\phi - \left(\frac{\phi_{zz}}{\phi_z} + \frac{2\,\phi_z \bar{\phi}}{1-|\phi|^2}\right) h_z + h_{zz} - h_z^2}{\frac{|\phi_z|^2}{(1-|\phi|^2)^2} + h_{z\bar{z}}}.$$

Algebraic simplification then leads to (7. 2). The transformed formula (7. 3) follows from (7. 2) by using the fact that

$$\mathscr{S}_\phi(\psi(\zeta)) = -\mathscr{S}_\psi(\zeta)\left(\frac{1}{\psi_\zeta}\right)^2$$

and the chain rule to evaluate the other terms. □

Remark. Formulae (6. 9), (6. 12), (7. 2) and (7. 3) result from a local calculation. Hence ψ need not be univalent in Lemma 7. 1. If ψ is conformal the image of ψ is a non-schlicht planar region and the inverse, ϕ is locally defined. The formula (7. 2) is only valid locally, however formula (7. 3) is valid globally (see proof of Theorem 7. 2).

The mapping $v(\zeta) = w \circ \psi(\zeta)$ satisfies the equation:

$$v_\zeta = \nu\, v_{\bar{\zeta}}.$$

The map, v carries the unit disk onto the image of w. If Σ is such that G_- is a homeomorphism onto the interior of Ω^c then v is a homeomorphism onto the interior of $\hat{\Omega}^c$; if, moreover,

$$\sup_{\theta \in \Omega} K(\Lambda, \theta) = K(\Lambda)$$

is finite then v is a quasiconformal map. As is well known (see [L-V]) the maximum dilatation of Λ, $K(\Lambda)$ can be expressed in terms of the sup norm of μ (or ν) by:

$$K(\Lambda) = \frac{1 + \|\mu\|_{L^\infty}}{1 - \|\mu\|_{L^\infty}} = \frac{1 + \|\nu\|_{L^\infty}}{1 - \|\nu\|_{L^\infty}}.$$

If Λ is continuous in $\bar{\Omega}$ and reduces to the identity on $\partial\Omega$ then we can extend ψ to $\mathbb{C}$ via:

$$\Psi(\zeta) = \begin{cases} \psi(\zeta) & \zeta \in D_1, \\ v\left(\dfrac{1}{\bar{\zeta}}\right) & \zeta \in D_1^c. \end{cases}$$

This gives a continuous extension of ψ to $\mathbb{C}$ which satisfies:

$$\Psi_{\bar{\zeta}} = \tilde{v}(\zeta)\, \Psi_{\zeta}$$

off of ∂D_1. The Beltrami coefficient $\tilde{v}(\zeta)$ is defined by:

$$\tilde{v}(\zeta) = \begin{cases} 0 & \zeta \in D_1 \\ v\left(\dfrac{1}{\bar{\zeta}}\right)\left(\dfrac{\bar{\zeta}}{\zeta}\right)^2 & \zeta \in D_1^c. \end{cases}$$

The unit circle is a removable set for a quasiconformal map and therefore Ψ is a $\dfrac{1 + \|v\|_{L^\infty}}{1 - \|v\|_{L^\infty}}$-quasiconformal extension of ψ.

This construction can be reversed to obtain a sufficient condition for ψ to be univalent and have a K-quasiconformal extension to $\mathbb{C}$:

Theorem 7.2. *Let ψ be a conformal map on D_1. Suppose there exists a real valued function, $g(\zeta)$ in $C^5(D_1)$ such that:*

1) $$\left| \frac{(1-|\zeta|^2)^2 \left[g_{\zeta\zeta} - g_\zeta^2 - \frac{1}{2}\mathcal{S}_\psi \right] - 2\bar{\zeta}(1-|\zeta|^2)\, g_\zeta}{1 + (1-|\zeta|^2)^2\, g_{\zeta\bar{\zeta}}} \right| \leqq k \quad \textit{for a} \quad k<1$$

2) $1 + (1-|\zeta|^2)^2\, g_{\zeta\bar{\zeta}} > 0$

and

3) $$|g_\zeta(\zeta)|\,(1-|\zeta|^2) \leqq k \max\left[|\zeta|, \frac{1}{2|\zeta|}\right].$$

then ψ is univalent and has a $\left(\dfrac{1+k}{1-k}\right)$-quasiconformal extension to $\mathbb{C}$.

Proof of Theorem 7.2. First we give an outline of the proof:

I) We first assume that ψ is conformal in a neighborhood of $\bar{D}_1$ and argue as follows:

a) Locally, we construct a surface using the function:

$$\varrho \cdot S^{-1}(z) = g(\phi(z)) + \log\left(\frac{|\phi'|}{1-|\phi|^2}\right) + \log(1+|z|^2)$$

where ϕ is a local inverse for $S^{-1} \circ \psi$. We show that the principal curvatures satisfy:

$$(7.4) \qquad |k_i| < 1 \quad i = 1, 2.$$

b) We piece together a complete surface, Σ which satisfies (7. 4) and conclude that $\operatorname{Im} S^{-1} \circ \psi$ is a Jordan domain and ψ is univalent.

c) Using the reflection Λ constructed from Σ we obtain a quasiconformal extension of ψ.

II) We use an approximation argument of Ahlfors' to remove the hypothesis that ψ be conformal in a neighborhood of D_1.

Let $K = \dfrac{1+k}{1-k}$.

Part I, a. Since we are assuming that ψ is conformal in neighborhood of $\bar{D}_1$, $|\psi'|$ is bounded above and below on $\bar{D}_1$. Thus there is a finite cover of $\bar{D}_1$, $\{U_1, \ldots, U_N\}$ such that $\psi|_{U_i}$ is invertible. Let $V_i = S^{-1} \circ \psi(U_i)$ and let ϕ_i denote the inverse of $S^{-1} \circ \psi|_{U_i}$.

Define:

$$\varrho_i \circ S^{-1}(z) = g \circ \phi_i(z) + \log\left[\frac{|\phi_i'(z)|}{1 - |\phi_i(z)|^2}\right] + \log(1 + |z|^2).$$

on $S(V_i)$. If $d\sigma^2$ is the round metric on S^2 then

$$e^{2\varrho_i} d\sigma^2 = \phi_i^* \left(\frac{e^{2g} |d\zeta|^2}{(1 - |\zeta|^2)^2}\right). \tag{7. 5}$$

From this it is evident that

$$\varrho_i(\theta) = \varrho_j(\theta) \quad \text{for} \quad \theta \in V_i \cap V_j. \tag{7. 6}$$

For each t we can construct the envelope $\Sigma(\varrho_i + t)$. Following the proof of Proposition 5. 2, we locally parametrize points on these surfaces by their Gauss images:

$$\{R_{\varrho_i + t}(\theta) \colon \theta \in V_i\}.$$

From (7. 6) it is evident that

$$R_{\varrho_i + t}(\theta) = R_{\varrho_j + t}(\theta) \quad \text{for} \quad \theta \in V_i \cap V_j. \tag{7. 7}$$

Using (7. 7) we can piece together the local representations to obtain a map:

$$i_t \colon D_1 \to \mathbb{H}^3.$$

Denote the image of this map by Σ_t. We will show that i_t is an immersion for every t by showing that the principal curvatures of Σ_t satisfy (7. 4). Locally the Gauss map G_+^i is defined by:

$$G_+^i(R_{\varrho_i + t}(\theta)) = \theta.$$

We can define the backward Gauss map via:

$$G^i_-(\varrho) = \lim_{t \to \infty} R_{\varrho_i + t}(G^i_+(p))$$

and define $\Lambda_i = G^i_- \circ (G^i_+)^{-1}$.

Since ϱ_i is a C^5 function it follows that for each θ, $R_{\varrho_i+t}(\theta)$ fails to be an immersion for at most two values of t (see § 5).

Fix a $\theta_0 \in V_i$ and choose t_0 such that $R_{\varrho_i+t}(\theta)$ is an immersion at $t = t_0$ and $\theta = \theta_0$. Let (k_1, k_2) denote the principal curvatures of $\Sigma(\varrho_i + t_0)$ at $p_0 = R_{\varrho_i+t_0}(\theta_0)$. From Proposition 5.2 it follows that $k_i(p_0) \neq 1$ for $i = 1, 2$.

Since Λ_i is given terms of ϱ_i by formula (3.11), formula (6.12) for the Beltrami coefficient of $w^i = S \circ \Lambda_i \circ S^{-1}$ and hypothesis 1) apply to show that $K(\Lambda_i, \theta_0) \leqq K$. On the other hand we can apply Proposition 5.1 to calculate $K(\Lambda_i, \theta_0)$ in terms of the principal curvatures $(k_1(p_0), k_2(p_0))$; putting together these observations we obtain:

$$(7.8)\quad \max\left\{\left|\frac{1-k_1(p_0)}{1+k_1(p_0)} \cdot \frac{1+k_2(p_0)}{1-k_2(p_0)}\right|, \left|\frac{1-k_1(p_0)}{1-k_1(p_0)} \cdot \frac{1-k_2(p_0)}{1+k_2(p_0)}\right|\right\} \leqq K^2.$$

From (7.8) it follows that $k_1 = -1$ and $k_2 \neq -1$ cannot occur for in this case the dilatation would be infinite. We now exclude the possibility $k_1 = k_2 = -1$. In this case the curvature of the metric $e^{2\varrho_i} d\sigma^2$, K_∞ would equal zero at $G^i_+(p_0)$. We can assert this for if ϱ_i is C^5 then the curvature of this metric is expressed in terms of k_1 and k_2 by:

$$(7.9)\qquad K_\infty(G^i_+(p_0)) = \frac{k_1(p_0)\, k_2(p_0) - 1}{(1-k_1(p_0))\,(1-k_2(p_0))}.$$

This is the statement of Proposition 5.3 in [Ep 1]. Formula (5.11) of [Ep 1] also expresses the curvature of this metric in terms of ϱ:

$$(7.10)\qquad K_\infty(\theta_0) = (1 - \Delta_{S^2}\varrho_i(\theta_0))\, e^{-2\varrho_i(\theta_0)}.$$

An elementary calculation shows that the right hand side of (7.10) is a negative multiple of

$$(1 + (1 - |\zeta_0|^2)^2\, g_{\zeta\bar{\zeta}}(\zeta_0))$$

where $\psi(\zeta_0) = S(\theta_0)$. From hypothesis 2) it follows that $K_\infty < 0$ and therefore from (7.9) it follows that $k_1 = k_2 = -1$ cannot occur. Thus we've shown that

$$|k_i| \neq 1 \quad i = 1, 2.$$

The principal curvatures of $\Sigma(\varrho_i + t_0)$ at p_0 satisfy two inequalities:

$$K(\Lambda_i, G_+(p_0)) \leqq K$$

and

$$(k_1(p_0)\, k_2(p_0) - 1)\,(1 - k_1(p_0))\,(1 - k_2(p_0)) < 0.$$

The region where these two inequalities are satisfied simultaneously is indicated by the shaded region in Figure 2:

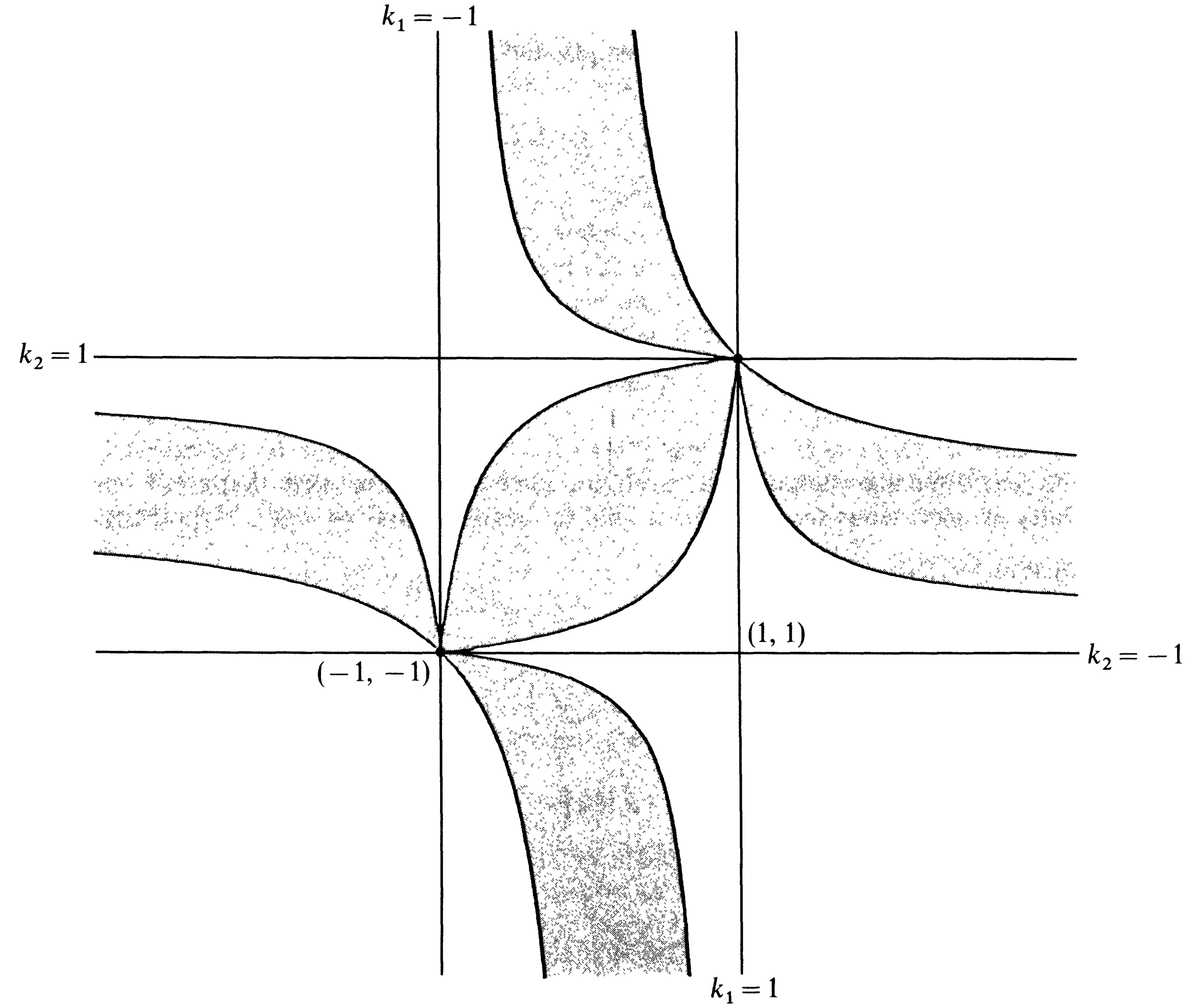

Figure 2

Region shown for $K = \frac{1}{2}$

By considerations of orientation we can exclude the possibility that (k_1, k_2) lies in one of the non-compact components. This is because w_i satisfies the Beltrami equation:

$$w^i_{\bar z} = \mu w^i_{z}.$$

By hypothesis 1), $\|\mu\|_{L^\infty} < \frac{K-1}{K+1}$. The linearization of $z \mapsto w^i(z)$ has the matrix:

$$d w^i = \begin{pmatrix} \mu w^i_{\bar z} & w^i_{\bar z} \\ \overline{w^i_{\bar z}} & \overline{\mu w^i_{\bar z}} \end{pmatrix}.$$

The determinant of this matrix is non-positive and therefore w^i reverses orientation. Note that as $|k_i| \neq 1$ the Jacobian determinant of w^i never vanishes. By the remarks after the derivation of the formula (5. 5) it follows that Λ_i and hence w^i preserve orientation if (k_1, k_2) lies in one of the noncompact shaded regions in Figure 2. Therefore we've established that

$$|k_i(p_0)| < 1 \quad i = 1, 2.$$

Since θ_0 and hence p_0 were arbitrary points it follows that this holds everywhere on $\Sigma(\varrho_i + t_0)$. The focal manifold of $\Sigma(\varrho_i)$ is empty and thus $\Sigma(\varrho_i)$ is a smoothly imbedded surface in $\mathbb{H}^3$ with

$$|k_i| < 1, \quad i = 1, 2.$$

Part I, b. From part a it follows that i_t is an immersion; to use the results of §§ 2—4 we need to show that Σ_t is a complete surface. Let $ds^2(t)$ denote the induced metric on Σ_t and let $d\tilde{s}^2(t) = i_t^*(ds^2(t))$ be the pullback to the disk. Clearly it suffices to show that $d\tilde{s}^2(t)$ is a complete metric on D_1. Since i_t is an immersion $d\tilde{s}^2$ is locally positive definite and smooth, hence it suffices to show that

$$\int_\gamma d\tilde{s}(t) = \infty$$

for any curve, γ: $[0, 1] \to D_1$ such that $\gamma(1) \in \partial D_1$. This will certainly be the case if $i_t(\gamma)$ tends to $\partial \mathbb{H}^3$. We may suppose that $\gamma \subset U_i$ for some i. If we can show that $e^{2\varrho_i(\theta)}$ tends to infinity as $\phi_i(\theta)$ tends to ∂D_1 then it follows that $i_t(\gamma)$ tends to $\partial \mathbb{H}^3$. Hypothesis 3) implies that for $|\zeta| > \frac{1}{\sqrt{2}}$:

$$|g_\zeta(\zeta)| \leqq \frac{k|\zeta|}{1-|\zeta|^2}$$

implies that

$$|g(\zeta)| \leqq C + k \log\left(\frac{1}{1-|\zeta|^2}\right).$$

Thus

$$\frac{e^{2g}}{(1-|\zeta|^2)^2} \geqq \frac{C}{(1-|\zeta|^2)^{\frac{4}{K}+1}}$$

and therefore tends to infinity on ∂D_1.

As remarked above

$$e^{2\varrho_i} d\sigma^2 = \phi_i^*\left(\frac{e^{2g}|d\zeta|^2}{(1-|\zeta|^2)^2}\right).$$

If X is a vector in $T_\theta S^2$ of unit $d\sigma^2$-length then the $|d\zeta|^2$ length of $\phi_{i_*}(X)$ is uniformly bounded from below for $\theta \in V_i$. Thus the $e^{2\varrho_i} d\sigma^2$-length of X tends to infinity as $\phi_i(\theta)$ tends to ∂D_i. Therefore $e^{2\varrho_i(\theta)}$ also tends to infinity as $\phi_i(\theta)$ tends to D_1. As a consequence we obtain that Σ_t is a complete surface whose curvatures satisfy (7. 4).

We can apply corollary 4. 5 to conclude that $\partial_\infty \Sigma_t$ is a Jordan curve. Let Ω_+ and Ω_- be the components of $S^2/\partial_\infty \Sigma_t$. The image of $S^{-1} \circ \psi$ lies in one of these sets. If this were not the case then we could find a disk $U \Subset D_1$ such that $S^{-1} \circ \psi(U) \cap \partial_\infty \Sigma_t \neq \emptyset$. If $\varrho_U(\theta)$ denotes the corresponding, locally defined support function then it follows from Lemma 2. 3 that:

$$\Sigma_t \cap \bigcup_{\theta \in S^{-1} \circ \psi(U)} B(\theta, \varrho_U(\theta)) = \emptyset.$$

Therefore $\partial_\infty \Sigma_t \cap S^{-1} \circ \psi(U) = \phi$, an obvious contradiction. We can suppose that

$$\operatorname{Im} S^{-1} \circ \psi \subset \Omega_+ .$$

In fact $\operatorname{Im} S^{-1} \circ \psi = \Omega_+$. This is because Σ_t is complete and so G_+ is a diffeomorphism. If a point in $\operatorname{int} \Omega_+$ were omitted by $S^{-1} \circ \psi$ then we would find a smooth curve $\gamma(s)$ in D_1 with $\gamma(1) \in \partial D_1$ and $i_t(\gamma(s))$ tending to a point in Σ_t. This is in contradiction to the argument given above and thus

$$S^{-1} \circ \psi(D_1) = \Omega_+ .$$

We've shown that $S^{-1} \circ \psi$ is a local homeomorphism from a closed disk to a closed disk hence ψ is one to one.

Part I, c. The reflection Λ generated by Σ_t is clearly K-quasiconformal and the map:

$$\Psi(\zeta) = \begin{cases} S^{-1} \circ \psi(\zeta) & \zeta \in D_1 \\ \Lambda \circ S^{-1} \circ \psi\left(\dfrac{1}{\bar{\zeta}}\right) & \zeta \in D_1^c \end{cases}$$

defines a homeomorphism from $\hat{\mathbb{C}}$ to S^2. This map is K-quasiconformal off of ∂D_1. As above the ∂D_1 is a removable set for quasiconformal maps and thus Ψ is a K-quasiconformal extension of $S \circ \psi$.

Part II. To complete the proof we need to remove the additional smoothness hypotheses made on ψ. Hypothesis 3) allows us to show that if $\psi(\zeta)$ and $g(\zeta)$ satisfy 1) then so do $\psi(r\zeta)$ and $g(r\zeta)$ for any $r < 1$. The argument in part I clearly applies to $\psi(r\zeta)$ for any $r < 1$. We obtain a sequence of quasiconformal reflections, $\{\Lambda_r\}$ defined by $\psi(r\zeta)$ and $g(r\zeta)$. These maps have uniformly bounded dilatation and thus we can use a standard normal families argument to extract a uniformly convergent subsequence:

$$\Lambda_{r_n} \to \Lambda .$$

All that remains is to verify hypothesis 1) for $\psi(r\zeta)$ and $g(r\zeta)$. We need to show:

$$(7.11) \qquad \left| r^2 \left[g_{\zeta\zeta}(r\zeta) - g_\zeta^2(r\zeta) - \frac{2\bar{\zeta} g_\zeta(r\zeta)}{1-|\zeta|^2} - \frac{1}{2}\mathscr{S}_\psi(r\zeta) \right] \right| \leqq \frac{K-1}{K+1} \left[r^2 g_{\zeta\bar{\zeta}}(r\zeta) + \frac{1}{1-|\zeta|^2} \right].$$

From hypothesis 1) we have:

$$(7.12) \qquad \left| r^2 \left[g_{\zeta\zeta}(r\zeta) - g_\zeta^2(r\zeta) - \frac{2r\bar{\zeta} g_\zeta(r\zeta)}{1-|r\zeta|^2} - \frac{1}{2}\mathscr{S}_\psi(r\zeta) \right] \right| \leqq r^2 \left(\frac{K-1}{K+1}\right) \left[g_{\zeta\bar{\zeta}}(r\zeta) + \frac{1}{1-|r\zeta|^2} \right].$$

We will show (7. 11) holds if g satisfies either:

A $$|g_\zeta(\zeta)|\,(1-|\zeta|^2)\leqq k\left(\frac{1}{2|\zeta|}\right)$$

or

B $$|g_\zeta(\zeta)|\,(1-|\zeta|^2)\leqq k\,|\zeta|.$$

Both of which are implied by hypothesis 3). The proofs begin in the same way:

Following Ahlfors' we apply the triangle inequality to show that (7. 12) implies (7. 11) provided:

$$r^2 k\left[\frac{1}{(1-|r\zeta|^2)^2}+g_{\zeta\bar\zeta}(r\zeta)\right]+|2r\bar\zeta g_\zeta(r\zeta)|\left[\frac{1}{1-|\zeta|^2}-\frac{r^2}{1-|r\zeta|^2}\right] \leqq k\left[\frac{1}{(1-|\zeta|^2)^2}+r^2 g_{\zeta\bar\zeta}(r\zeta)\right]. \tag{7. 13}$$

Rewriting (7. 13) we obtain:

$$|2r\zeta g_\zeta(r\zeta)|\left[\frac{1}{1-|\zeta|^2}-\frac{r^2}{1-|r\zeta|^2}\right]\leqq k\left[\frac{1}{(1-|\zeta|^2)^2}-r^2(1-|r\zeta|^2)^2\right]. \tag{7. 14}$$

First we prove A:

Rewriting (7. 14) we obtain

$$|2r\zeta g_\zeta(r\zeta)|\,(1-|r\zeta|^2)\leqq k\left(\frac{1-r^2|\zeta|^4}{1-|\zeta|^2}\right). \tag{7. 15}$$

An easy calculation shows that the right hand side of (7. 15) is strictly increasing for $|\zeta|\in(0,1)$. In order for (7. 15) to hold it suffices that A holds.

To prove B we go back to 7. 14 and apply Ahlfors' inequality:

$$|\zeta|^2\left[\frac{1}{1-|\zeta|^2}-\frac{r^4}{(1-|r\zeta|^2)^2}\right]\leqq\frac{1}{(1-|\zeta|^2)^2}-\frac{r^2}{(1-|r\zeta|^2)^2} \tag{7. 16}$$

on the right hand side of (7. 14) to see that it suffices to show that:

$$2r^2\frac{|g_\zeta(r\zeta)|}{|r\zeta|}\leqq\left(\frac{K-1}{K+1}\right)\frac{1+r^2-2r^2|\zeta|^2}{(1-r^2|\zeta|^2)(1-|\zeta|^2)}. \tag{7. 17}$$

For (7. 17) to hold it suffices that

$$(1-|r\zeta|^2)\frac{|g_\zeta(r\zeta)|}{|r\zeta|}\leqq\left(\frac{K-1}{K+1}\right)\frac{r^2+1}{2r^2}. \tag{7. 18}$$

If hypothesis B holds then (7. 18) is clearly satisfied for any $r<1$ and therefore the estimate in (7. 11) holds as well. This completes the proof of the theorem. □

Remarks. 1) It is clear that hypothesis 3) can be replaced by:

$$(1-|\zeta|^2)\,|g_\zeta(\zeta)| \leqq k'\left(\max\left(|\zeta|, \frac{1}{2}|\zeta|\right)\right)$$

for a $k'<1$. If $k'>k$ then at the end of the approximation argument we have a $\frac{1+k'}{1-k'}$-quasiconformal reflection, Λ. However Λ is given by formula (3.8) and so the Beltrami coefficient is given by (6.9) or equivalently (7.3), hence Λ is actually a $\frac{1+k}{1-k}$-quasiconformal reflection.

2) The conditions given in Theorem 7.2 reduce to known conditions for special choices of g. For instance if $g\equiv 0$ then we obtain the classical condition [Ahl-We]:

$$|\mathcal{S}_\psi(\zeta)\,(1-|\zeta|^2)^2| \leqq 2k.$$

If $g(\zeta)=c\log(1-|\zeta|^2)$ then we obtain the sufficient condition of Ahlfors' given in [Ahl 1]. In this case his condition is more general in that c is allowed to be complex.

3) A very interesting special case was pointed to the author by Prof. Pommerenke. He suggested taking g to be a harmonic function of the form $g=\operatorname{Re}\log\phi'$. Here ϕ is a locally univalent function which must satisfy:

$$(1-|\zeta|^2)\left|\frac{\phi''(\zeta)}{\phi'(\zeta)}\right| \leqq k'\max\left\{\frac{1}{2|\zeta|}, |\zeta|\right\}.$$

The univalence criterion takes the form:

$$\left|\mathcal{S}_\phi(\zeta)-\left(\mathcal{S}_\psi(\zeta)-\frac{2\bar\zeta}{(1-|\zeta|^2)}\frac{\phi''}{\phi'}(\zeta)\right)\right| \leqq \frac{2k}{(1-|\zeta|^2)^2}. \tag{7.19}$$

In a latter publication we will show that these conditions are sometimes sharp and produce the possible non-Jordan region which arises when (7.19) is satisfied with $k=1$.

Further specializing to $\phi=\psi$ we obtain:

$$\left|\zeta\frac{\phi''}{\phi'}(\zeta)\right| \leqq \frac{k}{(1-|\zeta|^2)}. \tag{7.20}$$

This is the sharp form due to Becker of a result of Duren, Shields and Shapiro, [Be]. By changing variables in (7.19) via $z=\frac{\zeta-1}{\zeta+1}$ we obtain

$$2\operatorname{Re} z\left|\frac{\phi''}{\phi'}(z)\right| \leqq k. \tag{7.21}$$

This was proved by Gehring [Ge].

In [Be-Po] (7. 20) and (7. 21) are shown to be sharp.

4) The reflections which are generated by surfaces in $\mathbb{H}^3$ satisfy an additional relation. In terms of the map

$$w = S \circ \Lambda \circ S^{-1}$$

it can be expressed by the equation

$$\operatorname{Re}(dw \wedge dz/(w-z)^2) = 0.$$

In [Ep 3] we investigate the consequences of such a relation on the topology of a line field in $\mathbb{H}^3$.

5) If ψ does not satisfy the hypotheses of the theorem but rather:

$$\left| \frac{(1-|\zeta|^2)^2 \left[g_{\zeta\zeta} - g_\zeta^2 - \frac{1}{2} \mathscr{S}_\psi \right] - 2\bar{\zeta}(1-|\zeta|^2)\, g_\zeta}{1+(1-|\zeta|^2)^2\, g_{\zeta\bar{\zeta}}} \right| \leqq 1$$

for a g which satisfies:

$$1+(1-|\zeta|^2)^2\, g_{\zeta\bar{\zeta}} > 0$$

and

$$(1-|\zeta|^2)\,|g_\zeta(\zeta)| \leqq \max\left\{\frac{1}{2|\zeta|}, |\zeta|\right\},$$

then $\psi(r\zeta)$ and $g(r\zeta)$ will satisfy the hypotheses of the theorem for some constant k_r. k_r tends to one as $r \to 1$. From this we conclude that ψ is univalent in the disk. In fact one can show that ψ is continuous in the closed disk and that $\psi(\partial D_1)$ has a logarithmic module of continuity. We will treat this in a latter publication.

If $g = 0$ then we would have:

$$(1-|\zeta|^2)^2\, |\mathscr{S}_\psi| < 2$$

and $\phi^*\left(\frac{|d\zeta|^2}{(1-|\zeta|^2)^2}\right)$ is the hyperbolic metric on $\operatorname{Im}\psi$. From the standard estimates on this metric we can conclude that the surface Σ_t generated by

$$\frac{\log|\phi'|}{1-|\phi|^2} + \log(1+|z|^2)$$

is complete and satisfies

$$|k_i| < 1 \quad i = 1, 2;$$

thus we obtain:

Corollary. *If ψ is conformal in D_1 and satisfies $(1-|\zeta|^2)^2\,|\mathscr{S}_\psi(\zeta)| < 2$ then ψ is univalent and $\psi(D_1)$ is a Jordan domain.*

This corollary was obtained in [Ge-Po] using different techniques.

Added in proof. After this work was completed it came to the author's attention that K. Uhlenbeck had obtained similar results to those in § 2 in his paper: Closed minimal surfaces in hyperbolic 3-manifolds, Ann. Math. Studies **103** (1985).

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