# Handbook of Complex Analysis:

# Geometric Function Theory

## Volume 1

*Edited by*
*R. Kühnau*

NORTH-HOLLAND

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# Preface

What is Geometric Function Theory (GFT)? Historically we mean by this the theory of conformal mappings; that is, mappings induced by analytic functions. In the main, these mappings are understood here as univalent (schlicht) mappings. Global univalence is an essential aspect of the theory. This carries much more significance than local univalence. Following Gauß, these conformal mappings are "in den kleinsten Teilchen ähnlich", in particular angle-preserving.

Lately, quasiconformal and related mappings have been included in the theory because they have many properties in common with conformal mappings. Furthermore, it turns out surprisingly that quasiconformal mappings are intimately connected with analytic functions in the form of quadratic differentials that arise in the theory of extremal quasiconformal ("möglichst konform" following H. Grötzsch) mappings.

Existence and uniqueness theorems, starting with the Riemann mapping theorem as a corner stone, constitute a central topic of GFT. Historically, these were followed by mapping theorems for multiply connected domains, mainly the work of P. Koebe.

Another central topic arises by asking for properties of these mappings, mainly in the form of so-called distortion theorems. These are a priori estimates of functionals. Perhaps the first such distortion theorem was the famous Schwarz lemma. This lemma, including its proof, at first appears to be an extremely simple thing, but it has turned out to be a very powerful and surprising tool – new aspects and generalizations having appeared again and again.

Afterwards, an essentially new idea appeared in the form of the Koebe distortion theorem which immediately yields a-priori estimates for many situations, although at first not necessarily in sharp form. Distortion theorems still represent an essential part of the theory. Therefore this is also a central theme in this Handbook.

Nowadays we have many powerful methods for solving extremal problems. Among them are the miraculous Löwner differential equation technique, the simple but surprisingly effective area method and Grunsky's method of contour integration, Grötzsch's strip method, followed by the method of extremal length of Beurling and Ahlfors, which is especially fruitful also in the multiply connected case, and the very general variational method of Schiffer. Up to now, the interrelations between these methods are still not completely clear.

In GFT "purely" geometric aspects are sometimes considered: for example area, length, perimeter, diameter (also in non-euclidean or other metrics). However there are also many questions in consideration of which the geometric aspect is not so evident. The Bieberbach conjecture is an example. Here, as in many other cases, geometry intervenes only in the form of the univalence of the mappings. (Therefore, nowadays GFT is often commonly referred to today as the "Theory of univalent functions".) Isn't it a common phenomenon nowadays to refer to many subareas of mathematics as "geometric", while there is very little "real geometry" involved?

There are a number of questions that lead to geometry even though their original formulation is not of a geometric character. One only has to think of the appearance of quadratic differentials in the characterization of solutions of extremal problems.

Questions of a not explicitly geometric nature appear in GFT nowadays mainly in the theory of univalent mappings of simply connected domains. The theory in the case of multiply connected domains is in some sense in peculiar contrast to the simply connected case to which the greatest part of the theory is devoted. In the simply connected case we have the power and Laurent series for functions, analytic, respectively, inside and outside the unit circle, leading to the classes $S$ and $\Sigma$.

Of course, even in this Handbook completeness is impossible, and many topics can only be intimated. We hope however that the many references provide a helpful guide for further studies.

The theory of conformal mappings is intimately connected with the theory of boundary value problems for harmonic functions. This is the reason for many applications in mathematical physics and the need for good numerical methods for the construction of conformal mappings. However this interplay only works in two dimensions because in three dimensions, due to the classical theorem of Liouville, there are only a few and trivial conformal mappings. In higher dimensions the powerful instrument of conformal mappings fails. Arnold Sommerfeld wrote in his "Vorlesungen über theoretische Physik, Band II: Mechanik der deformierbaren Medien", in § 19:

*"Das mächtige Werkzeug der Funktionentheorie läßt sich also in der dreidimensionalen Potentialtheorie nicht verwenden. David Hilbert äußerte gelegentlich, um die Fruchtlosigkeit aller dahin gehenden Versuche prägnant zu kennzeichnen: Die Zeit ist eindimensional, der Raum dreidimensional, die Zahl, d.h. die vollkommene komplexe Zahl, ist zweidimensional".*

The mathematical theory of quasiconformal mappings in space, while it has undergone much development in recent decades, is not related to harmonic functions at all.

A comprehensive history of GFT has so far not been written. When did the term "Geometric Function Theory" first appear? To answer this question it should be observed that the meaning of the term has changed a bit over time. For example, F. Klein already used the term ("Geometrische Funktionentheorie") in his "Gesammelte Abhandlungen", volume 3 (Springer-Verlag, Berlin 1923) pp. 477 ff., mainly to clarify the concept of analytic functions with the construction of Riemann surfaces (cf. also his lectures "Funktionentheorie in geometrischer Behandlungsweise" in Leipzig 1880/81, B.G. Teubner, Leipzig 1987). And in his supplement to A. Hurwitz's classical "Vorlesungen über allgemeine Funktionentheorie und elliptische Funktionen", R. Courant used the term "Geometrische Funktionentheorie" to contrast Riemann's conception of function theory with the Weierstrass development of function theory from power series.

The first comprehensive monograph on Geometric Function Theory is due to G.M. Golusin, and it bore precisely that title. The reader is referred to Volume 2 of this Handbook for a listing of books dealing with GFT or special parts thereof.

GFT is a living subject in which surprisingly new questions, perhaps with old roots, continue to appear; cf. the theory of circle packings which appeared at first in a partially forgotten paper of P. Koebe.

We would like to leave the reader of this Handbook with the impression that *"Konforme Abbildung ist immer modern"* (to quote Koebe as orally communicated to the editor of this Handbook by H. Grötzsch).

Reiner Kühnau

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CHAPTER 1

# Univalent and Multivalent Functions

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## 1. Univalent functions

### 1.1. *Introduction*

Suppose that $f(z)$ is analytic in a domain $\Delta$, i.e., an open connected set of the closed complex plane. The function $f(z)$ is said to be univalent (or schlicht) in $\Delta$ if $f(z)$ assumes different values $w$ for different values of $z$, so that the equation $f(z) = w$ has at most one root in $\Delta$ for every complex $w$.

Univalent functions provide a conformal mapping. If $D$ is the image of $\Delta$, i.e., the set of all values assumed by $f(z)$ in $\Delta$, then $D$ is also a domain and if $\Delta$ is simply connected, so is $D$. The map is conformal if, when two curves $\gamma_0, \gamma_1$ intersect at an angle $\alpha$ in $\Delta$, then the images $\Gamma_0, \Gamma_1$ intersect at the image point $w_0 = f(z_0)$ at the same signed angle $\alpha$. Conversely if

$$f = u + iv$$

where $u, v$ are real differentiable functions of $x, y$, and $z = x + iy$, then $f$ is conformal if and only if $f$ is analytic, i.e., $u, v$ obey the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

and further

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \neq 0$$

in $\Delta$. If

$$f(z) = a_0 + a_p(z - z_0)^p + \cdots, \qquad a_p \neq 0$$

then $f(z)$ assumes values close to $a_0$ exactly $p$ times near $z = z_0$. Thus $f(z)$ is not univalent nor conformal in $\Delta$ if $f'(z_0) = 0$, so that $p \geqslant 2$.

It was stated by Riemann [71, p. 40] and first rigorously proved by Koebe [51] that every simply connected plane domain (except for the whole plane) can be mapped (1, 1) conformally onto the unit disk $\Delta$. The map becomes unique if a given point $z_0$ is mapped onto a given point $w_0$ and the argument of $f'(z_0)$ is fixed.

Riemann's Theorem enables many problems in general domains $D$ to be reduced to problems in

$$\Delta\colon\ |z| < 1. \tag{1.1}$$

Thus the class of the corresponding conformal maps or functions univalent in $\Delta$, given by (1.1), acquires a special importance. We may normalize so that $f(0) = 0$, $f'(0) = 1$.

Otherwise we may consider $\phi(z) = (f(z) - f(0))/f'(0)$ instead of $f$, since $\phi$ is univalent if and only if $f$ is univalent. We accordingly denote by $S$ the class of functions

$$f(z) = z + \sum_{0}^{\infty} a_n z^n \tag{1.2}$$

univalent in $|z| < 1$.

It is sometimes convenient to allow the image domain $D$ to lie in the closed complex plane, i.e., to include the point at $\infty$. In this case the function $f$ may have one simple pole at a point $z_0$ in $\Delta$. We choose $z_0 = 0$ and normalize

$$f(z) = \frac{1}{z} + \sum_{0}^{\infty} b_n z^n. \tag{1.3}$$

The class of functions $f(z)$ given by (1.3) and univalent in $\Delta$ is denoted by $\Sigma$.

In this article we shall give a survey of some important inequalities for $S$ and to a lesser extent $\Sigma$, involving the maximum modulus, coefficients and mean values of functions in the class.

## 1.2. *The basic inequalities*

Many interesting results follow simply from the following Area-Theorem, discovered by Gronwall [29].

THEOREM 1.1. *Suppose that $f(z) \in \Sigma$, that $D$ is the image of $\Delta$ and that $E$ is the complement of $D$ in the complex plane. Then the area, i.e., the 2-dimensional Lebesgue measure of $E$ is*

$$\pi\left(1 - \sum_{1}^{\infty} n|b_n|^2\right).$$

*Thus $\sum_1^\infty n|b_n|^2 \leqslant 1$.*

It is not hard to show that the image of $|z| = r$ by $\Sigma$ encloses an area

$$A(r) = \pi\left\{\frac{1}{r^2} - \sum_{1}^{\infty} n|b_n|^2 r^{2n}\right\}$$

(see, e.g., Hayman [42, p. 2]). Letting $r$ tend to 1 we deduce Theorem 1.1.

From this result Bieberbach [7] deduced the following theorem.

THEOREM 1.2. *If $f(z) \in S$, then $|a_2| \leqslant 2$. If $d$ is the distance from the origin to the nearest point of the complement of the image $D$, then $d \geqslant 1/4$. Equality holds in both cases if and only if $f(z)$ is the Koebe function*

$$f_\theta(z) = \frac{z}{(1 - ze^{-i\theta})^2} = z + \sum_1^\infty nz^n e^{i(n-1)\theta} \tag{1.4}$$

*which maps $\Delta$ onto the complement of the ray*

$$w = -te^{i\theta}, \quad \frac{1}{4} \leqslant t \leqslant \infty. \tag{1.5}$$

The result with an absolute constant instead of $1/4$ had previously been obtained by Koebe [50], who conjectured the correct value $1/4$. The deduction of Theorem 1.2 from Theorem 1.1 is so simple and elegant that we give it here.

Suppose that $f(z) \in S$. Then so does

$$f_2(z) = \{f(z^2)\}^{1/2} = z + \frac{1}{2}a_2 z^3 + \cdots.$$

Conversely if $f_2(z)$ is odd and univalent in $\Delta$ then $f(z) = (f_2(z^{1/2}))^2 \in S$. Again

$$F(z) = \frac{1}{f_2(z)} = \frac{1}{z} - \frac{1}{2}a_2 z + \cdots \in \Sigma.$$

Thus, by Theorem 1.1, $b_1 = |\frac{1}{2}a_2| \leqslant 1$, so that $|a_2| \leqslant 2$. Also by Theorem 1.1 equality is possible if and only if $b_n = 0$, $n \geqslant 2$, so that

$$F(z) = \frac{1}{z} - ze^{-i\theta},$$

$$f_2(z) = \frac{z}{1 - z^2 e^{-i\theta}}, \qquad f(z) = \frac{z}{(1 - ze^{-i\theta})^2} = f_\theta(z).$$

We see easily that $f_\theta(z)$ maps $\Delta$ onto the complement of the ray (1.5).

We next prove that if $w \notin D$, then $|w| \geqslant 1/4$. To see this we consider

$$\phi(z) = \frac{w f(z)}{w - f(z)} = z + \left(a_2 + \frac{1}{w}\right)z^2 + \cdots.$$

Then since $f(z) \neq w$, $\phi(z) \in S$ so that $|a_2 + 1/w| \leqslant 2$. Thus

$$\left|\frac{1}{w}\right| \leqslant \left|a_2 + \frac{1}{w}\right| + |a_2| \leqslant 4$$

so that $|w| \geqslant 1/4$. Equality is possible only if $|a_2| = 2$, so that $f(z)$ is a Koebe function.

## 1.3. *Bounds for* $|f(z)|$, $|f'(z)|$ *and* $|f'(z)|/|f(z)|$

Theorem 1.2 leads very simple to a range of further inequalities giving sharp upper and lower bounds for $|f(z)|$, $|f'(z)|$.

THEOREM 1.3.[1] *If* $f(z) \in S$, *we have for* $|z| = r$ $(0 < r < 1)$

$$\frac{r}{(1+r)^2} \leqslant |f(z)| \leqslant \frac{r}{(1-r)^2}, \tag{1.6}$$

$$\frac{1-r}{(1+r)^3} \leqslant |f'(z)| \leqslant \frac{1+r}{(1-r)^3}, \tag{1.7}$$

*and*

$$\frac{1-r}{r(1+r)} \leqslant \left|\frac{f'(z)}{f(z)}\right| \leqslant \frac{1+r}{r(1-r)}. \tag{1.8}$$

*Strict inequality holds in all cases unless* $f(z) \equiv f_\theta(z)$ *for some* $\theta$.

We apply Theorem 1.2 to

$$\phi(z) = f\left(\frac{z_0 + z}{1 + \overline{z_0} z}\right) = b_0 + b_1 z + b_2 z^2 + \cdots,$$

where $|z_0| = r$. Then

$$b_0 = f(z_0), \qquad b_1 = \phi'(0) = \left(1 - |z_0|^2\right) f'(z_0),$$

and

$$b_2 = \frac{1}{2}\phi''(0) = \frac{1}{2}\left(1 - |z_0|^2\right)^2 f''(z_0) - \overline{z_0}\left(1 - |z_0|^2\right) f'(z_0).$$

Clearly $(\phi(z) - b_0)/b_1 \in S$, so that Theorem 1.2 yields $|b_2| \leqslant 2|b_1|$. From this Theorem 1.3 follows. (For details see Hayman [42, pp. 5–7].)

## 1.4. *Means and coefficients*

Having proved that $|a_2| \leqslant 2$, Bieberbach conjectured that $|a_n| \leqslant n$, if $n \geqslant 2$ and $f(z) \in S$, with equality only for the Koebe functions. This conjecture gave rise to much interesting mathematics and was only proved in full generality by de Branges [14]. However

[1] Bieberbach [7], Gronwall [30], Szegö [84].

Littlewood [54] obtained at least the right order of magnitude for the bounds of $|a_n|$. We write, for real $\lambda$ and $0 < r < 1$,

$$M(r, f) = \sup_{|z|=r} |f(z)|, \tag{1.9}$$

$$I_\lambda(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^\lambda \, d\lambda. \tag{1.10}$$

Littlewood [54] noted that while Theorem 1.3 yields the sharp bound

$$M(r, f) \leqslant \frac{r}{(1-r)^2},$$

the deduction

$$I_\lambda(r, f) \leqslant \{r/(1-r)^2\}^\lambda$$

no longer yields the correct order of magnitude. The sharp bounds for the means $I_\lambda(r)$ were obtained by Baernstein [3] by a technique involving his star function, a concept with applications in other areas of analysis. Once again the extremals are the Koebe functions. Littlewood developed a more elementary but also profound technique which yields

$$I_\lambda(r, f) = O(1-r)^{1-2\lambda}, \quad \text{if } \lambda > \frac{1}{2}.$$

We sketch the proof of his

THEOREM 1.4. *If $f \in S$, then*

$$I_1(r, f) < \frac{r}{1-r}, \quad 0 < r < 1, \tag{1.11}$$

*and*

$$|a_n| < e\, I_1\left(\frac{n-1}{n}, f\right) < en, \quad n \geqslant 2. \tag{1.12}$$

As in the proof of Theorem 1.2 we consider

$$\phi(z) = f_2(z) = \{f(z^2)\}^{1/2} = z + b_3 z^3 + \cdots$$

and note that, by (1.6),

$$M(r, \phi) \leqslant \frac{r}{1-r^2}.$$

Also

$$\frac{1}{2\pi}\int_0^{2\pi} \left|\phi'\left(re^{i\theta}\right)\right|^2 d\theta = \sum_1^\infty n^2 |b_n|^2 r^{2n-2}.$$

Thus

$$\begin{aligned}\pi\sum_1^\infty n|b_n|^2 r^{2n} &= \int_0^r \int_0^{2\pi} \left|\phi'\left(\rho e^{i\theta}\right)\right|^2 \rho\, d\theta\, d\rho \\ &= \text{area of transform of } |z| < r \text{ by } w = \phi(z) \\ &< \pi M(r,\phi)^2 \leqslant \frac{\pi r^2}{(1-r^2)^2},\end{aligned}$$

since $\phi$ is univalent. This is the crucial step. Integrating we obtain

$$\sum_1^\infty |b_n|^2 r^{2n} < \frac{r^2}{(1-r^2)},$$

i.e.,

$$I_1\left(r^2, f\right) = I_2(r,\phi) < \frac{r^2}{(1-r^2)},$$

which is (1.11), and (1.12) follows.

The technique has far reaching generalizations.

THEOREM 1.5. *Suppose that* $f(z) = \sum_0^\infty a_n z^n$ *is univalent in* $\Delta$ *(given by* (1.1)) *and that*

$$M(r,f) < C(1-r)^{-\beta}, \quad 0 < r < 1, \tag{1.13}$$

*where* $\beta > 1/2$. *Then we have*

$$|a_n| < A_1(\beta)\, C n^{\beta-1}, \quad n \geqslant 1, \tag{1.14}$$

*where the constant* $A_1$ *depends on* $\beta$ *only. If*

$$M(r,f) = \mathrm{o}(1-r)^{-\beta}, \quad \text{as } r \to 1 \tag{1.15}$$

*then*

$$|a_n| = \mathrm{o}\left(n^{\beta-1}\right), \quad \text{as } n \to \infty. \tag{1.16}$$

For details see Hayman [42, p. 71]. The implication from (1.15) to (1.16) is not explicitly stated there, but follows by the same method. However the argument breaks down completely when $\beta \leqslant 1/2$. Baernstein [4] has shown that the range can be extended, and that Theorem 1.5 remains valid for $\beta > \beta_0$, where $\beta_0 < 1/2$, in particular for $\beta = 1/2$.

## 1.5. *k-symmetric functions*

Suppose that $f(z) = z + \cdots$, is analytic in $\Delta$ and, for $k = 2, 3, \ldots$, write

$$f_k(z) = \left\{f\left(z^k\right)\right\}^{1/k} = z + b_{k+1}z^{k+1} + b_{2k+1}z^{2k+1} + \cdots. \tag{1.17}$$

The functions $f_k(z)$ satisfy

$$f_k(\omega z) = \omega f_k(z), \quad \text{where } \omega = \exp(2\pi i/k),$$

and are called $k$-symmetric. We see easily that $f(z) \in S$ if and only if $f_k(z) \in S$. Evidently, by applying (1.6) to $f(z)$, we obtain

$$M(r, f_k) \leqslant \frac{r}{(1 - r^k)^{2/k}},$$

with equality if and only if

$$f_k(z) = \frac{z}{(1 - z^k e^{i\theta})^{2/k}}.$$

On applying Theorem 1.5 including Baernstein's extension, we deduce

$$|b_n| < A(k)n^{2/k-1} \tag{1.18}$$

for the coefficients of $k$-symmetric functions, when $k = 1, 2, 3, 4$, where the constant $A(k)$ depends on $k$ only. Szegö conjectured[2] that (1.18) holds for all $k$. The results for $k = 1, 2, 3, 4$ are due to Littlewood [54], Littlewood and Paley [55], V.I. Levin [52] and Baernstein [4], respectively. However an example of Pommerenke [62] and [64, p. 133] shows that the conjecture is false for $k \geqslant 12$. The cases $5 \leqslant k \leqslant 11$ remain open.

## 1.6. *The class $\Sigma$ and bounded univalent functions*

While thanks to de Branges we know the exact bounds for the coefficients of $S$ the situation for $\Sigma$ is much less satisfactory. The first few sharp bounds are $|b_1| \leqslant 1$, $|b_2| \leqslant 2/3$ and $|b_3| \leqslant 1/2 + e^{-6}$ due to Gronwall [29], Prawitz [69] and Garabedian and Schiffer [21], respectively.

[2] Oral communication to V.I. Levin [52].

However for $\Sigma$ even the order of magnitude of the coefficients remains in doubt. There is a remarkable analogue between the bounds for the coefficients of $\Sigma$ and those of bounded univalent functions. This has been made precise recently by Carleson and Jones [11]. Suppose that $S_M$ is the class of functions

$$f(z) = \sum_{1}^{\infty} a_n z^n,$$

univalent in $\Delta$ and satisfying $|f(z)| < 1$ and let $A_n$ be the bound for $|a_n|$, when $f \in S_M$. Similarly let $B_n$ be the bound for $b_n$ when $f(z) \in \Sigma$ and is given by (1.3). Then Carleson and Jones prove the existence and equality of the limits

$$\gamma = \lim_{n\to\infty} \frac{-\log A_n}{n} = \lim_{n\to\infty} \frac{-\log B_n}{n}.$$

They call $\gamma$ the Clunie-constant, since Clunie and Pommerenke [13] first showed that $\gamma > 1/2$. Their methods suggest that $\gamma \leqslant 0.76$. However the best established upper bound remains $\gamma < 0.83$ of Pommerenke [64, p. 133]. Makarov [58] has shown that $\beta_0 = 1 - \gamma$. The best lower bound $\beta_0 = 1 - \gamma < 1/2 - 1/86$ is due to Grinshpan and Pommerenke [28].

Carleson and Jones conjecture that $\gamma = 3/4$. Their method does not seem to lead to $k$-symmetric functions with large coefficients, but, if it did, their conjecture might imply that $k = 8$ is the critical case for Szegö's conjecture and that the conjecture holds for $k \leqslant 7$ but not for $k \geqslant 9$.

## 1.7. *Coefficient differences*

Suppose that $f(z) \in S$. What can we say about the coefficient differences $||a_{n+1}| - |a_n||$? It turns out that both the Koebe functions and the odd Koebe functions $f_2(z)$ have the extreme behaviour. We have [41], [42, p. 180].

THEOREM 1.6. *If $f \in S$ then*

$$\big||a_{n+1}| - |a_n|\big| < A,$$

*where $A$ is an absolute constant.*

The best known bounds are

$$-2.97 < |a_{n+1}| - |a_n| < 3.61$$

due to Grinshpan [27].

A corresponding bound for the differences of successive coefficients of $k$-symmetric functions has been obtained by Lucas [57] and Hayman [42, p. 185]. Lucas proved

THEOREM 1.7. *Suppose that* $f(z) \in S$. *If* $f(z) = z + a_3 z^3 + \cdots$ *is odd univalent, then*

$$\big||a_{2n+1}| - |a_{2n-1}|\big| < A_2\, n^{1-\sqrt{2}}.$$

*If* $f(z) = z + a_4 z^4 + \cdots$ *is* 3*-symmetric, then*

$$\big||a_{3n+1}| - |a_{3n-2}|\big| < A_3\, n^{2/3-\sqrt{4/3}}.$$

Here $A_2$, $A_3$ are absolute constants.

If $k \geqslant 4$ no bound sharper than the immediate consequences of Theorem 1.5 are known. This yields for $k = 4$

$$\big||a_{4n+1}| - |a_{4n-3}|\big| < 2\max\big(|a_{4n+1}|, |a_{4n-3}|\big) < A\, n^{-1/2}.$$

If $k > 4$, we obtain the order of magnitude $n^{\beta_0 - 1}$ from Baernstein's extension of Theorem 1.5.

## 2. Asymptotic behaviour

### 2.1. *Introduction*

There is a general principle that if under certain hypotheses there is an upper (or lower) bound on the growth of a class of functions, then those functions having the extremal growth show a regularity of behaviour. Theorems of this nature are called regularity theorems. In this section we discuss some regularity theorems for univalent functions.

### 2.2. *The maximum modulus*

The simplest result of this nature is [39], [42, pp. 8, 9]

THEOREM 2.1. *Suppose that* $f(z) \in S$. *The limit*

$$\alpha = \lim_{r \to 1} (1 - r)^2 M(r, f) \tag{2.1}$$

*always exists and* $0 \leqslant \alpha \leqslant 1$. *Further* $\alpha = 1$ *if and only if* $f(z)$ *is a Koebe function* $f_\theta(z)$ *given by* (1.4).

The result is an almost immediate consequence of the inequality (1.8) of Theorem 1.3. We fix $\theta$ and write

$$l(r) = \log\big|f\big(re^{i\theta}\big)\big|.$$

Then

$$l'(r) \leqslant \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right| \leqslant \frac{1+r}{r(1-r)}. \tag{2.2}$$

Thus

$$\frac{d}{dr}\left\{ l(r) - \log \frac{r}{(1-r)^2} \right\} \leqslant 0.$$

Hence

$$\frac{(1-r)^2}{r} \left| f\left(re^{i\theta}\right) \right|$$

decreases with increasing $r$ and strictly, unless $f(z) \equiv f_\theta(z)$. We assume now that $f(z) \not\equiv f_\theta(z)$ for any $\theta$. Then if $0 < r_1 < r_2 < 1$, $0 \leqslant \theta < 2\pi$, we have

$$\frac{(1-r_2)^2}{r_2} \left| f\left(r_2 e^{i\theta}\right) \right| < \frac{(1-r_1)^2}{r_1} \left| f\left(r_1 e^{i\theta}\right) \right| \leqslant \frac{(1-r_1)^2}{r_1} M(r_1, f). \tag{2.3}$$

We choose $\theta$, so that the left-hand side assumes its maximum value and deduce that

$$\frac{(1-r_2)^2}{r_2} M(r_2, f) < \frac{(1-r_1)^2}{r_1} M(r_1, f).$$

Thus

$$\alpha(r) = \frac{(1-r)^2}{r} M(r, f)$$

is a positive strictly decreasing function of $r$. Since

$$\alpha(0) = \lim_{r \to 0} \frac{M(r, f)}{r} = 1$$

we deduce that the limit

$$\alpha = \alpha(1) = \lim_{r \to 1} \alpha(r) = \lim_{r \to 1} (1-r)^2 M(r, f)$$

exists and $0 \leqslant \alpha < 1$.

## 2.3. *Means and coefficients*

It turns out that the asymptotic behaviour of the means $I_\lambda(r, f)$, the coefficients $|a_n|$ and the maximum modulus of the derivatives $M(r, f^{(p)})$ among others can also be determined in terms of $\alpha$. It is convenient to distinguish 3 cases. If $\alpha = 1$, then $f(z) \equiv f_\theta(z)$ and so the behaviour of all the above quantities is known. If $\alpha = 0$, then

$$M(r, f) = \frac{o(1)}{(1-r)^2},$$

and it is not difficult to obtain corresponding estimates for means and coefficients, for instance by Theorem 1.5.

Most interesting is the case $0 < \alpha < 1$.

This seems a good time to introduce powers $\{f(z)/z\}^\lambda$ and their coefficients. We have seen that, if $k$ is a positive integer, then $f(z) \in S$ if and only if

$$f_k(z) = \{f(z^k)\}^{1/k} = z + b_{k+1}z^{k+1} + \cdots \in S. \tag{2.4}$$

The functions $f_k(z)$ are precisely the $k$-symmetric functions in $S$, i.e., the functions in $S$ of the form (1.17). We may also write $\lambda = 1/k$ and consider, with a new notation

$$F_\lambda(z) = \left(\frac{f(z)}{z}\right)^\lambda = \sum_{n=1}^{\infty} a_n(\lambda)z^{n-1}, \tag{2.5}$$

where

$$f(z) = z + \sum_2^\infty a_n z^n \in S.$$

The $a_n(\lambda)$ for $\lambda = 1/k$ are precisely the coefficients of the $k$-symmetric univalent functions, and $a_n(1) = a_n$. It is of interest to discuss the behaviour of the powers $F_\lambda(z)$ and their means and coefficients. We have [42, Theorem 5.5, p. 151, and Example 5.4, p. 164]

THEOREM 2.2. *If $\alpha$ is the constant defined in Theorem* 2.1, *and the $F_\lambda(z)$ are defined by* (2.5) *then, for $\lambda > 1/2$,*

$$(1-r)^{1-2\lambda} I_1\{r, F_\lambda(z)\} = (1-r)^{1-2\lambda} r^{-\lambda} I_\lambda(r, f) \to \alpha^\lambda \frac{\Gamma(\lambda - 1/2)}{2\sqrt{\pi}\,\Gamma(\lambda)}, \quad \textit{as } r \to 1. \tag{2.6}$$

*Also if $\lambda > 1/4$*

$$\frac{|a_n(\lambda)|}{n^{2\lambda-1}} \to \frac{\alpha^\lambda}{\Gamma(2\lambda)}, \quad \textit{as } n \to \infty. \tag{2.7}$$

We see for instance that if $f(z) \not\equiv f_\theta(z)$ for some $\theta$ then the means $I_\lambda(r)$ and the coefficients $|a_n(\lambda)|$ are finally less than the corresponding means and coefficients relating to $f_\theta(z)$. Thus taking $\lambda = 1$, we have in all cases

$$|a_n| \leqslant n, \quad n > n_0(f).$$

For if $f(z) \equiv f_\theta(z)$ we have $a_n = ne^{i(n-1)\theta}$ for all $n$ while in all other cases

$$\frac{|a_n|}{n} \to \alpha$$

where $\alpha < 1$. Similarly if

$$f(z) = z + a_3 z^3 + a_5 z^5 + \cdots$$

is odd and $f(z) \in S$, we have $|a_n| \leqslant 1, n > n_0(f)$. For if $\alpha = 1$, we have

$$f(z) \equiv \frac{z}{1 - z^2 e^{i\theta}} = z + z^3 e^{i\theta} + z^5 e^{2i\theta} + \cdots,$$

while in all other cases

$$\lim_{n\to\infty} |a_{2n+1}| = \alpha^{1/2} < 1.$$

However in this situation it is known (Schaeffer and Spencer [77], see also Duren [16, p. 107]) that for every $n$, with $n \geqslant 2$ an odd univalent function can be found such that $|a_{2n+1}| > 1$. It was shown be Fekete and Szegö [19] that the exact upper bound for $|a_5|$ is $1/2 + e^{-2/3} = 1.01\ldots.$

It follows from de Branges' theorem ([14], see also Hayman [42, p. 247]) that for $\lambda \geqslant 1$ and all $n$ we have

$$\bigl|a_n(\lambda)\bigr| \leqslant d_{n-1}(2\lambda) = \frac{\Gamma(n-1+2\lambda)}{\Gamma(n)\,\Gamma(2\lambda)}, \tag{2.8}$$

with equality if and only if $f(z) \equiv f_\theta(z)$. On the other hand this result fails for $-1 < \lambda < 1$ and $n = 3$. In this case the exact bound for $|a_3(\lambda)|$ is

$$|\lambda|\bigl\{1 + 2e^{2(\lambda-1)/(\lambda+1)}\bigr\} > d_2(2\lambda). \tag{2.9}$$

The case $\lambda = 1/2$ gives theorem of Fekete and Szegö.

### 2.4. *Ideas behind Theorem 2.2*

We sketch a proof of Theorem 2.2, at least for the case $\lambda = 1$. If $\alpha = 0$ the conclusion follows simply by an extension of Littlewood's technique for Theorem 1.4. The results are also a direct consequence of Theorem 1.5. If $\alpha = 1$, $f$ reduces to a Koebe function and the conclusion is trivial. We suppose now that $0 < \alpha < 1$. We return to the proof of Theorem 2.1 and recall from (2.3) that

$$\alpha(r) = \frac{(1-r)^2}{r} M(r, f)$$

decreases to $\alpha$. In particular for each positive integer $n$ and $r_n = 1 - 1/n$, we can find $\theta = \theta_n$, such that, for $r = r_n$,

$$(1-r)^2 \frac{|f(re^{i\theta})|}{r} \geqslant \alpha(r) > \alpha.$$

Also the left-hand side is a decreasing function of $r$ for fixed $\theta$, so that the inequality remains valid for $0 < r < r_n$. If $\theta_0$ is a limit point of the sequence $\theta_n$ we deduce that

$$(1-r)^2 \frac{|f(re^{i\theta_0})|}{r} \geqslant \alpha, \quad 0 < r < 1.$$

Using Theorem 2.1, we deduce that

$$\left|f\left(re^{i\theta_0}\right)\right| \sim \frac{\alpha}{(1-r)^2}.$$

Next we can show that in this case for $z = re^{i\theta_0}$

$$\Re \frac{zf'(z)}{f(z)}$$

is close to $2/(1-r)$ for most $r$, so that

$$\Im \frac{zf'(z)}{f(z)}$$

is small for such $r$, by (1.8). Thus $\arg f(re^{i\theta_0})$ is slowly varying. We now deduce that in a sector

$$S_K(\theta_0, r) = \begin{cases} \dfrac{1-r}{K} < \left|1 - ze^{-i\theta_0}\right| < K(1-r), \\ \left|\arg\left(1 - ze^{-i\theta_0}\right)\right| < \dfrac{\pi}{2} - \dfrac{1}{K}, \end{cases}$$

the function $(1 - ze^{-i\theta_0})^2 f(z)$ is close to $\alpha(r)$, when $K$ is a fixed large constant, $r$ is close to 1, and $|\alpha(r)| = r$.

Now area considerations show that $f(z)$ is relatively small outside $S_K(\theta_0, r)$. We write, with $r = r_n$,

$$a_n = \frac{1}{2\pi i r^n}\int_{\theta_0-\pi}^{\theta_0+\pi} f\left(re^{-in\theta}\right) d\theta,$$

$$ne^{-i(n-1)\theta} = \frac{1}{2\pi i r^n}\int_{\theta_0-\pi}^{\theta_0+\pi} \frac{r}{(1-re^{i(\theta-\theta_0)})^2}\, d\theta$$

and deduce eventually that

$$a_n - \alpha(r)\frac{n}{r}e^{-i(n-1)\theta_0} = o(n)$$

which yields (2.7) in this case. The argument extends to the case $\lambda > 1/2$ and also yields (2.6). However to obtain (2.7) for $\lambda > 1/4$ we need a number of refinements, and in particular we have to apply the Cauchy integral formula to $f'(z)$ instead of $f(z)$. (For details see Hayman [42, Section 1.7, pp. 15–26, and Theorem 5.6, pp. 155–158].)

## 2.5. *Bounds for the* $|a_n(\lambda)|$

What can we say about the bounds of the $|a_n(\lambda)|$ for varying $f$? We write

$$A_n(\lambda) = \sup_{f\in S}\left|a_n(\lambda)\right|.$$

It has been shown by Hayman and Hummel [43] that, at least for $\lambda > 1/4$,

$$\frac{A_n(\lambda)}{n^{2\lambda-1}} \to \frac{K(\lambda)}{\Gamma(2\lambda)},$$

where $K(\lambda)$ is a positive constant. It follows from (2.8) that $K(\lambda) = 1$ if $\lambda \geqslant 1$ and in all cases $K(\lambda) \geqslant 1$. We showed that, for $\lambda < 0.4998$, we have $K(\lambda) > 1$ and we conjectured that $K(\lambda) = 1$ for $\lambda \geqslant 1/2$, but $K(\lambda) > 1$ for $\lambda < 1/2$. The quantity $K(\lambda)$ is evidently infinite if $\lambda$ is small as the examples of Pommerenke [64, p. 133] show. For odd univalent functions, i.e., $\lambda = 1/2$, the best known result is

$$A_n\left(\frac{1}{2}\right) < 1.14, \quad n = 1, 2, \ldots,$$

due to V.I. Milin [60], so that $K(1/2) \leqslant 1.14$.

The asymptotics of coefficient differences are more complicated but Eke [18] has shown that if $f \in S$ then $|a_{n+1}| - |a_n| \to 0$ unless $\alpha > 0$ in Theorem 2.1, in which case $|a_{n+1}| - |a_n| \to \alpha$, or $f$ has 2 radii of greatest growth in which case $|a_{n+1}| - |a_n|$ oscillates. Finally Hamilton [33] proved that if $f \in S$ then

$$\limsup_{n\to\infty}\left||a_{n+1}| - |a_n|\right| < 1$$

unless there is a real $\phi$ such that

$$e^{i\phi} f\left(ze^{-i\phi}\right) \equiv \frac{z}{1 - 2z\cos(\theta) + z^2}$$

for some constant $\theta$. Further $||a_{n+1}| - |a_n|| < 1$ for large $n$ unless $\theta/\pi$ is rational.

## 3. Löwner's theory and de Branges' theorem

Many of the results mentioned so far can be obtained largely by semi-elementary, though sometimes quite complicated methods based on the idea that, since $w = f(z)$ is univalent, the area of the image $D_0$ of any region $\Delta_0$ in $\Delta$, counting multiplicity, does not exceed the area of the projection of $D_0$ onto the $w$-plane. For instance if $D_0$ lies in $|w| < R$, then the area of $D_0$ does not exceed $\pi R^2$.

In order to solve more difficult extremal problems a number of other techniques were developed by various authors. Of these the most successful one appears to be that of Löwner [56]. Löwner investigated mappings of $\Delta$ onto a subdomain consisting of $\Delta$ cut along a Jordan arc $\gamma$. Alternatively following Duren [16, p. 80] we may consider a domain $G$, consisting of the complement of a Jordan arc $\gamma$ going from a point $B$ to $\infty$.

It is convenient to assume that $\gamma$ takes the form $BP\infty$, where $P$ is the point $\rho e^{i\theta}$, $P\infty$ is a ray

$$w = re^{i\theta}, \quad \rho \leqslant r < \infty,$$

and the Jordan arc $BP$ lies entirely in $|z| \leqslant r$. We also assume that $\gamma$ is so chosen that the normalized map

$$w = f(z) = z + \cdots$$

maps $\Delta$ onto $G$, so that $f(z) \in S$. Further it is not hard to see that the class $S_1$ of all such maps $f$ is dense in $S$. In other words, given any function $f(z)$ in $S$ there exists a sequence $f_n(z)$ of functions in $S_1$ such that

$$f_n(z) \to f(z)$$

locally uniformly in $\Delta$. Thus the coefficients and other functionals $L(f_n)$, such as the value of $f_n$ or one of its derivatives at a fixed point $z_0$ in $\Delta$, tend to the value of the corresponding functional $L(f)$ on $f$. From this it follows that extremal values of $L(f)$ in the class $S_1$, such as the upper bound of $L(f)$ if $L$ is real, are the same as the extremals in the larger class $S$.

The deepest part of Löwner's theory is his proof that the functions $f$ in $S_1$ are solutions of Löwner's differential equation

$$\frac{\partial w}{\partial t} = -w\,\frac{1 + \kappa(t)w}{1 - \kappa(t)w}, \quad 0 \leqslant t \leqslant \infty. \tag{3.1}$$

Here $w = f(z,t)$, with $f(z,0) = z$, $e^t f(z,t) \in S$, and $|\kappa(t)| = 1$. Also

$$e^t f(z,t) \to f(z) \tag{3.2}$$

locally uniformly as $t \to \infty$.

It is not hard to see that given any measurable $\kappa(t)$, with $|\kappa(t)| = 1$, $0 \leqslant t < \infty$, there exists a function $w = f(z,t)$, satisfying (3.1) almost everywhere in $t$, absolutely continuous for $t$ in any finite interval $[0,T]$ and with the initial condition $f(z,0) = z$, where $|z| < 1$. We also find easily that $e^t f(z,t) \in S$. It is harder to construct the function $f(z,t)$ satisfying (3.1) and (3.2) when $f(z) \in S_1$. The rough idea is as follows.

We denote by $\gamma(t)$ the arc $t \leqslant \tau < \infty$ of $\gamma$ and by $G(t)$ the complement of $\gamma(t)$ in the complex plane. We can choose the parametrization of $\gamma(t)$ in such a way that a function

$$g_t(z) = e^t(z + \cdots)$$

maps $\Delta$ onto $G(t)$. Then

$$f(z,t) = g_t^{-1}\{f(z)\} = ze^{-t} + \cdots$$

turns out to have the required properties. Since $\gamma(t)$ is a ray near $\infty$ by hypothesis, we see that, for large $t$,

$$g_t(z) = \frac{e^t z}{(1 + ze^{-i\theta})^2},$$

and this yields (3.2). The proof that $f(z,t)$ satisfies the equation (3.1) lies deeper and we refer the reader to Duren [16, pp. 82–87] or Hayman [42, Chapter 7].

### 3.1. *Applications of Löwner's theory*

Löwner's theory has an astonishingly large number of applications. The method is to express a functional in $S$ in terms of the function $\kappa(t)$ which appears in (3.1). In many cases it is then possible to obtain directly sharp inequalities for the class $S_1$ and hence for $S$, since $S_1$ is dense in $S$. However extra work is needed to obtain all the extremals in $S$, since the method generally only yields extremals in $S_1$.

We sketch one example given by Löwner [56].

THEOREM 3.1. *Suppose that $f(z) \in S$ and that*

$$z = \phi(w) = f^{-1}(w) = w + \sum_2^\infty b_m w^m$$

*is the inverse function of $f$. Then*

$$|b_m| \leqslant \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2m-1) \cdot 2^m}{(m+1)!}, \quad m \geqslant 2, \tag{3.3}$$

*with equality when*

$$f(z) = \frac{z}{(1+z)^2}, \qquad \phi(w) = \left[1 - 2w - (1-4w)^{1/2}\right]/(2w).$$

It is enough to prove the inequality for the functions $e^{t_0} f(z, t_0)$ since these are dense in $S$. Let $z = \phi_t(w)$ be the inverse function of $f(z, t)$ so that

$$\phi_t\{f(z,t)\} = z. \tag{3.4}$$

We write $\beta = e^{-t_0}$ and

$$\phi_t(w) = e^t\left[w + \sum_{m=2}^{\infty} b_m(t) w^m\right], \quad 0 \leqslant t \leqslant t_0. \tag{3.5}$$

Then

$$\phi(w) = \phi_{t_0}(\beta w) = w + \sum_{m=2}^{\infty} b_m w^m$$

is inverse to $\beta^{-1} f(z, t_0)$ and it is enough to prove (3.3) for the coefficients $b_m$ of $\phi(w)$. We write $w = f(z, t)$ and note that (3.4) leads to

$$\frac{\partial}{\partial w}\phi_t(w)\frac{\partial w}{\partial t} + \frac{\partial \phi_t(w)}{\partial t} = 0.$$

Substituting this in (3.1) yields

$$\frac{\partial \phi_t(w)}{\partial t} = \frac{\partial \phi_t(w)}{\partial w} w \frac{1 + \kappa w}{1 - \kappa w}.$$

Substituting (3.5) in this, differentiating both sides and equating coefficients of $w^m$, we obtain

$$b_m'(t) + b_m(t) = m b_m(t) + 2\sum_{r=1}^{m-1} r b_r(t)\kappa(t)^{m-r}, \quad 0 \leqslant t \leqslant t_0, \ m \geqslant 2,$$

with the boundary conditions

$$b_1(t) \equiv 1, \quad 0 \leqslant t \leqslant t_0; \qquad b_m(0) = 0, \qquad b_m(t_0) = \beta^{-m+1} b_m, \quad m \geqslant 2.$$

These yield the inductive relation

$$b_m(t) = 2e^{(m-1)t}\int_0^t \left\{\sum_{r=1}^{m-1} r b_r(\tau)\kappa(\tau)^{m-1}\right\} e^{-(m-1)\tau}\, d\tau, \quad m \geqslant 2.$$

It is now obvious that, if $t_0 > 0$ and $m > 1$, $|b_m(t_0)|$ attains its maximum possible value if $\kappa(t) \equiv 1$. In this case all the $b_m(t)$ are real and positive. It is also evident that the corresponding value of

$$b_m = e^{-(m-1)t_0} b_m(t_0)$$

increases with increasing $t_0$ and so we obtain the upper bound for variable $t_0$ in the limit as $t_0 \to \infty$.

We now take $\kappa(t) \equiv 1$ in the differential equation (3.1) and obtain on integration

$$\frac{f(z, t_0)}{1 + \{f(z, t_0)\}^2} = e^{-t_0} \frac{z}{(1+z)^2},$$

so

$$\frac{w}{1+w^2} = \frac{\beta \phi_t(w)}{1 + \{\phi_{t_0}(w)\}^2}.$$

Writing $\phi(w) = \phi_{t_0}(\beta w)$ we deduce

$$\frac{\phi(w)}{[1+\phi(w)]^2} = \frac{w}{(1+\beta w)^2}.$$

Thus, as $t_0 \to \infty$ and so $\beta \to 0$, $\phi(w) \to \psi(w)$, where

$$\frac{\psi(w)}{[1+\psi(w)]^2} = w,$$

i.e.,

$$\psi(w) = \frac{1 - 2w - \sqrt{1-4w}}{2w} = \sum_{m=1}^{\infty} b_m w^m$$

where $b_m$ is given by (3.3). This proves Theorem 3.1.

There are many other applications of Löwner's theory and we give some of them without proof. Löwner [56] himself used the method to give the sharp bound $|a_3| \leqslant 3$ for the third coefficient. In other applications the Koebe function is no longer extreme. Thus we have Duren [16, p. 104] and Hayman [42, p. 217]

THEOREM 3.2 (Fekete and Szegö [19]). *If $f \in S$ and $0 < \alpha < 1$ then*

$$\left|a_3 - \alpha a_2^2\right| \leqslant 1 + 2e^{-2\alpha/(1-\alpha)}.$$

*Equality holds for a function $f(z)$ in $S$ and with real coefficients.*

COROLLARY 1. *If $f(z) \in S$ and*

$$F_\lambda(z) = \left(\frac{f(z)}{z}\right)^\lambda = \sum_{n=1}^{\infty} a_n(\lambda) z^{n-1}$$

*then we have the sharp inequality*

$$|a_3(\lambda)| \leqslant |\lambda|\{1 + 2e^{-2(1-\lambda)/(1+\lambda)}\}, \quad -1 < \lambda < 1,$$

*in accordance with* (2.9).

COROLLARY 2. *If*

$$f(z) = z + b_3 z^3 + b_5 z^5 + \cdots$$

*is odd univalent, then $|b_3| \leqslant 1$, but for $b_5$ be have the sharp inequality*

$$|b_5| \leqslant \frac{1}{2} + e^{-2/3} = 1.013\ldots.$$

In fact with the above notation $b_3 = a_2(1/2)$, $b_5 = a_3(1/2)$, so that Corollary 2 follows from Corollary 1. We note that

$$a_1(\lambda) = 1, \quad a_2(\lambda) = \lambda a_2 \quad \text{and} \quad a_3(\lambda) = \lambda a_3 + \frac{\lambda(\lambda-1)}{2} a_2^2,$$

so that Corollary 1 follows from Theorem 3.2.

The following sharp bounds can also be obtained by means of Löwner's theory.

THEOREM 3.3 (Grunsky [32], Duren [16, p. 95], Hayman [42, p. 224]).

$$-\log \frac{1+|z|}{1-|z|} \leqslant \arg \frac{f(z)}{z} \leqslant \log \frac{1+|z|}{1-|z|}.$$

THEOREM 3.4 (Golusin [23], Duren [16, p. 99], Hayman [42, p. 228]).

$$|\arg f'(z)| \leqslant \begin{cases} 4 \sin^{-1} |z|, & |z| \leqslant \dfrac{1}{\sqrt{2}}, \\[2ex] \pi + \log \dfrac{|z|^2}{1-|z|^2}, & \dfrac{1}{\sqrt{2}} < |z| < 1. \end{cases}$$

### 3.2. *De Branges' theorem*

Löwner's theory is also the key element in the proof by de Branges [14] of Bieberbach's conjecture. However a good deal more was required.

Fundamental additional ideas were supplied by I.M. Milin [59]. Suppose that $f(z) \in S$ and that

$$\log \frac{f(z)}{z} = \sum_{k=1}^{\infty} c_k z^k.$$

Milin conjectured that

$$\sum_{k=1}^{n} \left( \frac{4}{k} - k|c_k|^2 \right) (n - k + 1) \geqslant 0, \quad n = 1, 2, \ldots, \tag{3.6}$$

and showed that from this conjecture that of Bieberbach could be deduced. To do this he and Lebedev proved some subtle inequalities involving the coefficients of power series.

Suppose that

$$\omega(z) = \sum_{k=1}^{\infty} A_k z^k$$

is a formal power series, that

$$\phi(z) = \exp \omega(z) = \sum_{k=0}^{\infty} D_k z^k$$

and that the binomial coefficients $d_k(\lambda)$ are defined by (2.8). Then Milin and Lebedev proved [59, Theorem 2.4, p. 50], [42, p. 246]

THEOREM 3.5. *If $\lambda > 0$, we have for $n = 1, 2, \ldots$*

$$|D_n| \leqslant d_n(2\lambda) \exp \left\{ \frac{1}{2d_n(2\lambda)} \sum_{k=1}^{n} d_{n-k}(2\lambda) \left( \frac{k^2 |A_k|^2 - 4\lambda^2}{2k\lambda} \right) \right\}.$$

*Equality holds if and only if*

$$A_k = \frac{2\lambda}{k} \eta^k, \quad k = 1, 2, \ldots, n, \ \textit{where}\ |\eta| = 1.$$

We now suppose that (3.6) holds. It turns out that this implies

$$\sum_{k=1}^{n} \left\{ k^2 |c_k|^2 - \frac{4}{k} \right\} d_{n-k}(2\lambda) \leqslant 0$$

for $\lambda \geqslant 1$. (The case $\lambda = 1$ is (3.6).) We now apply Theorem 3.5 with $A_k = \lambda c_k$,

$$F_\lambda(z) = \left(\frac{f(z)}{z}\right)^\lambda = \sum_1^\infty a_n(\lambda) z^{n-1}$$

and $D_{n-1} = a_n(\lambda)$. We deduce that

$$|a_n(\lambda)| \leqslant d_{n-1}(2\lambda). \tag{3.7}$$

Writing $\lambda = 1$ we obtain

$$|a_n| \leqslant n$$

with equality only for the Koebe function.

The equality (3.7) also shows that, for $\lambda \geqslant 1$, $n \geqslant 2$, the $|a_n(\lambda)|$ attain their maximum value $d_{n-1}(2\lambda)$ if and only if $f(z)$ is a Koebe function. This conclusion fails for $-1 < \lambda < 1$ and $n = 3$ by Corollary 1 of Theorem 3.2. (3.7) also implies a number of other results including some conjectures of Rogosinski [74] and Robertson [72]. De Branges [14] proved Milin's conjecture (3.6). The proof is subtle and beyond the scope of this short article. We refer the reader to de Branges [14] or Hayman [42, Chapter 8].

## 4. Subclasses

Before de Branges [14] proved Bieberbach's conjecture in full generality the result had been established for a number of subclasses. The study of these has an independent interest.

### 4.1. *Convex and starlike functions*

Suppose that $E$ is a set in the complex plane. We say that $E$ is starlike with respect to the star centre $z_0$ if, for every $z$ in $E$ the segment

$$z_0 z = \{\zeta = z_0 + (1-t)z \colon 0 \leqslant t \leqslant 1\} \subseteq E.$$

A set $E$ is convex if every point of $E$ is a star centre for $E$, i.e., if the segment $z_1 z_2 \subseteq E$, whenever $z_1 \in E$ and $z_2 \in E$. Functions mapping the unit disk $\Delta$ onto convex or starlike domains are called convex or starlike functions respectively. The convex, starlike subclasses of $S$ are denoted by $C$ and $S^*$ respectively. The following result characterizes $C$ and $S^*$.

THEOREM 4.1. *Suppose that $f \in S$. Then $f \in C$ if and only if for all $z$ in $\Delta$*

$$\Re\left\{z\frac{f''(z)}{f'(z)}\right\} > -1.$$

*Further* $f \in S^*$ *if and only if*

$$\Re\left\{z\frac{f'(z)}{f(z)}\right\} > 0.$$

*In particular* $f \in C$ *if and only if* $zf'(z) \in S^*$.

The last remark is due to Alexander [2].

It follows from Alexander's theorem that results for $C$ and $S^*$ are closely related. It is not hard to see that

$$|a_k| \leqslant 1$$

if $f \in C$ and hence that $|a_n| \leqslant n$ if $f \in S^*$. For the proofs we refer the reader to Duren [16, p. 44, 45] or Hayman [42, p. 11, 14]. Much deeper is the following result of Ruscheweyh and Sheil-Small [76], see also Duren [16, p. 246 et seq.]. If

$$f(z) = \sum_0^\infty a_n z^n \quad \text{and} \quad g(z) = \sum_0^\infty b_n z^n$$

we define the convolution

$$f \circ g\,(z) = \sum_0^\infty a_n b_n z^n.$$

THEOREM 4.2. *If* $f \in C$ *and* $g \in C$ *then* $f \circ g \in C$.

The result had been conjectured by Pólya and Schoenberg [61].

It follows from Theorems 4.1 and 4.2 that if $f \in S^*$ and $g \in S^*$ then

$$\sum_1^\infty \frac{a_n b_n}{n} z^n \in S^*.$$

## **4.2.** *Typically real and close-to-convex functions*

The function $f$ in $\Delta$ is called typically real, $f \in T$, if is real if and only if $z$ is real. If

$$f(z) = u + iv = z + \sum_2^\infty a_n z^n$$

and $z = x + iy$, we see that $vy \geqslant 0$ in this case.

THEOREM 4.3 (Rogosinski [73], Dieudonné [15], Szász [83]). *If* $f(z) \in T$, *then* $|a_n| \leqslant n|a_1|$.

In fact we have

$$\begin{aligned} |a_n r^n| &= \left| \frac{2}{\pi} \int_0^\pi v(re^{i\theta}) \sin n\theta \, d\theta \right| \\ &\leqslant \frac{2n}{\pi} \int_0^\pi |v(re^{i\theta}) \sin \theta| \, d\theta = nr|a_1|, \quad 0 < r < 1, \end{aligned}$$

in this case. If $f \in S$ and $f$ has real coefficients then $f \in T$ and the above proof applies. On the other hand typically real functions need not be univalent.

Another interesting subclass is the class $K$ of close-to-convex functions introduced by Kaplan [49]. We say that $f \in K$ if there exists $g(z)$ such that $g(z) \in C$ and

$$\Re \frac{f'(z)}{g'(z)} > 0, \quad z \in \Delta.$$

Lewandowski [53] has shown that close-to-convex functions are those mapping $\Delta$ onto a domain $D$, whose complement is the union of rays going from a point $z_0$ to $\infty$. Thus $K$ contains the class of starlike functions. Reade [70] has shown that Bieberbach's conjecture holds for $K$.

## 5. Brennan's conjecture and related problems

Suppose that $f(z) \in S$ so that $w = f(z)$ maps $\Delta$ onto a simply connected domain $D$. Thus the inverse function $z = \phi(w)$ maps $D$ onto $\Delta$. Brennan [10] has asked for what values of $p$ the area integral

$$\iint_D |\phi'(w)|^p \, |dw|^2 = \iint_\Delta |f'(z)|^{2-p} \, |dz|^2$$

converges. The Koebe function shows that for $p \leqslant 4/3$ and $p \geqslant 4$ the integrals may diverge. Brennan conjectured that the integrals converge for the remaining values of $p$, i.e., $4/3 < p < 4$. This follows simply from classical inequalities such as Theorem 1.3 if $4/3 < p < 3$. The case $p \geqslant 3$ is harder. Brennan himself showed that his conjecture holds if $3 \leqslant p \leqslant 3 + \delta$ where $\delta > 0$ and Pommerenke [65] has proved the result with $\delta = 0.399$. He also showed [66] that the full conjecture is equivalent to

$$\int_0^{2\pi} |f'(re^{i\theta})|^{-\lambda} \, d\theta = O(1 - r)^{-1-\varepsilon} \tag{5.1}$$

whenever $\varepsilon > 0$, $0 < \lambda < 2$ and $f \in S$. It is thus an interesting open problem to find for what values of $\lambda$ (5.1) holds. For a discussion of related results and conjectures see Pommerenke [68].

The behaviour of the means and coefficients of $1/f'(z)$, when $f \in S$, is still not well understood. Ruscheweyh [75] has shown that, if $f$ is close-to-convex then $1/f'(z)$ has bounded coefficients $f_n$ but Pommerenke [67, Exercise 2, p. 194] has shown that there exists $f$ in $S$ for which

$$f_n \neq O(n^{0.064}).$$

Readers wishing to deepen their knowledge of univalent functions are recommended to study Duren [16], Hayman [42] and Pommerenke [64]. A very full biography is given by Bernardi [6]. Duren's book also contains an excellent historical account of the subject and a large and careful bibliography.

## 6. Multivalent functions

### 6.1. *Introduction*

The notion of univalent functions can be generalized. Suppose that $f(z)$ is analytic in a domain $\Delta$. For $w$ in $\Delta$ let $n(w)$ be the number of roots of the equation $f(z) = w$ in $\Delta$. We distinguish 4 cases.

(a) If $n(w) \leqslant p$ for all $w$, we say that $f$ is *p-valent* in $\Delta$. In this case $p$ must be a positive integer. If $f$ is $p$-valent with $p = 1$, then $f$ is univalent.

(b) In most cases a weaker average condition is sufficient to obtain results. We define

$$p(R) = \frac{1}{2\pi} \int_0^{2\pi} n\big(Re^{i\phi}\big)\, d\phi. \tag{6.1}$$

If

$$p(R) \leqslant p, \quad 0 < R < \infty, \tag{6.2}$$

we say that $f(z)$ is *circumferentially mean p-valent* (c.m. $p$-valent). The condition says that values $w$ lying on circles $|w| = R$ are assumed on the average at most $p$ times. Now $p$ can be any positive number. This definition is due to Biernacki [9].

(c) A still weaker condition is that

$$W(R) = \frac{1}{\pi} \int_0^R t\, dt \int_0^{2\pi} n\big(te^{i\phi}\big)\, d\phi = \int_0^R 2p(t)t\, dt \leqslant pR^2. \tag{6.3}$$

Functions satisfying the condition (6.3) are called *areally mean p-valent* (a.m. $p$-valent). Clearly (6.2) implies (6.3) so that c.m. $p$-valent functions are also a.m. $p$-valent The definition of a.m. $p$-valent functions is due to Spencer [80].

Various other generalizations are sometimes useful. We may for instance replace (6.3) by

$$\limsup_{R\to\infty} \frac{W(R)}{R^2} \leqslant p.$$

A more useful condition, applicable when $p$ is a positive integer is

(d) For $0 < R < \infty$, either $n(Re^{i\phi}) = p$, $0 \leqslant \phi < 2\pi$, or else there exists $\phi$, such that $0 \leqslant \phi < 2\pi$ and $n(Re^{i\phi}) < p$. Functions satisfying this condition are called *weakly $p$-valent* (Hayman [38]).

Evidently (b) implies (d), but neither of (c) and (d) implies the other.

In general $p$-valent functions behave a little like $p$'th powers of univalent functions. However most of the powerful techniques of the theory of univalent functions are not available. We quote the following useful theorem, see Hayman [42, p. 145].

THEOREM 6.1. *Suppose that $\eta > 0$, that $f(z)$ is analytic in a domain $\Delta$ and that $\psi(z) = f(z)^{\eta}$ is single-valued there. Then*

(i) *If $f(z)$ is c.m. $p$-valent in $\Delta$, then $\psi(z)$ is c.m. $(\eta p)$-valent there.*

(ii) *If $f(z)$ is a.m. $p$-valent in $\Delta$, then $\psi(z)$ is a.m.$(\eta_0 p)$-valent there where $\eta_0 = \max(\eta, \eta^2)$.*

(iii) *If $f(z)$ is weakly $p$-valent and $\eta = 1/p$ then $\psi(z)$ is weakly univalent.*

This result makes it possible in some cases to reduce the study of mean or weakly $p$-valent functions to that of the corresponding univalent functions. However when taking fractional powers the zeros of $f$ are apt to cause problems. They can be dealt with by a lemma in Hayman [37, Lemma 3, p. 152]. In what follows we shall assume that $\Delta$ is the unit disk $|z| < 1$, unless the contrary is explicitly stated.

## 6.2. *Sharp bounds*

There are a few cases where the classical sharp bounds for univalent functions can be extended to $p$-valent functions. We have the following theorems, see Hayman [38].

THEOREM 6.2. *Suppose that $p$ is a positive integer and that*

$$f(z) = z^p + a_{p+1} z^{p+1} + \cdots$$

*is weakly $p$-valent in $\Delta$: $|z| < 1$. Then*

$$|a_{p+1}| \leqslant 2p. \tag{6.4}$$

*Further we have for $|z| = r$, $0 < r < 1$,*

$$\frac{r^p}{(1+r)^{2p}} \leqslant |f(z)| \leqslant \frac{r^p}{(1-r)^{2p}}. \tag{6.5}$$

*and*

$$|f'(z)| \leqslant \frac{p(1+r)}{r(1-r)} |f(z)| \leqslant \frac{p r^{p-1}(1+r)}{(1-r)^{2p+1}}.$$

*Thus*

$$r^{-p}(1-r)^{2p}M(r,f)$$

*decreases strictly with increasing $r$ and so tends to a limit $\alpha$, as $r \to 1$, where $\alpha < 1$ unless $f(z) = f_\theta(z)^p$, where $f_\theta(z)$ is a Koebe function. Finally the equation $f(z) = w$ has precisely $p$ roots in $\Delta$ if $|w| < 4^{-p}$.*

The inequality (6.4) had been proved for areally mean $p$-valent functions by Spencer [81] and the lower bound in (6.5) was extended to the latter class by Garabedian and Royden [20]. We note that Theorem 6.2 extends Theorems 1.2, 1.3 and 2.1 to weakly $p$-valent functions, except that we have lost the lower bounds in (1.7) and (1.8) for $|f'(z)|$ and $|f'(z)/f(z)|$ respectively. This is inevitable, since the derivatives of $p$-valent functions, unlike those of univalent functions, may very well vanish in $\Delta$ if $p > 1$, and the same is true for mean $p$-valent functions for every positive $p$.

The analogue of Theorem 6.2 can also be proved for functions $f(z)$ weakly or c.m. $p$-valent in $\Delta$ and such that $f(z) \neq 0$ (Hayman [38, p. 145]).

At this point we may also mention the asymptotic bounds. The following result is relatively simple for c.m. $p$-valent functions but lies deeper when $f$ is a.m. $p$-valent.

THEOREM 6.3 (Eke [17,18]). *Suppose that*

$$f(z) = \sum_{0}^{\infty} a_n z^n$$

*is a.m. $p$-valent in $\Delta$. Then the limit*

$$\alpha = \lim_{r \to 1}(1-r)^{2p}M(r,f)$$

*always exists and, if $p > 1/4$,*

$$\lim_{n \to \infty} \frac{|a_n|}{n^{2p-1}} = \frac{\alpha}{\Gamma(2p)}. \tag{6.6}$$

EXAMPLE 6.1 [42, p. 76]. The identity (6.6) fails if $p < 1/4$. Let $n_k$ be a rapidly increasing sequence and let $\varepsilon$ be a positive number. Set

$$\begin{cases} a_n = \varepsilon\, 2^{-k} n^{-1/2}, & \text{if } n = n_k, \\ a_n = 0, & \text{otherwise.} \end{cases}$$

Then $f(z)$ is continuous in $\overline{\Delta}$ and, given a positive $p$, we can choose $\varepsilon$ so small that $f$ is a.m. $p$-valent. But the $a_n$ need satisfy nothing stronger than

$$a_n = o\left(n^{-1/2}\right).$$

This example makes Baernstein's extension of Theorem 1.5 all the more remarkable. It is not known whether similar examples exist for c.m. $p$-valent or strictly $p$-valent functions.

At this point we should mention that Jenkins [46, p. 159] has extended (6.4) to the next coefficient by proving the sharp inequalities

$$\left|\frac{a_{p+2}}{p} + \frac{1}{2p}\left(\frac{1}{p} - 1\right)a_{p+1}^2\right| \leqslant 3$$

and hence

$$|a_{p+2}| \leqslant 2p^2 + p \tag{6.7}$$

for c.m. $p$-valent functions in the case of Theorem 6.2. This result is remarkable in combining the depth of the Löwner theory with the generality of c.m. $p$-valent functions. Jenkins' argument uses his theory of modules together with symmetrization to obtain sharp estimates for

$$|f(r)| + |f(-r)|$$

when $f$ is c.m. 1-valent and his conclusion follows from that for univalent functions. An unpublished example of Spencer shows that (6.7) fails for a.m. 1-valent functions.

Garabedian and Royden [20] extended the lower bound in (6.5) and equivalently the last statement of Theorem 6.2 to such functions. Biernacki [9] proved that with the hypotheses of Theorem 6.2, $f(z)$ assumes every value in $|w| < 1/4$ at least once.

## 6.3. *Coefficient bounds and the Goodman conjecture*

Evidently polynomials of degree $p$ are $p$-valent in the plane. Thus we cannot hope to obtain a bound for $|a_p|$ in terms of the previous coefficients, when $f$ is $p$-valent in $\Delta$. However Cartwright [12] showed that the later coefficients can be bounded in terms of $a_0$ to $a_p$ for such functions. By looking at polynomials of degree $p$ of the Koebe function $f(z) = z(1-z)^{-2}$, Goodman [24] arrived at the conjecture

$$|a_n| \leqslant \sum_{k=1}^{p} \frac{2k(n+p)!}{(p+k)!\,(p-k)!\,(n-p-1)!\,(n^2-k^2)}|a_k|, \quad n > p. \tag{6.8}$$

Goodman and Robertson [26] proved the conjecture for the subclass of typically real functions of order $p$. This result is sharp. For an interesting historical account of what is known, we refer the reader to Goodman [25]. Unfortunately for general $p$-valent functions (6.8) is only known to be true in some very special cases, e.g., if $a_1 = \cdots = a_{p-1} = 0$ and $n = p+1$ (6.4) and $n = p+2$ (6.7). The simplest general case of (6.8) that is open is

$$|a_3| \leqslant 5|a_1| + 4|a_2|$$

for 2-valent functions.

**6.4.** *Orders of magnitude*

While sharp bounds for $p$-valent functions are hard to find, many of the order of magnitude results extend from univalent to a.m. $p$-valent functions. The key result is the following theorem.

THEOREM 6.1. *Suppose that $f$ is a.m. $p$-valent in $\Delta$. Then*

$$M(r, f) < A(p)\,\mu_p\,(1-r)^{-2p}, \tag{6.9}$$

*where $\mu_p = \sum_{0\leqslant\nu\leqslant p} |a_\nu|$, and the constant $A(p)$ depends only on $p$.*

Theorem 6.1 was proved for $p$-valent functions by Cartwright [12] and in the general case by Spencer [79]. We may take (Hayman [42, p. 38])

$$A(p) = e^{1/2}(47.2)^{2p}.$$

A bound with $A(p) = A^p$ is due to Jenkins and Oikawa [47].

To go from (6.9) to corresponding bounds for the means and coefficients, we follow essentially similar techniques. The key result is a length area principle in conformal mapping introduced by Ahlfors [1] in order to prove Denjoy's conjecture. This result is also at the basis of the theory of quasiconformal mapping. Grötzsch [31] had introduced such a length area principle a little earlier.

For the case of $p$-valent functions the principle is conveniently stated in the following form (Hayman [42, p. 29]).

THEOREM 6.2. *Suppose that $f(z)$ is analytic in an open set $\Delta$ of finite area $A$ and that $p(R)$ is defined by* (6.1). *Let $l(R)$ be the total length of the curves in $\Delta$ on which $|f(z)| = R$. Then*

$$\int_0^\infty \frac{l(R)^2\,dR}{R\,p(R)} \leqslant 2\pi A.$$

This inequality is sharp (Hayman [42, Examples 2.1 and 2.2, p. 32]).

For the proof and application of Theorem 6.2 we refer the reader to Hayman [42, Chapter 2]. The deduction from Theorem 6.3 of bounds for the means and coefficients of a.m. $p$-valent functions will be found in Hayman [42, Chapter 3]. For this purpose we require the following triple identity. We suppose now that $f(z)$ is analytic in $\Delta$ and for $0 < r < 1$ we define $p(r, R)$ as in (6.1) but with $\Delta$ replaced by $|z| < r$. We define $I_\lambda(r)$ as in (1.10). Then we have the following triple identity of which the first equation is due to Hardy [34] and Stein [82] and the second to Spencer [78]. For a proof see Hayman [42, p. 67].

THEOREM 6.3. *We have for $\lambda > 0$, and $0 < r < 1$*

$$r\frac{d}{dr}I_\lambda(r) = \frac{\lambda^2}{2\pi}\int_0^r \rho\,d\rho \int_0^{2\pi} \left|f\left(\rho e^{i\theta}\right)\right|^{\lambda-2}\left|f'\left(\rho e^{i\theta}\right)\right|^2 d\theta$$

$$= \lambda^2 \int_0^\infty p(r, R) R^{\lambda - 1}\, dR.$$

Suppose now that $f$ is c.m. $p$-valent and that (6.9) has been proved. Choosing $\lambda = 1$ we deduce that

$$r \frac{d}{dr} I_1(r) \leqslant \int_0^{M(r,f)} p\, dR = pM(r, f) \leqslant A(p)\, \mu_p (1 - r)^{-2p}.$$

This leads for $p > 1/2$ to

$$I_1(r) = \mathrm{O}(1 - r)^{1-2p}$$

and hence (see Biernacki [8] for the $p$-valent case) to

$$|a_n| = \mathrm{O}\big(n^{2p-1}\big) \tag{6.10}$$

by Cauchy's inequality. By considering $I_1(r, f')$ we can extend the conclusion to $p > 1/4$. It fails for $p < 1/4$ as Example 6.1 shows. The case $p = 1/4$ remains open. In this case we obtain

$$|a_n| = \mathrm{O}\big(n^{-1/2} \log n\big).$$

The arguments and conclusions extend to a.m. $p$-valent functions. Further Theorem 1.5 also extends to a.m. $p$-valent functions with constants depending on $p$ and $\beta$, but Baernstein's extension to the case $\beta < 1/2$ fails for a.m. $p$-valent functions by Example 6.1. The bounds for the coefficients of $k$-symmetric functions become

$$|a_n| = \mathrm{O}\big(n^{2p/k-1}\big)$$

provided that $2p/k > 1/2$, i.e., $k < 4p$. Again the conclusion fails for $k > 4p$ and remains in doubt for $k = 4p$.

Lucas [57] has also obtained bounds for coefficient differences for a.m. $p$-valent functions, namely

$$|a_{n+1}| - |a_n| = \mathrm{O}\big(n^{2p-2}\big), \qquad p \geqslant 1, \tag{6.11}$$

$$|a_{n+1}| - |a_n| = \mathrm{O}\big(n^{2p-2\sqrt{p}}\big), \quad \frac{1}{4} < p < 1. \tag{6.12}$$

The case $p = 1$ is due to Hayman [41]. (6.11) is certainly sharp as

$$f(z) = (1 - z)^{-2p}$$

shows. (6.12) may not be sharp, but is the best that is known. For $p < 1/4$ we cannot hope to improve on

$$|a_{n+1}| - |a_n| = \mathrm{o}\big(n^{-1/2}\big).$$

Corresponding conclusions for successive coefficients of $k$-symmetric a.m. $p$-valent functions also hold. For proofs we refer the reader to Chapter 6 of Hayman [42]. The argument extends to the mean $p$-valent case without difficulty.

### 6.5. *Lengths of level sets*

It follows from Theorem 6.2 that if $f$ is a.m. $p$-valent in a set of finite area then the length $l(R)$ of the level set $|f(z)| = R$ is bounded on the average in any interval $[R, 2R]$ say. It is natural to ask whether $l(R)$ is always finite. In fact Hayman and Wu [45] have shown that if $f$ is univalent in the unit disk, then $l(R)$ is bounded by an absolute constant. (See also Garnett, Gehring and Jones [22] for a simpler proof and some generalizations.) However no comparable result is known even for strictly 2-valent functions.

However some restriction on the valency is necessary. Jones [48] has given an example of a bounded analytic function in the unit disk for which all the non-empty level sets have infinite length.

### 6.6. *Valencies on sequences*

Suppose that $f(z)$ is weakly $p$-valent in $\Delta$. Thus, for $0 < R < \infty$, $f(z)$ assumes in $\Delta$ every value on $|w| = R$ exactly $p$ times or $f$ assumes some value on $|w| = R$ less than $p$ times. It turns out (Hayman [38, 42, p. 147]) that the former condition holds for $R < l_f$ and the latter for $R \geqslant l_f$. Here $l_f$ is bounded above by $A(p)\,\mu_p$ as in Theorem 6.1. Thus, for large $R$, there exists $w = w_R$, such that $|w| = R$ and the equation

$$f(z) = w \tag{6.13}$$

has at most $p - 1$ roots in $\Delta$.

This led Littlewood [54] to consider the growth of functions satisfying the above condition on a sequence $w = w_k$, where $w_0 = 0$,

$$|w_k| \leqslant |w_{k+1}| < K|w_k|, \quad 1 \leqslant k < \infty, \quad w_k \to \infty \quad \text{as } k \to \infty, \tag{6.14}$$

where $K$ is a constant, $K > 1$. He proved

THEOREM 6.4. *If $w_k$ is a sequence satisfying* (6.14) *and Equation* (6.13) *never has more than $p - 1$ roots for $w = w_k$, then*

$$M(r, f) < C_1 \mu_p (1 - r)^{-C_2}, \quad 0 < r < 1, \tag{6.15}$$

*where $C_1, C_2$ depend on $K$, $w_1$ and $p$ only.*

By considering $f(z)/w_1$ instead of $f(z)$ we may assume that $w_1 = 1$. We can also reduce the problem to $p = 1$. If $f(z) \neq 0$, we consider

$$\phi(z) = \{f(z)\}^{1/p}$$

instead of $f(z)$. Then $f(z)$ omits (i.e., fails to assume) at least one of the $p$ roots $W_k = w_k^{1/p}$, since if $\phi = W_k$ we deduce $f(z) = w_k$. In the general case when $f$ has $p-1$ zeros we can consider $f$ and hence $\phi$ in a suitable simply connected subdomain of $\Delta$ containing none of the zeros without loosing too much (Hayman [37, p. 152]). Thus we may consider the case when $f(z)$ omits a sequence of values.

The first sharpening of Littlewood's result is due to Cartwright [12] who proved that if in addition to (6.14)

$$\left|\frac{w_{k+1}}{w_k}\right| \to 1 \quad \text{as } k \to \infty \tag{6.16}$$

then (6.15) holds with $C_2 = 2p + \varepsilon$ for every positive $\varepsilon$, where $C_1$ now depends also on $\varepsilon$.

Further Hayman [36] showed that if

$$\sum \left\{\log\left|\frac{w_{k+1}}{w_k}\right|\right\}^2 < \infty, \tag{6.17}$$

we may replace $\varepsilon$ by 0, and the condition (6.17) is necessary for this if the $w_k$ are all positive. Hayman [35] also investigated how $C_2$ depends on $K$, when $K$ is large.

The behaviour of the coefficients is harder. We have no area principle at our disposal since $f(z)$ may well assume all values except the $w_k$ infinitely often. However Baernstein and Rochberg [5] have shown, that if $f$ omits a sequence satisfying (6.14) and (6.16), then the coefficients $a_n$ of $f$ satisfy

$$a_n = O\left(n^{1+\varepsilon}\right) \tag{6.18}$$

for every $\varepsilon$ and this result is sharp. A weaker result had been obtained by Pommerenke [63]. The corresponding problem for $\varepsilon = 0$ is still open. However Baernstein [3] and independently Hayman and Weitsman [44] have shown that if $f(z)$ is weakly univalent or more generally if $f$ omits a set of values whose moduli form a sufficiently thick subset of the positive axis, the conclusion

$$a_n = O(n) \tag{6.19}$$

holds. For the above result it is essential that the omitted set has positive capacity. To obtain a conclusion such as (6.19) when only a sequence of values $w_k$ is omitted, we need at present to impose restrictions on the arguments as well as the moduli of $w_k$. For instance Hayman [39] has shown that (6.19) holds if

$$\arg w_k = O\left(1/|w_k|\right)$$

and

$$w_{k+1} - w_k = O\left(|w_k|^{1/2}\right).$$

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CHAPTER 2

# Conformal Maps at the Boundary

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# 1. Introduction

## 1.1. *The scope of this article*

A *conformal map* is an injective meromorphic function, in other words an angle-preserving homeomorphism of some domain onto another.

We shall restrict ourselves to simply connected domains. The case of a domain of finite connectivity can easily be reduced to the simply connected case by making suitable cuts. The case of domains of infinite connectivity however presents new problems and will not be considered.

Let $F$ be a simply connected domain in $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with at least two boundary points. The Riemann mapping theorem states that there is a unique conformal map $f$ of the unit disk $\mathbb{D}$ onto $F$ such that $f(0)$ and $\arg f'(0)$ take given values.

Now let $G$ be another simply connected domain with at least two boundary points and consider a conformal map $\varphi$ of $F$ onto $G$. Then $g = \varphi \circ f$ is a conformal map of $\mathbb{D}$ onto $G$ and $\varphi = g \circ f^{-1}$. Hence problems about conformal maps from one domain onto another can, at least in principle, be reduced to problems about conformal maps from the unit disk onto a given domain.

Hence we have the following factorization by conformal maps:

$$\varphi = g \circ f^{-1} : F \to G \quad \text{with } f : \mathbb{D} \to F,\ g : \mathbb{D} \to G. \tag{1.1.1}$$

Thus we shall always consider conformal maps $f$ of $\mathbb{D}$ onto domains $F$ in $\mathbb{C}$; domains containing $\infty$ are converted to domains in $\mathbb{C}$ by the transformation $w \mapsto 1/(w-a)$ with $a \notin F$.

We have chosen $\mathbb{D} = \{|z| < 1\}$ as our standard domain. Often the upper halfplane $\{\operatorname{Im} \zeta > 0\}$ is used instead of $\mathbb{D}$; they are connected by

$$\zeta = \mathrm{i}\frac{1+z}{1-z}, \qquad z = \frac{\zeta - \mathrm{i}}{\zeta + \mathrm{i}}.$$

The choice of the upper halfplane often simplifies formulas but has the disadvantage of singling out the boundary point $\infty$.

We shall study what happens when $z$ approaches the unit circle $\mathbb{T} = \partial\mathbb{D}$ and thus $f(z)$ approaches the boundary $\partial F$. This knowledge is important in many applications of conformal mappings.

## 1.2. *Three introductory examples*

EXAMPLE 1. Let $F$ be the *rectangle* with vertices $\pm a \pm \mathrm{i}b$ and let $f$ map $\mathbb{D}$ conformally onto $F$ such that $f(0) = 0$ and $f'(0) > 0$. In this special case the function $f$ can be explicitly determined. The Schwarz–Christoffel formula [91, p. 193] yields

$$f(z) = \frac{2b}{K(\sin\alpha)} \int_0^z \frac{d\zeta}{\sqrt{1 - 2\cos(2\alpha)\zeta^2 + \zeta^4}}, \qquad \frac{b}{a} = \frac{K(\sin\alpha)}{K(\cos\alpha)}, \tag{1.2.1}$$

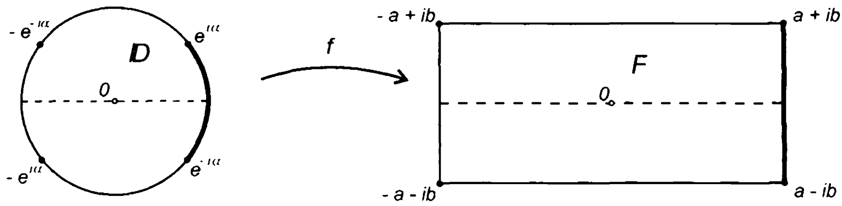


Fig. 1.

where $K$ is the complete elliptic integral of the first kind.

This example is typical for the situation studied in Section 3. The function $f$ is continuous and injective in $\overline{\mathbb{D}}$ and is analytic on $\mathbb{T}$ except at the preimages $\pm e^{\pm i\alpha}$ of the corners.

But the pathologies studied in Section 4 throw their shadow onto this simple case. It follows from (1.2.1) that

$$\alpha \sim 4\exp\left(-\frac{\pi a}{2b}\right) \quad \text{as } \frac{a}{b} \to \infty$$

and is thus very small indeed. For example if $b = a/24$ then $\alpha \approx 4e^{-12\pi}$. Thus the entire right-hand side (two percent of $\partial F$) corresponds to an arc of length $< 4 \times 10^{-16}$. This "crowding effect" presents severe numerical problems.

EXAMPLE 2. The *snowflake curve* is defined by the following infinite construction. We start with an equilateral triangle of side length 1. In the middle of each side we erect an equilateral triangle of side length $3^{-1}$. This gives a polygon with $3 \cdot 4$ sides. After $n$ generations we obtain a polygon with $3 \cdot 4^n$ sides of length $3^{-n}$ and thus of total length $3(4/3)^n$. As $n \to \infty$ the polygon tends to a Jordan curve $J$ of infinite length. It can be shown that $J$ does not have a tangent at any point.

Now let $f$ map $\mathbb{D}$ conformally onto the inner domain of $J$. Then $f$ has a continuous extension to $\overline{\mathbb{D}}$ but the derivative $f'$ cannot be extended to $\mathbb{T}$ in any sense. Moreover there is a partition $\mathbb{T} = A \cup B$ where $A$ has zero measure whereas $f(B)$ has linear measure 0. Thus almost nothing (the set $A$) is mapped onto almost everything and almost everything (the set $B$) is mapped onto almost nothing.

This example is typical for the pathologies that may occur for general domains; see Section 4. Most Julia sets that arise in iteration theory give rise to the same difficulties.

EXAMPLE 3. We consider the "comb domain" $F$ indicated in Figure 3. There is an infinite number of vertical segments that accumulate at $[0, i]$. Now the map $f$ of $\mathbb{D}$ onto $F$ is discontinuous at the point $\zeta_0$ that corresponds to the "prime end" $[0, i]$. Furthermore $f$ is not injective on $\mathbb{T} \setminus \{\zeta_0\}$. For instance each point on $[3, 3+i]$ has "two sides" and thus has two preimages on $\mathbb{T}$. We discuss these problems in Section 2.

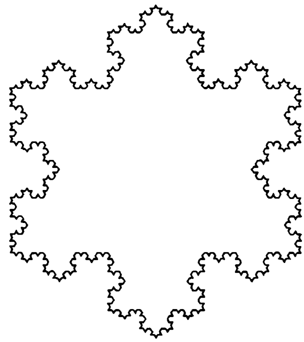

Fig. 2. Two generations in the construction of the snowflake curve.

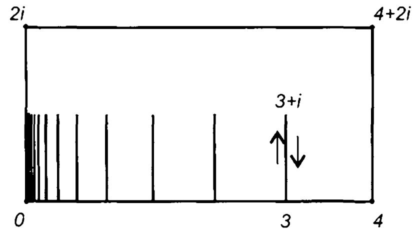


Fig. 3. A domain with a "prime end" at [0, i]. The point 0 is "not accessible from $F$".

Note that this domain is much nicer than the snowflake curve as far as $f'$ is concerned. The derivative exists at all points of $\mathbb{T}$ except at $\zeta_0$ and at the points that correspond to the countably many corners of $\partial F$.

## 2. Continuity at the boundary

We study the limit behaviour of a conformal map at the boundary, in particular the question whether it has a continuous extension. The behaviour of the derivative will be studied in Sections 3 and 4. Many results of the present section have a strong topological flavour.

### 2.1. *Jordan curves and locally connected sets*

The most important result in this area is Carathéodory's theorem [10]. By *Jordan arc* we mean the homeomorphic image of $[0, 1]$. By *Jordan curve* we mean the homeomorphic image of $\mathbb{T}$; it bounds two *Jordan domains*.

THEOREM 2.1. *A conformal map $\varphi$ of a Jordan domain $F$ onto a Jordan domain $G$ can be extended to a homeomorphism of $\overline{F}$ onto $\overline{G}$.*

By the factorization $\varphi = g \circ f^{-1}$ in (1.1.1) it is sufficient to show that a conformal map $f$ from $\mathbb{D}$ onto a Jordan domain $F$ has a continuous and injective extension to $\overline{\mathbb{D}}$. The continuity will follow from Theorem 2.2 (i) and the injectivity from Theorem 2.10 because Jordan curves have no cut points.

A consequence is the purely topological *Schoenflies theorem*: A bijective continuous map of $\mathbb{T}$ onto a Jordan curve in $\mathbb{C}$ can be extended to a homeomorphism of $\mathbb{C}$ onto $\mathbb{C}$.

First we consider continuous extensions which need not be injective. We understand continuity in the spherical metric if $F$ is unbounded.

A compact set $A$ in $\widehat{\mathbb{C}}$ is called *locally connected* if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $a, b \in A$ with $\operatorname{dist}(a, b) < \delta$, we can find a connected compact set $B$ with

$$a, b \in B \subset A, \quad \operatorname{diam} B < \varepsilon.$$

See, e.g., [93, p. 88], [125, p. 20]. The continuous image of a locally connected compact set is again locally connected and compact. A *curve* is, by definition, the continuous image of a segment and is therefore locally connected.

THEOREM 2.2. *Let $f$ map $\mathbb{D}$ conformally onto $F \subset \mathbb{C}$. Then the following four conditions are equivalent*:

(i) *$f$ has a continuous extension to $\overline{\mathbb{D}}$*;
(ii) *$\partial F$ is a curve*;
(iii) *$\partial F$ is locally connected*;
(iv) *$\mathbb{C} \setminus F$ is locally connected.*

The only difficult part is the implication (iv) $\Rightarrow$ (i). We now sketch the proof restricting ourselves to bounded domains.

Let $\zeta \in \mathbb{T}$ and consider the circular arc

$$C(r) = \{z \in \mathbb{D}: |z - \zeta| = r\} \quad (0 < r < 1).$$

By the Schwarz inequality, the length $\ell(r)$ of its image $f(C(r))$ satisfies

$$\ell(r)^2 = \left(\int_{C(r)} |f'(z)||dz|\right)^2 < \pi r \int_{C(r)} |f'(z)|^2 |dz|$$

because $C(r)$ has length $< \pi r$. It follows that

$$\int_0^1 \frac{\ell(r)^2}{\pi r} dr < \iint_{\mathbb{D}} |f'(z)|^2 \, dx\, dy = \text{area } F < \infty.$$

Hence there exist $r_n \to 0$ such that $\ell(r_n) \to 0$ as $n \to \infty$.

Since $\ell(r_n) < \infty$ the curve $f(C(r_n))$ has definite endpoints $w_n$ and $w_n'$ on $\partial F$. By (iv), there are connected compact sets $B_n$ with

$$w_n, w_n' \in B_n \subset \mathbb{C} \setminus F, \quad \operatorname{diam} B_n \to 0 \quad (n \to \infty). \tag{2.1.1}$$

Since $V_n := \{f(z)\colon z \in \mathbb{D},\ |z-\zeta| < r_n\}$ lies in the domain between $f(C(r_n))$ and $B_n$, it follows that

$$\operatorname{diam} V_n \leqslant \max\bigl(\ell(r_n), \operatorname{diam} B_n\bigr) \to 0 \quad (n \to \infty) \tag{2.1.2}$$

which implies that $f(z)$ tends to a limit as $z \to \zeta$ for every $\zeta \in \mathbb{T}$, and it is easy to deduce that (i) holds.

The reasoning that led from (2.1.1) to (2.1.2) was somewhat vague. Statements like this can often be made precise by a useful topological theorem [93, p. 110], [25, p. 362].

JANISZEWSKI'S THEOREM or ALEXANDER'S LEMMA. *Let $A$ and $B$ be closed sets in $\widehat{\mathbb{C}}$ whose intersection $A \cap B$ is connected. If two points are separated neither by $A$ nor by $B$, then they are not separated by the union $A \cup B$.*

We mention some further theorems from plane topology that are often useful in geometric function theory. See [128,85,73], [125, p. 108] for the first result. A compact set $T$ is called totally disconnected if each of its components is a point. This is, e.g., true if $T$ is countable.

PLANE SEPARATION THEOREM. *Let $A \subset \mathbb{C}$ and $B \subset \widehat{\mathbb{C}}$ be compact and let $T = A \cap B$ be totally disconnected. Given $a \in A \setminus T$, $b \in B \setminus T$ and $\varepsilon > 0$ there exists a Jordan curve $J$ with*

$$J \cap (A \cup B) \subset T$$

*that separates $a$ and $b$ and lies in an $\varepsilon$-neighbourhood of $A$.*

See [105, p. 36] for a "colour" version of the next result [86]. A *triod* consists of three Jordan arcs that intersect only at their common junction point.

MOORE TRIOD THEOREM. *Every collection of disjoint plane triods is countable.*

TORHORST THEOREM ([117], [125, p. 106]). *Let $E \subset \widehat{\mathbb{C}}$ be compact and locally connected and let $G$ be a component of $\widehat{\mathbb{C}} \setminus E$. Then $\partial G$ is connected and locally connected.*

## **2.2.** *Prime ends and cluster sets*

Now we turn to general simply connected domains.

THEOREM 2.3. *Let $f$ map $\mathbb{D}$ conformally onto $F$. Then a curve in $F$ ending at a point of $\partial F$ has as preimage a curve in $\mathbb{D}$ ending at a point of $\mathbb{T}$. Moreover curves with distinct endpoints on $\partial F$ have preimages with distinct endpoints on the unit circle $\mathbb{T}$.*

More precisely, we are given a halfopen curve $\Gamma$: $w(t)$, $0 \leqslant t < 1$, with $w(t) \in F$ such that $\lim_{t\to 1} w(t) \in \partial F$ exists. Then $\lim_{t\to 1} f^{-1}(w(t)) \in \mathbb{T}$ also exists. Note however that the image in $F$ of a smooth curve in $\mathbb{D}$ can oscillate wildly.

A *crosscut* $C$ of $F$ is a Jordan arc that lies in $F$ except for its (distinct) endpoints that lie on $\partial F$. A crosscut $C$ divides a simply connected domain into exactly two domains $U$ and $V$ with [105, p. 27]

$$F \cap \partial U = F \cap \partial V = F \cap C.$$

It follows from Theorem 2.3 that its preimage $f^{-1}(C)$ is a crosscut of $\mathbb{D}$.

Let $w_0$ be some fixed point of $F$ and let $(C_n)$ be a sequence of crosscuts of $F$ with $w_0 \notin C_n$. Let $V_n$ be the component of $F \setminus C_n$ that does not contain $w_0$. We say that $(C_n)$ is a *null-chain* if

$$C_n \cap C_{n+1} = \emptyset, \quad V_{n+1} \subset V_n \quad \text{for all } n, \tag{2.2.1}$$

$$\operatorname{diam} C_n \to 0 \quad \text{as } n \to \infty. \tag{2.2.2}$$

If $F$ is unbounded then we have to use the spherical diameter. Note that (2.2.2) does not imply that $\operatorname{diam} V_n \to 0$.

The null-chains $(C_n)$ and $(C'_n)$ are called *equivalent* if, for each $m$,

$$V_n \subset V'_m, \quad V'_n \subset V_m \quad \text{for } n > n_0(m). \tag{2.2.3}$$

The equivalence classes of null-chains are called the *prime ends* of $F$. This concept was introduced by Carathéodory. It is perhaps a complicated notion but it gives a full and completely geometric description of a complicated situation.

The *impression* of the prime end $p$ is defined by

$$I(p) = \bigcap_{n=1}^{\infty} \overline{V}_n. \tag{2.2.4}$$

The impression is a compact connected subset of $\partial F$ and does not depend on the choice of the null-chain representing the prime end $p$.

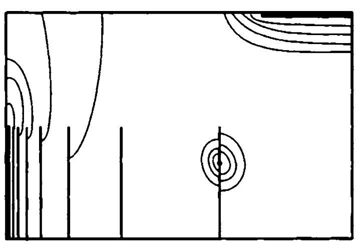

Fig. 4. Three null-chains and a further sequence (at top right) that does not satisfy (2.2.2). The three null-chains are all non-equivalent.

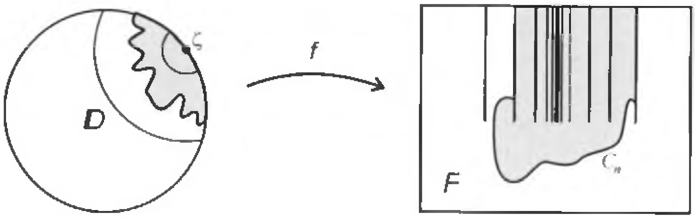


Fig. 5. The inclusion relation (2.2.5) in the Prime End Theorem.

We call $\omega \in I(p)$ a *principal point* of $p$ if there is a null-chain $(C_n)$ representing $p$ such that $C_n \to \{\omega\}$ as $n \to \infty$. The set $\Pi(p)$ of all principal points is a compact connected subset of $I(p)$.

The main result is the *Prime End Theorem* of Carathéodory [11]. See, e.g., [19, p. 172] or [105, p. 31] for a proof.

THEOREM 2.4. *Let $f$ map $\mathbb{D}$ conformally onto $F$. Then there is a bijective correspondence between $\mathbb{T}$ and the set of prime ends of $F$ with the following property:*

*If $\zeta \in \mathbb{T}$ and if $(C_n)$ is a null-chain representing the prime end $p$ corresponding to $\zeta$, then*

$$f^{-1}(C_n) \subset \{z \in \mathbb{D}\colon\ \delta_n < |z-\zeta| < \delta'_n\} \tag{2.2.5}$$

*with* $0 < \delta_n < \delta'_n \to 0$ *as* $n \to \infty$.

The *cluster set* $C(f, \zeta)$ of $f$ at the point $\zeta \in \mathbb{T}$ consists of all $\omega \in \widehat{\mathbb{C}}$ such that there exist sequences $(z_n)$ in $\mathbb{D}$ with

$$z_n \to \zeta, \quad f(z_n) \to \omega \quad \text{as } n \to \infty. \tag{2.2.6}$$

If $E \subset \mathbb{D}$ then the *cluster set* $C_E(f, \zeta)$ *along* $E$ is the set of all $\omega$ for which there are $(z_n)$ in $E$ satisfying (2.2.6). Thus

$$C_E(f,\zeta) = \bigcap_{r>0} \operatorname{clos}\{f(z)\colon\ z \in E,\ |z-\zeta| < r\}. \tag{2.2.7}$$

The following consequence of the Prime End Theorem is due to Carathéodory [11] and Lindelöf [72]. See, e.g., [105, p. 34] for the proof. A *Stolz angle* at $\zeta \in \mathbb{T} = \partial\mathbb{D}$ is the interior of a triangle $\Delta$ with a vertex at $\zeta$ and $\overline{\Delta} \subset \mathbb{D} \cup \{\zeta\}$.

THEOREM 2.5. *Let $f$ map $\mathbb{D}$ conformally onto $F$ and let $p$ be the prime end corresponding to $\zeta \in \mathbb{T}$. The impression satisfies*

$$I(p) = C(f, \zeta) \tag{2.2.8}$$

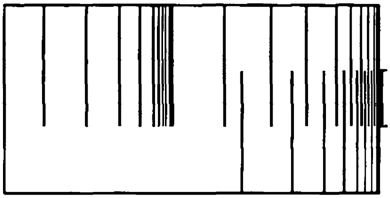

Fig. 6. Two unsymmetric prime ends; the left prime end has only one principal point, the other a whole segment of them.

*while the set of principal points satisfies*

$$\Pi(p) = \bigcap_{\Gamma} C_{\Gamma}(f,\zeta) = C_{[0,\zeta)}(f,\zeta) = C_{\Delta}(f,\zeta), \tag{2.2.9}$$

*where $\Gamma$ runs through all curves in $\mathbb{D}$ ending at $\zeta$ and where $\Delta$ is any Stolz angle at $\zeta$.*

The *Collingwood Category Theorem* [17], [19, p. 76] implies that $I(p) = \Pi(p)$ holds generically in topological terms:

THEOREM 2.6. *If $f : \mathbb{D} \to \widehat{\mathbb{C}}$ is continuous then*

$$C(f,\zeta) = C_{[0,\zeta)}(f,\zeta) \quad \textit{for } \zeta \in \mathbb{T} \setminus B,$$

*where $B$ is of first Baire category.*

The *one-sided cluster sets* $C^{\pm}(f,\zeta)$ consist of all $\omega \in \widehat{\mathbb{C}}$ for which there exist $(z_n)$ with

$$\arg z_n \gtrless \arg \zeta, \quad z_n \to \zeta, \quad f(z_n) \to \omega \quad \text{as } n \to \infty. \tag{2.2.10}$$

The *Collingwood Symmetry Theorem* [18], [19, p. 82], [105, p. 38] states:

THEOREM 2.7. *If $f : \mathbb{D} \to \widehat{\mathbb{C}}$ is any function then*

$$C(f,\zeta) = C^{+}(f,\zeta) = C^{-}(f,\zeta) \tag{2.2.11}$$

*except possibly for countably many $\zeta \in \mathbb{T}$.*

The prime end corresponding to $\zeta$ is called *symmetric* if (2.2.11) holds. Thus all but countably many prime ends are symmetric. The prime end in Figure 5 is symmetric.

The continuun $\partial F$ is indecomposable [87, p. 58] if and only if $C^{+}(f,\zeta)$ or $C^{-}(f,\zeta) = \partial F$ for some $\zeta$ [51]. There exists $F$ such that $C(f,\zeta) = \partial F$ for all $\zeta$ [51, Theorem 4], [71]. See, e.g., [81,100,19,94,112] for more information about prime ends and their classification.

## 2.3. *Limits and injectivity*

We say that the function $g : \mathbb{D} \to \widehat{\mathbb{C}}$ has the *angular limit* $\omega \in \widehat{\mathbb{C}}$ at $\zeta$ if

$$g(z) \to \omega \quad \text{as } z \to \zeta,\ z \in \Delta, \tag{2.3.1}$$

for every Stolz angle at $\zeta$. The angular limit (if it exists) will often be denoted by $g(\zeta)$. We say that $g$ is *continuous* at $\zeta$ if

$$g(z) \to g(\zeta) \quad \text{as } z \to \zeta,\ z \in \mathbb{D}, \tag{2.3.2}$$

so that there is an unrestricted limit. When we write $\omega = g(\zeta)$ we will only mean that $g$ has the *angular* limit $\omega$ at $\zeta$ and not necessarily an unrestricted limit.

The next result is an immediate consequence of Theorem 2.5.

THEOREM 2.8. *Let $\omega \in \widehat{\mathbb{C}}$. Let $f$ map $\mathbb{D}$ conformally onto $F$ and let $p$ be the prime end corresponding to $\zeta \in \mathbb{T}$. Then the following four conditions are equivalent:*

(i) *$f$ has the angular limit $\omega$ at $\zeta$;*
(ii) *$f$ has the radial limit $\omega$ at $\zeta$;*
(iii) *the point $\omega$ is accessible from $F$, i.e., there exists a curve $\Gamma \subset \mathbb{D}$ ending at $\zeta$ such that $f(z) \to \omega$ as $z \to \zeta$, $z \in \Gamma$;*
(iv) *the principal cluster set satisfies $\Pi(p) = \{\omega\}$.*

*Furthermore $f$ is continuous at $\zeta$ if and only if the impression of $p$ satisfies $I(p) = \{\omega\}$.*

Let $\operatorname{cap} E$ denote the logarithmic capacity of $E$; see, e.g., [63,47]. Capacity zero implies measure zero, even Hausdorff dimension zero. The next result [6,24,114,102] therefore shows that the angular limit exists and is finite almost everywhere on $\mathbb{T}$.

THEOREM 2.9. *Let $f$ map $\mathbb{D}$ conformally onto $F$. Then the angular limit $f(\zeta)$ exists for $\zeta \in \mathbb{T} \setminus E$ where $\operatorname{cap} E = 0$. Furthermore*

$$\operatorname{cap} f(A) \geqslant c(\operatorname{cap} A)^2 \quad \text{for } A \subset \mathbb{T} \setminus E, \tag{2.3.3}$$

*where $c = c(f)$ is a positive constant.*

See, e.g., [105, p. 215], [103, p. 344] for the proof of this result. Even more is true, namely that

$$\int_0^1 |f'(r\zeta)|\, dr < \infty \quad \text{for } \zeta \in \mathbb{T} \setminus E. \tag{2.3.4}$$

It follows from (2.3.3) that

$$\operatorname{cap}\{\zeta \in \mathbb{T} \setminus E:\ f(\zeta) = \omega\} = 0 \quad \text{for } \omega \in \partial F. \tag{2.3.5}$$

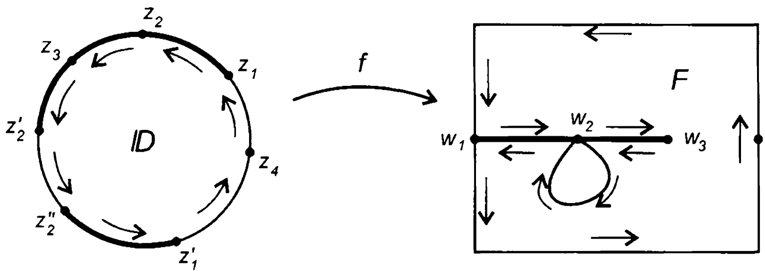


Fig. 7. In this example for boundary correspondence, the point $w_1$ has two preimages while $w_2$ has even three.

The limit in (2.3.1) even exists almost everywhere for rather tangential approach [88,119], but need not exist for unrestricted approach.

Now we turn to injectivity. Let $E$ be a connected compact set. We call $w \in E$ a *cut point* of $E$ if $E \setminus \{w\}$ is not connected. For instance every point of a Jordan arc except for the endpoints is a cut point whereas a (closed) Jordan curve has no cut points. See [87, p. 203], [125, p. 41] for a discussion of cut points.

THEOREM 2.10. *Let $f$ map $\mathbb{D}$ conformally onto $F$ and let $\omega \in \partial F$. Then the following three conditions are equivalent*:

(i) *$\omega$ is a cut point of $\partial F$*;

(ii) *there exists a Jordan curve $J$ with $J \cap \partial F = \{\omega\}$ whose inner and outer domains both contain points of $\partial F$*;

(iii) *there exist distinct $\zeta, \zeta' \in \mathbb{T}$ with angular limits*

$$f(\zeta) = f(\zeta') = \omega. \tag{2.3.6}$$

*Furthermore there is no third point $\zeta''$ with $f(\zeta'') = \omega$ except possibly for countably many cut points $\omega$.*

PROOF. The equivalence (i) $\Leftrightarrow$ (ii) follows from the Plane Separation Theorem; see Section 2.1. The final statement is a consequence of the Moore Triod Theorem formulated in Section 2.1.

(ii) $\Rightarrow$ (iii): It follows from Theorem 2.3 that $f^{-1}(J)$ is either a Jordan arc with distinct endpoints $\zeta, \zeta' \in \mathbb{T}$, or a Jordan curve $C$ with a point on $\mathbb{T}$. The second case is not possible because then [67] the normal function $f(z)$ would tend to $w$ as $z \to \zeta$ inside $C$ so that $J$ would not separate $\partial F \setminus \{\omega\}$.

(iii) $\Rightarrow$ (ii): By Theorem 2.9 there exist $\zeta_j \in \mathbb{T}$ $(j = 1, 2)$ on the two arcs of $\mathbb{T} \setminus \{\zeta, \zeta'\}$ such that $f(\zeta_j) \neq \omega$ exist. Then the Jordan curve

$$J = f([0, \zeta)) \cup f([0, \zeta')) \cup \{\omega\} \subset F \cup \{\omega\}$$

separates the points $f(\zeta_1)$ and $f(\zeta_2)$ of $\partial F$. □

THEOREM 2.11. *Let $f$ map $\mathbb{D}$ conformally onto $F$. If the radial limit $f(\zeta)$ is a cut point of $\partial F$ then $f$ is continuous at $\zeta$ except possibly for countably many $\zeta \in \mathbb{T}$.*

PROOF. Let (2.3.6) hold. By the Collingwood Symmetry Theorem, we may assume that also (2.2.11) holds. Let $\Gamma = [0, \zeta) \cup [0, \zeta')$ and let $G^{\pm}$ be the two components of $\mathbb{D} \setminus \Gamma$, furthermore let $H^{\pm}$ be the two complementary domains of the Jordan curve $f(\Gamma) \cup \{\omega\}$. Then $f(G^{\pm}) \subset H^{\pm}$ with a suitable choice of notation. Hence, by (2.2.11) and (2.2.10),

$$C(f, \zeta) = C^{+}(f, \zeta) \cap C^{-}(f, \zeta) \subset \overline{H}^{+} \cap \overline{H}^{-} \cap \partial F = \{\omega\}. \qquad \square$$

Finally we consider the continuous extension of the inverse function $f^{-1}$ to the boundary in the topology of $\widehat{\mathbb{C}}$.

THEOREM 2.12. *A conformal map of $F$ onto $\mathbb{D}$ has a continuous extension to $\overline{F}$ if and only if*

$$I(p) \cap I(p') = \emptyset \quad \text{for all prime ends } p \neq p' \text{ of } F. \tag{2.3.7}$$

If (2.3.7) holds then we can simply extend $f^{-1}$ to $\partial F$ by defining $f^{-1}(w) = \zeta$ for $w \in I(p)$ where $p$ corresponds to the point $\zeta \in \mathbb{T}$.

There exists [20,34] a domain $F$ such that $f^{-1}$ has a continuous extension to $\partial F$ whereas $f$ is discontinuous at every point of $\mathbb{T}$.

## 3. Domains with nice boundaries

Most domains that arise in the applications of conformal mapping in science are bounded by a finite number of smooth arcs. We shall see that the smoothness properties of the arcs are (with some exceptions) inherited by the function. The behaviour at the corners however can be more complicated. See, e.g., [91,60] for explicit maps.

### 3.1. *Free boundary arcs*

The open Jordan arc $B \subset \partial F$ is called a *free boundary arc* of the domain $F$ if there is a Jordan arc $C$ such that $B \cup C$ is a (closed) Jordan curve whose inner domain lies in $F$.

THEOREM 3.1. *Let $f$ map $\mathbb{D}$ conformally onto $F$. If $B$ is a free boundary arc of $F$ then there is an open arc $A \subset \mathbb{T} = \partial\mathbb{D}$ such that $f$ has a continuous injective extension to $\mathbb{D} \cup A$ and $f(A) = B$.*

The simplest case is that $B$ lies on a line or circle; see, e.g., [21]. Then the reflection principle shows that $f$ has an analytic extension to $(\mathbb{C} \setminus \mathbb{T}) \cup A$. The image domain is $F \cup B \cup F^*$ where $F^*$ is obtained by reflecting $F$ upon $B$; there is a pole if $\infty \in F^*$. The extended function is injective if and only if $F \cap F^* = \emptyset$. In any case $f'$ has no zeros.

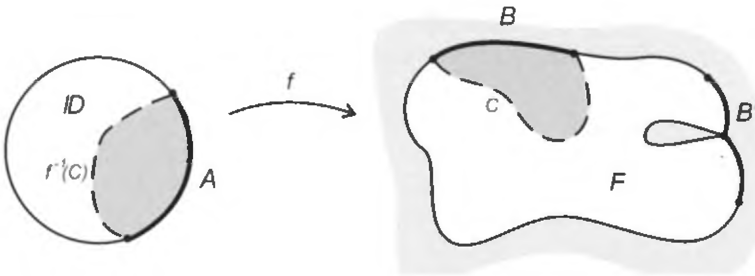


Fig. 8. $B$ is a free boundary arc but $B'$ is not. Theorem 3.1 can be proved by applying Theorem 2.1 to the shaded domains.

An open *analytic arc* is a Jordan arc $B$: $g(t)$, $\alpha < t < \beta$, where $g$ can be expanded in a Taylor series around each point $t \in (\alpha, \beta)$. It is easy to see that $g$ is conformal in some domain $V$ with $(\alpha, \beta) \subset V$ that is symmetric with respect to $\mathbb{R}$. A closed analytic arc is defined similarly replacing $(\alpha, \beta)$ by $[\alpha, \beta]$.

THEOREM 3.2. *Let $f$ map $\mathbb{D}$ conformally onto $F$. If $B$ is an analytic free boundary arc of $F$, then $f$ has an analytic continuation to some domain containing $\mathbb{D} \cup f^{-1}(B)$.*

PROOF. Since $B$ is a free boundary arc of $F$ we see from Theorem 3.1 that $f$ maps some Jordan domain $H \subset \mathbb{D}$ with $A \subset \partial H$ into $F$ such that $f(A) = B$. Let $G$ be the component of $f(H) \cap g(V)$ with $B \subset \partial G$. Then $g^{-1} \circ f$ maps $f^{-1}(G)$ conformally onto $g^{-1}(G)$ such that $A$ goes to $(\alpha, \beta)$, and the reflection principle shows that $g^{-1} \circ f$ has an analytic continuation $h$ to a domain $U$ with $A \cup f^{-1}(G) \subset U$. Thus $f$ has the analytic continuation $g \circ h$ to $U$. □

### 3.2. *Smooth boundary arcs*

Throughout this subsection we will assume that $f$ maps $\mathbb{D}$ conformally onto $F$ and that $B$ is an open free boundary arc of $F$. We see from Theorem 3.1 that $f$ is continuous in $\mathbb{D} \cup A$ where $A$ is an open arc of $\mathbb{T}$ and $f(A) = B$. The simplest case is that $F$ is a Jordan domain and $B = \partial F$ and thus $A = \mathbb{T}$.

The Jordan arc $B$ is called *smooth* (or of class $C^1$) if it has a parametric representation

$$B\colon\ h(t),\ \alpha < t < \beta,\ \text{with } h'(t) \neq 0, \tag{3.2.1}$$

where $h$ is continuously differentiable. This is true if and only if $B$ has a continuously turning tangent.

THEOREM 3.3. *The free boundary arc $B$ of $F$ is smooth if and only if* $\arg f'$ *has a continuous extension to $\mathbb{D} \cup A$ where again $A = f^{-1}(B)$.*

This theorem of Lindelöf [72] is proved, e.g., in [105, p. 44] for the case that $B = \partial F$. We show now how to obtain the continuous extension of $\arg f'$ for the case that $B \neq \partial F$.

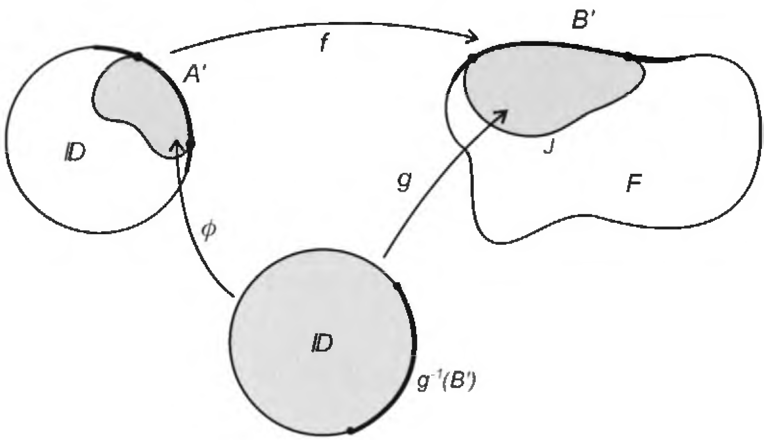


Fig. 9. How to reduce the "local" case $B \neq \partial F$ to the "global" case $B = \partial F$.

Let $B'$ be any closed subarc of $B$ and let $J$ be a smooth Jordan curve with $B' \subset J$ and $J \setminus B' \subset F$; smoothness is achieved by making $J$ tangential to $B$ at the endpoints of $B'$. Let $g$ map $\mathbb{D}$ conformally onto the inner domain of $J$. Then we know that $\arg g'$ is continuous on $\mathbb{T}$. Furthermore $\varphi = g^{-1} \circ f$ maps the arc $A' = f^{-1}(B') \subset A$ onto $g^{-1}(B') \subset \mathbb{T}$. Hence $\varphi$ is analytic on $A'$ by the reflection principle and $\varphi'(z) \neq 0$. Therefore

$$\arg f'(z) = \arg g'\big(\varphi(z)\big) + \arg \varphi'(z)$$

has a continuous extension to every $A'$ and thus to $A$.

Unfortunately it is not true that $|f'|$ always has a continuous extension. For instance $f_1(z) = 2z + (1-z)\log(1-z)$ maps $\mathbb{D}$ onto a domain bounded by a smooth Jordan curve but $|f_1'(x)| \to \infty$ as $x \to 1$.

The arc $B$ represented by (3.2.1) is called *Dini-smooth* if

$$\big|h'(t_1) - h'(t_2)\big| \leqslant \omega(\delta) \quad \text{for } |t_1 - t_2| \leqslant \delta, \tag{3.2.2}$$

where $\omega$ increases and

$$\int_0^{\pi} \frac{\omega(x)}{x}\, dx < \infty. \tag{3.2.3}$$

THEOREM 3.4. *If $B$ is a free Dini-smooth boundary arc then $f'$ has a continuous extension to $\mathbb{T} \cup A$ with $f'(z) \neq 0$.*

See, e.g., [123] for a proof of this theorem [59,121]. If $B = \partial F$ then

$$\big|f'(z_1) - f'(z_2)\big| \leqslant c_1 \int_0^{\delta} \frac{\omega(x)}{x}\, dx + \delta c_1 \int_{\delta}^{\pi} \frac{\omega(x)}{x^2}\, dx \tag{3.2.4}$$

for $z_1, z_2 \in \overline{\mathbb{D}}$, $|z_1 - z_2| \leqslant \delta$, where $c_1$ is a constant; see, e.g., [105, p. 48].

We say that $B$ has *Dini-continuous curvature* if the tangent angle $\vartheta(t) = \arg h'(t)$ satisfies a condition of the type (3.2.2) and (3.2.3). Then [121] the second derivative has a continuous extension to $\mathbb{T} \cup A$.

Let $n = 1, 2, \ldots$ and $0 < \alpha < 1$. We say that the smooth arc $B$ is of *class* $C^{n,\alpha}$ if the $n$-th derivative $h^{(n)}$ exists and satisfies the Hölder condition

$$h^{(n)}(t_1) - h^{(n)}(t_2) = \mathrm{O}\big(|t_1 - t_2|^{\alpha}\big). \tag{3.2.5}$$

The smooth arc $B$ is of class $C^{\infty}$ if all derivatives exist. If $B$ is of class $C^{1,\alpha}$ for some $\alpha > 0$ then $B$ is Dini-smooth.

THEOREM 3.5. *If the free boundary arc $B$ is of class $C^{n,\alpha}$ then $f'(z) \neq 0$ for $z \in \mathbb{T} \cup A$, and $f^{(n)}$ has an extension to $\mathbb{D} \cup A$ satisfying*

$$f^{(n)}(z_1) - f^{(n)}(z_2) = \mathrm{O}\big(|z_1 - z_2|^{\alpha}\big) \quad \textit{for } z_1, z_2 \in S, \tag{3.2.6}$$

*where $S = \{r\zeta\colon\ 0 \leqslant r \leqslant 1,\ \zeta \in A'\}$ and $A'$ is any closed subarc of $A$.*

This is the *Kellogg–Warschawski Theorem* [121]. It implies that all derivatives of $f$ are continuous in $\mathbb{T} \cup A$ if $B$ is of class $C^{\infty}$. See [122] for further results and [105, p. 49] for a proof in the case $B = \partial F$.

### 3.3. *Corners*

Throughout this subsection we assume that $f$ maps $\mathbb{D}$ conformally onto the domain $F$ with locally connected boundary. Thus $f$ has a continuous extension to $\overline{\mathbb{D}}$ by Theorem 2.2.

Let $\zeta = \mathrm{e}^{\mathrm{i}\vartheta} \in \mathbb{T}$. We say that $F$ has a *corner* at $\omega = f(\zeta) \neq \infty$ if

$$\arg\big[f(\mathrm{e}^{\mathrm{i}t}) - \omega\big] \to \beta^{\pm} \quad \text{as } t \to \vartheta\pm. \tag{3.3.1}$$

The *interior angle* $\pi\alpha = \beta^- - \beta^+$ satisfies $0 \leqslant \alpha \leqslant 2$. If $\alpha = 0$ we have an outward pointing cusp; if $\alpha = 2$ we have an inward pointing cusp. We have used the conformal representation $f(\mathrm{e}^{\mathrm{i}t}),\ 0 \leqslant t \leqslant 2\pi$, of the boundary. The definition remains unchanged if we use a different parametric representation. Note that "corner at $\omega$" is not quite precise because there may be different $\zeta \in \mathbb{T}$ with $f(\zeta) = \omega$.

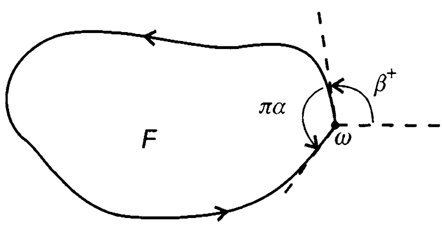


Fig. 10. The definition of corner.

THEOREM 3.6. *Let $\zeta = e^{i\vartheta}$. The domain $F$ has a corner of interior angle $\pi\alpha$ $(0 \leqslant \alpha \leqslant 2)$ at $f(\zeta) \neq \infty$ if and only if*

$$\arg \frac{f(z) - f(\zeta)}{(z-\zeta)^{\alpha}} \to \beta^{+} - \alpha\left(\vartheta + \frac{\pi}{2}\right) \quad \text{as } z \to \zeta,\ z \in \overline{\mathbb{D}}. \tag{3.3.2}$$

Here $\vartheta + \pi/2 < \arg(z-\zeta) < \vartheta + 3\pi/2$. See, e.g., [105, p. 51] for a proof of this theorem of Lindelöf. More is true for *Dini-smooth* corners [121], [105, p. 52]. In the excluded case $\alpha = 0$ of an outward pointing cusp, the behaviour depends on the order of tangency.

THEOREM 3.7. *Suppose that $f$ maps two closed arcs of $\mathbb{T}$ meeting at $\zeta$ onto Dini-smooth arcs. If $F$ has a corner of interior angle $\pi\alpha$ $(0 < \alpha \leqslant 2)$ at $f(\zeta) \neq \infty$ then the functions*

$$\frac{f(z) - f(\zeta)}{(z-\zeta)^{\alpha}} \quad \text{and} \quad \frac{f'(z)}{(z-\zeta)^{\alpha-1}} \tag{3.3.3}$$

*are continuous and $\neq 0, \infty$ near $\zeta$ in $\overline{\mathbb{D}}$.*

In practice the most important case is the *analytic corner*. This is completely described by the *Lewy–Lehman Theorem* [65].

THEOREM 3.8. *Suppose that $f$ maps two closed arcs of $\mathbb{T}$ meeting at $\zeta$ onto analytic arcs $C^{\pm}$ and let $F$ have a corner of interior angle $\pi\alpha$ $(0 < \alpha \leqslant 2)$ at $\omega = f(\zeta) \neq \infty$. If $\alpha$ is irrational then, as $z \to \zeta$,*

$$f(z) \sim \omega + \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} a_{kj}(z-\zeta)^{k+\alpha j}, \quad a_{01} \neq 0. \tag{3.3.4}$$

*If $\alpha = p/q$ with relatively prime $p$ and $q$ then*

$$f(z) \sim \omega + \sum_{k=0}^{\infty} \sum_{j=1}^{q} \sum_{m=0}^{[k/p]} a_{kjm}(z-\zeta)^{k+pj/q}\big(\log(z-\zeta)\big)^{m}, \quad a_{010} \neq 0. \tag{3.3.5}$$

These are asymptotic expansions in the sense that, if we truncate the expansion to a finite sum, the order of magnitude of the remainder is comparable to that of the neglected term of least order. We now discuss the first terms in more detail [105, p. 57]. Let $\kappa^{\pm}$ denote the curvature of the arcs $C^{\pm}$ at $\omega$.

(a) Let $0 < \alpha < 1$. Then (3.3.4) or (3.3.5) give

$$f(z) = \omega + e^{i\beta^{+}}\left(bs^{\alpha} - b^{2}\frac{\kappa^{-} + \kappa^{+}e^{-i\pi\alpha}}{2\sin\pi\alpha}s^{2\alpha}\right) + O\big(s^{1+\alpha}\big), \tag{3.3.6}$$

where $z = \zeta(1 + is)$, $s \to 0$, $\operatorname{Im} s > 0$. The quantity $b > 0$ is not determined by the geometry at $\omega$.

(b) Let $\alpha = 1$ so that the analytic arcs meet tangentially at $\omega$. Then (3.3.5) gives

$$f(z) = \omega + e^{i\beta^+}\left(bs + b^2\frac{\kappa^- - \kappa^+}{2\pi}s^2 \log\frac{1}{|s|}\right) + O(s^2), \tag{3.3.7}$$

where $z = \zeta(1 + is)$, $s \to 0$, $\operatorname{Im} s > 0$. Surprisingly there is always a logarithmic term except if the curvatures are the same.

(c) If $1 < \alpha \leqslant 2$ then the term with $s^{1+\alpha}$ dominates the term with $s^{2\alpha}$ stated in (3.3.6).

The situation is however more favourable if the corner at $\omega$ is formed by two *circular arcs* and if $\alpha \neq 0, 1, 2$. Then the corresponding circles intersect also at some point $\omega^* \neq \omega$ and

$$w \mapsto \left[\frac{w - \omega}{w - \omega^*}\right]^{1/\alpha}$$

transforms $\partial F$ near $\omega$ into a straight line. The reflection principle now shows that

$$f(z) = \omega + (z - \zeta)^\alpha g(z), \tag{3.3.8}$$

where $g$ is analytic and $\neq 0$ near $\zeta$. See [91, p. 198], [60, p. 114] for the determination of the conformal map onto domains bounded by circular arcs.

### 3.4. *Integral representations*

Let $f$ map $\mathbb{D}$ conformally onto the bounded domain $F$ with locally connected boundary. The domain $F$ is called *regulated* [95], [105, p. 59] if the three limits

$$\beta(t) = \lim_{\tau \to t+} \arg\left[f\left(e^{i\tau}\right) - f\left(e^{it}\right)\right], \tag{3.4.1}$$

$$\beta(t\pm) = \lim_{\tau \to t\pm} \beta(\tau) \tag{3.4.2}$$

exist for all $t$. Thus $\partial F$ has a forward tangent everywhere (possibly several at multiple points) and its angle $\beta(t)$ is a "regulated function".

Now let $F$ be regulated. Then $\beta(t+) = \beta(t)$, and $\beta(t-)$ is the angle of the backward tangent. Furthermore $\beta(t-) = \beta(t)$ with at most countably many exceptions. Hence a regulated domain has at most countably many corners which have the interior angles $\pi - \beta(t+) + \beta(t-)$. For $z \in \mathbb{D}$ we have the integral representation

$$\log f'(z) = \log\left|f'(0)\right| + \frac{i}{2\pi}\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z}\left(\beta(t) - t - \frac{\pi}{2}\right) dt. \tag{3.4.3}$$

A stronger condition is that $F$ has *bounded boundary rotation* [98], [27, p. 269], that is

$$\int_0^{2\pi} \left|d\beta(t)\right| < \infty, \tag{3.4.4}$$

so that $\beta$ is of bounded variation. Then we can integrate (3.4.3) by parts and obtain

$$\log f'(z) = \log f'(0) - \frac{1}{\pi}\int_0^{2\pi} \log\left(1 - e^{-it}z\right) d\beta(t). \tag{3.4.5}$$

The domain $F$ is convex if and only if $\beta(t)$ is increasing and thus $d\beta(t) \geqslant 0$.

Now we suppose that $F$ has corners of interior angles $\pi\alpha_\nu$ at $f(\zeta_\nu)$, $\zeta_\nu = e^{it_\nu}$ for $\nu = 1, \dots, n$ and that $\partial F$ has continuous curvature except at these corners. Then

$$\beta'(t) = 1 + \operatorname{Re}\left[z\frac{f''(z)}{f'(z)}\right] \quad \text{for } z = e^{it} \neq \zeta_\nu; \tag{3.4.6}$$

the curvature is $= \beta'(t)/|f'(z)|$. It follows from (3.4.5) that

$$\begin{aligned}\log f'(z) = \log f'(0) &- \sum_{\nu=1}^{n} \alpha_\nu \log\left(1 - \bar{\zeta}_\nu z\right) \\ &- \frac{1}{\pi}\int_0^{2\pi} \log\left(1 - e^{-it}z\right)\beta'(t)\,dt \end{aligned} \tag{3.4.7}$$

for $z \in \overline{\mathbb{D}}\setminus\{\zeta_1, \dots, \zeta_n\}$. If $F$ is a polygon then $\beta'(t) = 0$ and (3.4.7) reduces to the *Schwarz–Christoffel formula* [91, p. 93]

$$f'(z) = f'(0)\prod_{\nu=1}^{n}\left(1 - \bar{\zeta}_\nu z\right)^{-\alpha_\nu}. \tag{3.4.8}$$

The integral representation (3.4.7) looks explicit: The size $\pi\alpha_\nu$ of the angles and the tangent angles are geometric quantities. But the difficulty about the angles is that $\zeta_\nu \in \mathbb{T}$ is unknown, and the difficulty about $\beta(t)$ is that the definition (3.4.1) refers to the conformal representation

$$\partial F\colon\ f\left(e^{it}\right), \quad 0 \leqslant t \leqslant 2\pi.$$

See, e.g., [35,50] about numerical methods for the determination of the conformal maps.

## 4. General boundaries

Now we turn to simply connected domains $F$ with arbitrary boundaries. We study the derivative $f'$ of a conformal map of $\mathbb{D}$ onto $F$. Its argument is related to tangents and its absolute value to distortion and linear measure.

### 4.1. *Distortion near the boundary*

Let $f$ map $\mathbb{D}$ conformal onto $F \subset \mathbb{C}$. We define

$$d_f(z) = \operatorname{dist}\bigl(f(z), \partial F\bigr) \quad \text{for } z \in \mathbb{D}. \tag{4.1.1}$$

The most basic result is the invariant form of the Koebe One-Quarter Theorem,

$$\frac{1}{4}\bigl(1-|z|^2\bigr)\bigl|f'(z)\bigr| \leqslant d_f(z) \leqslant \bigl(1-|z|^2\bigr)\bigl|f'(z)\bigr| \quad \text{for } z \in \mathbb{D}, \tag{4.1.2}$$

which connects $|f'|$ to geometry. We can write (4.1.2) as

$$\frac{|dz|}{1-|z|^2} \leqslant \frac{|dw|}{\operatorname{dist}(w, \partial F)} \leqslant \frac{4|dz|}{1-|z|^2} \quad \text{with } w = f(z) \tag{4.1.3}$$

which establishes a strong relation between the analytically defined *hyperbolic metric* (on the left) and the geometrically defined *quasi-hyperbolic metric* (in the middle).

The *hyperbolic* (non-Euclidean) *segment* with endpoints $z, z' \in \overline{\mathbb{D}}$ is the arc between $z$ and $z'$ of the circle orthogonal to $\mathbb{T}$. Let $\Lambda(L)$ denote the Euclidean length of the curve $L \subset \mathbb{C}$ and let $c_1, c_2, \ldots$ denote suitable absolute constants.

THEOREM 4.1. *Let $f$ map $\mathbb{D}$ conformally onto $F$. If $S$ is a hyperbolic segment in $\overline{\mathbb{D}}$ and $\Gamma$ is a curve in $\overline{\mathbb{D}}$ with the same endpoints as $S$, then*

$$\Lambda\bigl(f(S)\bigr) \leqslant c_1 \Lambda\bigl(f(\Gamma)\bigr), \quad \operatorname{diam} f(S) \leqslant c_2 \operatorname{diam} f(\Gamma). \tag{4.1.4}$$

This is the *Gehring–Hayman Theorem* [40], [105, p. 88]. Since $f(S)$ is a geodesic of the hyperbolic metric of $F$, we can express this theorem as follows: Among all curves connecting two given points in a simply connected domain, the hyperbolic geodesic has also minimal Euclidean length up to a constant. This theorem and many related estimates carry over to the universal covering map of $\mathbb{D}$ onto multiply connected domains, provided that $\partial F$ is *uniformly perfect*, i.e., (4.1.3) holds with 4 replaced by some finite constant. See, e.g., [4,104,42,49,106].

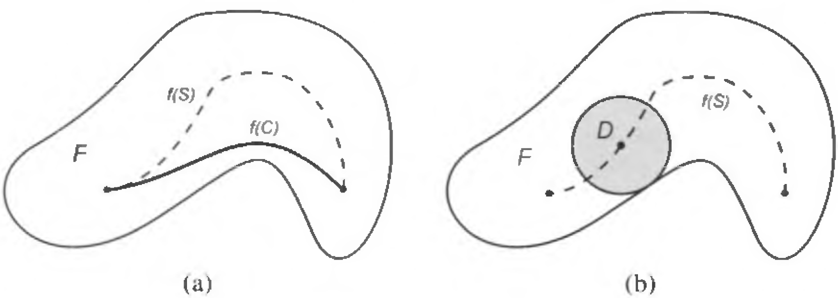


Fig. 11. The theorems of Gehring–Hayman and Jørgensen.

THEOREM 4.2. *Let $f$ map $\mathbb{D}$ conformally onto $F$. If $S$ is a hyperbolic segment in $\overline{\mathbb{D}}$ and if $D$ is a disk in $F$, then $D \cap f(S)$ is connected.*

This theorem of Jørgensen [58] carries over to hyperbolic geodesics in multiply connected domains. See [83,84] for generalizations and the connection with curvature.

THEOREM 4.3. *Let $f$ map $\mathbb{D}$ conformally onto $F$. If $L$ is any line or circle then*

$$\Lambda\big(f^{-1}(L \cap F)\big) \leqslant 4\pi.$$

This result is due to Hayman and Wu [48]; see [97] for the constant $4\pi$. It has been generalized to more general curves; see, e.g., [30,8] and Theorem 5.5.

The final result [105, p. 216], [3] shows that (with respect to "harmonic measure") most points on $\mathbb{T}$ near $z \in \mathbb{D}$ have images reasonably close to $f(z)$; see Figure 12.

THEOREM 4.4. *Let $f$ map $\mathbb{D}$ conformally onto $F$. If $z \in \mathbb{D}$ and $a \leqslant 2$, $b \leqslant 4$, then*

$$\big|f(z) - f(\zeta)\big| \leqslant b d_f(z) \leqslant b\big(1 - |z|^2\big)\big|f'(z)\big| \tag{4.1.5}$$

*for all $\zeta \in \mathbb{T} \setminus E$ with $|\zeta - z| \leqslant a(1 - |z|)$, where the exceptional set $E = E(z)$ satisfies*

$$\frac{\Lambda(E)}{2\pi} \leqslant \operatorname{cap} E \leqslant \frac{2(1 + a^2)(1 - |z|)}{\sqrt{b}}, \tag{4.1.6}$$

*and if $S$ is the hyperbolic segment from $z$ to $\zeta$ then*

$$\Lambda\big(f(S)\big) \leqslant c_3 b d_f(z). \tag{4.1.7}$$

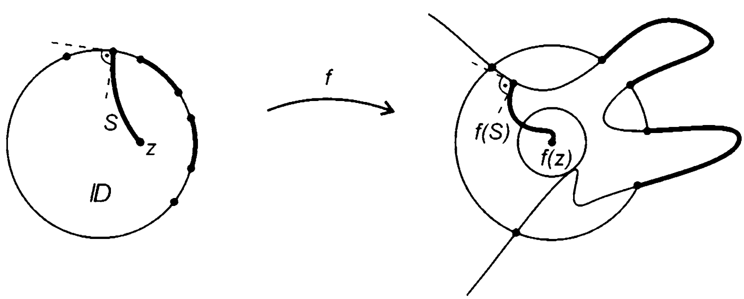


Fig. 12. The situation of Theorem 4.4. The heavily drawn arcs come from the exceptional set. The two circles around $f(z)$ have radii $d_f(z)$ and $bd_f(z)$.

**4.2.** *The angular derivative*

We say that $F$ has a *tangent* at the finite point $\omega = f(\zeta)$ of $\partial F$ if

$$\arg[f(e^{it}) - \omega] \to \begin{cases} \beta & \text{as } t \to \vartheta+, \\ \beta + \pi & \text{as } t \to \vartheta-; \end{cases} \tag{4.2.1}$$

compare (3.3.1). It follows from Theorem 3.6 that $F$ has a tangent at $\omega$ if and only if

$$\lim_{z \to \zeta,\ z \in \mathbb{D}} \arg \frac{f(z) - \omega}{z - \zeta} \tag{4.2.2}$$

exists and is finite.

A weaker condition is that the function $f$ is *isogonal* at $\zeta \in \mathbb{T}$. This means that $f(z)$ and $\arg[(f(z) - f(\zeta))/(z - \zeta)]$ have finite angular limits, that is, limits as $z \to \zeta$, $z \in \Delta$, in every Stolz angle $\Delta$. If $F$ is a John domain (Section 5.2) then isogonality implies the existence of a tangent [105, p. 101].

If $f$ is isogonal at $\zeta$ then every smooth arc ending at $\zeta$ non-tangentially to $\mathbb{T}$ is mapped into a smooth arc ending at $f(\zeta)$, and the angles between such arcs are preserved. Ostrowski [96,52], [105, p. 254] has given a geometric characterization:

THEOREM 4.5. *Let $f$ map $\mathbb{D}$ conformally onto $F$ and let the angular limit $\omega = f(\zeta) \neq \infty$ exist. Then $f$ is isogonal at $\zeta$ if and only if, for some $\gamma$,*

$$f(r\zeta) \in S(\varepsilon) = \left\{ w\colon \left|\arg(w - \omega) - \gamma\right| < \frac{\pi}{2} - \varepsilon,\ |w - \omega| < \delta(\varepsilon) \right\} \subset F$$

*for every $\varepsilon > 0$ and for some $\delta(\varepsilon) > 0$ and furthermore there are points $\omega^{\pm}(r) \in \partial F$ with*

$$\omega^{\pm}(r) = \omega \pm i e^{i\gamma} r + o(r) \quad \textit{as } r \to 0.$$

The analytic function $f$ has an *angular derivative* at $\zeta \in \mathbb{T}$, denoted by $f'(\zeta)$, if it has a finite angular limit $f(\zeta)$ and if

$$\frac{f(z) - f(\zeta)}{z - \zeta} \to f'(\zeta) \quad \text{as } z \to \zeta,\ z \in \Delta, \tag{4.2.3}$$

for every Stolz angle $\Delta$ at $\zeta$.

THEOREM 4.6. *Let the conformal map $f$ of $\mathbb{D}$ into $\mathbb{C}$ have the angular limit $\omega = f(\zeta) \neq \infty$ and let $b \neq \infty$. Then the following four conditions are equivalent:*

(i) *$f$ has the angular derivative $b$ at $\zeta$;*

(ii) *$f'$ has the angular limit $b$ at $\zeta$;*

(iii) *there is a curve $\Gamma$ ending at $\zeta$ such that*

$$\frac{f(z) - f(\zeta)}{z - \zeta} \to b \quad \textit{as } z \to \zeta,\ z \in \Gamma;$$

(iv) *there is a curve* $\Gamma$ *ending at* $\zeta$ *such that*

$$f'(z) \to b \quad \text{as } z \to \zeta,\ z \in \Gamma.$$

This theorem is an immediate consequence of a theorem [67] about normal functions because $\log f'$ and $\log(f - \omega)$ are Bloch functions and therefore normal functions; see, e.g., [105, p. 73]. If $F$ is a John domain (Section 5.2) then (i) implies that (4.2.3) holds as $z \to \zeta$, $z \in \overline{\mathbb{D}}$ [105, p. 101].

The existence of a finite angular derivative can be characterized [110,52] in terms of the moduli of curve families [105, p. 258]. See [121,69,124,28,111] and [105, p. 259] for characterizations in important special cases.

We say that $f$ is *conformal* at $\zeta \in \mathbb{T}$ if $f'(\zeta)$ exists and is nonzero and finite. If $f$ is conformal at $\zeta$ then $f$ is isogonal at $\zeta$.

There is a simple sufficient condition for conformality [12, p. 97], [105, p. 83], namely that there are circles $C_1$ and $C_2$ touching at $\omega = f(\zeta)$ such that

$$C_1 \subset F \cup \{\omega\}, \qquad C_2 \subset \mathbb{C} \setminus F.$$

The function $f_0$ defined by the gap series

$$\log f_0'(z) = \frac{\mathrm{i}}{3} \sum_{k=0}^{\infty} z^{2^k} \quad (z \in \mathbb{D}) \tag{4.2.4}$$

maps $\mathbb{D}$ conformally onto a Jordan domain (even a quasidisk) but is nowhere isogonal and therefore nowhere conformal on $\mathbb{T}$; see, e.g., [105, p. 193].

## 4.3. *The behaviour almost everywhere*

One of the basic theorems on boundary behaviour is *Plessner's Theorem*: If $g$ is meromorphic in $\mathbb{D}$ then, for almost all $\zeta \in \mathbb{T}$, either $g$ has a finite angular limit at $\zeta$, or $C_\Delta(g, \zeta) = \widehat{\mathbb{C}}$ for every Stolz angle $\Delta$ at $\zeta$. See (2.2.6) for the definition of cluster sets.

THEOREM 4.7. *Let* $f$ *map* $\mathbb{D}$ *conformally into* $\mathbb{C}$. *Then, for almost all* $\zeta \in \mathbb{T}$, *the only alternatives are*

(i) *the angular limit* $f'(\zeta) \neq 0, \infty$ *exists*;

(ii) $C_{[0,\zeta)}(f', \zeta) = C_{[0,\zeta)}(\log f', \zeta) = \widehat{\mathbb{C}}$.

This is a consequence of Plessner's Theorem; the angular cluster set $C_\Delta$ can be replaced by the radial cluster set $C_{[0,\zeta)}$ because $f'$ and $\log f'$ are normal functions [67].

THEOREM 4.8. *If* $f$ *maps* $\mathbb{D}$ *conformally into* $\mathbb{C}$ *then*

$$\limsup_{r \to 1} \frac{|\log f'(r\zeta)|}{\sqrt{\log(1/(1-r)) \log\log\log(1/(1-r))}} \leqslant 6 \quad \textit{for almost all } \zeta \in \mathbb{T}. \tag{4.3.1}$$

This is the *Makarov Law of the Iterated Logarithm* [75], [105, p. 188]. It is marvelously precise because there exist functions $f$ for which the above lim sup is greater than a positive constant for almost all $\zeta \in \mathbb{T}$; see [54] for a lower estimate. It follows from (4.3.1) that

$$(1-r)^{\varepsilon} < |f'(r\zeta)| < (1-r)^{-\varepsilon} \quad (r_0(\varepsilon,\zeta) < r < 1) \tag{4.3.2}$$

for $\varepsilon > 0$ and almost all $\zeta \in \mathbb{T}$. There exist $f$ such that case (i) never occurs in Theorem 4.7 so that $\liminf |f'(r\zeta)| = 0$ and $\limsup |f'(r\zeta)| = \infty$ for almost all $\zeta$, see (4.2.4). But we see from (4.3.2) that the rate tends to be slow. See [77] for the existence of $f'(\zeta)$.

We say that $f$ is *twisting* at $\zeta \in \mathbb{T}$ if the angular limit $f(\zeta) \neq \infty$ exists and

$$\begin{aligned} &\liminf_{z\to\zeta,\, z\in\Gamma} \arg[f(z) - f(\zeta)] = -\infty, \\ &\limsup_{z\to\zeta,\, z\in\Gamma} \arg[f(z) - f(\zeta)] = +\infty \end{aligned} \tag{4.3.3}$$

for every curve $\Gamma$ ending at $\zeta$. Thus the boundary point $f(\zeta)$ can be reached only by twisting in both directories infinitely often around it. Note that this definition is completely geometric.

THEOREM 4.9. *A conformal map of $\mathbb{D}$ into $\mathbb{C}$ is, for almost all $\zeta \in \mathbb{T}$, either conformal at $\zeta$ or twisting at $\zeta$.*

This is the *McMillan Twist Theorem* [82]; see, e.g., [105, p. 142] for a proof. Recall that we defined above that $f$ is conformal at $\zeta$ if the angular derivative $f'(\zeta) \neq 0, \infty$ exists.

Let $\Lambda(E)$ denote the *linear measure* of the set $E \subset \mathbb{C}$; see, e.g., [29]. This is the Lebesgue measure if $E \subset \mathbb{T}$ and the length if $E$ is a Jordan arc.

The McMillan Twist Theorem implies that the four sets of

(a) $\{\zeta \in \mathbb{T}$: the angular derivative $f'(\zeta) \neq \infty$ exists$\}$,
(b) $\{\zeta \in \mathbb{T}$: $f$ is conformal at $\zeta\}$,
(c) $\{\zeta \in \mathbb{T}$: $f$ is isogonal at $\zeta\}$,
(d) $\{\zeta \in \mathbb{T}$: $f(\mathbb{D})$ contains a triangle of vertex $f(\zeta)\}$

are the same except for sets of zero measure. Moreover [82], [105, p. 146], if $E$ is a subset to one of these sets, then

$$\Lambda(E) = 0 \quad \Leftrightarrow \quad \Lambda(f(E)) = 0. \tag{4.3.4}$$

Absolute continuity properties like this are related to problems of measurability; see, e.g., [105, p. 133].

THEOREM 4.10. *Let $f$ map $\mathbb{D}$ onto $F$. Then there is a partition $\mathbb{T} = A_1 \cup A_2 \cup A_3$ such that*

(i) *the set $A_1 \subset \mathbb{T}$ has zero measure;*
(ii) *$f$ is conformal at all $\zeta \in A_2$ and $f(A_2)$ has $\sigma$-finite linear measure;*
(iii) *the image $f(A_3) \subset \partial F$ has zero linear measure.*

This is the *Makarov Compression Theorem* [74,75]; see [105, p. 147] for a proof. It shows that, if $\partial F$ does not have $\sigma$-finite linear measure, then almost nothing, namely $A_1$, is mapped onto almost everything as far as linear measure is concerned. For any perfect set $A \subset \mathbb{T}$ and any totally disconnected set $B$, there even exists [101] a conformal map $f : \mathbb{D} \to \mathbb{C}$ with $f(A) \supset B$.

If $f$ is conformal almost nowhere on $\mathbb{T}$ then Theorem 4.10 implies that $\Lambda(A_3) = 2\pi$ but $\Lambda(f(A_3)) = 0$. Thus almost everything goes to almost nothing. The situation is however different for Hausdorff dimension because [76], [105, p. 231]

$$A \subset \mathbb{T},\ \dim A = 1 \quad \Rightarrow \quad \dim f(A) \geqslant 1 \tag{4.3.5}$$

for all conformal maps; see, e.g., [29] for Hausdorff measure and Hausdorff dimension. Bishop and Jones [8] have shown that, if $L$ is a rectifiable curve and $E \subset \mathbb{T}$, then

$$\Lambda(E) > 0,\ f(E) \subset L \cap \partial F \quad \Rightarrow \quad \Lambda\big(f(E)\big) > 0. \tag{4.3.6}$$

See [55] for a characterization of sets lying on a rectifiable curve.

Now let $J$ be a Jordan curve and let $f_1$ and $f_2$ map $\mathbb{D}$ conformally onto the two complementary domains of $J$. These maps are continuous and injective in $\overline{\mathbb{D}}$ by Theorem 2.1. Hence the "conformal welding"

$$\varphi = f_1^{-1} \circ f_2 : \mathbb{T} \to \mathbb{T} \tag{4.3.7}$$

is a homeomorphism. See [7] for the next theorem and, e.g., [113], [105, p. 152] for further results.

THEOREM 4.11. *Let $T$ be the set of points on the Jordan curve $J$ where $J$ has a tangent. Then*

$$J = T \cup B_1 \cup B_2 \quad \textit{with } \Lambda\big(f_k^{-1}(B_k)\big) = 0\ (k = 1, 2). \tag{4.3.8}$$

*The homeomorphism $\varphi$ is absolutely continuous if and only if $\Lambda(f^{-1}(T)) = 2\pi$, and it is singular if and only if $\Lambda(T) = 0$.*

The homomorphism $\varphi$ is, by definition, absolutely continuous if $\Lambda(E) = 0$ implies $\Lambda(\varphi(E)) = 0$; and it is singular if there exists $E$ with $\Lambda(E) = 0$ and $\Lambda(\pi \setminus \varphi(E)) = 0$.

### 4.4. *The average growth of the derivative and the power series*

Let $f$ map $\mathbb{D}$ conformally into $\mathbb{C}$. For $p \in \mathbb{R}$ we define

$$\beta_f(p) = \limsup_{r \to 1} \left( \log \int_0^{2\pi} \left| f'\left(r\, e^{it}\right) \right|^p dt \right) \Big/ \log \frac{1}{1-r}. \tag{4.4.1}$$

Thus $\beta_f(p)$ is the smallest exponent such that the integral means of the derivative satisfy

$$\frac{1}{2\pi}\int_0^{2\pi} |f'(re^{it})|^p\,dt = \mathrm{O}\big((1-r)^{-\beta_f(p)-\varepsilon}\big) \quad (r \to 1) \tag{4.4.2}$$

for every $\varepsilon > 0$. The function $\beta_f$ is called the *integral means spectrum* of $f$. It is a convex function. The *universal integral means spectrum* is defined by

$$B(p) = \sup\big\{\beta_f(p)\colon\ f \text{ maps } \mathbb{D} \text{ conformally onto a bounded domain}\big\}. \tag{4.4.3}$$

This important quantity is not yet fully known; see [78], [105, p. 176]. Here are some results:

$$B(p) = p - 1 \quad \text{for } p \geqslant 2 \quad \text{(classical)}, \tag{4.4.4}$$

$$B(p) = p - 1 + \mathrm{O}\big((p-2)^2\big) \quad \text{as } p \to 2 \quad [56], \tag{4.4.5}$$

$$B(1) < 0.4886 \quad [15,79], \tag{4.4.6}$$

$$B(p) < 3p^2 + \mathrm{O}(p^3) \quad \text{as } p \to 0 \quad [15], [105, \text{p. } 178], \tag{4.4.7}$$

$$B(-1) < 0.601, \quad B(-2) < 1.547 \quad [105, \text{p. } 178], [5], \tag{4.4.8}$$

$$B(p) = |p| - 1 \quad \text{for large negative } p \quad [14]. \tag{4.4.9}$$

Numerical evidence indicates that $B(p) \geqslant p^2/4$ for $|p| \leqslant 2$. The following conjecture was (in various stages) made by Brennan [9], Carleson and Jones [13] and Kraetzer [61]:

*BCJK Conjecture.* The universal integral means spectrum is given by

$$B(p) = \begin{cases} p^2/4 & \text{for } |p| \leqslant 2, \\ |p| - 1 & \text{for } |p| \geqslant 2. \end{cases} \tag{4.4.10}$$

Knowing $B(p)$ would solve problems also for some other families. Makarov [78, Theorem 5.4] considered possibly unbounded conformal maps and showed that

$$\sup \beta_f(p) = \max\big(B(p), 3p - 1\big). \tag{4.4.11}$$

Now we consider Hölder-continuous conformal maps, that is, we assume that

$$f(z) - f(z') = \mathrm{O}\big(|z - z'|^\alpha\big) \quad \big(z, z' \in \mathbb{D}\big) \tag{4.4.12}$$

with $0 < \alpha \leqslant 1$. Then [79]

$$\sup\big\{\beta_f(p)\colon\ f \text{ is } \alpha\text{-Hölder continuous}\big\} = \begin{cases} B(p) & \text{for } p \leqslant p_\alpha, \\ T_\alpha(p) & \text{for } p \geqslant p_\alpha, \end{cases} \tag{4.4.13}$$

where $T_\alpha(p) = (1-\alpha)(p-p_\alpha) + B(p_\alpha)$ is the tangent to $B(p)$ of slope $1-\alpha$. Using (4.4.5) it follows that [116,56]

$$\beta_f(2) \leqslant 1 - c\alpha^2 \tag{4.4.14}$$

for some universal constant $c > 0$. The BCJK conjecture would imply $c = 1$.

The integral means spectrum for negative $p$ gives lower bounds for Hausdorff dimensions [76]: If $A \subset \mathbb{T}$ then

$$\dim f(A) \geqslant \frac{q \dim A}{\beta_f(-q) + q + 1 - \dim A} \quad \text{for all } q > 0. \tag{4.4.15}$$

The BCJK conjecture would give $\dim f(A) \geqslant (\dim A)/(1+\sqrt{1-\dim A})$.

The integral means spectrum for $p = 1$ and $p = 2$ is connected with the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad (z \in \mathbb{D}). \tag{4.4.16}$$

De Branges [23] has proved the Bieberbach conjecture $|a_n| \leqslant n|a_1|$; see [27] for the history.

The coefficient problem for bounded conformal maps is connected with the value of $B(1)$ because the exponent in the trivial estimate

$$a_n = \mathrm{O}\big(n^{B(1)-1+\varepsilon}\big) \ (n \to \infty) \quad \text{for every } \varepsilon > 0 \tag{4.4.17}$$

cannot be improved [13]. If the BCJK conjecture turns out to be correct, then $a_n = \mathrm{O}(n^{-3/4+\varepsilon})$.

If area $f(\mathbb{D}) < \infty$ then the power series converges at $\zeta \in \mathbb{T}$ if and only if the radial limit exists [62, p. 65]; see [44] for generalizations. It is easy to see that $\beta(2) \equiv \beta_f(2) = \inf \lambda$ for all $\lambda$ satisfying

$$\iint_{\mathbb{D}} |f'(z)|^2 \big(1-|z|\big)^{\lambda-1} \, dx\, dy < \infty. \tag{4.4.18}$$

If $\beta(2) < 1$ it follows that

$$\sum_{k=n}^{\infty} |a_k| = \mathrm{O}\big(n^{(\beta(2)-1)/2+\varepsilon}\big) \ (n \to \infty) \quad \text{for } \varepsilon > 0. \tag{4.4.19}$$

Thus the power series converges absolutely in $\overline{\mathbb{D}}$ if $f(\mathbb{D})$ is a Hölder domain.

## 5. Some special types of domains

### 5.1. *Subdomains of the unit disk*

A common situation is that one has domains $G$ and $H$ with $G \subset H$ and is interested in $\partial G \cap \partial H$. For instance $G$ might be a domain under investigation whereas $H$ is a simpler domain with known properties. Let $g$ and $h$ be conformal maps of $\mathbb{D}$ onto $G$ and $H$ respectively. Then $f = h^{-1} \circ g$ maps $\mathbb{D}$ conformally onto $F = h^{-1}(G) \subset \mathbb{D}$ and there is a set $A$ with

$$A \subset \mathbb{T}, \quad f(A) \subset \mathbb{T} \tag{5.1.1}$$

such that $g(A) = \partial G \cap \partial H$ in a suitable sense; see Figure 13.

For $\zeta \in \mathbb{T}$ let $f(\zeta)$ denote the angular limit whenever it exists; it can fail to exist only on a set of zero capacity and thus of zero measure (Theorem 2.9). It is no essential restriction to assume that $f(0) = 0$.

THEOREM 5.1. *Let $f$ map $\mathbb{D}$ conformally into $\mathbb{D}$ and let $A \subset \mathbb{T}$ be a Borel set with $f(A) \subset \mathbb{T}$. If $f(0) = 0$ then*

$$\Lambda\big(f(A)\big) \geqslant \Lambda(A), \tag{5.1.2}$$

$$\operatorname{cap} f(A) \geqslant \frac{\operatorname{cap} A}{\sqrt{|f'(0)|}} \geqslant \operatorname{cap} A. \tag{5.1.3}$$

Thus $f$ increases the size of sets $A$ on $\mathbb{T}$ provided that $f(A)$ also lies on $\mathbb{T}$. The estimate (5.1.2) is *Löwner's Lemma* and is valid also if $f$ is not injective. Its invariant form is

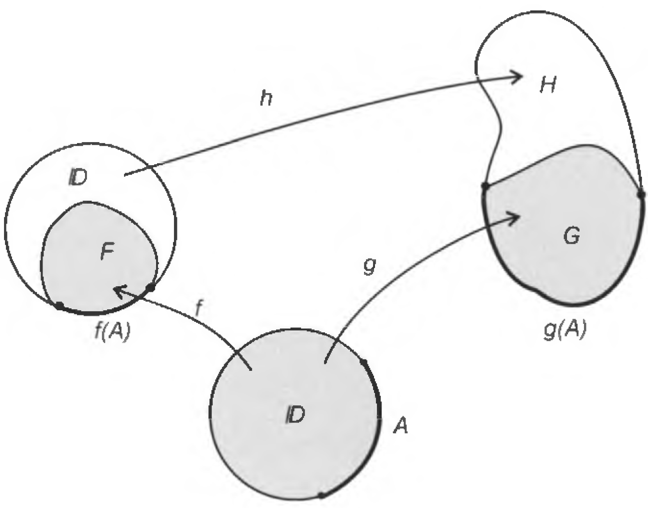


Fig. 13. How to reduce the case $G \subset H$ to $F \subset \mathbb{D}$.

Carleman's principle of domain extension for harmonic measure [92, p. 68]. See [105, p. 217] for (5.1.3).

These results were extended to Hausdorff measures by Makarov [76], [105, p. 234] and to generalized capacities by Hamilton [43]. Both results imply that

$$\dim f(A) \geqslant \dim A \quad \text{if } A \subset \mathbb{T},\ f(A) \subset \mathbb{T}, \tag{5.1.4}$$

where dim is the Hausdorff dimension [29].

THEOREM 5.2. *Let $f$ map $\mathbb{D}$ conformally into $\mathbb{D}$. If $f(\zeta)$ exists and lies on $\mathbb{T}$ then the angular derivative $f'(\zeta)$ exists and*

$$0 < |f'(\zeta)| \leqslant +\infty. \tag{5.1.5}$$

*Moreover $f'(\zeta) \neq \infty$ for almost all $\zeta$ with $f(\zeta) \in \mathbb{T}$.*

The first part follows from the Julia–Wolff Lemma [105, p. 82] valid for all analytic selfmaps of $\mathbb{D}$. The second part [82] follows at once from Theorem 4.9 because $f : \mathbb{D} \to \mathbb{D}$ cannot be twisting at any $\zeta$ with $f(\zeta) \in \mathbb{T}$.

## 5.2. *John domains and quasidisks*

Let $f$ map $\mathbb{D}$ conformally onto the bounded domain $F$. We call $F$ a *John domain* if there is a constant $c_0$ such that (see Figure 14)

$$\min[\operatorname{diam} F_1, \operatorname{diam} F_2] \leqslant c_0 \operatorname{diam} C \tag{5.2.1}$$

for every crosscut $C$ of $F$, where $F_1$ and $F_2$ are the components of $F \setminus C$. It is possible to limit oneself to rectilinear crosscuts. The boundary of John domains is locally connected

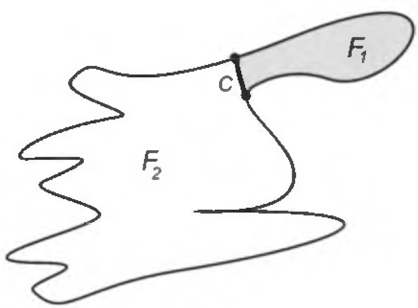


Fig. 14. A John domain; the condition is that the shaded part is not too narrow.

so that $f$ has a continuous extension to $\overline{\mathbb{D}}$. See [80,90], [105, p. 96] for a discussion of John domains.

Each of the following conditions (a)–(f) holds if and only if $F$ is a John domain. The positive constants $c_1, \ldots$ depend only on $F$ and on the parameters stated below. For $z \in \mathbb{D}$ we consider the "box"

$$B(z) = \{\zeta \in \mathbb{D}: \ |z| \leqslant |\zeta| < 1, \ |\arg\zeta - \arg z| \leqslant \pi(1 - |z|)\}. \tag{5.2.2}$$

The *inner distance* in $F$ is defined by

$$\mathrm{dist}_F(w, w') = \inf \Lambda(L) \quad \text{for } w, w' \in \overline{F}, \tag{5.2.3}$$

where $L$ runs through all curves that connect $w$ and $w'$ within $F$ and $\Lambda$ denotes the length.

(a) Any two points $w_1, w_2 \in F$ can be joined by a curve $L \subset F$ such that [90]

$$\mathrm{dist}(w, \partial F) \geqslant c_1 \min[\Lambda(L_1), \Lambda(L_2)] \quad (w \in L), \tag{5.2.4}$$

where $L_j$ is the part of $L$ between $w_j$ and $w$.

(b) If $I_j\,(j = 1, 2)$ are the arcs of $\mathbb{T} \setminus \{z, z'\}$ $(z, z' \in \mathbb{T})$ then

$$\min_j \mathrm{diam}\{f(\zeta): \ \zeta \in I_j\} \leqslant c_2 \, \mathrm{dist}_F(f(z), f(z')). \tag{5.2.5}$$

(c) With $d_f(z) = \mathrm{dist}(f(z), \partial F)$ we have

$$\mathrm{diam}\, f(B(z)) \leqslant c_3 d_f(z) \leqslant c_3(1 - |z|^2)|f'(z)| \quad \text{for } z \in \mathbb{D}. \tag{5.2.6}$$

(d) There exists $\alpha > 0$ such that

$$|f(\zeta) - f(\zeta')| \leqslant c_4 d_f(z) \left(\frac{|\zeta - \zeta'|}{1 - |z|}\right)^{\alpha} \quad \text{for } \zeta, \zeta' \in B(z), \ z \in \mathbb{D}. \tag{5.2.7}$$

(e) There exists $\alpha > 0$ such that

$$|f'(\zeta)| \leqslant c_5 |f'(z)| \left(\frac{1 - |\zeta|}{1 - |z|}\right)^{\alpha - 1} \quad \text{for } \zeta \in B(z), \ z \in \mathbb{D}. \tag{5.2.8}$$

(f) For every $\varepsilon > 0$ (or for some $\varepsilon < 1$) there exists $\delta > 0$ such that

$$\Lambda(A) \leqslant \delta \Lambda(I) \quad \Rightarrow \quad \mathrm{diam}\, f(A) \leqslant \varepsilon \, \mathrm{diam}\, f(I) \tag{5.2.9}$$

for all arcs $A \subset I \subset \mathbb{T}$.

It follows from (5.2.7) that $f$ satisfies the Hölder condition (4.4.12). Thus a John domain is a Hölder domain but not conversely.

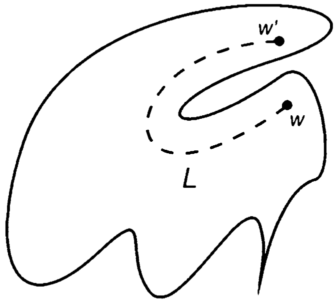


Fig. 15. A linearly connected domain.

We now turn to a "dual" concept. The simply connected domain $F$ is called *linearly connected* (see Figure 15) if any two points $w, w' \in F$ can be connected by a curve $L \subset F$ such that, for some constant $c$,

$$\operatorname{diam} L \leqslant c\left|w - w'\right|, \tag{5.2.10}$$

in other words that the inner distance (5.2.3) satisfies $\operatorname{dist}_F(w, w') \leqslant c|w - w'|$. See, e.g., [36,89]. If $f$ maps $\mathbb{D}$ conformally onto the linearly connected domain $F$, then $f$ has a continuous extension to $\overline{\mathbb{D}}$ and there exists $\beta < 2$ such that [105, p. 104]

$$\left|f(\zeta) - f(\zeta')\right| \geqslant c_1 d_f(z)\left(\frac{|\zeta - \zeta'|}{1 - |z|}\right)^{\beta} \quad \text{for } \zeta, \zeta' \in B(z),\ z \in \mathbb{D}, \tag{5.2.11}$$

$$\left|f'(\zeta)\right| \geqslant c_2\left|f'(z)\right|\left(\frac{1 - |\zeta|}{1 - |z|}\right)^{\beta - 1} \quad \text{for } \zeta \in B(z),\ z \in \mathbb{D}; \tag{5.2.12}$$

compare (5.2.7) and (5.2.8). It follows from (5.2.11) for $z = 0$ that the inverse map $f^{-1} : F \to \mathbb{D}$ satisfies the Hölder condition

$$\left|f^{-1}(w) - f^{-1}(w')\right| \leqslant c_3\left|w - w'\right|^{1/\beta} \quad \text{for } w, w' \in \overline{F}; \tag{5.2.13}$$

one can choose $\beta = 2 - (2/\pi)\arcsin[1/(2c)]$, see [89].

The Jordan curve $J$ is called a *quasicircle* if there is a constant $c$ such that

$$\min[\operatorname{diam} J_1, \operatorname{diam} J_2] \leqslant c\left|w - w'\right| \quad \text{for } w, w' \in J, \tag{5.2.14}$$

where $J_1$ and $J_2$ are the arcs into which $J$ is separated by $w$ and $w'$. A *quasidisk* is a domain whose boundary is a quasicircle. We shall only consider bounded quasidisks.

THEOREM 5.3. *Let $f$ map $\mathbb{D}$ conformally onto the bounded domain $F$. Then the following five conditions are equivalent.*

(i) *$F$ is a quasidisk;*

(ii) *$F$ is a linearly connected John domain;*

(iii) *any two points* $w_1, w_2 \in F$ *can be joined by a curve* $L \subset F$ *with* $\Lambda(L) \leqslant c_1|w_1 - w_2|$ *such that* (5.2.4) *holds*;
(iv) $f$ *has a quasiconformal extension to* $\widehat{\mathbb{C}}$;
(v) *there is a quasiconformal homeomorphism of* $\mathbb{C}$ *onto itself that maps* $\mathbb{T}$ *onto* $\partial F$.

This important theorem combines the work of many authors, above all Ahlfors [1,38,80, 41]; see the books [2,68,66].

Condition (ii) allows us to apply the results mentioned above to quasidisks. For example, combining (5.2.7) with (5.2.11) we obtain the estimate

$$c_2\left(\frac{|\zeta - \zeta'|}{1 - |z|}\right)^\beta \leqslant \frac{|f(\zeta) - f(\zeta')|}{d_f(z)} \leqslant c_3\left(\frac{|\zeta - \zeta'|}{1 - |z|}\right)^\alpha \tag{5.2.15}$$

for $\zeta, \zeta' \in B(z)$, $z \in \mathbb{D}$ with constants $0 < \alpha < \beta < 2$. Condition (iii) says that $F$ is a uniform domain [39,120].

On the other hand, condition (iv) allows us to apply the powerful theory of quasiconformal maps. For instance, the Lehto majorant principle [66, p. 77] can be used to show that we can choose $\beta = 1 + k$ and $\alpha = 1 - k$ in (5.2.15) if $f$ has a k-quasiconformal extension to $\widehat{\mathbb{C}}$, where $0 \leqslant k < 1$.

If $g$ is a conformal map of $\mathbb{D}$ onto a domain that contains the quasidisk $F$ then $g^{-1}(F)$ is also a quasidisk [30].

## 5.3. *Boundaries of finite length*

Let again $\Lambda$ denote the linear measure and let $H^1$ denote the Hardy space [26,37]. The basic result is the *Riesz* or *Riesz–Privalov Theorem* [109,107], see, e.g., [105, p. 134].

THEOREM 5.4. *Let* $f$ *map* $\mathbb{D}$ *conformally onto the bounded domain* $F$. *Then*

$$f' \in H^1 \quad \Leftrightarrow \quad \Lambda(\partial F) < \infty.$$

*If* $\Lambda(\partial F) < \infty$ *then* $f$ *is continuous in* $\overline{\mathbb{D}}$ *and*

$$f'(\zeta) = \lim_{z \to \zeta,\ z \in \overline{\mathbb{D}}} \frac{f(z) - f(\zeta)}{z - \zeta} \neq 0, \infty$$

*exists for almost all* $\zeta \in \mathbb{T}$, *furthermore, for* $E \subset \mathbb{T}$,

$$\Lambda(E) = 0 \quad \Leftrightarrow \quad \Lambda\big(f(E)\big) = 0.$$

If $\partial F$ has finite length we can thus write $f'$ as the Poisson integral of its boundary values. This is not always true for $\log|f'|$. We call $F$ a *Smirnov domain* if $\Lambda(\partial F) < \infty$ and if

$$\log|f'(z)| = \frac{1}{2\pi}\int_0^{2\pi} \frac{1 - |z|^2}{|\zeta - z|^2} \log|f'(\zeta)||d\zeta| \quad \text{for } z \in \mathbb{D}. \tag{5.3.1}$$

See, e.g., [26, p. 173], [105, p. 160] for some properties. If $F$ is a Smirnov domain then $\mathbb{C} \setminus \overline{F}$ need not be Smirnov [57].

The curve $L$ is called *Ahlfors-regular* if

$$\Lambda(L \cap D) \leqslant c_0 \operatorname{diam} D \quad \text{for every disk } D; \tag{5.3.2}$$

see, e.g., [22]. If $\partial F$ is Ahlfors-regular then $F$ is a Smirnov domain [126].

If every boundary point is the vertex of a triangle of fixed size that lies in $F$, then $\Lambda(\partial F) < \infty$ and $f$ satisfies a Hölder condition (4.4.12) [70,33].

The next result [8] gives the complete answer to the problem to which curves Theorem 4.3 of Hayman and Wu can be generalized.

THEOREM 5.5. *The curve $L$ has the property that*

$$\Lambda\big(f^{-1}(f(\mathbb{D}) \cap L)\big) < \infty \quad \textit{for all conformal maps } f$$

*if and only if $L$ is Ahlfors-regular.*

Now we consider a length version of quasi-circles; compare (5.2.14). The rectifiable Jordan curve $J$ is called a *Lavrentiev curve* or *chord-arc curve* if there is a constant $c$ such that

$$\min\big[\Lambda(J_1), \Lambda(J_2)\big] \leqslant c\big|w - w'\big| \quad \text{for } w, w' \in J, \tag{5.3.3}$$

where $J_1$ and $J_2$ are the arcs of $J \setminus \{w, w'\}$. The inner domain is called a *Lavrentiev domain.* The next theorem combines the work of many people [64,118,53,127].

THEOREM 5.6. *Let $f$ map $\mathbb{D}$ conformally onto the inner domain $F$ of a rectifiable Jordan curve. Then the following five conditions are equivalent.*

(i) *$F$ is a Lavrentiev domain;*

(ii) *$F$ is an Ahlfors-regular quasidisk;*

(iii) *there is a bi-Lipschitz map $h : \mathbb{C} \to \mathbb{C}$ with $h(\mathbb{T}) = J$, that is*

$$c_1\big|z - z'\big| \leqslant \big|h(z) - h(z')\big| \leqslant c_2\big|z - z'\big| \quad \textit{for } z, z' \in \mathbb{C};$$

(iv) *$F$ is linearly connected and*

$$\int_{-\pi(1-r)}^{\pi(1-r)} \big|f'(\zeta e^{it})\big|\, dt \leqslant c_3(1 - r)\big|f'(r\zeta)\big| \quad \textit{for } \zeta \in \mathbb{T},\ 0 \leqslant r < 1; \tag{5.3.4}$$

(v) *$F$ is a linearly connected Smirnov domain such that, for all arcs $I \subset \mathbb{T}$,*

$$\frac{1}{\Lambda(I)} \int_I \big|f'(\zeta)\big||d\zeta| \leqslant c_4 \exp\bigg(\frac{1}{\Lambda(I)} \int_I \log\big|f'(\zeta)\big||d\zeta|\bigg). \tag{5.3.5}$$

Inequality (5.3.5) is a reversed geometric–arithmetic means inequality. It says that $|f'|$ satisfies the *Coifman–Fefferman* $(A_\infty)$ *condition* [16]. It holds if and only if, for every $\varepsilon > 0$ (or for some $\varepsilon < 1$) there exists $\delta > 0$ such that

$$\Lambda(A) \leqslant \delta\Lambda(I) \quad \Rightarrow \quad \Lambda\big(f(A)\big) \leqslant \varepsilon\Lambda\big(f(I)\big) \tag{5.3.6}$$

for all arcs $I \subset \mathbb{T}$ and all measurable sets $A \subset I$; compare condition (5.2.9) for John domains.

This is related to the Muckenhoupt conditions $(A_p)$ which are reversed Hölder conditions. Lavrentiev domains thus tie in with the rich theory of functions of bounded mean oscillation (BMO). See, e.g., [108,37,53,115].

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CHAPTER 3

# Extremal Quasiconformal Mappings of the Disk

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## *Contents*



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HANDBOOK OF COMPLEX ANALYSIS: GEOMETRIC FUNCTION THEORY, VOLUME 1
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## Preface

The idea of what are now called extremal quasiconformal mappings is due to Grötzsch [47] who used what is actually the more descriptive term "most conformal". The mappings are required to satisfy side conditions that in general cannot be satisfied by conformal mappings. We want the deviation from conformality to be extremal (i.e., minimal) in a sense to be made precise, and want a quantitative measure of the deviation of the mapping from conformality. The extremal dilatation provides this measure. Our survey is devoted to selected topics connected with the problem of determining extremal quasiconformal mappings of regions with specified boundary values. It overlaps the excellent survey of Strebel's [165] dated 1986. During the last several years there has been important progress in characterizing the conditions under which *unique* extremality occurs, giving the subject a certain degree of closure, and a new survey therefore appears justified. On the other hand, due to space limitations, this article is limited essentially to mappings of a simply connected region which, without loss of generality, can be taken to be the disk. Admittedly, this restriction also reflects limitations of the author's personal expertise, and therefore in the main the article should only be considered as an introduction to the current status of the core of the subject. We have attempted to partially compensate for this defect by a fairly extensive bibliography covering generalizations and other important directions. For other surveys on our and closely related material we refer the reader to [10,31,65,74], and the texts [61] and [83].

## 1. Introduction

### 1.1. *Definition and basic properties of quasiconformality*

The idea of extremal quasiconformal mappings satisfying prescribed conditions on the boundary was introduced by Grötzsch in 1932 [47]. Grötzsch required the mappings to be $C^1$ homeomorphisms with positive Jacobian. This requirement was later relaxed as it had the disadvantage that compactness and normality properties did not hold. We summarize the basic notions and properties of the modern definition in this section. For detailed expositions of this background material, the reader is referred to the monographs by Ahlfors [3], Lehto and Virtanen [84], and Lehto [83]. In particular, see [84] for detailed discussions of the equivalent forms of the modern definition and references thereto.

A sense-preserving homeomorphism $f : \Omega \to \Omega'$ of a region $\Omega$ of the $z$-plane that is locally absolutely continuous as a function of $x$ for almost all $y$ and as a function of $y$ for almost all $x$ is called $K$-qc ($K$-quasiconformal), $K \geqslant 1$, if the partial derivatives $f_z = \frac{1}{2}(f_x - f_y)$, $f_{\bar z} = \frac{1}{2}(f_x + f_y)$ (which exist almost everywhere) are locally square-integrable, and if the directional derivatives satisfy

$$\max_\alpha \left|\partial_\alpha f(z)\right| \leqslant K \min_\alpha \left|\partial_\alpha f(z)\right| \tag{1.1}$$

for a.a. $z \in \Omega$. It follows that $f$ is totally differentiable a.e. Roughly speaking, (1.1) means that at almost all points $z$ of $\Omega$ infinitesimal circles are mapped onto infinitesimal ellipses with axis ratio $D_f(z) \leqslant K$. A mapping is *quasiconformal* if it is $K$-qc for some $K$. Equivalently to (1.1), the complex dilatation,

$$\mu_f(z) = \frac{f_{\bar{z}}}{f_z},$$

exists a.e. and belongs to the closed ball of radius $k = (K-1)/(K+1)$ of $L^\infty(\Omega)$. A computation shows that

$$D_f(z) = \frac{1 + |\mu_f(z)|}{1 - |\mu_f(z)|}.$$

Conversely, according to the theorem of Morrey [93], given $\mu \in L^\infty(\Omega)$, $\|\mu\|_\infty < 1$, there exists a quasiconformal mapping $f$ of $\Omega$, uniquely determined up to composition with a conformal mapping, such that $\mu_f = \mu$. $D_f(z)$ is referred to as the *dilatation* at $z$, in distinction to the *complex* dilatation $\mu_f(z)$.

According to the geometric characterization of $K$-quasiconformality due to Ahlfors [1], the sense-preserving homeomorphism $f$ of $\Omega$ is $K$-qc if and only if

$$M\big(f(Q)\big) \leqslant K\, M(Q) \tag{1.2}$$

for every quadrilateral (= Jordan domain with four distinguished boundary points) whose closure lies in $\Omega$. Here $M(\,)$ denotes the conformal modulus.

If $f$ is quasiconformal there is a smallest $K$ for which it is $K$-qc. This is referred to as the "maximal dilatation", and we denote its value by

$$K[f] = \operatorname{ess\,sup} D_f(z) = \frac{1 + k[f]}{1 - k[f]}, \quad k[f] = \|\mu_f\|_\infty.$$

It is clear that preceding or following $f$ by a conformal map does not change the value of $D_f$ at corresponding points, and the value of $K[f]$ therefore stays the same. Moreover, if $f$ and $g$ are quasiconformal, and $f \circ g$ makes sense, then $K[f \circ g] \leqslant K[f]K[g]$.

The *affine stretch* by the factor $A$ ($A =$ real const $\geqslant 1$),

$$f(z) = F_A(z) = Ax + \mathrm{i}y \quad (z = x + \mathrm{i}y),$$

is the simplest example of a quasiconformal mapping. For it,

$$\mu_f(z) \equiv \frac{A-1}{A+1}, \qquad D_f(z) \equiv A = K[f].$$

In this report we will restrict ourselves to *simply-connected* regions $\Omega$, and, without loss of generality, it will frequently suffice to consider the unit disk, $\Delta = \{|z| < 1\}$.

For a $K$-qc mapping, just like for the special case, $K = 1$, of a conformal mapping, every isolated singularity is removable. Moreover, the generalization of the Osgood–Carathéodory theorem holds: *A quasiconformal mapping between Jordan domains extends to a homeomorphism between their closures.*

EXAMPLE 1.1.1. Suppose $w = f(z) = r^s e^{i\theta}$ $(z = re^{i\theta})$, where $s > 0$ is a fixed parameter. This example is especially interesting, because $f$ turns out to be quasiconformal in spite of the bad distortion near $z = 0$. Introducing $\log z$, $\log w$, we see that for $z \neq 0$, $f$ is locally *conformal* $\circ$ *affine* $\circ$ *conformal*, where the affine mapping is a stretch by the factor $s$. Therefore $f$ is a $K$-qc map of $\{0 < |z| < \infty\}$ onto $\{0 < |w| < \infty\}$, with $K[f] = \max[s, 1/s]$. Since $z = 0$ is an isolated singularity, $f$ is actually a $\max[s, 1/s]$-qc map of the complex plane $C$ onto itself. For the complex dilatation we have

$$\mu_f(z) = \frac{s-1}{s+1}\frac{z}{\bar{z}}, \quad z \neq 0,$$

and, hence

$$k[f] = \operatorname{ess\,sup}\{|\mu_f(z)|: z \in C\} = \left|\frac{s-1}{s+1}\right|.$$

## 1.2. *Extension of boundary values – quasisymmetry, extremality*

Given two topologically equivalent regions, $\Omega$, $\Omega'$, under what condition can a homeomorphism between their boundaries be extended to a quasiconformal mapping of $\Omega$ onto $\Omega'$? The question can be answered most directly in the case of Jordan domains. In 1956, Beurling and Ahlfors [17] found that a necessary and sufficient condition that an increasing homeomorphism $u(x)$ of $R$ onto itself can be extended to a quasiconformal mapping between the upper-half planes is the *quasisymmetry* of $u$: There exists $\rho > 0$, such that

$$\frac{1}{\rho} \leqslant \frac{u(x+t) - u(x)}{u(x) - u(x-t)} \leqslant \rho, \quad x, t \in R. \tag{2.1}$$

To prove the sufficiency of (2.1), Beurling and Ahlfors used the following formula to define an extension $f$ of $u$ to $\{z = x + iy: y > 0\}$:

$$f(x+iy) = \frac{1}{2}\int_0^1 \big[h(x+ty) + h(x-ty)\big]\,dt + \frac{i}{2}\int_0^1 \big[h(x+ty) - h(x-ty)\big]\,dt. \tag{2.2}$$

By the use of compositions with logarithms, e.g., as in [158], the Beurling–Ahlfors condition (2.1) can be restated in terms the unit disk $\Delta = \{|z| < 1\}$: *Suppose $h(z)$ is a sense-preserving homeomorphism of $\partial\Delta$ onto itself. A necessary and sufficient condition that $h$ can be extended to a quasiconformal mapping of $\Delta$ onto itself is the quasisymmetry of $h$: There exists $\rho > 0$, such that*

$$\frac{1}{\rho} \leqslant \left|\frac{h(ze^{it}) - h(z)}{h(z) - h(ze^{-it})}\right| \leqslant \rho, \quad \textit{whenever } |z| = 1, \textit{ and } t \textit{ is real}. \tag{2.3}$$

The idea of *extremality* is to make $K[f]$ *as small as possible*, that is, to make $f$ as "nearly conformal" as possible, subject to the side condition under consideration, which in our case will consist in a specification of the mapping on the boundary of the region. Namely, suppose $\Omega$ is a Jordan region and the function $h$, given on $\partial\Omega$, has a quasiconformal extension to $\Omega$. Let

$$\mathcal{Q}[h] = \{f\colon\ f \text{ is a quasiconformal extension of } h \text{ to } \Omega\},$$

and

$$K_0 = K_0[h] = \inf\bigl\{K[f]\colon\ f \in \mathcal{Q}[h]\bigr\}.$$

By a compactness argument, it follows that *there exists a* $K_0$-qc *extension* $f_0$ *of* $h$ *to* $\Omega$. We refer to $f_0$ as an *extremal mapping* (corresponding to the boundary values $h$), and to $K_0$ as the *extremal dilatation.*[1] It is easy to see that $K_0$ and $f_0$ are conformal invariants. Also, since the inverse of a $K$-qc mapping is also a $K$-qc mapping, it follows that $h^{-1}$ determines the same value $K_0$ as $h$, and that $f_0^{-1}$ is an extremal mapping for the boundary values $h^{-1}$.

EXAMPLE 1.2.2 (*Grötzsch* [47]). With $A \geqslant 1$, let

$$\mathcal{R} = \{z = x + \mathrm{i}y\colon\ 0 < x < a,\ 0 < y < 1\},$$
$$\mathcal{R}' = \{w = u + \mathrm{i}v\colon\ 0 < u < Aa,\ 0 < v < 1\}.$$

Let $h$ be a homeomorphism of $\partial\mathcal{R}$ onto $\partial\mathcal{R}'$ so that each vertex of $\mathcal{R}$ is mapped onto the similarly located vertex of $\mathcal{R}'$, and the values of $h$ are free between the vertices. Then *$K_0 = A$, and the affine stretch $F_A(z) = Ax + \mathrm{i}y$ is uniquely extremal.*

PROOF. This is a core example of the entire theory, and the proof is a classic example of the "length-area" method. Let $w = f(z)$ be a quasiconformal extension of $h$. For a.a. $y$, $0 < y < 1$, a horizontal section of $\mathcal{R}$ is mapped onto a curve of length at least the width of $\mathcal{R}'$; that is, since $dw = f_z\,dz + f_{\bar z}\,\overline{dz}$,

$$Aa \leqslant \int |dw| \leqslant \int_0^a |f_z + f_{\bar z}|\,dx.$$

Integrating with respect to $y$, $0 < y < 1$,

$$Aa \leqslant \iint_{\mathcal{R}} |f_z + f_{\bar z}|\,dx\,dy. \tag{2.4}$$

[1] If $h$ has a quasiconformal extension, then it is clear that $K_0$ is uniquely determined by $h$, but, as we will see in Example 1.4.4, extremal extensions are not necessarily unique.

The Jacobian of $f$ is $J_f(z) = |f_z|^2 - |f_{\bar z}|^2 > 0$ a.e. Introducing the factor $\sqrt{J_f(z)}/\sqrt{J_f(z)}$ in the integrand of (2.4), and using Schwarz's Inequality,

$$A^2 a^2 \leqslant Aa \iint_{\mathcal{R}} \frac{|f_z + f_{\bar z}|^2}{|f_z|^2 - |f_{\bar z}|^2}\, dx\, dy.$$

Therefore,

$$Aa \leqslant \iint_{\mathcal{R}} \frac{|1 + \mu_f(z)|^2}{1 - |\mu_f(z)|^2}\, dx\, dy \leqslant \frac{1 + k[f]}{1 - k[f]}\, a, \tag{2.5}$$

or,

$$K[f] \geqslant A.$$

Thus, $F_A$ is extremal. To prove that $F_A$ is uniquely extremal, one follows the implication of equality back from (2.5). □

EXAMPLE 1.2.3. We denote the two-dimensional Lebesgue measure of a set $S$ by $|S|$. Suppose $|\Omega| < \infty$. Let $h$ be the restriction of $F_A$ to $\partial\Omega$. Then, as in the preceding example, $K_0 = A$, *and* $F_A$ *is a uniquely extremal extension.* The proof proceeds very much as in Example 1.2.1, by considering the lengths of images of the components of horizontal sections of $\Omega$. Note that, in this example, $\Omega$ can be an arbitrary region. It need not be a Jordan domain, nor need it be bounded.

NOTES. The Beurling–Ahlfors extension (2.2), and, more generally, an extension that is defined by a linear operation on the boundary function, can *not*, except for a restricted class of boundary functions, be used to construct an extremal extension (see Example 4.2.3).

In the problem of quasiconformal reflection that is treated extensively by Kühnau, the boundary values for which we wish to determine an extremal quasiconformal extension to $\{z\colon |z| < 1\}$ are the boundary values on $\{|z| = 1\}$ of a *conformal* mapping of $\{z\colon |z| > 1\}$.

### 1.3. *The extremal problem for n-gons. Teichmüller mappings*

Let $Q$ and $Q'$ be quadrilaterals, that is, Jordan domains with respective cyclically arranged "vertices" $\{z_1, z_2, z_3, z_4\}$, $\{w_1, w_2, w_3, w_4\}$. We can arrange to map $Q$ quasiconformally onto $Q'$ by finding conformal maps, $\Phi : Q \to \mathcal{R}$, $\Psi : Q' \to \mathcal{R}'$, onto rectangles, $\mathcal{R}$, $\mathcal{R}'$, and defining $f : Q \to Q'$ as $f = \Psi^{-1} \circ F_K \circ \Phi$. By Example 1.2.1, we see that if we ask for a mapping of $Q$ onto $Q'$ so that $z_j \mapsto w_j$ and that is free on the edges between the vertices, then $f$ is the uniquely extremal quasiconformal mapping satisfying these requirements. (One can evidently arrange matters so that $\mathcal{R}$, $\mathcal{R}'$ each have unit height, and that $\mathcal{R}'$ is at least as wide as $\mathcal{R}$, as in the example.) Setting $k = (K-1)/(K+1)$, the complex dilatation of $f$ turns out to be

$$\mu_f(z) = k \frac{\overline{\varphi(z)}}{|\varphi(z)|}, \quad z \in Q, \tag{3.1}$$

where $\varphi(z) = [\Phi'(z)]^2$. Assume that $Q$ consists of $\Delta$ with the four distinguished boundary points. Rewriting the integral of inequality (2.5) of Example 1.2.1 as an integral over $\Delta$, we are led to the following:

*Let $f$ be a quasiconformal mapping of $\Delta$, and let $Q$ denote $\Delta$ with four distinguished boundary points. Let*

$$K_Q = \frac{1+k_Q}{1-k_Q} = \frac{M(f(Q))}{M(Q)},$$

*and let $\Phi_Q$ map $Q$ conformally onto a rectangle $\mathcal{R}$ of unit area, $\varphi_Q = \Phi_Q'^2$. Then*

$$\frac{k_Q}{1-k_Q} \leqslant \Re \iint_\Delta \frac{\mu_f}{1-|\mu_f|^2} \varphi_Q \, dx\, dy + \iint_\Delta \frac{|\mu_f|^2}{1-|\mu_f|^2} |\varphi_Q| \, dx\, dy. \tag{3.2}$$

If we vary $z$ on the arcs of $\partial Q$ between the vertices, we have either $\Phi'(z)\,dz$ real or purely imaginary, depending on whether the arc goes to a horizontal or vertical edge of $\mathcal{R}$; thus, in either case, on these arcs,

$$\varphi(z)\,dz^2 = \text{real}. \tag{3.3}$$

One refers to $\varphi(z)$ (more precisely to $\varphi(z)\,dz^2$) as a *quadratic differential*. The condition (3.3) states that this quadratic differential is *real* on the arcs $(z_j, z_{j+1})$, $z_5 = z_1$ of $Q$.

The preceding has far-reaching generalizations. Suppose $n \geqslant 4$. Let $E_n$, $E_n'$ be $n$-gons; that is, Jordan domains that we can assume to be unit disks $\Delta$ in the $z$ and $w$ planes, respectively, with $n$ distinguished boundary points $\{z_j\}$, $\{z_j'\}$.

THEOREM 3.1. *There is a uniquely extremal quasiconformal mapping $f$ of $E_n$ onto $E_n'$. It has a complex dilatation of the form* (3.1), *where $\varphi(z)\,dz^2$ is a quadratic differential, holomorphic in $\Delta$, and real on the arcs $(z_j, z_{j+1})$, $z_{n+1} = z_1$.*

THEOREM 3.2. *Assume that $E_n$ is an $n$-gon consisting of $\Delta$ with $n \geqslant 4$ distinguished boundary points. Let $K_n = (1+k_n)/(1-k_n)$ denote the maximal dilatation of the extremal mapping of $E_n$ onto $f(E_n)$ of Theorem* 3.1, *and $\varphi_n$ the associated quadratic differential, normalized by*

$$\iint_\Delta |\varphi_n(z)|\, dx\, dy = 1.$$

*Then*

$$\frac{k_n}{1-k_n} \leqslant \Re \iint_\Delta \frac{\mu_f}{1-|\mu_f|^2} \varphi_n \, dx\, dy + \iint_\Delta \frac{|\mu_f|^2}{1-|\mu_f|^2} |\varphi_n| \, dx\, dy.$$

REMARKS. Theorem 3.1 is due to Teichmüller. For a proof, see Strebel [155]. For a proof of Theorem 3.2, see [132]. Of course, in (3.1), and in Theorem 3.2 both $k$ and $\varphi$ depend on the location of the vertices of $E_n$ and $E'_n$, not just on the value of $n$. When $n > 5$, the function $\varphi(z)$ may have zeroes in $\Delta$. When $n = 5$, Teichmüller [174] showed that $\Phi(z)$ is single-valued; namely, $\Phi(E_5)$ is a pentagon with horizontal and vertical edges. On the other hand, when $n > 5$, $\varphi(z)$ need not have a single-valued square root. While $f$ itself is single-valued, it will have *local* representations of the form

$$f = \Psi^{-1} \circ F_K \circ \Phi$$

where $\Phi$, $\Psi$ are branches of square roots of holomorphic functions $\varphi(z)$, $\psi(w)$. These functions $\varphi$ and $\psi$ both belong to $L^1(\Delta)$, $\psi(w)\,dw^2$ being a quadratic differential that is real on the arcs between the vertices of $E'_n$. With the help of the reflection principle one sees that both $\varphi(z)$ and $\psi(w)$ are *rational* functions with at worst first-order poles at the vertices of $E_n$ and $E'_n$ respectively. (In the case $n = 4$, this is just the Schwarz–Christoffel formula.)

In studying extremal quasiconformal mappings of a region, the $L^1$ norms of functions analytic in that region play a special role. We let

$$L^1_a(\Omega) = \{\varphi\colon\ \varphi(z) \text{ is analytic in } \Omega,\ \varphi \in L^1(\Omega)\}.$$

When speaking of the norm of a quadratic differential $\varphi$ that is defined in $\Omega$, it will be understood that we are referring to the $L^1$ norm,

$$\|\varphi\| = \iint_\Omega |\varphi(z)|\,dx\,dy.$$

A mapping $f$ of the type (3.1), where $\|\varphi\| < \infty$, is said to be a *Teichmüller mapping with finite norm.* For example, the uniquely extremal mapping of $E_n$ onto $E'_n$ is of this sort, where, when $n = 4$, $\|\varphi\| = |\mathcal{R}|$, $\mathcal{R}$ being the auxiliary rectangle. On the other hand, if $s > 0$, $s \neq 1$, then the $f$ of Example 1.1.1 is a Teichmüller mapping with *infinite* norm, since for it, $\varphi(z) = \pm 1/z^2$, which does not belong to $L^1$ over $C$.

If $f$ is a Teichmüller mapping with $\mu_f(z) = k\overline{\varphi(z)}/|\varphi(z)|$, and if the mapping $z \leftrightarrow \tilde{z}$ effects a conformal change of domain from $\Omega$ to $\tilde{\Omega}$, then $f(z)$ is replaced by $\tilde{f}(\tilde{z})$, where

$$\mu_{\tilde{f}}(\tilde{z}) = \tilde{k}\,\frac{\overline{\tilde{\varphi}(\tilde{z})}}{|\tilde{\varphi}(\tilde{z})|}, \quad \tilde{k} = k, \quad \tilde{\varphi}(\tilde{z})\,d\tilde{z}^2 = \varphi(z)\,dz^2, \quad \|\tilde{\varphi}\| = \|\varphi\|. \tag{3.4}$$

So, $\tilde{f}$ is also a Teichmüller mapping, and the norm stays invariant.

## 1.4. *The extremal problem for given boundary values for the disk. Early results*

Ideally, one would like to start with a quasisymmetric homeomorphism $h$ of $\partial\Delta$ onto $\partial\Delta$, and "determine" $K_0$ and an extremal extension $f_0$ more or less effectively from it. A theoretical way of doing this is as follows. Choose a dense set of points $\{z_j\}$ on $\partial\Delta$, and use it to determine $E_n$, and $E'_n = h(E_n)$ $(n = 4, 5, \ldots)$. For given $n$, there is, as per Theorem 3.1, a corresponding uniquely extremal $K_n$-qc Teichmüller mapping $f_n$ with an associated quadratic differential $\varphi_n$ of finite norm. It is not difficult to show that $K_n \nearrow K_0$, and that the sequence $\{f_n\}$ will have a subsequence converging uniformly on the closed disk. The limit of such a subsequence will be an extremal mapping $f_0$. While this procedure is theoretically important it has not led to any kind of direct representation of $f_0$ in terms of $h$, the major hindrance being the lack of a hold on how $\varphi_n$ depends on $n$. The theory has sidestepped this problem in two different ways. In the theoretical development, instead of specifying the boundary values explicitly, one usually specifies a *complex dilatation*, and asks whether or not a quasiconformal mapping with that complex dilatation is or is not extremal or respectively uniquely extremal. In this case, it is the boundary correspondence induced by the given complex dilatation that is thought of as given.[2] In explicit examples, on the other hand, it is only by luck that an extremal extension for a particular boundary correspondence is determined, the fact of the case being that the result often has been obtained by working backwards. [3]

In the development of the theory of extremal quasiconformal mappings, the Teichmüller mappings have constituted a major focus. For extremal mappings of $n$-gons as well as mappings subject to other special types of side conditions for which variational approaches have been successful (see Belinskii [11], Schiffer [143], Renelt [138]). Teichmüller mappings are, in fact, the only extremal mappings. When we look for extremal mappings with given boundary values, Teichmüller mappings still play a very important role, but they are not the only possibility.

After affine mappings of regions of finite area, the next most elementary case of a Teichmüller mapping with quadratic differential $\varphi$ occurs when $\sqrt{\varphi(z)}$ is the derivative of a univalent function, or, more generally, single-valued. We will denote by $L^1_{ss}(\Delta)$ the subclass of $L^1_a(\Delta)$ consisting of functions with single-valued square roots. If $\varphi \in L^1_{ss}(\Delta)$, then $\Phi(z) = \int \sqrt{\varphi(z)}\, dz$ is also single-valued, and $\Phi(\Delta)$ can be thought of as a one- or several-sheeted surface that is stretched horizontally by $F_K$. Since $f$ is given locally as $\Psi^{-1} \circ F_K \circ \Phi$, $\Psi$ also turns out to be single-valued. Applying the length-area method used for Example 1.2.2, one concludes in this elementary case that the Teichmüller mapping is uniquely extremal for its boundary values. It turns out, however, that the conclusion holds whether or not we are in the elementary case. By using the *Teichmüller metric* (Teichmüller [173], Jenkins [58], Strebel [154]), $\int |\varphi(z)|^{1/2}|dz|$ in $\Delta$, in place of where the Euclidean metric in $\Phi(\Delta)$ is used in the elementary case, Strebel proved the following:

THEOREM 4.1. *Suppose $f$ is a Teichmüller mapping of $\Delta$ with a quadratic differential of finite norm. Then $f$ is uniquely extremal for its boundary values.*

EXAMPLE 1.4.1. $\mu_f(z) = k\bar{z}/|z|$, $z \in \Delta$. Since $\varphi(z)$ has a simple zero, $\varphi \in L^1_a(\Delta) \setminus L^1_{ss}(\Delta)$.

[2] It is convenient to consider two boundary values as equivalent if they are induced by mappings $f_1$, $f_2$, where $f_1^{-1} \circ f_2$ is conformal.

[3] For mappings close to the identity (cf. Section 6) a more direct procedure as described in [123] is sometimes available.

A Teichmüller mapping with infinite norm need not be extremal for its boundary values. The simplest example is the affine stretch $F_A$ of the upper-half plane $H^+ = \{z = x + \mathrm{i}y\colon\ y > 0\}$. The conformal map $w = Az$ has the same boundary values.

On the other hand, finiteness of the norm of a Teichmüller mapping is not necessary for unique extremality of Teichmüller mappings. This was already shown by Beurling and Ahlfors in [17]:

EXAMPLE 1.4.2.[4] For $s \neq 0$, $z \in H^+$, let $w = f(z)$ be the mapping of Example 1.1.1. Since

$$\iint_{H^+} \left|\frac{1}{z^2}\right| dx\, dy = \infty,$$

$f$ is a Teichmüller mapping of $H^+$ onto $H^+$ with infinite norm. By estimating moduli of quadrilaterals inscribed in $H^+$, Beurling and Ahlfors showed that $f$ was extremal for its boundary values on the real axis. Now, if one replaces $z$, $w$, by $\zeta = \log z$, $\omega = \log w$, respectively, the mapping is transformed to a horizontal stretch of the strip $\{0 < \Im\zeta < \pi\}$ onto the strip $\{0 < \Im\omega < \pi\}$ (if $0 < s < 1$ the "stretch" is actually a compression). It turns out (Strebel [153]; this is also a corollary of the next example) that $f$ is actually uniquely extremal.

EXAMPLE 1.4.3 [153]. One forms a region $\Omega$ by attaching up to 4 "arms" (i.e., half-strips) whose sides are parallel to the coordinate axes to a region $\mathcal{D}$, at most one right arm of the type $\{z\colon \Re z > x_1,\ y_1 < \Im z < y_2\}$, at most one left arm, at most one upper arm, and at most one lower arm. It is required that there is at least one arm, and that cross cuts of any arm, located sufficiently far out constitute cross cuts of $\Omega$. *Let $f$ be a quasiconformal mapping of $\Omega$ with the same boundary values on $\partial\Omega$ as $F_A$. Then $K[f] \geqslant A$; thus $K_0 = A$. If $|\mathcal{D}| < \infty$ then $K_0 = A$ and $F_A$ is uniquely extremal.* In the proof an inequality of Grötzsch is used to estimate the moduli of quadrilaterals one side of which is far out on an arm.

EXAMPLE 1.4.4 (*"Strebel's Chimney"*.). This is a famous example, namely the first example of a quasiconformal mapping that is extremal for its boundary values, but not uniquely extremal. Form $\Omega$ as in the preceding example, using $\mathcal{D} = \{z\colon\ \Im z < 0\}$, and attaching the upper arm $\{z\colon\ 0 < \Re z < 1,\ \Im z > -1\}$. According to Example 1.4.3, $F_A$ is extremal for the boundary values it induces. On the other hand, the quasiconformal mapping

$$f_1(z) = \begin{cases} F_A(z) & (0 < \Re z < 1,\ \Im z \geqslant 0),\\ Az & (\Im z < 0), \end{cases}$$

has the same boundary values as $F_A$ and $K[f_1] = K[F_A] = A$; that is, $f_1$ is also extremal, $f_1 \neq F_A$. Moreover, if we define

$$f_2(x + \mathrm{i}y) = \begin{cases} Ax + \mathrm{i}y & (0 < x < 1,\ y \geqslant 0),\\ A(x + \mathrm{i}Ay) & (y < 0), \end{cases}$$

[4]We will come back to some of the examples that follow by other methods in Sections 2 and 3.

then $f_2$ has the same boundary values as $F_A$, $f_2$ and $F_A$ are both extremal, $f_2 \neq F_A$, even though we have $D_{f_2} = D_{F_A}$.

EXAMPLE 1.4.5 (*Affine stretch of angular regions* [153]). Let

$$\Omega_\alpha = \{z\colon\ -\alpha/2 < \arg z < \alpha/2\} \quad (0 < \alpha < 2\pi).$$

It turns out that the mapping $w = F_A(z)$ is not extremal. For $\alpha = \pi$, this is clear, as the identity has the same boundary values as $F_A$. For $\alpha \neq \pi$, one is able to identify the extremal mapping by luck. Let $\zeta = \log z$, $\lambda = \log w$. A computation shows that the strip $S = \{\zeta\colon\ |\Im\zeta| < \alpha/2\}$ is mapped onto the strip $S' = \{\lambda\colon\ |\Im\lambda| < \beta/2\}$, where

$$\beta = 2\tan^{-1}\left[\frac{1}{A}\tan\frac{\alpha}{2}\right].$$

The boundary values $\zeta \to \lambda$ amount to a shift to the right in the amount of

$$s = \log\sqrt{A^2\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2}}.$$

By Example 1.4.2, there is a uniquely extremal mapping from $S$ to $S'$. It is an affine mapping consisting of a shift to the right by the amount $s$, and a compression by the factor $\alpha/\beta$. Going back to $z \to w$, we conclude that there is a uniquely extremal Teichmüller mapping with infinite norm in this case, and

$$K_0 = \frac{\alpha}{\beta} = \frac{\alpha}{2\tan^{-1}[\frac{1}{A}\tan\frac{\alpha}{2}]} < A.$$

EXAMPLE 1.4.6 (*Affine stretch of parabola-type regions*). We consider the regions

$$G_\beta = \{z = x + \mathrm{i}y\colon\ x > |y|^\beta\} \quad (1 \leqslant \beta \leqslant \infty\},$$

with the boundary values of the affine stretch $F_A$. By $G_\infty$ we mean the half-strip

$$G_\infty = \{z = x + \mathrm{i}y\colon\ x > 0,\ |y| < 1\}.$$

This is a very useful and instructive family of boundary value problems as it is possible to make very precise answers regarding extremality and unique extremality, and the results cover the entire spectrum of possibilities. Partial results were obtained by Sethares [144] and Blum [20], and completed by Reich and Strebel [129]. Referring to Example 1.4.5, we have $G_1 = \Omega_{\pi/2}$. Therefore, for $\beta = 1$, $F_A$ is *not* extremal. As $\beta$ increases, the opening of $G_\beta$ at infinity decreases. One can therefore expect the chances that $F_A$ is extremal to increase. In fact, Sethares showed that $F_A$ is extremal when $\beta > 1$. For $\beta = \infty$, we know by Example 1.4.3, that $F_A$ is uniquely extremal. In 1969, Blum proved that $F_A$ was actually uniquely extremal when $\beta > 3$. This was then also established in the more delicate case

$\beta = 3$ in [129]. The techniques for all this were extensions and refinements of those of [153]. Finally [129], when $1 < \beta < 3$ it was possible to construct explicit mappings $f$ as variations of $F_A$ but still with the same boundary values as $F_A$, for which $D_f(z) < A$, $(z \in G_\beta)$. In summary,

$$F_A \text{ is } \begin{cases} \text{not extremal} & \text{when } \beta = 1, \\ \text{extremal but not uniquely extremal} & \text{when } 1 < \beta < 3, \\ \text{uniquely extremal} & \text{when } 3 \leqslant \beta \leqslant \infty. \end{cases}$$

NOTE. The numbers $K_n$ of Section 1.3 can be related quantitatively to $K_0$. To clarify this let us temporarily replace the symbol $K_n$ by the more informative notation $K(E_n; h)$; that is, $K(E_n; h)$ denotes the dilatation of the extremal Teichmüller mapping of the $n$-gon $E_n$ onto the $n$-gon $h(E_n)$. For a fixed quasisymmetric $h$, and a fixed $n$, $(n \geqslant 4)$, let

$$\mathcal{K}_n[h] = \sup\{K(E_n; h)\colon\ E_n \text{ is an } n\text{-gon}\}, \quad n = 4, 5, 6, \ldots.$$

It is of course clear that

$$\mathcal{K}_n[h] \leqslant \mathcal{K}_{n+1}[h] \leqslant K_0[h],$$

and, in line with the remarks at the beginning of this section,

$$\lim_{n \to \infty} \mathcal{K}_n[h] = K_0[h].$$

As is to be expected,[5] it does not in general happen that $\mathcal{K}_n[h] = K_0[h]$ for some finite $n$. Conditions on $h$ under which this *could* occur have been investigated by Anderson and Hinkkanen [7], Reich [124], Wu [176], Strebel [172], and others. On the other hand, the construction of Beurling and Ahlfors [17] implies that knowledge of $\mathcal{K}_4[h]$ can be used to provide an upper bound for $K_0[h]$. This can be done by first bounding [17] the quasisymmetry constant $\rho$ of (2.1) in terms of $\mathcal{K}_4$, and then combining this with a bound for $K_0$ in terms of $\rho$. A good bound of the latter type is due to Lehtinen [81]; namely,

$$K_0 \leqslant \min\{\rho^{3/2}, 2\rho - 1\}.$$

## 1.5. *Relationship to harmonic mappings*

If $h$ is a homeomorphism of $\partial\Delta$ onto itself then, according to a result of Kneser and Radó [60], the complex-harmonic Poisson integral extension $P(h; z)$ of $h$ to $\Delta$ is a homeomorphism of $\Delta$. Conditions under which $P(h; z)$ is quasiconformal (in general, it won't be) were found by Martio [92]. Except in very unusual circumstances, $P(h; z)$ will, however, not be extremal. The possibility that harmonic mappings are extremal mappings does however present itself if by "harmonic" we mean harmonic with respect to a *weight* function.

[5] Kühnau [80] points out that this follows, e.g., from an inequality in [74] involving Fredholm eigenvalues.

Let $\rho(w)$, denote a measurable weight function defined for $|w| < 1$; that is, $\rho(w) \geqslant 0$, $\iint \rho(w)\,du\,dv = 1$ $(w = u + iv)$. Let $\mathcal{W}(\Delta)$ denote the class of all such weight functions over $\Delta$. Let $\mathcal{D}_\rho[f]$ denote the Douglas–Dirichlet functional

$$\mathcal{D}_\rho[f] = \iint_\Delta \left(|f_z|^2 + |f_{\bar z}|^2\right)\rho\big(f(z)\big)\,dx\,dy, \tag{5.1}$$

where we assume that $f \in \mathcal{Q}[h]$, and $\rho \in \mathcal{W}(\Delta)$. It turns that our extremal dilatation $K_0[h]$ can be determined by a variational problem for $\mathcal{D}_\rho[f]$.

THEOREM 5.1.

$$\sup_{\rho\in\mathcal{W}(\Delta)} \inf_{f\in\mathcal{Q}[h]} \mathcal{D}_\rho[f] = \frac{1}{2}\left(K_0[h] + \frac{1}{K_0[h]}\right).$$

PROOF. We can rewrite the inequality of Theorem 3.2 as

$$\iint_\Delta D_f(z)\big|\varphi_n(z)\big|\,dx\,dy = \iint_\Delta \frac{1+|\mu_f(z)|}{1-|\mu_f(z)|}\big|\varphi_n(z)\big|\,dx\,dy \geqslant K_n. \tag{5.2}$$

Let $z = g(w)$ be the mapping inverse to $w = f(z)$. Setting $G(t) = (1/2)(t + 1/t)$, we have

$$\mathcal{D}_\rho[f] = \iint_\Delta G\big(D_g(w)\big)\rho(w)\,du\,dv. \tag{5.3}$$

Since $G(t)$ is an increasing convex function of $t$, Jensen's inequality applied to (5.2) gives

$$\iint_\Delta G\big(D_f(z)\big)\big|\varphi_n(z)\big|\,dx\,dy \geqslant G(K_n),$$

for all $f \in \mathcal{Q}[h]$, and $n = 4, 5, \ldots$. Since $|\varphi_n| \in \mathcal{W}(\Delta)$, this implies that

$$\sup_{\rho\in\mathcal{W}(\Delta)} \inf_{f\in\mathcal{Q}[h]} \iint_\Delta G\big(D_f(z)\big)\rho(z)\,dx\,dy \geqslant G(K_n).$$

Since the left side is independent of $n$, the right side can be replaced by $G(K_0[h])$. Moreover, if $f_0$ is an extremal extension of $h$,

$$\iint_\Delta G\big(D_{f_0}(z)\big)\rho(z)\,dx\,dy \leqslant \iint_\Delta G\big(K_0[h]\big)\rho(z)\,dx\,dy = G\big(K_0[h]\big).$$

Thus,

$$\sup_{\rho\in\mathcal{W}(\Delta)} \inf_{f\in\mathcal{Q}[h]} \iint_\Delta G\big(D_f(z)\big)\rho(z)\,dx\,dy = G\big(K_0[h]\big).$$

Interchanging the roles of $z$ and $w$, and applying (5.3), the theorem follows. □

NOTES. Theorem 5.1 [114] verified a general heuristic principle formulated in 1954 by Gerstenhaber and Rauch [46] in the special case of boundary values on the circumference of the disk. For further results, see [105,135,90,98]. Since harmonic mappings will not enter the main part of this exposition, we shall, however, not be pursuing the topic in this article further.

## 2. The Hamilton–Krushkal condition – necessity

### 2.1. *Statement of the condition. Consequences. Examples*

Suppose $\mu(z)$ is a complex dilatation, $z \in \Delta$; that is, $\mu \in L^{\infty}(\Delta)$, and $\|\mu\|_{\infty} < 1$. If $f$ is a quasiconformal mapping of $\Delta$ with complex dilatation $\mu$, then whether or not $f$ is extremal for its boundary values on $\partial\Delta$ is, in line with the discussion in Section 1.4, completely determined by $\mu$. For brevity, we shall refer to a complex dilatation as extremal or uniquely extremal, as the case may be, to mean that a mapping with that complex dilatation is extremal or uniquely extremal.

In order to formulate the H–K (Hamilton–Krushkal) [6] condition, introduce the linear functional

$$\Lambda_{\mu}[\varphi] = \iint_{\Delta} \mu(z)\varphi(z)\,dx\,dy \quad \big(\varphi \in L^1_a(\Delta)\big). \tag{1.1}$$

$L^1_a(\Delta)$ is a Banach space. As usual, we denote the norm of $\Lambda_{\mu}$ over it by

$$\|\Lambda_{\mu}\| = \sup\big\{\big|\Lambda_{\mu}[\varphi]\big| \colon \varphi \in L^1_a(\Delta),\ \|\varphi\| = 1\big\}.$$

In this expression, $\|\varphi\|$ is our notation for the $L^1$ norm, as introduced in Section 1.3.

THEOREM 1.1 (H–K). *If $\mu$ is extremal, then*

$$\|\Lambda_{\mu}\| = \|\mu\|_{\infty}. \tag{1.2}$$

One speaks of a sequence, $\varphi_n \in L^1_a(\Delta)$, $n = 1, 2, \ldots$, as a *Hamilton sequence* for $\mu$, if $\|\varphi_n\| = 1$, $\{\varphi_n\}$ is loc. unif. convergent in $\Delta$, and $\lim \Lambda_{\mu}[\varphi_n] = \|\Lambda_{\mu}\|$. Since $L^1_a(\Delta)$ is a normal family, it is a consequence of (1.2) that a Hamilton sequence exists, and, by choosing a subsequence if necessary, one can assume that the Hamilton sequence has a limit function

$$\lim_{n\to\infty} \varphi_n(z) = \varphi_0(z) \in L^1_a(\Delta), \qquad \|\varphi_0\| \leqslant 1.$$

[6] The condition was given by Hamilton [48] in the form stated here (for arbitrary regions $\Omega$, not just $\Delta$). Independently, it was found by Krushkal [62] under the restriction that $|\mu(z)| = \text{const}$. As the affine stretch of the chimney, Example 1.4.4, shows, it is possible for $f$ to be extremal even though $\mu_f(z)$ has non-constant absolute value. So the fact that one should a-priori allow $\mu$ to have non-constant absolute value is not merely an apparent advantage.

We proceed to an important consequence [49,104,158] of the H–K condition:

THEOREM 1.2. *Suppose $\mu$ is extremal, and $\varphi_0$ is a limit function of a Hamilton sequence $\{\varphi_n\}$. Then either $\varphi_0(z) \equiv 0$ or $\mu$ is a complex dilatation of Teichmüller type with finite norm:*[7]

$$\mu(z) = \|\mu\|_\infty \frac{\overline{\varphi_0(z)}}{|\varphi_0(z)|}, \quad \textit{for a.a. } z \in \Delta.$$

In case $\varphi_0(z) \equiv 0$, one says that the Hamilton sequence *degenerates*.

COROLLARY [48]. *Suppose $\mu$ is extremal. Then for every compact subset $S$ of $\Delta$,*

$$\operatorname{ess\,sup}\{|\mu(z)|: z \in \Delta \setminus S\} = \|\mu\|_\infty.$$

For further necessary conditions on extremal complex dilatations that follow from Theorem 1.1, see Belna and Ortel [12] and Ortel and Smith [97].

We proceed with some examples.

EXAMPLE 2.1.1 (*Affine stretch of angular regions*). We illustrate the application of Theorem 1.1 by giving an alternative proof [103] of the fact (Example 1.4.5) that the affine stretch of the angular region $\Omega_\alpha$ is not extremal. Since the complex dilatation of an affine stretch is constant, it follows from Theorem 1.1 that for the affine stretch of a simply connected region $\Omega$ to be extremal, it is necessary that

$$\|\Lambda_1\| = \sup\left\{ \frac{|\iint_\Omega \varphi(z)\,dx\,dy|}{\iint_\Omega |\varphi(z)|\,dx\,dy}: \varphi \in L_a^1(\Omega)\right\} = 1. \tag{1.3}$$

Let $S = \{w = u + \mathrm{i}v: -\alpha/2 < v < \alpha/2\}$. Then

$$I = \iint_{\Omega_\alpha} \varphi(z)\,dx\,dy = \iint_S \mathrm{e}^{-2\mathrm{i}v} f(w)\,du\,dv, \quad f(w) = \mathrm{e}^{2w}\varphi(\mathrm{e}^w),$$

and

$$J = \iint_{\Omega_\alpha} |\varphi(z)|\,dx\,dy = \iint_S |f(w)|\,du\,dv$$
$$= \int_{-\alpha/2}^{\alpha/2} dv \int_{-\infty}^{\infty} |f(u+\mathrm{i}v)|\,dv.$$

Since $J < \infty$, $\int_{-\infty}^{\infty} f(u+\mathrm{i}v)\,du$ exists for a.a. $v$, and since $f(w)$ is holomorphic in $S$,

$$\int_{-\infty}^{\infty} f(u+\mathrm{i}v)\,du = c = \text{const} \quad \text{for a.a. } v.$$

[7]The norm refers to the $L^1$ norm of $\varphi_0$ over $\Delta$.

Therefore,

$$I = c\int_{-\alpha/2}^{\alpha/2} e^{-2iv}\,dv = c\sin\alpha, \quad \alpha|c| \leqslant J.$$

Hence, $\|\Lambda_1\| \leqslant |\sin\alpha|/\alpha < 1$, which shows again that the affine stretch of $\Omega_\alpha$ is not extremal.[8]

In connection with the question whether or not (1.2) holds for a particular $\mu$, the "null" class,

$$\mathcal{N}(\Omega) = \left\{ \nu \in L^\infty(\Omega)\colon \iint_\Omega \nu(z)\varphi(z)\,dx\,dy = 0 \text{ for all } \varphi \in L_a^1(\Omega) \right\},$$

plays an important role. If there exist $\nu \in \mathcal{N}(\Omega)$ such that $\|\mu - \nu\|_\infty < \|\mu\|_\infty$, then (1.2) evidently does not hold. We illustrate this by outlining the procedure with an example:

EXAMPLE 2.1.2. Let $k$ be a constant, $0 < k < 1$. We define $\mu(z)$, for a.a. $z \in \Delta$, as $\mu(z) = k\kappa(z)$, where

$$\kappa(z) = \begin{cases} +1, & z \in \Delta^+ = \{z \in \Delta\colon \Im z > 0\}, \\ -1, & z \in \Delta^- = \{z \in \Delta\colon \Im z < 0\}. \end{cases}$$

The problem as to whether or not (1.2) is satisfied reduces to determining whether or not

$$\|\Lambda_\kappa\| = \sup \frac{\left|\iint_{\Delta^+} \varphi(z)\,dx\,dy - \iint_{\Delta^-} \varphi(z)\,dx\,dy\right|}{\iint_\Delta |\varphi(z)|\,dx\,dy},$$

where the sup is over all $\varphi \in L_a^1(\Delta)$, is strictly less than 1 or not.[9] The question is not at all trivial as, for example, by Runge's Theorem (see, e.g., [18]), there exists a sequence of polynomials in $z$ with limit $\kappa(z)$, $z \in \Delta^+ \cup \Delta^-$, suggesting that $\|\Lambda_\kappa\|$ might be 1.

SKETCH OF PROOF THAT $\|\Lambda_\kappa\| < 1$. It suffices to find a function $\nu \in L^\infty(\Delta)$, such that

$$\Re\nu(z) \geqslant \delta > 0 \quad (z \in \Delta^+), \qquad \Re\nu(z) \leqslant -\delta \quad (z \in \Delta^-), \tag{1.4}$$

as with such a $\nu$ we get $\|\kappa - t\nu\|_\infty < 1$ when $t > 0$ is sufficiently small. Let

$$\nu_0(z) = -ie^{i\theta} = \sin\theta - i\cos\theta, \quad z = re^{i\theta} \in \Delta.$$

Since,

$$\iint_\Delta \nu_0(z)z^n\,dx\,dy = 0, \quad n = 0, 1, \ldots,$$

[8] Actually, $\|\Lambda_1\| = |\sin\alpha|/\alpha$.

[9] Note that the sup could be taken just over polynomials $\varphi(z)$ as these are dense in $L_a^1(\Delta)$ in the norm of $L^1(\Delta)$.

we have $\nu_0 \in \mathcal{N}(\Delta)$. In view of the fact that $\Re\nu_0(z) \geqslant 0$ for $z \in \Delta^+$, and $\Re\nu_0(z) \leqslant 0$ for $z \in \Delta^-$, we see that (1.4) is "almost" satisfied. With some modifications of $\nu_0$, [104], it is possible to transform "almost" into "actual".

Our $\mu(z)$ is the complex dilatation of the mapping

$$f(z) = \begin{cases} x + \mathrm{i}y/K, & z = x + \mathrm{i}y \in \Delta^+, \\ x + \mathrm{i}Ky, & z = x + \mathrm{i}y \in \Delta^-, \end{cases} \quad \left(K = \frac{1+k}{1-k}\right).$$

The conclusion that $\mu$ is not the complex dilatation of an extremal mapping was originally carried out directly by means of a geometric variation of $f$ [103]. A third proof is due to Harrington and Ortel [49], and a fourth proof follows from a general theorem of these authors in [50]. □

EXAMPLE 2.1.3. Suppose $\mu(z)$, $z \in \Delta$, is extremal, $k = \|\mu\|_\infty > 0$, and for some $\rho$, $0 < \rho < 1$,

$$\mu(z) = s(z)\frac{\overline{\varphi(z)}}{|\varphi(z)|}, \quad z \in A = \{z\colon\ \rho \leqslant |z| < 1\},$$

where

(i) $s(z)$ is a non-negative measurable function in $A$,
(ii) $\varphi(z)$ is holomorphic in $A$, $|\varphi(z)| \geqslant m > 0$ $(z \in A)$,
(iii) $J = \iint_A |\varphi(z)|\,dx\,dy < \infty$.

*It follows that $\varphi$ has a holomorphic extension to all of $\Delta$, $s(z) \equiv k$, and*

$$\mu(z) = k\frac{\overline{\varphi(z)}}{|\varphi(z)|}, \quad z \in \Delta.$$

PROOF. Let

$$S = \{z \in A\colon\ |\mu(z)| \leqslant k/2\}.$$

By Theorem 1.1 we know that there is a Hamilton sequence $\{\varphi_n\}$ such that

$$\delta_n = \iint_\Delta \big[k|\varphi_n| - \Re(\mu\varphi_n)\big]\,dx\,dy = \mathrm{o}(1), \quad n \to \infty. \tag{1.5}$$

Thus,

$$\iint_S |\varphi_n|\,dx\,dy = \mathrm{o}(1), \tag{1.6}$$

and, therefore, in view of (ii),

$$\iint_S |\varphi_n/\varphi|\,dx\,dy = \mathrm{o}(1). \tag{1.7}$$

Suppose the Hamilton sequence degenerated. Then, by (1.6) and (1.7),

$$\delta_n = \iint_{A\setminus S} \left[k|\varphi_n| - \Re(\mu\varphi_n)\right] dx\,dy + o(1)$$
$$= \iint_{A\setminus S} \left[\frac{k}{s}\left|\frac{\varphi_n}{\varphi}\right| - \Re\left(\frac{\varphi_n}{\varphi}\right)\right] s|\varphi|\,dx\,dy + o(1).$$

Since the integrand is non-negative,

$$\delta_n \geqslant \frac{km}{2} \iint_{A\setminus S} \left[\frac{k}{s}\left|\frac{\varphi_n}{\varphi}\right| - \Re\left(\frac{\varphi_n}{\varphi}\right)\right] dx\,dy + o(1). \tag{1.8}$$

Since we are assuming that $\{\varphi_n\}$ degenerates,

$$\iint_A \frac{\varphi_n}{\varphi}\,dx\,dy = \frac{1-\rho^2}{2}\int_0^{2\pi} \frac{\varphi_n(\rho e^{i\theta})}{\varphi(\rho e^{i\theta})}\,d\theta = o(1). \tag{1.9}$$

By Schwarz's Inequality,

$$\iint_A \left|\frac{\varphi_n}{\varphi}\right| dx\,dy \geqslant \frac{\left(\iint_A |\varphi_n|\,dx\,dy\right)^2}{J} = \frac{1+o(1)}{J}.$$

By (1.7), (1.8), (1.9), therefore,

$$\delta_n \geqslant \frac{km}{2}\iint_{A\setminus S} \frac{k}{s}\left|\frac{\varphi_n}{\varphi}\right| dx\,dy + o(1)$$
$$\geqslant \frac{km}{2}\iint_{A\setminus S} \left|\frac{\varphi_n}{\varphi}\right| dx\,dy + o(1)$$
$$\geqslant \frac{km}{2J} + o(1),$$

which contradicts (1.5). The conclusion now follows easily by Theorem 1.2. □

NOTES. It was already pointed out by Hamilton [48] that for extremal complex dilatations $\mu(z)$, the essential maximum of $|\mu(z)|$ must occur on the boundary. Example 2.1.3 is based on a communication from V. Božin, V. Marković and M. Mateljević. For other corollaries of the Hamilton–Krushkal condition see Ortel [95] and Ortel and Smith [97].

### **2.2.** *Proof of the Hamilton–Krushkal theorem for the disk*

We outline a proof following the method of Krushkal [62] as modified in [131]. Krushkal's proof uses a key lemma that we state here for the case of the unit disk:

LEMMA 2.1 (Fundamental variational lemma). *Suppose $g(z,t), z \in \Delta$, is a quasiconformal mapping for $0 < t < \delta$, and $\mu_g(z,t) = t\nu(z) + \varepsilon(z,t)$, where $\nu \in \mathcal{N}(\Delta)$, and $\|\varepsilon(\cdot,t)\|_\infty = \mathrm{o}(t)$ $(t \to 0)$. There exists a quasiconformal mapping $h(z,t)$ of $\Delta$ with the same boundary values as $g(z,t)$, such that*

$$\|\mu_h(\cdot,t)\|_\infty = \mathrm{o}(t) \quad (t \to 0). \tag{2.1}$$

REMARKS. The lemma, for the case of general regions, not just the disk, is usually credited to Bers, e.g., [13]. If the H–K is proved without using the lemma, then the lemma itself is also a *consequence* of the H–K theorem (see Reich and Strebel [132], Theorem 13, where the derivation of the H–K theorem is by means of limiting polygonal mappings, sidestepping the lemma. Also see footnote 12). For a fairly elementary constructive derivation of the lemma, see [120].

If $\mu_g(z,t) = t\nu(z)$, $\nu \in \mathcal{N}(\Delta)$, $\|\nu\|_\infty = 1$, then [117] one can determine $h$ so that $\|\mu_h(\cdot,t)\|_\infty = t^2 + \mathrm{O}(t^3)$; the coefficient of $t^2$ is best possible.

PROOF OF THEOREM 1.1. By the Hahn–Banach theorem, $\Lambda_\mu$ can be extended to a linear functional on $L^1(\Delta)$ without increase in norm. By the Riesz representation theorem, there therefore exists a function $\mu_1 \in L^\infty(\Delta)$, with $k_1 = \|\mu_1\|_\infty = \|\Lambda_\mu\| \leqslant \|\mu\|_\infty = k$, such that the extended functional has the representation

$$\Lambda_\mu[\varphi] = \iint_\Delta \mu_1(z)\varphi(z)\,dx\,dy, \quad \varphi \in L^1(\Delta).$$

Thus,

$$\nu = \mu - \mu_1 \in \mathcal{N}(\Delta). \tag{2.2}$$

We show that the assumption $k_1 < k$ leads to a contradiction.

Suppose $f: \Delta_z \to \Delta_w$ is a quasiconformal mapping with complex dilatation $\mu(z)$, and $g: \Delta_z \to \Delta_\zeta$ is a quasiconformal mapping with complex dilatation $t\nu(z)$. Consider $w = F(\zeta,t) = (f \circ g^{-1})(\zeta)$, $t > 0$. The complex dilatation of $F(\zeta,t)$ is

$$\mu_F(\zeta,t) = \frac{g_z}{\overline{g_z}} \frac{\mu(z) - t\nu(z)}{1 - t\overline{\nu(z)}\mu(z)}, \quad \zeta = g(z,t).$$

We will show that there exist $t_0(k_1,k) > 0$, $\delta(k_1,k) > 0$, such that

$$|\mu_F(\zeta,t)| \leqslant k - \delta t, \quad 0 \leqslant t \leqslant t_0,\ z \in \Delta. \tag{2.3}$$

First consider

$$S_1 = \left\{ z \in \Delta\colon |\mu(z)| < \frac{k_1 + k}{2} \right\}.$$

By comparing the value of $\mu_F(\zeta,t)$ with that of $\mu_F(\zeta,0)$, it is clear that there exist $\delta_1>0$, $t_1>0$, such that

$$\left|\mu_F(\zeta,t)\right|\leqslant k-\delta_1 t,\quad 0\leqslant t\leqslant t_1,\ z\in S_1.$$

Next, consider

$$S_2=\left\{z\in\Delta\colon\ \frac{k_1+k}{2}\leqslant\left|\mu(z)\right|<k\right\}.$$

As

$$\left|\mu_F(\zeta,t)\right|^2=\frac{|\mu|^2-2t\Re(\nu\bar{\mu})+t^2|\nu|^2}{1-2t\Re(\nu\bar{\mu})+t^2|\nu|^2|\mu|^2},$$

we get for $z\in S_2$ the development

$$\left|\mu_F(\zeta,t)\right|=\left|\mu(z)\right|-t\frac{1-|\mu(z)|^2}{|\mu(z)|}\Re\big(\nu(z)\overline{\mu(z)}\big)+\mathrm{O}\big(t^2\big),\tag{2.4}$$

where the $\mathrm{O}(t^2)$ term is uniform with respect to $z$. For $z\in S_2$,

$$\Re(\nu\bar{\mu})=\Re\big[(\mu-\mu_1)\bar{\mu}\big]\geqslant|\mu|^2-|\mu_1\mu|\geqslant\left(\frac{k_1+k}{2}\right)^2-k_1k=\frac{(k-k_1)^2}{4}.$$

In $S_2$, the coefficient of $t$ in (2.4) is therefore bounded below by

$$\frac{1-|\mu|^2}{|\mu|}\Re(\nu\bar{\mu})\geqslant\frac{1-k^2}{k}\cdot\frac{(k-k_1)^2}{4}.$$

Therefore we can find $\delta_2>0$ and $t_2>0$ such that for $z\in S_2$,

$$\left|\mu_F(\zeta,t)\right|\leqslant k-\delta_2 t,\quad 0\leqslant t\leqslant t_2.$$

Thus, for $t_0=\min(t_1,t_2)$, $\delta_0=\min(\delta_1,\delta_2)$, (2.3) is valid.

Now, by (2.2) and Lemma 2.1, there exists a mapping $\zeta=h(z,t)$ with the same boundary values as $g(z,t)$, such that (2.1) holds.

Consider

$$\tilde{f}(z,t)=\big(f\circ g^{-1}\circ h\big)(z,t)=(F\circ h)(z,t),$$

which has the same boundary values as $f(z)$, but by (2.1) and (2.3) a strictly smaller maximal dilatation for all sufficiently small $t>0$, contradicting the hypothesis that $f$ is extremal. □

## 2.3. *Admissible variations*

If $\kappa \in L^\infty(\Delta)$, we call $\eta \in L^\infty(\Delta)$ an *admissible variation of* $\kappa$ if $\|\eta\|_\infty = \|\kappa\|_\infty$, and if for some set $S \subset \Delta$ (possibly, $S = \emptyset$), and some real number $s$, $|\kappa(z)| < s < \|\kappa\|_\infty$ a.e. in $S$, and $\eta(z) = \kappa(z)$ a.e. in $\Delta \setminus S$. The notion is introduced in [23]. Obviously, the definition only makes sense if $\Delta \setminus S$ has positive measure. In the special case when $|\kappa(z)|$ is a.e. constant, $\kappa$ itself is evidently the only admissible variation of $\kappa$.

THEOREM 3.1. *Suppose $\mu_g$ is an admissible variation of $\mu_f$. If $f$ is extremal, then so is $g$.*[10]

PROOF. Following Section 2.1, we have

$$\|\Lambda_{\mu_f}\| = \|\mu_f\|_\infty = \|\mu_g\|_\infty. \tag{3.1}$$

Suppose $\|\Lambda_{\mu_g}\| < \|\mu_g\|_\infty$. Then there exists $\tilde{\eta} \in L^\infty(\Delta)$, such that $\tilde{\eta} - \mu_g \in \mathcal{N}(\Delta)$, and $\|\tilde{\eta}\|_\infty = \|\mu_g\|_\infty - \delta = \|\mu_f\|_\infty - \delta$, for some $\delta > 0$. Let

$$\tilde{\mu}(z) = \mu_f(z) + \frac{\|\mu_f\|_\infty - s}{2\|\mu_f\|_\infty}\big[\tilde{\eta}(z) - \mu_g(z)\big],$$

so that $\tilde{\mu} - \mu_f \in \mathcal{N}(\Delta)$. When $z \in S$, we have $|\mu_f(z)| < s$, and therefore,

$$\big|\tilde{\mu}(z)\big| < s + \frac{\|\mu_f\|_\infty - s}{2\|\mu_f\|_\infty}\big[2\|\mu_f\|_\infty - \delta\big] = \|\mu_f\|_\infty - \frac{\|\mu_f\|_\infty - s}{2\|\mu_f\|_\infty}\delta.$$

On the other hand, when $z \in \Delta \setminus S$, then $\mu_g(z) = \mu_f(z)$, and therefore

$$\tilde{\mu}(z) = \left(\frac{1}{2} - \frac{s}{2\|\mu_f\|_\infty}\right)\tilde{\eta}(z) + \left(\frac{1}{2} + \frac{s}{2\|\mu_f\|_\infty}\right)\mu_f(z),$$

and, thus,

$$\big|\tilde{\mu}(z)\big| \leqslant \left(\frac{1}{2} - \frac{s}{2\|\mu_f\|_\infty}\right)\big(\|\mu_f\|_\infty - \delta\big) + \left(\frac{1}{2} + \frac{s}{2\|\mu_f\|_\infty}\right)\|\mu_f\|_\infty.$$

Hence, $\|\tilde{\mu}\|_\infty < \|\mu_f\|_\infty$, contradicting (3.1). It follows that $\|\Lambda_{\mu_g}\| = \|\mu_g\|_\infty$; that is, $g$ is extremal. □

REMARK. The import of the theorem is, roughly, that it is the character of $\mu_f$ when $|\mu_f(z)|$ is close to $\|\mu_f\|_\infty$ that determines whether $f$ is extremal. There is a simpler proof of the theorem, using Hamilton sequences, but the above proof has the advantage of carrying over (Theorem 5.4.2) to the case of *unique* extremality.

[10] Of course, $g$ and $f$ will in general not have the same boundary values.

## 3. The main inequality

### 3.1. *The basic inequality – elementary version*

Roughly speaking, the fundamental variational lemma, Lemma 2.2.1, tells us that if $\mu_f \in \mathcal{N}(\Delta)$ and if $k[f]$ is "small", then $f$ has approximately the boundary values of a conformal mapping, the approximation error being small compared to $k[f]$. It is natural to ask what can be said in the other direction: If a quasiconformal mapping has the boundary values of a conformal mapping, what can be said about its complex dilatation? Since $\mu_f$ determines $f$ up to composition with a conformal mapping, this is equivalent to asking what can be said about $\mu_f$ if we know that $f$ is the identity on the boundary. In principle, there is an exact answer through the determination of $f$ by singular integrals, but a more useful *necessary* condition [128] will be found by a different approach below.

Assume that $w = f(z)$ is a quasiconformal mapping of $\Delta_z$ onto $\Delta_w$ whose restriction to $\partial\Delta_z$ is the identity. Let $\zeta = \xi + i\eta = G(z)$ be an *arbitrary* function univalent (i.e., holomorphic and one-to-one) in the closure of $\Delta$, and consider also the same univalent function $G$ applied in $\Delta_w$ by means of the formula $w^* = G(w)$. Denoting $G(\Delta_z) = \Omega$, $G(\Delta_w) = \Omega^*$, and $w^* = (G \circ f \circ G^{-1})(\zeta) = f^*(\zeta)$, we will start off by applying the length-area method to the mapping $f^* : \Omega \to \Omega^*$, extending the procedure of Example 1.2.1.

With $p = f_z$, $q = f_{\bar z}$, $p^* = f^*_\zeta$, $q^* = f^*_{\bar\zeta}$, we have

$$dw^* = p^*\, d\zeta + q^*\, \overline{d\zeta} = G'(w)\, dw = G'(w)(p\, dz + q\, \overline{dz}).$$

If $\gamma(\eta) = \{\zeta \in \Omega\colon \Im\zeta = \eta\}$, then, the hypothesis on the boundary values of $f$ implies that the length of $f^*(\gamma(\eta))$ is at least that of $\gamma(\eta)$; that is,

$$\int_{\gamma(\eta)} d\xi \leqslant \int_{\gamma(\eta)} |p^* + q^*|\, d\xi.$$

Integrating with respect to $\eta$,

$$\|g\| = |\Omega| \leqslant \iint_\Omega |p^* + q^*|\, d\xi\, d\eta = \iint_\Delta |p^* + q^*| \big|g(z)\big|\, dx\, dy,$$

where $g(z) = [G'(z)]^2$. Using Schwarz's Inequality,

$$\begin{aligned}\|g\|^2 &\leqslant \iint_\Delta J(w^*/z)\, dx\, dy \iint_\Delta |p^* + q^*|^2 \frac{|g(z)|^2}{J(w^*/z)}\, dx\, dy \\ &= \|g\| \iint_\Delta |p^* + q^*|^2 \frac{|g(z)|}{J(w^*/\zeta)}\, dx\, dy,\end{aligned}$$

or

$$\|g\| \leqslant \iint_\Delta \frac{|p^* + q^*|^2}{|p^*|^2 - |q^*|^2} |g(z)|\, dx\, dy. \tag{1.1}$$

Since $G(f(z)) = f^*(G(z))$, we have $G'(w)p = p^*G'(z)$, $G'(w)q = q^*\overline{G'(z)}$. Therefore,

$$\frac{q^*}{p^*} = \mu_f(z)\frac{g(z)}{|g(z)|}.$$

Substituting this into (1.1), and simplifying, we obtain

$$-\Re \iint_\Delta \frac{\mu_f(z)}{1-|\mu_f(z)|^2} g(z)\,dx\,dy \leqslant \iint_\Delta \frac{|\mu_f(z)|^2}{1-|\mu_f(z)|^2}|g(z)|\,dx\,dy.$$

Replacing $G$ by $e^{i\tau}G$, with appropriately chosen real $\tau$, we conclude that

$$\left|\iint_\Delta \frac{\mu_f(z)}{1-|\mu_f(z)|^2} g(z)\,dx\,dy\right| \leqslant \iint_\Delta \frac{|\mu_f(z)|^2}{1-|\mu_f(z)|^2}|g(z)|\,dx\,dy. \tag{1.2}$$

The above inequality has been established for $g(z) = [G'(z)]^2$, where $G(z)$ was an arbitrary function univalent in the closed disk. But if $G$ is just a single-valued holomorphic function, continuous in the closed disk, the proof still works although $\Omega$ (and also $\Omega^*$) may have several sheets. Finally, by exhausting $\Delta$ from the inside by disks of increasing radii, we see that (1.2) is valid as long as $g(z) = [G'(z)]^2$, where $G(z)$ is holomorphic in $\Delta$, and $\|g\| < \infty$. Referring to the class $L^1_{ss}(\Delta)$ introduced in Section 1.4, we summarize: *If $f$ is a quasiconformal mapping of $\Delta$ that keeps the boundary points fixed, then* (1.2) *holds for all $g \in L^1_{ss}(\Delta)$.*

### **3.2.** *The basic inequality – general version*

Inequality (1.2) holds under considerably weaker assumptions than in the elementary version of Section 1.

THEOREM 2.1. *If $f$ is a quasiconformal mapping of $\Delta$ that keeps the boundary points fixed, then*

$$\left|\iint_\Delta \frac{\mu_f(z)}{1-|\mu_f(z)|^2} \varphi(z)\,dx\,dy\right| \leqslant \iint_\Delta \frac{|\mu_f(z)|^2}{1-|\mu_f(z)|^2}|\varphi(z)|\,dx\,dy,$$

*holds for all $\varphi \in L^1_a(\Delta)$.*

For the proof of Theorem 2.1 see [128]. In view of approximability it is enough to restrict the proof to $\varphi$ that are holomorphic in the closed disk or even to polynomials in $z$. In place of considering the curves $\gamma(\eta)$ in $G(\Delta)$, as in the elementary case, the proof uses the so-called *trajectories* of $\varphi$ in $\Delta$, that is the curves in $\Delta$ on which $\varphi(z)\,dz^2 > 0$, and in place of the curves $f^*(\gamma(\eta))$, the proof uses the images of the trajectories under $f$. The use of the Euclidean metric in $G(\Delta)$ is replaced by the use of the Teichmüller metric in $\Delta$.

REMARKS. If we apply Theorem 2.1 to the case $\mu_f(z) = t\nu(z) + o(t)$, $t \to 0$, where $f$ keeps the points of $\partial\Delta$ fixed, then we conclude that $\nu \in \mathcal{N}(\Delta)$. In this sense, Theorem 2.1 represents a rough converse of Lemma 2.2.1.

The question arises whether one could conclude *directly* from the fact that (1.2) holds for all $g \in L^1_{ss}(\Delta)$, for some $\mu \in L^\infty(\Delta)$, $\|\mu\|_\infty < 1$, that (1.2) holds for that same $\mu$ for all $g \in L^1_a(\Delta)$. If this were the case, the use of facts about quadratic differentials and the Teichmüller metric might be avoided in the proof of Theorem 2.1. For example, in the theory of Hardy spaces there are a number of inequalities that are first established for holomorphic functions with single-valued square roots that can be extended to the general case with the help of Blaschke products. We show that the answer in our case, however, is *no*. Let $\mu(z) = k\bar{z}/|z|$, with $0 < k < 1$, to be chosen below. If $g \in L^1_{ss}(\Delta)$, we can write

$$g(z) = \left(\sum_0^\infty b_n z^n\right)^2.$$

This gives

$$\iint_\Delta \frac{\mu(z)}{1-|\mu(z)|^2} g(z)\,dx\,dy = \frac{4\pi}{3}\frac{k}{1-k^2} b_0 b_1,$$

and

$$\iint_\Delta \frac{|\mu(z)|^2}{1-|\mu(z)|^2} |g(z)|\,dx\,dy = \frac{\pi k^2}{1-k^2}\sum_0^\infty \frac{|b_n|^2}{n+1}.$$

Therefore, (1.2) will be assured for arbitrary $g \in L^1_{ss}(\Delta)$ if and only if

$$\frac{4\pi}{3}\frac{k}{1-k^2}|b_0 b_1| \leqslant \frac{\pi k^2}{1-k^2}\left(|b_0|^2 + \frac{|b_1|^2}{2}\right).$$

Hence (1.2) will hold for arbitrary $g \in L^1_{ss}(\Delta)$ if we choose $2\sqrt{2}/3 \leqslant k < 1$. If, however, we choose $g(z) = z$, which belongs to $L^1_a(\Delta)$, but not to $L^1_{ss}(\Delta)$, then (1.2) will obviously be violated irrespective of the choice of $k$.

In contrast to the situation with (1.2), we observe the following:

PROPOSITION. *If for some $\nu \in L^\infty(\Delta)$, $\iint_\Delta \nu(z)\varphi(z)\,dx\,dy = 0$ for all $\varphi \in L^1_{ss}(\Delta)$, then $\nu \in \mathcal{N}(\Delta)$.*

PROOF. Since

$$z^n = \frac{1}{2}\left[(1+z^n)^2 - 1 - z^{2n}\right], \quad n = 1, 2, \ldots,$$

it is clear that every polynomial in $z$ is a linear combination of elements of $L^1_{ss}(\Delta)$. Hence, the set of linear combinations of elements of $L^1_{ss}(\Delta)$ is dense in $L^1_a(\Delta)$. □

### 3.3. *Versions of the main inequality*

Assume that $f(z)$ and $\tilde{f}(z)$ are quasiconformal mappings of $\Delta$, agreeing on $\partial\Delta$. Let $F(z) = (\tilde{f}^{-1} \circ f)(z)$ $(z \in \Delta)$. The complex dilatation of $F$ is

$$\mu_F = \frac{((\tilde{f}^{-1})_w \circ f)q + ((\tilde{f}^{-1})_{\bar{w}} \circ f)\bar{p}}{((\tilde{f}^{-1})_w \circ f)p + ((\tilde{f}^{-1})_{\bar{w}} \circ f)\bar{q}} \quad (p = f_z,\ q = f_{\bar{z}}).$$

Since $F$ is the identity on $\partial\Delta$, we can apply Theorem 2.1 to $\mu_F(z)$. We set

$$\begin{aligned} &\kappa(z) = \mu_f(z) = q/p, \qquad \alpha(z) = (\mu_{f^{-1}} \circ f)(z) = -q/\bar{p}, \\ &\beta(z) = (\mu_{\tilde{f}^{-1}} \circ f)(z). \end{aligned} \tag{3.1}$$

(Note that $|\alpha(z)| = |\kappa(z)|$ a.e.) The result is the following:

THEOREM 3.1 (Main inequality). *Suppose $f(z)$ and $\tilde{f}(z)$ are quasiconformal mappings of $\Delta$, agreeing on $\partial\Delta$. Let $\kappa(z)$, $\alpha(z)$, $\beta(z)$ be defined as in* (3.1). *Then*

$$\begin{aligned} &\Re \iint_\Delta \left(\frac{\kappa}{\alpha} - \bar{\beta}\kappa\right) \frac{(\alpha - \beta)\varphi}{(1 - |\kappa|^2)(1 - |\beta|^2)}\, dx\, dy \\ &\quad \leqslant \iint_\Delta \frac{|\alpha - \beta|^2 |\varphi|}{(1 - |\kappa|^2)(1 - |\beta|^2)}\, dx\, dy \end{aligned} \tag{3.2}$$

*holds for all $\varphi \in L_a^1(\Delta)$.*

As (3.3) and (3.4), below, we list two further versions of the Main Inequality. They are each completely equivalent to (3.2), and can be obtained from (3.2) by simple algebraic manipulation.[11]

$$\begin{aligned} &\iint_\Delta \frac{(|\alpha|^2 - |\beta|^2) + (1 - |\kappa|)(|\alpha| - \Re\frac{\bar{\beta}\alpha}{|\alpha|})}{(1 + |\kappa|)(1 - |\beta|^2)} |\varphi|\, dx\, dy \\ &\quad \leqslant \Re \iint_\Delta \frac{\bar{\alpha}}{|\alpha|}\left(|\varphi| - \frac{\kappa}{|\kappa|}\varphi\right) \frac{(1 - \bar{\beta}\alpha)(\alpha - \beta)}{(1 - |\kappa|^2)(1 - |\beta|^2)}\, dx\, dy, \end{aligned} \tag{3.3}$$

$$\iint_\Delta |\varphi|\, dx\, dy \leqslant \iint_\Delta |\varphi| \frac{|1 - \kappa\frac{\varphi}{|\varphi|}|^2}{1 - |\kappa|^2} \cdot \frac{\left|1 + \frac{\kappa\beta}{\alpha} \cdot \frac{\varphi}{|\varphi|} \cdot \left[\frac{1 - \bar{\kappa}\frac{\bar{\varphi}}{|\varphi|}}{1 - \kappa\frac{\varphi}{|\varphi|}}\right]\right|^2}{1 - |\beta|^2}\, dx\, dy. \tag{3.4}$$

The Main Inequality was first obtained in the form (3.4) by Reich and Strebel [132] by a proof independent of Theorem 2.1. It was extended to arbitrary $\Omega$ by Strebel [161] Another proof is due to Bers [15]. Expression (3.3) appears in [111]. For an extension to the case when the complex dilatations are not necessarily bounded below 1, see [90].

[11] In the expressions the fraction $\kappa/\alpha$ is to be interpreted as $-\bar{p}/p$. This is unambiguous a.e., as $p \neq 0$ a.e.

### 3.4. *Bounds on $K_0[f]$. Characterization of extremal mappings*

Let $f$ be a quasiconformal mapping of $\Delta$ and let $K_0[f]$ denote the extremal dilatation corresponding to the restriction of $f$ to $\partial\Delta$, $k_0[f] = (K_0[f]-1)/(K_0[f]+1)$. Obviously, $k_0[f] \leqslant k[f]$. For the purpose of obtaining a *lower* bound for $k_0[f]$, we define

$$I[\mu] = \sup\left\{\Re \iint_\Delta \frac{\mu}{1-|\mu|^2}\varphi\,dx\,dy:\ \varphi \in L_a^1(\Omega),\ \|\varphi\| \leqslant 1\right\}, \tag{4.1}$$

and

$$Q[\mu] = \sup\left\{\iint_\Delta \frac{|\mu|^2}{1-|\mu|^2}|\varphi|\,dx\,dy:\ \varphi \in L_a^1(\Omega),\ \|\varphi\| \leqslant 1\right\}, \tag{4.2}$$

whenever $\mu \in L^\infty(\Delta)$, $\|\mu\|_\infty < 1$.

THEOREM 4.1. *If $f$ is a quasiconformal mapping of $\Delta$, then*

$$\frac{k_0[f]}{1+k_0[f]} \geqslant I[\mu_f] - Q[\mu_f].$$

PROOF. Let $\tilde{f}$ be an extremal mapping. In the notation for Theorem 3.1, with $\kappa = \mu_f$, we have $\|\beta\|_\infty = k_0[f]$. Therefore (3.4) implies

$$\|\varphi\| = \iint_\Delta |\varphi|\,dx\,dy \leqslant K_0[f] \iint_\Delta \frac{|1-\mu_f\frac{\varphi}{|\varphi|}|^2}{1-|\mu_f|^2}|\varphi|\,dx\,dy,$$

for all $\varphi \in L_a^1(\Delta)$. Expanding the integrand, and rearranging terms, we get

$$\begin{aligned}\Re \iint_\Delta \frac{\mu_f}{1-|\mu_f|^2}\varphi\,dx\,dy &\leqslant \iint_\Delta \frac{|\mu_f|^2}{1-|\mu_f|^2}|\varphi|\,dx\,dy + \frac{k_0[f]}{1+k_0[f]} \\ &\leqslant Q[\mu_f] + \frac{k_0[f]}{1+k_0[f]}.\end{aligned}$$

Therefore, $I[\mu_f] \leqslant Q[\mu_f] + k_0[f]/(1+k_0[f])$. □

Using Theorem 1.3.2, it is easy to obtain an upper bound on $k_0[f]$. The result[12] is as follows:

THEOREM 4.2. *If $f$ is a quasiconformal mapping of $\Delta$, then*

$$\frac{k_0[f]}{1-k_0[f]} \leqslant I[\mu_f] + Q[\mu_f].$$

[12] This can also be used for an alternative proof of Theorem 2.1.1 for the case of the disk [132].

Suppose now

$$\sup\left\{\Re\iint_{\Delta}\mu_f\varphi\,dx\,dy\colon\ \varphi\in L_a^1(\Delta),\ \|\varphi\|\leqslant 1\right\}=\|\mu_f\|_\infty.$$

It is easy to see that this implies that

$$I[\mu_f]=\frac{\|\mu_f\|_\infty}{1-\|\mu_f\|_\infty^2}.$$

Since, obviously,

$$Q[\mu_f]\leqslant\frac{\|\mu_f\|_\infty^2}{1-\|\mu_f\|_\infty^2},$$

it follows by Theorem 4.1 that $k_0[f]\geqslant k[f]$, and, therefore, $k_0[f]=k[f]$; that is, $f$ is extremal. Thus, we see that the Hamilton–Krushkal condition (2.1.2) is both necessary and sufficient for extremality:

THEOREM 4.3. *Suppose $f$ is a quasiconformal mapping of $\Delta$. A necessary and sufficient condition that $f$ is an extremal extension of its boundary restriction to $\partial\Delta$ is that*

$$\|\Lambda_{\mu_f}\|=\|\mu_f\|_\infty;$$

*that is, $\iint_\Delta \mu_f\varphi\,dx\,dy$ is its own Hahn–Banach extension from $L_a^1(\Delta)$ to $L^1(\Delta)$.*

The Main Inequality has through Theorem 4.3 provided us with what may be called an "analytic" method to test for extremality, in distinction to the earlier methods that were more "geometric" in character in that they were more closely tied to the original geometric concept of quasiconformality. If a Hamilton sequence exists for $\Lambda_\mu$, then and only then, according to Theorem 4.3, is a mapping $f$ with complex dilatation $\mu$ extremal. There is a *natural Hamilton sequence* $\{\varphi_n\}$, determined by the boundary restriction of $f$. Namely, as in Section 1.3, we inscribe $n$-gons $\{E_n\}$ with vertices $\{z_{in}\}$, with the set of vertices dense on $\partial\Delta$, as $n\to\infty$. Then a normalized sequence $\{\varphi_n\}$ determined through the extremal mappings of Theorem 1.3.1 is guaranteed to have a subsequence that constitutes a Hamilton sequence for an extremal extension of the boundary values of $f$. As we have seen, boundary values on $\partial\Delta$ may have more than a single extremal extension; the natural Hamilton sequence is determined completely by a specification of the boundary values of $f$, and therefore is a Hamilton sequence for every extremal extension. There is a generalization of this fact to arbitrary $\Omega$:

THEOREM 4.4 [30]. *Let $h$ be a function defined on $\partial\Omega$ possessing a quasiconformal extension to $\Omega$. There exists a universal Hamilton sequence $\{\varphi_n\}$ which is a Hamilton sequence for every extremal extension of $h$.*

In the case of $\Omega = \Delta$, we have

THEOREM 4.5 [177]. *Let h be a function defined on $\partial\Delta$, possessing a quasiconformal extension to $\Delta$. A Hamilton sequence for any extremal extension of h is a Hamilton sequence for every extremal extension of h.*

### 3.5. *Test for extremality – examples*

EXAMPLE 3.5.1 (*Affine stretches with a general axis*). If $a$ is a complex constant, $0 < |a| < 1$, then $f(z) = z + a\bar{z}$, $z \in \Omega$, is an affine stretch of $\Omega$ by the factor $K[f] = (1+|a|)/(1-|a|)$ in the direction of $\frac{1}{2}\arg a$. Since $\mu_f(z) \equiv a$, *the necessary and sufficient condition that $f$ is extremal for its restriction to $\partial\Omega$ is that*

$$\|\Lambda_1\| = \sup_{\varphi \in L_a^1(\Omega)} \frac{\left|\iint_\Omega \varphi(z)\,dx\,dy\right|}{\iint_\Omega |\varphi(z)|\,dx\,dy} = 1. \tag{5.1}$$

When $|\Omega| < \infty$, condition (5.1) is of course satisfied, but in any case, even when $|\Omega| = \infty$, *whether or not $f$ is extremal is independent of the value of $a$*, and, in particular, independent of the direction of stretch.

The question arises how to try to verify condition (5.1) in a concrete case. Theoretically, in case $\Omega$ is simply-connected, one could transfer the problem to the unit disk $\Delta$, and then attempt to construct a sequence $\{\varphi_n\}$ by means of $n$-gon approximations, but of course this is in general not practical. If $|\Omega|$ is finite, then $\varphi_n(z) \equiv 1$ works; so, if $|\Omega| = \infty$, one tries, as a compromise, $\varphi_n \in L_a^1(\Omega)$ with $\lim \varphi_n(z) = 1$.

We illustrate the above with the regions $G_\beta$, $\beta \geqslant 1$, of Example 1.4.6. For $\beta = 1$, we know (Example 2.1.1) that $\|\Lambda_1\| = 2/\pi$; so, when $\beta = 1$, the affine stretch is not extremal. When $\beta > 1$, we try $\varphi(z) = e^{-tz}$, $t > 0$. It is easy to verify that

$$\lim_{t\to 0} \frac{\left|\iint_{\{x>|y|^\beta\}} e^{-tz}\,dx\,dy\right|}{\iint_{\{x>|y|^\beta\}} |e^{-tz}|\,dx\,dy} = 1.$$

So, we can conclude that the affine stretch of $G_\beta$ is extremal when $\beta > 1$.

EXAMPLE 3.5.2. We know from Example 1.4.3, that the affine stretch of the strip

$$\Sigma_0 = \{z = x + iy\colon\ -1 < y < 1\}$$

is extremal. To prove this by condition (5.1), let

$$\sigma_n(z) = (1/n)\exp\left(-z^2/n^2\right).$$

One finds that

$$\iint_{\Sigma_0} \sigma_n(z)\,dx\,dy = 2\sqrt{\pi},$$

while

$$\iint_{\Sigma_0} |\sigma_n(z)|\,dx\,dy = \sqrt{\pi}\int_{-1}^{1} \exp\left(\frac{y^2}{n^2}\right) dy \to 2\sqrt{\pi}.$$

Thus, $\|\Lambda_1\| = 1$.

EXAMPLE 3.5.3. Proof using (5.1) that the affine stretch of the chimney region (Example 1.4.4) is extremal.

We orient and size the chimney as

$$C = \{z\colon \Re z < 0\} \cup \Sigma_0,$$

where $\Sigma_0$ is the strip defined in Example 3.5.2. Let $T(z)$ map $C$ conformally onto $\Sigma_0$, $T(\mathrm{i}) = \mathrm{i}$, $T(-\mathrm{i}) = -\mathrm{i}$, $T(+\infty) = +\infty$, and let $\{\sigma_n(z)\}$ be as defined in Example 3.5.2. One verifies [118] that

$$\varphi_n(z) = \frac{1}{2\sqrt{\pi}}\sigma_n\big(T(z) - n^3\big)$$

is a Hamilton sequence for $\|\Lambda_1\|$. Qualitatively, the idea this time is as follows. As $n$ increases, there is a broadening "wave" shaped like $\sigma_n(z)$, moving to $+\infty$ along the "chimney" arm of $C$ on which $\varphi_n(z) \approx \text{const} \neq 0$, but the term $n^3$ in the formula for $\varphi_n(z)$ insures that the wave moves so fast compared to the rate at which it broadens that we have $\lim_{n\to\infty} \varphi_n(z) = 0$ for all $z \in C$. This behavior brings out the fact that it is the shape of the arm near $+\infty$ that makes the affine stretch of the region as a whole extremal.

## 4. Local versus global effects

### 4.1. *Substantial boundary points. Boundary dilatation*

Suppose $h$ is a quasisymmetric mapping of $\partial\Delta$ onto itself. To what extent is $K_0[h]$ determined by local properties of $h$? In order to examine this question, the author [102] introduced the concept of *substantial boundary point* (relative to the boundary correspondence $h$) as follows. If $\Gamma \subset \partial\Delta$ is a closed arc consisting of more than a single point, let

$$K^*_\Gamma[h] = \inf K[f], \tag{1.1}$$

where the inf is over all quasiconformal mappings $f$ of $\Delta$ onto $\Delta$ whose boundary values on $\Gamma$ agree with those of $h$. Thus, $K^*_\Gamma[h] \leqslant K^*_{\partial\Delta}[h] = K_0[h]$. The determination of $K^*_\Gamma$ is obtained as a free boundary value problem characterized by the criterion of Theorem 3.4.3, except that the class $L^1_a(\Delta)$ is replaced by the subclass of functions $\varphi \in L^1_a(\Delta)$ for which

$\varphi(z)\,dz^2 = \text{real}$ on $\partial\Delta \setminus \Gamma$. Evidently, $K^*_\Gamma[h]$ decreases as $\Gamma$ is shortened. We define the *local dilatation* of $h$ at the point $\zeta \in \partial\Delta$ as

$$K^*_\zeta[h] = \inf K^*_\Gamma[h], \tag{1.2}$$

where the inf is over all arcs $\Gamma$ containing $\zeta$ as interior point. Finally, we say that $\zeta$ *is a substantial boundary point, if*

$$K^*_\zeta[h] = K_0[h]. \tag{1.3}$$

The idea is that if $\zeta$ is a substantial boundary point, then the local behavior of $h$ in arbitrarily small neighborhoods of $\zeta$ already determines $K_0[h]$. Recalling the concept of degenerating Hamilton sequence (Section 2.1), one proves the following:

THEOREM 1.1. *Suppose $f$ is a quasiconformal mapping of $\Delta$ onto $\Delta$. A necessary and sufficient condition that $f$ is extremal and that $\zeta$ is a substantial boundary point in the sense of* (1.3) *is that there exists a degenerating Hamilton sequence $\varphi_n(z)$, such that*

$$\lim_{n\to\infty} \varphi_n(z) = 0 \quad \textit{uniformly on every compact subset of } \overline{\Delta} \setminus \{\zeta\}, \tag{1.4}$$

*for which*

$$\lim_{n\to\infty} \iint_\Delta \mu_f(z)\varphi_n(z)\,dx\,dy = \|\mu_f\|_\infty.$$

There is also a slightly different approach to a possible definition of substantial boundary point. We start with the *boundary dilatation* of $h$, defined by Strebel [157,158] as

$$H[h] = \inf\bigl\{K[f]\colon\ f \text{ is a qc mapping of } U(\partial\Delta) \text{ into } \Delta,\ f|_{\partial\Delta} = h\bigr\}, \tag{1.5}$$

and define, by analogy [32], the *local dilatation* at $\zeta \in \partial\Delta$ as

$$\begin{aligned} H_\zeta[h] = \inf\bigl\{K[f]\colon\ & f \text{ is a qc mapping of } U(\zeta) \text{ into } \Delta, \\ & f|_{U(\zeta)\cap\partial\Delta} = h|_{U(\zeta)\cap\partial\Delta}\bigr\}, \end{aligned} \tag{1.6}$$

where the infimum in (1.5) is over all neighborhoods $U(\partial\Delta)$ of $\partial\Delta$ in $\Delta$, and the infimum in (1.6) over all neighborhoods $U(\zeta)$ of $\zeta$ in $\Delta$. It is clear that $H_\zeta[h] \leqslant H[h] \leqslant K_0[h]$ for all $\zeta \in \partial\Delta$. With this setup one now defines $\zeta$ to be a substantial boundary point if

$$H_\zeta[h] = K_0[h]. \tag{1.7}$$

For any given boundary correspondence $h$, it is immediate that any boundary point that is substantial in the sense of (1.7) is also substantial in the sense of (1.3). Conversely, if $\zeta$ is a substantial boundary point in the sense of (1.3), then, by Theorem 1.1 there exists a degenerating Hamilton sequence satisfying (1.4), and by results of Fehlmann's ([32], Satz 4.1, Satz 5.2) there exists a substantial boundary point in the sense of (1.7). *The definitions* (1.3) *and* (1.7) *are therefore equivalent.* Note that the concept of substantial boundary point is conformally invariant; that is, if in the definitions $\Delta$ is replaced by an arbitrary simply-connected region $\Omega$, and if a boundary value problem is transferred by a conformal mapping between $\Delta$ and $\Omega$, then substantial boundary points correspond. Of course, for a general simply-connected region, "boundary point" has to be interpreted as prime end.

As a consequence of Theorem 2.1.2 and the above Theorem 1.1, we have

THEOREM 1.2. *Suppose $f$ is an extremal mapping. Then either $\mu_f$ is of Teichmüller type with finite norm or there is a substantial boundary point* (*or both*).

The following result of Fehlmann and Sakan [38] provides information about *all* the degenerating Hamilton sequences on the basis of knowledge about the set of substantial boundary points.

THEOREM 1.3. *Let $h:\partial\Delta \to \partial\Delta$ be quasisymmetric. and let $S$ be the set of substantial boundary points. Then for every set $U$ open in $\overline{\Delta}$, $U \supset S$, and for every degenerating Hamilton sequence $\{\varphi_n\}$ associated to an extremal extension of $h$,*

$$\lim_{n\to\infty} \iint_{\Delta\setminus U} |\varphi_n(z)|\,dx\,dy = 0.$$

For some applications, a comparison of the quantities $H$ and $H_\zeta$ is useful. A first result makes use of the Beurling–Ahlfors extension. We define $\mathcal{P}(r)$, $0 < r < \infty$, as the conformal modulus of the upper half-plane $H^+$ with vertices at $-1, 0, r, \infty$, chosen so as to be continuous and strictly increasing over $[0, \infty]$, $\mathcal{P}(0) = 0$, $\mathcal{P}(\infty) = \infty$. If $h:\partial\Delta \to \partial\Delta$ is quasisymmetric, then [32]

$$H[h] \leqslant \left(\mathcal{P}^{-1}\left(\max_{\zeta\in\partial\Delta} H_\zeta[h]\right)\right)^2.$$

By a bootstrapping procedure, with the help of the Main Inequality, Fehlmann succeeded in obtaining what is evidently the best possible result:

THEOREM 1.4 [33, Theorem 4]. *Suppose $h:\partial\Delta \to \partial\Delta$ is quasisymmetric. Then*

$$H[h] = \max_{\zeta\in\partial\Delta} H_\zeta[h].$$

For further relations between the various constants, see the work of Fehlmann, in particular Section 6 of [32].

EXAMPLE 4.1.1 (*Affine stretch*, $K_\zeta^* = 1$). If $h = F_A|_{\partial\Omega}$, then at points $\zeta \in \partial\Omega$ where $\partial\Omega$ is "sufficiently" smooth, we get $K_\zeta^*[h] = 1$. Suppose $\partial\Omega$ has a tangent with tangent angle

$\tau(s)$ ($s$ = arc length) in a neighborhood of $\zeta$. Suppose $\tau(s)$ has modulus of continuity $\omega(s)$, where $\int_0 \frac{\omega(t)}{t} dt < \infty$. This is sufficient [102] to guarantee that $K_\zeta^* = 1$, and therefore also that $H_\zeta[h] = 1$.

EXAMPLE 4.1.2 (*Affine stretch of chimney*). For the chimney $C$ as defined as in Example 3.5.3, $h = F_A|_{\partial C}$, one obtains

$$H_\zeta = \begin{cases} 1, & \zeta \in \partial C, \zeta \neq -\mathrm{i}, +\mathrm{i}, +\infty, \\ B, & \zeta = -\mathrm{i}, +\mathrm{i}, \\ A, & \zeta = +\infty, \end{cases}$$

where $B$ has at most the value of the extremal dilatation of the restriction of $F_A$ to the angular region $\{0 < \arg z < 3\pi/2\}$. Since the affine stretch is never extremal for an angular region, we know that $1 < B < A$. Thus, there is one and only one substantial boundary point; namely the prime end $+\infty \in \partial C$. For a more general situation, see Fehlmann [37].

EXAMPLE 4.1.3 (*Teichmüller mapping with finite norm and substantial boundary point*). We consider the boundary values $h = F_A|_{\partial\Omega}$, where

$$\Omega = \left\{ z = x + \mathrm{i}y\colon\ 0 < y < x^\alpha,\ 0 < x < 1 \right\},$$

with $\alpha > 1$. The region $\Omega$ has a cusp at $z = 0$. Consider

$$g_n(z) = c_n \mathrm{e}^{-nz}, \quad n = 1, 2, \ldots,$$

where

$$c_n = \left( n^{\alpha+1} \Big/ \int_0^\infty t^\alpha \mathrm{e}^{-t}\, dt \right) + \mathrm{o}\left(n^{\alpha+1}\right) \quad \text{as } n \to \infty$$

is chosen so as to make

$$\|g_n\| = \iint_\Omega |g_n(z)|\, dx\, dy = 1.$$

Since this makes

$$\lim \iint_\Omega g_n(z)\, dx\, dy = 1,$$

while

$$\lim_{n\to\infty} g_n(z) = 0 \quad \text{locally uniformly in } \overline{\Omega} \setminus \{0\},$$

we can conclude that there is a substantial boundary point at $z = 0$. If we like, we can transfer this situation to the unit disk. Namely, suppose $w = \Phi(z)$, $\Phi(-1) = 0$, maps

$\{|z| < 1\}$ conformally onto $\Omega$. For $K = (1+k)/(1-k) > 1$, let $\Psi(w)$ map $\{|w| < 1\}$ conformally onto $F_K(\Omega)$. Then $f = \Psi^{-1} \circ F_K \circ \Phi$ is a Teichmüller mapping with complex dilatation

$$\mu_f(z) = k\frac{\overline{\varphi_0(z)}}{|\varphi_0(z)|}, \qquad \varphi_0(z) = \Phi'(z)^2, \qquad \|\varphi_0\| = |\Omega|.$$

The point $-1 = \Phi^{-1}(0)$ becomes a substantial boundary point of $\partial\Delta$ for the boundary values $h = f|_{\partial\Delta}$, and the Hamilton sequence,

$$\varphi_n(z) = g_n\big(\Phi(z)\big)\Phi'(z)^2 = c_n e^{-n\Phi(z)}\varphi_0(z)$$

is the one described by Theorem 1.1. One moral of this example, referring to Theorem 1.2, is that *the conditions that $\mu_f$ is of Teichmüller type with finite norm and that there is a substantial boundary point can occur simultaneously.*

EXAMPLE 4.1.4 (*Every boundary point substantial*). For $0 < k < 1$, $0 < r_n < r_{n+1}$, $r_n \to 1$, define

$$\mu_f\big(re^{i\theta}\big) = ke^{-in\theta}, \quad r_n < r < r_{n+1},\ n = 1, 2, \ldots,\ 0 \leqslant \theta < 2\pi.$$

With appropriate choice of $\{r_n\}$, $\mu_f$ can be shown [38] to be extremal, and a subsequence of the sequence $\{(n+2)z^n/2\pi\}$ can be identified as a degenerating Hamilton sequence. With the help of Theorem 1.3 it turns out that every point $z$, $|z| = 1$, is substantial. For another example with this property, see [32].

## 4.2. *The frame mapping condition*

It was suggested[13] by Teichmüller [173, p. 185], that boundary values on $\partial\Delta$ that allow quasiconformal extensions to $\Delta$ always allow an extremal extension to what we now call a Teichmüller mapping. As pointed out in connection with Example 1.4.2, it has been known since Strebel's paper [153] that sometimes the only extremal extensions were Teichmüller mappings with infinite norm. (See also Section 5.3 below.) Moreover, as we will see (Example 5.3.1) it may occur that quasisymmetric boundary values possess no extremal extensions of Teichmüller-type with either finite *or* infinite norm. In fact [22], there exist uniquely extremal mappings $f$ such that $D_f(z) < K[f]$ for a.a. $z$. Nevertheless, we know from Theorem 1.3.1 that changing the given boundary data just slightly does result in having as extremal mapping a Teichmüller mapping with finite norm. Also, as we will see in Section 5.5, for any quasisymmetric $h$, there exist extensions $g$ for which $D_g(z)$ is a.e. a constant whose value is arbitrarily close to $K_0[h]$.

A sufficient condition that a quasisymmetric $h$ can be extended to a Teichmüller mapping with finite norm, the so-called *frame mapping condition* was found by Strebel [157,158] as

[13] Teichmüller states deliberately that he intended this more as a stimulus for research than a formal conjecture.

a fairly simple consequence of the Main Inequality. The criterion makes use of the quantity $H$ of the preceding section, and has been found (see, e.g., [22,31]) to have important applications. The result is the following:

THEOREM 2.1 (Frame mapping criterion). *Let $h:\partial\Delta \to \partial\Delta$ be quasisymmetric. Suppose $H[h] < K_0[h]$. Then there is a uniquely extremal mapping $f_0$. It is a Teichmüller mapping with finite norm.*

PROOF. Choose $\widetilde{H}$ with $H[h] < \widetilde{H} < K_0[h]$. Let $\tilde{h}$ be an $\widetilde{H}$-qc extension of $h$ into an annulus $\{\tilde{r} < |z| < 1\}$. By an extension theorem of Lehto and Virtanen [84, p. 96], one can conclude that for some $r$ ($\tilde{r} < r < 1$), there exists a quasiconformal mapping $\tilde{f}$ of $\Delta$, such that $\tilde{f}(z) = \tilde{h}(z)$ for $r < |z| < 1$.

Let $f_0$ be an extremal mapping for the boundary values $h$, and set $\kappa = \mu_{f_0}$. Referring to Theorem 2.1.2, let $\varphi_n$ be a Hamilton-sequence for $\kappa$ that we suppose degenerates. We will see that this leads to a contradiction. From the Main Inequality in the form 3.3.4, with $f = f_0$, we have

$$\|\varphi_n\| \leqslant \iint_\Delta |\varphi_n| \frac{|1-\kappa\frac{\varphi_n}{|\varphi_n|}|^2}{1-|\kappa|^2} \cdot E(z)\,dx\,dy, \tag{2.1}$$

where

$$E(z) = \frac{1+|\mu(z)|}{1-|\mu(z)} = D_{\tilde{f}^{-1}}\bigl(f_0(z)\bigr).$$

We choose $\rho < 1$ so that $E(z) \leqslant \widetilde{H}$ when $\rho < |z| < 1$, and break up the integral on the right side of (2.1) into

$$\|\varphi_n\| \leqslant \iint_\Delta |\varphi_n| \frac{|1-\kappa\frac{\varphi_n}{|\varphi_n|}|^2}{1-|\kappa|^2} \cdot E(z)\,dx\,dy = \iint_{|z|<\rho} + \iint_{\rho<|z|<1}.$$

Since we are assuming that $\{\varphi_n(z)\}$ is a degenerating Hamilton sequence for $\kappa$, we therefore have

$$\begin{aligned} 1 &\leqslant \varepsilon + \widetilde{H} \iint_{\rho<|z|<1} |\varphi_n| \frac{|1-\kappa\frac{\varphi_n}{|\varphi_n|}|^2}{1-|\kappa|^2}\,dx\,dy \\ &\leqslant \varepsilon + \widetilde{H} \iint_\Delta |\varphi_n| \frac{|1-\kappa\frac{\varphi_n}{|\varphi_n|}|^2}{1-|\kappa|^2}\,dx\,dy, \end{aligned}$$

for any given $\varepsilon > 0$, and $n$ sufficiently large. Expanding the integral, we obtain as $n \to \infty$, by the Hamilton–Krushkal condition, Theorem 2.1.1,

$$1 \leqslant \varepsilon + \widetilde{H} \cdot \frac{1}{K_0[h]}$$

and, hence, $K_0[h] \leqslant \widetilde{H}$, a contradiction. By Theorem 2.1.2, the conclusion therefore follows. □

In practice, the simplest situation occurs when $H[h] = 1$, and this case can be characterized as follows [158].

COROLLARY. *Suppose $h : \partial\Delta \to \partial\Delta$ is a sense-preserving homeomorphism with $H[h] = 1$ that is not the boundary value of a Möbius transformation. Then $k_0[h] > 0$, and there is a Teichmüller mapping of finite norm that is the uniquely extremal extension of $h$ to $\Delta$.*

In view of the above, it is evidently of interest to know whether or not $H[h] = 1$. Here is the solution:

THEOREM 2.2 [158]. *Suppose $w = h(z)$ is an orientation-preserving homeomorphism of $\{|z| = 1\}$ onto $\{|w| = 1\}$. Then $H[h] = 1$ if and only if*

$$\lim_{\tau\to 0} \frac{h(z+\tau) - h(z)}{h(z) - h(z-\tau)} = 1 \quad \textit{uniformly in } \tau \ (|z| = 1, |z+\tau| = 1, \ |z-\tau| = 1).$$

EXAMPLE 4.2.1 (*Sufficient condition for $H[h] = 1$* [158]). If $w = h(\mathrm{e}^{\mathrm{i}\theta})$ is an orientation-preserving homeomorphism of $\{|z| = 1\}$ onto $\{|w| = 1\}$ for which $h'(\mathrm{e}^{\mathrm{i}\theta}) \neq 0$, and $h''(\mathrm{e}^{\mathrm{i}\theta})$ is bounded, then $H[h] = 1$. (Compare with Example 4.1.1 and Theorem 1.4.)

EXAMPLE 4.2.2 (*Quasiconformal extensions of conformal mappings*). Suppose

$$w = W(z) = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots \tag{2.2}$$

is univalent for $|z| > 1$. Does there exist a quasiconformal extension of $W(z)$ to the plane? The simplest situation occurs when the series (2.2) converges to a function univalent for $|z| > \rho$, $\rho < 1$. In that case $W(\partial\Delta)$ is an analytic curve, and by the preceding example and the Corollary of Theorem 2.1, either $W(z) \equiv z$, or there is a Teichmüller mapping with finite norm that provides the unique extension.

EXAMPLE 4.2.3 (*Extremal extensions not a linear operation on the boundary values*). Suppose $f_1(z)$ and $f_2(z)$ are Teichmüller mappings with finite norm of the upper half-plane $H^+$ onto itself with boundary functions $u_1(x)$, $u_2(x)$ that are $C^2$ on the extended real axis. According to the frame mapping criterion, the boundary function $u_1(x) + u_2(x)$ has a Teichmüller mapping with finite norm as the unique extremal extension. Since $f_1 + f_2$ is not in general a Teichmüller mapping, this shows that extremal extensions cannot be determined by a linear operation on the boundary function (as, e.g., the Beurling–Ahlfors extension).

NOTES. For practical applications of Theorem 2.1 to situations where $H[h] > 1$, see [166].

There has been extensive work on the problem of quasiconformal extensions of conformal mappings and vice versa, motivated in part by the theory of Teichmüller spaces, in part by what can be looked upon as successful attempts to understand univalent functions better, and in part by other applications, e.g., quasiconformal "reflections". See the survey articles by Becker [10], and by Kühnau [74], the references there, and the later and ongoing work by Kühnau.

## 5. Unique extremality

### 5.1. *The classical case*

The main tool in this chapter will be the main inequality, Theorem 3.3.1, in the version of the inequality (3.3.3).

The fact that a Teichmüller mapping of $\Delta$ with finite norm was uniquely extremal for its boundary values (Theorem 1.4.1) was, as we know, proved [154] long before the discovery of the main inequality. However, as an appetizer, we derive it now from (3.3.3), following the notation of that section.

PROOF OF THEOREM 1.4.1. We assume that

$$\kappa(z) = \mu_f(z) = k\frac{\overline{\varphi_0(z)}}{|\varphi_0(z)|}, \quad \varphi_0 \in L_a^1(\Delta),\ k > 0. \tag{1.1}$$

Let $\tilde{f}(z)$ be an extremal extension of $f|_{\partial\Delta}$. Thus, $|\alpha(z)| = |\kappa(z)| \equiv k$, $|\beta(z)| \leqslant k$. We use (3.3.3) with $\varphi = \varphi_0$. Since, by (1.1),

$$|\varphi_0| - \frac{\kappa}{|\kappa|}\varphi_0 = 0,$$

the right side, and hence the left side of (3.3.3) vanish. Therefore, $\beta(z) = \alpha(z)$; thus

$$\mu_{\tilde{f}^{-1}} = \mu_{f^{-1}}.$$

Since $\tilde{f}^{-1}$ has the same boundary values as $f^{-1}$, it follows that $\tilde{f}^{-1} = f^{-1}$. Therefore $\tilde{f} = f$. We conclude that $f$ is uniquely extremal. □

### 5.2. *Some necessary conditions*

The following shows that if $f$ is uniquely extremal, then, roughly speaking, $D_f(z)$ must equal $K[f]$ at "most" places of $\Delta$.

THEOREM 2.1. *Suppose $f : \Delta \to \Delta$ is uniquely extremal for its boundary values. Then for any open subset $U$ of $\Delta$, we must have* $\sup\{D_f(z)\colon\ z \in U\} = K[f]$.

PROOF BY CONTRADICTION. We can assume that $K[f] > 1$. Suppose there exists $\varepsilon > 0$, and a disk $\{|z - z_0| < \rho\} \subset \Delta$, such that

$$\sup_{|z-z_0|<\rho} D_f(z) < K[f] - \varepsilon,$$

where, by sup we mean ess sup, that is, we neglect sets of area-measure zero. We proceed to construct a variation of $f$.

For $|\delta| < 1$, $|z| < 1$, consider the homeomorphism,

$$T(\delta; z) = z + (1 - |z|)\delta,$$

of $\{|z| < 1\}$ onto itself, which shifts the origin to the point $z = \delta$, and is the identity for $\{|z| = 1\}$. A computation shows that

$$\mu_T(z) = \frac{\delta z}{\delta \bar{z} - 2|z|}, \qquad k[T] = \frac{|\delta|}{2 - |\delta|}.$$

Let

$$V(\zeta) = \begin{cases} z_0 + \rho T\left(\delta; \dfrac{\zeta - z_0}{\rho}\right), & |\zeta - z_0| < \rho, \\ \zeta, & |\zeta - z_0| \geqslant \rho, |\zeta| < 1, \end{cases}$$

and define $\tilde{f} = f \circ V$. Then $\tilde{f}$ has the same boundary values as $f$, and, if $|\delta|$ is sufficiently small, $K[\tilde{f}] = K[f]$, contradicting the fact that $f$ is uniquely extremal. □

If a quasiconformal mapping $f$ of a region is uniquely extremal for its boundary values then it is clear that the restriction of $f$ to any subregion is uniquely extremal, and, in particular, extremal, for its boundary values on that subregion. Accordingly, the following holds.

THEOREM 2.2. *Suppose $f : \Delta \to \Delta$ is uniquely extremal. Then for any simply-connected subregion $\Omega \subset \Delta$,*

$$\sup\left\{ \frac{\left|\iint_\Omega \mu_f(z)\varphi(z)\,dx\,dy\right|}{\iint_\Omega |\varphi(z)|\,dx\,dy} \colon \varphi \in L_a^1(\Omega) \right\} = \operatorname{ess\,sup}\big\{|\mu_f(z)| \colon z \in \Omega\big\}. \tag{2.1}$$

EXAMPLE 5.2.1 (*The necessary condition of Theorem 2.2 is not sufficient for unique extremality*). That is, even if the restriction of a mapping to every subregion is extremal for the boundary values on that subregion, that does not necessarily make the mapping uniquely extremal on the original region. In fact, we show that even if (2.1) held for every measurable set $\Omega$ with $|\Omega| > 0$, and $L_a^1(\Omega)$ is replaced by the smaller class $L_a^1(\Delta)$ in (2.1), that is still not sufficient to make $f$ uniquely extremal.

Instead of working in $\Delta$, we consider the parabolic region $G_2$ of Example 1.4.6 and Example 3.5.1. As we know, the affine stretch of $G_2$ is extremal but not uniquely extremal for its boundary values on $\partial G_2$. The function $e^{-sz}$ belongs to $L^1_a(G_2)$ whenever $s > 0$. As may be verified (see [112, pp. 307–308] for the details),

$$\lim_{s\searrow 0} \frac{\left|\iint_E e^{-sz}\,dx\,dy\right|}{\iint_E e^{-sx}\,dx\,dy} = 1,$$

for any set $E \subset G_2$ with $|E| > 0$.

Evidently any complex dilatation $\mu(z)$ can be written in the form

$$\mu(z) = |\mu(z)| \frac{\overline{\varphi(z)}}{|\varphi(z)|}, \quad \text{a.a. } z \in \Delta, \tag{2.2}$$

for *some* (in general non-holomorphic) function $\varphi(z)$.

THEOREM 2.3.[14] *Suppose $f$ is a uniquely extremal quasiconformal mapping of $\Delta$, and $\mu_f$ has a representation of the form* (2.2) *where $\varphi(z)$ equals a holomorphic function $\psi(z)$ in the part of $\Delta$ outside a compact subset of $\Delta$. Then $\psi(z)$ has a holomorphic extension to all of $\Delta$, and*

$$\mu_f(z) = \|\mu_f\|_\infty \frac{\overline{\psi(z)}}{|\psi(z)|}, \quad \textit{a.a. } z \in \Delta.$$

PROOF. Let $\{A_n\}$, $A_n = \{z \in \Delta\colon \rho_n \leqslant |z| \leqslant \rho_n'\}$, be a sequence of closed annuli in $\Delta$, with $\rho_n < \rho_n' < \rho_{n+1}$, $n = 1, 2, \ldots$, $\lim \rho_n = 1$, such that $\psi(z) \neq 0$ in $A_n$, $n = 1, 2, \ldots$. In line with the remarks above, the restriction of $f$ to $\{z\colon |z| < \rho_n\}$ is extremal for its boundary values on $\{|z| = \rho_n\}$. Applying Example 2.1.3 to the restriction of $f$ for successive $n$, the theorem follows. □

## 5.3. *Some sufficient conditions*

If $f$ is a quasiconformal mapping of $\Delta$ onto itself, we introduce the real-valued functional $\delta_f$ over $L^1_a(\Delta)$, defined by

$$\delta_f\{\varphi\} = \iint_\Delta \bigl\{k[f]|\varphi(z)| - \Re[\mu_f(z)\varphi(z)]\bigr\}\,dx\,dy, \quad \varphi \in L^1_a(\Delta).$$

Also, for convenience, let

$$\lambda_\mu[\varphi] = \Re \Lambda_\mu[\varphi] = \Re \iint_\Delta \mu(z)\varphi(z)\,dx\,dy, \quad \varphi \in L^1_a(\Delta).$$

[14] Communication from V. Božin, V. Marković and M. Mateljević.

THEOREM 3.1. *Let $A$ be a compact subset of $\Delta$ with empty interior that does not disconnect $\Delta$. Assume that*

$$|\mu_f(z)| \equiv k = k[f] \quad \text{when } z \in \Delta \setminus A, \tag{3.1}$$

$$\operatorname{ess\,sup}\{|\mu_f(z)|:\ z \in A\} < k. \tag{3.2}$$

*Suppose there exists a sequence $\{\varphi_n\}$, $\varphi_n \in L^1_a(\Delta)$, with the following properties:*

$$\lim_{n\to\infty} \delta_f\{\varphi_n\} = 0, \tag{3.3}$$

*and*

$$0 < \liminf_{n\to\infty} |\varphi_n(z)| \leqslant +\infty, \quad \text{a.e. in } \Delta \setminus A. \tag{3.4}$$

*Then $f$ is uniquely extremal.*

PROOF. Suppose $g$ competes with $f$ in the sense that $g$ is also a quasiconformal mapping of $\Delta$ onto itself, and

$$g|_{\partial\Delta} = f|_{\partial\Delta}.$$

Let

$$\alpha(z) = \mu_{f^{-1}}\big(f(z)\big), \qquad \beta(z) = \mu_{g^{-1}}\big(f(z)\big). \tag{3.5}$$

We can write the Main Inequality in the form (3.3.3) as

$$\iint_\Delta L(z)\big|\varphi(z)\big|\,dx\,dy \leqslant \Re \iint_\Delta R(z)\left(\big|\varphi(z)\big| - \frac{\mu_f(z)}{|\mu_f(z)|}\varphi(z)\right)dx\,dy$$
$$\big(\varphi \in L^1_a(\Delta)\big), \tag{3.6}$$

where

$$L = \frac{(|\alpha|^2 - |\beta|^2) + (1-|\alpha|)(|\alpha| - \Re\frac{\bar{\beta}\alpha}{|\alpha|})}{(1+|\alpha|)(1-|\beta|^2)},$$

and

$$R = \frac{\bar{\alpha}}{|\alpha|}\frac{(1-\bar{\beta}\alpha)(\alpha-\beta)}{(1-|\alpha|^2)(1-|\beta|^2)}.$$

It is easy to see from the definition of $\delta_f$ that the assumptions (3.1)–(3.3) imply that

$$\iint_A |\varphi_n|\,dx\,dy \to 0, \tag{3.7}$$

which, again by (3.3), implies that

$$\delta_n' = \iint_{\Delta\setminus A} \{k|\varphi_n| - \Re[\mu_f\varphi_n]\}\,dx\,dy \to 0. \tag{3.8}$$

In view of (3.6), (3.7), we have

$$\begin{aligned}\iint_{\Delta\setminus A} & L(z)|\varphi_n(z)|\,dx\,dy \\ &\leqslant \Re \iint_{\Delta\setminus A} R(z)\left(|\varphi_n(z)| - \frac{\mu_f(z)}{|\mu_f(z)|}\varphi_n(z)\right)dx\,dy + \tau_n,\end{aligned} \tag{3.9}$$

with $\tau_n \geqslant 0$, $\lim \tau_n = 0$. Suppose now that $\|\mu_g\|_\infty \leqslant \|\mu_f\|_\infty$. In view of our hypothesis this means that

$$|\mu_f(z)| = |\alpha(z)| = k, \quad |\beta(z)| \leqslant k \qquad (z \in \Delta\setminus A). \tag{3.10}$$

It follows from (3.10) that an upper bound for the right side of (3.9) is

$$\frac{1+k^2}{(1-k^2)^2 k}\iint_{\Delta\setminus A} |\alpha-\beta|\cdot\big|k|\varphi_n| - \mu_f\varphi_n\big|\,dx\,dy + \tau_n. \tag{3.11}$$

By (3.10),

$$L(z) \geqslant \frac{(1-k)|\alpha(z)|\left(1-\Re\overline{\frac{\beta(z)}{\alpha(z)}}\right)}{(1+k)} = \frac{(1-k)k\left(1-\Re\overline{\frac{\beta(z)}{\alpha(z)}}\right)}{(1+k)},$$

when $z \in \Delta\setminus A$. On the other hand,

$$|\alpha-\beta|^2 = |\alpha|^2 + |\beta|^2 - 2\Re(\bar\beta\alpha) = k^2 + |\beta|^2 - 2k^2\Re\frac{\bar\beta}{\bar\alpha} \leqslant 2k^2\left(1-\Re\frac{\bar\beta}{\bar\alpha}\right),$$

when $z \in \Delta\setminus A$. Therefore,

$$\frac{(1-k)}{2(1+k)k}\iint_{\Delta\setminus A} |\alpha-\beta|^2|\varphi_n|\,dx\,dy = \frac{(1-k)}{2(1+k)k}I_n \tag{3.12}$$

is a lower bound for the left side of (3.9). Since $||w| - w|^2 = 2|w|\,(|w| - \Re w)$ for any complex number $w$, we have, using Schwarz's inequality,

$$\iint_{\Delta\setminus A} |\alpha-\beta|\cdot\big|k|\varphi_n| - \mu_f\varphi_n\big|\,dx\,dy \leqslant \sqrt{2kI_n\delta_n'}. \tag{3.13}$$

So, by (3.9) and (3.11)–(3.13),

$$\frac{(1-k)}{2(1+k)k} I_n \leqslant \frac{\sqrt{2}(1+k^2)}{(1-k^2)^2\sqrt{k}} \sqrt{I_n \delta_n'} + \tau_n.$$

Since $\lim \tau_n = 0$, and $\lim \delta_n' = 0$, it follows that $\lim I_n = 0$. Using (3.4) and Fatou's Lemma, we therefore conclude that

$$\alpha(z) = \beta(z) \quad \text{for a.a. } z \in \Delta \setminus A.$$

From (3.5) it therefore follows that

$$\mu_{f^{-1}}(w) = \mu_{g^{-1}}(w) \quad \text{for } w \in \Delta \setminus f(A).$$

Thus, $g^{-1} = \Upsilon \circ f^{-1}$, with $\Upsilon = g^{-1} \circ f$ conformal in $\Delta \setminus A$. Since $\Upsilon$ has the boundary values of the identity on $\partial\Delta$, we see that $\Upsilon(z) = z$, $z \in \Delta \setminus A$; that is,

$$f(z) = g(z), \quad z \in \Delta \setminus A.$$

Since every point of $\Delta$ is a limit point of $\Delta \setminus A$, it follows by continuity, that $g = f$, thus proving that $f$ is uniquely extremal. □

EXAMPLE 5.3.1 (*Example of a uniquely extremal mapping $f$ of $\Delta$, such that $|\mu_f(z)| < k_0[f]$ on a set of positive measure* [22]). To carry out the construction of $\mu_f$ we choose an arbitrary compact subset $A$ of $\Delta$, containing at least two points and such that $\Delta \setminus A$ is doubly connected, and an arbitrary number $k$, $0 < k < 1$. We will construct a function $\mu \in L^\infty(\Delta)$ and a sequence $\varphi_n \in L_a^1(\Delta)$ $(n = 1, 2, \ldots)$, satisfying the following conditions (3.14)–(3.16):

$$|\mu(z)| = \begin{cases} 0, & z \in A, \\ 1, & \text{for a.a. } z \in \Delta\setminus A, \end{cases} \tag{3.14}$$

$$\lim\{\|\varphi_n\| - \lambda_\mu[\varphi_n]\} = 0, \tag{3.15}$$

$$\lim|\varphi_n(z)| = \infty \quad \text{a.e. in } \Delta\setminus A. \tag{3.16}$$

Once this is done, we choose $A$ as a connected subset of $\Delta$ with empty interior and with $|A| > 0$, and we set $\mu_f(z) = k\mu(z)$. As a result, $k[f] = k$, and $|\mu_f(z)| < k[f]$ for $z \in A$, while, by Theorem 3.1, $f$ is uniquely extremal; that is, $k_0[f] = k[f]$.

*Construction of $\mu(z)$*. Let $\{J_n\}$, $\{A_n\}$ be closed Jordan domains with the following (see Figure 1) properties:[15]

$$J_n \subset \Delta, \quad A_n \subset \Delta, \quad J_n \subset \operatorname{Int} J_{n+1}, \quad A_{n+1} \subset \operatorname{Int} A_n, \quad J_n \cap A_n = \emptyset,$$

[15] Such $\{J_n\}$, $\{A_n\}$ are easily constructed for the case when $A$ is a closed disk $\{|z| \leqslant a\}$ $(0 < a < 1)$, and hence in view of the conformal equivalence of $\Delta\setminus A$ with a ring domain, in general.

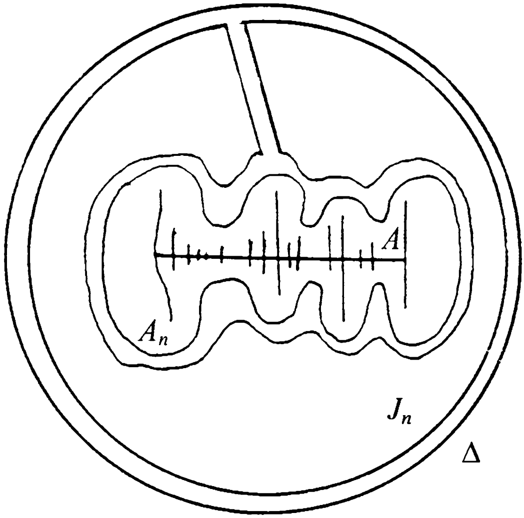


Fig. 1.

$$\left|\bigcup_1^\infty J_n\right| = |\Delta \backslash A|, \qquad \bigcap_1^\infty A_n = A.$$

We proceed with the construction of a sequence of polynomials $\{Q_n(z)\}$, a sequence of polynomials $\{\varphi_n(z)\}$, and a subsequence $\{K_n\}$ of $\{J_n\}$.

($n = 1$). Let $K_1 = J_1$. Let $Q_1(z)$ be a polynomial formed with the help of Runge's theorem [18] such that $|Q_1(z)| > 2$ when $z \in K_1$, $|Q_1(z)| < \frac{1}{2}$ when $z \in A$. Set $\varphi_1(z) = Q_1(z)$.

($n = 2$). Choose $K_2$ from among $\{J_2, J_3, \ldots, \}$ such that

$$\iint_\Delta |\varphi_1(z)|\, dx\, dy - \iint_{A \cup K_2} |\varphi_1(z)|\, dx\, dy < 1.$$

(This is possible since $A \cup \bigcup_1^\infty J_n$ has measure $\pi$.) Let

$$\alpha_1 = \max\left\{|\varphi_1(z)| : z \in K_2\right\}.$$

We next use Runge's theorem[16] to find a polynomial $Q_2(z)$ such that

$$|Q_2(z)| > 2 \quad \text{and} \quad \alpha_1 \left|1 - \frac{|Q_2(z)|}{Q_2(z)}\right| < \frac{1}{2} \quad \text{when } z \in K_2,$$

[16]Namely, if $K_2 = J_m$, we use Runge's theorem to uniformly approximate the function equalling the constant 3 in $J_{m+1}$ and 0 in $A_{m+1}$ by a polynomial. This is possible because, since $J_n$ and $A_n$ are disjoint Jordan domains, neither separates the point at infinity from the other.

and

$$|Q_2(z)| < \frac{1}{2} \quad \text{for } z \in A.$$

Set

$$\varphi_2(z) = \varphi_1(z) Q_2(z).$$

($n \to n+1$). Proceeding as above, we find increasing Jordan domains $K_3, K_4, \ldots$ from among $\{J_n\}$, and polynomials $Q_3, Q_4, \ldots$, such that $K_{n+1}$ satisfies

$$\iint_\Delta |\varphi_n(z)|\, dx\, dy - \iint_{A \cup K_{n+1}} |\varphi_n(z)|\, dx\, dy < \frac{1}{n}, \tag{3.17}$$

and that $Q_{n+1}$ satisfies

$$|Q_{n+1}(z)| > 2 \quad \text{and} \quad \alpha_n \left| 1 - \frac{|Q_{n+1}(z)|}{Q_{n+1}(z)} \right| < \frac{1}{2^n}, \quad \text{when } z \in K_{n+1}, \tag{3.18}$$

and

$$|Q_{n+1}(z)| < \frac{1}{2}, \quad \text{when } z \in A, \tag{3.19}$$

where

$$\alpha_n = \max\{ |\varphi_n(z)| \colon z \in K_{n+1} \}. \tag{3.20}$$

We set

$$\varphi_{n+1}(z) = \varphi_n(z) Q_{n+1}(z) \quad (z \in \Delta). \tag{3.21}$$

For $z \in \bigcup_1^\infty K_n$ it is clear that

$$\lim |\varphi_n(z)| = \infty;$$

thus,

$$\lim |\varphi_n(z)| = \infty \quad \text{for a.a. } z \in \Delta \backslash A, \quad \text{and} \quad \lim \|\varphi_n\| \equiv \infty. \tag{3.22}$$

Moreover, $|\varphi_n(z)| > 2^n$ for $z \in K_1$. Thus,

$$\alpha_n > 2^n, \quad n = 1, 2, 3, \ldots. \tag{3.23}$$

Let

$$\mu_n(z) = \begin{cases} \dfrac{|\varphi_n(z)|}{\varphi_n(z)}, & z \in \bigcup_1^\infty K_n, \text{ i.e., a.e. in } \Delta\backslash A, \\ 0, & z \in A. \end{cases} \tag{3.24}$$

By (3.21), then,

$$\mu_{n+1}(z) = \mu_n(z)\frac{|Q_{n+1}(z)|}{Q_{n+1}(z)} \quad \text{a.e. in } \Delta\backslash A,$$

and, by (3.18),

$$\alpha_n\left|1 - \frac{\mu_{n+1}(z)}{\mu_n(z)}\right| < \frac{1}{2^n} \quad \text{for } z \in K_{n+1}.$$

Hence,

$$\alpha_n\left|\mu_{n+1}(z) - \mu_n(z)\right| < 2^{-n}\left|\mu_n(z)\right| = 2^{-n} \quad \text{for } z \in K_{n+1}, \tag{3.25}$$

and therefore, by (3.23),

$$\left|\mu_{n+1}(z) - \mu_n(z)\right| < 4^{-n}, \quad \text{for } z \in A \cup K_{n+1}. \tag{3.26}$$

Thus, for every $z \in A \cup \bigcup_1^\infty K_n$, $\{\mu_j(z)\}$ is a Cauchy sequence; that is, $\mu(z) = \lim \mu_n(z)$ exists a.e. in $\Delta$. In particular,

$$\left|\mu(z)\right| = 1 \quad \text{a.e. in } \Delta\backslash A, \quad \text{and} \quad \mu(z) = 0 \text{ in } A. \tag{3.27}$$

It remains to show that relation (3.15) holds. Since the sets $K_n$ are increasing, (3.26) implies that

$$\begin{aligned} \left|\mu(z) - \mu_n(z)\right| &\leqslant 4^{-n} + 4^{-(n+1)} + 4^{-(n+2)} + \cdots \\ &= \frac{1}{3 \cdot 4^{n-1}}, \quad \text{for } z \in A \cup K_{n+1}. \end{aligned} \tag{3.28}$$

Moreover, since $\{\alpha_n\}$ is obviously an increasing sequence, (3.25) implies that

$$\alpha_n|\mu - \mu_n| \leqslant \alpha_n \sum_{j=n}^\infty |\mu_{j+1} - \mu_j| \leqslant \sum_{j=n}^\infty \alpha_j|\mu_{j+1} - \mu_j| \leqslant \sum_{j=n}^\infty 2^{-j} = 2^{-(n-1)}$$

for $z \in A \cup K_{n+1}$; that is,

$$\left|\mu(z) - \mu_n(z)\right| \cdot \left|\varphi_n(z)\right| \leqslant 2^{-(n-1)}, \quad z \in A \cup K_{n+1}. \tag{3.29}$$

Now, by (3.17), we have

$$\begin{aligned} \|\varphi_n\| &= \iint_{A\cup K_{n+1}} |\varphi_n|\,dx\,dy + o(1), \\ \Lambda_\mu[\varphi_n] &= \iint_{A\cup K_{n+1}} \mu\varphi_n\,dx\,dy + o(1). \end{aligned} \tag{3.30}$$

On the other hand, by (3.29), (3.24),

$$\iint_{A\cup K_{n+1}} |\varphi_n|\,dx\,dy = \iint_{A\cup K_{n+1}} \mu\varphi_n\,dx\,dy + o(1).$$

From (3.30) and the above it follows that

$$\|\varphi_n\| = \Lambda_\mu[\varphi_n] + o(1).$$

This completes the construction of $\mu(z)$.

NOTE. Although $|\mu(z)| \equiv \|\mu\|_\infty$ a.e. in $\Delta \setminus A$, Theorem 2.3 tells us that $\mu_f(z)$ cannot be of Teichmüller type in $\Delta \setminus A$.

In a useful special case of Theorem 3.1, $A = \emptyset$, and $\mu_f$ is of Teichmüller type, but not necessarily of finite norm:

THEOREM 3.2. *Suppose*

$$\mu_f(z) = k\frac{\overline{\varphi_0(z)}}{|\varphi_0(z)|} \quad \textit{at a.a. } z \in \Delta,$$

*where* $0 < k < 1$, *and* $\varphi_0(z)$ *is a measurable function on* $\Delta$ *that vanishes on at most a set of measure zero. If there exists a sequence* $\{\varphi_n\}$, $\varphi_n \in L_a^1(\Delta)$, *such that both following conditions* (3.31) *and* (3.32) *hold,*

$$\lim_{n\to\infty} \varphi_n(z) = \varphi_0(z) \quad \textit{pointwise a.e. in } \Delta, \tag{3.31}$$

*and*

$$\lim_{n\to\infty} \Re \iint_\Delta \left[ |\varphi_n(z)| - \frac{\overline{\varphi_0(z)}}{|\varphi_0(z)|}\varphi_n(z) \right] dx\,dy = 0. \tag{3.32}$$

*Then $f$ is uniquely extremal.*[17]

[17]Since we do not a-priori assume in the hypotheses of Theorem 3.2 that $\varphi_0(z)$ is holomorphic, the theorem allows $f$ to be a-priori more general than of Teichmüller type. The author has been advised by V. Božin that there in fact exist non-holomorphic functions $\varphi_0(z)$ satisfying the hypotheses. He states that the construction of $\mu_f$ can be carried out on a simply-connected region obtained by starting with a modification of the type of construction used for Example 5.3.1, followed by slitting $\Delta$ from $\partial\Delta$ to the set $A$.

An alternative to Theorem 3.2 in which part of the hypothesis is strengthened and another part weakened is sometimes technically useful. It is derived directly from the main inequality in [111], and is as follows:

THEOREM 3.3. *Suppose*

$$\mu_f(z) = k\frac{\overline{\varphi_0(z)}}{|\varphi_0(z)|} \quad \text{at a.a. } z \in \Delta,$$

*where* $0 < k < 1$, *and* $\varphi_0(z)$ *is a locally* $L^1$ *function on* $\Delta$ *that vanishes on at most a set of measure zero.*

*Suppose there exists a sequence* $\{\varphi_n\}$, $\varphi_n \in L^1_a(\Delta)$, *such that the following conditions hold,*

$$\lim_{n\to\infty} \varphi_n(z) = \varphi_0(z) \quad \text{pointwise a.e. in } \Delta,$$

*the sequence*

$$Re \iint_\Delta \left[ |\varphi_n(z)| - \frac{\overline{\varphi_0(z)}}{|\varphi_0(z)|}\varphi_n(z) \right] dx\, dy, \quad n = 1, 2, 3, \ldots, \tag{3.33}$$

*is bounded, and*

$$\liminf_{M\to\infty} \iint_{\Omega(n,M)} |\varphi_n(z)|\, dx\, dy = 0$$

*uniformly with respect to n, where*

$$\Omega(n, M) = \{z \in \Delta: |\varphi_n(z)| > M|\varphi_0(z)|\}.$$

*Then* $f$ *is uniquely extremal.*

A computation shows that the expression (3.32) of Theorem 3.2 and the analogous expression in the statement of Theorem 3.3 remain invariant if $f$ is transferred to an arbitrary simply-connected region $\Omega$ by means of a conformal mapping. Both theorems therefore remain valid for quasiconformal mappings of $\Omega$ providing the domains of integration are replaced by $\Omega$, and $\varphi_n \in L^1_a(\Omega)$, $n = 1, 2, 3, \ldots$.

EXAMPLE 5.3.2 (*Affine stretch of regions* $G_\beta$). (Cf. Examples 1.4.6 and 3.5.1.) By Theorem 3.2, a sufficient condition for the affine stretch of $G_\beta$ to be uniquely extremal is that there exist $\varphi_n \in L^1_a(G_\beta)$ such that

$$\lim_{n\to\infty} \varphi_n(z) = 1 \text{ a.e.} \quad \text{and} \quad I_n = \lim_{n\to\infty} \Re \iint_{G_\beta} \left[ |\varphi_n(z)| - \varphi_n(z) \right] dx\, dy = 0.$$

Using $\varphi_n(z) = \mathrm{e}^{-tz}$, $t = 1/n$, one finds that

$$\lim_{n\to\infty} I_n = \begin{cases} \infty, & 1 < \beta < 3, \\ 1/6, & \beta = 3, \\ 0, & \beta > 3. \end{cases}$$

By Theorem 3.2 it follows that the affine stretch is uniquely extremal when $\beta > 3$. With the help of Theorem 3.3, the same conclusion follows for $\beta = 3$.

EXAMPLE 5.3.3 (*Criterion based on mean growth of $\varphi(z)$*). If $\varphi(z)$ is holomorphic in $\Delta$, then, for any $0 < t < 1$, $\varphi(tz)$ belongs to $L_a^1(\Delta)$. If one uses $\varphi_n(z) = \varphi(t_n z)$, with $t_n \nearrow 1$ with Theorem 3.3, it is possible to prove the following [51]: Suppose

$$\int_0^{2\pi} |\varphi(re^{i\theta})|\, d\theta = \mathrm{O}\left(\frac{1}{1-r}\right), \quad 0 \leqslant r < 1.$$

*Then the Teichmüller-type mapping with $\mu_f = k\bar{\varphi}/|\varphi|$ is uniquely extremal.*

A classical example is obtained when $\varphi$ is holomorphic for $|z| \leqslant 1$ except for at worst second-order poles on $\{|z| = 1\}$ (Sethares [144]). For example, the affine stretch of a half-strip (cf. Example 1.4.3) falls into this category.

NOTES. Theorem 3.1 is a special case of the necessary and sufficient condition of Božin, Lakic, Marković, and Mateljević [22], discussed below, and examples like Example 5.3.1 were first constructed by these authors. The above exposition follows Reich [126]. Theorem 3.2 is due to Reich [110,111]. For further results connected with Example 5.3.2 and Example 5.3.3, see the work of Huang XinZhong and coauthors [55–57].

### 5.4. *Necessary and sufficient conditions for unique extremality*

The decisive discoveries that closed a chapter in the search for necessary and sufficient conditions for unique extremality are due to Božin, Lakic, Marković, and Mateljević in the papers [89,22,23]. In Theorem 4.1, below, we state a part of their results. We first introduce a definition.

If $\kappa \in L^\infty(\Delta)$, we refer to $X(\kappa) = \{z \in \Delta\colon |\kappa(z)| = \|\kappa\|_\infty\}$ as the *extremal set* of $\kappa$. The extremal set of $\kappa$ may of course be empty.

Recall also the definition of *admissible variation* of an element of $L^\infty(\Delta)$ from Section 2.3.

THEOREM 4.1. *The following conditions are equivalent.*

(1) *The quasiconformal mapping $f$ of $\Delta$ onto itself is uniquely extremal for its boundary values.*
(2) *The formula $\iint_\Delta \mu_f \varphi\, dx\, dy$, $\varphi \in L^1(\Delta)$, provides the unique Hahn–Banach extension of $\Lambda_{\mu_f}$ from $L_a^1(\Delta)$ to $L^1(\Delta)$.*

(3) *For any admissible variation $\eta$ of $\mu_f$, there exists a sequence $\{\varphi_n\}$, $\varphi_n \in L_a^1(\Delta)$, with the following properties*:

$$\lim_{n\to\infty} \iint_\Delta \{k[f]|\varphi_n(z)| - \Re[\eta(z)\varphi_n(z)]\}\, dx\, dy = 0, \tag{4.1}$$

*and*

$$0 < \liminf_{n\to\infty} |\varphi_n(z)| \leqslant +\infty, \quad \text{for a.a. } z \in X(\eta). \tag{4.2}$$

NOTE. In Theorem 3.1 the extremal set is $\Delta \setminus A$. Theorem 3.1 is a fairly direct consequence of the characterization (3) above. Since, by (2), unique extremality for the quasiconformal mapping problem is equivalent to that for the Hahn–Banach extension of $\Lambda_\mu$ from $L_a^1(\Delta)$ to $L^1(\Delta)$, Example 5.3.1 provides *eo ipso* an example for the latter where the extension is unique but where $\mu(z)$ does not have a.e. constant absolute value. Indeed, the original construction of the $\mu(z)$ of Example 5.3.1 occurred in [23] where a necessary and sufficient condition for uniqueness of the Hahn–Banach extension was derived from first principles by examining the standard proof [139] of the Hahn–Banach theorem (see also [148]). This resulted in recognizing that conditions essentially like those of Theorem 3.1 were sufficient for uniqueness of the Hahn–Banach extension.

EXAMPLE 5.4.1 (*Integration identities for holomorphic functions*). Given a simply-connected region $\Omega$, consider the area integral

$$\Lambda_\Omega[\varphi] = \iint_\Omega \varphi(z)\, dx\, dy, \quad \varphi \in L_a^1(\Omega).$$

If there exists $\tau \in L^\infty(\Omega)$ with the properties $\|\tau\|_\infty \leqslant 1$, $|\{z\colon \tau(z) \neq 1\}| > 0$, such that

$$\iint_\Omega \varphi(z)\, dx\, dy = \iint_\Omega \tau(z)\varphi(z)\, dx\, dy, \quad \text{for all } \varphi \in L_a^1(\Omega),$$

we speak of an integration identity for holomorphic functions over $\Omega$. It is easy to see that an integration identity cannot exist if $|\Omega| < \infty$. By Theorem 3.4.3 and part (2) of Theorem 5.4.1, such an identity exists if and only if the affine stretch of $\Omega$ is not uniquely extremal. Thus, for the regions $G_\beta$ ($\beta \geqslant 1$), of Example 1.4.6, integration identities will exist if and only if $1 \leqslant \beta < 3$. For explicit formulas see [109]. In another context, integration identities have also occurred in the work of H.S. Shapiro.

The following result [22,23] which is implied by (and actually included in) the statement of Theorem 4.1 provides an interesting complement to Theorem 2.3.1.

THEOREM 4.2. *Suppose $\mu_g$ is an admissible variation of $\mu_f$. If $f$ is uniquely extremal, then so is $g$.*

PROOF. By Theorem 2.3.1, $g$ is extremal. Therefore,

$$\|\Lambda_{\mu_g}\| = \|\mu_g\|_\infty = \|\mu_f\|_\infty.$$

Suppose there exists $\tilde{\eta} \in L^\infty(\Delta)$, such that $\tilde{\eta} - \mu_g \in \mathcal{N}(\Delta)$, and $\|\tilde{\eta}\|_\infty = \|\mu_g\|_\infty$.
Define $\tilde{\mu}$ as in the proof of Theorem 2.3.1. Replacing $\delta$ by 0 in that proof, we obtain

$$\|\tilde{\mu}\|_\infty = \|\mu_f\|_\infty, \qquad \tilde{\mu} - \mu_f \in \mathcal{N}(\Delta).$$

By Theorem 4.1, part (2), it follows from our hypothesis that $f$ is uniquely extremal, that $\tilde{\mu} = \mu_f$. Hence, $\tilde{\eta} = \mu_g$; that is, the Hahn–Banach extension of $\Lambda_{\mu_g}$ from $L_a^1(\Delta)$ to $L^1(\Delta)$ is unique. Again, by Theorem 4.1, part (2), it now follows that $g$ is uniquely extremal. □

### 5.5. *Point-shift mappings. Variability sets*

Although we have seen (Example 5.3.1) that there are quasisymmetric boundary values $h$ of the disk for which no extremal mapping of Teichmüller type or any other extremal mappings with constant dilatation exist, we will now see (Corollary of Theorem 5.1) that there *are* quasiconformal mappings $f$ with boundary values $h$ with (a.e.) constant $D_f(z)$ with value $D_f(z)$ *arbitrarily close* to $K_0[h]$. These developments are largely due to Strebel [158,169,170]. Even though this article restricts itself in principle to quasiconformal mappings of the disk, we need to make an exception in this section, and also allow the *punctured* disk as a domain of the mapping. It is known [161] that the Basic Inequality theorem, Theorem 3.2.1 holds for *arbitrary* planar regions $\Omega$. Hence, also the Main Inequality, Theorem 3.3.1, as well as its corollaries,[18] including the Frame Mapping Criterion, Theorem 4.2.1, hold for arbitrary $\Omega$, and, in particular, for the punctured disk.

EXAMPLE 5.5.1 (*Teichmüller's shift mapping* [175]). Given $0 < s < 1$, we wish to find an extremal quasiconformal mapping $T(z)$ of $\Delta$ onto itself that is the identity on $\partial\Delta$ and satisfies $T(0) = s$. This is equivalent to finding an extremal quasiconformal mapping $T(z)$ of the punctured disk $\Delta_0 = \{z \in \Delta\colon\ z \neq 0\}$ onto the punctured disk $\Delta_s = \{z \in \Delta\colon\ z \neq s\}$ that is the identity on $\partial\Delta$, as for this mapping, $z = 0$ is a removable singularity ([84] 1.8.1). It is obvious that a neighborhood of $\partial\Delta_0$ can be mapped onto a neighborhood of $\partial\Delta_s$ so as to be conformal at each point, namely by using the identity mapping near $\{|z| = 1\}$, and the mapping $z + s$ near $z = 0$. By the frame mapping criterion, applied to the region $\Omega = \Delta_0$, it therefore follows that there is a uniquely extremal mapping $T(z)$ with the required boundary values, and that $T(z)$ is a Teichmüller mapping with a quadratic differential $\varphi(z)$ that is holomorphic in $\Delta_0$ and has finite norm over $\Delta_0$. Since $\varphi$ has an isolated singularity at $z = 0$, and $\varphi \in L^1(\Delta_0)$, it is not difficult to show that, as a consequence, $\varphi$ has at worst a simple pole at $z = 0$, but the singularity must actually

[18] Among the more important corollaries is the fact that the necessary and sufficient condition for extremality, Theorem 3.4.3, holds for arbitrary $\Omega$, because the necessary condition, Theorem 2.1.1, was also proved by Hamilton for arbitrary $\Omega$. Similarly, the necessary and sufficient conditions for unique extremality of Theorem 4.1 are valid for arbitrary $\Omega$.

be a simple pole, because if $\varphi$ were holomorphic at $z = 0$, then, by Theorem 1.4.1, $T(z)$ would be extremal for its boundary values on the *non*-punctured disk $\Delta$, which would mean that $T(z)$ would have to be the identity in $\Delta$. There seems to be no practical systematic procedure for finding $\varphi$ more or less explicitly other than by "luck". It turns out [117] that the problem has a rather elementary solution if instead of starting with the disk $\Delta$ and looking for an extremal mapping $T$ that is the identity on the unit circle and shifts 0 to $s$, we start with the region $\mathcal{E}$, bounded by an ellipse with eccentricity $\tau$, and foci at $z = 0$ and $z = z_2 < 0$, and and look for an extremal mapping $F$ that is the identity on $\partial\mathcal{E}$ and maps 0 onto $z_2$. The solution falls into our hands essentially unexpectedly if we decide to study quasiconformal mappings $F$ with the complex dilatation

$$\mu_F(z) = kz/|z|, \tag{5.1}$$

that is, Teichmüller mappings with the quadratic differential $1/z$. Since $1/z \in L_a^1(\mathcal{E}_0)$, where $\mathcal{E}_0$ denotes $\mathcal{E}$ punctured at $z = 0$, we know that any quasiconformal mapping $F$ satisfying (5.1) is the uniquely extremal mapping of $\mathcal{E}_0$ for the boundary values that it induces. In particular, consider

$$F(z) = \frac{1}{1-\tau^2}\left(z + 2\tau|z| + \tau^2\bar{z}\right) + z_2, \tag{5.2}$$

which satisfies (5.1), with $k = \tau$. One verifies that (5.2) is a homeomorphism of $\mathcal{E}$ onto itself that keeps all points of the boundary ellipse fixed and sends $z = 0$ to $z = z_2$. Suppose now $z = G(\zeta)$ maps $\{|\zeta| < 1\}$ conformally onto $\mathcal{E}$, $G(0) = 0$, $G^{-1}(z_2) = s$. Then

$$T = G^{-1} \circ F \circ G$$

is the mapping we are looking for, and

$$\mu_T(\zeta) = \frac{\overline{G'(\zeta)}}{G'(\zeta)}\mu_F\big(G(\zeta)\big) = \tau\frac{\overline{\varphi(\zeta)}}{|\varphi(\zeta)|},$$

with $\varphi(\zeta) = [G'(\zeta)]^2/G(\zeta)$ $(|\zeta| < 1)$.

A given quasisymmetric boundary correspondence $h: \partial\Delta \to \partial\Delta$ may or may not give rise to a uniquely extremal mapping. In order to obtain more insight into these two alternatives, the *variability set* and the *point shift* mapping were introduced by Strebel [158]. If $z_0 \in \Delta$, the variability set of $z_0$ is defined as

$$V[h; z_0] = \big\{ f_0(z_0)\colon\ f_0 \text{ is extremal for the boundary values } h \big\}.$$

Evidently, unique extremality occurs if and only if $V[h; \zeta]$ consists of a single point for all $\zeta \in \Delta$.

The idea of the point shift mapping is simply a generalization of the Teichmüller shift mapping of Example 5.5.1. Namely, suppose $z_0 \in \Delta$, $\omega \in \Delta$, $\omega \notin V[h; z_0]$. By the same reasoning as in Example 5.5.1, there exists a uniquely extremal mapping $g$ of $\Delta$ which takes $z_0$ to $\omega$ and which has the given boundary values $h$ on $\partial\Delta$. The mapping $g$ is of Teichmüller type with a quadratic differential of finite norm that is holomorphic in $\Delta$ except for a simple pole at $z = z_0$. We refer to $g$ as a point shift mapping, and denote its dilatation $K[g]$ by

$$K[g] = K_0[h, z_0; \omega],$$

or, for a fixed quasisymmetric mapping $h$, and fixed $z_0 \in \Delta$, just by $K(\omega)$, for short. (So, for the mapping $T$ of Example 5.5.1, $K[T] = (1+\tau)/(1-\tau) = K_0[\textit{identity}, 0; s] = K(s)$.) To extend the definition of $K(\omega)$ to all of $\Delta$, one defines

$$K(\omega) = K_0[h], \quad \text{for } \omega \in V[h; z_0].$$

THEOREM 5.1. *For fixed $h$, $z_0$, the functions $K(\omega)$ have the following properties:*

(1) *$K(\omega)$ is a continuous function of $\omega$, $\omega \in \Delta$.*
(2) *For $t > K_0[h]$, the set $\{\omega \in \Delta\colon\ K(\omega) = t\}$ is a Jordan curve.*
(3) $\lim_{|\omega|\to 1} K(\omega) = \infty$.

COROLLARY. *The sets $V[h; z_0]$ and $\Delta \setminus V[h; z_0]$ are both connected. If $\omega_0 \in \partial V[h; z_0]$, $\omega_n \in \Delta \setminus V[h; z_0]$, $\lim \omega_n = \omega_0$, then the corresponding point shift mappings $g_n$ are of Teichmüller type with boundary values $h$, and such that $K[g_n] \to K_0[h]$.*

For the proof of Theorem 5.1 and further properties of the variability sets and the point shift mappings the reader is referred to the papers of Strebel cited above.

## 6. The case of infinitesimal dilatations

### 6.1. *LNA extensions*

Suppose $\kappa \in L^\infty(\Delta)$. (We do not require that $\|\kappa\|_\infty < 1$, but merely that $\|\kappa\|_\infty < \infty$.) If $t$ is a real or complex number with $|t|$ close to zero, we can think of $t\kappa(z)$ as an "infinitesimal" complex dilatation of a mapping $f(z,t)$ which is an (infinitesimal) deformation of the identity; that is,

$$f(z,t) \approx z + tg(z) \quad (z \in \Delta). \tag{1.1}$$

(Note that in order for $f(z,t)$ to be a homeomorphism for sufficiently small $|t|$ it is not necessary that $g(z)$ is a homeomorphism, but it suffices for example that $g_z$ and $g_{\bar z}$ are bounded.) If we differentiate (1.1) with respect to $z$ and $\bar z$ we have, formally,

$$(\partial f)(z,t) = f_z(z,t) \approx 1 + \mathrm{O}(t), \qquad (\bar\partial f)(z,t) = f_{\bar z}(z,t) \approx tg_{\bar z}(z),$$

or, $\mu_f(z,t) \approx tg_{\bar z}(z)$. So, so far as terms of order $t$ are concerned, finding an extremal deformation of the identity which is the extension of a homeomorphism of $\partial\Delta$ that is itself close to the identity is equivalent to minimizing $\|\bar{\partial} g\|_\infty$, given the values of $g$ on $\partial\Delta$. These heuristic ideas lead to the following formulation.

PROBLEM. Let $g(z)$, $z \in \partial\Delta$, be a continuous complex-valued function. Let $\mathcal{V}[g]$ denote the class of continuous complex-valued extensions $G$ of $g$ to $\Delta \cup \partial\Delta$, possessing bounded generalized $\bar{\partial}$-derivatives; that is, $\|\bar{\partial} G\|_\infty < \infty$. If $\mathcal{V}[g]$ is non-empty, let

$$m_0[g] = \inf\{\|\bar{\partial} G\|_\infty : G \in \mathcal{V}[g]\}.$$

If $G_0 \in \mathcal{V}[g]$ is such that $\|\bar{\partial} G_0\|_\infty = m_0[g]$, then we refer to $G_0$ as an *LNA* (*least non-analytic*) *extension of* $g$, and we refer to $\bar{\partial} G_0$ as *extremal for the LNA problem.* Given $q \in L^\infty(\Delta)$, a function $G$ such that $\bar{\partial} G = G_{\bar{z}} = q$ is determined up to an additive function holomorphic in $\Delta$ by

$$G(z) = -\frac{1}{\pi} \iint_\Delta \frac{q(\zeta)}{\zeta - z}\, d\xi\, d\eta.$$

Thus, the problem of determining LNA extensions is equivalent to the problem of characterizing elements $q$ of class $L^\infty(\Delta)$ that are extremal for the LNA problem. This is very much analogous to the situation with extremal quasiconformal mappings where the problem of finding extremal extensions is equivalent to characterizing their complex dilatations. In fact, the analogy goes much further as we will now see.

For $\kappa \in L^\infty(\Delta)$, let

$$\Lambda_\kappa[\varphi] = \iint_\Delta \kappa(z)\varphi(z)\, dx\, dy \qquad \bigl(\varphi \in L_a^1(\Delta)\bigr).$$

Thus, $\iint_\Delta \kappa\varphi\, dx\, dy$ is a Hahn–Banach extension of $\Lambda_\kappa$ from $L_a^1(\Delta)$ to $L^1(\Delta)$ if and only if $\|\Lambda_\kappa\| = \|\kappa\|_\infty$. For extremality for the LNA problem the situation is *verbatim* the same as was the case in Section 3.4 and Section 3.5 where $\kappa$ played the role of a complex dilatation; namely:

THEOREM 1.1. *Suppose $q \in L^\infty(\Delta)$. Then $q$ is extremal for an LNA problem if and only if $\iint_\Delta q\varphi\, dx\, dy$ is a Hahn–Banach extension of $\Lambda_q$ from $L_a^1(\Delta)$ to $L^1(\Delta)$. Furthermore, $q$ is uniquely extremal for an LNA problem if and only if $\iint_\Delta q\varphi\, dx\, dy$ is the unique Hahn–Banach extension of $\Lambda_q$ from $L_a^1(\Delta)$ to $L^1(\Delta)$.*

The frame mapping criterion (Section 4.2) also has its counterpart for infinitesimal dilatations. To formulate it, one introduces the Zygmund class $\lambda_*(\partial\Delta)$ of continuous complex-valued functions satisfying

$$g(z+h) - 2g(z) + g(z-h) = \mathrm{o}(h)$$

uniformly with respect to $z$, whenever $z, z+h, z-h \in \partial\Delta$.

THEOREM 1.2. *Suppose $g \in \lambda_*(\partial\Delta)$. Then there is a unique LNA extension $G_0$, and $\bar{\partial}G_0$ must be of the form*

$$\bar{\partial}G_0(z) = \|\bar{\partial}G_0\|_\infty \frac{\overline{\varphi_0(z)}}{|\varphi_0(z)|} \quad a.e.\ in\ \Delta \tag{1.2}$$

*for some $\varphi_0 \in L^1_a(\Delta)$.*

NOTES. Infinitesimal dilatations were already considered extensively by Teichmüller in [173]. For criteria to guarantee that $\mathcal{V}[g]$ is non-empty and proofs of Theorem 1.1 and Theorem 1.2, see [127].

Many results for extremal quasiconformal mappings have their counterparts for LNA extensions (see, e.g., [121–123,125]), and a mutual relationship also exists in the direction of *generating* quasiconformal mappings by means of deformations of the identity, using the so-called *parametric representation* [149,45,8]. In addition to various distortion theorems for quasiconformal mappings, the Fundamental Variational Lemma 2.2.1 can be derived by such a procedure [120].

### 6.2. *An example*

We consider [125] the boundary values

$$g(z) = \bar{z}^n, \quad z \in \partial\Delta, \tag{2.1}$$

where $n$ is a positive integer. Since $g \in \lambda_*(\partial\Delta)$, the LNA extension $G_0$ is unique; by Theorem 1.2, $\bar{\partial}G_0$ is of the form (1.2). Evidently, $\|\bar{\partial}G_0\|_\infty = m_0[g] > 0$ as there is no holomorphic extension of $g$ to $\Delta \cup \partial\Delta$. In view of the uniqueness of $G_0$, the symmetry of $g$ implies that $G_0(z)$ has the form

$$G_0(z) = p(r^2)\bar{z}^n \quad (z = re^{i\theta}), \tag{2.2}$$

where

$$G_0(0) = \lim_{r\to 0} r^n p(r^2) = 0, \quad \lim_{r\to 1} p(r^2) = 1. \tag{2.3}$$

Since $\varphi_0$ has at most finitely many zeroes in any compact subset of $\Delta$, one sees that $p$ is a $C^\infty$ function of $r^2$, $0 < r < 1$. By (2.2),

$$|\bar{\partial}G_0(z)| = nr^{n-1}p(r^2) + r^{n+1}p'(r^2).$$

So, in view of (1.2),

$$nr^{n-1}p(r^2) + r^{n+1}p'(r^2) \equiv \text{const} = A = m_0[g].$$

The only solution of this differential equation satisfying (2.3) is

$$p(t) = \frac{2}{n+1} A t^{\frac{1-n}{2}}.$$

Hence

$$A = m_0[g] = \frac{n+1}{2}, \qquad G_0(re^{i\theta}) = re^{-in\theta}.$$

For $\bar{\partial} G_0(z)$ we obtain

$$\bar{\partial} G_0(z) = \frac{n+1}{2} \frac{\overline{\varphi_0(z)}}{|\varphi_0(z)|}, \quad \varphi_0(z) = z^{n-1}.$$

### 6.3. *Explicit extension operators*

As is the case for extremal quasiconformal extensions, there does not exist a linear operator which when acting on $g$ ($\mathcal{V}[g] \neq \emptyset$), produces LNA extensions. (The proof is analogous to the one in Example 4.2.3.) However, there are some simple linear integral operators which when applied to a function $g$ on $\partial\Delta$ produce an extension whose $\bar{\partial}$-derivative has a sup norm comparable to $m_0[g]$. An example is the class of operators $T_\alpha$ ($\alpha > 2$), defined by (3.1) below.[19]

$$(T_\alpha g)(z) = \frac{(1-|z|^2)^{\alpha-1}}{2\pi i} \int_{\partial\Delta} \frac{g(\zeta)\, d\zeta}{(1-\bar{z}\zeta)^{\alpha-1}(\zeta - z)}, \quad z \in \Delta. \tag{3.1}$$

THEOREM 3.1 [127,125]. *Suppose $g$ is a continuous complex-valued function on $\partial\Delta$, $\mathcal{V}[g] \neq 0$. Then*

$$\|\bar{\partial} T_\alpha g\|_\infty \leqslant C_\alpha m_0[g],$$

*where*

$$C_\alpha = \frac{(\alpha-1)\Gamma(\alpha-2)}{[\Gamma(\alpha/2)]^2} \quad (\alpha > 2).$$

The minimum value of $C_\alpha$ is 2.52710.... It is obtained for approximately $\alpha = 3.14$.

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[19] These operators are related to the class of operators considered by Forelli and Rudin [40]. The case $\alpha = 4$ also occurs in the work of Earle [29] as well as in [119,120,122]. See also [21].

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JFM = Jahrbuch über die Fortschritte der Mathematik
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CHAPTER 4

# Conformal Welding

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*Contents*



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## 1. Introduction

In conformal welding (or sewing or glueing) one uses conformal mappings of the inside and outside of the unit disk $\mathbf{U} = \{|z| < 1\}$ to represent homeomorphisms $\phi$ of the unit circle $\mathbf{T} = \{e^{it}\colon\ 0 \leqslant t \leqslant 2\pi\}$. These homeomorphisms need not be differentiable, let alone analytic. The theory has important applications including Teichmüller space (i.e., the space of all Riemann surfaces), first imagined by Riemann in the nineteenth century. Its importance carries forward into physics as String Theory (i.e., the grand unified theory) which is based on Teichmüller space, see Witten's 1986 address to the ICM [34].

The following provides a simple example of the conformal welding. We assume $\phi : \mathbf{T} \to \mathbf{T}$ is analytic on some neighbourhood of $\mathbf{T}$. Construct the abstract sphere $\Omega$ by joining the unit disk $\mathbf{U}$ to its exterior $\mathbf{L} = \{|z| > 1\} \cup \{\infty\}$ along the circle $\mathbf{T}$ by the correspondence

$$z \to 1/\overline{\phi(1/\bar{z})}.$$

However the abstract Riemann surface $\Omega$ is conformally equivalent to the ordinary Riemann sphere. Translating: this means there are conformal maps $f : \mathbf{U} \to A$ and $g : \mathbf{L} \to B$ so that

$$f \circ \phi = g, \quad z \in \mathbf{T},$$

where $A$ and $B$ are disjoint domains with common boundary $\gamma$ which is an analytic curve. (In particular, $f$, $g$ are conformal on $\mathbf{T}$.) This observation is an immediate consequence of Koebe's (1905) Uniformisation Theorem. It is one of a host of basic ideas of conformal pasting developed early in the century which go back to Schwarz's conformal representation of polygonal domains, see Carathéodory [7] or Kühnau [19]. Koebe [18] even gave conformal welding for multiple domains. As an application Courant (in the late 30s) used a variational method based on conformal welding in his solution of the Plateau–Douglas problem of minimal surfaces, see his 1950 book [8] for an account (without attribution). In the late 40s the welding theorem was proved by Schaeffer and Spencer [29] by a variational technique (another variational problem which yields $f$, $g$ was given by Grunsky [11]), in both cases $\phi$ is analytic. Some early results may also be found in the 1952 book of Goluzin [10]. Actually in the early 30s Lavrentieff [20] went beyond analytic $\phi$ and introduced quasiconformal mapping (in the same paper) although it was not until 1946 that Volkovskykii [33] used quasiconformal mapping to obtain conformal welding. (One notes that conformal welding seems analogous to the classical Riemann–Hilbert problem but is different, and solved by different means. However the Hilbert–Haseman problem with its Carleman shifts generalizes both (classical) problems, see [21].)

Consider an arbitrary Jordan curve $\gamma$ with complementary domains $A$, $B$ and corresponding conformal mappings $f : \mathbf{U} \to A$ and $g : \mathbf{L} \to B$. Now from Carathéodory (1912) the mappings extend to homeomorphisms of $\mathbf{T}$ so that

$$\phi = f^{-1} \circ g$$

is a homeomorphism of $\mathbf{T}$. Thus it is natural to conjecture that any homeomorphism $\phi : \mathbf{T} \to \mathbf{T}$ can be so represented. In fact this is not true, indeed there are counterexamples for which $\phi$ is analytic except at one point (e.g., $|\phi(e^{it}) - 1| = t^3$ $(t \to 0+)$, $t^2$ $(t \to 0-)$). There are several ways to understand this. One way is to realize that the relationship between $f$ and $g$ implies estimates on the harmonic measures of adjacent subarcs of $\gamma$ and consequently bounds on what $\phi$ can do to adjacent subarcs of $\mathbf{T}$, see [23].

The connection with quasiconformal mappings has been a central theme of conformal welding. A homeomorphism $\Phi : \mathbf{C} \to \mathbf{C}$ is quasiconformal, see [22], if it has generalized $L^2$ derivatives which satisfy the Beltrami equation

$$\bar{\partial}\Phi = \mu \partial \Phi$$

for some measurable $\mu$ satisfying $\|\mu\|_\infty \leqslant k < 1$. Geometrically this means that $\Phi$ maps small disks to small ellipses of bounded eccentricity.

It is easier to understand the main result of conformal welding if we transform $\mathbf{T}$ to the line $\mathbf{R} \cup \{\infty\}$ so that $\mathbf{U}$ is now the upper half plane and $\mathbf{L}$ the lower half plane. A famous result of Beurling and Ahlfors [2] characterizes $\phi : \mathbf{R} \to \mathbf{R}$ which extend to quasiconformal mappings $\Phi$ by the property that $\phi$ is quasisymmetric, i.e., there exists a constant $c$ such that for any real $x$ and $h$

$$c^{-1} \leqslant \frac{\phi(x+h) - \phi(x)}{\phi(x) - \phi(x-h)} \leqslant c.$$

This condition just means that the family of rescalings and translations of $\phi$ is equicontinuous, so for example any bilipschitz $\phi$ is quasisymmetric, but so is $\phi(x) = x^{1/3}$ (as are more exotic singular examples). *The fundamental theorem of conformal welding is that conformal welding is possible for arbitrary quasisymmetric functions.* This was proved by Pfluger [27] in 1960. Lehto and Virtanen [22] shortly afterwards gave a different proof.

These are statements of the classical results of the field. In the rest of the article we shall discuss the general problem of the existence and uniqueness of conformal welding. Next we mention applications to Teichmüller space where there is Bers' theorem of Simultaneous Uniformisation, one of the major achievements of twentieth century mathematics. Finally we consider the problem of how the regularity $\phi$ of determines the regularity of $f$, $g$.

## 2. Existence

We begin with the proof that any quasisymmetric mapping $\phi : \mathbf{R} \to \mathbf{R}$ has conformal welding, by conformal mappings $f$, $g$ which extend to quasiconformal mappings of the plane. As mentioned before, the central ingredient is the Ahlfors and Beurling extension of $\phi$ to a quasiconformal mapping $\Phi$ of the whole plane with complex dilatation $\mu$. The other ingredient is the solution of the Beltrami equation by quasiconformal mappings for any measurable $\mu$. (In the final form due to Bojarski, see [24], using the Calderón–Zygmund $L^p$ estimates on the Beurling transform, but earlier authors had less general cases including Gauss who did the real analytic case.) In any case one now solves the Beltrami equation

$$\bar{\partial} g = \begin{cases} \mu \partial g, & z \in \mathbf{U}, \\ 0, & z \in \mathbf{L}. \end{cases}$$

The quasiconformal solution $g$ is therefore analytic on $\mathbf{L}$ while $f = g \circ \Phi^{-1}$ is analytic on $\mathbf{U}$. In particular, as $\Phi = \phi$ on $\mathbf{R}$

$$g = f \circ \phi, \quad z \in \mathbf{R}.$$

Also one can now characterize the Jordan boundary $\gamma$ as a "quasicircle": a Jordan curve through $\infty$ satisfying the Ahlfors "3-point" condition that there is a constant $c$ with

$$\frac{|z_1 - z_2| + |z_2 - z_3|}{|z_1 - z_3|} \leqslant c$$

for any ordered points $z_1, z_2, z_3$ on $\gamma$, see [24].

One might try to give general necessary and sufficient conditions for $\phi$ to admit conformal welding. There are conditions (weaker than quasisymmetry) due to Lehto [23], see also [32], which are known to be sufficient:

$$c^{-1}(h) \leqslant \frac{\phi(x+h) - \phi(x)}{\phi(x) - \phi(x-h)} \leqslant c(h),$$

where $c(h) = \mathrm{O}(\log(1/h))$ as $h \to 0$. On the other hand, there are counterexamples with $c(t) = \mathrm{O}(h^{\varepsilon})$. The sharp result is not known.

At the core of existence is obtaining conditions when $\phi$ is analytic except at one point $\zeta$. The problem is that the topological plane $\Omega - \{\zeta\}$ we first constructed may be hyperbolic rather than parabolic. In this case $A$ and $B$ are still disjoint domains but their common boundary is an open Jordan arc clustering at a nontrivial continuum. The theory of modulus provides the simplest test for when an isolated point of a Riemann surface is removable. An annullus $\{r < |z| < 1\}$ has capacity $m = 1/\log(1/r)$ which is equal to the infimum of the Dirichlet integrals

$$\iint_{r<|z|<1} |\nabla u|^2 \, dx \, dy$$

where $u = 0$ on $|z| = 1$ and $u = 1$ on $|z| = r$. In particular, if the capacity is zero then $r = 0$. Thus $\Omega - \{\zeta\}$ is parabolic if and only if the modulus of annulli surrounding $\zeta$ on the abstract sphere can be made small. So for every $\varepsilon > 0$ we consider continuous functions $\psi$ on $\mathbf{R}$ with compact support so that $\psi(\zeta) = 1$. Then consider the harmonic function $u$ on $\mathbf{U}$ with boundary value $\psi$ on $\mathbf{R}$ and the harmonic function $v$ on $\mathbf{L}$ with boundary value $\psi(\phi)$ on $\mathbf{R}$. The condition we obtain is that the Dirichlet integrals satisfy

$$(\mathcal{D}) \quad \iint_{\mathbf{U}} |\nabla u|^2 \, dx \, dy + \iint_{\mathbf{L}} |\nabla v|^2 \, dx \, dy \leqslant \varepsilon.$$

These conditions are not as impossible as one might think. Now for any Dirichlet integrals on some domain $D$ and quasiconformal mapping $\Phi$ of $D$:

$$\iint_{\Phi(D)} \left|\nabla\big(u(\Phi)\big)\right|^2 dx\,dy \leqslant K \iint_D |\nabla v|^2 dx\,dy.$$

One immediately sees that quasisymmetric mappings satisfy condition $(\mathcal{D})$. On the other hand quasisymmetry is too strong as we only need certain Dirichlet integrals to be bounded, in fact any test functions will give a corresponding test for parabolic. The $\mathcal{D}$ test shows that we have conformal welding in this case. This approach, doing it as a type problem is seen in Volkovyskii [33], Oikawa [26] (although Courant is already clear about the problem).

Of course one wishes to generalize the above results away from the special case that $\phi$ be analytic except at one point. Now we switch back to the unit circle $\mathbf{T}$. The general condition will be a uniform version of condition $\mathcal{D}$ namely: for every $\varepsilon > 0$ there exits a $\delta > 0$ that for every annullus $\{r < |z - \zeta| < 1\}$, $\zeta \in \mathbf{T}$ of small capacity $m < \delta$ the corresponding Dirichlet integrals, i.e., the capacity of the abstract annullus has capacity $m' < \varepsilon$. To obtain conformal welding in this case one simply approximates $\phi$ by the piecewise-linear homeomorphism $\phi_n$, ensuring that the $\mathcal{D}$ condition holds uniformly for the $\phi_n$. The corresponding conformal weldings $f_n$, $g_n$ are normalized so that the capacity condition ensures that small rings map to small rings, uniformly. Thus we have an equicontinuous family $f_n$, $g_n$ on the unit circle from which we extract a subsequence which converges to a pair $f$, $g$ which is a welding for $\phi$. It would seem that our uniform $\mathcal{D}$ condition is also necessary. However in the next section we show that this is not true. This is because of the various types of nonuniqueness associated with conformal welding.

Thus we almost have necessary and sufficient conditions for conformal welding. In other situations (see [12,13]) one requires a generalized form of conformal welding where the boundary between $A$ and $B$ need no longer be a Jordan curve but nevertheless the conformal maps $f$, $g$ represent the homeomorphism $\phi$. One way to do this is to use the angular limits $f(e^{it})$, $g(e^{it})$ (keeping with the unit disk again) which for conformal mappings are not only defined almost everywhere but in fact everywhere except for a set of zero (log) capacity, a result of Beurling (1940). In [12] one uses the Hausdorff dimension dim and defines $\phi$ to be *regular* if

$$\dim(E) > 0 \quad \Leftrightarrow \quad \dim\big(\phi(E)\big) > 0, \quad \forall E \subset \mathbf{T}.$$

Then it is shown that for *regular* $\phi$ there exist conformal mappings $f$, $g$ so that

$$f\big(\phi(e^{it})\big) = g\big(e^{it}\big),$$

except for a set of $e^{it}$ of zero arc length. To prove this one takes approximate conformal weldings and ensures convergence. Once again one obtains compactness of the family of approximations but this time not in the space of continuous functions but instead in the Banach space of boundary functions of Dirichlet functions. There are fairly simple $\phi$ which have no conformal welding in the classical sense but do in the generalized sense.

## 3. Uniqueness

For many applications it is important that the conformal weldings $f$, $g$ of $\phi$ be essentially unique (up to a bilinear transformation). Clearly there is no uniqueness if conformal welding fails in the classical sense, for example, if $\gamma$ clusters on some continuum $K$ say. For then any conformal mapping $h$ on $\mathbf{C} - K$ gives another conformal welding $h \circ f$, $h \circ g$ of $\phi$. However even if $\phi$ has classical conformal welding with a Jordan curve there need not be uniqueness. The easiest case is when $\gamma$ has positive area, then one defines a nontrivial quasiconformal mapping $\Phi$ with dilatation supported on $\gamma$ so that $\Phi = h$ is conformal off $\gamma$ and once again we get another conformal welding, see [4,12].

To understand Jordan curves $\gamma$ for which there are nontrivial homeomorphisms of $\mathbf{C}$ which are analytic off $\gamma$ we need concepts from the theory of null sets developed by Ahlfors and Beurling [1]. A compact set $E$ belongs to $\mathcal{N}(\mathcal{D})$ if every function $h$ analytic and with finite Dirichlet integral on $\mathbf{C} - E$ has analytic extension to $E$. It is a main result of this theory that this is equivalent to there being NO nontrivial conformal maps on $\mathbf{C} - E$. Another related result is that if $E$ is NOT $\mathcal{N}(\mathcal{D})$ there exists a conformal mapping $h$ on $\mathbf{C} - E$ so that $\mathbf{C} - h(\mathbf{C} - E)$ has positive area. However such an $h$ need not be continuous (yes indeed point components of $E$ can be stretched to continua and vice versa).

A parallel concept is for bounded conformal mappings. The requirement that all conformal $h$ on $\mathbf{C} - E$ preserve point components is denoted by $\mathcal{N}(\mathcal{BS})$ ($BS$ meaning bounded schlicht). For example a totally disconnected closed set $E \in \mathbf{R}$ belongs to $\mathcal{N}(\mathcal{D})$ if and only if it belongs to $\mathcal{N}(\mathcal{BS})$. (In particular, no such $h$ can be constructed for $E \subset \mathbf{R}$.) However there are more general sets $E$ which are in $\mathcal{N}(\mathcal{BS})$ but not in $\mathcal{N}(\mathcal{D})$. This means that there are nontrivial functions $h$ conformal on $\mathbf{C} - E$ which necessarily extend to homeomorphisms of $\mathbf{C}$. Then given such a set $E$ it is easy to construct a Jordan curve which contains $E$.

Consequently if a curve $\gamma$ contains a set $E$ in $\mathcal{N}(\mathcal{BS})$ but not in $\mathcal{N}(\mathcal{D})$ there exists a nontrivial homeomorphism $h$ which is analytic off $\gamma$. There can be no unique conformal mapping for $\phi = f^{-1} \circ g$. Here we constructed examples by the theory of null sets, another approach is given by Bishop [4].

In other examples if $\gamma$ contains a totally disconnected compact set $E$ which is not on $\mathcal{N}(\mathcal{BS})$ even, then there exist $h$ conformal on $\mathbf{C} - E$ so that at least one point component is stretched to a continuum. Thus $\phi$ cannot satisfy the uniform $\mathcal{D}$ criterion which ensures that this does not happen, although we have conformal welding. But $\phi$ has classical conformal welding by $f$, $g$ and generalized conformal welding by $h \circ f$, $h \circ g$ of $\phi$. Therefore the uniform $\mathcal{D}$ cannot be a necessary condition of conformal welding. One might ask if the converse is true, that is, if $\gamma$ is a Jordan curve and there exists a (nonlinear) homeomorphism $h$ which are conformal off $\gamma$ then does $\gamma$ contain a set $E$ in $\mathcal{N}(\mathcal{BS})$ but not in $\mathcal{N}(\mathcal{D})$.

On the other hand if $\phi$ is quasisymmetric, even though the $\gamma$ need not be rectifiable, one can prove their are no (nonbilinear) homeomorphisms which are conformal off $\gamma$. *Thus we have the very important result that conformal welding is unique for quasisymmetric functions.*

These nonuniqueness results bespoke a certain kind of nonstability of the problem. Conformal welding is obviously unstable in the uniform norm on $\phi$. However in the

$c$-quasisymmetric category, as $K$-quasiconformal mappings form a compact family, there is stability with respect to the uniform norm, see Huber and Kühnau [17] (in which they even have an explicit formula for the conformal welding functions in the category of diffeomorphisms).

## 4. Fuchsian groups

By the Uniformisation Theorem any (hyperbolic) Riemann surface $R$ is conformally equivalent to the unit disk $\mathbf{U}$ modulo a discontinuous group $G$ of bilinear mappings $\beta: \mathbf{U} \to \mathbf{U}$. Therefore any homeomorphism $\Theta$ of $R$ onto another Riemann surface $R'$ is equivalent to a homeomorphism $\theta: \mathbf{U} \to \mathbf{U}$ so that $\theta \circ G \circ \theta^{-1}$ is a Fuchsian group $G'$ uniformizing $R'$. If $G$ and $G'$ are of the first kind (i.e., the Limit set of orbits of 0 is dense in $T$) then $\theta$ extends to a homeomorphism $\phi: \mathbf{T} \to \mathbf{T}$ which is equivariant with respect to $G$, i.e., $\phi \circ \beta \circ \phi^{-1} = \beta' \in G'$ for all $\beta \in G$. In the case of a finitely generated group of the first kind (e.g., any compact Riemann surface) the map $\phi$ is quasisymmetric. We now apply conformal welding and obtain conformal mappings $f, g$ onto domains $A, B$ bounded by a quasicircle $\gamma$. Uniqueness means that both $f$, $g$ are equivariant. Consequently $\mathcal{G} = f \circ G \circ f^{-1}$ is a discontinuous group acting on $A$ (which has limit set $\gamma$). This is conformally equivalent to $G$, i.e., $A/\mathcal{G}$ is another uniformization of $R$. On the other hand $\mathcal{G} = g \circ G' \circ g^{-1}$ is a discontinuous group acting on $B$ which is conformally equivalent to $G'$ acting on $L$. Therefore the two Fuchsian groups $G$, $G'$ have been simultaneously uniformized by $\mathcal{G}$ acting on $A$, $B$. This is Bers' theorem on simultaneous uniformization. The group $\mathcal{G}$ is said to be quasi-Fuchsian and it has limit set $f(\mathbf{T})$ which is a quasicircle.

In general any $G$ equivariant homeomorphism $\phi: \mathbf{T} \to \mathbf{T}$ can be extended to a quasiconformal mapping $\Phi$. Here the problem is that $\Phi$ should also be equivariant, a property not given by the original Ahlfors Beurling extension but obtained by Tukia and later by Earle and Hubbard, see [25]. Thus the space of Riemann surfaces (quasiconformal images of a fixed surface $R$) is realized as the space of $G$ equivariant quasisymmetrics $\phi$. To each of these conformal welding assigns an equivariant conformal mapping $f$ on $U$. This is used to construct the Universal Teichmüller Space $\mathcal{T}$, i.e., those $f$ arising from conformal welding of a quasisymmetric $\phi$. These are results of Ahlfors. (The same results hold if one restricts oneself to a fixed Fuchsian group $G$.) Any further discussion is properly the subject of Teichmüller space, the whole point is to show that conformal welding lies at the basis for its construction. A fine exposition of this theory is Lehto's 1986 book [25].

Until now we restricted our attention to quasisymmetric $\phi$. However for infinitely generated groups the $\phi$ need not be quasisymmetric, indeed nonhomeomorphisms are possible (say if a group of the first kind is transformed to a group of the second). Nevertheless it is possible to obtain a theory of simultaneous uniformization for arbitrary topological transformations of Riemann surfaces, see [13], a theory that depends on generalized conformal welding. The latter depends on special properties of the $\phi$ associated with a group. A general theory of conformal welding for monotone $\phi$ which may be nonhomeomorphic has yet to be written down.

In the opposite direction other Teichmüller Spaces based on conformal welding have been considered. There is the model due to Gardiner and Sullivan [9] based on "asymptotically conformal" quasimappings (introduced by Strebel [31], see also Pommerenke [28]) in which the dilatation is continuous. This has been of interest in Dynamics. An even smoother class was considered by Semmes [30] who used "chord arc" curves, i.e., uniformly rectifiable at all scales.

## 5. Regularity

It is a result going back to Privalov (1919) that for any rectifiable closed Jordan curve $\gamma$ the harmonic measure taken from the $A$-side of $\gamma$ is absolutely continuous with respect to the harmonic measure taken from the $B$-side, i.e., $\phi$ is absolutely continuous. By Cauchy's representation theorem for rectifiable $\gamma$ it is easy to see that there are no (nonbilinear) homeomorphisms which are conformal off $\gamma$, so we have uniqueness (up to bilinear mappings). However nothing like the converse is true. In particular, an absolutely continuous $\phi$ need not have conformal welding. Indeed there are no good necessary and sufficient conditions on $\phi$ for $\gamma$ to be rectifiable. For sufficient conditions on the complex dilatation $\mu$ for $\Phi(\mathbf{R})$ for $\gamma$ to be rectifiable see Carleson [5] and also [14] (where a meromorphic function with a rectifiable Julia set is constructed). The requirement that $\phi$ is absolutely continuous does not suffice, even if $\phi$ is already quasisymmetric, see Huber [15,16]. Semmes [30] and Bishop [3] showed that even $A_p$ conditions do not suffice. In general there is a loss of regularity between $\phi$ and the $f$, $g$. So if $\phi$ has continuous $k$-derivatives (and nonzero first derivative), then $f$, $g$ have $k-1$ derivatives, which are $\alpha$ Holder continuous for $\alpha < 1$.

It is interesting that the examples of $\phi$ arising in Teichmüller Theory are often highly irregular. In the case of a finitely generated group of the first kind Tukia proved that the map $\phi$ has the important property of being either bilinear or totally singular (i.e., zero derivative a.e.) but nevertheless quasisymmetric, see [25]. Furthermore Bowen proved that the limit set $\gamma$ of a quasi-Fuchsian group is either a circle/line or a Jordan curve with fractal dimension $\mathrm{Dim}(\gamma) > 1$. The analogous result was proved for the Julia set of a rational function, as conformal welding can be used in Complex Dynamics, see [14]. These results are a large part of the interest in fractals at the end of the century. All of this means that the natural applications of conformal welding are for $\phi$ which are not absolutely continuous even and thus very far removed from the initial observations of which started the subject early in the century.

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CHAPTER 5

# Area Distortion of Quasiconformal Mappings

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*Contents*



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## 1. Introduction

A homeomorphism $f : \mathbf{C} \to \mathbf{C}$ is called $K$-quasiconformal if its directional derivatives satisfy $\max_\alpha |\partial_\alpha f(x)| \leqslant K \min_\alpha |\partial_\alpha f(x)|$ a.e. (with respect to area measure). It is also required that $f$ be area preserving, i.e., if we use $|E|$ to denote the area of any (measurable) planar set $E$:

$$|E| = 0 \quad \Leftrightarrow \quad |f(E)| = 0.$$

For quasiconformal mappings this means that $f$ is in the Sobolev class $W^{1,2}_{\mathrm{loc}}$, i.e., functions whose first derivatives are square integrable. There is the equivalent geometric definition that $f$ have "bounded distortion". This is measured by two measures of stretching:

$$L_{f(x,r)} = \max_{|h|=r} \big| f(x+h) - f(x) \big|, \qquad l_{f(x,r)} = \min_{|h|=r} \big| f(x+h) - f(x) \big|.$$

The infinitesimal distortion of $f$ at $x$ is $H_{f(x)} = \limsup_{r \to 0} L_{f(x,r)}/l_{f(x,r)}$. If $f$ is conformal, then $H_{f(x)} = 1$ (and the converse is true). This reflects the fact that infinitesimally conformal mappings preserve circles. Unfortunately this elegant geometric definition is tricky to work with so the analytic definition is more common. The equivalence between the geometric definition and the analytic definition was shown by Pesin [21] in 1956. If quasiconformal mappings were merely generalizations of conformal maps their theory would be a curiosity. In fact they are important for some of the most significant mathematical theory of the century.

It has been known since the work of Ahlfors [1] and Mori [18] in 1955 that $K$-quasiconformal mappings are locally Hölder continuous with Hölder exponent $1/K$. The mapping $z \to z|z|^{1/K-1}$ shows that this is best possible. Bojarski [7], as a consequence of his fundamental existence theorem, showed that $K$-quasiconformal mappings actually belong $W^{1,p}_{\mathrm{loc}}$ for some $p = p(K) > 1$. This is equivalent to the fact that $f$ distorts area by a power depending only on $K$. The above example shows one might guess that the optimal exponent in area distortion is also $1/K$. The following theorem was conjectured and formulated by Gehring and Reich [9]. Let $\mathbf{U}$ be the unit disk $\{|z| < 1\}$. In 1992 Astala [4] proved.

THEOREM 1. *Suppose $f : \mathbf{U} \to \mathbf{U}$ is a $K$-quasiconformal mapping with $f(0) = 0$. Then we have $|f(E)| \leqslant M|E|^{1/K}$ for all Borel measurable sets $E \subset \mathbf{U}$. Moreover, the constant $M = M(K)$ depends only on $K$ with $M(K) = 1 + \mathrm{O}(K-1)$.*

Although this theorem is only for self maps of the disk a simple trick allows a factorization which proves similar bounds depending only on the normalization. As a straightforward consequence of Astala's theorem:

$$\sup\{p\colon\ f \in W^{1,p}_{\mathrm{loc}}\} = \frac{2K}{K-1}.$$

Also there is the optimal result for the distortion of Hausdorff dimension:

$$\frac{1}{K}\left(\frac{1}{\mathrm{Dim}(E)}-\frac{1}{2}\right)\leqslant\frac{1}{\mathrm{Dim}(f(E))}-\frac{1}{2}\leqslant K\left(\frac{1}{\mathrm{Dim}(E)}-\frac{1}{2}\right).$$

The original proof of Astala depended on deep ideas from Dynamics: relating Hausdorff dimension of holomorphically varying Cantor type sets. Shortly afterwards a short direct proof was obtained by A.E. Eremenko and D.H. Hamilton [8], as a direct consequence of the holomorphy of the class of all quasiconformal mappings. In this case the normalization is $f$ is conformal off $\mathbf{U}$ with $f(z)=z+o(1)$ near $\infty$ (i.e., the classical class $\Sigma$).

THEOREM 2. *Suppose $f\in\Sigma$ is a $K$-quasiconformal mapping. Then we have $|f(E)|\leqslant K^{1/K}\pi^{1-1/K}|E|^{1/K}$ for all Borel measurable sets $E\subset\mathbf{U}$.*

We shall see that this bound is a combination of Theorem 5 (the case where $f$ is conformal on $E$ and Theorem 6 (where $f$ is conformal off $E$), both of which are sharp. Theorem 2 corrects a typo in [8] where the $1/K$ power of $K$ was omitted. (Also it was stated that Theorem 1 in [8] had best possible constants, whereas it was meant that inequalities of Theorems 5 and 6 (only) were sharp.)

As well as having applications to analysis (we shall later give sharp bounds for the Beurling Ahlfors Transform) these inequalities, in particular, the constant of Theorem 5, have proved important in applied mathematics: namely to the theory of determining optimal bounds of physical properties (heat/electrical conductivity, magnetic permeability, elastic stiffness) of compound solids consisting of two or more different materials combined together with some microstructure.

## 2. From Grötzsch to Bojarski

Diffeomorphisms with uniform bounded distortion were first studied around 1928 by H. Grötzsch [11]. In 1939 O. Teichmüller [25] found a fundamental connection between quasiconformal mappings and quadratic differentials in his studies of extremal mappings between Riemann surfaces. However, the class of quasiconformal diffeomorphisms is not closed under uniform limits. Thus the generalization to Sobolev spaces is necessary if one is to solve extremal problems. We then find the limit of a bounded sequence of quasiconformal mappings is either quasiconformal or constant.

Shortly after Grötzsch, Lavrentieff [14] showed the importance of quasiconformal mapping for problems in partial differential equations. Here we study elliptic equations of the form

$$\mathrm{div}(A\nabla u)=0\quad \text{a.e. in }\Omega$$

where $A=A(z)$ is uniformly elliptic matrix field measurable in $z$. If one can find a homeomorphism $f$ with Jacobian matrix $Df$ satisfying the equations

$$(Df)^t(Df)=\det(Df)^2A^{-1}$$

then it turns out that $u = v(f)$ where $v$ is harmonic. These problems were solved by Lavrentieff for continuous $A$ and finally by C.B. Morrey [19], in general. It took another 20 years before Bers recognized that homeomorphic solutions are quasiconformal mappings.

Nowadays it is fundamental that the correct approach is to use complex notation. From the analytic definition of a quasiconformal mapping we see that there is a measurable function $\mu$ defined in $\Omega$ such that

$$\bar{\partial} f(z) = \mu(z) \partial f(z).$$

Indeed $\|\mu\|_\infty = (K-1)/(K+1) < 1$. This is what we call the complex Beltrami equation. Notice that when $\mu = 0$, or equivalently $K = 1$, we obtain the usual Cauchy–Riemann equations. The function $\mu_f = \partial f / \bar{\partial} f$ is called the Beltrami coefficient of $f$ or the complex dilatation of $f$.

The Beltrami equation has a long history. Gauss first studied the equation, with (real) analytic $\mu$, in the 1820s while investigating the problem of existence of isothermal coordinates on a given surface. In studying solutions to the Beltrami equation an operator now known as the Beurling Ahlfors transform has proved to be important. It is defined by a singular integral of Calderón–Zygmund type:

$$g(z) = S\omega(z) = \frac{-1}{i\pi} \iint_{\mathbf{C}} \frac{\omega(\zeta)}{(\zeta - z)^2} \, dx \, dy.$$

Beurling observed that this was a unitary transformation of $L^2(\mathbf{C})$. It was one of the early successes of Calderón–Zygmund theory that $S$ is a bounded operator of $L^p(\mathbf{C})$. Now, in the sense of distributions, $S$ is the $\partial$ derivative of the Cauchy Transform

$$h(z) = T\omega(z) = \frac{-1}{i\pi} \iint_{\mathbf{C}} \frac{\omega(\zeta)}{(\zeta - z)} \, dx \, dy.$$

However $T$ is not singular but for $p > 2$ transforms functions in $L^p(\mathbf{C})$ to the Hölder continuous functions $h$:

$$|h(z) - h(w)| \leqslant C|z - w|^\alpha,$$

with $\alpha = \alpha(p)$ by Sobolev's Theorem. These two operators allow a surprisingly explicit solution to the complex Beltrami equation. Beginning with $\mu$ with compact support we see that $\bar{\partial} f = \rho$ is of compact support and lies in $L^p(\mathbf{C})$ if $f$ has integrable derivatives in $L^p(\mathbf{C})$. As the mapping $f$ is conformal at $\infty$ we can assume it is asymptotic to $z$ at $\infty$. Thus by the generalized form of Cauchy Theorem $f(z) = z + T(\rho)$ and differentiating $\partial f = 1 + S(\rho)$. So if $f$ is to satisfy the Beltrami equation we obtain

$$\rho = \bar{\partial} f = \mu + \mu S(\rho).$$

Now suppose that $\|\mu\|_\infty \|S\|_p < 1$. Then the equation may be solved for $\rho$ to give $\rho = (I - \mu S)^{-1} \mu$ and hence

$$f = z + T(I - \mu S)^{-1} \mu = z + T(\mu) + T(\mu S \mu) + \cdots.$$

Since $\|S\|_2 = 1$ and $\|S\|_p$ is logarithmically convex in $p$ then for every $\mu$ there is a $p > 2$ so that

$$\|\mu\|_\infty \|S\|_p < 1.$$

Thus the representing series is convergent. It is not too difficult to go from the representation formula to the existence theorem. The existence theorem for quasiconformal mappings, more recently called the "measurable Riemann mapping theorem", is one of the most fundamental results in the theory and has come to play a central role in modern complex analysis, Teichmüller theory and complex dynamics.

THEOREM 3. *Let $\mu$ be a measurable function in a domain $\Omega \subset \mathbf{C}$ and suppose $\|\mu\|_\infty < 1$. Then there is a quasiconformal mapping $f : \Omega \to \mathbf{C}$ such that $\mu$ is the complex dilatation of $f$.*

The proof sketched above was done by Ahlfors [2] in 1955 in the case that $\mu$ is Hölder, the all important measurable case is due to Bojarski (1957). However it was Ahlfors and Bers [3] in 1960 who recognized the crucial fact that $f$ depends holomorphically on $\mu$ and used this in Teichmüller Theory.

Let $f, g : \Omega \to \mathbf{C}$ be quasiconformal mappings. The transformation formula for the Beltrami coefficient of a composition of quasiconformal mappings is

$$\mu_{f \circ g^{-1}}(\zeta) = \frac{\mu_{f(z)} - \mu_{g(z)}}{1 - \mu_{f(z)}\bar{\mu}_{g(z)}} \left( \frac{\partial g(z)}{|\partial g(z)|} \right)^2, \quad \zeta = g(z).$$

Thus if $\mu_f = \mu_g$ we conclude that $f \circ g^{-1} : g(\Omega) \to f(\Omega)$ is a conformal mapping since it is analytic and injective. So we get uniqueness up to a conformal mapping. This formula is a crucial step in the proof of Theorem 1.

## 3. Holomorphy

Astala's original proof made use of notions that appeared in a paper [16] of R. Mañé, P. Sad and D. Sullivan in 1983 that has been dubbed "holomorphic motions". Basically the idea is that a holomorphic family of injections $A \to \mathbf{C}$ of a set $A \subset \mathbf{C}$ is necessarily a quasiconformal mapping. Here is the precise definition.

DEFINITION 1. A holomorphic motion of a set $A \subset \overline{\mathbf{C}}$ is a map $f : \mathbf{U} \times A \to \overline{\mathbf{C}}$ such that
(i) for each fixed $z \in A$, the map $\lambda \to f(\lambda, z)$ is holomorphic in $\mathbf{U}$,
(ii) for each fixed $\lambda \in \mathbf{U}$, the map $z \to f(\lambda, z) = f_\lambda(z)$ is an injection, and
(iii) the mapping $f_0$ is the identity on $A$.

Note that there is no assumption regarding the continuity of $f$ as a function of $z$ or the pair $(\lambda, z)$. That such continuity occurs is a consequence of the $\lambda$-lemma of Mañé, Sad and Sullivan, given here as extended by Slodkowski [24]

THEOREM 4. *If* $f : \mathbf{U} \times A \to \overline{\mathbf{C}}$ *is a holomorphic motion of* $A \subset \mathbf{C}$, *then* $f$ *has an extension to* $F : \mathbf{U} \times \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ *such that*

(i) $F$ *is a holomorphic motion of* $\overline{\mathbf{C}}$,
(ii) $F$ *is continuous in* $\mathbf{U} \times \overline{\mathbf{C}} \to \overline{\mathbf{C}}$, *and*
(iii) $F_\lambda : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ *is* $K$*-quasiconformal with* $K \leqslant (1 + |\lambda|)/(1 - |\lambda|)$ *for each* $\lambda \in \mathbf{U}$.

Astala constructs a holomorphic motion of a Cantor type set then obtains his result with the Ruelle–Bowen thermodynamic formalism.

In fact holomorphy was important right from the beginning of quasiconformal theory when Teichmüller used quasiconformal mapping to consider the space of Riemann surfaces of some fixed compact topological type. He showed that so called Teichmüller space has a natural holomorphic structure. In fact the modern form of this theory was used by Bers and Royden in their proof of one form of the $\lambda$-lemma. Actually for the planar quasiconformal mappings the appearance of holomorphy is completely obvious. We have only to think of the space of admissible $\mu$ as the open unit ball $\mathbf{B}$ in $L^\infty$ and realize that the mapping

$$\Phi : \mu \to f = z + T(\mu) + T\big(\mu S(\mu)\big) + \cdots$$

is a holomorphic mapping from $\mathbf{B}$ to $W^{1,2}_{\mathrm{loc}}(\mathbf{C})$. This is a central fact in much of the Teichmüller theory as well as applications to Complex dynamics.

## 4. The class $\Sigma(k)$

We define the class $\Sigma(k)$ functions $f$ which are quasiconformal on $\mathbf{C}$ with dilatation $\mu$ supported on unit disk $\mathbf{U} = \{|z| < 1\}$ satisfying $|\mu| \leqslant k$, normalized by $f(z) = z + o(1)$ at $\infty$. Variational problems for classes of quasiconformal mappings were considered, from 1960 (see [23] for references). It was proved that $\Sigma(k)$ is exactly the class of $K$-quasiconformal mappings represented by $f = z + T(\mu) + T(\mu S(\mu)) + \cdots$ with dilatation supported on $\mathbf{U}$ with $k = (K-1)/(K+1)$. As $k \to 1$ in the limit we obtain the class $\Sigma$ the classical family of maps conformal on $\{|z| > 1\}$ with the above normalization. This class is famous for providing the classical distortion theorems for conformal mappings from the famous area theorem: i.e., for $f(z) = z + b_1 z^{-1} + \cdots$

$$\sum_{n=1}^{\infty} n|b_n|^2 \leqslant 1.$$

The other side of the area inequality also shows $|\mathbf{C} - f(|z| > 1)| \leqslant \pi$ with equality only for $f = z$. Somewhat surprisingly we shall find that this first theorem of geometric function theory plays a role in proving the area theorem for quasiconformal mappings.

There is a straightforward way of obtaining estimates on $\Sigma(k)$ from bounds on $\Sigma$. This was first done by Kühnau [15], see [23] for complete references. From the area theorem we have $|b_1| \leqslant 1$ in $\Sigma$. Now we apply Schwarz lemma to the holomorphic functional $\mu \to b_1$ and obtain the bound $|b_1| \leqslant k$ in $\Sigma(k)$. This is the germ of the idea which enabled a proof of the area distortion theorem by holomorphy. This discussion is for quasiconformal mappings with dilatation supported on **U**, however it is valid for the analogous class of quasiconformal mappings with dilatation supported on any compact set $\Delta$ of span $\sigma = 1$, i.e., for functions $f(z) = z + b_1 z^{-1} = \cdots$ conformal off $\Delta$ the bound $|\mathbf{C} - f(\mathbf{C} - \Delta)| \leqslant \pi$ is best possible. The case where $\Delta$ is connected (and so has transfinite diameter 1) is essential to the proof of Theorem 1.

## 5. A strange Harnack inequality

The area distortion theorem requires more than Schwarz lemma:

LEMMA 1. *Let $a_1, \ldots, a_n$ be positive functions on the unit disk, such that $\log a_j$ is harmonic and*

$$\sum_{j=1}^{n} a_j(\lambda) \leqslant 1, \quad |\lambda| < 1.$$

*Then for* $|\lambda| < 1$

$$\log\left(\sum_{j=1}^{n} a_j(\lambda)\right) \leqslant \frac{1 - |\lambda|}{1 + |\lambda|} \log\left(\sum_{j=1}^{n} a_j(0)\right).$$

The proof is based on the following well known "entropy" inequality occurring in statistical mechanics. This also occurs in the work of Astala and provides a common ground.

Let $p_j, q_j > 0$ be probability distributions on $\{1, 2, \ldots, n\}$. Then

$$-\sum_{j=1}^{n} p_j \log q_j + \sum_{j=1}^{n} p_j \log p_j \geqslant 0.$$

The proof is trivial: using the convex function $\phi(x) = x \log(x)$, the left of the inequality becomes $\sum q_j \phi(p_j/q_j)$. So the inequality follows from

$$\sum q_j \phi(p_j/q_j) \geqslant \phi\left(\sum q_j p_j/q_j\right) = 0.$$

To prove the lemma, for $|\lambda| < 1$ and $|z| < 1$ define

$$p_j = \frac{a_j(\lambda)}{\sum a_j(\lambda)} \quad \text{and} \quad p_j = \frac{a_j(z)}{\sum a_j(z)}.$$

Then for fixed $\lambda$ the function

$$H(z) = -\sum p_j \log a_j(z) + \sum p_j \log p_j$$

is harmonic in $z$. By the "entropy" inequality

$$H(z) \geqslant -\log \sum a_j(z) \geqslant 0.$$

So that the classical Harnack inequality gives

$$H(z) \geqslant \frac{1-|z|}{1+|z|} H(0).$$

Finally putting $z = \lambda$ and using the "entropy" inequality again

$$H(\lambda) = -\log \sum a_j(\lambda) \geqslant \frac{1-|\lambda|}{1+|\lambda|}\Big(-\sum p_j \log a_j(0) + \sum p_j \log p_j\Big)$$
$$\geqslant \frac{1-|\lambda|}{1+|\lambda|}\Big(-\log \sum \log a_j(0)\Big),$$

which proves Lemma 1.

## 6. Theorem 1, part I

Actually Theorem 1 has two distinct cases. The first part is the heart of the problem. Let $\Delta$ be a compact set of span 1. Define $\Sigma^*$ to be the normalized conformal mappings and $\Sigma^*(k)$ the corresponding quasiconformal mappings $f$ conformal off $\Delta$ (if there are any).

THEOREM 5. *Suppose $f \in \Sigma^*$ is a $K$-quasiconformal mapping. Then for all Borel measurable sets $E \subset \Delta$ such that $f$ is conformal on $E$ (i.e., the dilatation $\mu = 0$ a.e. on $E$):*

$$|f(E)| \leqslant \pi^{1-1/K}|E|^{1/K}.$$

Without loss of generality $f$ is smooth, since smooth quasiconformal mappings are $W^{1,2}_{\text{loc}}(\mathbf{C})$ dense in the space of all $K$-quasiconformal mappings. (This is not true for dimension $n \geqslant 4$.) In particular, we may assume the dilatation $p$ is smooth and supported on $\Delta$. So proving the theorem for the smooth case gives a uniform bound for the general case and proves the theorem. Now for $|\lambda| < 1$ define $K_\lambda = (1+|\lambda|)/(1-|\lambda|)$ and

$$\mu_\lambda(z) = \lambda \frac{K+1}{K-1} \mu(z),$$

so with $\lambda = (K-1)/(K+1)$ we have $\mu_\lambda = \mu$. Now let $f_\lambda(z) \in \Sigma^*$ have dilatation $\mu_\lambda$ by using the standard solution of the Beltrami equation:

$$f_\lambda = z + T(\mu_\lambda) + T(\mu_\lambda S \mu_\lambda) + \cdots.$$

The function $f_\lambda$ has Jacobian

$$J_\lambda = |\partial_z f_\lambda|^2(1-|\mu_\lambda|^2)$$

which is everywhere nonzero as $\mu$ is smooth. However by Holomorphy $\partial_z f_\lambda$ is holomorphic in $\lambda$. Therefore the function

$$a(z,\lambda) = \frac{1}{\pi}|\partial_z f_\lambda|^2$$

has the property that $\log a(z,\lambda)$ is harmonic in $\lambda$. Furthermore if $f$ is conformal on $E$ we have $\mu_\lambda = 0$ on $E$ and hence $J_\lambda/\pi = a(z,\lambda)$. Also by the Area theorem for $\Sigma^*$ (i.e., definition of span)

$$\iint_E a(\lambda,z)\,dx\,dy = \iint_E \frac{J_\lambda}{\pi}\,dx\,dy \leqslant \iint_\Delta \frac{J_\lambda}{\pi}\,dx\,dy = \frac{|f_\lambda(\Delta)|}{\pi} \leqslant 1.$$

Therefore $a(z,\lambda)$ satisfies the continuous version of Lemma 1 (i.e., we integrate over $E$ instead of sum over $j$) and so

$$\log\left(\iint_E \frac{J_\lambda}{\pi}\,dx\,dy\right) \leqslant \frac{1-|\lambda|}{1+|\lambda|}\log\left(\frac{|E|}{\pi}\right).$$

Setting $\lambda = (K-1)/(K+1)$ then proves the theorem.

## 7. Theorem 1, part 2

The complementary result we need to prove is

THEOREM 6. *Suppose $f \in \Sigma^*$ is a $K$-quasiconformal mapping. Then for all Borel measurable sets $E \subset \Delta$ such that $f$ is conformal on $\mathbf{C} - E$:*

$$|f(E)| \leqslant K|E|.$$

The argument here is due to Gehring and Reich. It begins with the observation that as the Beurling transform $S$ is unitary then for any set $G$, by Cauchy–Schwarz,

$$\iint |S(\chi_G)|\,dx\,dy \leqslant |G|.$$

Also $S$ is (almost) self-adjoint so for any function $\rho$ supported on $G$ we have

$$\iint |S(\rho)|\,dx\,dy \leqslant \|\rho\|_\infty |G|,$$

which can be regarded as the basic lemma.

The main idea is to set up a deformation family of quasiconformal mappings and integrate this inequality. As before the uniform bound is proved for sufficiently smooth mappings and this is enough. Here the deformation parameter is a real variable $t \in [0, 1)$. For fixed function $\mu$, supported on $E$ define $\mu_t = t\mu$ with corresponding normalized mappings $f_t \in \Sigma^*(t)$. In particular $f_0 = z$ and $f$ is obtained for $t = (K-1)/(K+1)$. Taking the infinitesimal form of the composition formula for dilatations we may write

$$\frac{\partial f_t}{\partial t} = g_t \circ f_t,$$

where $g_t$ is the infinitesimal deformation given by

$$g_t(z) = z + T(\rho_t), \qquad p_t = \frac{\mu \circ f_t^{-1}}{1 - t^2|\mu \circ f_t^{-1}|^2} e^{2i \arg(\partial_z f_t^{-1})}.$$

Now as $\partial T = S$ we see that

$$\frac{d|f_t(E)|}{dt} = 2\Re\left(\iint_{f_t(E)} S(\rho_t)\, dx\, dy\right).$$

The above bound on $S$ easily implies

$$\frac{d|f_t(E)|}{dt} \leqslant 2\frac{|f_t(E)|}{1-t^2},$$

which integrates to give

$$\left|f_t(E)\right| \leqslant \frac{1+t}{1-t}|E|,$$

proving the theorem.

It remains to complete the proof of Theorem 1. Thus $f$ is quasiconformal with dilatation $\mu$ supported on compact set $\Delta$ of span 1. We decompose $f$ into $g \circ h$ where $h$ is conformal on $\mathbf{C} - E$ and $g$ is conformal on $h(E)$ (and $\mathbf{C} - h(\Delta)$). Thus by the composition formula $h$ has dilatation $\mu$ on $E$ and zero elsewhere. While $g$ has dilatation $\nu(z)$ on $\mathbf{C} - h(E)$ and zero elsewhere. Now $|\mu(h^{-1}(z))| = |\nu(z)|$. This both $g$ and $h$ are $K$-quasiconformal. We may take $g$ and $h$ to be normalized at $\infty$, in particular $h(\Delta)$ has span 1. Therefore applying Theorem 5 to $h$ yields

$$\left|h(E)\right| < K|E|$$

while Theorem 4 gives

$$\left|g \circ h(E)\right| \leqslant \pi^{1-1/K} K^{1/K} |E|^{1/K},$$

which is what we wanted.

## 8. Bounds on the Beurling–Ahlfors transform

It is well known that Astala's theorem and the consequent regularity theory for quasiconformal mappings would follow, in the sharpest possible form, from the conjectured values of the $p$-norms of the Beurling–Ahlfors transform. The following is the natural conjecture (see T. Iwaniec and G.J. Martin [13]).

The $p$-norms of the Beurling–Ahlfors transform $S: L^p \to L^p$ satisfy $\|S\|_p = p - 1$ if $p \geqslant 2$ and $(p-1)^{-1}$ if $p \leqslant 2$.

The calculation of the $p$-norms of the Beurling–Ahlfors transform remains one of the outstanding problems in the area. For the case $p = \infty$ we have $S$ is an operator of *BMO*. Thus the maximal function of a function $\omega \in L^\infty$ satisfies

$$m(t) = \left|\left\{z \in U\colon \Re(S\omega) > t\right\}\right| \leqslant \exp(-Ct).$$

Sharp bounds come from considering any measurable $E \subset \mathbf{U}$ and proving

$$\iint_{\mathrm{U}-E} \left|S(\chi_E)\right| dx\, dy \leqslant |E| \log \frac{\pi}{|E|}.$$

This sharp bound due to Eremenko and Hamilton refines an earlier result of Astala. The proof is also by holomorphic deformation which like the other proofs in this article is simple enough to include:

For any function $\mu$ supported on $\mathbf{U} - E$ with $\|\mu\|_\infty \leqslant 1$ we define $\mu_\lambda = \lambda\mu$ and the corresponding family of normalized mappings $f_\lambda$. This time we let positive $\lambda \to 0$ to find that

$$\begin{aligned} |f_\lambda| &= |E| + 2\Re\left(\lambda \iint S(\mu)\, dx\, dy\right) + \mathrm{o}(\lambda) \leqslant \pi^{2\lambda+\mathrm{o}(\lambda)}|E|^{1-2\lambda+\mathrm{o}(\lambda)} \\ &= |E| + 2|\lambda||E| \log \frac{\pi}{|E|} + \mathrm{o}(\lambda), \end{aligned}$$

by Theorem 4. Hence we obtain

$$\left|\iint S(\mu)\, dx\, dy\right| \leqslant |E| \log \frac{\pi}{|E|},$$

for all $\mu$ supported on $\mathbf{U} - E$ and bounded by 1. This gives our result as $S$ is unitary.

REMARK. In the twentieth century the operator here referred to as the Beurling Ahlfors transform has been also called complex (or the two dimensional) Hilbert transform, in harmonic analysis it is usually just the Beurling transform.

## 9. Further applications

Astala's theorem has applications to the $L_1$-theory of analytic functions, quadratic differentials and critical values of harmonic functions, see Iwaniec [13]. Also, by results of Lavrentieff, Bers and others the solutions to the elliptic differential equations $\operatorname{div}(A(x)\nabla u) = 0$ can also be considered. Therefore Astala's theorem yields sharp exponents of integrability on the gradient $\nabla u$; note that the dilatation of $f$ and so necessarily the optimal integrability exponent depends in a complicated manner on all the entries of the matrix $A$ rather than just on its ellipticity coefficient.

We conclude by sketching some results of applied mathematics where the sharp constants of Theorem 5 are needed to produce optimal bounds. Composite materials consist of two (or more) phases of materials with different physical properties (conductivity, stiffness etc.) with some fine scale structure (e.g., laminate). The problem is to determine the large scale physical properties.

The material can be realized as a measurable matrix field

$$\sigma = \chi_E \sigma_1 + \chi_{E^c} \sigma_2$$

where $E$, $E^c$ are complementary subsets of fundamental square $[0,1]^2$ and $\sigma_j$ are measurable positive definite matrices subject only to the restriction that $\sigma_j$ has prescribed eigenvalues $\lambda_{j,1}$, $\lambda_{j,2}$ and also that $E$ has prescribed volume fraction $|E| = p$. Then $\sigma$ is extended to $\mathbf{C}$ by periodicity. For smooth $f$ we solve the pde

$$\operatorname{div}\left(\sigma\left(\frac{z}{\varepsilon}\right)\nabla u_\varepsilon\right) = f$$

and letting $\varepsilon \to 0$ seek the weak limit $u_\varepsilon \to u_0$ which satisfies

$$\operatorname{div}(\sigma_0 \nabla u_0) = f$$

for some constant matrix $\sigma_0$. The problem is to determine the region of possible values for the eigenvalues $\lambda_1$, $\lambda_2$ of $\sigma_0$ in terms of $\lambda_{j,1}$, $\lambda_{j,2}$ and $p$. Using the sharp bounds of the Eremenko–Hamilton inequality, the first results were given by Nesi [20], extended by Astala and Mietten [6], before the optimal bounds were completed by Milton and Nesi [17].

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CHAPTER 6

# Siegel Disks and Geometric Function Theory in the Work of Yoccoz

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*Contents*



HANDBOOK OF COMPLEX ANALYSIS: GEOMETRIC FUNCTION THEORY,
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## 1. Introduction

Complex Dynamics, in the century since Julia and Fatou first applied Montel's Principle to the family of iterates $f^n = f \circ f \circ \cdots \circ f$ of rational functions, has been the last major specialized area of geometric function theory to develop. Sullivan [8] revived the theory with his application of ideas from Teichmüller Theory, and the century has culminated with Fields medals for Yoccoz (1994) and McMullen (1998). The central result of Yoccoz [9] concerns a problem which however does not use the ideas from Dynamics or Topology that infuses much of the field. This result can be understood in classical terms, furthermore its proof makes heavy use of geometric function theory, in particular the classical distortion theorems for univalent functions.

From the late 19th century mathematicians have considered the problem of linearizing a function $f(z) = \lambda z + a_2 z^2 + \cdots$ analytic in $|z| < r$: i.e., find mapping $h = z + b_2 z^2 + \cdots$ univalent in $|z| < R$ such that

$$h^{-1} \circ f \circ h = \lambda z, \quad |z| < R.$$

This is called the Schröder equation. H. Poincaré and G. Koenigs established that for $|\lambda| \neq 0$ and $|\lambda| \neq 1$, $f$ is always linearizable. When $\lambda = 0$, $f$ is not linearizable unless $f$ is identically equal to 0. If $\lambda$ is a root of unity, then $f$ is linearizable if and only if some iterate of $f$ is the identity.

Thus the only case of interest is when $\lambda = e^{2\pi i \alpha}$, with $\alpha$ irrational. In the 20s, H. Cremer showed that the arithmetic nature of $\alpha$ plays a fundamental role. Indeed, let $(p_n/q_n)_{n \geqslant 0}$ be the sequence of the $n$th convergent of the continued fraction of $\alpha$. Cremer proved that if

$$\sup \frac{\log q_{n+1}}{q_n} = \infty$$

then $f$ is not linearizable. Positive results were not obtained until 1942 when C.L. Siegel [6] showed that the condition $\log q_{n+1} = O(\log q_n)$ implies that $f$ is linearizable. This result is the first hard theorem of Complex Dynamics.

In 1965, in the spirit of Siegel, A.D. Brjuno [2] refined the result. He established that $f$ is linearizable whenever

$$\sum_0^\infty \frac{\log q_{n+1}}{q_n} < \infty.$$

From now on, when this condition is met, we will say that $\alpha$ is a Brjuno number and write $\alpha \in \mathcal{B}$.

Finally, in 1987, J.C. Yoccoz [9] proved the optimality of Brjuno's condition. Using powerful analytic techniques, he gave an alternate proof of Siegel–Brjuno's theorem (in dimension 2) and constructed nonlinearizable holomorphic germs whenever $\alpha \notin \mathcal{B}$. Here is a formal statement of Yoccoz' result:

THEOREM 1. *Let $\lambda$ be a complex number of modulus* 1 *that is not a root of unity. Then Brjuno's condition*

$$\sum_0^\infty \frac{\log q_{n+1}}{q_n} < \infty$$

*is necessary and sufficient for every $f$ satisfying $f(0) = 0$ and $f'(0) = \lambda$ to be linearizable.*

It is very easy to construct examples of linearizable functions $f$ with $\alpha \notin \mathcal{B}$. For instance, we can conjugate $\lambda z$ with any conformal map. However, it is not obvious that there exist linearizable rational functions $f$ with $\alpha \notin \mathcal{B}$. In fact, A. Douady [4] conjectured that a rational function $R$ is linearizable iff $\alpha \in \mathcal{B}$. J.C. Yoccoz proved that Douady's conjecture is true at least for quadratic polynomials (since refined by R. Perez-Marco [5] to structurally stable polynomials).

## 2. The theorem of Yoccoz

In order to normalize the problem Yoccoz uses the class $S_\alpha$ of functions $f(z) = e^{2\pi i\alpha}z + a_2z^2 + \cdots$ univalent in the unit disk. For each $f \in S_\alpha$ there is the number $R = R(f)$ equal to the radius of convergence of the linearizing map $h$ ($R = 0$ if $f$ is not linearizable). Yoccoz then uses the classical distortion bounds in an intricate estimate interacting with arithmetical properties of $\alpha$. The continued fraction expansion is used to define an arithmetical function $\Phi : \mathbf{R} \to [0, \infty)$ which has the property that $\Phi(\alpha)$ is finite iff $\alpha \in \mathcal{B}$. The fundamental problem of Yoccoz is to prove there is an absolute constant $C$ so that

$$\left| \inf_{f \in S_\alpha} \log\bigl(R(f)\bigr) + \Phi(\alpha) \right| < C.$$

In other words there are uniform lower bounds for the radius $R$ where $h$ is univalent depending solely on $\alpha$. Although the result seems to call for number theory or dynamics Yoccoz essentially works from scratch making use of the compactness of $S_\alpha$ and some explicit distortion theorems. He begins with the conformal capacity $\kappa$ of a simply connected domain $U$ containing 0. This is defined to be $\kappa = |g'(0)|$ where $g$ is the conformal mapping of the unit disk $\mathbf{D}$ onto $U$ so that $g(0) = 0$. Next for the mapping $f \in S_\alpha$ he considers

$$V = \bigl\{ z \in \mathbf{D}\colon\ f^n(z) \in \mathbf{D},\ \forall n = 1, 2, \ldots \bigr\}.$$

The set $U$ will be the component of the interior $V$ (if any) containing 0. It is standard that $U$ is simply connected and nonempty if and only if $f$ is linearizable. (For this $f^n/\lambda^n \to h$ on $U$.)

Next Yoccoz relates the conformal capacity $\kappa$ to the radius of convergence $R$. From the Koebe distortion theorem he notes $R \geqslant \kappa$. Furthermore $\kappa$ is lower semicontinuous on $S_\alpha$:

$$f_n \to f \ \Rightarrow\ \kappa(f) = \limsup \kappa(f_n).$$

Now then $\kappa = R$ is equivalent to $U$ being compactly contained in $\mathbf{D}$ and $f$ has no analytic continuation. Hence he concludes

$$\inf\{\kappa(f) \colon f \in S_\alpha\} = \inf\{R(f) \colon f \in S_\alpha\}.$$

Thus for the rest of the proof he works with the conformal radius. The proof consists of intricate calculations to establish that $\kappa > 0$ if $\alpha \in \mathcal{B}$. Naturally he relies heavily on such distortions bounds as the well known

$$\left|\frac{zf'}{f} - 1\right| \leqslant \frac{2|z|}{1-|z|}.$$

However he also uses new distortion bounds such as:

$$\left|f(z) - f(-z) - 2\lambda z\right| \leqslant 2|z|^3 \frac{3-|z|^2}{(1-|z|^2)^2}$$

which he observes is established by the Bieberbach conjecture (proved by De Brange).

Conversely, from the method, when $\alpha \notin \mathcal{B}$ one finds (rather implicitly) a germ $f \in S_\alpha$ which is not linearizable. The details are far too technical to sketch here. However we will sketch Yoccoz' amazing result regarding the equivalence of linearizability of $p = \lambda(z - z^2)$ and $\alpha \in \mathcal{B}$. Notice first that although we are considering only $\lambda(z - z^2)$, by conjugation with linear functions, this is any quadratic except for $z^2$. Assume $\alpha \notin \mathcal{B}$. So there exists a univalent function $f(z) = \lambda z + a_2 z^2 \cdots$ which is not linearizable. Define $f_a(z) = f(az)/a$ and $p(z) = \lambda(z - z^2)$. We shall assume, contrary to hypothesis, that $p$ is linearizable. Moreover, we conjugate $p$ with $bz$ to get a new polynomial

$$p_b(z) = \lambda\left(z - bz^2\right).$$

Notice that $p_b$ is linearizable iff $p$ is linearizable. Let $\eta$ be a $\mathbf{C}^\infty$ function such that $\eta\colon [0,\infty) \to [0,1]$ with $\eta(|z|) = 1$ on $|z| < r$ and $\eta(|z|) = 0$ on $|z| > r + \delta$. We define $\phi$ by

$$\phi(z) = \eta\left(|z|\right) f_a(z) + \left(1 - \eta\left(|z|\right)\right) z.$$

With this definition $\phi$ coincides with $f_a$ on $|z| < r$ and with $z$ on $|z| > r + \delta$. Using the distortion bounds for $f$ it is easy to see that $\phi$ is a quasiconformal pasting of $f_a$ and $z$, provided $a$ is small enough, $r < 1$. Let $A$ be the annular region $\{z\colon r < |z| < r + \delta\}$. Define a mapping $\phi_b = \phi \circ p_b$. Then $\phi_b$ is a quasiregular mapping, i.e., the composition of a quasiconformal homeomorphism and a holomorphic mapping, with complex dilatation $\mu_{\phi_b}$ on $p_b^{-1}(A)$. We shall show that there is a quasiconformal mapping $\psi$ close to $z$ such that

$$\psi \circ \phi_b \circ \psi^{-1} = p_\beta,$$

where $p_\beta$ is a polynomial. The idea is that, for large $b$, as $\phi_b$ is expanding on $p_b^{-1}(A)$, we may use $\mu_\phi$ to define a $\phi_b$-invariant dilatation. Let $E$ be the outside of $p^{-1}(A)$. Then we have the following:

(1) $\phi_b(E)$ is strictly contained in $E$.

(2) $E \cup A^*$ is strictly contained in $\phi_b^{-1}(E)$, where $A^*$ is an annular region that is approximately the same as $p_b^{-1}(A)$.

(3) $\phi_b$ is analytic on both $E$ and $\mathbf{C}_\infty$ in $\phi_b^{-1}(E)$.

In fact, just as $\phi$ is analytic everywhere except on $A$, $\phi_b$ must analytic on any region not containing $p_b^{-1}(A)$, which is the case for both $E$ and $\mathbf{C}_\infty$ in $\phi_b^{-1}(E)$. Application of the composition formula for dilatations to backward iteration allows the construction of a $\phi_b$ invariant dilatation and hence

LEMMA 1. *There exists a quasiconformal mapping $\psi$ such that*

$$\psi \circ \phi_b \circ \psi^{-1} = p_\beta,$$

*where $p_\beta$ is a polynomial.*

Now the linearization of $p$ implies that of its conjugate $p_b$. In general $p_\beta$ is a polynomial close to $p$. (This is the heart of the argument of Perez Marco: if $p$ is stable than $p_\beta$ must therefore be linearizable.) But quadratics of the form $\lambda(z - bz^2)$ conjugate to the equivalent $\lambda(z - \beta z^2)$. Now going back, we see that as $\phi_b$ and $p_\beta$ are conjugate, $\phi_b$ must also be linearizable. Let $h_\beta$ be the formal linearizing map of $p_\beta$, which satisfies $h_\beta(0) = 0$ and $h'_\beta(0) = 1$. Let $R_\beta$ be the radius of convergence of $h_\beta$. We claim that:

LEMMA 2. *There exists $\rho > 0$ such that $R_\beta \geqslant \rho$.*

Suppose that $h_\beta = \sum h_n z^n$. Then we know that $1/R_\beta = \limsup |h_n|^{1/n}$. We observe that $h_\beta$ is an analytic function of $p_\beta$, i.e., of $\beta$ so the $H_n$ are polynomials in $\beta$. Thus $R_\beta$ is subharmonic and bounded. Hence $R_\beta$ is subharmonic in a neighborhood of $p$. By the maximum principle for subharmonic functions, $R_\beta$ attains its maximum, i.e., $\sup 1/R_\beta$ is attained. Thus there exists $\rho$ such that

$$\frac{1}{R_\beta} \leqslant \frac{1}{\rho}, \quad \rho \leqslant R_\beta.$$

Therefore $\rho > 0$.

Similarly, $\phi_b$ has linearizing map $H_b$ with radius of convergence $R_b \geqslant \rho > 0$. The coefficients $H_n^*$ of $H_b$ are polynomials in $b$. Thus for $|b| = C$,

$$|H_n^*| \leqslant \frac{1}{\rho^n}.$$

Thus by the maximum principle, for $b = 0$, the coefficients $H_n^*$ of $H_0$ satisfy

$$H_n^* \leqslant \frac{1}{\rho^n}.$$

So $\phi_0$ is linearizable.

Recall that for $|z| < r$, we have $\phi_0 = f_a$. We thus find that $f$ is linearizable, this is the desired contradiction.

This proof sketched is essentially Yoccoz with a subtle twist that displays the stronger results that if $p$ is stable (and almost all polynomials are) then it is linearizable iff $\alpha \in \mathcal{B}$. Also one sees that even such unstable polynomials such as $\lambda z(1-z)^n$ have this property.

Further applications of geometric function theory to dynamics may be found in Carleson's book [3], as well as Steinmetz [7].

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CHAPTER 7

# Sufficient Conditions for Univalence and Quasiconformal Extendibility of Analytic Functions

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HANDBOOK OF COMPLEX ANALYSIS: GEOMETRIC FUNCTION THEORY, VOLUME 1
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## 1. Introduction and classification of univalence conditions

A function $w = f(z)$ of a complex variable $z$ is called univalent (or schlicht) in a domain $D$ if its range $f(D)$ covers a part of a single sheet of the $w$-plane. This is equivalent to the fact that the inverse function is single-valued. To begin with, let us present a preliminary classification of univalence conditions for regular (i.e., single-valued and $\mathbb{C}$-differentiable) or meromorphic (having polar singularities) functions.

For a regular function $f(z)$ to be univalent in a small neighborhood of the point $a$ it is necessary and sufficient that $f'(a) \neq 0$. Such (local) univalence at all points of a domain is however insufficient for the univalence in the domain. A counterexample is provided by the function $e^z$ which is not univalent in the disk $|z| \leqslant R$, $R > \pi$, while being locally univalent at every point of the plane.

The only functions, univalent in the whole plane $z$, are the Möbius transformations $w = (az + b)/(cz + d)$, $ad - bc \neq 0$. If a meromorphic function is given in a domain $D \subset \mathbb{C}$ and has an injective and conformal extension onto the whole plane, then it reduces to a Möbius transformation. Therefore, the conformal extension of a meromorphic in $D$ function is not so interesting as its quasiconformal extension to be considered in the sequel. All other functions, regular in the whole plane except for isolated singularities, map it onto many-sheeted Riemann surfaces. For any such function there exist maximal domains of univalence which cannot be extended without losing either the univalence or the regularity of the mapping. Typical examples of maximal disks of univalence are: $|z| < \pi$ for the function $e^z$ (in view of the non-injectivity for $|z| = \pi$: $e^{i\pi} = e^{-i\pi} = -1$), the unit disk $|z| < 1$ for any branch of the logarithmic function $\ln(z + 1)$ (there is a branch point $z = -1$ on $|z| = 1$) and $|z| < 1$ for $(z + 1)^2$ (the derivative vanishes at $z = -1$).

A first very simple but useful sufficient condition for a function to be univalent is the following.

*If the function $f(z)$ is analytic in the convex domain $G$ and if there is a complex number $a \neq 0$ such that $|f'(z) - a| < |a|$ in $G$ then $f(z)$ is univalent in $G$.*

PROOF. $|f'(z) - a| < |a| \Rightarrow \Re(e^{i\alpha} f'(z)) > 0 \Rightarrow |f(z_2) - f(z_1)| = |\int_{z_1}^{z_2} f'(z)\,dz| \geqslant |z_2 - z_1| \int_0^1 \Re[e^{i\alpha} f'(z_1 + t(z_2 - z_1))]\,dt > 0$.

A necessary condition for univalence is any property of univalent functions, in particular, any inequality that holds for all functions, univalent in some domain, e.g., in a disk.

As a rule, the criteria (i.e., necessary and sufficient conditions) for global univalence are rather cumbersome and unwieldy when finding explicit classes of univalent functions. State, for instance, the Grunsky criterion (1939) (see [20,62]).

*Let the function*

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

*be regular in the disk* $E = \{z\colon |z| < 1\}$ *and let*

$$\ln\big[\big(f(t) - f(z)\big)/(t - z)\big] = \sum_{p,q=1}^{\infty} \omega_{p,q} t^p z^q.$$

*Then for* $f(z)$ *to be univalent in* $E$ *it is necessary and sufficient that the inequality*

$$\left|\sum_{p,q=1}^{N} \omega_{p,q} x_p x_q\right| \leqslant \sum_{p=1}^{N} \frac{1}{p} |x_p|^2$$

*holds for any positive integer* $N$ *and for all complex numbers* $x_p$, $p = 1, \dots, N$.

The Grunsky criterion for the functions $f(z)$, regular in $E$ and having a quasiconformal extension onto $\overline{E^-} = \widehat{\mathbb{C}} \setminus E$, as well as for the functions $f(z)$, univalent in $E^-$, having a simple pole at $\infty$ and a quasiconformal extension onto $\overline{E}$, is given in [39, pp. 137–138, 134–135, 87–88].

The sharpened Grunsky inequalities are also sufficient for the existence of a quasiconformal extension, but with an unknown dilatation bound [62, p. 286]. Until now there is no necessary and sufficient criterion for the existence of a $K$-quasiconformal extension.

Recall that the mapping $f : D \to \mathbb{C}$ is called quasiconformal in the domain $D$ (of the extended complex plane), if $f$ is a homeomorphic $L^2$-solution of the Beltrami equation $w_{\bar{z}} - \mu(z) w_z = 0$, $z \in D$, where $\mu(z)$ is a measurable function in $D$ with $\|\mu\|_\infty = k < 1$.

The constant $K(f) = (1 + \|\mu\|_\infty)/(1 - \|\mu\|_\infty) = (1 + k)/(1 - k) \geqslant 1$ is called the maximal dilatation and serves as a measure for the deviation of $f$ from a conformal mapping, moreover, $K(f) = 1$ only for conformal mappings. A mapping $f$ with $K(f) = K$ is called $K$-quasiconformal or $k$-quasiconformal [20,63].

Recently R. Kühnau [43] obtained the following results in this connection.

Let $f(z)$ be analytic for $|z| > 1$ with hydrodynamic normalization at $z = \infty$. If $C_{p,q}$ are the corresponding Grunsky coefficients, then

$$\left|\sum_{p,q=1}^{N} C_{p,q} x_p x_q\right| \leqslant k \cdot \sum_{p=1}^{N} \frac{1}{p} |x_p|^2$$

for all complex systems $\{x_p\}$ with $k = 1$ is necessary and sufficient for schlichtness of $f(z)$ for $|z| > 1$ [20,62,63].

If additionally $f(z)$ has a $K$-quasiconformal extension for $|z| < 1$, then Grunsky coefficient inequality is necessary with $k = (K - 1)/(K + 1) < 1$. But with this constant $k$, this inequality is not sufficient for the existence of a $K$-quasiconformal extension for $|z| < 1$. Then the author gives an explicit function $\Phi(K) < 1$ such that the Grunsky coefficient inequality with $k = \Phi(K)$ is indeed sufficient for the existence of a $K$-quasiconformal extension of $f(z)$.

The question for the "best" function $\Phi$ remains open. These questions are closely related to the theory of Fredholm eigenvalues of quasicircles and to the problem $\sup \frac{M'}{M}$. The author considers, for a fixed homeomorphism of the unit circle, all quadrilaterals in form of the unit disk with four marked boundary points and conformal modulus $M$, and additionally to this, the corresponding images with modulus $M'$ (see [43]).

In what follows when mentioning some results we (as a rule) either directly refer to the original paper or the book or to the bibliography in one of the surveys (by writing: see [11]).

An exceptionally simple statement has the following criterion called the boundary correspondence principle.

*Let the function $w = f(z)$, regular in the domain $d$ bounded by a Jordan curve $l$ and continuous in the closed domain $\bar{d}$, map $l$ continuously and injectively onto a closed Jordan curve $L_f$. Then $f(z)$ maps $d$ onto the inner domain $D$ of $L_f$ and is univalent in $\bar{d}$.*

This principle can be briefly written in the form:

$$L_f \text{ is a Jordan curve} \Leftrightarrow f(z) \text{ is a univalent function.} \tag{1}$$

The class of the curves $L_f$ contains the curves with clear geometric properties, namely, convex, starlike, spirallike (and their limit positions). The same terms refer to the domains $D$ having appropriate boundaries $L_f$ as well as to functions $f(z)$, regular in some domain $d$, with $f(d) = D$.

The conditions for convexity, starlikeness and spirallikeness with respect to 0 for the function

$$f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots \tag{2}$$

in the disk $E$ can be written, respectively, in the form

$$\Re z\frac{f''(z)}{f'(z)} + 1 \geqslant 0, \quad \Re z\frac{f'(z)}{f(z)} \geqslant 0, \quad \Re\left(e^{i\alpha} z\frac{f'(z)}{f(z)}\right) \geqslant 0, \quad |\alpha| < \pi/2. \tag{3}$$

All the conditions in (3) are deduced by using the criterion (1) for $f(z)$ in $|z| < r$ and by passing to the limit as $r \to 1$.

To check that some closed smooth curve $L$ is Jordan, one can use the boundary rotation $a(D)$ of the domain $D$ which equals the total variation of the tangent angle to $L$, i.e., $a(D) = \int_L |d \arg dw|$. The domain $D$ is univalent, provided that $a(D) \leqslant 4\pi$. By applying it to the function (2) we obtain an extension of the convexity condition, namely, the Paatero condition (see [11])

$$a(D) = \lim_{r\to 1}\int_0^{2\pi}\left|1 + \Re z\frac{f''(z)}{f'(z)}\right| d\theta \leqslant 4\pi, \quad z = re^{i\theta} \in E. \tag{4}$$

Condition (4) can be deduced by another method. For the function $f(z)$, regular in a convex domain, there is a test for univalence in the form $\Re f'(z) \geqslant 0$ (the Noshiro–

Warschawski condition). In the case of the disk this test can be written in the form $\Re[f'(z)/\varphi'(z)] \geqslant 0$, where $\varphi(z)$ maps the disk onto a convex domain. The latter condition reduces to the form

$$\int_{\theta_1}^{\theta_2} \left[1 + \Re z \frac{f''(z)}{f'(z)}\right] d\theta \geqslant -\pi, \quad z = re^{i\theta}, \tag{5}$$

for any $0 \leqslant \theta_1 < \theta_2 \leqslant 2\pi$ and $r < 1$. This integral condition is valid whenever (4) holds. Condition (5) determines the Kaplan class of the close-to-convexfunctions [37].

We note, that [63,30,31] contain the considered and other sufficient conditions of univalency.

For the further study it is helpful to consider some special Jordan curves $L_f$ called quasicircles which are images of usual circles under quasiconformal homeomorphisms of the plane. If $\partial f(E)$ is a quasicircle then it is possible to extend $f(z)$ from the disk $E$ onto the whole plane, the coefficient of the quasiconformal extendibility being determined by the properties of the curve $\partial f(E)$. The choice of the classes of quasicircles $\partial f(E)$ is closely connected with sufficient conditions for univalence.

The question of sufficient conditions for univalency is closely related to the question of the radius of univalency of classes of analytic functions (the greatest disk with fixed center in which all functions of the class are univalent) [54].

Given a set $M$ of functions and a property $P$ which the functions may or may not have in a disk $|z| < r$, the radius for the property $P$ in the set $M$ is denoted by $R_P(M)$ and is the largest $R$ such that every function in the set $M$ has the property $P$ in each disk $E_R$ for every $r < R$. As was mentioned in the book [30] it is relatively easy matter to call attention to 20 different sets of functions of varying degrees of interest, and 20 different properties, and thus create 400 new problems: find $R_P(M)$ for each pair $P$ and $M$. Some of the problems will be trivial ($R_S(S) = 1$), and some may be meaningless, but a thorough treatment of the remaining problems could by itself occupy a large book.

Chapter 13 (radius problems and Koebe domains) of the book [30] contains the broad classification survey of results on various geometrical radia.

The questions we concerned are stated in the monographs [20,30–32,62].

The great body of references concerning this subject can be found in [16].

## 2. A review of sufficient conditions for quasiconformal extendibility in canonical domains

We shall survey the results on the sufficient conditions for quasiconformal extendibility of functions, regular in the disk $E = \{z\colon |z| < 1\}$, in its exterior $E^- = \{z\colon |z| > 1\}$ (with a simple pole at $\infty$) or in the halfplane $H = \{z\colon \Re z > 0\}$.

**1.** In order to single out subclassesof univalent functions in $E$ the following constructive idea proves to be useful. Let $f(z)$ be a regular function in $E$. This function is univalent if there exists either a Löwner chain or a quasiconformal extendibility, which can be described according to the following scheme:

$$f(z) \to \begin{cases} f(z,t), & \dfrac{\partial f}{\partial t} = zh(z,t)\dfrac{\partial f}{\partial z}, \quad 0 \leqslant t < \infty,\ z \in E; \\ f(z,0) = f(z), & \underline{\Re h(z,t) > 0}, \\ f(z,t) = a_0(t) f_0(z) + O(1), & \\ f_0(z) = z + a_2 z^2 + \cdots, & \lim_{t\to\infty} a_0(t) = \infty, \end{cases}$$

$$f(z) \to \begin{cases} \hat{f}(z,\bar{z}), & \hat{f}|_{\partial E} = f|_{\partial E}, \\ \lim_{z\to\infty} \hat{f}(z,\bar{z}) = \infty, & \underline{|\hat{f}_{\bar{z}}/\hat{f}_z| \leqslant k < 1}. \end{cases}$$

Here the underlined inequalities are the central ones.

The scheme shows that under certain assumptions the function $f(z)$ can be included either into a chain $f(z,t)$ generating embedded domains, or into a quasiconformal homeomorphism $\hat{f}(z,\bar{z})$ of the two planes. In the case when the boundary properties of $f(z) = z + a_2 z^2 + \cdots$ are not regular enough, consider the family $f(rz)/r$ and then tend $r$ to 1.

The upper part of the scheme is related to the Löwner–Kufarev equation and is characterized in this form in a paper by Pommerenke (1965) [61]. The lower part of the scheme was created by Ahlfors and Weill (1962) [1].

In fact, these two approaches are equivalent. The approach with the Löwner–Kufarev equation is slightly more algorithmic, besides the chains $f(z,t)$ given by it suggest the form of the quasiconformal extensions. Given a quasiconformal extension, it is easy to obtain the chain $f(z,t)$ satisfying the Löwner–Kufarev equation. Let us show how it works. By substituting $z = \zeta e^t$, $\bar{z} = e^t/\zeta$ into the expression for $\hat{f}(z,\bar{z})$, we construct the chain $f(\zeta,t) = \hat{f}(\zeta e^t, e^t/\zeta)$ for $|\zeta| = 1$ and $0 \leqslant t < \infty$, which can be analytically continued to the disk $E$. Since $\partial\hat{f}/\partial\bar{\zeta} = 0$, then the function $f(\zeta,t)$ is analytic in $\zeta$. The Löwner–Kufarev equation is valid in view of the following calculations:

$$\left.\begin{aligned} \frac{\partial f}{\partial \zeta} &= \hat{f}_z \cdot e^t - \hat{f}_{\bar{z}} \cdot \frac{e^t}{\zeta^2} \\ \frac{\partial f}{\partial t} &= \hat{f}_z \cdot \zeta e^t + \hat{f}_{\bar{z}} \cdot \frac{e^t}{\zeta} \end{aligned}\right\} \quad \Rightarrow \quad \frac{\partial f/\partial t}{\zeta\,\partial f/\partial \zeta} = \frac{1 - \hat{f}_{\bar{z}}/\hat{f}_z \cdot \zeta^{-2}}{1 + \hat{f}_{\bar{z}}/\hat{f}_z \cdot \zeta^{-2}} = h(\zeta,t).$$

From $|\hat{f}_{\bar{z}}/\hat{f}_z| \leqslant k < 1$ it follows that $\Re h(\zeta,t) > 0$ for $|\zeta| = 1$, hence, $\Re h(\zeta,t) > 0$ for $|\zeta| \leqslant 1$.

Conversely, given a chain $f(\zeta,t)$ ($|\zeta| \leqslant 1$, $0 \leqslant t < \infty$), satisfying the equation $f'_t/f'_\zeta = \zeta h(\zeta,t)$ with $\Re h(\zeta,t) > 0$, we obtain a quasiconformal extension by the formula

$$\hat{f}(z,\bar{z}) = f\big(\sqrt{z/\bar{z}}, \log|z|\big), \quad |z| \geqslant 1.$$

The ratio $\hat{f}_{\bar{z}}/\hat{f}_z$ can be written as

$$\frac{\hat{f}_{\bar{z}}}{\hat{f}_z} = \frac{z}{\bar{z}}\frac{f_t - \zeta f_\zeta}{f_t + \zeta f_\zeta} = -\frac{z}{\bar{z}}\frac{1 - f_t/\zeta f_\zeta}{1 + f_t/\zeta f_\zeta}$$

and estimated by

$$\left|\frac{\hat{f}_{\bar{z}}}{\hat{f}_z}\right| \leqslant k < 1 \quad \Leftrightarrow \quad \left|\frac{f_t}{\zeta f_\zeta} - \frac{1+k^2}{1-k^2}\right| \leqslant \frac{2k}{1-k^2} \underset{k\to 1}{\longrightarrow} \Re\frac{f_t}{\zeta f_\zeta} > 0.$$

We shall briefly dwell upon the recent interactions of the two approaches.

In 1955 and 1964 Bazilevich (see [11]) discovered the cases of integrability for the Löwner–Kufarev equation with $h(z,t) = 1/[h(z) + th_0(z)]$, $\Re h(z) > 0$, $\Re h_0(z) > 0$. The corresponding integral representation determines the Bazilevich class of functions. We demonstrate how the class emerges.

The characteristic equation for the Löwner–Kufarev equation

$$\frac{\partial f}{\partial t} = \frac{z}{h(z) + th_0(z)}\frac{\partial f}{\partial z}$$

is a linear inhomogeneous first order equation

$$z\frac{dt}{dz} + th_0(z) = -h(z).$$

The general integral of the equation is easily determined as

$$\varphi(z,t) \equiv t\exp\int\frac{h_0(z)}{z}\,dz + \int\frac{h(z)}{z}\left(\exp\int\frac{h_0(z)}{z}\,dz\right)dz = C,$$

and the general solution of the original equation can be written in the form $\Phi[\varphi(z,t)]$ with an arbitrary differentiable function $\Phi(\varphi)$. By introducing real parameters $\alpha$ and $\beta$, $\alpha > 0$, in the representation

$$h_0(z) = \mathrm{i}\beta + \alpha z g'(z)/g(z)$$

(where $g(z) = z + c_2z^2 + \cdots$ is starlike in $E$), we have $\exp\int\frac{h_0(z)}{z}\,dz = z^{\mathrm{i}\beta}g^{\alpha}(z)$. Therefore, in the disk $E$ we get $\varphi(z,t) = z^{\alpha+\mathrm{i}\beta}\psi(z,t)$, $\psi(z,t)$ being a regular function with respect to $z$. For the composition $\Phi[\varphi(z,t)]$ to be regular, too, we need to take $\Phi(\varphi) = (A\varphi)^{1/(\alpha+\mathrm{i}\beta)}$, the factor $A$ being determined by the normalization $f(0) = f'(0) - 1 = 0$. For $t = 0$ we obtain the representation

$$f(z) = \left[\frac{\alpha+\mathrm{i}\beta}{1+\mathrm{i}a}\int_0^z h(\zeta)\zeta^{\mathrm{i}\beta-1}g^{\alpha}(\zeta)\,d\zeta\right]^{1/(\alpha+\mathrm{i}\beta)} = z + a_2z^2 + \cdots, \quad z \in E.$$

These integral representations constitute the Bazilevich class $B_{\alpha,\beta}$. The class $B_{1,0}$ coincides with the Kaplan class, while the subclass of $B_{\alpha,\beta}$ with $h(z) = 1 + ia$ is the class of spirallike functions.

Prokhorov and Sheil-Small (see [11]) proved the equivalence of the class $B_{\alpha,\beta}$ with $\alpha > 0$ and the class of functions characterized by the condition

$$\int_{\theta_1}^{\theta_2} \Re F\big(re^{i\theta}\big)\, d\theta > -\pi, \quad 0 < r < 1,\ 0 < \theta_2 - \theta_1 < 2\pi, \tag{6}$$

$$F(z) = 1 + z\frac{f''(z)}{f'(z)} + (\alpha - 1 + i\beta)z\frac{f'(z)}{f(z)},$$

under the additional assumption that $f(0) = 0$, $f(z)f'(z)/z \neq 0$, $z \in E$. We observe also that the boundary $\partial f(E)$ is accessible from the outside by the curves $w = a(1+bt)^{1/(\alpha+i\beta)}$. Finally, the class $B_{\alpha,\beta}$ can be generalized from the case of the positive $\alpha$ to $-1/2 \leqslant \alpha \leqslant 0$ (for $\alpha < -1/2$ condition (6) becomes meaningless), and there is an extension of the Bazilevich class to the cases of an annulus and the exterior of the disk (see [11]).

In 1962 Ahlfors and Weill [1] applied a quasiconformal extension to obtain the Nehari condition for the halfplane $H$. At first, such condition for the disk $E$ in the form

$$\big|\{f, z\}\big| \leqslant \frac{2}{(1 - |z|^2)^2}, \tag{7}$$

with the Schwarzian derivative $\{f, z\} = \left(\dfrac{f''}{f'}\right)' - \dfrac{1}{2}\left(\dfrac{f''}{f'}\right)^2$

was proved in 1949 by Nehari [57] who used non-oscillating solutions of differential equations. We shall return to the condition (7) and present its detailed proof in the sequel.

In 1970 Becker (see [14]) gave two proofs for the condition

$$\big|zf''(z)/f'(z)\big| \leqslant 1/\big(1 - |z|^2\big) \tag{8}$$

both by the Löwner–Kufarev chains and on the basis of a quasiconformal extension, thus showing the equivalence of these approaches for the condition (8).

In the 70s various univalence conditions were obtained to generalize the conditions (7) and (8). Their authors are Ahlfors (1974), Becker (1976, for the domain $E^-$), Ruscheweyh (1976), V. Singh and Chichra (1977), Lewandowski (1985) ([9,10]). In the proofs of more complicated conditions in the disk the method of Löwner–Kufarev chains is preferred as a more algorithmic one. At present new effects are achieved both in the conditions themselves and in their applications to integral representations (see, e.g., the papers by Pascu, Radomir and others (1983–1998) [9,10]).

**2.** We present here three proofs of the Nehari condition (7) for univalence of $f(z) = z + a_2z^2 + \cdots + a_nz^n + \cdots$ in $E$.

(1) The first proof is a minor modification of the original one given by Nehari in 1949 [57].

Consider a two-parametric family of functions

$$\omega(z; c_1, c_2) = \frac{c_1 f(z) + c_2}{\sqrt{f'(z)}}. \tag{9}$$

A second order differential equation whose general solution is (9) can be obtained after excluding the parameters $c_1$ and $c_2$ by repeated differentiation

$$\omega' = c_1\sqrt{f'} - \frac{1}{2}\frac{c_1 f + c_2}{f'^{3/2}} f'' = c_1\sqrt{f'} - \frac{1}{2}\frac{f''}{f'}\omega;$$

$$\omega'' = \frac{1}{2}c_1\frac{f''}{\sqrt{f'}} - \frac{1}{2}\left(\frac{f''}{f'}\right)'\omega - \frac{1}{2}\frac{f''}{f'}\omega'$$

$$= \frac{1}{2}c_1\frac{f''}{\sqrt{f'}} - \frac{1}{2}\frac{f''}{f'}\left(c_1\sqrt{f'} - \frac{1}{2}\frac{f''}{f'}\omega\right) - \frac{1}{2}\left(\frac{f''}{f'}\right)'\omega$$

$$= -\frac{1}{2}\left[\left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2\right]\omega;$$

$$\omega'' + p(z)\omega = 0, \qquad p(z) = \frac{1}{2}\{f, z\}. \tag{10}$$

In the formula (9) we use that $f'(z) \neq 0$, $z \in E$, is implied by (7). Indeed, if $f'(z) = (z - z_0)^m \varphi(z)$, $\varphi(z_0) \neq 0$, $m \geqslant 1, m$ being an integer, then we have

$$\frac{f''}{f'} = \frac{m}{z - z_0} + \frac{\varphi'}{\varphi},$$

$$\{f, z\} = -\frac{m}{(z - z_0)^2} + \left(\frac{\varphi'}{\varphi}\right)' - \frac{1}{2}\frac{m^2}{(z - z_0)^2} - \frac{m}{z - z_0}\frac{\varphi'}{\varphi} - \frac{1}{2}\left(\frac{\varphi'}{\varphi}\right)^2,$$

i.e., the Schwarzian derivative has a second order pole with a coefficient at $1/(z - z_0)^2$ in the form $-m(1 + m/2) \neq 0$ for $m \geqslant 1$. The fact that the Schwarzian derivative has a pole at $z_0 \in E$ contradicts to its boundedness at this point in view of (7).

The univalence of $f(z)$ is equivalent to non-oscillation of the solutions of (10). Indeed, if the function $f(z)$ were not univalent in $E$, i.e., $f(z_1) = f(z_2) = a$ for $z_1, z_2 \in E$, $z_1 \neq z_2$, then the oscillating solution $\omega(z; c_1, -ac_1)$ should have two zeros: $\omega(z_1; c_1, -ac_1) = \omega(z_2; c_1, -ac_1) = 0$. Conversely, if a solution of (10) has two zeros at distinct points $z_1, z_2$ in $E$, then the function $f(z)$ has the same values at these points, i.e., is not univalent. By virtue of the local univalence of $f(z)$, under the condition (7) and on the assumption of its multivalence two distinct points $z_1, z_2$ can be found lying on the circle $|z| = r < 1$, such that $f(z_1) = f(z_2)$, besides the function $f(z)$ is univalent in $|z| < r$. By a rotation, it is possible to achieve, if necessary, that $\Re z_1 = \Re z_2 \geqslant 0$. Draw a circle passing through the points $z_1$, $z_2$ and orthogonal to $|z| = 1$. Denote the intersection point of the circle and the real axis which lies in $|z| < 1$ by $r_0$, $0 \leqslant r_0 < 1$. The Möbius transformation

$$z = z(t) = \frac{it + r_0}{1 + ir_0 t}$$

maps the disk $|t| < 1$ onto $|z| < 1$, the arc $z_1 r_0 z_2$ corresponding to the segment

$$[-\rho, \rho]; \quad z_1 = \frac{i\rho + r_0}{1 + i\rho r_0}, \quad z_2 = \frac{-i\rho + r_0}{1 - i\rho r_0} = \bar{z}_1.$$

Write the equation of the form (10) applied to the function

$$w(t; c_1, c_2) = \frac{c_1 F(t) + c_2}{\sqrt{F'(t)}}, \quad F(t) = f \circ z(t).$$

We get

$$w'' + P(t)w = 0, \quad P(t) = \frac{1}{2}\{F, t\}, \tag{11}$$

where $\{F, t\} = \{f, z\}|_{z=z(t)} z'(t)^2$ with the estimate following from (7),

$$\big|\{F, t\}\big| \leqslant \frac{2}{(1 - |z(t)|^2)^2}|z'(t)|^2 = \frac{2}{(1 - |t|^2)^2}. \tag{12}$$

A particular solution of (11) having the form

$$w_0(t) = \frac{F(t) - a}{\sqrt{F'(t)}}, \quad a = f(z_1) = f(z_2),$$

vanishes at the points $\pm\rho$, i.e.,

$$w_0(-\rho) = w_0(\rho) = 0, \quad 0 < \rho < 1. \tag{13}$$

Show that (13) implies a result contradicting to (12).

We substitute $w_0(t)$ into (11), multiply the resulting identity by $\overline{w_0(t)}\,dt$ and integrate it along the segment $[-\rho, \rho]$. We obtain the equality

$$\int_{-\rho}^{\rho} P(t)\big|w_0(t)\big|^2\,dt = -\int_{-\rho}^{\rho} w_0''(t)\overline{w_0(t)}\,dt.$$

The right-hand side of the identity after integration by parts takes the form

$$-\int_{-\rho}^{\rho} w_0''(t)\overline{w_0(t)}\,dt = -w_0'(t)\overline{w_0(t)}\big|_{-\rho}^{\rho} + \int_{-\rho}^{\rho} w_0'(t)\overline{w_0'}(t)\,dt$$
$$= \int_{-\rho}^{\rho} \big|w_0'(t)\big|^2\,dt.$$

Therefore, in view of (12) we have

$$\int_{-\rho}^{\rho} \left|w_0'(t)\right|^2 dt = \frac{1}{2}\left|\int_{-\rho}^{\rho} \{F,t\}\left|w_0(t)\right|^2 dt\right| \leqslant \int_{-\rho}^{\rho} \frac{|w_0(t)|^2\,dt}{(1-|t|^2)^2}. \tag{14}$$

We present an elementary proof of an inequality which contradicts to (14). Starting from an obvious inequality for a real-valued continuously differentiable function $u(t) \not\equiv 0$ on $[-\rho, \rho]$ (provided that $u(-\rho) = u(\rho) = 0$), we have

$$\begin{aligned} 0 &< \int_{-\rho}^{\rho} \left(u'(t) + \frac{tu(t)}{1-t^2}\right)^2 dt \\ &= \int_{-\rho}^{\rho} u'^2(t)\,dt + 2\int_{-\rho}^{\rho} \frac{tu(t)u'(t)\,dt}{1-t^2} + \int_{-\rho}^{\rho} \frac{t^2u^2(t)\,dt}{(1-t^2)^2}. \end{aligned} \tag{15}$$

The equality in (15) may be attained only for the function $u(t) = C\sqrt{1-t^2}$, which is a solution of the equation $u'(t)/u(t) = -t/(1-t^2)$. However, the case $C = 0$ is excluded because $u(t) \not\equiv 0$, and for $C \neq 0$ the function $C\sqrt{1-t^2}$ does not satisfy the condition $u(-\rho) = u(\rho) = 0$. Since we have

$$\int_{-\rho}^{\rho} \frac{tu(t)u'(t)\,dt}{1-t^2} = \frac{t}{1-t^2}\frac{u^2(t)}{2}\bigg|_{-\rho}^{\rho} - \frac{1}{2}\int_{-\rho}^{\rho} \frac{(t^2+1)u^2(t)\,dt}{(1-t^2)^2},$$

then from (15) it follows that

$$0 < \int_{-\rho}^{\rho} u'^2(t)\,dt - \int_{-\rho}^{\rho} \frac{u^2(t)\,dt}{(1-t^2)^2}. \tag{16}$$

If $w_0(t) = u_0(t) + \mathrm{i}v_0(t)$ is a regular function in $|t| < 1$, then on $[-\rho, \rho]$ we have $w_0'(t) = u_0'(t) + \mathrm{i}v_0'(t)$. Write inequality (16) for the functions $u_0(t), v_0(t)$. By summing the two inequalities, in view of $u_0^2 + v_0^2 = |w_0|^2$, ${u_0'}^2 + {v_0'}^2 = |w_0'|^2$, we get

$$\int_{-\rho}^{\rho} \left|w_0'(t)\right|^2 dt > \int_{-\rho}^{\rho} \frac{|w_0'(t)|^2\,dt}{(1-t^2)^2}.$$

Consequently, assumption (13) is false and $f(z)$ is univalent under the condition (7).

(2) Ahlfors and Weill [1] instead of the condition $|\{f,z\}| \leqslant 2/(2\Im z)^2$ in the halfplane or the condition (7) use the condition $|\{f,z\}| \leqslant 2k/(2\Im z)^2$ or its equivalent in the disk $E$

$$\left|\{f,z\}\right| \leqslant \frac{2k}{(1-|z|^2)^2}, \quad 0 < k < 1, \tag{17}$$

and prove the univalence of $f(z)$ in $\{z\colon \Im z > 0\}$ or in $E$ with a $K$-quasiconformal extension onto the whole plane, $K = (1+k)/(1-k)$. The proof employs the mapping

$$\hat{f}(z) = \begin{cases} f(z) = \dfrac{u_1(z)}{u_2(z)}, & z \in E; \\ g(z) = \dfrac{u_1 \circ \lambda(z) + [z - \lambda(z)]u_1' \circ \lambda(z)}{u_2 \circ \lambda(z) + [z - \lambda(z)]u_2' \circ \lambda(z)}, & z \in \mathbb{C} \setminus \overline{E}, \end{cases} \tag{18}$$

where $\lambda(z) = 1/\bar{z}$ is a conformal reflection in $\partial E$, $u_1(z)$, $u_2(z)$ are linear independent solutions of $u'' + p(z)u = 0$, $p(z) = \{f, z\}/2$. The function (18) gives an injective $K$-quasiconformal mapping of the whole plane $\overline{\mathbb{C}}$ onto itself, because

(a) $g(z)$ satisfies the Beltrami equation

$$g_{\bar{z}} = \mu g_z, \quad g_z \neq 0, \ |\mu| \leqslant k \quad \text{(by (17))};$$

(b) the mapping $\hat{f}$ is continuous by virtue of the equality

$$\lim_{z \to t \in \partial E} f(z) = \lim_{z \to t \in \partial E} g(z);$$

(c) $\lim_{z\to\infty} \hat{f}(z) = \infty$.

In view of (a)–(c) and the monodromy theorem we deduce from the local univalence of $\hat{f}(z)$ its global univalence. Hence the function $f(z)$ is univalent, too.

(3) By setting $\chi(z,t) = z^2(1 - e^{-2t})^2 p(e^{-t}z)$, $p(z) = \{f, z\}/2$, $h(z,t) = (1 - \chi(z,t))/(1 + \chi(z,t))$ and using the inequality (7), we have $\Re h(z,t) > 0$. Write the partial differential equation for the subordination chain $f(z,t)$ in the form

$$\frac{\partial f}{\partial t} = z h(z,t) \frac{\partial f}{\partial z}.$$

To solve it, let us compose the characteristic equation

$$\frac{dt}{1 + \chi(z,t)} = \frac{-dz/z}{1 - \chi(z,t)} \quad \Rightarrow \quad \frac{dt - dz/z}{2} = \frac{dt + dz/z}{2\chi(z,t)}$$

and write the equation as

$$du = -\frac{dv}{(v-u)^2 p(u)}, \quad u = e^{-t}z, \ v = e^{t}z.$$

In view of the representation $f_1(z)/f_2(z) = f(z)$, where $f_k'' + p(z)f_k = 0$, $k = 1, 2$, we arrive at an integrable combination

$$\frac{(v-u)f_k'' \, du}{f_k(v-u)f_k''/f_k} = \frac{f_k' \, dv}{f_k'(v-u)^2 f_k''/f_k} = \frac{d[f_k + (v-u)f_k']}{[f_k + (v-u)f_k'](v-u)f_k''/f_k}$$

$$\Rightarrow \quad d\ln\big[f_1 + (v-u)f_1'\big] = d\ln\big[f_2 + (v-u)f_2'\big].$$

After the integration we have

$$\frac{f_1(u)+(v-u)f_1'(u)}{f_2(u)+(v-u)f_2'(u)}=c,$$

therefore,

$$f(z,t)=\frac{f_1(\mathrm{e}^{-t}z)+(\mathrm{e}^{t}-\mathrm{e}^{-t})zf_1'(\mathrm{e}^{-t}z)}{f_2(\mathrm{e}^{-t}z)+(\mathrm{e}^{t}-\mathrm{e}^{-t})zf_2'(\mathrm{e}^{-t}z)}=\mathrm{e}^{t}z+\cdots.$$

Consequently, all these functions $f(z,t)$ are univalent, in particular, the function $f(z,0)=f_1(z)/f_2(z)=f(z)$ is univalent.

For the condition (7) the last finishing touches are given in the paper [28] by Gehring and Pommerenke, where the condition and its counterparts for the exterior of the disk and the halfplane are shown to be sufficient for univalence in the closed domains alongside with explicitly stating the exceptional cases and evaluating the coefficients for the relevant quasiconformal extensions.

The boundary of each domain $f(E)$ for the function $f(z)$ satisfying (7) is a Jordan curve. The only exceptions are the domains $f(E)$ obtained from a strip by Möbius transformations. As $\beta\to 0$, the images $f_{\mathrm{i}\beta}(E)$ which are infinitely sheeted for $\beta>0$ converge to a strip as to the kernel. These functions are found by Hille [34] and have the form

$$f_{\mathrm{i}\beta}\left(\frac{1+z}{1-z}\right)=\left[\left(\frac{1+z}{1-z}\right)^{\mathrm{i}\beta}-1\right]/(\mathrm{i}\beta)$$

in $E$, where $\{f_{\mathrm{i}\beta}((1+z)/(1-z)),z\}=2(1+\beta^2)/(1-z^2)^2$, and $f_{\mathrm{i}\beta}(z)=(z^{\mathrm{i}\beta}-1)/(\mathrm{i}\beta)$ in $H=\{z\colon \Re z>0\}$, where

$$\{f_{\mathrm{i}\beta}(z),z\}=\left(1+\beta^2\right)/2z^2\quad\Rightarrow\quad\left|\{f_{\mathrm{i}\beta}(z),z\}\right|\leqslant 2\left(1+\beta^2\right)/(2\Re z)^2.$$

For the domain $E^-$ the exceptional function is $\Phi_0(z)=2(\ln\frac{z+1}{z-1})^{-1}$, and its Schwarzian derivative equals $\{\Phi_0(z),z\}=2/(z^2-1)^2$.

General conditions for univalence with a quasiconformal extension in terms of the Schwarzian derivative are obtained in [19,60]. In these papers for any admissible in $E$ metric a univalence condition is accompanied by extremal function showing its sharpness. For example, given the metric $g(z)|dz|^2=|dz|^2/(1-|z|^2)^{2t}$, $0<t<1$, and the function $\delta=2\int_0^1\sqrt{g(x)}\,dx=\sqrt{\pi}\,\Gamma(1-t)/\Gamma(3/2-t)$, we state the condition in the form

$$\left|\{f(z),z\}-\frac{2t(1-t)\bar{z}^2}{(1-|z|^2)^2}\right|\leqslant\frac{2t}{(1-|z|^2)^2}+\frac{2\pi^2}{\delta^2}\frac{1}{(1-|z|^2)^{2t}}.$$

For $1/2\leqslant t<1$ determine the extremal function as

$$F(z)=\frac{\delta}{\pi}\tan\left[\frac{\pi}{\delta}\int_0^z\sqrt{g(\zeta)}\,d\zeta\right].$$

**3.** We now consider sufficient conditions for univalence of analytic functions in the form of bounds on the functional $\sup_{z\in E}(1-|z|^2)|f''(z)/f'(z)|$. At first the condition of the form

$$\left|\frac{f''(z)}{f'(z)}\right| \leqslant \frac{a}{1-|z|^2}, \quad z \in E,$$

was proved in a paper (see [11]) by Duren, Shapiro and Shields with a coefficient $a \leqslant 2(\sqrt{5}-2)$ as a consequence of (7). Within this trend noticeable results are obtained by Becker [14], Kühnau [41], Avkhadiev [7], Becker and Pommerenke [15].

THEOREM 1 [15]. *The following conditions are sufficient for relevant functions to be univalent*:

$$\text{(1)} \quad \sup_{z\in E}\left|\left(1-|z|^2\right)f''(z)/f'(z)\right| \leqslant 1,$$

$$\text{(2)} \quad \sup_{z\in E}\left|\left(1-|z|^2\right)zf''(z)/f'(z)\right| \leqslant 1,$$

$$\text{(3)} \quad \sup_{z\in H}\left|2(\Re z)g''(z)/g'(z)\right| \leqslant 1,$$

$$\text{(4)} \quad \sup_{z\in E^-}\left|\left(|z|^2-1\right)zF''(z)/F'(z)\right| \leqslant 1,$$

*moreover,*

(a) *the constant* 1 *for all the cases is sharp, i.e., it cannot be increased with the condition still guaranteeing the univalence*;

(b) $f(E)$ *is a Jordan domain,* $g(H)$ *and* $F(E^-)$ *are also Jordan domains in* $\mathbb{C}$ *except for the cases*

$$g = (S \circ g_0 \circ T)^{-1},$$

$$(T \circ F \circ S_0)^{-1}(w) = \left[(1+\beta)w - 1\right]^{(1-\beta)/2} \cdot \left[(1-\beta)w + 1\right]^{(1+\beta)/2},$$

*where*

$$0 < \beta < 1, \quad T(w) = aw + b \quad (a, b \in \mathbb{C},\ a \neq 0),$$

$$g_0(w) = w + \ln(w-1) \quad (w \in \mathbb{C} \setminus (-\infty, 1]),$$

$$S(z) = cz + d \quad (c > 0,\ \Im d = 0), \qquad S_0(z) = \varepsilon z \quad (\varepsilon \in \mathbb{C},\ |\varepsilon| = 1).$$

Conditions (1), (2), (4) belong to Becker, assertion (b) is pertaining to Becker and Pommerenke, while (3) together with (4) follow from the results by Kühnau [41] and Avkhadiev [7].

The sharpness of the constant 1 for the case (3) was shown by Mañé, Sad and Sullivan, and also by Astala, Gehring (see [11]); for (4) – by Pommerenke, for (1) and (2) – by Becker and Pommerenke (see [15] and the references). We make an important observation.

If in (1)–(4) the constant 1 is replaced by some $k$, $0 < k < 1$, then we get sufficient conditions for $k$-quasiconformal extendibility of the relevant analytic functions onto $\overline{\mathbb{C}}$.

The following calculations are useful for the examples which are extremal for Becker's condition in $H$, in $E$ and in $E^-$. The function $g_0(z)$, given in the theorem, is the inverse of $w = f_0(z)$ , i.e., $f_0(z)$ satisfies in $H$ the equation

$$z = f_0(z) + \ln[f_0(z) - 1]. \tag{19}$$

Choosing a branch of $\ln(w - 1)$ in the plane with a slit along the real axis from 1 to $-\infty$ so that $\ln(w-1)|_{w=2} = 0$, we get the normalization $f_0(2) = 2$ for $f_0(z)$.

We illustrate $f_0(H)$ by considering in the $\omega$-plane a superposition of an unperturbed two-dimensional flow of fluid and a vortex around the point i. Then the complex potential of the resulting flow takes the form $\Omega = \omega + \mathrm{i}\ln(\omega - \mathrm{i})$. Since the conjugate velocity

$$\overline{\vec{v}(\omega)} = \Omega' = 1 + \frac{\mathrm{i}}{\omega - \mathrm{i}} = \frac{\omega}{\omega - \mathrm{i}}$$

vanishes for $\omega = 0$, the point $\omega = 0$ is critical. It is a double point on the streamline passing through it, where $\Omega|_{\omega=0} = \mathrm{i}\ln(-\mathrm{i}) = \mathrm{i}(-\mathrm{i}\pi/2) = \pi/2$. Now, the substitution $\omega = \mathrm{i} f_0(z)$, $\Omega = \mathrm{i}(z + \pi/2)$, yields

$$\mathrm{i}\left(z + \mathrm{i}\frac{\pi}{2}\right) = \mathrm{i} f_0(z) + \mathrm{i}\ln[\mathrm{i} f_0(z) - \mathrm{i}] \quad \Leftrightarrow \quad (19).$$

For the correspondence between the planes $z$ and $f_0$ see Figure 1.

The image of the imaginary axis under the mapping $f_0$ has a double point 0, corresponding to the points $\pm \mathrm{i}\pi$. Indeed,

$$f_0(z) = 0 \quad \Rightarrow \quad (19)|_{f_0=0} \quad \Rightarrow \quad z = \ln(-1) = \pm\pi,$$

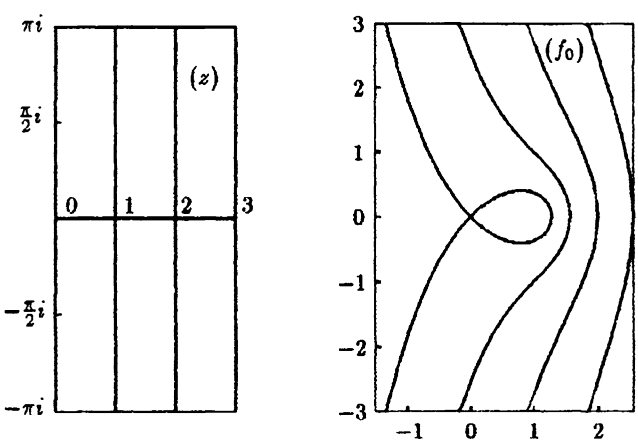


Fig. 1.

because $|\arg(f_0(z)-1)| \leqslant \pi$, by the choice of the logarithm's branch.

Now we follow [15] to show that

$$\max_{\Re z>0}\left(\left|\frac{f_0''(z)}{f_0'(z)}\right| 2\Re z\right) = 1.$$

Indeed, the differentiation of (19) by $z$ yields

$$1 = \frac{f_0}{f_0 - 1} f_0' \quad \Rightarrow \quad f_0' = 1 - \frac{1}{f_0}$$

and after repeated differentiation ($f_0'' = f_0'/f_0^2$) we estimate the ratio $|f_0''/f_0'|$ by using the equality $\Re z = \Re f_0(z) + \ln|f_0(z) - 1|$ which follows from (19). We have

$$\left|\frac{f_0''(z)}{f_0'(z)}\right| = \frac{1}{|f_0(z)|^2}\,\frac{2\Re f_0(z) + 2\ln|f_0(z)-1|}{2\Re z} \leqslant \frac{1}{2\Re z},$$

because

$$\begin{aligned}
&2\Re f_0(z) + 2\ln\big|f_0(z) - 1\big| \leqslant \big|f_0(z)\big|^2\\
&\quad \Leftarrow\ \big|f_0(z)\big|^2 - 2\Re f_0(z) + 1 = (f_0 - 1)(\overline{f_0} - 1) = |f_0 - 1|^2 \geqslant 1 + 2\ln|f_0 - 1|\\
&\quad \Leftarrow\ \mathrm{e}^{\tau} \geqslant 1 + \tau,\\
&\tau = \ln|f_0 - 1|^2
\end{aligned}$$

(the calculations are readily performed by following the arrows). The equality in these estimates is attained for $\tau = 0$, i.e., for $|f_0(z) - 1| = 1$ when $\ln|f_0(z) - 1| = 0$ and $2\Re f_0(z) = |f_0(z)|^2$. Thus,

$$\max_{\Re z>0}\left|\frac{f_0''(z)}{f_0'(z)} 2\Re z\right| = 1$$

is attained at the points of $|f_0 - 1| = 1$, the latter corresponding to a smooth arc in $H$ connecting the points $\pm \mathrm{i}\pi$.

The function $f_0(\gamma, z) = (f_0(z) - 1)^{1+\gamma} + 1$, normalized by $f_0(\gamma, 2) = 2$, is two-valent for any $\gamma$, $0 < \gamma \leqslant 1$, and satisfies the condition $|f_0''(\gamma, z)/f_0'(\gamma, z)| \leqslant (1 + 4\gamma)/(2\Re z)$, whose coefficient can be made as close as possible to 1 by choosing $\gamma$. We check the two assertions.

The mapping $(w - 1)^{1+\gamma} + 1$, $0 < \gamma \leqslant 1$, carries the points lying on $|w - 1| = 1$ to the points on the same circle by doubly covering the arc, whose endpoints have central angles $\pm\pi(1 + \gamma)$. This double arc turns to the doubly covered circle for $\gamma = 1$. If $\gamma > 1$, then the third sheet appears.

The inequality with the coefficient $1 + 4\gamma$ is obtained as follows:

$$\ln\big[f_0(z) - 1\big] \prec 2\ln z - 2\ln 2 \quad \Rightarrow \quad \ln\big[f_0(z) - 1\big] = 2\ln\omega(z) - 2\ln 2,$$

where $\omega(z)$ is analytic in $H$ and $\omega(H) \subset H$. By the invariant form of the Schwarz lemma [32] ($|d\omega|/2\Re\omega \leqslant |dz|/2\Re z$) we determine

$$\left|\frac{f_0'(z)}{f_0(z)-1}\right| = \frac{2|\omega'(z)|}{|\omega(z)|} \leqslant \frac{2|\omega'(z)|}{\Re\omega(z)} \leqslant \frac{2}{\Re z}$$

and, therefore,

$$f_0'(\gamma, z) = (1+\gamma)(f_0(z)-1)^\gamma f_0'(z)$$
$$\Rightarrow \quad \left|\frac{f_0''(\gamma, z)}{f_0'(\gamma, z)}\right| \leqslant \frac{\gamma|f_0'(z)|}{|f_0(z)-1|} + \left|\frac{f_0''(z)}{f_0'(z)}\right| \leqslant \frac{4\gamma+1}{2\Re z}.$$

Thus, the examples show that the Becker functional is 1-admissible but not 2-admissible (according to the definitions by Avkhadiev [7]). However, we can prove a stronger assertion: the Becker functional is $p$-admissible for no $p \geqslant 2$, i.e., for no $p \geqslant 2$ there is a sufficient condition for $p$-valence having the form $|f''(z)/f'(z)| \leqslant a(p)/(1-|z|^2)$ with $a(p) > 1$. In this sense the functional behaves itself like the Nehari functional from the condition (7). This result applied to the disk are published by Avkhadiev and Kayumov [12], and its application to the halfplane $H$ by Aksent'ev [4].

For the disk we can construct a two-valent function satisfying the Becker condition with the coefficient close to 1 and having the form $f_0[\gamma, \Phi_\rho(\zeta)] = F(\rho, \gamma; \zeta)$, where $\Phi_\rho(\zeta) = \frac{1+\rho^2}{1-\rho^2} + \frac{2\rho}{1-\rho^2}\zeta$ maps the disk $E$ onto a part of the halfplane $H$. If $\rho$ is close to 1, then the function $F(\rho, \gamma; \zeta)$ is two-valent for $0 < \gamma \leqslant 1$. Find a bound for the required coefficient. To this end, deduce the equality

$$\frac{F''(\rho, \gamma; \zeta)}{F'(\rho, \gamma; \zeta)} = \frac{f_0''[\gamma, \Phi_\rho(\zeta)]}{f_0'[\gamma, \Phi_\rho(\zeta)]}\Phi_\rho'(\zeta)$$

and estimate it

$$(1-|\zeta|^2)\left|\frac{F''(\rho, \gamma; \zeta)}{F'(\rho, \gamma; \zeta)}\right| \leqslant \frac{(1+4\gamma)(1-|\zeta|^2)2\rho}{2\Re\Phi_\rho(\zeta)(1-\rho^2)} \leqslant \frac{(1+4\gamma)\rho(1-|\zeta|^2)}{1+\rho^2-2\rho|\zeta|}$$
$$\leqslant (1+4\gamma)\rho,$$

because $1-|\zeta|^2 \leqslant 1+\rho^2-2\rho|\zeta|$.

For the Becker condition in the exterior of the disk the following two-valent function serves as the 'borderline':

$$F_\varepsilon(z) = \int_1^z \left(\frac{z}{\sqrt{1-z^2}}\right)^{1+\varepsilon} dz \quad \Leftrightarrow \quad \left|\frac{F_\varepsilon''(z)}{F_\varepsilon'(z)}\right| \leqslant \frac{1+\varepsilon}{|z|^2-1}.$$

The function $F_0(z) = \sqrt{z^2-1}$ is univalent in $E^-$, while the image of the boundary has a double point $F_0(-1) = F_0(1) = 0$.

Among the results of Theorem 1 we observe the univalence test 2) deduced in [14] by embedding into the subordination chain with

$$h(z,t) = \frac{1-(1-\mathrm{e}^{-2t})\mathrm{e}^{-t}zf''(\mathrm{e}^{-t}z)/f'(\mathrm{e}^{-t}z)}{1+(1-\mathrm{e}^{-2t})\mathrm{e}^{-t}zf''(\mathrm{e}^{-t}z)/f'(\mathrm{e}^{-t}z)},$$

where by 2) we have $\Re h(z,t) \geqslant 0,\ z \in E,\ t \geqslant 0$.

By the substitution $\mathrm{e}^{-t}z = u,\ \mathrm{e}^{t}z = v$, the characteristic equation for the related Löwner–Kufarev equation can be written in the form

$$(v-u)f''(u)\,du + f'(u)\,dv = 0$$

with general integral $(v-u)f'(u) + f(u) = C$. The left-hand side of the integral in the original variables

$$f(z,t) = f\left(\mathrm{e}^{-t}z\right) + \left(\mathrm{e}^{t} - \mathrm{e}^{-t}\right)zf'\left(\mathrm{e}^{-t}\right)$$

is regular in $z$ and continuously differentiable in $t$. Besides, $f(z,t) = z\mathrm{e}^{t} + \mathrm{O}(1)$ for large $t$. Therefore, the functions $f(z,t)$ are univalent in $E$ for all $t \geqslant 0$. Hence, by condition 2), the function $f(z,0) = f(z)$ is univalent, too.

**4.** Description of an approach by Rahmanov to the proof of univalence for functions analytic in $E$ (see [9,11]). Suppose that the boundary $L = f(\partial E)$ lies in some domain $G$, covered by a one-parametric family $\{\Gamma_a\}$ of Jordan arcs $\Gamma_a$, the curves $\Gamma_{a_1}$ and $\Gamma_{a_2}$ being disjoint for distinct $a_1, a_2$. In this case the intersection points of $\Gamma_a$ and $L$ determine a single-valued function $a(\theta),\ 0 \leqslant \theta \leqslant 2\pi$. By the boundary correspondence principle it is clear that $f(z)$ is univalent whenever $a(\theta)$ is strictly monotonous, e.g., increasing.

Thus, by giving $\{\Gamma_a\}$, we determine a subclass of analytic and univalent in $E$ functions, characterized by the inequality

$$\frac{da}{d\theta} = \frac{da(\theta;f)}{d\theta} > 0.$$

Let $f(z)$ be analytic in $E$, $f(0) = 0$, $f'(0) \neq 0$. Set $\Gamma_a$ to be a level curve of a multi-valued harmonic in $G$ function $\varphi(u,v)$, namely, $\Gamma_a = \{(u,v)\colon\ \varphi(u,v) = a\}$. The function $\partial\varphi/\partial v + \mathrm{i}\partial\varphi/\partial u$ is analytic in $w = u + \mathrm{i}v$. Assume that the range of $f(z)$ lies in $G$, $\partial\varphi/\partial v + \mathrm{i}\partial\varphi/\partial u = 1/\Psi(w)$, besides $\Psi(w)$ is analytic in $G$, $\Psi(0) = 0$, $\Re\Psi'(0) > 0$. The arcs $\Gamma_a$ near the point $w = 0$ behave asymptotically as level curves of $\Re(\lambda \ln w)$, $\lambda =$ const. Thus, there holds the following

THEOREM 2 (see [9]). *An analytic in $E$ function $f(z)$ is univalent in the disk if and only if the condition*

$$\Re\frac{zf'(z)}{\Psi[f(z)]} > 0, \quad z \in E,$$

*holds for some analytic function $\Psi(w)$, where $\Psi(0) = 0$, $\Re\Psi'(0) > 0$.*

**5.** Kühnau proved the following result.

THEOREM 3 [42]. *If a meromorphic in $E^-$ function $f(z)$ has a $K$-quasiconformal extension onto $\mathbb{C}$, then the relation*

$$\left|\ln \frac{f'(z)f'(\zeta)(z-\zeta)^2}{(f(z)-f(\zeta))^2}\right| \leqslant k \ln \frac{|z\bar{\zeta}-1|^2}{(|z|^2-1)(|\zeta|^2-1)} \tag{20}$$

*holds for all $z, \zeta \in E^-$.*

Clearly, for any $k > 0$ (20) is a sufficient condition for univalence of $f(z)$ in $E^-$. Therefore, it looks naturally to ask whether (20) is sufficient for $f$ to have a quasiconformal extension at least for small $k$. The answer is given by the following

THEOREM 4 (see [39, p. 95]). *In order that a meromorphic in $E^-$ function $f(z)$ has a quasiconformal extension onto $\mathbb{C}$ it is necessary and sufficient that there exists a constant $q$, $0 \leqslant q < 1$, such that*

$$\left|\frac{f'(z)f'(\zeta)(z-\zeta)^2}{(f(z)-f(\zeta))^2}\right| \leqslant \left(\frac{|z\bar{\zeta}-1|^2}{(|z|^2-1)(|\zeta|^2-1)}\right)^q. \tag{21}$$

*In addition, if $f(z)$ has a $k$-quasiconformal extension, then in* (21) *we can set $q$ to equal $k$. If* (21) *holds, then the function $f(z)$ has a $k^*$-quasiconformal extension with some $k^*$ depending only on $q$.*

The proof is based on the estimate of the cross-ratio of an arbitrary ordered quadruple of points from $\partial f(E^-)$ which implies that the curve $\partial f(E^-)$ is quasiconformal.

The next result can be proved in another way. Set

$$U_f(z,\zeta) = \frac{f'(z)f'(\zeta)}{(f(z)-f(\zeta))^2} - \frac{1}{(z-\zeta)^2}, \quad z,\zeta \in E,$$

$$d_E(U_f) = \sup_{\zeta \in E}(1-|\zeta|^2)\left(\frac{1}{\pi}\iint_E |U_f(z,\zeta)|^2\,dx\,dy\right)^{1/2}.$$

THEOREM 5 [13]. *For the regular function $f(z)$ to be univalent it is necessary and sufficient that*

$$d_E(U_f) \leqslant 1 \tag{22}$$

*for all $z$, $|z| < 1$. The equality in* (22) *is attained if and only if the function $w = 1/f(z)$ maps $E$ onto the domain whose complement to the $w$-plane has a zero area.*

The sufficiency of (22) is obvious while the necessity follows from the Area theorem for regular in $|\zeta| < 1$ functions

$$\Phi(\zeta; z) = \frac{f'(z)(1-|z|^2)}{f(\frac{z+\zeta}{1+\bar{z}\zeta}) - f(z)} = \frac{1}{\zeta} + a_0(z) + a_1(z)\zeta + \cdots.$$

As a refinement of the result we present the following

THEOREM 6. *In order that the function $f(z)$, analytic in $E$, has a $k$-quasiconformal extension onto $\overline{\mathbb{C}}$ it is necessary that the inequality $d_E(U_f) \leqslant k$ holds and sufficient that $d_E(U_f) \leqslant k/3$ holds.*

The second part of the theorem belongs to Zhuravlev [69] and is obtained by the non-trivial inequality $|\{f, z\}|(1-|z|^2)^2 \leqslant 6d_E(U_f)$, for the first one see [46].

**6.** Due to an idea by R. Mañé, P. Sad, D. Sullivan from [51], the quasiconformal mappings find another original application in the stability theory for dynamic systems generated by iterated rational mappings of the Riemann sphere into itself with varying coefficients.

The paper [51] is dealing with analytic families $f: W \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of endomorphisms, where $W$ is connected complex manifold and the function $f = f(z, t)$ is analytic with respect to both its variables. By means of classical and modern results concerning dynamical properties of that mappings the authors construct topological relations between certain close endomorphisms from the given family. The last construction uses essentially the following

λ-LEMMA 1 [51]. *Let $A$ be a subset of $\overline{\mathbb{C}}$, $D$ the open unit disk of $\overline{\mathbb{C}}$ and $i_\lambda : A \to \overline{\mathbb{C}}$ a family of injections depending analytically on $\lambda \in D$ (i.e., the function $\lambda \to i_\lambda(z)$ is analytic for all $z \in A$). Suppose that $i_0$ is the inclusion map $A \hookrightarrow \overline{\mathbb{C}}$. Then every $i_\lambda$ has a quasi-conformal extension $i_\lambda : \overline{A} \to \overline{\mathbb{C}}$ which is a topological embedding depending analytically on $\lambda \in D$ and so that the map $D \times A \ni (\lambda, z) \to i_\lambda(z) \in \overline{\mathbb{C}}$ is continuous.*

The proof of the λ-Lemma is based in the following: any analytic map of the unit λ-disk into the triply punctured sphere $\overline{\mathbb{C}} - \{0, 1, \infty\}$ is distance non-increasing for the complete Poincaré metrics on the unit disk and punctured sphere.

The ideas of the papers [51] and [67] get the furthest development in [17] and [65]. Thus, the work [65] gives the following generalization of the λ-Lemma:

λ-LEMMA 2 [65]. *Every holomorphic motion $f: D \times A \to \overline{\mathbb{C}}$ of an arbitrary subset $A$ of $\overline{\mathbb{C}}$ can be extended to a holomorphic motion $F: D \times \overline{\mathbb{C}}$ of the whole of $\overline{\mathbb{C}}$, parametrized by the same unit disc.*

Here the holomorphic motion of the set $A \subset \overline{\mathbb{C}}$, is a function $f(z, t): A \times D \to \overline{\mathbb{C}}$ satisfying the following conditions: for any $t \in D$ the mapping $z \to f(z, t)$ is injective on $A$; for any $z \in A$ the function $t \to f(z, t)$ is holomorphic on $D$; $f(z, 0) = z$ for $z \in A$.

The described above results have a surprising connection with the following Kühnau problem. Let us define the quasiconformal reflection with respect to arbitrary subset $E \subset \overline{\mathbb{C}}$ as quasiconformal mapping of the sphere $\overline{\mathbb{C}}$ onto itself with negative Jacobian, which keeps the points of $E$ fixed. Kühnau [44,45] formulates the following conjecture: a set $E \subset \overline{\mathbb{C}}$, admits the quasiconformal reflection if and only if it is subset of some quasicircle. Krushkal' [40] proved this conjecture. In addition, he obtained certain sufficient condition on the set $E$ for its embedding into a quasicircle. His proof of the Kühnau conjecture and the embedding condition uses the main results on homotopies with holomorphic dependence on complex time parameter from a subdomain of $\overline{\mathbb{C}}$ (see [21,65,17]).

## 3. Conditions for quasiconformal extension in non-canonical domains

In the 70s a natural trend starts to develop which is the generalization of (7), (8) in the form

$$|\{f, z\}| \leqslant a(D)\rho_D^2(z), \tag{23}$$

$$\left|\frac{f''(z)}{f'(z)}\right| \leqslant b(D)\rho_D(z), \tag{24}$$

$\rho_D(z)$ being the Poincaré density of the domain $D$ at $z$. In papers by Ahlfors (1963), Lehto (1977–1979), Gehring (1977), Osgood (1980), Martio and Sarvas (1979), Gehring and Astala (1985) (see [11,59]) the following remarkable effect is gradually developed. It appears that the mere existence of sufficient conditions for univalence in the form (23), (24) with non-zero constants $a(D)$, $b(D)$ characterizes the domain: it must be quasicircular, i.e., an image of a disk or a multiply connected circular domain (if $D$ is multiply connected) under quasiconformal homeomorphism of the planes. In the survey by Gehring [27] the property is presented as one of the 15 equivalent characteristics of the quasidisk.

Among another achievements relating to the two last research trends we should mention an extensive generalization of existence theorems for sufficient conditions of univalence obtained by Krushkal' in 1985 and Avhadiev results on admissible functionals in 1987.

Let $D$ be a simply connected domain in $\mathbb{C}$ bounded by quasicircle so that $\infty \in \partial D$, and $P_n(f) = f^{(n)}/f' - F(f''/f', \dots, f^{(n-1)}/f')$, where $n \geqslant 2$ and $F$ is an analytic function of its arguments. It is assumed that the upper bound $\sup_{z\in D} \operatorname{dist}(z, \partial D)^{n-1}|F|$ is finite for any analytic and univalent in $D$ function $f(z)$.

THEOREM 7 [38]. *Let $f(z)$ be an analytic in $D$ function and $|f(z) - \alpha z| = O(1)$ for $z \to \infty$ $(z \in D)$ and for certain $\alpha \neq 0$. There exists a constant $c = c(P_n, D) > 0$ such that the function $f(z)$ is univalent in $D$ if it satisfies condition*

$$\sup_{z\in D}\left|\operatorname{dist}(z, \partial D)^{(n-1)} P_n(f)\right| < c.$$

Let $M(D)$ be a set of analytic in $D$ functions, $t = I(f)$ be a non-negative functional defined on $M(D)$.

The functional $I(f)$ is called *p-admissible* for $M(D)$ in $D$ if there exists a constant $a > 0$ such that the conditions $f \in M(D)$, $I(f) < a$, imply that the function $f(z)$ is no more than $p$-valent in $D$.

Set

$$p_0(f, D) = \sup_{w \in \mathbb{C}} \sum_{f(z)=w} p(z, f, D),$$

where $p(z, f, D)$ is the local valency of $f$ at the point $z$.

We say that the functional $I(f)$ has a *stable kernel* in the class $M(D)$ if it satisfies the following conditions:

(1) the set $\ker I = \{f \in M(D)\colon\ I(f) = 0\}$ is not empty, any mapping $f \in \ker I$ is finite-valent and continuous in spherical metric on $\overline{D}$, and

$$p_0 = \sup\{p_0(f, \overline{D})\colon\ f \in \ker I\} < \infty;$$

(2) any sequence $f_k \in M(D)$ satisfying condition $\lim_{k\to\infty} I(f_k) = 0$ contains a subsequence $f_{k_j}$ convergent to $f_0 \in \ker I$ in spherical metric and such that for any point $z \in \overline{D}$ one can find a number $N = N(z)$ and a neighborhood $U = U(z, \varepsilon)$ of the point $z$ where the estimate $p(f_{k_j}, U) \leqslant p(z, f_0, \overline{D})$ holds for all $j > N$.

THEOREM 8 [7]. *If the functional $I$ has a stable kernel in the class $M(D)$, then it is $p_0$-admissible for $M(D)$.*

The prevalent technique in proofs of conditions like (23), (24) is the method of quasiconformal extension. The evaluation of particular constants for special domain or for special class of domains seems to be an urgent problem. Lehto [48] (1977) and Lehtinen [47] (1988) found the best possible constants in the condition (23) for polygons and ellipse. Let us dwell on construction of sufficient conditions for univalence and quasiconformal extendibility in the form (24) by means of subdomains. This approach uses monotonicity of hyperbolic metric $\rho_D(z)|dz|$. Indeed, $D_k \subset D \Rightarrow \rho_{D_k}(z) \geqslant \rho_D(z)$. Sufficient condition for univalence in the form $|\Phi(z, f)| \leqslant A\rho_D^n(z)$ is valid if one can construct sufficient conditions for quasiconformal extendibility in the same form $|\Phi(z, f)| \leqslant A_k\rho_{D_k}^n(z)$ for subdomains $D_k$ so that any two points of the domain $D$ belong to a common subdomain $D_k$; $k = 1, 2, \ldots$ and $\inf_k A_k = A > 0$.

The main steps in the proof of extendibility conditions are the study of quasiconformal reflection $\lambda(z)$ in the boundary of the domain and the estimation of the product $(|\lambda_z| + |\lambda_{\bar z}|)|z - \lambda(z)|$.

The first way for construction of the reflection is connected with quasiconformal extension of the Riemann mapping of $D$ onto $\mathbb{C} \setminus D$. To this end, a number of authors used both explicit forms of the Riemann mappings for special domains (for angular sector – Lehto [50]; for a circular lune – Aksent'ev and Shabalin [6]; for rhombus and its conformal images – Maier, 1988 [11]) and general properties of conformal mappings of certain classes of domains connected with the Ahlfors extension of homeomorphism of the real axis onto the upper halfplane (class of domains satisfying the chord–arc condition – Chuev and Shabalina; class of domains with Lyapunov's boundaries).

The second way for construction of quasiconformal reflection takes into account geometrical properties of the class of boundary curves. This method is applicable to rectifiable $\alpha$-starlike curves (Fait, Krzyż and Zygmund [24]), $\alpha$-starlike local rectifiable curves (Aksent'ev and Shabalin [5]), spiral and $\Phi$-like curves (Sevodin, 1989).

The absence of the Riemann mapping in the reflection formula creates additional difficulties for estimating the difference $|z-\lambda(z)|$ by the Poincaré density, which can be overcome by partitioning into subdomains and applying the monotonicity of the hyperbolic metric.

We present a new result obtained in this way. Let $\overline{D_+ \cup D_-} = \overline{\mathbb{C}}$. Assume that $\partial D_+ = \partial D_-$ is the Lyapunov curve, i.e., $M_+^{-1} < |F_+'(\zeta)| < M_+$ for $\zeta \in D_+$ and $M_-^{-1} < |F_-'(\zeta)| < M_-$. Then there holds

THEOREM 9. *If a regular in $D_+$ function $f(z)$ satisfies the condition*

$$\left|\frac{zf''(z)}{f'(z)}\right| \leqslant \frac{\rho_{D^+}(z)}{M_+^4 M_-^2 \max(M_+, M_-)},$$

*then this function is univalent and admits quasiconformal extension onto the whole plane.*

Analogous theorems are valid for functions which are regular in the domain $D_-$ or in domains with boundary containing $\infty$. If $D_+ = E$, $D_- = E_-$ or $D_\pm$ are halfplanes, then the condition of the theorem turns into the Becker condition.

The proof of Theorem 9 follows the known scheme. We construct extension of the function $f(z)$ to mapping of the whole plane onto itself with subsequent application of the Hadamard theorem. The extension is determined by formula

$$\hat{f}(z) = \begin{cases} f(z), & z \in \overline{D}_+; \\ f \circ \lambda(z) + (z - \lambda(z)) f' \circ \lambda(z), & z \in D_-. \end{cases}$$

Here we put $\lambda(z) = F_+ \circ H \circ F_-^{-1}(z)$, $H(\zeta) = h(\arg\zeta)/|\zeta|$, $h(\arg\zeta) = F_+^{-1} \circ F_-(e^{i\arg\zeta})$. After laborious consecutive estimates of $|\lambda_{\bar{z}}|$, $|\lambda_z|$, $|H_{\bar{\zeta}}|$, $|H_\zeta|$ and $|z-\lambda(z)|$ we obtain inequality

$$\big(|\lambda_{\bar{z}}| + |\lambda_z|\big)|z - \lambda(z)|\left|\lambda(z)\frac{f''(\lambda)}{f'(\lambda)}\right| < 1,$$

which implies $|\hat{f}_{\bar{z}}/\hat{f}_z| < 1$.

COROLLARY 1. *If the range of the function $\omega = \log f'(z)$, $z \in D_+$, lies in the domain $\Omega_0$ satisfying inequality*

$$\rho^{-1}(\Omega_0, \omega) \leqslant 1/B, \quad B = M_+^4 M_-^2 \max(M_+, M_-)$$

*at any its point $\omega$, then $f(z)$ is univalent in the closed domain $\overline{D}_+$.*

COROLLARY 2. *A function $f(z)$ is univalent in closed domain $\overline{D}_+$ with Lyapunov boundary if* $|\log|f'(z)|| < \pi(4B)^{-1}$, $z \in D_+$.

The Ahlfors–Weill method enables us to determine explicitly the constant $b(D)$ from the condition (23) for domains with known quasiconformal reflection in the boundary. Let us cite certain results concerning this approach. Lehto [50] studies circular lunes, angular sectors and the exterior of an ellipse as domains $D$.

THEOREM 10 [50]. *We have*:

(a) $a(D) = 2\alpha^2$ *for angular sector* $D = \{z\colon\ 0 < \arg z < \alpha\pi\}$, $0 < \alpha \leqslant 1$;

(b) $a(D) = 4\alpha - 2\alpha^2$ *for the sector* $D = \{z\colon\ 0 < \arg z < \alpha\pi\}$, $1 \leqslant \alpha < 2$;

(c) $a(D) \geqslant 8q^2/(1+q)^2$ *if $D$ is the exterior of the ellipse with semi-axes $a$ and $b$*, $0 < a/b = q < 1$.

The quasiconformal reflection is constructed here in terms of functions which map the domain $D$ conformally onto a disk or a halfplane.

THEOREM 11 [66]. *If $D$ is convex, then $b(D) \leqslant 1$. If $D$ is not convex, then $b(D) < 1$.*

Fait, Krzyż and Zygmunt [24] apply the geometrical approach to the construction of a quasiconformal reflection for $\alpha$-starlike ($0 \leqslant \alpha < 1$) curve $L_1 = \{z\colon\ z = R(\varphi)e^{i\varphi},\ 0 \leqslant \varphi \leqslant 2\pi\}$, where function $R(\varphi)$ is absolutely continuous and satisfies condition $|R'(\varphi)/R(\varphi)| < \coth\beta$, $\beta = (1-\alpha)\pi/2$, and obtain reflection function $\lambda(z) = R^2(\arg z)/\bar{z}$. Similar considerations concerning the image $L_0$ of real axis under the conformal mapping of upper halfplane by function $z(\zeta)$ is applied in [5] for evaluation of $\lambda(z)$ in the following cases.

Let $D_\pm(\alpha)$ be the interior and exterior of a closed curve $L_1$, and $D_0(\alpha)$ be a domain with the boundary $L_0$ ($\infty \in L_0$). Assume that defined in $D_j(\alpha)$ function $f(z)$ is regular there for $j = 1$ and satisfies the condition $\lim_{z\to\infty}[f(z)/z] > 0$ for $j = 0, -1$. In what follows $A[D_j(\alpha)]$ stands for $\pi(1+\exp(\pi|j|\cot\beta))/(4\beta\tan(\beta/2))$ and $\beta = (1-\alpha)\pi/2$ as above.

THEOREM 12 [5]. *If the function $f(z)$ satisfies inequality*

$$\left|\left(\frac{z}{R(\varphi)}\right)^{|j|}\frac{f''(z)}{f'(z)}\right| < \frac{k\rho_{D_j(\alpha)}(z)}{A[D_j\alpha]}, \qquad \varphi = \arg z,\ z \in D_j(\alpha),\ k < 1,$$

*then it is univalent in $D_j(\alpha)$ and has a $K$-quasiconformal extension onto the whole plane with $K = (1+k)/(1-k)$.*

An important step in this proof is the estimate of $|z - \lambda(z)|$ in terms of the Poincaré density. It is performed by inscribing of lunes into $D_j(\alpha)$.

Recall another sample of constructive estimation of the constant $b(D)$ in (24). Let $D(d, k_1)$ be a convex domain with the smooth boundary $\partial D(d, k_1)$ of diameter $d$ such that its curvature is bounded below by $k_1$. Then any two of its points $z_1, z_2$ can be covered by a closed convex circular lune with the inner angle $\beta\pi = 2\arcsin(dk_1/2)$ at the vertex. This property is applied to prove the following theorem.

THEOREM 13 [6]. *A regular in $D(d, k_1)$ function $f(z)$ is univalent if*

$$\left|\frac{f''(z)}{f'(z)}\right| < \beta\rho_{D(d,k_1)}(z), \quad z \in D(d, k_1).$$

## 4. Univalence conditions in multiply connected domains

Due to the following Osgood's idea [58] the Ahlfors–Weill method appeared to be fruitful for proving sufficient conditions of univalence in multiply connected domains.

A family $\mathcal{D}$ of domains $d \subset D$ is said to be a *$K$-quasiconformal decomposition* of $D$ if any $d \in \mathcal{D}$ is bounded by a $K$-quasiconformal curve and any two points $z_1, z_2 \in D$ lie in closure of some $d \in \mathcal{D}$. We say that domain $D$ *admits quasiconformal decomposition* if there exists $K$-quasiconformal decomposition for certain finite $K$.

Osgood obtained the following useful result [58].

THEOREM 14 [58]. *If $D$ is a finitely connected domain and every component of $\partial D$ is either point or quasiconformal curve then $D$ admits quasiconformal decomposition.*

This result together with the monotonicity of the Poincaré density with respect to extension of domain and the Ahlfors theorem [2] imply the following statement.

THEOREM 15 [58]. *Suppose that $D$ is a finitely connected domain and any component of $\partial D$ is either a point or a quasiconformal curve. Then there exists a positive constant $a = a(D)$ depending only on $D$ such that $f$ is univalent in $D$ if it is analytic in $D$ and* $\sup_{z\in D} \rho_D^{-2}(z)|\{f, z\}| \leqslant a(D)$.

The converse is valid, too.

THEOREM 16 [26]. *Let $D$ be a proper subdomain of $\mathbb{C}$. If there exists a constant $a = a(D) > 0$ such that for any analytic in $D$ function $f(z)$ the inequality $\sup_{z\in D} \rho_D^{-2}(z)|\{f, z\}| \leqslant a(D)$ implies the univalence of $f(z)$ in $D$, then any component of $\partial D$ is either a quasiconformal curve or a point.*

Similar assertions are valid also for condition

$$\sup_{z\in D}\left|\rho_D^{-1}(z) f''(z)/f'(z)\right| < b(D).$$

The existence theorems for constants $b(D)$ are proved by Gehring and Osgood [29,59]. Martio and Sarvas [53] proved them for $\rho_D^{-1}(z)$ instead of $\operatorname{dist}(z, \partial D)$.

Clearly, if the domain $D$ admits quasiconformal decomposition onto simply connected domains with known explicit expressions for quasiconformal reflection in the boundaries, then the decomposition enables us to evaluate the constants $a(D)$, $b(D)$ for multiply connected domain $D$.

Let us cite results obtained by covering the domain by circular lunes. This approach is developed in the works of Aksent'ev and Shabalin, Sevodin, Sagitova, Zinov'ev, Maier, Nasyrov (see [11,56]).

We introduce a class of finitely connected domains $G(\alpha, \beta)$ as follows. A domain $D$ belongs to the class $G(\alpha, \beta)$ if for any two points $t_1, t_2 \in L \subset \partial D$ (here $L$ is any component of $\partial D$) one can find a circular lune lying in the domain $D$ with the inner angle $\omega_1 \geqslant \alpha\pi$ between its boundary circles and with "supporting" angle $\omega_2 \leqslant \beta\pi$. Here "supporting" angle is the minimal non-negative angle between segment $[t_1, t_2]$ and circular arc belonging to the lune and passing through the points $t_1$, $t_2$. We state a result of Aksent'ev and Shabalin in generalized form.

THEOREM 17 (cf. [5,6]). *If the function $f(z)$ is analytic in domain $D \in G(\alpha, \beta)$ and satisfies condition*

$$\sup_{z \in D} \left| \rho_D^{-1}(z) f''(z)/f'(z) \right| \leqslant c(\alpha, \beta) \min(\alpha, 2 - \alpha),$$

$$c(\alpha, \beta) = \max_{\theta} \left| \frac{e^{i\theta(\alpha-1)/\alpha} + e^{-i\beta\pi}}{e^{i\theta} + e^{-i\beta\pi}} \right|,$$

*then $f(z)$ is univalent in $D$.*

We note one comparatively easy sufficient condition of univalence in an annulus.

THEOREM 18 [64]. *A meromorphic in the annulus $D = E(1, Q) = \{z\colon\ 1 < |z| < Q\}$ function $f(z)$ is univalent in $D$ if it satisfies one of the following inequalities*:

$$\sup_{z \in D} \left| \rho_D^{-1}(z) f''(z)/f'(z) \right| \leqslant 3\bigl(1 + \pi/(2\beta)\bigr)^{-1},$$

$$\sup_{z \in D} \left| \rho_D^{-2}(z) \{f, z\} \right| \leqslant \bigl(1 + \pi/(2\beta)\bigr)^{-2}/2,$$

*where* $\beta = \arcsin[(Q - 1)/(Q + 1)]$.

Martio and Sarvas [53] propose another method for the proof of univalence conditions in multiply connected domains. They introduce the following definition.

Let $0 < \alpha \leqslant \beta < \infty$. We call domain $D$ by $(\alpha, \beta)$-domain from class $I(\alpha, \beta)$ if there exists a point $z_0 \in D$ such that any other point $z \in D$ can be connected with $z_0$ by a rectifiable arc $\gamma \subset D$ so that its natural parametrization $\gamma(t)\colon [0, d] \to D$ satisfies the conditions $\gamma(0) = z$, $\gamma(d) = z_0$, $d \leqslant \beta$ and $\operatorname{dist}(\gamma(t), \partial D) \geqslant (\alpha/d)t$ for $t \in [0, d]$. A domain $D \subset \mathbb{C}$ is called $(\alpha, \beta)$-uniform if for any two points $z_1, z_2 \in D$, $z_1 \neq z_2$ it contains a domain $D(z_1, z_2) \in I(\alpha|z_1 - z_2|, \beta|z_1 - z_2|)$ such that $z_1, z_2 \in D(z_1, z_2) \subset D$.

THEOREM 19 [53]. *Let $f : D \to \mathbb{C}$ be analytic in $(\alpha, \beta)$-uniform domain $D \subset \mathbb{C}$, $0 < \alpha \leqslant \beta < \infty$ and $f'(z) \neq 0$ for all $z \in D$. If*

$$\text{(i)} \quad \sup_{z \in D} \left| \frac{f''(z)}{f'(z)} \right| \operatorname{dist}(z, \partial D) < \frac{\alpha}{\beta}(2\beta + 1)^{-1},$$

*or*

$$\text{(ii)} \quad \sup_{z \in D} \left|\{f, z\}\right| \operatorname{dist}^2(z, \partial D) < \frac{1}{2}\left(\frac{\alpha}{\beta}\right)^2 (2\beta + 1)^{-1},$$

*then $f$ is injective in $D$.*

The proof is done by reducing to a contradiction of the assumption on the existence of two points $z_1, z_2 \in D$ such that $z_1 \neq z_2$ but $f(z_1) = f(z_2)$, and on the following $M$-approximating property [53]: for any two points $z_1 \neq z_2$ there exists a domain $D(z_1, z_2) \in I(\alpha|z_1 - z_2|, \beta|z_1 - z_2|)$, $D(z_1, z_2) \subset D$ and a Möbius transformation $T \in M$ such that

$$\left|T \circ f(z) - z\right| < |z_1 - z_2|/2 \tag{25}$$

for any $z \in D(z_1, z_2)$.

To prove the key inequality (25) we integrate the equation $f''(z)/f'(z) = g(z)$ or $\{f, z\} = g(z)$ along the path $\gamma$ connecting the point $z_0$ with an arbitrary point of the domain $D(z_1, z_2)$ and use the conditions (i), (ii) and properties of $\gamma$.

The present method is applicable for research of injectivity of mappings in general metric spaces (see [11]).

The idea of reducing to a contradiction of the assumption of non-univalence of function (i.e., equality $f(z_1) - f(z_2) = 0$ for certain two points $z_1, z_2 \in D$, $z_1 \neq z_2$) is efficiently applied by other authors, too. For example, Avhadiev [8] uses the technique for proving sufficient univalence conditions in terms of the boundedness of $|\{f, z\}|$ for functions analytic in special domains. In particular, the development of a method from [8] yields the following results.

Let $A > 0$, $B > 0$, $\alpha, \beta \in (0, 1]$ be constants. We say that $D \in G_\alpha(A)$ if for any $z_1, z_2 \in D$ there exists an arc $\gamma(z_1, z_2) \subset D$ such that its length does not exceed $A|z_1 - z_2|^\alpha$, and $D \in G_{\alpha,\beta}(A, B)$ if the arc $\gamma(z_1, z_2)$ in addition satisfies the following condition: any point $z \in \gamma(z_1, z_2)$ divides $\gamma(z_1, z_2)$ into parts so that the minimal of their lengths does not exceed $B \operatorname{dist}^\beta(z, \partial D)$.

Now we cite a simplified version of F.G. Avhadiev result [7].

THEOREM 20. *Let $D \in G_\alpha(A)$, $d = \operatorname{diam} D$. A meromorphic and locally univalent in $D$ function $f(z)$ is univalent in $D$ if even one of the conditions holds*:

$$\text{(i)} \quad \alpha \in [1/2, 1], \quad \left|f''(z)/f'(z)\right| < 4k_0/A^2 d^{2\alpha - 1}, \quad z \in D,$$

*where constant* $k_0 = 0,854$ *is the root of* $2k \int_0^1 e^{kt^2}\,dt = e^k$;

$$\text{(ii)} \quad \alpha \in [1/3, 1], \quad \left|\{f, z\}\right| < 16/A^3 d^{3\alpha - 1}.$$

The proof is based on an estimate of the right-hand side of the identity

$$f(z_2) - f(z_1) - (z_2 - z_1)f'(w) = \int_{z_1}^{z_2} \left( \int_{w}^{\zeta} f''(z)\, dz \right) d\zeta,$$

where integrals are taken along $\gamma(z_1, z_2)$. By the assumptions of the theorem and properties of domain from the class $G_\alpha(A)$ we obtain a contradictory relation $|z_2 - z_1| < |z_2 - z_1|$.

Note that the lower bounds $\alpha = 1/2$ in (i) and $\alpha = 1/3$ in (ii) are sharp.

Realization of the same idea in the proof of following theorem is more cumbersome.

THEOREM 21 [7]. *Let $f(z)$ be meromorphic and locally univalent in domain $D \in G_{\alpha,\beta}(A, B)$. There exist positive constants $k_1 = k_1(\alpha)$ and $k_2 = k_2(\alpha)$ such that the function $f(z)$ is univalent in $D$ if it satisfies one of following requirements*:

(i) $\delta = \beta(2 - 1/\alpha) \in [0, 1]$, $\quad \left|\operatorname{dist}^\delta(z, \partial D)\dfrac{f''(z)}{f'(z)}\right| \leqslant \dfrac{k_1(\alpha)}{B^2}\left(\dfrac{B}{A}\right)^{1/\alpha}, \quad z \in D;$

(ii) $\delta = \beta(3 - 1/\alpha) \in [0, 2]$, $\quad \left|\operatorname{dist}^\delta(z, \partial D) S_f(z)\right| \leqslant \dfrac{k_2(\alpha)}{B^3}\left(\dfrac{B}{A}\right)^{1/\alpha}, \quad z \in D.$

## 5. Certain relations with universal Teichmüller spaces

Certain mentioned above results have interesting applications in description of structure of Teichmüller space of holomorphic functions.

Let $D$ be a domain in the extended plane which is conformally equivalent to a disk. We introduce the following norm for holomorphic in $D$ functions $\varphi$:

$$\|\varphi\|_D = \sup_{z \in D} |\varphi(z)|\, \rho_D(z)^{-2}, \tag{26}$$

where $B(D)$ stands for the Banach space of all holomorphic in $D$ functions with finite norm (26).

Let $f$ be a locally one-to-one meromorphic function in $D$ and $S_f = \{f, z\}$ be its Schwarzian derivative which is determined at $\infty$ and at the poles of $f$ by inversion. Any holomorphic in $D$ function is the Schwarzian derivative of a meromorphic function. One can easily verify the following useful equalities:

$\|S_f\|_D = \|S_{\lambda \circ f \circ \mu}\|_{\mu^{-1}(D)}$, if $\lambda$ and $\mu$ are Möbius transformations;

$\|S_f - S_g\|_D = \|S_{f \circ g^{-1}}\|_{g(D)}$, if $g$ is conformal on $D$;

$\|S_f\|_D = \|S_{f^{-1}}\|_{f(D)}$.

We relate to the domain $D$ the following constants:

$\sigma_1 = \|S_f\|_D$, where $f$ is conformal mapping of $D$ onto $E$,

$$\sigma_2 = \sup\{\|S_f\|_D\colon\ f \text{ is univalent in} D\}.$$
$$\sigma_3 = \sup\{a\colon \text{ if } \|S_f\|_D < a \text{ then } f \text{ is univalent in } D\}.$$

It is well known that $\sigma_1 \leqslant 6$ and the inequality is sharp. Additional information concerning the boundary of $D$ enables us to improve this bound for special cases. For example, if $\partial D$ is $K$-quasiconformal curve, then [49]

$$\sigma_1 \leqslant 6(K^2-1)/(K^2+1).$$

If $D$ is a convex domain, then $\sigma_1 \leqslant 2$ [49]. A relation between $\sigma_1$ and $\sigma_2$ is established in [48]: $\sigma_2 = \sigma_1 + 6$ for any domain $D$.

The constant $\sigma_3$ is positive if and only if $\partial D$ is a quasicircle. The sufficiency of the condition is proved by Ahlfors [2], and necessity by Gehring [26]. Nehari [57] and Hille [34] prove that $\sigma_3(E) = 2$. Lehtinen [47] establishes inequality $\sigma_3(D) \leqslant 2$ for all simply connected domains in $\overline{\mathbb{C}}$ and proves that it is attained if and only if $D$ is a disk or a halfplane. The results of Section 3 of the present paper give sharp values and lower bounds of $\sigma_3$ for certain domains.

Assume now that the domain $D$ is bounded by a quasiconformal curve. Set

$$S(D) = \{\varphi = S_f\colon\ f \text{ is univalent in } D\},$$
$$T(D) = \{\varphi = S_f\colon\ f \text{ is extendible to a quasi conformal mapping of the whole plane}\}.$$

$T(D)$ is called the universal Teichmüller space. Clearly, $T(D) \subset S(D) \subset B(D)$.

Ahlfors [2] proves that $T(D)$ is open in the norm (26) (see also [26,48]). Gehring [26] establishes the inclusion $\operatorname{int} S(D) \subset T(D)$. He also gives negative solution of a well-known Bers problem on identity between the closure of the universal Teichmüller space in the norm $B(E)$ and the set of Schwarzian derivatives of all univalent in the disk functions. The paper [26] (see also [39]) contains examples of functions $\varphi \in S(E) \setminus \overline{T}(E)$ which are Schwarzian derivatives of conformal mappings of $E$ onto domains bounded by slits along certain spirals.

As Lehto [48] shows, the constant $\sigma_3$ can be treated as the distance between the set $S(E) \setminus \overline{T}(E)$ and the point $\varphi_D = S_h$ where $h$ is a conformal mapping of $E$ onto $D$. Therefore $\sigma_3 \geqslant 2 - \sigma_1$. This bound is sharp. Lehto [50] and Lehtinen [47] evaluate these constants for the angular sector

$$A_k = \{z\colon\ z \in \mathbb{C},\ 0 < \arg z < k\pi\}, \quad 0 < k \leqslant 1,$$

and obtain that $\sigma_3(A_k) = 2k^2$ and $\sigma_1 = 2 - 2k^2$, i.e., $\sigma_3 = 2 - \sigma_1$; $\sigma_3(A_k) = 4k - 2k^2$, $1 < k < 2$.

If the inner univalence radius $\sigma_3$ of some domain satisfies equality $\sigma_3 = 2 - \sigma_1$ then this domain is called a *Nehari circle*. In the paper [55] one can find examples of the Nehari circles in the form of rectangles and hexagons.

## 6. Mechanical and physical applications

### 6.1. *Integral representations. Aerohydrodynamical inverse boundary value problem*

The Löwner–Kufarev equation enables us to reduce the study of properties of univalent functions $f(z,t)$ to investigation of regular functions with positive real part $h(z,t)$: any such function $h(z,t)$ determines a univalent subordination chain $f(z,t)$. A similar (although easier) interaction of properties of functions $f(z)$ and $p(t)$ is determined by representation

$$f(z) = I\big[p(t), z\big], \quad z \in D, \tag{27}$$

which defines a regular function in $D$ by known density $p(t)$. A typical problem concerning representation (27) is the following one. We partition the class $A$ of functions $p(t)$ into two subclasses $A_1$ and $A_2$ so that a function $f(z)$ is univalent in $D$ for $p(t) \in A_1$ and multivalent for $p(t) \in A_2$.

As a rule, important representations (27) have an integral form, and the problem consists in description of subclasses of $A_1$ corresponding to certain sufficient conditions for univalence. For example, if function $u(t)$ is non-negative for $a < t < b$, does not decrease on $(a,c)$ and does not increase on $(c,b)$, then the Cauchy type integral

$$f(z) = \frac{1}{\pi \mathrm{i}} \int_a^b \frac{u(\tau)\,d\tau}{\tau - z}$$

is univalent outside of the segment $[a,b]$. The same result is valid for the Schwarz integral

$$\frac{1}{2\pi} \int_0^{2\pi} u(\theta) \frac{\mathrm{e}^{\mathrm{i}\theta} + z}{\mathrm{e}^{\mathrm{i}\theta} - z}\, d\theta$$

in $E$: under the restrictions it belongs to Kaplan class of close-to-convex functions.

The important place among integral representations of the form (27) belongs to solutions of *inverse boundary value problems* (I.B.V.P.) for regular functions.

The first and the simplest I.B.V.P. of aerohydrodynamics is considered by Mangler [52] and Tumashev [68]. It consists in determining the form of an isolated airfoil profile in plane flow of ideal incompressible fluid, given the velocity distribution along the profile $v(s)$, $0 \leqslant s \leqslant l$ (here $s$ is the length of variable arc of the boundary, $l$ being the length of the whole boundary). Mangler and Tumashev proposed certain ways for solving the problem based on the idea of juxtaposition of planes and obtained integral representation of solutions in the form

$$\ln z'(\zeta) = -\frac{1}{2\pi} \int_0^{2\pi} p(\theta) \frac{\mathrm{e}^{\mathrm{i}\theta} + \zeta}{\mathrm{e}^{\mathrm{i}\theta} - \zeta}\, d\theta \quad (|\zeta| > 1), \tag{28}$$

where density $p(\theta)$ of the Schwarz integral is constructively determined by $v(s)$. The function $z(\zeta)$ from this representation maps domain $E^-$ onto the required domain. But the formula does not ensure such important properties of the desired boundary as its closedness and simplicity. Therefore a number of authors elaborated various approaches for solving these problems. A way to satisfy the closedness condition is proposed by Mangler himself [52]. The state-of-art research in aerohydrodynamical I.B.V.P., the relevant closedness and univalence conditions, questions of optimization of profiles, the technique of quasi-solutions can be found in monographs [23,22].

In the I.B.V.P. theory the problem of decomposition of the class of admissible densities $p(\theta)$ into two subclasses according to whether the function $z(\zeta)$ is univalent or multivalent in closed domain is called the *weak problem of univalence*. The known results on it are based on sufficient conditions mentioned in the present survey. There exist also certain achievements on the *strong problem of univalence* concerning analogous decomposition of the set of boundary values $v(s)$.

In the so-called interior I.B.V.P. the boundary values of required analytic function $w(z)|_L = \varphi(s) + i\psi(s)$, $0 \leqslant s < l$, are given on the unknown boundary $L$. The required domain (with its boundary $L$) is determined by the same formula (28) (here $\zeta \in E$ and the minus sign must be omitted) [68,25]. Basing on sufficient conditions of univalence in the disk $E$, bounds are obtained for coefficient $A$ of the Hölder condition where $p(\theta) \in H(A, \alpha)$ and for constants $A_1, A_2$ in conditions $\max p(\theta) - \min p(\theta) < A_1$ and $\max p'(\theta) \leqslant A_2$ guaranteeing univalence of the function $z(\zeta)$.The Zygmund condition for the function $\nu(\theta) = \int_0^\theta p(\theta)\, d\theta$ is studied, too. An important fact of existence of two separating constants in this condition is discovered by Shabalin. Namely, there exist constants $k_0$ and $k_1$ such that the formula (28) with $p(\theta) = \nu'(\theta)$ maps the class of all functions $\nu$ satisfying the Zygmund condition with coefficient $k < k_0$ into the class of univalent functions $z(\zeta)$, and the Zygmund class with $k > k_1$ – into the class of multivalent functions $z(\zeta)$. For $k_0 \leqslant k < k_1$ the image contains both univalent and multivalent functions. The proof of this result uses the Becker condition and necessary condition of univalence in the form of restriction on the functional $|zf''(z)/f'(z)|$. The best bounds for constants $k_0 > \pi^2/20$ and $k_1 < 12$ belong to Avhadiev [7].

### 6.2. *Exterior inverse boundary value problem and conformal radius*

The solution of exterior I.B.V.P. (where the unknown curve $L$ bounds the domain $D^-$ containing $\infty$) by passing to the disk can be rewritten in the form

$$z(\zeta) = \int_a^\zeta f'(\zeta)\left(\frac{1 - \bar{\zeta}_0\zeta}{\zeta - \zeta_0}\right)^2 d\zeta, \quad \zeta \in E,$$

where the function $f(\zeta)$ is a solution of interior I.B.V.P. with the same boundary values $w(z)|_L = \varphi(s) + i\psi(s)$. The pole of $z(\zeta)$ is determined by equation

$$f''/f' = 2\bar{\zeta}/(1 - |\zeta|)^2$$

which simultaneously is necessary condition for extremum of conformal radius

$$R(f(\zeta), f(E)) = |f'(\zeta)|(1 - |\zeta|^2).$$

This relation enables us to conclude that the number of solutions of exterior I.B.V.P. does not exceed the number of stationary points of conformal radius, and that uniqueness of stationary point of conformal radius means uniqueness of solution of exterior I.B.V.P. The uniqueness of stationary point of the function $R(f(\zeta), f(E))$ can be ensured in certain subclasses of univalent functions, e.g., in Nehari class with condition (7).

## 6.3. *An inversion problem for logarithmic potential*

The theorems on uniqueness of solution of exterior I.B.V.P. have much in common with uniqueness theorems for solutions of inversion problem for logarithmic potential. The problem can be stated as follows. Given the expansion $w = \sum_{k=0}^{\infty} c_k/z^{k+1}$ of a regular function $w(z)$ at $\infty$ and constant density $\mu > 0$, we seek a bounded simply connected domain $D$ such that

$$-\frac{1}{\pi}\iint_D \frac{\mu\, d\xi\, d\eta}{\zeta - z} = w(z), \quad \zeta = \xi + i\eta,$$

for any $z$ outside of $D$. A known uniqueness class for this problem consists of domains which are starlike with respect to a common inner point (Novikov's theorem).

We note certain conditions for solvability of the problem found by Cherednichenko [18].

(1) If $c_l = 0$ for $l \leqslant n$, then a necessary condition for solvability is that the inequalities $|c_k/c_0| \leqslant (c_0/\mu)^{k/2} l_k^{(n)}$, $k = 1, 2, \ldots, n-1$, hold, where $l_k^{(n)}$ are known and depend, in particular, on bounds for coefficients of univalent polynomials.

This result implies unsolvability of the inversion problem for sufficiently large $\mu$.

(2) If $|u(z) - c_0/z| \leqslant \lambda$ for $|z| = R$, then a solution of the inversion problem exists for $\mu \leqslant \mu_0(R, \lambda)$ (the explicit form of $\mu_0$ is known).

(3) For any $\alpha$, $0 < \alpha < 1$, there exist a number $\mu$ and domain $D_\mu^*$ starlike of order $\alpha$ (i.e., $\alpha = \min_{|t|=1} \Re[tz'(t)/z(t)]$ where $z(t)$ maps $E$ on $D_\mu^*$) solving the inversion potential problem for $w(z) = c_0/z + c_1/z^2 + \cdots + c_{n-1}/z^n$, $c_0 > 0$, $n > 1$.

The condition (1) shows the relation of the inverse potential problem with the coefficient problem in theory of univalent functions.

## 6.4. *Flows in domains with variable boundaries*

The Hele–Shaw flows in viscous liquids and shrinking of the water–oil interface are samples of non-stationary problems of mathematical physics solvable by complex analysis technique.

Let us state the problem. Suppose that $D_t \subset \mathbb{C}$ is a simply connected domain filled by ideal incompressible fluid and depending on parameter $t \geqslant 0$ (time). At the origin it has a source (sink) of given capacity $q(t)$. The influence of the surrounding media on $D_t$ is characterized by constancy of pressure on its boundary. The initial domain $D_0$ is given. The problem is to determine the form of $D_t$ for all $t > 0$.

This problem reduces to a boundary value problem of finding the evolution family $f(z,t): E \to D_t$ (with normalization $f(0,t)=0$, $\frac{\partial f}{\partial z}(0,t)>0$) which satisfies the nonlinear boundary condition

$$\Re\left[\frac{\partial f}{\partial t}\overline{\left(z\frac{\partial f}{\partial z}\right)}\right] = q(t), \quad |z|<1, \tag{29}$$

and initial condition $f(z,0) = f_0(z)$ with known $f_0(z): E \to D_0$. Equation (29) can be easily written as

$$\Re\left[\frac{\partial f/\partial t}{z\partial f/\partial z}\right] = q(t)\left|z\frac{\partial f}{\partial z}\right|^{-2} \quad \Rightarrow \quad \frac{\partial f}{\partial t} = q(t)z\frac{\partial f}{\partial z}H\big[f(z,t),z\big], \quad |z|<1,$$

where

$$H\big[f(z,t),z\big] = \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\partial f}{\partial z}\right|^{-2}\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta.$$

It is similar to Löwner–Kufarev equation.

Starting from Polubarinova-Kochina and Galin in 1945, a lot of authors constructed explicit evolution families in quadratures. Recently Hohlov [35] obtained close-to-convex evolution families. Classification of solutions of Equation (29) is given in [36] (see also [33]).

### 6.5. *Problems of boundary conjugation*

Numerous questions of continuum mechanics are reduced to conjugation problems for analytic functions. The study of these problems uses various extensions. Here we give a brief description of certain problems of this kind.

Let $D^+$ be a simply connected Jordan domain in $\mathbb{C}$, $\partial D^+ = L$, $D^- = \overline{\mathbb{C}} \setminus \overline{D}^+$. The simplest conjugation problem consists in determining the functions $\Phi^+(z)$ and $\Phi^-(z)$, regular in the domains $D^+$ and $D^-$, respectively, by the boundary condition

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in L, \tag{30}$$

where function $G(t)$ and $g(t)$ are defined on $L$. Thus, it is a problem of piecewise-analytic extension through the curve $L$ with linear conjugation condition (30).

The conjugation problem with a shift (it is also called *Haseman problem*) is a generalization of the above problem. Its boundary condition has the form

$$\widetilde{\Phi}^+\big(\alpha(\tau)\big) = \widetilde{G}(t)\widetilde{\Phi}^-(\tau) + \tilde{g}(\tau), \quad \tau \in \Gamma, \tag{31}$$

where the closed Jordan curve $\Gamma$ divides the plane into domains $\widetilde{D}^+$ and $\widetilde{D}^-$, the functions $\widetilde{G}$ and $\tilde{g}$ are defined on $\Gamma$, and given homeomorphism $\alpha$ maps this curve onto itself and preserves its orientation.

Problem (31) can be reduced to the conjugation problem (30) by using the theory of univalent functions. If we find conformal maps $\omega^+ : \widetilde{D}^+ \mapsto D^+$ and $\omega^- : \widetilde{D}^- \mapsto D^-$ satisfying the boundary condition

$$\omega^+\big(\alpha(\tau)\big) = \omega^-(\tau), \quad \tau \in \Gamma,$$

then the substitution $\widetilde{\Phi}^{\pm}(z) = \Phi^{\pm}(\omega^{\pm}(z))$ reduces the problem (31) to (30). This method is known as *conformal sewing*. One of best known construction of the sewing function $\omega$ is based on quasiconformal extension of $\alpha(t)$. By the method Lyubarskii obtained interesting results concerning completeness and minimality of certain functional systems. Zverovich (see [25]) successfully applied the conformal sewing and local conformal sewing for studying the problems of boundary conjugation on Riemann surfaces.

Kats and his successors (see [11]) apply extensions for solving the conjugation problem (30) for a non-rectifiable curve $L$. We describe here the simplest case $G(t) = 1$. The conjugation problem (30) reduces to a so-called *gap problem*

$$\Phi^+(t) - \Phi^-(t) = g(t), \quad t \in L. \tag{32}$$

A quasi-solution of this problem is a function $\varphi(z)$ which continues $g(t)$ into $D^+$ and vanishes in $D^-$. Then the solution itself can be found in the form $\Phi(z) = \varphi(z) - \psi(z)$ where $\psi$ must be chosen so that $\Phi$ is holomorphic function. Consequently,

$$\psi(z) = \frac{1}{2\pi \mathrm{i}} \iint_{D^+} \frac{\partial \varphi}{\partial \bar{\zeta}} \frac{d\zeta\, d\bar{\zeta}}{\zeta - z}.$$

The further study reduces to interpretation of continuity conditions for $\psi(z)$ in $\mathbb{C}$. In this way Kats proves that the gap problem (32) has a solution if the function $g(t)$ satisfies Hölder condition with exponent $\delta > d/2$, where $d$ is the box dimension of the curve $L$. This result is also applicable to fractal curves.

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CHAPTER 8

# Bounded Univalent Functions


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*Contents*




HANDBOOK OF COMPLEX ANALYSIS: GEOMETRIC FUNCTION THEORY, VOLUME 1
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## 1. Introduction

It is interesting to observe the role of the nonlinear univalence property in different problems for the class $S$ of holomorphic univalent functions $f$ in the unit disk $E = \{z\colon |z| < 1\}$ normalized by the expansion

$$f(z) = z + a_2 z^2 + \cdots, \quad z \in E. \tag{1}$$

The Koebe functions

$$K_\alpha(z) = \frac{z}{(1 - e^{i\alpha} z)^2}, \quad \alpha \in \mathbb{R},$$

give extrema to numerous extremal problems in $S$.

It is more intriguing to study the influence of the property of boundedness within the class $S$. Denote by $S(M)$, $M > 1$, the class of functions $f \in S$ satisfying $|f(z)| < M$ in $E$. Let $S_R(M)$ be the class of functions $f \in S(M)$ such that $f(\bar{z}) = \overline{f(z)}$. The similar role to the Koebe functions $K_\alpha$ in $S$ is played by the Pick functions $P_\alpha^M$ in $S(M)$ given by the equation

$$\frac{M^2 P_\alpha^M(z)}{(M - P_\alpha^M(z))^2} = K_\alpha(z), \quad z \in E,\ M > 1,\ P_\alpha^\infty = K_\alpha,$$

or, equivalently,

$$P_\alpha^M(z) = M K_\alpha^{-1}\left(\frac{1}{M} K_\alpha(z)\right).$$

The functions $P_\alpha^M$ map $E$ onto the disk $E^M$ of radius $M$ centered at the origin minus a radial segment.

The classes $S$ and $S(M)$ have a lot of common features but there are many differences of principal character. So $S(M)$ has to be studied individually as well as in comparison with $S$.

Sometimes univalent functions maximize functionals in the whole class of bounded holomorphic functions. The excellent example is given by the Schwarz [66] lemma which states that if a function $\omega$, $\omega(0) = 0$, is holomorphic and bounded in $E$, $|\omega(z)| < 1$, then $|\omega(z)| \leqslant |z|$, $z \in E$, and $|\omega'(0)| \leqslant 1$ with the equality sign in both inequalities only for rotations of the identity function, i.e., $\omega(z) = e^{i\alpha} z$, $\alpha \in \mathbb{R}$. This lemma was generalized in many directions and gave impulses to wide researches.

Extremal problems and other topics for bounded univalent functions were investigated by many authors. We suggest [23,46,36,37,52,1,78,79,17] as references.

## 2. Coefficient growth in $S(M)$

According to the area principle a bounded univalent function $f$ has finite Dirichlet integral

$$\iint_E |f'(z)|^2\,dx\,dy = \pi\sum_{n=1}^{\infty} n|a_n|^2 < \infty,$$

and so $a_n = o(n^{-1/2})$. Clunie and Pommerenke [15] established the existence of an absolute constant $\gamma > 1/2$ such that $a_n = O(n^{-\gamma+\varepsilon})$, $n \to \infty$, for every $\varepsilon > 0$ and for every bounded function $f \in S$, see also [17] and [53]. This problem has a close connection with the asymptotic coefficient problem for the class $\Sigma$ of univalent functions

$$F(z) = z + b_0 + b_1 z^{-1} + \cdots, \qquad |z| > 1.$$

The maximal growth of $b_n$ is of the same order as for $a_n$ in $S(M)$, see [11,15,51].

The best value of $\gamma$ is unknown. The argument of Clunie and Pommerenke gives $\gamma > 0.5090$. Makarov and Pommerenke [42] improved this to $\gamma > 0.5114$. As for the estimates from above Littlewood [39] demonstrated that $\gamma < 1$ and Clunie [13] used the similar construction for the coefficients $b_n$ of functions $F \in \Sigma$. Pommerenke [50–52] sharpened these results. At present the best upper bound $\gamma < 1/2 + 1/86$ due to Grinshpan and Pommerenke [24] improves the previous estimates $\gamma < 0.83$ of Pommerenke [50], [52, p. 133]. Carleson and Jones [11], see also [32], experimentally established that $\gamma < 0.76$ and conjectured that $\gamma = 3/4$.

## 3. Parametric representations of bounded univalent functions

The variational method is of decisive importance for extremal problems in the class $S$. The nonlinear univalence condition produces a lot of complications in constructing of variational formulas for a rich subclass of functions from a neighbourhood of the extremal function. Applications to the class $S(M)$ require saving the boundedness of neighbouring functions. Such difficulties restrict the power of variational methods in the class $S(M)$.

Nevertheless a lot of successful attempts to apply variational methods in extremal problems for bounded univalent functions were made by many authors. We will point for example at the paper of Charzyński and Janowski [12] who generalized the quantitative and qualitative properties of the coefficient region of $S$ to $S(M)$.

In this section we will concentrate on another powerful investigation method for $S(M)$.

**3.1.** *The Löwner differential equation.* The parametric representation of $S(M)$ by means of the Löwner differential equation is almost the same as of $S$. The simple idea of Carathéodory was realized in the parametric method owing to Löwner's skill in describing a piecewise smooth deformation $w(z,t)$, $0 \leqslant t < \infty$, between the identity $w = z$ and the given mapping $w = f(z)$ with the help of the ordinary differential equation

$$\frac{dw}{dt} = -w\frac{e^{iu} + w}{e^{iu} - w}, \quad w(z,0) = z, \tag{2}$$

$$z \in E, \ 0 \leqslant t < \infty, \quad \lim_{t \to \infty} e^t w(z,t) = f(z).$$

The function $u = u(t)$ is piecewise continuous on $[0, \infty)$. Every solution $w(z, t)$ and hence the limit are univalent in $E$. Being integrated on the segment $[0, \log M]$, Equation (2) parametrically describes the class $S(M)$. The set of all functions $Mw(z, \log M)$ is dense in $S(M)$. In particular, if $u(t)$ is constant on $[0, \log M]$, e.g., $u(t) = \alpha$, then $Mw(z, \log M) = P^M_{\alpha+\pi}(z)$.

Equation (2) is generalized to the Löwner–Kufarev differential equation

$$\frac{dw}{dt} = -wp(w, t) \tag{3}$$

with the same additional conditions as in (2) and with a function $p$, $p(0, t) = 1$, in its right-hand side which is analytic in $E$ with respect to the first variable and measurable on $[0, \infty)$ with respect to the second variable and has the positive real part. Solutions of (3) on $[0, \log M]$ parametrize the whole class $S(M)$.

It is known that the special choice of $p$ in (3) leads after integrating to representations of some classes of univalent functions. For instance, if $p$ does not depend on $t \in [0, \infty)$ and $w(z, t)$ are solutions of (3), then $f(z) = \lim_{t\to\infty} e^t w(z, t)$ represent the class $S^*$ of starlike functions $f \in S$. It is interesting to notice that in this case the functions

$$f^M(z) = Mw(z, \log M) = Mf^{-1}\left(\frac{1}{M} f(z)\right), \quad f \in S^*,$$

form the subclass of $S(M)$ of the so-called quasi-starlike functions [18] which are not necessary starlike.

**3.2.** *Estimates of functionals in $S(M)$.* Equations (2) or (3) are important tools in extremal problems for $f \in S(M)$. For example, let us show the way to estimate $|f(z)|$ in $S(M)$ (compare with the Schwarz lemma).

THEOREM 1. *If $f \in S(M)$, $M > 1$, then the following estimates*

$$P^M_\pi(|z|) \leqslant |f(z)| \leqslant P^M_0(|z|), \quad z \in E,$$

*hold.*

PROOF. The real part of (2) after dividing by $w$ is equivalent to

$$\frac{d \log|w|}{dt} = -\frac{1 - |w|^2}{|e^{iu} - w|^2}, \quad |w(z, 0)| = |z|,$$

which is easily estimated by a suitable choice of $u$ as

$$-\frac{1 + |w|}{1 - |w|} \leqslant \frac{1}{|w|} \frac{d|w|}{dt} \leqslant -\frac{1 - |w|}{1 + |w|}$$

or

$$d\log K_\pi\big(|w|\big)\geqslant -dt,\qquad d\log K_0\big(|w|\big)\leqslant -dt.$$

Note that $K_\pi$ and $K_0$ are increasing functions on $(0,1)$ and $|w(z,t)|$ as a function of $t$ decreases on $[0,\infty)$. Integrate the last inequalities on $[0,\log M]$ and obtain

$$K_\pi\left(\frac{1}{M}|f(z)|\right)\geqslant\frac{1}{M}K_\pi\big(|z|\big),\qquad K_0\left(\frac{1}{M}|f(z)|\right)\leqslant\frac{1}{M}K_0\big(|z|\big)$$

or finally

$$|f(z)|\geqslant MK_\pi^{-1}\left(\frac{1}{M}K_\pi\big(|z|\big)\right)=P_\pi^M\big(|z|\big),$$

$$|f(z)|\leqslant MK_0^{-1}\left(\frac{1}{M}K_0\big(|z|\big)\right)=P_0^M\big(|z|\big),$$

which ends the proof. □

It is possible to show similarly the extremal character of the Pick functions $P_\alpha^M$ in the estimates of $|f'(z)|$ and some other functionals in $S(M)$.

Theorem 1 could be proved in another way, e.g., by the area principle, the minimizing property of the Dirichlet integral etc. Anyway it was generalized to inequalities which appear to be necessary and sufficient univalence conditions. Developing the Löwner theory approach analogous to Theorem 1 Shlionskii [67] obtained general inequalities which imply in particular the following theorems.

THEOREM 2. *Let $f\in S(M)$, $M>1$, and $z$ and $\zeta$ be arbitrary points from $E$. Then the inequality*

$$\left|\frac{f'(z)f'(\zeta)}{(f(z)-f(\zeta))^2}-\frac{1}{(z-\zeta)^2}\right|\leqslant\big(T(z)T(\zeta)\big)^{1/2}$$

*holds where*

$$T(u)=\left|\frac{1}{(1-|u|^2)^2}-\frac{M^2|f'(u)|^2}{(M^2-|f(u)|^2)^2}\right|.$$

For a holomorphic function $f$ normalized by (1) and $z$ and $\zeta$ from a neighbourhood of the origin, let

$$\log\frac{f(z)-f(\zeta)}{z-\zeta}\,\frac{z\zeta}{f(z)f(\zeta)}=\sum_{k,l=1}^{\infty}a_{k,l}z^k\zeta^l,$$

$$\log\left(1-\frac{f(z)\overline{f(\zeta)}}{M^2}\right)=-\sum_{k,l=1}^{\infty}b_{k,l}z^k\bar{\zeta}^l.$$

THEOREM 3. *A holomorphic function $f$ normalized by* (1) *belongs to $S(M)$ iff the inequality*

$$\left|\sum_{k,l=1}^{n} a_{k,l}x_k x_l\right| \leqslant \sum_{k=1}^{n}\frac{|x_k|^2}{k} - \sum_{k,l=1}^{n} b_{k,l}x_k\bar{x}_l$$

*holds for arbitrary complex numbers $x_1,\dots,x_n$ and natural $n$.*

Theorem 3 reduces to the known Grunsky inequalities if $M=\infty$ and hence $b_{k,l}=0$ for all natural $k$ and $l$.

Putting $z=\zeta$ in Theorem 2 one can obtain the inequality due to Alenicyn [3]

$$\frac{1}{6}\left|\{f,z\}\right| + \frac{M^2|f'(z)|^2}{(M^2-\left|f(z)\right|^2)^2} \leqslant \frac{1}{(1-|z|^2)^2}, \qquad z\in E,$$

where $\{f,z\}$ is the Schwarzian derivative.

If $M\to\infty$, then the Alenicyn inequality reduces to the known necessary univalence condition proved by Kraus [33] and Nehari [48].

## 4. Control theory methods in extremal problems for $S(M)$

**4.1.** *Optimization methods.* Typical extremal problems to estimate continuous functionals on $S(M)$ are naturally widened to the problem to describe value sets for systems of functionals. The most attractive is the value set

$$V_n^M = \left\{(a_2,\dots,a_n):\ f\in S(M)\right\}$$

of the coefficient system in $S(M)$.

The deep penetration of the variational principles into the parametric method is connected with the point of view that the differential Löwner equation (2) is a control equation for $f(z)$ in the dense subclass of $S(M)$. Moreover, after differentiating (2) with respect to the initial data $z$ we obtain a control system for $\{f(z), f'(z),\dots, f^{(n)}(z)\}$ at some points $z_1,\dots,z_m$. If $z=0$, then dividing over the corresponding factorials and excluding the two initial degenerate functionals we come to the control system for $V_n^M$.

The Pontryagin maximum principle in the control theory is an interpretation of the classical necessary conditions in the calculus of variations: the Euler equation and the Weierstrass inequalities. It is important also to understand geometrically the transversality conditions as the property of the conjugate vector to be orthogonal or support to the boundary hypersurface of the value set.

Consider the coefficient problem for $S(M)$. Let $w(z,t)$ be an integral of the Löwner differential equation (2). Put

$$e^t w(z,t) = \sum_{n=1}^{\infty} a_n(t)z^n, \quad a_1(t)=1.$$

Determine the quadratic matrix $A(t)$

$$A(t)=\begin{pmatrix} 0 & 0 & \dots & 0 & 0\\ a_1(t) & 0 & \dots & 0 & 0\\ a_2(t) & a_1(t) & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ a_{n-1}(t) & a_{n-2}(t) & \dots & a_1(t) & 0\end{pmatrix}.$$

Substituting $w(z,t)$ in (2) and equating the coefficients at $z^k$, $2\leqslant k\leqslant n$, we obtain a phase control system of equations which may be written in the vector form with respect to the vector $a(t)=(a_1(t),\dots,a_n(t))^{\mathrm{T}}$

$$\frac{da(t)}{dt}=-2\sum_{s=1}^{n-1}\mathrm{e}^{-s(t+\mathrm{i}u)}A^s(t)a(t),\quad a(0)=a^0=(1,0,\dots,0)^{\mathrm{T}}. \tag{4}$$

The boundary hypersurface $\partial V_n^M$ of $V_n^M$ is a boundary of the reachable set for the control system (4) on the segment $[0,\log M]$. Every point of this boundary is reached by a certain optimal trajectory $a(t)$ corresponding to a choice of the optimal control $u(t)$ which satisfies all the necessary extremum conditions. Namely, if $(A^s a)^{\mathrm{T}}\overline{\Psi}$ is a scalar product of the vector $A^s a$ and the conjugate complex valued vector of the Lagrange multipliers $\Psi=(\Psi_1,\Psi_2,\dots,\Psi_n)^{\mathrm{T}}$, then $u(t)$ maximizes the Hamilton function

$$H(t,a,\overline{\Psi},u)=-2\Re\sum_{s=1}^{n-1}\left[\mathrm{e}^{-s(t+\mathrm{i}u)}(A^s a)^{\mathrm{T}}\overline{\Psi}\right]$$

for all $t\in[0,\log M]$.

The coordinates of the vector $\overline{\Psi}$, except for the first one which is nonessential, are the solutions of the conjugate Hamiltonian system written in the vector form where we put, for instance $\Psi_1=0$,

$$\frac{d\overline{\Psi}}{dt}=2\sum_{s=1}^{n-1}\mathrm{e}^{-s(t+\mathrm{i}u)}(s+1)\left(A^{\mathrm{T}}\right)^s\overline{\Psi},\quad \Psi(0)=\xi. \tag{5}$$

Since the system (5) and the Hamilton function $H$ are linear with respect to $\overline{\Psi}$, then without loss of generality we may normalize the conjugate vector multiplying it by a positive constant number. Moreover, it follows from the last coordinate equation of (5) that $\Psi_n=\text{const}$. Hence, if $\Psi_n\neq 0$, then we may put for instance $|\Psi_n|=1$.

Along with a boundary function $f$ delivering a point at the boundary hypersurface $\partial V_n^M$ its rotation is also a boundary function. The mapping $f(z)\to \mathrm{e}^{-\mathrm{i}\alpha}f(\mathrm{e}^{\mathrm{i}\alpha}z)$, $0\leqslant\alpha\leqslant 2\pi$, determines a certain curve on $\partial V_n^M$ and establishes a certain symmetry of $\partial V_n^M$ with respect to rotation. At the same time this mapping induces an evident change of the phase vector $a(t)$, the changes $\Psi(t)\to \mathrm{e}^{\mathrm{i}\alpha}\Psi(t)$ of the conjugate vector and $u(t)\to u(t)+\alpha$ of the optimal control. That is why to understand the structural properties of $\partial V_n^M$ invariant under rotation in the case $\Psi_n \neq 0$ it is possible for instance to normalize the conjugate vector by the condition $\Im\Psi_n = 0$, i.e., $\Psi_n = \pm 1$. The accepted normalization corresponds to the projection $V_n^{M'}$ of the set $V_n^M$ onto the hyperspace $\{(a_2, \ldots, a_n)\colon \Im a_n = 0\}$ of dimension $(2n-3)$. Its boundary hypersurface $\partial V_n^{M'}$ is $(2n-4)$-dimensional. To be definite let us consider the condition $\Psi_n = 1$, the second condition leads to the analogous description of another part of the hypersurface $\partial V_n^{M'}$.

For every admissible value $(t, a, \overline{\Psi})$ the Hamilton function $H$ is a trigonometrical polynomial of degree $(n-1)$ with respect to $u$. Its coefficients polynomially depend on $e^t$ and on coordinates of $a(t)$ and linearly depend on coordinates of $\overline{\Psi}(t)$. Hence the function $H$ has at most $(n-1)$ maximum points on $[0, 2\pi)$ which satisfy the equation

$$H_u(t, a, \overline{\Psi}, u) = 0. \tag{6}$$

Equation (6) determines a many valued function $u = u(t, a, \overline{\Psi})$. It is shown in [54] that if the function $H(0, a^0, \bar{\xi}, u)$ at $t = 0$ and certain $\xi$ has a unique maximum point $u = u(0, a^0, \bar{\xi})$ on $[0, 2\pi)$, then $\xi$ determines a single valued continuous branch $u = u_1(t, a, \overline{\Psi})$ of a multifunction which maximizes the Hamilton function at $a(t)$ and $\overline{\Psi}(t)$ given by (4)–(5) for all $t \in [0, \log M]$. This case represents the most general situation and it may be called the nonsingular regime. A corresponding extremal function $f \in S(M)$ generated by the Löwner differential equation maps the unit disk onto the disk $E^M$ minus a piecewise analytic curve having one finite tip.

The matter is different when the Hamilton function $H(0, a^0, \bar{\xi}, u)$ at $t = 0$ and a certain $\xi$ has $m$, $2 \leqslant m \leqslant n-1$, maximum points on $[0, 2\pi)$. This case is called the sliding regime, it is provided by $(m-1)$ equations with respect to $\xi$ which determine the equality of the values of the Hamilton function at $m$ critical points. The sliding regime requires a generalization of the phase control system. Instead of Equation (2) with the Schwarz kernel $P(w, u) = (e^{iu} + w)/(e^{iu} - w)$ in its right-hand side we should introduce the generalized Löwner differential equation

$$\frac{dw}{dt} = -w \sum_{k=1}^{m} \lambda_k P(w, u_k), \quad \lambda_1 \geqslant 0, \ldots, \lambda_m \geqslant 0, \ \sum_{k=1}^{m} \lambda_k = 1, \tag{7}$$

with the same additional conditions as in (2). All the functions $u_k = u_k(t)$, $\lambda_k = \lambda_k(t)$, $k = 1, \ldots, m$, play the role of control functions.

The difference between Equation (2) and its generalization (7) in the conformal mappings theory means that under the conditions of smoothness of all control functions integrals of Equation (7) map the unit disk $E$ onto the disk $E^M$ minus smooth curves having $m$ finite tips. However, this representation is not unique.

The following theorem eliminates the multivalence of extremal mapping representations by integrals of the generalized Löwner differential equation.

THEOREM 4 [54]. *Let a function $f \in S(M)$ deliver a nonsingular boundary point of the set $V_n^M$ and $f$ map the unit disk $E$ onto the disk $E^M$ minus piecewise analytic slits having $m$ finite tips. Then there exist real functions $u_1, \ldots, u_m$ continuous on $[0, \log M]$ and* *positive numbers* $\lambda_1, \ldots, \lambda_m$, $\sum_{j=1}^{m} \lambda_j = 1$, *such that a solution* $w = w(z,t)$ *of the Cauchy problem for the generalized Löwner differential equation* (7) *represents* $f$ *according to the formula* $f(z) = Mw(z, \log M)$. *This representation is unique.*

Theorem 4 has been proved firstly for the class $S$ but it is evidently true for $S(M)$. The proof of Theorem 4 is based on analytical and geometrical properties of coefficients $\lambda_1(t), \ldots, \lambda_m(t)$ which regulate the reciprocal lengths of truncated parts of the above slits under the mapping $\zeta = F(z,t)$, $z \in E$, $t \in [0, \log M]$, determined by the functional equation $F(w(z,t),t) = f(z)$. It was shown that a choice of constant numbers $\lambda_1, \ldots, \lambda_m$ provides the simultaneous truncation of all the slits only at the last moment $t$.

So, the sliding regime of the optimal control problem with $m$ maximum points of the Hamilton function is realized at $\xi \in \mathfrak{M}_m$ where the manifold $\mathfrak{M}_m$ of dimension $(2n-3-m)$ is determined by $(m-1)$ equations of equality for the values of the Hamilton function at $m$ critical points for $t = 0$.

It is shown in [54] that the sliding regime with $m$ maximum points of function $H$ at $t = 0$ preserves this property for $t > 0$. The number of maximum points can only decrease and only because of joining of some of them at certain moments $t$. If the number of maximum points of the Hamilton function is equal to $m'$ at the moment $t = t'$, then it can only decrease for $t > t'$, i.e., it will not exceed $m'$.

To obtain all the boundary points of the coefficient region $V_n^{M'}$ we have to solve the system (4)–(5) or its generalization according to (7) with $u = u(t, a, \overline{\Psi})$ given by (6) in its right-hand side. Solutions of the Cauchy problem for this system form the boundary hypersurface $\partial V_n^{M'}$ parametrized by $\xi \in \mathfrak{M}_m$ and $\lambda_1, \ldots, \lambda_{m-1}$.

An interesting approach to optimization methods was given by Roth [62] who also analyzed and compared different methods including the above scheme, Schiffer's variational method and control theory applications due to Friedland and Schiffer [21,22].

**4.2.** *Linear extremal problems.* Pay attention at the relation between the conjugate system (5) and the phase equations. Let again $w(z,t)$ be an integral of the differential equation (2), $w_z$ be its partial derivative with respect to $z$ and $q(z)$ be a holomorphic function in $E$, $q(0) = 0$. Put

$$g(z,t) = \frac{q(z)}{e^t w_z(z,t)} = \sum_{n=1}^{\infty} b_n(t) z^n. \tag{8}$$

Differentiate (2) with respect to $z$, obtain the differential equation for $w_z(z,t)$ and deduce from here the differential equation for $g(z,t)$. Substitute the power expansions for $g(z,t)$ and $w(z,t)$ and write the vector differential equation for the vector $b(t) = (b_1(t), b_2(t), \ldots, b_n(t))^{\mathrm{T}}$

$$\frac{db}{dt} = 2 \sum_{s=1}^{n-1} e^{-s(t+iu)} (s+1) A^s b \tag{9}$$

which differs from (5) for $\overline{\Psi}(t)$ only by the absence of the transposition symbol at the matrix $A$. Hence, if the initial data in (9) are correctly chosen, then the coordinates of the vector $b(t)$ are connected with the coordinates of the vector $\overline{\Psi}(t)$ by the relations $b_k(t) = \overline{\Psi}_{n+1-k}(t)$, $k = 1, \dots, n-1$. The choice of the initial data in (9) is connected with the option of function $q(z)$.

The function $q(z)$ is determined by the solved extremal problem. For instance, if we are looking for the maximum $\Re L$ in the class $S(M)$ of a linear functional

$$L(f) = \sum_{k=2}^{n} \bar{\mu}_k a_k$$

defined by the vector $\mu = (\mu_2, \dots, \mu_n)$, then the conjugate vector $\Psi(t)$ satisfies the transversality conditions $\Psi(\log M) = (0, \mu_2, \dots, \mu_n)$. Hence we ought to put in (8)

$$q(z) = zf'(z)\left(\bar{\mu}_n + \bar{\mu}_{n-1} z + \cdots + \bar{\mu}_2 z^{n-2}\right) \tag{10}$$

which provides the carrying-out of the conditions

$$b_k(\log M) = \overline{\Psi}_{n+1-k}(\log M), \quad k = 1, \dots, n-1.$$

Now the initial data in (5) are defined from (8), (10)

$$\bar{\xi}_{n+1-k} = \overline{\Psi}_{n+1-k}(0) = b_k(0) = \sum_{j=1}^{k} j a_j \bar{\mu}_{n-k+j}, \quad k = 1, \dots, n-1. \tag{11}$$

**4.3.** *The two-functional conjecture.* The boundary hypersurface of the coefficient region can be described in different ways, e.g., with the help of support or normal vector. In this connection the so-called two-functional conjecture sounds very attractively.

Let $L$ and $N$ be linear continuous functionals in $S$ which are different from constant and $L \neq cN$ for any $c > 0$. The two-functional conjecture supposes that if a function $f \in S$ maximizes $\Re L$ and $\Re N$, then $f$ is a Koebe function (see [17]).

Only partial results concerning the two-functional conjecture are known. This conjecture was generalized for the classes $S(M)$ and $S_R(M)$ with the more wide set of functions giving maxima for two functionals. Namely, Jakubowski and Majchrzak [27] proved the following theorem.

THEOREM 5. *If a function $f \in S_R(M)$ maximizes $a_n$ and $a_{n+1}$ in the class $S_R(M)$, then $w = f(z)$ satisfies the following equation*

$$\frac{M^2 w}{(\varepsilon M - w)(\bar{\varepsilon} M - w)} = \frac{z}{(\varepsilon - z)(\bar{\varepsilon} - z)}, \quad |\varepsilon| = 1.$$

The extremal functions of Theorem 5 map $E$ onto $E^M$ minus segments on the real axis. Theorem 5 remains true also in the case when $f$ maximizes $a_{p+1}$ and $a_k$ for a prime number $p$ and arbitrary $k$, $2 \leqslant k \leqslant p$ [27].

Starkov [71] generalized Theorem 5 for arbitrary numbers $n$ and $m$.

THEOREM 6. *Let $n \neq m$ and a function $f \in S_R(M)$ give the local extremum for $a_n$ and $a_m$ in the class $S_R(M)$. Then $w = f(z)$ satisfies the following equation*

$$\frac{M^2 w}{[(\varepsilon M^d - w^d)(\bar{\varepsilon} M^d - w^d)]^{1/d}} = \frac{z}{[(\varepsilon - z^d)(\bar{\varepsilon} - z^d)]^{1/d}},$$

$\sqrt[d]{1} = 1$, $|\varepsilon| = 1$, *$d$ is the common divisor of $(n-1)$ and $(m-1)$.*

Let

$$L(f) = \sum_{k=2}^{n} \bar{\mu}_k a_k, \qquad N(f) = \sum_{k=2}^{n} \bar{\nu}_k a_k.$$

The simultaneous maximization of $\Re L$ and $\Re N$ in $S(M)$ characterizes an angular point of the boundary hypersurface $\partial V_n^M$. Suppose

$$\max_{f \in S(M)} \Re L(f) = \Re L(f_0), \qquad \max_{f \in S(M)} \Re N(f) = \Re N(f_0),$$

and a point $A \in \partial V_n^M$ is delivered by $f_0$. Then there exist two support hyperplanes for $\partial V_n^M$ at $A$ with the normal vectors $\mu = (\mu_2, \ldots, \mu_n)$ and $\nu = (\nu_2, \ldots, \nu_n)$. Evidently there is a family of support hyperplanes at $A$ realizing a homotopy between $\mu$ and $\nu$. Analytically it is confirmed by the fact that $f_0$ maximizes also $\lambda \Re L + (1-\lambda)\Re N$ for all $\lambda \in [0, 1]$. Hence $A$ is an angular point of $\partial V_n^M$.

Let us show that the angular character of $A$ remains on the whole trajectory $a(t)$, $0 < t \leqslant \log M$, corresponding to $f_0$ and generated by the Löwner differential equation. Indeed, let $f_0(z) = Mw(z, \log M)$ where $w(z,t)$ is a solution of Equation (2) with a control function $u = u_0(t)$. Denote a solution of (4) with $u = u_0(t)$ by $a_0(t)$. According to (11) the conjugate vector $\overline{\Psi}_0(t)$ can be determined as a solution of (5) with $u = u_0(t)$ and $a = a_0(t)$ in its right-hand side and with the initial data

$$\bar{\xi}_{n+1-k} = \sum_{j=1}^{k} j a_j \big(\lambda \bar{\mu}_{n-k+j} + (1-\lambda)\bar{\nu}_{n-k+j}\big), \quad 0 \leqslant \lambda \leqslant 1, \ k = 1, \ldots, n-1.$$

Since the control $u_0$ is determined and the system (5) is linear, its solution is also a convex linear combination $\overline{\Psi}_0(t) = \lambda \overline{\Psi}_\mu(t) + (1-\lambda)\overline{\Psi}_\nu(t)$ where $\overline{\Psi}_\mu$ and $\overline{\Psi}_\nu$ are solutions of (5) with the initial data

$$\bar{\xi}_{n+1-k} = \sum_{j=1}^{k} j a_j \bar{\mu}_{n-k+j} \quad \text{and} \quad \bar{\xi}_{n-k+j} = \sum_{j=1}^{k} j a_j \bar{\nu}_{n-k+j}, \qquad k = 1, \ldots, n-1,$$

respectively. The Hamilton function is also a convex linear combination of two functions

$$H(t, a, \overline{\Psi}, u) = \lambda H(t, a, \overline{\Psi}_\mu, u) + (1-\lambda) H(t, a, \overline{\Psi}_\nu, u).$$

The optimal control $u_0$ maximizes simultaneously $H(t, a, \overline{\Psi}_\mu, u)$ and $H(t, a, \overline{\Psi}_\nu, u)$.

Thus the angular character of the point $A = a_0(\log M) \in \partial V_n^M$ is preserved on the whole trajectory $a_0(t)$, $0 < t \leqslant \log M$, generated by (4) with $u = u_0(t)$, i.e., $a(t)$ is an angular point of $\partial V_n^{e^t}$, $0 < t \leqslant \log M$.

## 5. Coefficient estimates

The qualitative description of the coefficient region $V_n^M$ does not allow us to obtain concrete coefficient estimates because of enormous technical difficulties. So the problem of estimating of functionals depending on coefficients of bounded univalent functions is the central one in various extremal problems.

Let $P_0^M(z) = z + \sum_{n=2}^{\infty} p_n^M z^n$. Pick [49] estimated the second coefficient for $f \in S(M)$

$$|a_2| \leqslant p_2^M = 2(1 - 1/M), \quad M > 1. \tag{12}$$

The third and the next coefficients are estimated differently on segments of $M$ [64,75, 28]

$$|a_3| \leqslant 1 - 1/M^2, \quad 1 < M \leqslant \mathrm{e}, \tag{13}$$

$$|a_3| \leqslant 1 + 2x^2 - 4x/M + 1/M^2, \quad M \geqslant \mathrm{e}, \tag{14}$$

where $x$ is the maximal positive root of the equation $Mx \log x = -1$.

Sharp estimates of $|a_4|$ in $S(M)$ are more complicated and not known up to now for some values $M \in (1, \infty)$. It is proved [65,78] that

$$|a_4| \leqslant (2/3)\left(1 - 1/M^2\right), \quad 1 < M \leqslant 34/19, \tag{15}$$

$$|a_4| \leqslant p_4^M, \quad M \geqslant 700. \tag{16}$$

Estimates (12), (13) and (15) as well as (14) and (16) made reasonable the two following conjectures.

Charzyński and Tammi supposed that for every $n \geqslant 2$ there existed $M_n^- > 1$ such that

$$|a_n| \leqslant \frac{2}{n-1}\left(1 - \frac{1}{M^{n-1}}\right), \quad 1 < M \leqslant M_n^-, \ f \in S(M),$$

with the equality sign for the functions $\sqrt[n-1]{P_\alpha^{M^{n-1}}(z)}$ which map $E$ onto $E^M$ slit along $(n-1)$ radial segments symmetric under rotation by $2\pi/(n-1)$. This conjecture was proved by Siewierski [68,69] who used the variational method and by Schiffer and Tammi [65] with the help of the Grunsky–Nehari inequalities.

The second conjecture due to Jakubowski concerned even coefficients for large $M$ and supposed that for every even $n$ there existed $M_n^+ > 1$ such that

$$|a_n| \leqslant \left|p_n^M\right|, \quad M \geqslant M_n^+, \ f \in S(M).$$

The Jakubowski conjecture was proved by Prokhorov [55]. The proof was based on the control theory approach and the de Branges result $|a_n| \leqslant n$ for $f \in S$ with the equality sign only for the Koebe functions $K_\alpha$. It was shown that in a neighbourhood of $K_\alpha$ there is a single function $f \in S(M)$ satisfying the necessary extremum conditions. One can easily check that $P_\alpha^M$ satisfies these conditions.

Moreover, the same reasonings led to the statement that the Pick functions $P_\alpha^M$ do not maximize odd coefficients of $f \in S(M)$ for large $M$.

Tammi [78,79] proposed his own approach to the coefficient problem in $S(M)$ and $S_R(M)$ based on the Löwner theory. He obtained a lot of general and concrete results. He described all the extremal functions of the coefficient region $V_3^M$. Besides Tammi found the coefficient region $\{(a_2, a_3, a_4)\colon\ f \in S_R(M)\}$ (see also [30]).

Different functionals (linear, nonlinear, homogeneous) depending on initial coefficients were investigated in $S(M)$, $S_R(M)$ and its subclasses of symmetric functions, see, e.g., [8,70,76,77,80,81,2,56–58,60].

## 6. Subclasses of bounded univalent functions

The main geometrical characteristics of univalent functions such as starlikeness, convexity, close-to-convexity, convexity in the direction of imaginary axis and others could be restricted from $S$ to $S(M)$ but the methods for extremal problems in these subclasses should be essentially modified. For instance, an interesting technique for estimating coefficients of bounded convex functions was developed by Wirths [82]. For bounded close-to-convex functions, Clunie and Pommerenke [14] stated that $a_n = \mathrm{O}(1/n)$, a best possible result. To show the essential differences between investigating geometrical subclasses of $S$ and $S(M)$ we will focus on the estimates of initial coefficients of starlike bounded functions.

Besides there is an interest to the special classes of bounded univalent functions which have no analogy with $S$, e.g., hyperbolically convex functions, bounded nonvanishing functions discussed below, univalent polynomials.

**6.1.** *Estimates of $|a_3|$ for starlike bounded functions.* Let $S^*(M)$ be the set of all starlike functions $f \in S(M)$. The fact that the estimate (12) for $|a_2|$ in $S(M)$ is attained by the Pick functions which are evidently starlike implies that (12) is sharp also in $S^*(M)$. Traditionally the variational methods were important in determining the extremal domains for extremal problems in geometrical subclasses of bounded univalent functions. Tammi attacks [75] on estimating $|a_3|$ in $S(M)$ which led to the inequalities (13)–(14) used Schiffer's variational methods. Similarly to $|a_2|$ the estimate (13) for $|a_3|$ attained by the symmetrized Pick function is sharp in $S^*(M)$ for $1 < M \leqslant \mathrm{e}$. To the contrary, the nonstarlike character of the extremal functions in (14) ensures that the estimates of $|a_3|$ in $S(M)$ and $S^*(M)$ are different for $M > \mathrm{e}$.

Because of difficulties in modifying Schiffer's variational method to allow for preservation of both boundedness and starlikeness at the same time, Barnard and Lewis developed a local variational technique that preserved these properties. They combined the Julia variational formula with the Löwner theory [4,7] (see also the survey [5]). It was shown in [4] that the extremal domain maximizing $|a_3|$ in $S^*(M)$ is the disk $E^M$ minus at most two symmetric radial slits. Denote this domain where $2\theta$ is the angle between the radial slits by $D^M(\theta)$. Let $A_3(M,\theta)$ be the third coefficient for the function $f: E \to D^M(\theta)$. Barnard [4] (see also [47,10]) conjectured that, for all $f \in S^*(M)$,

$$|a_3| \leqslant A_3(M, \pi/2), \quad 1 < M \leqslant 3, \tag{17}$$

$$|a_3| \leqslant A_3(M, 0), \quad 3 \leqslant M < \infty. \tag{18}$$

It follows from Tammi's results that (17) holds for $1 < M \leqslant \mathrm{e}$ and it was shown by Barnard and Lewis [6] that (18) holds for $5 \leqslant M < \infty$. Having the computer software available, Pearce was able to compute $A_3(3,\theta)$ and discovered that $A_3(3,\theta)$, as a function of $\theta$ from 0 to $\pi/2$, took its minimum at the endpoints. Thus the above conjecture was false. The further computations state that there exists a $\theta(M)$, $0 < \theta(M) < \pi/2$, such that, for some $M_0 > 3$,

$$\max\{A_3(M,0), A_3(M,\pi/2)\} < A_3(M,\theta(M))$$

for $2.83 < M < M_0 < 5$.

**6.2.** *Hyperbolically convex functions.* A domain $\Omega \subset E$ is called hyperbolically convex if the hyperbolic segment between any two points of $\Omega$ also belongs to $\Omega$. A conformal map $f: E \to E$ is called hyperbolically convex if $f(E)$ is a hyperbolically convex domain. These functions have been studied, e.g., in [35,36,20,43]. A systematic treatment of such functions has been begun from the article of Ma and Minda [40] (see, e.g., [44,41]).

Since hyperbolically convex functions are invariant under the group of conformal automorphisms of the unit disk, it is always possible to achieve the normalization

$$f(z) = \alpha z + \alpha_2 z^2 + \cdots, \quad 0 < \alpha \leqslant 1.$$

Denote the set of all such functions by $H_\alpha$.

Set

$$D_{h1} f(z) = \frac{(1-|z|^2) f'(z)}{1-|f(z)|^2}$$

and

$$D_{h2} f(z) = \frac{(1-|z|^2)^2 f''(z)}{1-|f(z)|^2} + \frac{2(1-|z|^2)^2 \overline{f(z)} f'(z)^2}{(1-|f(z)|^2)^2} - \frac{2\bar{z}(1-|z|^2) f'(z)}{1-|f(z)|^2}.$$

The next theorem gives criteria for hyperbolic convexity.

THEOREM 7 [40]. *Suppose $f$ is holomorphic and locally univalent in $E$ with $f(E) \subset E$. Then the following are equivalent.*

(i) *$f$ is hyperbolically convex.*

(ii) *For* $z \in E$,

$$\left|\frac{D_{h2} f(z)}{D_{h1} f(z)}\right| < 2.$$

(iii) *For* $z \in E$,

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)} + \frac{2z\overline{f(z)}f'(z)}{1 - |f(z)|^2}\right\} > 0.$$

(iv) *For each* $a \in E$, *the function*

$$F_a(z) = \frac{f(\frac{z+a}{1+\bar{a}z}) - f(a)}{1 - \overline{f(a)}f(\frac{z+a}{1+\bar{a}z})}$$

*is starlike with respect to the origin in* $E$.

(v) *For* $(z, a) \in E \times E$,

$$\Re\left\{\frac{(z-a)(1-\bar{a}z)f'(z)}{(f(z)-f(a))(1-\overline{f(a)}f(z))}\right\} > 0.$$

An important example of a hyperbolically convex function is

$$k_\alpha(z) = \frac{2\alpha z}{1 - z + \sqrt{(1-z)^2 + 4\alpha^2 z}} = \alpha z + \alpha\big(1 - \alpha^2\big)z^2 + \cdots$$

which maps $E$ conformally onto

$$E_\alpha = E \Big\backslash \left\{\left|w + \frac{1}{\alpha}\right| \leqslant \frac{1}{\alpha}\sqrt{1-\alpha^2}\right\}.$$

This function is extremal for many problems, e.g., the covering theorem, estimates of $|f(z)|$, $|a_2|$, $|f'(z)|$ for some values of $|z|$ etc. The following theorem contains the results proved by Ma and Minda [40] (parts (i) and (iii)), Kühnau [35,36] (part (ii)) and Mejia and Pommerenke [44] (part (iv)).

THEOREM 8. *Let* $f \in H_\alpha$. *Then*

(i) $-k_\alpha(-|z|) \leqslant |f(z)| \leqslant k_\alpha(|z|)$, $z \in E$;

(ii) *either* $f(E)$ *contains the closed disk*

$$\left\{w\colon\ |w| \leqslant \frac{\alpha}{1+\sqrt{1-\alpha^2}}\right\}$$

*or else* $f$ *is a rotation of* $k_\alpha$;

(iii) $|\alpha_2| \leqslant \alpha(1-\alpha^2)$;

(iv) $\Re(f(z)/z) > \alpha/2$, $z \in E$.

Among open problems for the class $H_\alpha$ we would mention the sharp upper bounds on $|f^{(n)}(0)|$ for $n \geqslant 3$, on the Schwarzian derivative and on $|f'(z)|$ for all $|z| < 1$. For $\alpha = 1/2$ it should be noted that $k_{1/2}$ cannot be the extremal function for the sharp upper bound on $|f^{(3)}(0)|$ because $k_{1/2}^{(3)}(0) = 0$.

**6.3.** *Bounded nonvanishing functions.* Let $H(E)$ denote the class of functions holomorphic in the disk $E$. Denote

$$B = \{f \in H(E)\colon\ f(z) = c_0 + c_1 z + \cdots,\ 0 < |f(z)| < 1,\ z \in E\},$$

$$B_s = \{f \in B\colon\ f \text{ is univalent in } E\}.$$

Similarly to the class of hyperbolically convex functions, the classes $B$ and $B_s$ are invariant under the group of conformal automorphisms of $E$ both at the complex planes of the variable and the function.

Hummel, Scheinberg and Zalcman [26] began wide investigations of $B$ and $B_s$. They posed some problems and showed the ways for solutions. However, the most active research was directed to $B$ where the coefficient problem as usually became the central one. Krzyż [34] formulated the conjecture that

$$\max_{f \in B} |c_n| = \frac{2}{\mathrm{e}}, \qquad n \geqslant 1,$$

with the extremal function

$$F_n(z) = \exp\left(\frac{z^n + 1}{z^n - 1}\right)$$

and its rotations. Note that $F_n$ is not univalent. The Krzyż conjecture was proved by several authors for $n = 1, 2, 3, 4$.

As for the class $B_s$, the coefficient estimates have the shorter history. Besides the easy sharp estimates

$$|c_0| < 1, \qquad |c_1| \leqslant \frac{4|c_0|(1 - |c_0|)}{1 + |c_0|} \leqslant 12 - 8\sqrt{2},$$

Prokhorov and Szynal [59] used some approaches to obtain the estimate

$$|c_2| \leqslant \frac{8d^*(1 - d^*)(1 - 2d^* - d^{*2})}{(1 + d^*)^3} = 0.45538\ldots,$$

where $d^* = 0.1414\ldots$ is the root of the equation

$$d^4 + 4d^3 + 6d^2 - 8d + 1 = 0.$$

Similarly to estimating of $|c_1|$, the extremal function up to rotation maps $E$ onto $E$ slit along the interval $(0, 1)$. Naturally, in the both cases of the estimates of $|c_1|$ and $|c_2|$ the values of $f(0) = c_0$ for the extremal functions are different.

Later on Ermers [19] modified somehow the proof of Prokhorov and Szynal eliminating from it all the applications to the numerical procedure of finding roots for polynomials of low degrees.

**6.4.** *Univalent polynomials.* The univalent polynomials are dense in the full class $S$ (see, e.g., [17, p. 25]). Suffridge [73] showed that even the subclass of polynomials with highest coefficient $a_n = 1/n$ is dense in $S$. The analogous result is true for polynomials and univalent functions with real coefficients. Note that before the proof of de Branges the Bieberbach conjecture $|a_k| \leqslant k$, $k \geqslant 2$, has been verified by Horowitz [25] for all univalent polynomials of degree up to 27. It is also interesting to observe the connection between starlike, convex, close-to-convex and typically real polynomials and the corresponding univalent functions.

Begin with the polynomials which are the $n$-th partial sums

$$s_n(z) = s_n(z; f) = z + \sum_{k=2}^{n} a_k z^k$$

of the power series (1) of $f \in S$. It is a consequence of Rouché's theorem that for $f \in S$, the radius of univalence of $s_n(z; f)$ tends to 1 as $n \to \infty$. A remarkable theorem of Szegö [74] establishes a uniform univalence radius of $s_n$.

THEOREM 9 [74]. *Every partial sum $s_n(z; f)$ of $f \in S$ is univalent in the disk $E^{1/4}$. The radius $1/4$ is best possible.*

The largest radius of univalence $\rho_n$ for $s_n$ is unknown. Jenkins [29] observed that his modification of Szegö's argument shows that

$$\rho_n \geqslant 1 - \frac{4+\varepsilon}{n} \log n$$

for each $\varepsilon > 0$ and for all large $n$. This improves an earlier result of Levin [38].

If $f$ is convex, starlike or close-to-convex, then all $s_n$ are also convex, starlike or close-to-convex in $E^{1/4}$, see [74] for the convexity and starlikeness properties. All these assertions follow from the general convolution theorem due to Ruscheweyh and Sheil-Small [63]. Robertson [61] proved that for an arbitrary starlike function $f$, $s_n(z; f)$ is starlike in the disk of radius $1 - 4n^{-1} \log n$, and for the Koebe function $K_\alpha$, $s_n(z; K_\alpha)$ is starlike in the disk of radius $1 - 3n^{-1} \log n$. According to the general theorems on convolution 4 here can be replaced by 3 for all starlike functions. The corresponding result holds for the classes of convex and close-to-convex functions in the same disk of radius $1 - 3n^{-1} \log n$.

It is interesting to notice that the case $n = 3$ in Theorem 9 is far from triviality. This circumstance confirms that the coefficient problem for univalent polynomials

$$p_n(z) = z + \sum_{k=2}^{n} a_k z^k$$

is not easy already for $n \geqslant 3$.

The question about the univalence property for polynomials $p_n$ has not been answered in full generality although some partial results are available. For $n = 2$ the problem is trivial: $p_2(z) = z + a_2 z^2$ is univalent in $E$ iff $|a_2| \leqslant 1/2$, and in this case $p_2$ is locally univalent. For $n > 2$ the necessary univalence condition $|a_n| \leqslant 1/n$ is far from sufficient. The full description of the coefficient set

$$W_n = \{(a_2, \ldots, a_n)\colon\ p_n \text{ is univalent in } E\}$$

for $n = 3$ was given by Kössler [31], by Cowling and Royster [16] and by Brannan [9]. The extremal univalent polynomial $p_3$,

$$p_3(z) = z + \frac{2\sqrt{2}}{3} z^2 + \frac{1}{3} z^3,$$

simultaneously maximizes $|a_2|$ and $|a_3|$.

Suffridge [72] showed that if $p_n$ has real coefficients and $a_n = 1/n$, then

$$|a_k| \leqslant \frac{n-k+1}{n} \frac{\sin \frac{k\pi}{n+1}}{\sin \frac{\pi}{n+1}}, \qquad 2 \leqslant k \leqslant n.$$

The inequality is sharp. In particular, for $n = 4$,

$$|a_2| \leqslant A = \frac{3(1+\sqrt{5})}{8} = 1.21352\ldots, \qquad |a_4| \leqslant \frac{2}{3} A = 0.80901\ldots.$$

Michel [45] proved that in the case $n = 4$ Suffridge's estimates hold for all polynomials with real coefficients, without the assumption that $a_4 = 1/4$. He found also that for polynomials $p_4$ with complex coefficients the sharp bound for $|a_3|$ is slightly larger,

$$|a_3| \leqslant B = \frac{\sqrt{3\sqrt{15} - 9}}{2} = 0.80915\ldots,$$

and similarly the sharp bound for $|a_2|$ is larger than $A$. Suffridge [72] showed that for polynomials $p_n$, $n > 5$, with complex coefficients all the sharp bounds for $|a_k|$ are larger than those for polynomials with real coefficients.

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CHAPTER 9

# The *-Function in Complex Analysis

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## *Contents*



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HANDBOOK OF COMPLEX ANALYSIS: GEOMETRIC FUNCTION THEORY, VOLUME 1
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## 1. Introduction

Let

$$A(R_1, R_2) = \{z \in \mathbb{C}\colon\ R_1 < |z| < R_2\}, \quad 0 \leqslant R_1 < R_2 \leqslant \infty,$$

denote an annulus, and let $u : A(R_1, R_2) \to \mathbb{R}$ be a Lebesgue measurable function such that $u(re^{i\theta}) \in L^1[-\pi, \pi]$ for each $r \in (R_1, R_2)$. Write

$$A^+(R_1, R_2) = \{z \in A(R_1, R_2)\colon\ \operatorname{Im} z > 0\},$$
$$A^{++}(R_1, R_2) = \{z \in A(R_1, R_2)\colon\ \operatorname{Im} z \geqslant 0\},$$

and define a new function $u^* : A^{++}(R_1, R_2) \to \mathbb{R}$ by

$$u^*(re^{i\theta}) = \sup_E \int_E u(re^{it})\,dt, \quad 0 \leqslant \theta \leqslant \pi,\ R_1 < r < R_2, \tag{1.1}$$

where the supremum is taken over all Lebesgue measurable sets $E \subset [-\pi, \pi]$ with Lebesgue measure $|E| = 2\theta$.

$u^*$ is called the *-function of $u$. This construction was introduced by the author in the 1970s. It has played a role in the solution of some extremal problems for univalent functions, meromorphic functions, and other subjects within complex function and potential theories. Versions of the *-function exist for functions defined on domains in Euclidean spaces, on spheres, and in hyperbolic spaces of arbitrary dimension, where the theory of the *-function comprises one aspect of a more general theory called symmetrization. Section 7 contains a short discussion of the general setting; a detailed account is in [15].

This article is meant to be a brief introduction to the theory and application of the *-function in the plane. Via exact statements of results and semi-detailed proofs of some of them, I hope to convey some idea of what the *-function can do and how it does it. The bibliography is intended to be representative rather than exhaustive.

There is large overlap between this article and Chapter 9 of Hayman's book [81], to which we will often refer. The proofs there are sometimes like the ones sketched here, sometimes not. Duren's book [40] also contains a chapter on the *-function which I found very helpful for the preparation of this article.

Here, in very general form, is a preview of the argument used to solve extremal problems later in this paper. We are given an extremal problem in the plane in which the competing functions $u$ are subharmonic or are the difference of two subharmonic functions. The expected extremal function $v$ is harmonic, at least in a certain region, and is a symmetric decreasing function on circles $|z| = r$. To prove that $v$ really does beat $u$, one shows that the desired conclusion would be a consequence of $u^* \leqslant V$, where $V(re^{i\theta}) = \int_{-\theta}^{\theta} v(re^{it})\,dt$. The "subharmonicity properties" of the *-function, to be stated in Section 3, imply that $u^* - V$ is subharmonic in a relevant domain, so that $u^* \leqslant V$ on the boundary implies $u^* \leqslant V$ inside. To finish, one must prove $u^* \leqslant V$ on the boundary and possibly elsewhere in the complement of the domain.

The idea to form the *-function, and to use an argument like the one in the preceding paragraph, occurred in 1971 when I was contemplating a conjecture of Edrei in Nevanlinna theory, about deficiencies of meromorphic functions, called the "spread conjecture". This conjecture and its proof (after it was proved it became known as the spread relation) is the subject of Section 10. Here I'll just note the main sources of the idea. A general source was the notion that to solve extremal problems about growth of entire functions

$$f(z)=\prod_{n=1}^{\infty}(1-z/a_n)$$

with $a_n \in \mathbb{C}$, $\sum_{n=1}^{\infty}|a_n|^{-1}<\infty$, it is sometimes helpful to pass to the entire function $F$ obtained by sweeping the zeros of $f$ to a ray, the negative axis, say:

$$F(z)=\prod_{n=1}^{\infty}(1+z/|a_n|).$$

We have then, for instance,

$$|F(-r)| \leqslant \min_{|z|=r}|f(z)| \leqslant \max_{|z|=r}|f(z)| \leqslant F(r).$$

See [23, Chapter 3] for applications of these inequalities.

More particular evidence for thinking the *-function might possess subharmonicity properties came from an argument of Edrei and Fuchs [46, p. 239] in which they use the fact that if $f$ is meromorphic in the plane with negative zeros and positive poles, then

$$U(re^{i\theta})=\int_{-\theta}^{\theta}\log|f(re^{it})|\,dt+2\pi N(r,\infty,f)$$

is harmonic in the upper half plane, where $N$ is Nevanlinna's weighted counting function.

*A word on Notation.* The meaning of the superscripts $*$, #, and ˜ in works involving symmetrization and rearrangement of functions is, sad to say, not standard. In this paper we follow [8], so that $u^*(re^{i\theta})$ is defined by (1.1), $\tilde{u}(re^{i\theta})$, to be defined in Section 2, denotes the symmetric decreasing rearrangement of $u$ on the circle $|z|=r$, and $u^{\#}(re^{i\theta})$, to be defined in Section 3, equals $u^*(re^{i\theta})$ plus a certain mean value which depends on $r$. Many authors, for example [95], use $u^*$ to denote symmetric decreasing rearrangement in $\mathbb{R}^n$ or Steiner symmetrization, while for others, e.g., [20] and [15], $u^*$ denotes the decreasing rearrangement of $u$, which is defined on $[0,\infty)$ or a subinterval. In [15], symmetric decreasing rearrangements, Steiner symmetrizations and cap symmetrizations are denoted by $u^{\#}$, while functions like (1.1) are denoted by $u^{I}$.

## 2. General properties of the *-function

In this section, we record some properties of $u^*(re^{i\theta})$ as a function of $\theta$ which will be used in the sequel. For $0 < a < \infty$, let $g \in L^1([-a,a],\mathbb{R})$. Define $g^* : [0,a] \to \mathbb{R}$ by

$$g^*(\theta) = \sup_E \int_E g(s)\,ds,$$

where the supremum is over all $E \subset [-a,a]$ with $|E| = 2\theta$. Proofs of most of the assertions below can be found in [8], [40], or [81].

PROPOSITION 2.1. *For each $\theta \in [0,a]$, there exists $E \subset [-a,a]$ such that $g^*(\theta) = \int_E g(s)\,ds$.*

The function $\lambda(t) = \lambda_g(t)$ defined by

$$\lambda(t) = \left|\left\{s \in [-a,a]\colon\ g(s) > t\right\}\right|, \qquad t \in \mathbb{R},$$

is called the *distribution function* of $g$. Functions $g$ and $h : [-a,a]$ are said to be equimeasurable, or to be rearrangements of each other, if $\lambda_g(t) = \lambda_h(t)$ for every $t \in \mathbb{R}$. Define $\tilde{g} : [-a,a] :\to \mathbb{R}$ by

$$\tilde{g}(s) = \inf\left\{t \in \mathbb{R}\colon\ \lambda_g(t) \leqslant 2s\right\}, \qquad s \in [0,a],$$

and $\tilde{g}(s) = \tilde{g}(-s)$ for $s \in [-a,0]$. It turns out that $\tilde{g}$ is a rearrangement of $g$. Moreover, $\tilde{g}(s) \searrow$ as $s \nearrow$ on $[0,a]$, and $\tilde{g}(s) = \tilde{g}(-s)$ for $s \in [-a,a]$. Accordingly, $\tilde{g}$ is called the symmetric decreasing rearrangement of $g$. If $g$ is continuous on $[-a,a]$ and the level sets $\{s \in [-a,a]\colon\ g(s) = t\}$ have measure zero for every $t \in \mathbb{R}$, then $\lambda_g(t)$ is continuous and strictly decreasing for $t \in [\operatorname{ess\,inf}_{[-a,a]} g, \operatorname{ess\,sup}_{[-a,a]} g]$, and for $s \in [0,a]$, $\tilde{g}(s)$ equals the inverse function of $\lambda$ evaluated at $2s$.

Our definitions of distribution function and symmetric decreasing rearrangement differ from that of many authors, for whom $g$ is replaced by $|g|$ in the definitions of $\lambda_g$ and $\tilde{g}$. For our purposes, it is better to consider the rearrangement of the full function $g$ rather than that of its absolute value.

PROPOSITION 2.2. $g^*(\theta) = \int_{-\theta}^{\theta} \tilde{g}(s)\,ds$, $\theta \in [0,a]$.

PROPOSITION 2.3.
(a) $g^*(0) = 0$, $g^*(a) = \int_{-a}^{a} g(s)\,ds$.
(b) $\frac{\partial g^*}{\partial \theta}(0) = 2\operatorname{ess\,sup}_{[-a,a]} g$.
(c) $\frac{\partial g^*}{\partial \theta}(a) = 2\operatorname{ess\,inf}_{[-a,a]} g$.

The following comparison principle is due essentially to Hardy, Littlewood and Pólya [78, p. 170, Misc. Theorems 249, 250]. See also [20, p. 88], as well as [8], [40], or [81].

PROPOSITION 2.4. *For $g, h \in L^1[-a,a]$, the following are equivalent:*
(a) *For every convex increasing function $\Phi : \mathbb{R} \to \mathbb{R}$ holds*

$$\int_{-a}^{a} \Phi\big(g(s)\big)\,ds \leqslant \int_{-a}^{a} \Phi\big(h(s)\big)\,ds.$$

(b) $\int_{-a}^{a}(g(s)-t)^+\,ds \leqslant \int_{-a}^{a}(h(s)-t)^+\,ds$, *for every $t \in \mathbb{R}$.*
(c) $g^*(s) \leqslant h^*(s)$, *for every $s \in [0,a]$.*
*Moreover, if $\int_{-a}^{a} g(s)\,ds = \int_{-a}^{a} h(s)\,ds$, then* (a), (b), *and* (c) *are equivalent to*
(d) (2.1) *holds for every convex $\Phi : \mathbb{R} \to \mathbb{R}$.*

Since $\Phi$ convex $\Rightarrow \Phi(x) \geqslant L(x)$ for some linear function $L$, and $g \in L^1[-a,a]$ with $a < \infty$, the integrals in (2.1) exist, perhaps as $+\infty$.

PROPOSITION 2.5. *If $g, h \in L^1[-a,a]$ and if $g^*(s) \leqslant h^*(s)$ for every $s \in [0,a]$, then* $\operatorname{ess\,sup}_{[-a,a]} g \leqslant \operatorname{ess\,sup}_{[-a,a]} h$.

Proposition 2.5 follows from (c) $\Rightarrow$ (b) of Proposition 2.4, or from Proposition 2.3(a) and (b), or by other means.

## 3. Subharmonicity properties of the *-function

We return now to functions $u : A(R_1, R_2) \to \mathbb{R}$ for which $u(re^{i\theta}) \in L^1([-\pi,\pi])$ for each $r \in I$. For such $u$, define a new function $Ju : A^{++}(R_1,R_2) \to \mathbb{R}$ by

$$Ju\big(re^{i\theta}\big) = \int_{-\theta}^{\theta} u\big(re^{it}\big)\,dt, \quad re^{i\theta} \in A^{++}(R_1,R_2). \tag{3.1}$$

Proposition 2.2 implies the relation $u^* = J\tilde{u}$, where $\tilde{u}(re^{i\theta})$ is the function obtained by passing from $u(re^{i\theta})$ to its symmetric decreasing rearrangement in $\theta$ for each fixed $r$.

The following commutation relation, in which $\Delta$ denotes the Laplace operator in the plane, is one source of the subharmonicity properties of the *-function. In the rest of this section, we'll write $A = A(R_1,R_2)$, $A^+ = A^+(R_1,R_2)$, $A^{++} = A^{++}(R_1,R_2)$.

PROPOSITION 3.1. *Let $u \in C^2(A)$. Then $\Delta Ju = J\Delta u$ on $A^+$.*

To prove this, one writes $\Delta = \partial_{rr} + r^{-1}\partial_r + r^{-2}u_{\theta\theta} \equiv \Delta_r + r^{-2}u_{\theta\theta}$. Then

$$\begin{aligned}
\Delta_r(Ju) &= \int_{-\theta}^{\theta} \Delta_r u\big(re^{it}\big)\,dt = \int_{-\theta}^{\theta} \big[\Delta u\big(re^{it}\big) - r^{-2}u_{tt}\big(re^{it}\big)\big]\,dt \\
&= J(\Delta u)\big(re^{i\theta}\big) - \big(\partial_\theta u(re^{i\theta}) - \partial_\theta u\big(re^{-i\theta}\big)\big)r^{-2} \\
&= J(\Delta u)\big(re^{i\theta}\big) - r^{-2}\partial_{\theta\theta}(Ju)\big(re^{i\theta}\big).
\end{aligned}$$

From Proposition 3.1 follows

PROPOSITION 3.2. *If $h$ is harmonic in $A$, then $Jh$ is harmonic in $A^+$.*

A function $u$ in a plane domain $\Omega$ is said to be $\delta$-subharmonic if it has a representation $u = u_1 - u_2$, where $u_1$ and $u_2$ are subharmonic in $\Omega$. For the theory of $\delta$-subharmonic functions, see [83] and [81]. We shall often make use of the following proposition. For continuous subharmonic functions it is proved in [40]; the case of general $\delta$-subharmonic functions requires just a few technical adjustments.

PROPOSITION 3.3. *If $u$ is $\delta$-subharmonic in $A$, then $u^*$ is continuous on $A^{++}$.*

Here now are three subharmonicity-like properties of the *-function.

THEOREM 3.1. *If $u$ is subharmonic in $A$, then $u^*$ is subharmonic in $A^+$.*

THEOREM 3.2. *If $u = u_1 - u_2$, with $u_1, u_2$ subharmonic in $A$, then*

$$u^{\#}(re^{i\theta}) \equiv u^*(re^{i\theta}) + \int_{-\pi}^{\pi} u_2(re^{it})\,dt \tag{3.2}$$

*is subharmonic in $A^+$.*

If $u$ is subharmonic in a plane domain $\Omega$, then $\Delta u$, taken in the sense of distributions, is a non-negative regular Borel measure on $\Omega$. Following the normalization of [83], $\frac{1}{2\pi}\Delta u$ will be called the Riesz measure of $u$. If $u$ is $\delta$-subharmonic in $\Omega$, then the distribution $\Delta u$ is a locally finite signed measure on $\Omega$. The measure $\Delta u$ is one kind of generalized Laplacian. In addition, we'll consider another generalized Laplacian $\Delta_1 u$, which is the function defined at each point of $\Omega$ by

$$\Delta_1 u(z) = \liminf_{\delta\to 0} \frac{4}{\delta^2}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} u(z+\delta e^{i\theta})\,d\theta - u(z)\right). \tag{3.3}$$

THEOREM 3.3. *Let $u \in C^2(A)$ satisfy $\Delta u \geqslant -f - \psi(u)$, where $f \in C(A)$ and $\psi \in C(\mathbb{R})$. Then $u^*$ is $\delta$-subharmonic in $A^+$, and*

$$\Delta(u^*) \geqslant -f^* - J\psi(\tilde{u}), \quad \textit{as measures in } A^+, \tag{3.4}$$

$$\Delta_1(u^*)(z) \geqslant -f^*(z) - J\psi(\tilde{u})(z), \quad z \in A^+. \tag{3.5}$$

The function on the right hand side of (3.4), call it $g$, is identified with the locally finite signed measure $g\,dx\,dy$. A variant of Theorem 3.3, called Theorem 3.4, will be stated in Section 12.

Theorem 3.1 is contained in Theorem 3.2, which can in turn be easily derived from Theorem 3.3. We have stated these results in increasing order of complexity for ease of digestion. Theorem 3.2 appears already in the first paper [7] involving the *-function. More complicated results like Theorem 3.3 are needed to prove comparison theorems for pde's, such as Weitsman's theorem [120] about symmetrization and the hyperbolic metric. See Section 12.

To obtain Theorem 3.2 from Theorem 3.3, we may assume that each $u_i$ is in $C^2(A)$. Let $f = \Delta u_2$. Then $\Delta u \geqslant -f$. Set

$$p(re^{i\theta}) = \int_{-\pi}^{\pi} u_2(re^{it})\,dt.$$

By (3.4) with $\psi = 0$,

$$\Delta u^{\#} = \Delta(u^* + p) \geqslant -f^* + \Delta p.$$

But

$$\Delta p(re^{i\theta}) = \Delta_r p(re^{i\theta}) = \int_{-\pi}^{\pi} \Delta_r u_2(re^{it})\,dt = \int_{-\pi}^{\pi} \Delta u_2(re^{it})\,dt$$
$$= \int_{-\pi}^{\pi} f(re^{it})\,dt \geqslant f^*(re^{i\theta}),$$

since $f \geqslant 0$. Thus, $\Delta u^{\#} \geqslant 0$, as asserted by Theorem 3.2.

We will sketch two proofs of Theorem 3.1. Method 1 [7,36], the original method, is more direct. Method 2 permits wider generalization. Proofs employing various other devices are in [8,81,18,106,54].

METHOD 1. Let $u$ be subharmonic in $A$. For $n \geqslant 1$, define $U_n$ on $A^{++}$ like $u^*$ was defined in (1.1), but take the sup only over sets $E$ of measure $2\theta$ which can be represented as unions of at most $n$ closed arcs on the unit circle. Then $U_n \nearrow u^*$ pointwise on $A^{++}$. Since $u^*$ is continuous on $A^{++}$, to prove that $u^*$ is subharmonic in $A^+$, it suffices to prove that each $U_n$ is subharmonic in $A^+$. Continuity of $U_n$ is easily established, so it suffices to prove that $U_n$ satisfies the sub-mean value property.

Fix $n \geqslant 1$, $z = re^{i\theta} \in A^+$, and $0 < \rho < r$. For $t \in [-\pi, \pi]$, define $r(t) > 0$ and $\alpha(t) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ by $r + \rho e^{it} = r(t)e^{i\alpha(t)}$. Define also $I(\theta, \phi)$ to be the arc on the unit circle with center $e^{i\phi}$, length $2\theta$, and set

$$v(r, \theta, \phi) = \int_{I(\theta,\phi)} u(re^{it})\,dt.$$

Let $E = \bigcup_{j=1}^{m} I(\theta_j, \phi_j)$ be a set for which the supremum in the definition of $U_n(re^{i\theta})$ is attained, where $1 \leqslant m \leqslant n$, $\sum_{j=1}^{m} \theta_j = \theta$, and the $I(\theta_j, \phi_j)$ are disjoint. For $t \in [-\pi, \pi]$, let

$$E(t) = I(\theta_1 + \alpha(t), \phi) \cup \left[\bigcup_{j=2}^{m} I(\theta_j, \phi_j + \alpha(t))\right],$$

where $\rho$ is assumed to be small enough so that the arcs are disjoint. Then $|E(t)| = 2\theta + 2\alpha(t)$, so

$$U_n\big(re^{i\theta} + \rho e^{i(\theta+t)}\big) = U_n\big(r(t)e^{i(\theta+\alpha(t))}\big) \geqslant \int_{E(t)} u\big(r(t)e^{is}\big)\,ds$$
$$= v\big(r(t), \theta_1 + \alpha(t), \phi_1\big) + \sum_{j=2}^{m} v\big(r(t), \theta_j, \phi_j + \alpha(t)\big).$$

It turns out that

$$\int_{-\pi}^{\pi} v\big(r(t), \theta_1 + \alpha(t), \phi_1\big)\,dt = \int_{-\pi}^{\pi} v\big(r(t), \theta_1, \phi_1 + \alpha(t)\big)\,dt.$$

Moreover, the subharmonicity of $u$ implies $v(r, \theta, \phi)$ is a subharmonic function of $re^{i\phi}$, so that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} v\big(r(t), \theta_j, \phi_j + \alpha(t)\big)\,dt \geqslant v(r, \theta_j, \phi_j), \quad 1 \leqslant j \leqslant m.$$

Together, these relations yield

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} U_n\big(re^{i\theta} + \rho e^{i(\theta+t)}\big)\,dt \geqslant \sum_{j=1}^{m} v(r, \theta_j, \phi_j) = \int_E u\big(re^{is}\big)\,ds = U_n\big(re^{i\theta}\big).$$

The mean value property of $U_n$ is established. □

METHOD 2. The main ingredients are a rearrangement inequality for convolutions on the circle, and a version for circular symmetrization in annuli of a theorem of J. Ryff [109,20]. Ryff's Theorem asserts that measurable functions $f$ on $[0, 1]$ can be factored as $f = f_d \circ T$, where $f_d$ is the decreasing rearrangement of $f$, and $T$ is a Lebesgue measure-preserving transformation of $[0, 1]$ onto itself.

CONVOLUTION INEQUALITY. *Let $f, g, h$ be real functions on the unit circle, two of which are in $L^1$, and the third in $L^\infty$. Then, with $\tilde{}$ denoting symmetric decreasing rearrangement,*

$$\int_{-\pi}^{\pi} h\big(e^{-i\theta}\big)\,d\theta \int_{-\pi}^{\pi} f\big(e^{it}\big)g\big(e^{i(\theta-t)}\big)\,dt$$
$$\leqslant \int_{-\pi}^{\pi} \tilde{h}\big(e^{-i\theta}\big)\,d\theta \int_{-\pi}^{\pi} \tilde{f}\big(e^{it}\big)\tilde{g}\big(e^{i(\theta-t)}\big)\,dt. \tag{3.7}$$

The analogous result in $\mathbb{R}$ is due to F. Riesz in 1930, see [78, p. 279]. Riesz's theorem was extended to $\mathbb{R}^n$ by Sobolev in 1938, see, e.g., [95, p. 79]. For the circle, (3.7) was proved in [13]. In the special case when one of the three functions is already symmetric decreasing, which suffices for application to proofs of subharmonicity of the *-function, (3.7) was proved in [18] along with an analogue for higher dimensional spheres. The special case on the circle was independently proved by Friedberg and Luttinger [63]. It was stated and applied to a problem about logarithmic capacity in [3, p. 34]. The proof in [3] seems incomplete, but it provided the main ideas for the proof in [18].

Consider now a subharmonic function $u$ in an annulus $A$. Let $G$ be a nonnegative compactly supported $C^\infty$ function in $A^+$. Define $g : A \to [0, \infty)$ by

$$g(re^{i\theta}) = \int_{|\theta|}^{\pi} G(re^{it})\, dt.$$

Then $g \in C^\infty(A)$, $g$ is compactly supported in $A$, and $g$ is symmetric decreasing on circles. Writing $u^* = J\tilde{u}$ and reversing the integration, one finds that

$$\int_{A^+} u^* \Delta G = \int_A \tilde{u} \Delta g. \tag{3.8}$$

In (3.8), and in the rest of this section, all integrals over plane regions are with respect to two-dimensional Lebesgue measure. Let $K$ be a nonnegative compactly supported $C^\infty$ symmetric decreasing function in $\mathbb{C}$, i.e., $K(z) = K_1(|z|)$ for some decreasing function $K_1 : [0, \infty) \to [0, \infty)$, with $\int_{\mathbb{C}} K = 1$. Write $K_\delta(z) = \delta^{-2} K(z\delta^{-1})$, and let $*$ denote convolution in $\mathbb{C}$. Using the Taylor approximation, one sees that

$$\Delta g = \lim_{\delta \to 0} c\delta^{-2}(K_\delta * g - g), \tag{3.9}$$

with uniform convergence in $A$, where $c = \frac{1}{4} \int_{\mathbb{C}} |z|^2 K(z)$. From (3.8) and (3.9), it follows that

$$\int_{A^+} u^* \Delta G = \lim_{\delta \to 0} c\delta^{-2} \int_A \tilde{u}(K_\delta * g - g). \tag{3.10}$$

By the Ryff-type theorem, there exists a measurable mapping $T : A \to A$ such that $T$ maps each circle $\{|z| = r\} \subset A$ onto itself, and the restriction of $T$ to each circle is one-dimensional Lebesgue measure-preserving. Define $h = g \circ T$. Then $\tilde{h} = g$. Moreover, $h$ has compact support in $A$. In the argument to follow, we assume that $\delta$ is small enough so that the integrands will all have compact support in $A$.

The convolution inequality on the circle implies [15, Cor. 5, p. 62]

$$\int_A \tilde{u} K_\delta * g \geqslant \int_A u K_\delta * h,$$

where the $*$ still denotes convolution on $\mathbb{C}$.

Since $uh = (\tilde{u}g) \circ T$, we have also $\int_A \tilde{u}g = \int_A uh$. Thus,

$$\int_A \tilde{u}(K_\delta * g - g) \geqslant \int_A u(K_\delta * h - h) = \int_A h(K_\delta * u - u). \tag{3.11}$$

The subharmonicity of $u$ implies that $K_\delta * u \geqslant u$ on the support of $h$. Since $h \geqslant 0$, it follows from (3.11) and (3.10) that $\int_{A^+} u^* \Delta G \geqslant 0$. Since $G$ is an arbitrary nonnegative smooth compactly supported function in $A^+$, it follows (see, e.g., [30]), that $u^*$ is subharmonic in $A^+$. □

The argument in Method 2 is mostly condensed from the proof of Theorem 5 in [15]. In [15] though, by means of an approximation step, one needs the Ryff-type result only for $u$ satisfying extra conditions, in which case its existence is immediate.

## 4. Nevanlinna's $N$ function

Let $f$ be holomorphic and nonconstant in a disk $\mathbb{D}(R) \equiv \{z \in \mathbb{C}: |z| < R\}$, $0 < R \leqslant \infty$. For $w \in \mathbb{C}$ and $0 < r \leqslant R$, define

$$N(r, w, f) = \sum_{z \in f^{-1}(w)} \log^+ \frac{r}{|z|} = \int_0^r \frac{n(t, w, f)}{t}\, dt, \tag{4.1}$$

where the sum is taken over all solutions $z$ of $f(z) = w$, counting multiplicity, and $n(t, w, f)$ is the number of solutions of $f(z) = w$ in $\mathbb{D}(t)$, counting multiplicity. Note that according to our definition $N(r, w, f) = \infty$ for $w = f(0)$. In Section 10 we shall adopt a different convention for $w = f(0)$.

The weighted and unweighted counting functions $N$ and $n$ are central players in Rolf Nevanlinna's value distribution theory of meromorphic functions. As will be seen in the following sections, applications of *- functions sometimes rely on considerations involving $N$. In particular, we will use the following "change of variable" formula. Recall that the Riesz measure of a subharmonic function was introduced just before the statement of (3.3).

PROPOSITION 4.1. *Let $f$ be holomorphic in $\mathbb{D}(R)$, and let $u$ be subharmonic in $\Omega \equiv f(\mathbb{D}(R))$, with Riesz measure $\mu$. If $u(f(0))$ is finite, then, for* $0 < r < R$,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} u\big(f(re^{i\theta})\big)\, d\theta = \int_\Omega N(r, w, f)\, d\mu(w) + u\big(f(0)\big). \tag{4.2}$$

Special cases of this formula are folklore. For example, the case $u(w) = |w|^p$, $p > 0$, $d\mu(w) = \frac{1}{2\pi} p^2 |w|^{p-2}\, d\mu(w)$, was used in [11]. A general version is stated and proved by C.S. Stanton in his doctoral dissertation (University of Wisconsin–Madison, 1982). Proofs, with applications in sundry directions, are in [60,110,14].

A proof of the formula, minus technicalities, goes as follows. Decompose $u$ as

$$u(w) = \int_{\Omega_1} \log|w - \zeta|\, d\mu(\zeta) + h(w), \tag{4.3}$$

where $\Omega_1$ is a domain with $\overline{f(\mathbb{D}(r))} \subset \Omega_1 \subset \overline{\Omega_1} \subset \Omega$, and $h$ is harmonic in $\Omega_1$. In (4.3), set $w = f(re^{i\theta})$ and integrate w.r.t $d\theta/(2\pi)$. By Jensen's formula, Fubini's theorem and the mean value theorem for harmonic functions, the right hand side becomes $\int_{\Omega_1}((N(r,\zeta,f)+\log|f(0)-\zeta|)\,d\mu(\zeta)+h(f(0))$, which equals the right hand side of (4.2).

Proposition 4.1 is still true, under reasonable hypotheses, if $f$ is meromorphic and $u$ is $\delta$-subharmonic on $f(\mathbb{D}(R))$. Complete statements and proofs are in [60] and [14].

We discuss here a special case of (4.2). For a domain $\Omega \subset \mathbb{C}$ and $a \in \Omega$, the Green function of $\Omega$ with pole at $a$ will be denoted by $g(\cdot,a,\Omega)$. If $z \in \mathbb{C} \setminus \Omega$, we define $g(z,a,\Omega) = 0$. Thus, our Green functions $g(z,a,\Omega)$ will always be defined for all $z \in \mathbb{C}$. If $f$ is univalent in $\mathbb{D}(R)$, then $g(w, f(0), f(\mathbb{D}(R))) = \log\frac{R}{|f^{-1}(w)|}$, and $[g(z, f(0), f(\mathbb{D}(R)) + \log\frac{r}{R}]^+$ is the Green function of $f(\mathbb{D}(r))$ with pole at $f(0)$, from which follows

$$N(r,w,f) = \left[g\big(w, f(0), f\big(\mathbb{D}(R)\big)\big) + \log\frac{r}{R}\right]^+ \quad 0 < r \leqslant R,\ w \in \mathbb{C}. \tag{4.4}$$

Let us return now to holomorphic not necessarily univalent $f$, but specialize $u$ to be a radial function in the plane. Thus, $u(w)$ depends only on $|w|$. For $s \in \mathbb{R}$, define $\Phi(s) = u(e^s)$. If $u \in C^2(\mathbb{C})$, then

$$\Delta u(w) = |w|^{-2}\Phi''(\log|w|).$$

Proposition 4.1 implies

PROPOSITION 4.2. *Let $f$ be holomorphic in $\mathbb{D}(R)$, and $\Phi : \mathbb{R} \to \mathbb{R}$ be convex and $C^2$. Then, for* $0 < r < R$,

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi\big(\log\big|f\big(re^{i\theta}\big)\big|\big)\,d\theta\\ &\quad = \frac{1}{2\pi}\int_{\mathbb{C}}|w|^{-2}\Phi''\big(\log|w|\big)N(r,w,f)|dw|^2 + \Phi\big(\log|f(0)|\big).\end{aligned} \tag{4.5}$$

In general, a radial function $u$ is subharmonic if and only if $\Phi$ is convex, in which case the Riesz measure of $u$ is rotationally symmetric, but might have a singular component. (4.5) remains true if the right hand side is appropriately interpreted. For us, an important example is $u(w) = \log^+\frac{|w|}{t}$, $t > 0$, for which $\Phi(s) = (s - \log t)^+$ and the Riesz measure is Lebesgue probability measure on $|w| = t$. Formula (4.5) becomes *Cartan's Formula*:

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\pi}^{\pi}\log^+\frac{|f(re^{i\theta})|}{t}\,d\theta\\ &\quad = \frac{1}{2\pi}\int_{-\pi}^{\pi}N\big(r, te^{i\phi}, f\big)\,d\phi + \log^+\frac{|f(0)|}{t}, \quad 0 < s < \infty.\end{aligned} \tag{4.6}$$

For discussion of Cartan's formula, see, for example, [85], [79], or [97].

## 5. Integral means of univalent functions

Let $S$ denote the class of all analytic univalent functions $f:\mathbb{D}\to\mathbb{C}$ with $f(0)=0$, $f'(0)=1$. The Koebe function $k$, defined by

$$k(z)=\frac{z}{(1-z)^2}=\sum_{n=1}^{\infty} nz^n,$$

conformally maps $\mathbb{D}$ onto the slit plane $\mathbb{C}\setminus\{x\in\mathbb{R}:\ x\leqslant -\frac{1}{4}\}$. Along with its rotations, $k$ is known to be extremal for many problems involving $S$. In [8], *-functions were used to prove that $k$ is extremal for a large class of problems about integral means. A function $\Phi:\mathbb{R}\to\mathbb{R}$ is said to be strictly convex if it is convex and is not linear on any nondegenerate subinterval.

THEOREM 5.1. *Let $\Phi:\mathbb{R}\to\mathbb{R}$ be convex. Then for $f\in S$ and $0<r<1$ holds*

$$\int_{-\pi}^{\pi}\Phi\big(\log\big|f\big(re^{i\theta}\big)\big|\big)\,d\theta\leqslant\int_{-\pi}^{\pi}\Phi\big(\log\big|k\big(re^{i\theta}\big)\big|\big)\,d\theta. \tag{5.1}$$

*If equality holds for some strictly convex $\Phi$ and some $r\in(0,1)$, then $f(z)=e^{-i\alpha}k(e^{i\alpha}z)$ for some real $\alpha$.*

Taking $\phi(x)=e^{px}$, we obtain

$$\int_{-\pi}^{\pi}\big|f\big(re^{i\theta}\big)\big|^p\,d\theta\leqslant\int_{-\pi}^{\pi}\big|k\big(re^{i\theta}\big)\big|^p\,d\theta,\quad -\infty<p<\infty,\ 0<r<1. \tag{5.2}$$

Letting $p\to\infty$ and $p\to-\infty$, we recover the inequalities

$$\big|k(-r)\big|\leqslant\inf_{|z|=r}\big|f(z)\big|\leqslant\sup_{|z|=r}\big|f(z)\big|=k(r)$$

first proved by Bieberbach in 1916.

Nonsharp inequalities for $L^p$ norms of functions in $S$ were proved by Prawitz in 1931. See, e.g., [40].

For $f\in S$, write $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ and $I_1(r,f)=\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(re^{i\theta})|\,d\theta$. Littlewood proved in 1925 that $I_1(r,f)\leqslant r(1-r)^{-1}$. From $r^n|a_n|\leqslant I_1(r,f)$, he deduced that $|a_n|\leqslant en$. Taking $p=1$ in (5.2), one calculates that the sharp estimate for $I_1$ is $I_1(r,f)\leqslant\frac{r}{1-r^2}$, which leads to

$$|a_n|\leqslant\frac{e}{2}n. \tag{5.3}$$

The choice $p=2$ in (5.2) gives

$$\sum_{n=1}^{\infty}|a_n|^2r^{2n}\leqslant\sum_{n=1}^{\infty}n^2r^{2n},\quad 0<r<1. \tag{5.4}$$

For $f \in S$, estimates (5.3) and (5.4) are subsumed by de Branges' [29] estimates $|a_n| \leqslant n$. There is, though, an unsolved coefficient problem for a class of functions which includes $S$. A not necessarily univalent analytic function $f : \mathbb{D} \to \mathbb{C}$ is said to be weakly univalent, in the sense of Hayman, [82,81], if there exists $R_0 \in (0, \infty)$ such that $f(\mathbb{D})$ contains the circle $|w| = R$ for each $R < R_0$, but for each $R \geqslant R_0 f(\mathbb{D})$ does not contain $|w| = R$. Theorem 5.1 and its consequences (5.2)–(5.4) are still true for weakly univalent $f$ with $f(0) = 0$, $f'(0) = 1$ [8,81]. Apparently, (5.3) is the best known uniform coefficient estimate for normalized weakly univalent functions. In particular, it is not known if $|a_n| \leqslant n$ holds for all such $f$ and for all $n$.

It appears that Theorem 5.1 cannot lead to the sharp estimates for individual coefficients of functions in $S$. On the other hand, in [17], inequalities (5.2) for negative $p$ are used to prove Loewner's sharp estimates for the coefficients of inverse functions of functions in $S$. See also [122].

SKETCH OF PROOF OF THEOREM 5.1. For $f \in S$ and $0 < r < 1$ holds

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log\left|f\left(re^{i\theta}\right)\right| d\theta = \log r.$$

Thus, $\log|f|$ and $\log|k|$ have the same mean values on circles $|z| = r$. From Proposition 2.4, it follows that that the conclusion of Theorem 5.1 is equivalent to validity of the inequality $(\log|f|)^* \leqslant (\log|k|)^*$ in the upper half of the unit disk, and also equivalent to validity of

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ \frac{|f(re^{i\theta})|}{s} d\theta \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ \frac{|k(re^{i\theta})|}{s} d\theta \tag{5.5}$$

for each $0 < r < 1$ and $0 < s < \infty$.

Define

$$u(w) = g(w, 0, \Omega), \qquad v(w) = g(w, 0, \Omega_1),$$

where $g$ stands for Green function, $\Omega = f(\mathbb{D})$, and $\Omega_1 = k(\mathbb{D})$. Recall that $u$ and $v$ are defined to be zero outside of $\Omega$ and $\Omega_1$, respectively. By (4.6) and (4.4), and setting $t = -\log r$, (5.5) is equivalent to

$$\int_{-\pi}^{\pi} \left(u\left(se^{i\theta}\right) - t\right)^+ d\theta \leqslant \int_{-\pi}^{\pi} \left(v\left(se^{i\theta}\right) - t\right)^+ d\theta$$

for each $s \in (0, \infty)$, $t > 0$. Since $u$ and $v$ are $\geqslant 0$, if this last inequality holds for $t = 0$ then it holds for $t \leqslant 0$. From the equivalence of (b) and (c) in Proposition 2.4, and denoting the open upper half plane by $\mathbb{H}$, it thus follows that Theorem 5.1 is equivalent to the *-function inequality

$$u^* \leqslant v^* \quad \text{in } \mathbb{H} \cup (\mathbb{R} \setminus 0). \tag{5.6}$$

Set $p = u^* - V$, where $V(se^{i\theta}) = Jv(re^{i\theta}) = \int_{-\theta}^{\theta} v(se^{it})\,dt$. Then $V \leqslant v^*$, by the definition (1.1) of the *-function, so to prove (5.5), it suffices to prove $p \leqslant 0$ in $\mathbb{H} \cup (\mathbb{R} \setminus 0)$. Now $u$ is subharmonic in $\mathbb{C} \setminus 0$, and $v$ is harmonic in the slit plane $\Omega_1$. By Proposition 3.3, $u^*$ and $V$ are continuous on $\mathbb{H} \cup (\mathbb{R} \setminus 0)$, while by Theorem 3.1 and Proposition 3.2, $u^*$ is subharmonic and $V$ is harmonic in $\mathbb{H}$. Thus, $p$ is subharmonic in $\mathbb{H}$ and is continuous on $\mathbb{H} \cup (\mathbb{R} \setminus 0)$.

From Proposition 2.3(c) and the definition of $V$, it follows that

$$\frac{\partial p}{\partial \theta}\left(se^{i\pi}\right) = 2 \inf_{\theta \in [-\pi, \pi]} u\left(se^{i\theta}\right) - 2v\left(se^{i\pi}\right). \tag{5.7}$$

Define $p$ in the lower half plane by setting $p(w) = p(\overline{w})$. Then $p$ is continuous in $\mathbb{C} \setminus 0$, and is subharmonic in the open upper and open lower half planes. Let $d$ denote the radius of the largest disk contained in $\Omega$. Then, by the one-quarter theorem, $d \geqslant \frac{1}{4}$. If $s \geqslant d$, there is a point of $|w| = s$ not in $\Omega$. Since $u = 0$ outside $\Omega$ and $v(-s) = 0$ for $s > 1/4$, (5.7) shows that $p_\theta(se^{i\pi}) = 0$ for $s \geqslant d$. As shown, for example, in [8], this implies that $p$ satisfies the sub-mean value property at points of $(-\infty, -d)$, and thus $p$ is in fact subharmonic in $\mathbb{C} \setminus [-d, \infty)$.

Let $M = \sup_{\mathbb{C} \setminus 0} p$. There is sequence $\{w_n\}$ in $\mathbb{C} \setminus \{0\}$ and a point $w_0$ in $\mathbb{C} \cup \{\infty\}$ such that $w_n \to w_0$ and $p(w_n) \to M$. If $w_0$ is positive real, then $u^*(w_0) = v_1(w_0) = 0$, so that $M = 0$. If $w_0 \in \mathbb{C} \setminus [-d, \infty)$, the strong maximum principle for subharmonic functions implies $p$ is constant in $\mathbb{C} \setminus [-d, \infty)$. The continuity of $p$ and vanishing of $p$ on the positive real axis then imply that again $M = 0$.

The remaining possibilities for $w_0$ are: (a) $w_0 = 0$, or (b) $w_0 = \infty$, or (c) $w_0 \in [-d, 0]$. From $f'(0) = k'(0) = 1$, it follows that near $w = 0$, both $u(w)$ and $v(w)$ have the form $-\log|w| + o(1)$. From this, it follows that $w_0 = 0 \Rightarrow M = 0$. Moreover, $u(w) + \log|w|$ is harmonic in $|w| < d$ and $v(w) + \log|w|$ is subharmonic in $|w| < d$, so that the mean and sub-mean value properties imply $u^*(-s) = -2\pi \log s$, $V(-s) \geqslant -2\pi \log s$ for $s \in (0, d]$. Thus, if either (a) or (c) hold, then $M = 0$. Finally, $u(w) \to 0$ and $v(w) \to 0$ as $w \to \infty$, so that $M = 0$ in case (b).

We've shown that $M = 0$, so that $p \leqslant 0$ in $\mathbb{C} \setminus 0$ and the integral means inequality (5.1) is proved. For the equality statement and for more detail in the argument above, see [8], [40] or [81]. □

Via similarly structured arguments, a number of other comparison theorems for integral means of univalent functions have been proved in various settings. For example, Theorem 6.2 in the next section contains a comparison of conformal mappings onto simply connected domains $\Omega$ with mappings onto their circularly symmetrized domains $\Omega^*$. Other integral means comparisons involve meromorphic univalent functions, functions in annuli, or functions satisfying constraints of various kinds. In addition to [8], a partial list of papers includes [89,72,112,92,122,16].

Leung [94] used *-functions to solve extremal problems for integral means of derivatives of starlike and other special classes of univalent functions.

## 6. Circular symmetrization, Green functions, and harmonic measures

Let $\Omega \subset \mathbb{C}$ be a domain. We define a new domain $\Omega^*$ which has the same "size" as $\Omega$ but more symmetry, as follows: For $0 < r < \infty$, let $\Omega(r)$ be the intersection of $\Omega$ with the circle $|z| = r$. If $\Omega(r)$ is empty then so is $\Omega^*(r)$, and if $\Omega(r)$ is the full circle then so is $\Omega^*(r)$. If $\Omega(r)$ is a nonempty proper subset of $|z| = r$, with $|\Omega(r)| = 2\theta$, then $\Omega^*(r)$ is the arc $\{re^{it}\colon |t| < \theta\}$. Moreover, $0 \in \Omega^*$ if and only if $0 \in \Omega$.

It is easy to see that $\Omega$ and $\Omega^*$ have the same area. Of course one can circularly symmetrize a domain with any given half-line playing the role of the positive real axis.

In the next section, we shall take up another symmetrization process, called *Steiner symmetrization,* in which one symmetrizes a domain with respect to a full line. For information about some uses of circular and Steiner symmetrization in the plane the reader may consult [101,82,81]. The article [15] treats symmetrization theory in a wider context.

Numerous domain functionals vary monotonically under symmetrization. We shall state three such results, from [8], involving Green functions, holomorphic functions and harmonic measures under circular symmetrization. Proofs may also be found in [81]. Section 7 contains some discussion of their Steiner analogues, while Section 12 contains theorems about circular symmetrization and solutions of pde's.

As in Section 5, $g(z, a, \Omega)$ denotes the Green function of $\Omega$ with pole at $a \in \Omega$, extended to be zero outside $\Omega$. We do not assume that $\Omega$ is simply connected.

THEOREM 6.1. *Let $\Omega \subset \mathbb{C}$ be a domain such that both $\Omega$ and $\Omega^*$ have Green functions. Then for each convex increasing $\Phi : \mathbb{R} \to \mathbb{R}$ and each $a \in \Omega$, $r \in (0, \infty)$ holds*

$$\int_{-\pi}^{\pi} \Phi\big(g(re^{i\theta}, a, \Omega)\big)\, d\theta \leqslant \int_{-\pi}^{\pi} \Phi\big(g(re^{i\theta}, |a|, \Omega^*)\big)\, d\theta. \tag{6.1}$$

If $\Omega$ is contained in another domain $\Omega_1$, then the maximum principle implies $g(z, a, \Omega) \leqslant g(z, a, \Omega_1)$ for each $z \in \mathbb{C}$ and $a \in \Omega$. The domains $\Omega$ and $\Omega^*$ have the same size in some senses, but Theorem 6.1 says that when $\Omega^*$ is viewed from $|a|$ and $\Omega$ from $a$, then $\Omega^*$ looks larger on the average.

Write

$$u(z) = g(z, a, \Omega), \qquad v(z) = g(z, |a|, \Omega^*).$$

By Proposition 2.4, the conclusion of Theorem 6.1 may be stated as $u^* \leqslant v^*$ in the upper half plane $\mathbb{H}$. The proof of this *-function inequality follows the same general plan as the proof of the corresponding inequality in Theorem 5.1, but some additional features are required. For example, if $a \neq 0$, then $u$ is not subharmonic in $\mathbb{C} \setminus \{0\}$, and $u^*$ fails to be subharmonic in any domain which intersects the upper half of $|z| = |a|$. However, from Theorem 3.2 one can deduce that $u^*(re^{i\theta}) + 2\pi \log^+ \frac{r}{|a|}$ is subharmonic in $\mathbb{H}$. It turns out also that $V(re^{i\theta}) + 2\pi \log^+ \frac{r}{|a|}$ is harmonic in $\Omega^* \cap \mathbb{H}$, where $V$ has the same meaning as in the proof of Theorem 5.1. Thus, $u^* - V$ is subharmonic in $\Omega^* \cap \mathbb{H}$, which is what one needs for the proof of Theorem 5.1 to carry over.

If $\Omega = \Omega^*$ and $a \geqslant 0$, then the proof of Theorem 6.1 just sketched shows that $v^* \leqslant V$. The opposite inequality follows from the definition of the *-function, so $v^* = V$. This implies that $v(re^{i\theta})$ is a symmetric decreasing function of $\theta$ for each $r > 0$. Thus, as a by-product of the proof of Theorem 6.1, we see that:

*The Green function of a circularly symmetric domain $\Omega^*$ with pole $a \geqslant 0$ is a symmetric decreasing function on each circle $|z| = r$.*

When $\Omega^*$ is simply connected and $a = 0$, this is equivalent to a theorem of Jenkins [87].

From Proposition 2.2, one sees that $u^* \leqslant v^*$ in $\mathbb{H}$ may be restated as

$$\int_{-\theta}^{\theta} \tilde{u}(re^{it})\,dt \leqslant \int_{-\theta}^{\theta} \tilde{v}(re^{it})\,dt, \quad re^{i\theta} \in \mathbb{H},$$

where $\tilde{}$ denotes symmetric decreasing rearrangement on circles. As just noted, $\tilde{v} = v$. In [80, Problem 5.17], the question was posed, in equivalent form, if we always have $\tilde{u} \leqslant v$ in $\Omega^*$. This conjectural stronger inequality was disproved by Pruss [102].

From $u^* \leqslant v^*$ and Proposition 2.5, we deduce that

$$\sup_{|z|=r} g(z, a, \Omega) \leqslant \sup_{|z|=r} g(z, |a|, \Omega^*) = g(r, |a|, \Omega^*), \quad 0 < r < \infty.$$

As noted in [81, Theorem 9.4], this inequality implies the result of Pólya and Szëgo that circular symmetrization increases the mapping radius. For $\Omega$ simply connected and $a = 0$, the Green function inequality is due to Krzyż [91].

Uniqueness theorems associated with the theorems of this section are in [56]. See also [12,114].

Next, suppose that $\Omega^*$ is simply connected and not the whole plane. Let $f$ be holomorphic in $\mathbb{D}$ with $f(\mathbb{D}) \subset \Omega$, and let $F$ be a conformal map of $\mathbb{D}$ onto $\Omega^*$ with $F(0) = |f(0)|$. Then Theorems 5.1 and 6.1 have a close relative:

THEOREM 6.2. *Suppose that $\Omega^*$, $f$ and $F$ are as described in the previous paragraph. Then for each convex increasing $\Phi : \mathbb{R} \to \mathbb{R}$ holds*

$$\int_{-\pi}^{\pi} \Phi(\log|f(re^{i\theta})|)\,d\theta \leqslant \int_{-\pi}^{\pi} \Phi(\log|F(re^{i\theta})|)\,d\theta. \tag{6.2}$$

If also $\Omega$ is simply connected and $f$ is a conformal map of $\mathbb{D}$ onto $\Omega$, then the argument in the proof of Theorem 5.1 shows that Theorem 6.2 is in fact equivalent to Theorem 6.1. If $f$ is not necessarily univalent but $\Omega$ is simply connected, then $f$ is subordinate to the conformal map $f_1$ onto $\Omega$ with $f(0) = f_1(0)$. The means in (6.2) increase when $f$ is replaced by $f_1$, see, e.g., [83, p. 76] or [11, p. 840]. These two results prove Theorem 6.2 when $\Omega$ is simply connected.

For general $\Omega$, not necessarily simply connected, (but $\Omega^*$ still simply connected) we use a theorem of Lehto [93] which asserts that the weighted counting function $N(r, w, f)$ is a subharmonic function of $w$ in $\mathbb{C}$, except for a logarithmic pole at $w = f(0)$. The argument used to prove Theorem 6.1 shows that

$$N^*(r, w, f) \leqslant \big(g(w, F(0), \Omega^*) + \log r\big)^*, \quad w \in \mathbb{H},\ 0 < r < 1. \tag{6.3}$$

To deduce (6.2) from (6.3), use (4.6) and the argument in the proof of Theorem 5.1 which gave (5.1) from (5.6), with $v(w) = g(w, F(0), \Omega^*)$, $u(w) = N(R, w, f)$, $0 < R < 1$, then let $R \to 1$.

(6.2) may be restated as $(\log|f|)^* \leqslant (\log|F|)^*$ in $\mathbb{D}^+$. From Proposition 2.5, it follows that

$$M(r, f) \leqslant M(r, F), \qquad |f'(0)| \leqslant |F'(0)|, \tag{6.4}$$

where $M$ denotes maximum modulus. The inequality for the derivatives follows from the inequality for the maximum moduli.

Hayman, in [80, p. 32], observed that (6.4) is a consequence of the Pólya-Szëgo theorem about increase of the mapping radius under symmetrization, and proposed the problem of extending (6.4) to the case of multiply connected $\Omega^*$, with $F$ a universal covering map of $\mathbb{D}$ onto $\Omega^*$ with $F(0) = |f(0)|$. The problem was solved by Weitsman [120]; we shall return to this question in Section 12 of this article. It is not known if the full analogue of (6.2) holds when $\Omega^*$ is multiply connected.

We turn now to harmonic measure. For a Borel set $E \subset \mathbb{C} \setminus \Omega$, $\omega(z, E, \Omega)$ will denote the harmonic measure of $E$ at $z$ with respect to $\Omega$. Roughly, $\omega(z, E, \Omega)$ is the function harmonic in $\Omega$ with boundary values 1 on $E \cap \partial\Omega$, 0 on $\partial\Omega \setminus E$. For a precise definition, see [83]. In the next theorem, $\Omega$ is a subdomain of $\mathbb{D}$, not necessarily simply connected, and $E = \partial\mathbb{D}$.

THEOREM 6.3. *Let $\Omega$ be a subdomain of $\mathbb{D}$. Then, for every convex increasing $\Phi : \mathbb{R} \to \mathbb{R}$ and $0 < r < 1$ holds*

$$\int_{-\pi}^{\pi} \Phi\big(\omega(re^{i\theta}, \partial\mathbb{D}, \Omega)\big)\, d\theta \leqslant \int_{-\pi}^{\pi} \Phi\big(\omega(re^{i\theta}, \partial\mathbb{D}, \Omega^*)\big)\, d\theta. \tag{6.5}$$

If $\partial\Omega^* \cap \mathbb{D}$ has inner logarithmic capacity zero, then $\omega(\cdot, \partial\mathbb{D}, \Omega^*) \equiv 1$, and (6.5) is trivially true. If $\partial\Omega^* \cap \mathbb{D}$ has positive inner logarithmic capacity, then so does $\partial\Omega \cap \mathbb{D}$. Letting $u$ and $v$ denote the harmonic measures of $\Omega$ and $\Omega^*$ respectively, extended to be zero in $\mathbb{D} \cap \Omega$, resp. $\mathbb{D} \cap \Omega^*$, the argument used to prove Theorem 6.1 can be adapted to prove $u^* \leqslant v^*$ in $\mathbb{D}^+$, which is equivalent to the conclusion of Theorem 6.3. As with the Green functions, the proof shows also that $v(z)$ is a symmetric decreasing function on each $|z| = r$. From $u^* \leqslant v^*$ and Proposition 2.5, we deduce a result of Haliste [76]:

$$\sup_{|z|=r} \omega(z, \partial\mathbb{D}, \Omega) \leqslant \sup_{|z|=r} \omega(z, \partial\mathbb{D}, \Omega^*) = \omega(r, \partial\mathbb{D}, \Omega^*). \tag{6.6}$$

Let $A$ be the set of $r \in [0, 1)$ such that $\Omega$ does not contain the full circle $|z| = r$, and let $\Omega^{**} = \mathbb{D} \setminus \{-r \colon r \in A\}$. Then $\Omega^* \subset \Omega^{**}$. From (6.6) and the maximum principle, we deduce

$$\sup_{|z|=r} \omega(z, \mathbb{D}, \Omega) \leqslant \omega(r, \mathbb{D}, \Omega^{**}), \quad 0 < r < 1.$$

This is the *Beurling–Nevanlinna Projection Theorem* [22,96,97, p. 107].

Further works along the lines of this section include [38,16,113,103,21,104,105]. The two last-mentioned papers provide discrete analogues and companions for the results and methods discussed here. Theorems 6.1 and 6.3 are special cases of a general theorem [15, Theorem 7] about comparison of solutions of pde's under symmetrization. For a broad survey of symmetrization results in function theory, see [37].

An interesting recent development in symmetrization theory is the rise to the fore of *polarization*. Polarization may be viewed as a very primitive form of symmetrization. Let $L$ be a line in $\mathbb{C}$. Then $\mathbb{C} \setminus L$ consists of two open half planes. Denote one of them by $H^+$, the other by $H^-$, and denote by $z^*$ the reflection of $z$ in $L$. Given a domain $\Omega \subset \mathbb{C}$, the polarization $\Omega_L$ of $\Omega$ with respect to $L$ is the new open set, not necessarily connected, defined as follows: For $z \in \mathbb{C}$, if both $z$ and $z^*$ are in $\Omega$ then both are in $\Omega_L$. If neither is in $\Omega$ then neither is in $\Omega_L$. If one is in $\Omega$ and the other is not, then the member of the pair which is in $H^+$ is in $\Omega_L$, and the member which belongs to $H^-$ is not in $\Omega_L$.

There is a corresponding concept of polarization of functions. Polarization was applied to a problem about symmetrization and capacity by Wolontis [121]. The proof in [18] of the convolution inequality on $S^n$ extending (3.7) is based on polarization, and in [15], polarization in $n$-dimensional Euclidean space, spheres, and hyperbolic spaces lies at the base of a general theory of symmetrization. [36] presents another general theory based on polarization. [114] and [21] contain results asserting that, under appropriate hypotheses, harmonic measures and Green functions change monotonically when domains are polarized. Brock and Solynin [24] showed that for $1 \leqslant p < \infty$, Steiner symmetrizations of a nonnegative function $f \in L^p(\mathbb{R}^n)$ can be realized as $L^p$ limits of sequences of successive polarizations starting with $f$. The analogous result is also true for circular symmetrization in the plane. Combined with [114] or [21], this result leads to new proofs of Theorems 6.1 and 6.3.

Monotonicity of Green functions under polarization in circles is a key ingredient in some lovely theorems of Aharonov, Shapiro and Solynin [1,2]. In the former paper they identify the univalent function in $\mathbb{D}$ whose image has least area among all functions $f(z) = z + a_2 z^2 + \cdots \in S$ with specified $|a_2|$. In the latter paper, they solve the analogous minimal area problem for functions in $S$ with $|f(r)| = b$, for fixed $0 < r < 1$ and $r(1+r)^{-2} < b < r(1-r)^{-2}$.

## 7. Vertical *-functions and Steiner symmetrization

We consider now an analogue of the *-function, also to be denoted $u^*$, in which integrals over sets on concentric circles are replaced by integrals over sets on parallel lines. For definiteness we'll take the lines to be parallel to the imaginary axis, and thus will call $u^*$ the vertical *-function.

For $-\infty \leqslant x_1 < x_2 \leqslant \infty$, set

$$B(x_1, x_2) = \{x + \mathrm{i}y \in \mathbb{C}\colon\ x_1 < x < x_2\}, \quad B^+(x_1, x_2) = B(x_1, x_2) \cap \mathbb{H},$$

and for $u: B(x_1, x_2) \to \mathbb{R}$, define $u^*: B^+(x_1, x_2)$ by

$$u^*(x + \mathrm{i}y) = \sup_E \int_E u(x + \mathrm{i}t)\, dt, \tag{7.1}$$

where the sup is taken over all $E \subset \mathbb{R}$ with $|E| = 2y$. The definition makes sense provided $u(x + \mathrm{i}t) \in L^1(E)$ as a function of $t$ for each fixed $x \in (x_1, x_2)$ and each $E \subset \mathbb{R}$ with finite measure. But to make the theory work as in the circular case, additional assumptions on $u$ must be imposed to overcome the added complication that lines have infinite measure. The simplest such assumptions are that

$$u \geqslant 0 \quad \text{in } B(x_1, x_2), \quad \text{and} \quad \lim_{t \to \pm\infty} u(x + \mathrm{i}t) = 0, \quad \forall x \in (x_1, x_2). \tag{7.2}$$

The propositions in Section 2 about functions $g$ and $h$ carry over to this new context, provided $g, h: \mathbb{R} \to \mathbb{R}^+$ and each tends to zero at $\pm\infty$. The subharmonicity theorems 3.1 and 3.3 also carry over, provided $u$ satisfies (7.2). Theorem 3.2 carries over when (7.2) holds and $u(x + \mathrm{i}t) \in L^1(\mathbb{R})$ for each fixed $x$. Here, for example, is the analogue of Theorem 3.1:

THEOREM 7.1. *Let $u$ be subharmonic $B(x_1, x_2)$, and satisfy* (7.2). *Then $u^*$, as defined by* (7.1), *is subharmonic in $B^+(x_1, x_2)$.*

To prove Theorem 7.1, and the analogues to Theorems 3.2, 3.3, one may repeat the proofs for the circular case with small changes, may deduce the vertical case from the circular case by exponential transformation as in [11], or may invoke the general result [15, Theorem 5].

The *Steiner symmetrization* $\Omega^s$ of $\Omega$, with respect to the real axis, is defined thus: For $x \in \mathbb{R}$, let $\Omega(x)$ be the intersection of $\Omega$ with the line $\{x + \mathrm{i}y\colon\ y \in \mathbb{R}\}$. If $\Omega(x)$ is empty, then so is $\Omega^s(x)$, and if $|\Omega(x)| = \infty$ then $\Omega^s(x)$ is the full line $\{x + \mathrm{i}y\colon\ y \in \mathbb{R}\}$. If $0 < |\Omega(x)| < \infty$, then $\Omega^s(x)$ is the vertical line segment $\{x + \mathrm{i}y\colon\ |y| < \frac{1}{2}|\Omega(x)|\}$. The Steiner symmetrization of $\Omega$ with respect to any line $L$ is defined in analogous fashion. Unless otherwise noted, $\Omega^s$ will denote Steiner symmetrization with respect to the real axis.

Under appropriate hypotheses on $\Omega$, subharmonicity of the vertical *- function can be used to prove that, as with the circular symmetrization considered in Section 6, Steiner symmetrization of domains increases integral means of Green functions and harmonic measures. Two such theorems are stated below. Under a variety of circumstances the hypotheses on $\Omega$ can be modified.

THEOREM 7.2. *Let $\Omega \subset \mathbb{C}$ be a domain with $|\Omega(x)| < \infty$ for each $x \in \mathbb{R}$. Then, for each convex increasing $\Phi : \mathbb{R} \to \mathbb{R}$, each $a \in \Omega$, and each $x \in \mathbb{R}$ holds*

$$\int_{\mathbb{R}} \Phi\big(g(x+\mathrm{i}y, a, \Omega)\big)\,dy \leqslant \int_{\mathbb{R}} \Phi\big(g\big(x+\mathrm{i}y, \operatorname{Re} a, \Omega^s\big)\big)\,dy.$$

THEOREM 7.3. *Let $\Omega$ be a domain with $\Omega \subset B(-\infty, x_2)$, where $x_2 < \infty$. Set $L(x_2) = \{z \in \partial\Omega\colon \operatorname{Re} z = x_2\}$. Then, for each convex increasing $\Phi : \mathbb{R} \to \mathbb{R}$ and each $x \in (-\infty, x_2)$ holds*

$$\int_{\mathbb{R}} \Phi\big(\omega(x+\mathrm{i}y, L(x_2), \Omega)\big)\,dy \leqslant \int_{\mathbb{R}} \Phi\big(\omega\big(x+\mathrm{i}y, L^s(x_2), \Omega^s\big)\big)\,dy. \tag{7.3}$$

In Theorem 7.3, $\omega(z, L(x_2), \Omega) \equiv 0$ for $z \in B(-\infty, x_2) \setminus \Omega$, and analogously for $\omega(z, L^s(x_2), \Omega^s)$. The consequence of (7.3):

$$\begin{aligned}\sup_{y\in\mathbb{R}} \omega\big(x+\mathrm{i}y, L(x_2), \Omega\big) &\leqslant \sup_{y\in\mathbb{R}} \omega\big(x+\mathrm{i}y, L^s(x_2), \Omega^s\big)\\ &= \omega\big(x, L^s(x_2), \Omega^s\big), \quad -\infty < x < x_2,\end{aligned}$$

is due to Haliste [76].

*Other symmetrizations.* Steiner and circular symmetrizations in the plane are members of a family of symmetrizations living in $\mathbb{R}^n$, $n$-dimensional spheres $S^n$, and $n$-dimensional hyperbolic spaces. For example, circular symmetrization in the plane can be extended to higher dimensions as follows: Given a domain $\Omega \subset \mathbb{R}^n$, for $r \in (0, \infty)$ let $\Omega(r) = \{x \in S^{n-1}\colon rx \in \Omega\}$. Define $\Omega^*(r)$ to be empty if $\Omega(r)$ is empty, to be $S^{n-1}$ if $\Omega(r) = S^{n-1}$, and to be a spherical cap on $S^{n-1}$ centered at the east pole $e_1 = (1, 0, \ldots, 0) \in S^{n-1}$ with the same surface measure as $\Omega(r)$ if $\Omega(r)$ is neither empty nor the full sphere. Then $\Omega^*$ is defined to be the union over $r > 0$ of all the $\Omega^*(r)$, together with the point 0 if $0 \in \Omega$.

In [15] this $\Omega^*$ is called the $(n-1, n)$-cap symmetrization of $\Omega$. The $n-1$ signifies that sets of dimension $n-1$ are being symmetrized into geodesic balls on the manifold $S^{n-1}$. For each $1 \leqslant k \leqslant n-1$ there are $(k, n)$-cap symmetrizations, associated with foliations of $\mathbb{R}^n$ into $k$-spheres, and also $(k, n)$-Steiner symmetrizations. The Steiner symmetrizations are associated with foliations of $\mathbb{R}^n$ into parallel $k$-planes, that is, into decompositions of of the form $\mathbb{R}^n = H + H^{\perp}$, where $H$ is a subspace of dimension $n-k$. To form the $(k, n)$-Steiner symmetrization of $\Omega$ with respect to $H$, for each $x \in H$ replace the slice $\Omega(x) = \Omega \cap (x + H^{\perp})$ with the $k$-dimensional ball in $x + H^{\perp}$ centered at $x$ whose $k$-dimensional Lebesgue measure equals that of $\Omega(x)$. For $\Omega \subset \mathbb{C}$, the set $\Omega^s$ discussed earlier in this section is thus a $(1, 2)$-Steiner symmetrization. In all dimensions, the Steiner case most frequently encountered is $k = 1$, so that the symmetrization is done within lines orthogonal to some fixed hyperplane.

In this connection, we remark that symmetrizations of "condensers" are important tools for quasiconformal analysis in $\mathbb{R}^n$. The book by Anderson, Vamanamurthy and Vuorinen [4] contains discussion and a number of references.

In $\mathbb{R}^n$, the simplest way to symmetrize $\Omega$ is to change it into a ball of the same measure centered at the origin. This process goes by various names, among them symmetric decreasing rearrangement (with respect to $\mathbb{R}^n$), point symmetrization, and Schwarz symmetrization. It has evident analogues on spheres and in hyperbolic spaces.

With each of the symmetrizations mentioned above one can associate a *-function [15]. Much of the theory works like it does for Steiner and circular symmetrization in the plane. One significant change, though, is that if $u$ is subharmonic with respect to the Laplace operator, then $u^*$ will be subharmonic with respect to a possibly different operator, which depends on the symmetrization process. For example, let $u$ be defined in a shell $A = \{x \in \mathbb{R}^n\colon\ R_1 < |x| < R_2\}$. Then the $u^*$ associated with $(n-1, n)$-cap symmetrization is defined in the two-dimensional set $A' = \{re^{i\theta}\colon\ r \in (R_1, R_2), 0 \leqslant \theta \leqslant \pi\}$ by

$$u^*\left(re^{i\theta}\right) = \sup_E \int_E u(rx)\, d\sigma(x),$$

where $\sigma$ is surface measure and the sup is over all $E \subset S^{n-1}$ with $\sigma(E) = \sigma(K(\theta))$, with $K(\theta) = \{x \in S^{n-1}\colon\ x \cdot e_1 > \cos\theta\}$, the cap on $S^{n-1}$ centered at the east pole with geodesic radius $\theta$. If $u$ is subharmonic in $A$ with respect to the $n$-dimensional Laplace operator, it was proved in [18] that for $n \geqslant 3$, $u^*$ is subharmonic in the interior of $A'$ with respect to the operator

$$L = \frac{\partial^2}{\partial r^2} + (n-1)\frac{\partial}{\partial r} + r^{-2}\left(\sin^{n-2}\theta\right)\frac{\partial}{\partial\theta}\left(\left(\sin^{2-n}\theta\right)\frac{\partial}{\partial\theta}\right).$$

The $n$-dimensional Laplace operator acting on functions of $r = |x|$ and the spherical coordinate $\theta$, defined for $x \in \mathbb{R}^n \setminus \{0\}$ by $x \cdot e_1 = r\cos\theta$, has the same radial part as $L$, but has $n-2$ in the exponent inside $\frac{\partial}{\partial\theta}$ and $2-n$ in the exponent outside $\frac{\partial}{\partial\theta}$.

## 8. Conjugate harmonic functions

Vertical *-functions have been applied to prove $L^p$-inequalities involving holomorphic functions and their real and imaginary parts in the unit disk. The following theorem is in [11, p. 839]. Related papers include [56,60,61,103,62].

THEOREM 8.1. *Let $\mu$ be a signed regular Borel measure on $\partial\mathbb{D}$ with total variation $\|\mu\| = 1$ and $|\mu(\partial\mathbb{D})| = b \in [0, 1]$. Set*

$$f(z) = \int_{\partial\mathbb{D}} \frac{1 + z\bar{\zeta}}{1 - z\bar{\zeta}}\, d\mu(\zeta),$$

$$F_b(z) = \frac{1}{2}(1+b)\left(\frac{1+z}{1-z}\right) - \frac{1}{2}(1-b)\left(\frac{1-z}{1+z}\right) = \frac{2z}{1-z^2} + b\frac{1+z^2}{1-z^2}.$$

*Then, for* $0 < r < 1$ *and* $0 < p \leqslant 2$ *holds*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(re^{i\theta})|^p\,d\theta \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}|F_b(re^{i\theta})|^p\,d\theta. \tag{8.1}$$

The integral defining $f$ is the analytic completion of the Poisson integral of $\mu$ which is real at the origin. It is sometimes called the Herglotz integral of $\mu$. Thus $F_b$ is the Herglotz integral of the measure with total variation one which assigns measure $\frac{1}{2}(1+b)$ to $z=1$, measure $-\frac{1}{2}(1-b)$ to $z=-1$. Theorem 8.1 says that to extremize $L^p$ norms on circles for Herglotz integrals of signed measures of fixed total variation and fixed net variation, one should "polarize" the measure, at least when $0 < p \leqslant 2$. It is not known to what extent this theorem can be extended to $p > 2$.

Let $u$ be a real valued function in $L^1(\mathbb{D})$. The Herglotz integral $f$ of $u$, more precisely, of the measure $u(e^{i\theta})\,d\theta/(2\pi)$, has a nontangential limit $f(e^{i\theta})$ for almost all $\theta$. Write $v(e^{i\theta}) = \operatorname{Im} f(e^{i\theta})$. Then $v$ is called the conjugate function of $u$. Kolmogorov, in 1925, see, e.g., [39], proved that $\|v\|_p \leqslant C_p\|u\|_1$ for $0 < p < 1$ and some constant $C_p$, where $\|\cdot\|_p$ is the $L^p$ norm on $\partial\mathbb{D}$ with respect to normalized Lebesgue measure. These bounds still hold, with the same constants, if, more generally, we permit $u$ to be a signed measure $\mu$ and replace $\|u\|_1$ by $\|\mu\|$. Davis [28] discovered the sharp value of $C_p$ via a brilliant argument with Brownian motion. His result may be stated as

$$\|v\|_p \leqslant C_p\|\mu\|,$$
$$\text{where } C_p = \frac{1}{2\pi}\int_{-\pi}^{\pi}|\sin\theta|^{-p}\,d\theta. \tag{8.2}$$

Equality holds when $\mu = \frac{1}{2}(\delta_1 - \delta_{-1})$, so that $f$ is the function $F_0$ of Theorem 8.1. If we let $r \to 1$ in (8.1) and maximize over $b$, then (8.2) follows. Thus, Theorem 8.1 yields as corollary a "classical" proof of Davis's theorem.

In [11, p. 842], Theorem 8.1 appears as a limiting case of a theorem about analytic mappings $f$ of $\mathbb{D}$ into a fixed bounded Steiner symmetric domain $\Omega$ with specified $f(0) \in \mathbb{R}$ and specified $\int_{-\pi}^{\pi}|\operatorname{Re} F(e^{i\theta})|\,d\theta$. It turns out that conformal mappings onto $\Omega$ minus a pair of symmetric vertical slits on the imaginary axis maximize $L^p$ norms on circles $|z| = r$ when $0 < p \leqslant 2$. The functions $F_b$ map $\mathbb{D}$ conformally onto the plane minus symmetric vertical slits on the imaginary axis, and thus are limit extremal functions when, for example, $\Omega$ runs through a sequence of disks centered at the origin with radii tending to infinity.

Here is another theorem from [11, p. 848], with similar proof, which we will sketch. Recall that $\tilde{u}$ denotes the symmetric decreasing rearrangement on circles of a real-valued function $u$.

THEOREM 8.2. *Let* $u \in L^1(\partial\mathbb{D})$. *Denote the Herglotz integrals of* $u$ *and* $\tilde{u}$ *by* $f = u + iv$ *and* $F = U + iV$ *respectively. Then, for* $0 < r < 1$ *holds*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|u(re^{i\theta})|^p\,d\theta \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}|U(re^{i\theta})|^p\,d\theta, \quad 1\leqslant p<\infty, \tag{8.3a}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|v(re^{i\theta})|^p\,d\theta \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}|V(re^{i\theta})|^p\,d\theta, \quad 1\leqslant p\leqslant 2, \tag{8.3b}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(re^{i\theta})|^p\,d\theta \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}|F(re^{i\theta})|^p\,d\theta, \quad 0< p\leqslant 2. \tag{8.3c}$$

Part (a) was known long before [11]. In fact, the inequality corresponding to (8.3a) is true when $\Phi(x)=|x|^p$ is replaced by an arbitrary convex function $\Phi$. This follows from the convolution inequality (3.7) – noting that the Poisson kernel $P(r,\theta)$ is a symmetric decreasing function of $\theta$, and taking the third function to be the characteristic function of a set $E$, – together with (c) $\Rightarrow$ (d) of Proposition 2.4. For $u\geqslant 0$, this strong form of (8.3a) is due to Gabriel [71]. For $p=1$ and $u\geqslant 0$ (8.3a) becomes equality. For $0<p<1$ and $u\geqslant 0$ the inequality in (8.3a) reverses. This follows from the convexity of $\Phi(x)=-x^p$ on $[0,\infty)$, together with a variant of Proposition 2.4 which asserts that if $g$ and $h$ are nonnegative with $g^*\leqslant h^*$ and $\int_{-a}^{a} g=\int_{-a}^{a} h$, then the $\Phi-$ means of $g$ are $\leqslant$ those of $h$ for every convex $\Phi:[0,\infty)\to\mathbb{R}$. The variant can be obtained from Proposition 2.4 via an approximation argument.

For $r=1$ (8.3b) becomes equality for $p=2$, and reverses for $2<p<\infty$. The reversal was discovered by Essén and Shea [56]. (8.3b) is probably false for $0<p<1$, but I do not know a proof. Examples exist [11, p. 849] with $f$ unbounded but $F$ bounded, so (8.3c) can fail for some large $p$ and some $r$.

SKETCH OF PROOF OF THEOREM 8.2. For simplicity, we consider just the case when $u$, and hence $U$, has mean value zero on $\partial\mathbb{D}$. Then $f(0)=F(0)=0$. By Proposition 4.1, if $P$ is subharmonic in $\mathbb{C}$ with Riesz measure $\mu$ and $P(0)=0$, then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}P(f(re^{i\theta}))\,d\theta=\int_{\mathbb{C}}N(r,w,f)\,d\mu(w), \quad 0<r<1. \tag{8.4}$$

The same equation holds when $f$ is replaced by $F$. To attack (8.3c), take $P(w)=|w|^p$. Then $d\mu(w)=\frac{1}{2\pi}p^2|w|^{p-2}|dw|^2$. Write $w=s+it$. If $0<p\leqslant 2$, then $|s+it|^{p-2}$ is a symmetric decreasing function of $t$. On the right hand side of (8.4), integrate first with respect to $t$, and do an integration by parts. One sees that (8.3c) will follow provided

$$\int_{-t}^{t}N(r,s+i\tau,f)\,d\tau\leqslant\int_{-t}^{t}N(r,s+i\tau,F)\,d\tau \tag{8.5}$$

for all $s+it\in\mathbb{H}$ and $0<r<1$. Let

$$Q(r,s+it)=N^*(r,s+it,f)-\int_{-t}^{t}N(r,s+i\tau,F)\,d\tau,$$

where * denotes the vertical *-function. Then (8.5) will be true if $Q\leqslant 0$ in $\mathbb{H}$.

A priori we may assume that $F$ is holomorphic in the closed unit disk. An argument with the argument principle then shows that $F$ univalently maps $\mathbb{D}$ onto a Steiner-symmetric domain $\Omega$. By (4.4), $N(r, w, F) = (g(w, 0, \Omega) + \log r)^+$. It turns out that $Q(1, w)$ is a subharmonic function of $w$ in $\Omega^+ = \Omega \cap \mathbb{H}$. Moreover, for $a \in \mathbb{R}$ the Riesz measure of $P(s + \mathrm{i}t) = (s - a)^+$ is $\frac{1}{2\pi}$ times Lebesgue measure $dt$ on the line $a + \mathrm{i}t$. From (8.4), it follows that

$$\int_{-\infty}^{\infty} N(1, a + \mathrm{i}t, f)\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(u\left(\mathrm{e}^{\mathrm{i}\theta}\right) - a\right)^+ d\theta.$$

The same equation holds when $f$ and $u$ are replaced by $F$ and $U$. Since $u$ and $U$ have the same distribution on $\partial\mathbb{D}$, the right hand sides are equal. Equality of the left hand sides implies that $Q(1, w) \leqslant 0$ for $w \in \mathbb{H} \setminus \Omega^+$. An argument like the one in Section 5 now shows that $Q(r, w) \leqslant 0$ for $0 < r \leqslant 1$ and $w \in \mathbb{H}$, thereby proving (8.3c).

Inequalities (8.3a) and (8.3b) are also consequences of (8.5). One just needs to know that the Riesz measures of the corresponding functions $P$ are absolutely continuous with respect to Lebesgue area measure with densities symmetric decreasing in the vertical direction, or are limits of such. For (8.3b), we have $P(s + \mathrm{i}t) = |t|^p$, for which $d\mu(w) = \frac{1}{2\pi} p(p-1)|t|^{p-2} ds\, dt$ for $p > 1$, while for $p = 1$, $d\mu$ is $1/\pi$ times Lebesgue measure on the real line. Thus, the required conditions are fulfilled when $1 \leqslant p \leqslant 2$. For (8.3a), $P(s + \mathrm{i}t) = |s|^p$, and $d\mu(w) = \frac{1}{2\pi} p(p-1)|s|^{p-2} ds\, dt$ for $p > 1$, $d\mu(w) = 1/\pi$ times Lebesgue measure on the imaginary axis for $p = 1$. The vertical symmetric decrease condition is trivially satisfied for $1 \leqslant p < \infty$. □

In addition to inequalities $\|v\|_p \leqslant C_p \|u\|_1$ for $0 < p < 1$, Kolmogorov proved also a so-called weak 1-1 inequality:

$$\left|\left\{\theta \in [-\pi, \pi]\colon \left|v\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right| \geqslant t\right\}\right| \leqslant C t^{-1} \|u\|_1, \quad t > 0.$$

The sharp value of this $C$ was also found by Davis [27] using Brownian motion, and a classical proof of Davis's result can again be found in [11]. For some related sharp weak 1-1 inequalities, see [25], [26, Remark 13.1].

For $1 < p < \infty$, there are inequalities $\|v\|_p \leqslant C_p \|u\|_p$, due to M. Riesz. The sharp constants were found by S. Pichorides [100]. Proofs along the same lines as Pichorides' have subsequently been given by other authors; [74] contains a particularly short one. Subharmonicity considerations supply the decisive ingredient in these proofs, but the proofs have not involved *-functions.

The literature on sharp Riesz-type inequalities is substantial. Here we'll just cite [118, 55,86]. The last paper is notable in that the method introduced by Pichorides is enriched to encompass plurisubharmonic considerations, which are used to find sharp constants for inequalities

$$\|u + \mathrm{i}v\|_p \leqslant C_p \|u\|_p, \quad 1 < p < \infty,$$

when $u$ is complex valued. This solves a problem discussed in [98, p. 143].

## 9. Variants of the *-function

The *-function of a function $u$ in an annulus was, in (1.1), defined in the upper half of that annulus by

$$u^*(re^{i\theta}) = \sup\left\{\int_E u(re^{it})\,dt\colon\ E \subset [-\pi,\pi],\ |E| = 2\theta\right\}. \tag{9.1}$$

To treat some extremal problems, it is beneficial to introduce auxiliary functions of the same general type as $u^*$, but defined differently to take into account particular features of the problem. This section contains a few examples.

First, we will look at problems for entire or subharmonic functions in the plane descended from the $\cos\pi\rho$ theorem. For an entire function $f$, denote by $M(r, f) = M(r)$ and $L(r, f) = L(r)$ the respective maximum and minimum moduli of $f$ on the circle $|z| = r$. We shall assume throughout this section that $f$ is nonconstant. The $\cos\pi\rho$ theorem, due independently to Wiman and to Valiron (1915), see, for example [23,81] and [54] for some history, asserts that if $f$ has order $\rho < 1$, then

$$\limsup_{r\to\infty} \frac{\log L(r)}{\log M(r)} \geqslant \cos\pi\rho. \tag{9.2}$$

The "Lindelöf functions"

$$f_\rho(z) = \prod_{n=1}^{\infty} \left(1 + zn^{-1/\rho}\right),$$

which satisfy

$$\log\left|f_\rho(z)\right| \approx \frac{\pi\rho}{\sin\pi\rho} r^\rho \cos\rho\theta,$$

see the analysis in [79, p. 117] or [97, p. 229], show that the $\cos\pi\rho$ inequality is sharp for $0 < \rho < 1$. For $\rho = 0$ the inequality is obviously sharp, since $L \leqslant M$.

Kjellberg [90], proved a significantly stronger theorem: Let $f$ be entire, order unspecified. Then for each $\lambda \in (0, 1)$ either $\log L(r) > \cos\pi\lambda \log M(r)$ holds for a sequence of $r$ tending to infinity, or else $\lim_{r\to\infty} r^{-\lambda}\log M(r)$ exists, and is positive or infinite. If $f$ has order $\rho < 1$, we recover (9.2) by taking $\lambda$ slightly larger than $\rho$.

A. Weitsman conjectured that Kjellberg's theorem could be "localized". His conjecture was confirmed in [9], which contains the following result.

THEOREM 9.1. *Let $f$ be entire, and $\lambda$ and $\beta$ be numbers with $0 < \lambda < \infty$, $0 < \beta \leqslant \pi$, $\beta\lambda < \pi$. Then either there exists a sequence of r tending to infinity for which the set*

$$\left\{\theta \in [-\pi,\pi]\colon\ \log\left|f\left(re^{i\theta}\right)\right| > \cos\beta\lambda \log M(r)\right\}$$

*contains an interval of length at least $2\beta$, or else $\lim_{r\to\infty} r^{-\lambda}\log M(r)$ exists, and is positive or infinite.*

Theorem 9.1 is sometimes called the "$\cos\beta\lambda$ theorem". For $\beta = \pi$, the $\cos\beta\lambda$ theorem reduces to Kjellberg's theorem.

The results above are still valid, with obvious changes, when $\log|f|$ is replaced by an arbitrary subharmonic function $u$ in the plane. We'll conduct the rest of the discussion in the subharmonic context.

To prove the subharmonic version of Theorem 9.1, one introduces, for given subharmonic $u$ in $\mathbb{C}$, two new functions of *-function type, which we'll call $U_1$ and $U_2$. The domain of $U_1$ is the closed upper halfplane, with the origin deleted, while $U_2$, which depends on the parameter $\beta \in (0, \pi]$, is defined in the angle $0 \leqslant \arg z \leqslant \frac{1}{2}\beta$, except at the origin.

$$U_1(re^{i\theta}) \equiv \sup_I \int_I u(re^{it})\,dt, \quad 0 \leqslant \theta \leqslant \pi,\ 0 < r < \infty,$$

where the sup is over all arcs $I$ on the unit circle with $|I| = 2\theta$.

$$U_2(re^{i\theta}) \equiv \sup_E \int_E u(re^{it})\,dt, \quad 0 \leqslant \theta \leqslant \frac{1}{2}\beta,\ 0 < r < \infty,$$

where the sup is over all sets $E$ of the form $E = [a_1, b_1] \cup [a_2, b_2] \cup [a_3, b_3]$ with $a_1 \leqslant b_1 \leqslant a_2 \leqslant b_2 \leqslant a_3 \leqslant b_3 \leqslant a_1 + 2\pi$, and

$$b_2 - a_2 = 2\theta, \qquad b_1 - a_1 + b_3 - a_3 = 2\theta, \qquad a_2 - b_1 = a_3 - b_2 = \beta - 2\theta.$$

The functions $U_1$ and $U_2$ turn out to be subharmonic in the interiors of their respective domains. For each of $U_1$ and $U_2$ and for each $r > 0$ there exist competing sets for which the sups are attained. The maximal intervals $I$ of length $2\beta$ for $U_1$ furnish candidates, some ultimately successful, for the intervals in the conclusion of Theorem 9.1. The role of the curious looking $U_2$ is less easily described. Suffice it to say that its subharmonicity and various other properties enable one to localize some of Kjellberg's arguments involving integrals and harmonic functions from the case $\beta = \pi$ to the case $\beta \in (0, \pi)$.

Further work related to $\cos\pi\rho$ and $\cos\beta\lambda$ theorems includes [107,53,57] and [66].

Fix again $\beta \in (0, \pi]$. For a real function $u$ defined in the annulus $A(R_1, R_2)$, define a function $U_3$, which depends on $\beta$, in the sector $A(R_1, R_2, \beta) = \{re^{i\theta} \in A(R_1, R_2)\colon 0 \leqslant \theta \leqslant \beta\}$ by

$$U_3(re^{i\theta}) = \sup_E \int_E u(re^{it})\,dt,$$

where the sup is over all $E \subset [-\pi, \pi]$ with $|E| = 2\theta$ such that $E$ is contained in a circular arc of length $2\beta$. One may call $U_3$ the *longest arc *-function*. It satisfies

$$U_3(re^{i\beta}) = U_1(re^{i\beta}), \qquad U_3(e^{2i\theta}) \geqslant U_2(re^{i\theta}).$$

If $u$ is subharmonic in $A(R_1, R_2)$, then $U_3$ is subharmonic in $A(R_1, R_2, \beta)$.

The longest arc *-function was introduced independently by A.E. Fryntov [64] and the author [12] to prove a conjecture of B.Ya. Levin. The application to Levin's conjecture in fact involved a horizontal version of $U_3$, in which one starts with a function $u$ in a rectangle $R = (x_1, x_2) \times (y_1, y_2)$ with $x_2 - x_1 \geqslant 2\beta$ and forms a new function in $R' \equiv (0, \beta) \times (y_1, y_2)$, call it $U_4(x + \mathrm{i}y)$, by maximizing integrals over sets $E$ on the horizontal line through $y$ which are contained in some interval of length $2\beta$ and have measure $2x$. In contrast with the circular case, sets $[x_1, a] \cup [b, x_2]$, where $x_1 < a < b < x_2$, are not considered to be single intervals.

If $u$ is subharmonic in $R$, then $U_4$ is not always subharmonic in $R'$. It will be subharmonic, however, if we impose some extra conditions on $u$ which prevent the maximal sets for $U_4$ from hitting the horizontal boundary of $R$.

Let $K$ be a closed subset of the real axis which is "$\delta$-dense" in the sense that $|K \cap I| \geqslant \delta$ for every interval $I \subset \mathbb{R}$ of length 1. Here $0 < \delta < 1$. There is a unique function $u = u(\cdot, K)$ which is continuous and symmetric with respect to the real axis in $\mathbb{C}$, is harmonic off $K$, vanishes on $K$, and satisfies $u(x + \mathrm{i}y) \sim y$ for each fixed $x$ as $y \to \infty$. Levin conjectured that among all $\delta$-dense sets $K$, the $u$ with maximal supremum on the real axis should come from

$$K_\delta = \bigcup_{n=-\infty}^{\infty} \left[n - \frac{1}{2}\delta, n + \frac{1}{2}\delta\right].$$

For $B > 0$, let $S(B)$ denote the horizontal strip $|y| < B$. Set

$$u(z, K, B) = \omega\bigl(z, \partial S(B), S(B) \setminus K\bigr),$$

where $\omega$ denotes harmonic measure. Then $Bu(z, K, B) \sim u(z, K)$ as $B \to \infty$. Thus, Levin's conjecture is a corollary of the following theorem.

THEOREM 9.2 ([64,12]). *If $K \subset \mathbb{R}$ is $\delta$-dense, then, for each convex increasing $\Phi : \mathbb{R} \to \mathbb{R}$, each $y \in [-B, B]$, each $0 < B < \infty$, and each interval $I \subset \mathbb{R}$ of length 1 holds*

$$\int_I \Phi\bigl(u(x + \mathrm{i}y, K, B)\bigr)\,dx \leqslant \int_I \Phi\bigl(u(x + \mathrm{i}y, K_\delta, B)\bigr)\,dx.$$

The proof follows the same general scheme as those in Section 6, the key element being subharmonicity of the $U_4$ corresponding to $u(\cdot, K, B)$. A feature not encountered in the earlier theorems is that here we must compare a generally non-periodic $K$ with the periodic set $K_\delta$.

Fryntov [65] applied the circular longest-arc *-function to prove a conjecture of Weitsman about Green functions. See also [68,69]. In [67], Fryntov proved a conjecture of Velling involving longest arcs and harmonic measure, but here his main tool is the clever construction, by a glueing procedure, of an auxiliary subharmonic function quite different from the *-functions.

## 10. The spread relation

As noted in Section 1, the spread relation provided an opportunity to discover the *-function. To state the spread relation, we must introduce some notations from Nevanlinna's theory of value distribution of meromorphic functions. See, for example, [79] or [97] for a detailed account of this theory, and [32] for some subsequent developments. Let $f$ be meromorphic and nonconstant in $\mathbb{C}$. For $r \in (0, \infty)$ and $w \in \mathbb{C} \cup \{\infty\}$, the weighted counting function $N(r, a, f)$ was defined in (4.1) by

$$N(r, a, f) = \sum_{z \in f^{-1}(a)} \log^+ \frac{r}{|z|},$$

with the convention there that $N(r, a, f) = \infty$ for $a = f(0)$. In this section we adopt a different convention: For $a = f(0)$, $N(r, a, f)$ equals the above sum taken over all nonzero $z \in f^{-1}(a)$.

The proximity function $m(r, \infty, f)$ is defined by

$$m(r, \infty, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ \left| f\left(re^{i\theta}\right)\right| d\theta,$$

while for $a \in \mathbb{C}$, $m(r, a, f) \equiv m(r, 1/(f - a), \infty)$. The Nevanlinna characteristic $T(r, f)$ is defined by

$$T(r, f) = m(r, \infty, f) + N(r, \infty, f),$$

and the deficiency $\delta(a, f)$ of the value $a \in \mathbb{C} \cup \{\infty\}$ by

$$\delta(a, f) = \liminf_{r \to \infty} \frac{m(r, a, f)}{T(r, f)} = 1 - \limsup_{r \to \infty} \frac{N(r, a, f)}{T(r, f)}. \tag{10.1}$$

It turns out that $T(r, f)$ is an increasing unbounded function of $r$. The equality of the second and third expressions in (10.1) is a consequence of Nevanlinna's First Fundamental Theorem, which implies that $m(r, a, f) + N(r, a, f) = T(r, f) + O(1)$ as $r \to \infty$ for each $a \in \mathbb{C}$. Nevanlinna's Second Fundamental Theorem implies his *deficiency relation*:

$$\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta(a, f) \leqslant 2. \tag{10.2}$$

If $f$ omits $a$ then $\delta(a, f) = 1$. Thus, the deficiency relation is a generalization of Picard's theorem, according to which a nonconstant meromorphic function in $\mathbb{C}$ can omit at most two values in $\mathbb{C} \cup \{\infty\}$.

The order $\rho$ of $f$ is defined by

$$\rho = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r}.$$

The lower order $\lambda$ of $f$ is defined by replacing lim sup by lim inf.

Edrei [41], [43, p. 155] proved that if $0 \leqslant s < \infty$ with $\lambda(f) \leqslant s \leqslant \rho(f)$, then there exist sequences $\{r_n\}_{n=1}^{\infty}, \{A_n\}_{n=1}^{\infty}, \{\varepsilon_n\}_{n=1}^{\infty}$ with $r_n \nearrow \infty$, $A_n \nearrow \infty$ and $\varepsilon_n \searrow 0$ such that for $n \geqslant 1$ holds

$$T(r, f) \leqslant (1+\varepsilon_n)(r/r_n)^s T(r_n, f), \quad {A_n}^{-1} r_n \leqslant r \leqslant A_n r_n.$$

Edrei called such $\{r_n\}$ *Pólya peaks* of $T(r, f)$ of order $s$. They have become a valuable tool in the study of asymptotic problems for entire and meromorphic functions. See [33] for more about Pólya peaks.

If $\delta(a, f) > 0$, then from (10.1) one sees that $m(r, a, f)$ must be large for large $r$, which implies that $f$ must often be close to $a$ on large circles $|z| = r$. How often? The spread relation provides a sharp answer for functions of finite lower order. We state the result for $a = \infty$.

THEOREM 10.1 (Spread relation). *Let $f$ be meromorphic in $\mathbb{C}$ with lower order $\lambda \in (0, \infty)$, $\{r_n\}$ be a sequence of Pólya peaks of $T(r, f)$ of order $\lambda$, and $\{\eta_n\}$ be a sequence of nonnegative numbers with* $\lim_{n\to\infty} \eta_n = 0$. *Then*

$$\liminf_{n\to\infty} \left|\left\{\theta \in [-\pi, \pi]\colon \log\left|f\left(r_n e^{i\theta}\right)\right| > \eta_n T(r_n, f)\right\}\right| \geqslant \min\left(\frac{4}{\lambda} \sin^{-1}\left(\left(\frac{\delta(\infty, f)}{2}\right)^{1/2}\right), 2\pi\right).$$

The spread relation was conjectured by Edrei [42]. A weaker form had been conjectured by Teichmüller [117]. In [41, p. 83], Edrei had proved his conjecture when the minimum on the right is $2\pi$. The general conjecture was proved in [7].

Theorem 10.1 is still true, of course, if $\delta(\infty, f)$ is replaced by $\delta(a, f)$ for any $a \in \mathbb{C}$. If the $\eta_n$ are chosen so that $\lim_{n\to\infty} \eta_n T(r_n, f) = \infty$, then for distinct values of $a$ the sets of $\theta$ where $\log|f(r_n e^{i\theta}) - a| < -\eta_n T(r_n, f)$ are disjoint for large $n$. This enabled Edrei [44] to deduce the following corollary, which provides a sharp form of the deficiency relation for functions of lower order $\leqslant 1$.

COROLLARY. *Let $f$ be a meromorphic function of lower order $\lambda \in (0, 1)$. Then*

$$\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta(a, f) \leqslant \begin{cases} 1, & 0 < \lambda \leqslant \frac{1}{2}, \\ 2 - \sin \pi\lambda, & \frac{1}{2} \leqslant \lambda < 1. \end{cases} \tag{10.3}$$

For $0 < \lambda \leqslant \frac{1}{2}$, (10.3) had already been proved in [41]. For $\lambda = 0$ there is no spread relation, but, as noted in [41, p. 85], a result of Edrei and Fuchs [45, p. 297] implies that a meromorphic function of lower order zero can have at most one deficient value, so that the sum of its deficiencies is also $\leqslant 1$. For $\lambda = 1$, there are many functions, such as $e^z$, for which the sum of the deficiencies equals two.

For $0 \leqslant \lambda \leqslant \frac{1}{2}$, equality in (10.3) is achieved by each entire function of lower order $\lambda$. For $\lambda \in [\frac{1}{2}, 1)$, the Lindelöf function $f_\lambda$ of order $\lambda$, introduced in Section 9, satisfies

$$\delta(\infty, f_\lambda) = 1, \qquad \delta(0, f) = 1 - \sin \pi\lambda.$$

Thus, (10.3) is sharp for each $\lambda \in [0, 1]$.

The spread relation itself is sharp. Meromorphic extremal functions involving products and quotients of Lindelöf functions, due to Edrei, are presented in [7]. Below, we shall construct extremal $\delta$-subharmonic functions. By approximation, see, e.g., [81, p. 834], meromorphic extremals can be constructed from the $\delta$-subharmonic ones.

For meromorphic $f$ with finite lower order $\lambda > 1$, it is known [31], see also [49,50], that the sum of the deficiencies is strictly less than two unless $\lambda$ is an integer or half-integer. It is an open problem to find the precise upper bound for the sum in terms of $\lambda$, when $\lambda > 1$ is not an integer or half-integer. Some non-sharp inequalities, and conjectures for sharp ones, are in [34]. See also [35].

The spread relation is really a theorem about $\delta$-subharmonic functions in $\mathbb{C}$. For a function $u$ defined on a circle $|z| = r$, write

$$N(r, u) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u\left(re^{i\theta}\right) d\theta,$$

for the mean value of $u$ on the circle, and for a function $u = u_1 - u_2$ with $u_1$ and $u_2$ subharmonic in $\mathbb{C}$, define the Nevanlinna characteristic of $u$ to be

$$T(r, u) = N(r, u^+) + N(r, u_2).$$

If $f$ is meromorphic in $\mathbb{C}$ then $u(z) = \log|f|$ is $\delta$-s.h. in $\mathbb{C}$. Via Jensen's formula, the characteristics $T(r, f)$ and $T(r, u)$ are seen to differ at most by a factor of $\log r$, which is negligible when considering asymptotic behavior of transcendental meromorphic functions. The evident extension of Theorem 10.1 to general $\delta$-s.h. functions in $\mathbb{C}$ is true. In fact, the asymptotic result Theorem 10.1 can be deduced from a corresponding non-asymptotic theorem. The general idea of the reduction is to consider the sequence $u_n(z) \equiv u(zr_n)/T(r_n, u)$, where $\{r_n\}$ is a Pólya peak sequence for $T(r, u)$. A theory of normal families exists for $\delta$-s.h. functions, see [5] or [6], from which one finds that if $u$ satisfies the hypotheses of the subharmonic version of Theorem 10.1, then a subsequence of $\{u_n\}$ converges in an appropriate sense to a $\delta$-s.h. function $u_\infty$, which satisfies the hypotheses of Theorem 10.2 below. The conclusion of Theorem 10.2 for $u_\infty$ implies the conclusion of Theorem 10.1 for $u$. Papers in which such arguments are applied to related problems include [5,49,52,50,53]. In particular, [52] contains a proof of the second fundamental theorem for meromorphic functions based on a non-asymptotic version for subharmonic functions followed by a subharmonic normal families argument.

THEOREM 10.2 (Non-asymptotic spread relation). *Suppose that $u = u_1 - u_2$ is $\delta$-subharmonic in $\mathbb{C}$, that $0 < \delta \leqslant 1$, $0 < \lambda < \infty$, and that*

$$T(r, u) \leqslant r^\lambda, \quad 0 < r < \infty, \tag{10.4a}$$

$$T(1,u)=1, \tag{10.4b}$$

$$N(r,u_2)\leqslant(1-\delta)T(r,u), \quad 0<r<\infty. \tag{10.4c}$$

*Then*

$$\left|\left\{\theta\in[0,2\pi]:\ u\left(e^{i\theta}\right)>0\right\}\right|\geqslant\min\left(\frac{4}{\lambda}\sin^{-1}\left(\left(\frac{\delta}{2}\right)^{1/2}\right),2\pi\right). \tag{10.5}$$

Before proving Theorem 10.2, we first construct a two-parameter family of functions which will furnish extremals. Let $\delta\in(0,1]$, $\lambda\in(0,\infty)$. Set

$$\beta=\frac{2}{\lambda}\sin^{-1}\left(\left(\frac{\delta}{2}\right)^{1/2}\right),\quad \text{i.e.,}\quad 1-\delta=\cos\beta\lambda. \tag{10.6}$$

Then $\beta>0$ and $0<\beta\lambda\leqslant\frac{\pi}{2}$. We assume also that $\beta\leqslant\pi$. For $z=re^{i\theta}$, define

$$v(z)=\begin{cases}\pi\lambda r^\lambda\sin\lambda(\beta-|\theta|), & 0\leqslant|\theta|\leqslant\beta,\\ 0, & \beta\leqslant|\theta|\leqslant\pi.\end{cases}$$

Assume also that $\lambda$ is not an integer. Then $v=v_1-v_2$, where $v_2$ and $v_1$ are defined by

$$v_2(z)=Ar^\lambda\cos\lambda(\pi-|\theta|),$$

$$v_1(z)=\begin{cases}Br^\lambda\cos\lambda\theta, & 0\leqslant|\theta|\leqslant\beta,\\ v_2(z), & \beta\leqslant|\theta|\leqslant\pi,\end{cases}$$

with

$$A=\pi\lambda\frac{\cos\beta\lambda}{\sin\pi\lambda},\qquad B=\pi\lambda\frac{\cos\lambda(\pi-\beta)}{\sin\pi\lambda}.$$

Now $v$ is an even function of $\theta$ which is harmonic in $0<\arg z<\beta$ and is zero in $\beta\leqslant\arg z\leqslant\pi$. Moreover, since

$$\lim_{\theta\to\beta-}\frac{v(re^{i\theta})}{\theta-\beta}=-\pi\lambda^2r^\lambda<0\quad\text{and}$$

$$\lim_{\theta\to0+}\frac{v(re^{i\theta})-v(r)}{\theta}=-\pi\lambda^2r^\lambda\sin\beta\lambda\leqslant0,$$

it follows that $v$ is subharmonic in $0<\arg z<\pi$ and is superharmonic in $|\arg z|<\beta$. Also, $v_2$ is harmonic in $\mathbb{C}\setminus\{\arg z=0\}$ and $v_1$ is harmonic in $\mathbb{C}\setminus\{\arg z=\pm\beta\}$. Since $v=v_1-v_2$ in $\mathbb{C}$, we deduce that $v_1$ and $v_2$ are each subharmonic in $\mathbb{C}$, so that $v$ is $\delta$-s.h. in $\mathbb{C}$. Calculation gives

$$N(r,v_2)=\frac{1}{2\pi}\int_{-\pi}^{\pi}v_2\left(re^{i\theta}\right)d\theta=r^\lambda\cos\beta\lambda,$$

$$T(r,v)=\frac{1}{2\pi}\int_{-\beta}^{\beta} v\big(re^{i\theta}\big)\,d\theta+N(r,v_2)=r^{\lambda}. \tag{10.7}$$

Thus,

$$\frac{N(r,v_2)}{T(r,v)}=1-\delta,\quad 0<r<\infty, \tag{10.8}$$

and

$$\big|\{\theta\in[-\pi,\pi]:\ v(e^{i\theta})>0\}\big|=2\beta=\frac{4}{\lambda}\sin^{-1}\left(\left(\frac{\delta}{2}\right)^{1/2}\right). \tag{10.9}$$

When $\lambda$ is an integer, $v$ is still $\delta$-s.h. in $\mathbb{C}$ and has a decomposition $v=v_1-v_2$ for which (10.7)–(10.9) hold. To obtain $v_1$ and $v_2$, take $\alpha=0, a=\beta$ in [5, p. 525].

The functions $v$ constructed above satisfy (10.4a)–(10.4c) and (10.5) with equality. They show that the spread relation is sharp for each $\delta\in(0,1]$ and $\lambda\in(0,\infty)$ such that the corresponding $\beta$ is $\leqslant\pi$. If $\beta\geqslant\pi$ then it's obvious that the spread relation cannot be improved. But to prove (10.5), we will again need, for each given $\lambda$ and $\delta$, $\delta$-s.h. functions $v$ which satisfy (10.4a)–(10.4c) with equality. For $\beta\geqslant\pi$ this is easy: Set $v=v_1-v_2$, where

$$v_1\big(re^{i\theta}\big)=r^{\lambda}\cos\lambda\theta,\qquad v_2\big(re^{i\theta}\big)=(1-\delta)r^{\lambda}\cos\lambda(\pi-|\theta|).$$

From $\beta\geqslant\pi$ and $\beta\lambda\leqslant\frac{\pi}{2}$ we see that $\lambda\leqslant\frac{1}{2}$. The reader may also verify that $\beta\geqslant\pi\Rightarrow v>0$ for $r>0$, $|\theta|<\pi$, and that $v$ satisfies (10.4a)–(10.4c) with equality.

SKETCH OF PROOF OF THEOREM 10.2. Let $0<\delta\leqslant 1$ and $\lambda\in(0,\infty)$ be given. Define $\beta$ by (10.6). We need to show that $|\{\theta\in[0,2\pi]:\ u(e^{i\theta})>0\}|\geqslant\min(2\beta,2\pi)$. First, we'll sketch the proof when $\beta\leqslant\pi$.

According to Theorem 3.2, $u^{\#}(re^{i\theta})=u^{*}(re^{i\theta})+2\pi N(r,u_2)$ is subharmonic in the upper half plane. Let $v$ be the function constructed above corresponding to $\delta$ and $\lambda$. Define

$$\begin{aligned}V\big(re^{i\theta}\big)&=\int_{-\theta}^{\theta} v\big(re^{it}\big)\,dt+2\pi N(r,v_2)\\&=\int_{-\theta}^{\theta} v\big(re^{it}\big)\,dt+\int_{|\theta|\leqslant t\leqslant\pi} v_2\big(e^{it}\big)\,dt.\end{aligned} \tag{10.10}$$

Let $S$ denote the sector $0<\arg z<\beta$. Since $v_1$ is harmonic in $S$ and $v_2$ is harmonic in the plane minus the positive real axis, it follows from Proposition 3.2 that V is harmonic in $S$. From their definitions, $v$ and $V$ are nonnegative in $\mathbb{C}$. From the relation $\sup_{\theta\in[-\pi,\pi]}u^{\#}(re^{i\theta})=2\pi T(r,u)$ and (10.4a), (10.7), we see that $u^{\#}\leqslant V$ on $\partial S$ and that $u^{\#}(z)=O(r^{\lambda})$ as $z=re^{i\theta}\to\infty$ in $S$. The total angular width of $S$ is $\beta$, so the Phragmén–Lindelöf principle will apply if $\beta\lambda<\pi$. But in fact, $\beta\lambda\leqslant\frac{\pi}{2}$. Thus, we conclude that

$$u^{\#}\leqslant V\quad\text{in } S. \tag{10.11}$$

By Proposition 2.2,

$$u^*(re^{i\theta}) = \int_{-\theta}^{\theta} \tilde{u}(re^{it})\,dt, \tag{10.12}$$

where $\tilde{u}$ denotes the symmetric decreasing rearrangement of $u$ on circles. Since

$$\left|\{\theta \in [0,2\pi]:\ u(e^{i\theta}) > 0\}\right| = \left|\{\theta \in [0,2\pi]:\ \tilde{u}(e^{i\theta}) > 0\}\right|$$

and $\tilde{u}$ is symmetric decreasing, if $\tilde{u}(e^{i\theta}) > 0$ for every $\theta \in (0,\beta)$, then (10.5) is true.

Suppose that $\tilde{u}(e^{i\theta}) \leqslant 0$ for some $\theta \in (0,\beta)$. Let $\theta_0$ be the infimum of all such $\theta$. From (10.4b), (10.12) and (10.11), it follows that

$$1 = T(1,u) = u^{\#}(e^{i\theta_0}) \leqslant V(e^{i\theta_0}).$$

But $v > 0$ on $|\theta| < \beta$, so from (10.10) we see that $\theta_0 < \beta$ implies

$$V(e^{i\theta_0}) < V(e^{i\beta}) = T(1,v) = 1.$$

This contradiction shows that $\tilde{u}(e^{i\theta}) > 0$ for each $\theta \in (0,\beta)$. Thus, (10.5) is true when $\beta \leqslant \pi$.

If $\delta \in (0,1]$ and $\lambda \in (0,\infty)$ are such that $\beta \geqslant \pi$, then the argument just given, using the $v$ corresponding to $\delta$ and $\lambda$, shows that $\tilde{u}(e^{i\theta}) > 0$ for all $\theta \in [0,\pi)$, which implies $|\{\theta \in [0,2\pi]:\ u(e^{i\theta}) > 0\}| = 2\pi$. □

Functions for which equality holds in Theorem 10.1 are studied in [10,47,48]. In [5, 111,59], one finds results about "$\alpha$-spreads", that is, about the size of the sets of $\theta$ where $u(re^{i\theta}) > \alpha T(r,u)$, for real constants $\alpha$. [75] contains an application of the spread relation to ordinary differential equations.

## 11. Paley's conjecture

In addition to the spread relation and $\cos\pi\rho$-type theorems, the *-function is well-suited to attack other problems about growth of subharmonic, $\delta$-subharmonic, entire and meromorphic functions. To illustrate, we sketch a proof of a conjecture of Paley from 1932. Paley's conjecture was first proved by Govorov [73].

THEOREM 11.1 [73]. *Let $f$ be an entire function of order $\rho \in [0,\infty)$. Then*

$$\liminf_{r\to\infty} \frac{\log M(r,f)}{T(r,f)} \leqslant \begin{cases} \dfrac{\pi\rho}{\sin\pi\rho}, & 0 \leqslant \rho \leqslant \dfrac{1}{2}, \\ \pi\rho, & \rho \geqslant \dfrac{1}{2}. \end{cases} \tag{11.1}$$

The inequality is sharp: Equality holds for Lindelöf functions $f_\rho(z) = \prod_{n=1}^{\infty}(1 + zn^{-1/\rho})$ for $0 < \rho < 1$, and for Mittag-Leffler functions

$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{\Gamma(1 + (n/\rho))}$$

when $0 < \rho < \infty$. The asymptotic behavior of the Lindelöf functions was stated in Section 9; for the Mittag-Leffler functions, see [77, p. 197]. Since $\log M(r) \geqslant T(r)$, every entire function of order zero is extremal for $\rho = 0$.

Theorem 11.1 is still true when order is replaced by lower order, and when $\log|f|$ is replaced by an arbitrary subharmonic function $u$ in $\mathbb{C}$. Moreover, inequality (11.1) is achieved when $r$ runs through a Pólya peak sequence for $T(r, f)$. For positive lower orders, these statements follow from Theorem 11.2 below in the same way that the asymptotic spread relation Theorem 10.1 follows from the non-asymptotic spread relation Theorem 10.2. In the deduction of Theorem 11.1 from Theorem 11.2, the function $u$ will satisfy $T(1, u) = 1$ in addition to (11.2).

THEOREM 11.2 (Non-asymptotic version of Govorov's theorem). *Suppose that $u$ is subharmonic in $\mathbb{C}$, that $0 < \lambda < \infty$, and that*

$$T(r, u) \leqslant r^\lambda, \quad 0 < r < \infty. \tag{11.2}$$

*Then, for* $0 < r < \infty$,

$$M(r, u) \leqslant \begin{cases} \frac{\pi\lambda}{\sin \pi\lambda} r^\lambda, & 0 < \lambda \leqslant \frac{1}{2}, \\ \pi\lambda r^\lambda, & \lambda \geqslant \frac{1}{2}. \end{cases} \tag{11.3}$$

We remind the reader that for subharmonic $u$, $T(r, u) \equiv N(r, u^+)$, the mean value of $u^+$ on $|z| = r$, and $M(r, u) \equiv \max_{\theta \in [-\pi, \pi]} u(re^{i\theta})$. For $0 < \lambda \leqslant 1/2$, extremals for Theorem 11.2 are furnished by $u(re^{i\theta}) = \frac{\pi\lambda}{\sin\pi\lambda} r^\lambda \cos\lambda\theta$, $|\theta| \leqslant \pi$, and for $1/2 \leqslant \lambda < \infty$, by

$$v(re^{i\theta}) = \begin{cases} \pi\lambda r^\lambda \cos\lambda\theta, & |\theta| \leqslant \frac{\pi}{2\lambda}, \\ 0, & \frac{\pi}{2\lambda} \leqslant |\theta| \leqslant \pi. \end{cases} \tag{11.4}$$

SKETCH OF PROOF OF THEOREM 11.2. The proof is like that of Theorem 10.2, but simpler. Suppose first that $\frac{1}{2} \leqslant \lambda < \infty$. Let $v$ be the function in (11.4) corresponding to $\lambda$. Define $V$ in the upper half plane by $V = Jv$, so that

$$V(re^{i\theta}) = \int_{-\theta}^{\theta} v(re^{it})\, dt.$$

Then $V$ is harmonic in the sector $S = \{z \in \mathbb{C}\colon\ 0 < \arg z < \frac{\pi}{2\lambda}\}$. By Theorem 3.1, $u^*$ is subharmonic in the upper half plane. Thus, $p \equiv u^* - V$ is subharmonic in $S$. From (11.2) and the definition of $v$, it follows that $p(re^{i\pi/2\lambda}) \leqslant 0$, while $u^*(r) = V(r) = 0$. Thus $p \leqslant 0$

on $\partial S$. Moreover, (11.2) implies also that $p(z) = O(|z|^\lambda)$ at $\infty$ in $S$. So, by Phragmén–Lindelöf, $u^* \leqslant V$ in $S$. Using again that $u^*(r) = V(r) = 0$, it follows that

$$\frac{\partial u^*}{\partial \theta}(r) \leqslant V_\theta(r) = 2v(r) = 2\pi\lambda r^\lambda, \quad 0 < r < \infty.$$

By Proposition 2.3(b), $\frac{\partial u^*}{\partial \theta}(r) = 2M(r,u)$. Conclusion (11.3) follows when $1/2 \leqslant \lambda < \infty$. When $0 < \lambda \leqslant 1/2$, the same argument works if we replace $S$ by the upper half plane and in the definition of $V$ replace $v(re^{it})$ by $\frac{\pi\lambda}{\sin\pi\lambda} r^\lambda \cos\lambda t$. □

Petrenko [99] gave an extension of Govorov's Theorem to meromorphic functions, which was in turn extended by Shea (see [70]). Rossi and Weitsman [108], see also [81], used *-functions in a different way from those sketched above to prove Petrenko-type theorems and other results about growth and distribution of values such as the Edrei–Fuchs "ellipse theorem" [46] and the spread relation. Other papers with applications of *-functions to growth and value distribution problems include [58] and [59].

## 12. Symmetrization and the hyperbolic metric

Let $\Omega$ be a domain in $\mathbb{C}$ such that $\mathbb{C} \setminus \Omega$ contains at least two points. Then $\Omega$ possesses a hyperbolic metric (= Poincaré metric) $\rho = \rho_\Omega : \Omega \to (0,\infty)$, of constant Gaussian curvature $-1$. It may be defined by

$$\rho(z) = 2\big((1-|w|^2)|F'(w)|\big)^{-1},$$

where $F$ is a universal covering map of $\mathbb{D}$ onto $\Omega$ and $w \in \mathbb{D}$ is any point with $F(w) = z$. The function $\rho$ belongs to $C^\infty(\Omega)$, and satisfies the pde

$$\Delta \log \rho = \rho^2 \tag{12.1}$$

in $\Omega$. It also satisfies the boundary condition

$$\lim_{z\to\zeta} \rho(z) = \infty, \quad \zeta \in \partial\Omega \cap \mathbb{C}. \tag{12.2}$$

$\rho$ may be characterized as the maximal solution of (12.1) in $\Omega$ [84], [81, Theorem 9.11]. Strictly speaking, $\rho$ should be called the density of the hyperbolic metric $\rho(z)|dz|$ on $\Omega$, but we shall opt for brevity and identify the metric with its density. Background on hyperbolic metrics may be found, for example, in [81].

Let, as in Section 6, $\Omega^*$ denote the circular symmetrization of $\Omega$. The main aim of this section is to state and partially prove the following theorem of Weitsman [120]. Set

$$u_\Omega = -\log \rho_\Omega, \qquad \Omega(r) = \{\theta \in [-\pi,\pi] : re^{i\theta} \in \Omega\}.$$

THEOREM 12.1. *Let $\Omega \subset \mathbb{C}$ be a domain such that $\Omega$ and $\Omega^*$ each have at least two complementary points in $\mathbb{C}$. Then for each convex increasing $\Phi : \mathbb{R} \to \mathbb{R}$ and each $r \in (0, \infty)$ such that $\Omega(r)$ is nonempty holds*

$$\int_{\Omega(r)} \Phi\big(u_\Omega(re^{i\theta})\big)\, d\theta \leqslant \int_{\Omega^*(r)} \Phi\big(u_{\Omega^*}(re^{i\theta})\big)\, d\theta. \tag{12.3}$$

Since $\rho_\Omega$ and $\rho_{\Omega^*}$ are bounded away from zero on $\Omega(r)$ and $\Omega^*(r)$ respectively, the integrals in (12.3) exist, but possibly equal $-\infty$. By Proposition 2.4, (12.3) can be restated in terms of *- functions as

$$u_\Omega{}^*(z) \leqslant u_{\Omega^*}{}^*(z), \quad z \in \Omega^{*+} \equiv \Omega^* \cap \mathbb{H}. \tag{12.4}$$

Here are two corollaries of Theorem 12.1. In each case $\Omega$ and $\Omega^*$ are as in the statement of Theorem 12.1.

COROLLARY 1. *For each $r \in (0, \infty)$ such that $\Omega(r)$ is nonempty holds*

$$\min_{z\in\Omega(r)} \rho_\Omega(z) \leqslant \min_{z\in\Omega^*(r)} \rho_{\Omega^*}(z) = \rho_{\Omega^*}(r).$$

The inequality follows from (12.4) and Proposition 2.5. The equality follows from another theorem of Weitsman [119], which asserts that $\rho_{\Omega^*}$ is symmetric increasing on each $\Omega^*(r)$. The symmetric increase of $\rho_{\Omega^*}$ can also be obtained by taking $\Omega = \Omega^*$ in the proof of Theorem 12.1 to be discussed below. The argument in that proof produces the inequality

$$u_{\Omega^*}{}^*\big(re^{i\theta}\big) \leqslant \int_{-\theta}^{\theta} u_{\Omega^*}\big(re^{it}\big)\, dt, \quad re^{i\theta} \in \Omega^{*+}.$$

The opposite inequality follows from the definition of the *-function. Symmetric decrease of $u_{\Omega^*}$ then follows from the equality of $u_{\Omega^*}{}^*$ with the function on the right hand side.

COROLLARY 2. *Let $f$ be holomorphic in $\mathbb{D}$ with $f(\mathbb{D}) \subset \Omega$, and let $F$ be a universal covering map of $\mathbb{D}$ onto $\Omega^*$ with $F(0) = |f(0)|$. Then*

$$|f'(0)| \leqslant F'(0), \quad \text{and} \quad M(r, f) \leqslant M(r, F), \quad 0 < r < 1.$$

For a deduction of Corollary 2 from Corollary 1, see [120] or [81, p. 700]. When $\Omega^*$ is simply connected, so that $F$ is univalent, Corollary 2 coincides with inequality (6.4). Theorem 6.1 tells us that when $\Omega^*$ is simply connected much more is true: convex integral means of $\log|f|$ on circles $|z| = r$ are $\leqslant$ those of $\log|F|$. It is not known if this is always true when $\Omega^*$ is multiply connected.

To prove Theorem 12.1 we need a variant of Theorem 3.3, which will be called Theorem 3.4. Theorem 3.3 asserts, roughly, that if a function $u$ defined in an annulus satisfies a differential inequality $\Delta u \geqslant -f - \psi(u)$, then $u^*$ satisfies $\Delta u^* \geqslant -f^* - J\psi(\tilde{u})$, where

$Jg(re^{i\theta}) = \int_{-\theta}^{\theta} g(re^{it})\,dt$ and $\tilde{u}$ denotes the symmetric decreasing rearrangement of $u$ on circles. In Theorem 3.4, we shall assume that $u$ is defined in a general plane domain $\Omega$. To obtain subharmonicity-type results when $\Omega$ is not an annulus, we need to make assumptions about the boundary behavior of $u$. A sufficient condition is that $u$ be a constant $C$ on $\partial\Omega$ and $u \geqslant C$ in $\Omega$. More formally, we shall assume that $\Omega$ is bounded and that

$$\lim_{z\to\zeta, z\in\Omega} u(z) = \inf_{\Omega} u, \quad \forall\zeta \in \partial\Omega. \tag{12.5}$$

It is permitted that $\inf_{\Omega} u = -\infty$. If $u \in C(\Omega)$ and $u$ satisfies (12.5), then $\tilde{u}$ is defined in $\Omega^*$, $u^*$ is defined in $\Omega^{*+}$, and we still have $u^* = J\tilde{u}$.

THEOREM 12.2. *Let $\Omega$ be a bounded domain in $\mathbb{C}$ and $\psi \in C(\mathbb{R})$. Suppose that $u \in C^2(\Omega)$ satisfies $\Delta u \geqslant -f - \psi(u)$ in $\Omega$ and the boundary condition* (12.5). *Then*

$$\Delta_1(u^*)(z) \geqslant -f^*(z) - J\psi(\tilde{u})(z), \quad z \in \Omega^{*+}.$$

The generalized Laplace operator $\Delta_1$ was defined in (3.3). An analogue of the Riesz measure inequality (3.4) also holds in the setting of Theorem 3.4. Proofs of Theorem 3.4 are implicit in [120] and [81, §9.6]. The result can also be obtained by keeping track of the terms with $\psi$ and $f$ in Methods 1 and 2 of Section 3. Everything carries over nicely as long as maximal sets for $u^*$ do not hit $\partial\Omega$, and this is a consequence of (12.5).

PARTIAL PROOF OF THEOREM 12.1. Write $\Omega$ as the increasing union of bounded domains $\Omega_n$ such that the closure of $\Omega_n$ is contained in $\Omega_{n+1}$, and write $u_n = u_{\Omega_n}$, $u = u_{\Omega}$. Then $u_n(z) \nearrow u(z)$, for each $z \in \Omega$ [81, (9.4.8) and (9.4.26)]. From Proposition 2.1 and the monotone convergence theorem, it follows that $u^*(z) \leqslant \liminf_{n\to\infty} u_n{}^*(z)$ for each $z \in \Omega^{*+}$. Thus, to prove (12.4), it suffices to prove that (12.4) holds for $z \in \Omega_n{}^{*+}$ when $u$ is replaced by $u_n$.

Write $v = u_{\Omega^*}$, and define $V$ in $\Omega^{*+}$ by $V(re^{i\theta}) = Jv(re^{i\theta}) = \int_{-\theta}^{\theta} v(re^{it})\,dt$. Then $V \leqslant v^*$. For fixed $n$, set $D = \Omega_n{}^{*+}$, $Q = u_n{}^* - V$, and $M = \sup_D Q$. To prove (12.4), it suffices to prove that $M \leqslant 0$. Let $\{z_k\}$ be a sequence in $D$ with $Q(z_k) \to M$ and $z_k$ tending to some point $z_0 \in \overline{D}$ as $k \to \infty$. As in the proofs of Theorems 5.1 and 6.1, one must examine a number of cases, depending on whether $z_0$ is an interior or a boundary point of $D$, and if the latter, the nature of the boundary point. Here, we'll just show that $Q$ cannot achieve a positive maximum at an interior point of $D$, and refer the reader to [120] or [81] for the boundary phase of the proof.

Write $\psi(x) = e^{-2x}$. Each of $u$ and $v$ satisfy the pde $\Delta u = -\psi(u)$. From Proposition 3.1 and Theorem 3.4 with $f = 0$, it follows that for $z = re^{i\theta} \in D$,

$$\Delta_1 Q(z) \geqslant \int_{-\theta}^{\theta} \left[\psi\left(v(re^{it})\right) - \psi\left(\tilde{u}_n(re^{it})\right)\right] dt.$$

Since $\psi$ is convex, we have $\psi(y)-\psi(x)\geqslant\psi'(x)(y-x)$, and hence, writing $F(re^{it})=\psi'(\tilde{u}_n(re^{it}))$,

$$\begin{aligned}\Delta_1 Q(z) &\geqslant \int_{-\theta}^{\theta} F(re^{it})\big[v(re^{it})-\tilde{u}_n(re^{it})\big]\,dt\\ &=2\int_0^{\theta} F(re^{it})\big[v(re^{it})-\tilde{u}_n(re^{it})\big]\,dt\\ &=\int_0^{\theta} F(re^{it})\frac{\partial}{\partial t}\big[Q(re^{i\theta})-Q(re^{it})\big]\,dt\\ &=-F(r)Q(re^{i\theta})-\int_0^{\theta}\big[Q(re^{i\theta})-Q(re^{it})\big]\frac{\partial F}{\partial t}(re^{it})\,dt. \end{aligned}\tag{12.6}$$

Since $\psi$ is convex and strictly decreasing, we have $F(r)<0$ and $\frac{\partial F}{\partial t}\leqslant 0$ for $0\leqslant t\leqslant\theta$. Let $z_0=re^{i\theta}$ be an alleged point at which $Q$ has a positive maximum in $D$. Then $Q(re^{i\theta})>0$ and $Q(re^{i\theta})-Q(re^{it})\geqslant 0$ for $0\leqslant t\leqslant\theta$. From (12.6), it follows that $\Delta_1 Q(z_0)>0$. But this cannot happen at a maximum. We conclude that no such $z_0$ can exist. □

Solynin proves in [114] that the counterpart of Theorem 12.1 for polarization is true. In [115], he proves another interesting comparison theorem for hyperbolic metrics.

Theorem 12.1 is a comparison theorem for solutions of pde's $\Delta u=-e^{-2u}$ under symmetrization. Its proof, with adaptations, can be used to prove comparison theorems in all dimensions under various symmetrizations, such as Steiner, for solutions or subsolutions of linear or semilinear equations $\Delta u=-f-\psi(u)$ under various hypotheses and boundary conditions. Theorem 7 of [15] is one rather general such result. Here we'll state perhaps its simplest special case: a comparison theorem for linear Poisson equations under circular symmetrization in the plane.

THEOREM 12.3. *Let $\Omega$ be a bounded domain in $\mathbb{C}$, $\Omega^*$ denote its circular symmetrization, and $f\in C(\Omega)$. Suppose that $u$ and $v$ are nonnegative $C^2$ solutions of*

$$\begin{aligned}&\Delta u=-f \quad \text{in } \Omega, \qquad u=0 \quad \text{on } \partial\Omega,\\ &\Delta v=-\tilde{f} \quad \text{in } \Omega^*, \qquad v=0 \quad \text{on } \partial\Omega^*.\end{aligned}$$

*Then for each convex increasing $\Phi:\mathbb{R}\to\mathbb{R}$ and each $r\in(0,\infty)$ such that $\Omega(r)$ is nonempty holds*

$$\int_{\Omega(r)}\Phi\big(u(re^{i\theta})\big)\,d\theta\leqslant\int_{\Omega^*(r)}\Phi\big(v(re^{i\theta})\big)\,d\theta.$$

For symmetric decreasing rearrangement in $\mathbb{R}^n$ there are comparison theorems with conclusions stronger than Theorem 12.2, going back to Talenti [116]. See [19,88,15] for discussion and related work.

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CHAPTER 10

# Logarithmic Geometry, Exponentiation, and Coefficient Bounds in the Theory of Univalent Functions and Nonoverlapping Domains

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HANDBOOK OF COMPLEX ANALYSIS: GEOMETRIC FUNCTION THEORY,
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## 1. Introduction

Some results inspired by elementary geometric observations are of great importance in the univalent function theory. They are often expressed or can be naturally restated in terms of compositions of the form $\log \circ \Phi$, and this provides the stage for applications of the exponentiation method. Here $\Phi$ is some nonvanishing operator on a class of univalent functions. Our point is well illustrated by such examples as the Löwner partial differential equation [197], Grunsky univalence criterion [120], Goluzin inequalities [69], Nehari inequalities for bounded functions [219], Bazilevich logarithmic inequality [21, 22], Garabedian–Schiffer inequalities [61], Milin theorem on logarithmic means [204], de Branges theorem on Löwner's chains [32], and some area inequalities of the Lebedev–Milin type for one and several domains ('50s–'80s).

The described group of results, related exponentiation techniques, and applications will be the main theme of our survey. This material occupies a special place in the branch of function theory dealing with logarithmic and exponential structures and can be found in the books by P.L. Duren [49], G.M. Goluzin [72], N.A. Lebedev [181], I.M. Milin [208], and Ch. Pommerenke [230], and in more recent publications, e.g., [8,32,51,52,58,90,91,93,100, 101,107–109,129,138,140,141,156,157,168,211,213,214,263]. Also consult the books by I.A. Aleksandrov [6], G. Sansone and J. Gerretsen [243], and W.K. Hayman [128] for the Löwner method, by A.W. Goodman [76] for special classes of univalent functions, and by S.L. Krushkal and R. Kühnau [158] for sharpened Grunsky and Goluzin inequalities. Other interesting instances of the topic, such as the Baernstein theorem on convex compositions [14], exponential convolutions with integral operators [96] (see also [144,217,241]), and estimates for approximate conformal mapping [110,111] (see also [60,260]), have roots of a different nature and will not be discussed here.

We will begin with the area method and Löwner's parametric method which have proved to be the most productive ones in establishing the basic results of geometric origin for univalent functions on simply connected domains. After a brief discussion of these two methods we turn to some exponentiation approaches (Milin's approach and others) which allow one to use conditions in a logarithmic form for solving extremal problems on traditional classes of univalent functions. Many of these problems are either coefficient ones or can be stated as such (growth and distortion inequalities, estimates for integral means, etc.). The best known problem of this kind is the now-settled Bieberbach conjecture, posed in 1916. We shall show some interesting properties of functions and, more generally, systems of functions, that were established by combining a geometric condition in a logarithmic form (or its consequence) with an exponentiation procedure.

The following notation is used in the sequel. Let $\{f\}_n$ stand for the coefficient of $z^n$ in the Taylor series expansion about $z = 0$ of a function (or formal power series) $f(z)$. The class of functions $w = f(z)$, $f(0) = f'(0) - 1 = 0$, that are analytic and univalent in the unit disk $E = \{z\colon |z| < 1\}$, is traditionally denoted by $S$. For $f \in S$, the Taylor coefficients of the function $\log[f(z)/z]$, i.e., $\{\log[f(z)/z]\}_n$, $n = 1, 2, \ldots$ (we take the (analytic) branch which vanishes at $z = 0$), are called the *logarithmic coefficients* of $f$. An important example of a function in $S$ is the *Koebe function* $K(z) = z/(1 - z)^2 = \sum_{n=1}^{\infty} nz^n$, which maps $E$ onto the plane $\mathbf{C}$ slit along $(-\infty, -1/4]$. Note that $\{K\}_n = n$ and $\{\log[K(z)/z]\}_n = 2/n$, $n = 1, 2, \ldots$. The Koebe function happens to be extremal for many basic functionals on $S$ and so it is of special interest. Sometimes it is preferable to work with the class $\Sigma$ of functions $w = F(z)$ that are analytic and univalent in the domain $U = \{z\colon |z| > 1\}$ exterior to $E$, except for a simple pole at infinity with residue 1, i.e., $\lim F(z)/z = 1$ as $z \to \infty$. For each $f \in S$, the function $F(z) = [f(1/z)]^{-1}$, $z \in U$, belongs to $\Sigma$. This allows one to restate a result established for the class $\Sigma$ in terms of functions in $S$.

With the class $S$ being of the first priority, its subclasses (odd functions, starlike functions, quasiconformally extendible functions, bounded functions, etc.) and associated classes ($\Sigma$, Bieberbach–Eilenberg functions, Gel'fer functions) have been intensively studied for many years. In some cases, as we shall see for the Bieberbach–Eilenberg functions, Gel'fer's ones, and bounded functions, a basic geometric condition can be a natural consequence of more general condition for the *Goluzin–Lebedev class* $\mathcal{M}(a_0, a_1, \dots, a_n)$, i.e., for a collection of all systems of $(n+1)$ functions $\{f_k(z)\}_0^n$, with $f_k(0) = a_k$, that map $E$ conformally and univalently onto nonoverlapping domains.

## 2. Full mappings and slit mappings

Full mappings and slit mappings are two important types of univalent functions. Full mappings include extremal functions for all sharp estimates established by the area method in the classes $S$ and $\Sigma$ (Sections 3 and 6), and slit mappings are crucial for the Löwner parametric method and its applications (Section 5).

A function $F \in \Sigma$ is said to be a *full mapping* if the complement of its image $F(U)$ has zero area (two-dimensional Lebesgue measure). A function $f \in S$ is a *full mapping* if the function $F(z) = [f(1/z)]^{-1}$, $z \in U$, is a full mapping in $\Sigma$. Thus, as one might expect from its name, a full mapping covers almost everything in the target plane. The Koebe function $K(z)$ is an example of a full mapping.

A system of functions $\{f_k(z)\}_0^n \in \mathcal{M}(a_0, a_1, \dots, a_n)$, of which one has a pole in $E$, is called a *full system of mappings* if the complement in $\overline{\mathbf{C}}$ of the union of all images $f_k(E)$ has zero area. A system of analytic functions $\{f_k(z)\}_0^n \in \mathcal{M}(a_0, a_1, \dots, a_n)$ is a *full system of mappings* if the system $\{[(f_k(z) - a_0]^{-1}\}_0^n$ is a full system of mappings. More generally, by a *full system in a (bounded) domain $D$* we mean a full system $\{f_k(z)\}_0^n$ as above but with $\overline{\mathbf{C}}$ replaced by $D$, i.e., $f_k(E) \subset D$ $(k = 0, 1, \dots, n)$ and the area of the complement in $\overline{D}$ of the union of all images $f_k(E)$ is zero. Full systems include extremal ones for all sharp estimates obtained by the area method for the systems of functions in classes $\mathcal{M}(a_0, a_1, \dots, a_n)$ (Sections 3 and 7).

A function which maps $E$ (or $U$) conformally onto the plane slit along a finite number of Jordan arcs is called a *slit mapping*. Thus the slit mappings in $S$ and $\Sigma$ are also full mappings. A slit mapping is called *single-slit* if it omits just one arc. Clearly, the Koebe function is a single-slit mapping. The totality of single-slit mappings in $S$ is dense with respect to the uniform convergence on compact subsets of $E$. This is a key property of slit mappings and its proof based on the Carathéodory convergence theorem (1912) (see [36] and, e.g., [49, Section 3.1] or [72, Chapter 2]) can be found in [49, Section 3.2] or [72, Chapter 3]. A slight refinement of the classical argument shows that something stronger is true (see [100]).

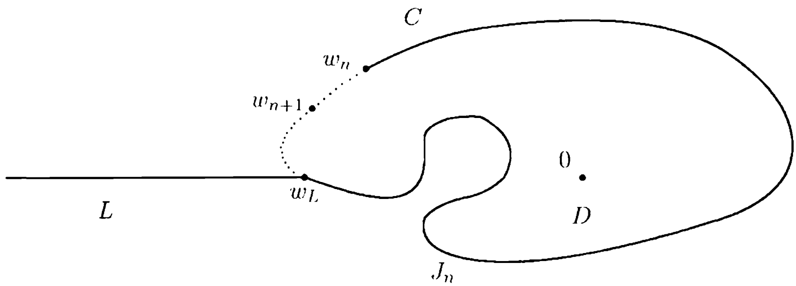


Fig. 1.

LEMMA 1. *To each $f \in S$ there corresponds a sequence of single-slit mappings $f_n \in S$, $n = 1, 2, \ldots$, such that $f_n \to f$ uniformly on compact subsets of $E$ as $n \to \infty$, and the boundary of each $f_n(E)$, $n \geqslant 1$, contains a subray of the negative real axis.*

PROOF. Each function $f \in S$ can be approximated uniformly on closed subdisks of $E$ by the functions $r^{-1} f(rz) \in S$, $0 < r < 1$. Thus, it is sufficient to prove the assertion for a function $f \in S$ that maps $E$ onto a domain $D$ bounded by an analytic Jordan curve $C$. In this case there exists a subray $L$ of the negative real axis that belongs to the complement of $\overline{D}$ except for its endpoint $w_L \in C$. Let $J_n$ be a Jordan arc that runs from infinity along $L$ to the point $w_L$ and then along a portion of $C$ to a point $w_n$ (Figure 1). Let $G_n$ be the complement of $J_n$ and let $g_n$ denote the unique one-to-one analytic map of $E$ onto $G_n$ such that $g_n(0) = 0$ and $g_n'(0) > 0$. Choose the endpoints $w_n$, $n = 1, 2, \ldots$, so that $J_n \subset J_{n+1}$ and $w_n \to w_L$. Then $D$ is the component of $\bigcap_{n=1}^{\infty} G_n$ containing the origin. According to the Carathéodory convergence theorem, $g_n \to f$ uniformly on compact subsets of $E$ as $n \to \infty$. Hence $g_n'(0) \to f'(0) = 1$ and we may take $f_n = g_n / g_n'(0)$, $n \geqslant 1$. □

Thus every function in $S$ can be approximated by a single-slit mapping that omits a subray of the negative real axis. This density result can still be obtained without Carathéodory's theorem, or by using only certain components of its proof (cf. [128, Chapter 7] and [100]). Similar slit mappings have been considered by G.M. Goluzin, P.P. Kufarev, W.K. Hayman, and other authors (cf. [70], [72, Chapter 3], [161], [128, Chapter 7]).

## 3. The area method

The property of a univalent function $w = F(z)$ to omit a part of the $w$-plane – which is of course of nonnegative area – is known as the (elementary) *area principle* (cf. L. Bieberbach [28]). This principle and its advanced versions give rise to the *area theorems* of the *area method* as follows. Given one or several univalent functions in $E$ with nonoverlapping image domains in the $w$-plane, define some mapping $Q(w)$ on a neighborhood of the complement $D$ of all their images. Then the fact that the image of this complement under $Q(w)$ has nonnegative area, which equals

$$\iint_D |Q'(w)|^2\, du\, dv, \quad w = u + iv,$$

for $Q$ analytic on $D$, will serve as a source of fruitful inequalities and applications if one uses skill in choosing the parameter function $Q(w)$. The arising inequalities are usually written in terms of the Laurent coefficients of certain compositions of involved functions. In the case of a full mapping or a full system of mappings every such inequality turns into equality.

The earliest area theorem in the elementary case $Q(w) = w$ was obtained by T.H. Gronwall for the class $\Sigma$ in 1914 [117], see Section 6. In 1932, H. Grunsky introduced a new approach to extremal problems, *the method of contour integration* [118], allowing one to obtain inequalities via a contour integral representation of nonnegative double integrals (e.g., area integrals). This method lies in close connection with the area method and theory of orthonormal systems. It is based on the Green formula, properties of canonical mappings and residue theorem, and it yields effective inequalities with an appropriate choice of the initial double integral. H. Grunsky discovered his famous univalence criterion (Section 6) in this way.

The area method and the method of contour integration were used and developed by L. Bieberbach, G. Faber, H. Prawitz, H. Grunsky, G.M. Goluzin, and later by Yu.E. Alenitsyn, S. Bergman, P.R. Garabedian, J.A. Hummel, J.A. Jenkins, N.A. Lebedev, H. Meschkowski, I.M. Milin, Z. Nehari, Ch. Pommerenke, M. Schiffer, O. Tammi, and others (see the books [49,72,181,208,230] for details and references). In the course of the century the area principle has been repeatedly generalized and applied to various classes of univalent and multivalent functions and systems of functions. The rich modern arsenal of the area method includes the Prawitz polar area theorem [234] (see also [208, Chapter 1]), Goluzin area theorem for $p$-valent functions [67], [72, Chapter 11] (see also [181, Chapter 1]), Milin area theorem for class $\Sigma$ and an arbitrary regular function $Q(w)$ [183, §1], [201], [208, Chapter 1], Lebedev (generalized) area theorem for classes $\mathcal{M}(a_0, a_1, \ldots, a_n)$ and $Q(w)$ with a regular single-valued derivative [177], [181, Chapter 3], Milin area theorem for finitely connected domains [202,206], [208, Part 2], Lebedev area theorem for systems of finitely connected domains [178], [181, Chapter 4] (see [181] and the works of L.L. Gromova and N.A. Lebedev [113,115] for further development), and other theorems.

Z. Nehari used the maximum principle for subharmonic functions to prove a general result implying some area type inequalities for univalent functions [220] (see also [208, Chapter 1]). O. Lehto [186] (see also R. Kühnau [163]) and then V.Ya. Gutlyanskii [122] and Yu.E. Alenitsyn [10] used a modified area principle to obtain some area theorems for quasiconformally extendible functions in $\Sigma$. Alternative arguments and generalizations can be found in the works of L.V. Ahlfors [5], V.Ya. Gutlyanskii and V.A. Shchepetev [123], V.S. Belikov [27], V.G. Sheretov [249], R. Kühnau [166], Yu.E. Alenitsyn [11], V.A. Shchepetev [248], E. Hoy [133–135], and the author [90,91,93]. In particular, a generalization based on the Dirichlet principle was applied to the pairs of functions with nonoverlapping image domains and a homeomorphic assembling [248,90]. R. Kühnau, Ch. Pommerenke, and the author considered closely related aspects of the Grunsky norm [163], [230, Chapter 9], [167,168,91,93] and its generalization ($\tau$-norm) [93].

The area method permits one to prove a number of important theorems which for the time being are not accessible by other means. Many results originally obtained by different methods can be naturally and with less effort proved by the area method. Moreover, the question of equality for inequalities proved by the classic area method can usually be settled, since the fact that the ignored area is zero for full mappings or for full systems of mappings only is of extra help in search for extremal functions (see, e.g., [49, Chapters 2 and 4], [181], [208, Chapter 1]). According to the nature of this survey our main examples of the area theorems will be the Milin area theorem for class $\Sigma$ (Section 6), Lebedev (generalized) area theorem (Section 7), and some of their stronger versions for functions with homeomorphic (quasiconformal) extensions and pairs of functions with homeomorphic assemblings respectively (Sections 9 and 10).

## 4. The Bieberbach theorem and conjecture

In 1916, L. Bieberbach published a short elegant theorem [29]. He proved a sharp bound for the second coefficient $\{f\}_2$ of functions $f \in S$. We give a compact version of his proof based on the elementary area principle (see [100]). A more traditional version uses the Gronwall outer area theorem (see Section 6 and, e.g., [72, Chapter 2] or [49, Chapter 2]).

THEOREM A [29]. *If $f \in S$, then $|\{f\}_2| \leqslant 2$, with equality if and only if $f$ is a rotation of the Koebe function $K(z) = z/(1-z)^2$.*

PROOF. The function $F(z) = [f(z^2)]^{-1/2} = z^{-1} - \frac{1}{2}\{f\}_2 z + \cdots$ is analytic and univalent in $E\backslash\{0\}$. For $r \in (0,1)$, let $C_r$ be the image under $F$ of the circle $|z| = r$. Clearly, $C_r$ is a simple closed curve. Switching to polar coordinates, write $F(re^{i\alpha}) = Re^{i\Psi}$, $0 \leqslant \alpha < 2\pi$. Since the area enclosed by $C_r$ is positive, we have

$$\frac{1}{2}\int_{C_r} R^2\, d\Psi > 0,$$

where the integration is performed along $C_r$ in the counterclockwise sense. Using $R\Psi_\alpha = rR_r$, one of the Cauchy–Riemann equations in polar coordinates, we find that $d\Psi = (r/R)R_r\, d\alpha$. Consequently

$$-\frac{d}{dr}\int_0^{2\pi} \left|F\left(re^{i\alpha}\right)\right|^2 d\alpha = 4\pi\left(r^{-3} - \left|\frac{1}{2}\{f\}_2\right|^2 r - \cdots\right) > 0.$$

As $r \to 1-$ we deduce that $|\{f\}_2| \leqslant 2$. Equality is possible only if $F(z) = z^{-1} - \lambda z$, where $|\lambda| = |\frac{1}{2}\{f\}_2| = 1$, and thus $f(z) = \bar{\lambda} K(\lambda z)$. □

Theorem A became the first supporting evidence for one of the most famous problems of analysis, Bieberbach's conjecture, which appeared as a footnote in [29].

*The Bieberbach conjecture asserts that*

$$\left|\{f\}_n\right| \leqslant n, \quad n=2,3,\ldots \tag{1}$$

*for each $f \in S$, and that equality holds for any given n only for the Koebe function $K(z)$ and its rotations $\bar{\lambda}K(\lambda z)$, $|\lambda|=1$.*

For the next seven decades this easily stated conjecture stood as a challenge for analysts throughout the world (see, e.g., [15,49,208] for details). It motivated the development of the Löwner parametric method (Section 5), Littlewood–Prawitz integral approach (Section 11), Grunsky coefficient approach (Section 6), area method (Sections 3, 6, and 7), Milin exponentiation method (Section 11), and other powerful tools in the theory of univalent functions.

The Bieberbach conjecture remained open until in 1984 L. de Branges [32] proved a stronger conjecturel for certain logarithmic functionals on $S$, which was proposed by I.M. Milin in 1971 [208, Chapter 3] (see Sections 12 and 13).

## 5. The Löwner method: parametric representation of slit mappings

The *method of parametric representations* was introduced and applied by K. Löwner (C. Loewner) in 1923 [197]. It was further developed and used by G.M. Goluzin [65,66, 68,70,71], I.E. Bazilevich [18,19,24], P.P. Kufarev [159–161], Ch. Pommerenke [226,227], V.Ya. Gutlyanskii [121], L. de Branges [32], and others (cf. E. Peschl [224], Y. Komatu [153], M.R. Kuvaev and P.P. Kufarev [172], N.A. Lebedev [174,175], G.G. Shlionskii [251,252], M.R. Kuvaev [171], J. Janikowski [145], V.I. Popov [233], J. Becker [25,26], C. FitzGerald [57], Z. Nehari [222], R.W. Barnard [17], Z. Charzyński and J. Lawrynowicz [38], O. Tammi [255,256], C. FitzGerald and Ch. Pommerenke [58], V.V. Goryainov [80], D.V. Prokhorov [235]). The books by I.A. Aleksandrov [6], P.L. Duren [49, Chapter 3], G.M. Goluzin [72, Chapters 3 and 4], W.K. Hayman [128, Chapter 7], Ch. Pommerenke [230, Chapter 6], and G. Sansone and J. Gerretsen [243, Chapter 11] contain a treatment of Löwner's parametric method, as well as the detailed bibliography.

Löwner's method permits one to solve many extremal problems on class $S$ and related classes. However, unlike the area method, one can often find it very difficult to identify all the extremal functions. The idea behind the method is to reduce a given problem to the one on a dense subclass associated with a partial differential equation. In a number of cases this subclass is the class of single-slit mappings in $S$ or $\Sigma$ (Section 2). Sometimes it suffices to deal just with a special subset of these mappings (cf. Lemma 1 in Section 2, Section 13, and [100]). The following Löwner representation theorem for single-slit mappings is at the heart of the matter. We use our exposition in [100].

THEOREM B [197]. *Let $f \in S$ map $E$ onto the complement of a given Jordan arc $J=\{w(t)\colon\ 0 \leqslant t \leqslant \infty\}$ ($w$ is one-to-one and continuous) extending from a finite point $w(0)$ to infinity. For each $t>0$ let $f(z,t)$ denote the unique one-to-one analytic map of $E$ onto the plane less the portion of $J$ from $w(t)$ to infinity such that $f(0,t)=0$ and $f_z(0,t)>0$, and let $f(z,0)=f(z)$.*

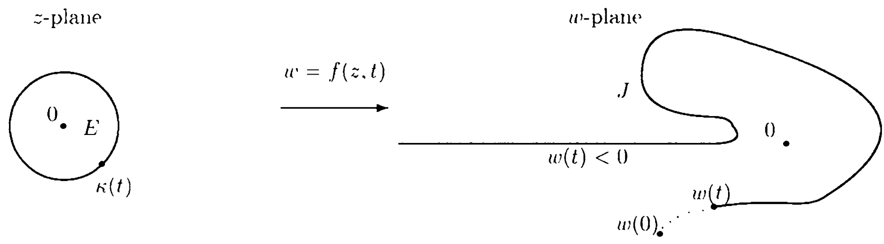


Fig. 2.

*The parametrization $w(t)$ can be chosen so that $f_z(0,t) = e^t$, $t > 0$. In this case $f(z,t)$ satisfies the partial differential equation*

$$f_t(z,t) = z f_z(z,t) \frac{\kappa(t) + z}{\kappa(t) - z}, \quad z \in E,\ t \geqslant 0,$$

*where $\kappa(t)$ is a continuous complex-valued function on* $[0, \infty)$ *with* $|\kappa| = 1$.

The proof of this classic result is given in [49, Section 3.3 and Exercise 8, p. 117] and [72, Chapter 3]; see also [128, Chapter 7] for the case of piecewise analytic cuts. The differential equation in Theorem B is called *Löwner's equation.* It is effective for applications (Section 13) in its logarithmic form:

$$\left[\log \frac{f(z,t)}{z}\right]_t = \left(z\left[\log \frac{f(z,t)}{z}\right]_z + 1\right)\frac{\kappa(t) + z}{\kappa(t) - z}, \quad z \in E,\ t \geqslant 0. \tag{2}$$

The family $\{f(z,t)\colon\ t \geqslant 0\}$ is an example of so-called *Löwner* (*subordination*) *chain* starting at $f(z)$ and generated by a continuously increasing family of simply connected domains (Figure 2). The point $w(t)$ corresponds to $\kappa(t)$ under the map $f(z,t)$. One can use the Schwarz lemma to show that if $w(t) < 0$ for $t \geqslant T \geqslant 0$ then $f(z,t) = e^t K(z)$ and $\kappa(t) = -1$ for these values of $t$. The Taylor coefficients of $f(z,t)$ (and $\log[f(z,t)/z]$) are differentiable in the parameter $t$ as can be shown by differentiating Cauchy's integral with respect to $t$.

K. Löwner (followed by many others) tried to use the parametric approach to prove the Bieberbach conjecture (Section 4). His original paper [197] contains a proof of (1) for $n = 3$ (and $n = 2$, cf. Theorem A in Section 4). However a proof for $n = 4$ based solely on Löwner's method was given (by Z. Nehari [222]) only 50 years later, when the cases $n = 4, 5, 6$ had been settled by other means (see, e.g., [49, Sections 3.5, 4.6 and Notes, pp. 69, 139] for details). Despite heroic efforts, no one was able to use Löwner's method in a direct proof of any case $n > 4$. In 1984, L. de Branges discovered that Löwner's method could be applied to Milin's functionals to prove Milin's, and thus Bieberbach's, conjecture ([32], see Sections 12 and 13). Fortunately, the difficulties in identifying the extremal functions can be avoided in the Bieberbach–Milin case.

There are several generalizations of the Löwner differential equation. The most productive of them is the Löwner–Kufarev equation for univalent functions (see P.P. Kufarev

[159,160], the books by Ch. Pommerenke [230, Chapter 6] and I.A. Aleksandrov [6], and, e.g., [19,20,25,79,121,218]). It was shown by Ch. Pommerenke that such an equation (see Theorem B1) holds for subordination chains of analytic functions which may not be univalent. In general, $g(z,t)$ $(z \in E,\ t \in [0,T])$, is called a *subordination chain* over $[0,T]$ if $g(z,t)$ is analytic for $z \in E$ for each fixed $t \in [0,T]$, $g_z(0,t)$ is a continuous and nonvanishing function of $t$, and $0 \leqslant t_1 \leqslant t_2 \leqslant T$ implies that $g(z,t_1)$ is subordinate to $g(z,t_2)$, $g(z,t_1) \prec g(z,t_2)$. A subordination chain is called *normalized* if $g(0,t) = 0$ and $g_z(0,t) = e^t$ for all $t \in [0,T]$.

THEOREM B1 [226]. *Let $g(z,t) = e^t z + \cdots$ be analytic in $E$ for each $t \in [0,T]$. Then $g(z,t)$ is a normalized subordination chain over the interval $[0,T]$ if and only if $g(z,t)$ is absolutely continuous as a function of $t$, uniformly for $z$ in compact subsets of $E$, and there exists a function $\mathcal{P}(z,t)$ analytic for $z \in E$ with $\mathcal{P}(0) = 1$, $\Re\{\mathcal{P}(z,t)\} > 0$ for all $z \in E$, measurable for $t \in [0,T]$ and such that*

$$g_t(z,t) = z g_z(z,t)\mathcal{P}(z,t)$$

*for each $z \in E$ and almost all $t \in [0,T]$.*

We also mention Komatu's work on univalent functions in an annulus [153], [6, Chapter 5]. However such and some other developments (cf. [71,171,172,175]) are complicated and have not been as fruitful so far.

## 6. The Grunsky univalence criterion and Milin area theorem

In 1939, H. Grunsky used his method of contour integration (Section 3) to derive important necessary and sufficient conditions of univalence for functions $F(z) = z + \alpha_0 + \alpha_1 z^{-1} + \cdots$ analytic in $U$, except for a simple pole at infinity (see Theorem C1 below; in fact, Grunsky has proved a more general result [120]). The Grunsky conditions (univalence criterion) are written in terms of certain coefficients, the *Grunsky coefficients*, that arise in the following way. Let $F \in \Sigma$ and $\zeta$ be an arbitrary finite point in $U$. Define a function of $z$ by taking the (analytic) branch of $\log[(z-\zeta)/(F(z)-F(\zeta))]$, $z \in U$, which vanishes at $z = \infty$. If $\zeta = \infty$ we take this function to be identically zero. The expansion

$$\log \frac{z-\zeta}{F(z)-F(\zeta)} = \sum_{n=1}^{\infty} A_n(\zeta) z^{-n}, \quad z \in U, \tag{3}$$

generates a sequence of single-valued functions $A_n(\zeta)$ $(n = 1, 2, \ldots)$ in the domain $U$. Using Cauchy's formula one can show that each function $A_n(\zeta)$ is analytic in $U$ (see I.M. Milin [208, Chapter 1]). Write

$$A_n(\zeta) = \sum_{n=1}^{\infty} \alpha_{nk} \zeta^{-k}, \quad \zeta \in U. \tag{4}$$

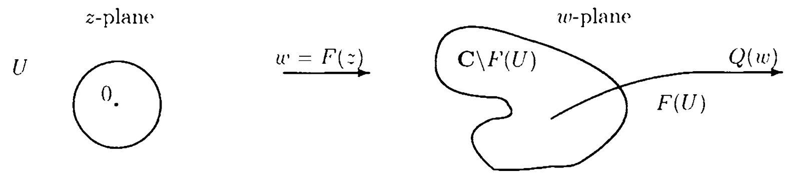


Fig. 3.

Then the coefficients $\alpha_{nk}$ $(n, k = 1, 2, \ldots)$ are the Grunsky coefficients. The following Milin area theorem for class $\Sigma$, established 12 years later than the Grunsky criterion, turns out to be a convenient tool to prove Grunsky's result and many others as well.

THEOREM C ([183, §1], [201], [208, Chapter 1]). *Let $w = F(z) \in \Sigma$, and let $Q(w)$ be an arbitrary nonconstant function, regular in the complement of $F(U)$. Suppose that the Laurent series expansion of the function $Q \circ F(z)$, which is regular in an annulus $1 < |z| < r$, has the form*

$$Q \circ F(z) = \sum_{n=-\infty}^{\infty} a_n z^n.$$

*Then*

$$\sum_{n=1}^{\infty} n|a_n|^2 \leqslant \sum_{n=1}^{\infty} n|a_{-n}|^2. \tag{5}$$

*Equality holds if and only if $F$ is a full mapping.*

Inequality (5) expresses the fact that the image of the complement of $F(U)$ under the mapping $Q(w)$ has nonnegative area (Figure 3). Consider three special cases of Theorem C.

(i) The case when $Q(w) = w$ is known as the Gronwall outer area theorem [117]. Its immediate consequence is the inequality $|\alpha_1| \leqslant 1$ for each function $F(z) = z + \alpha_0 + \alpha_1 z^{-1} + \cdots \in \Sigma$, with equality if and only if $F(z) = z + \alpha_0 + \alpha_1 z^{-1}$, $z \in U$, $|\alpha_1| = 1$. This inequality easily leads to Bieberbach's theorem on the second coefficient (Theorem A, Section 4).

The outer area theorem was generalized by G.M. Goluzin for meromorphic $p$-valent functions in $U$ $(p \geqslant 2)$ [67], [72, Chapter 11]. A function is $p$-valent in a given domain if it does not assume any value more than $p$ times there. Naturally, Goluzin's result contains the polynomial area theorem for univalent functions, i.e., Theorem C when $Q$ is a polynomial (see [265], [181, Introduction and Chapter 1], [49, Chapter 4]). Our second example is a well known case of the polynomial area theorem.

(ii) For $F \in \Sigma$ and any finite $w$, we have the expansion

$$\log \frac{z}{F(z) - w} = \sum_{n=1}^{\infty} P_n(w) z^{-n}$$

in a neighborhood of $z = \infty$. Polynomials $n P_n(w)$ $(n = 1, 2, \ldots)$ are called the *Faber polynomials* of the function $F$ (see, e.g., [245], [49, Chapter 4], [208, Chapter 1], [230, Chapter 3]). Now define the function $Q$ as a linear combination of polynomials $P_n$ and thus of Faber polynomials:

$$Q(w) = \sum_{n=1}^{N} x_n P_n(w).$$

This choice of $Q$ in Theorem C gives the following inequalities for the Grunsky coefficients of $F$ and arbitrary complex numbers $x_1, x_2, \ldots, x_N$:

$$\sum_{k=1}^{\infty} k \left| \sum_{n=1}^{N} x_n \alpha_{nk} \right|^2 \leqslant \sum_{n=1}^{N} \frac{|x_n|^2}{n} \quad (N = 1, 2, \ldots). \tag{6}$$

Inequalities (6) are known as *strong Grunsky inequalities* (see, e.g., [49, Chapter 4]). An application of the Cauchy–Schwarz inequality to (6) yields the classical Grunsky inequalities for the class $\Sigma$.

THEOREM C1 [120]. *Let $F \in \Sigma$ and let coefficients $\alpha_{nk}$ be defined by* (3) *and* (4). *Then*

$$\left| \sum_{n,k=1}^{N} \alpha_{nk} x_n x_k \right| \leqslant \sum_{n=1}^{N} \frac{|x_n|^2}{n} \tag{7}$$

*for each natural $N$ and any complex numbers $x_1, x_2, \ldots, x_N$.*

The Grunsky inequalities (7) provide necessary and sufficient conditions of univalence, thus making the coefficient conditions (6) and (7) equivalent. Though it is not difficult to derive (6) from Theorem C1 using Schur's transformation (cf. [181, Chapter 1]), strong Grunsky inequalities (6) were discovered only some twenty years later than (7) (first by N.A. Lebedev [177] in a general form and then independently by J.A. Jenkins, I.M. Milin, and Ch. Pommerenke; see, e.g., [181,208,230]).

The inequalities below were established by G.M. Goluzin in 1947. The Goluzin inequalities, which are also equivalent to the Grunsky inequalities and can rather easily be derived from them, are very effective for certain applications (cf. [57,228], [181, Chapters 1 and 2], [230, Chapter 9], [49, Chapter 4]).

THEOREM C2 ([69], [72, Chapter 4]). *Let $F \in \Sigma$, $N \geqslant 1$, and let $z_n \in U$, $n = 1, \ldots, N$. Then*

$$\left| \sum_{n,k=1}^{N} x_n x_k \log \frac{F(z_n) - F(z_k)}{z_n - z_k} \right| \leqslant - \sum_{n,k=1}^{N} x_n \overline{x_k} \log\left[1 - (z_n \overline{z_k})^{-1}\right] \tag{8}$$

*for all complex numbers $x_1, x_2, \ldots, x_N$.*

Note that the Goluzin inequalities can be formulated as an inequality between a positive definite Hermitian form and the absolute value of a complex symmetric form (see FitzGerald's approach, Section 11 and the book by R.A. Horn and Ch.R. Johnson [131, Chapter 4]).

(iii) In this last example let $Q(w) = Q_\zeta(w)$ be a branch of the function $\log[w - F(\zeta)]$, $\zeta \in U$, defined in the complement of $F(U)$. Using Theorem C and expansion (3) we obtain (see I.M. Milin [183, §1])

$$\sum_{n=1}^{\infty} n \left| A_n(\zeta) \right|^2 \leqslant \log \frac{1}{1 - |\zeta|^{-2}}, \qquad \zeta \in U. \tag{9}$$

This fact is a simple consequence of (6) as well. Inequality (9) and the inequalities obtained by taking linear combinations of functions $Q_\zeta$ as well as combinations of their integrals or the same order derivatives in $\zeta$ were used in applications by I.M. Milin, N.A. Lebedev, Ch. Pommerenke, and others (cf. [183, §1], [201], [208, Chapters 1 and 3], [225], [181, Chapter 1], [230, Chapter 3], [84,86], [49, Chapter 5], [108]). In particular, the proofs of the Bazilevich logarithmic inequality [22] and Milin's theorem on logarithmic means [204] are based on inequality (9). We now discuss these results of I.E. Bazilevich (Theorem C3) and I.M. Milin (Theorem C4) in connection with some deep properties of the Taylor coefficients of univalent functions.

Recall that the *Hayman index $\alpha_f$* of a function $f \in S$ is defined by the formula

$$\alpha_f = \lim_{r \to 1-} (1 - r)^2 \max_{|z|=r} \left| f(z) \right| \leqslant 1.$$

The equality $\alpha_f = 1$ holds if and only if $f$ is a rotation of the Koebe function. If $\alpha_f > 0$, there is a unique direction of maximal growth $e^{i\theta_0}$ for which

$$\lim_{r \to 1-} (1 - r)^2 \left| f\left(re^{i\theta_0}\right) \right| = \alpha_f .$$

THEOREM C3 [22]. *Let $f \in S$ have Hayman index $\alpha_f > 0$ and direction of maximal growth $e^{i\theta_0}$. Then*

$$\sum_{n=1}^{\infty} n \left| \left\{ \log \frac{f(z)}{z} \right\}_n - \frac{2}{n} e^{-i\theta_0 n} \right|^2 \leqslant 2 \log \frac{1}{\alpha_f}.$$

I.E. Bazilevich proved that equality occurs here for functions $f \in S$ which map $E$ onto the complement of an analytic arc [21], i.e., for special slit mappings (Section 2). A relatively simple proof of the Bazilevich inequality belongs to I.M. Milin [208, Chapter 3].

Functions $f \in S$ with $\alpha_f > 0$ are known as functions of *maximal* (*largest*) *growth*. Properties of functions in $S$ related to Hayman's index can be found in the works of W.K. Hayman [126], [128, Chapters 1 and 6] and also of I.E. Bazilevich, I.M. Milin, P.L. Duren, N.A. Lebedev, and others, e.g., [21–23,207], [208, Chapter 3], [46,47,180], [181, Chapter 2], [48,88,150,151], [49, Chapter 5], [51,63,139,262] (see Section 9 for the connection with the Grunsky norm).

In 1955, W.K. Hayman established a general result characterizing the asymptotic behavior of the coefficients of $p$-valent functions [126]. In the case of univalent functions, his best known asymptotics is the following.

HAYMAN'S REGULARITY THEOREM ([126], [128, Chapter 1]). *For each* $f \in S$,

$$\lim_{n\to\infty} \frac{|\{f\}_n|}{n} = \alpha_f.$$

Hayman's result describes the behavior of large coefficients in a very elegant form. At the time it was of much interest appearing to be in favor of the Bieberbach conjecture (Section 4). However the convergence here is not uniform in the class $S$ (N.A. Shirokov even showed that the convergence in each subclass of functions in $S$ with real coefficients and a fixed Hayman index fails to be uniform [250]), and some analogous examples show that a precise but nonuniform asymptotics may not guarantee the coefficient bounds it suggests (e.g., odd functions [126], [128, Chapter 5], Section 14, and Gel'fer functions [81], Section 16). I.M. Milin combined his Tauberian theorem (Section 11) with Bazilevich's logarithmic inequality to give a simple proof [207] of Hayman's coefficient asymptotics for univalent functions. See also [208, Chapter 3] and [49, Chapter 5].

In 1964, Ch. Pommerenke [225] applied the maximum principle for subharmonic functions to estimate the Faber polynomials on level curves. This important idea was used by I.M. Milin to prove Theorem C4 and later by the author (cf. [87,90,93]).

THEOREM C4 ([204], [208, Chapter 3]). *Let* $f(z) \in S$. *Then*

$$\sum_{m=1}^{n}\left(m\left|\left\{\log\frac{f(z)}{z}\right\}_m\right|^2 - \frac{4}{m}\right) \leqslant 4\delta, \quad n \geqslant 1,$$

*where* $\delta > 0$ *is an absolute constant* (*the Milin constant*). *The estimate*

$$\delta < \delta_m = \frac{1}{2}\left[\sum_{n=1}^{\infty}\frac{(\log 2)^n}{n!n} - \log\log 2 - \gamma\right] = 0.3118\ldots,$$

*where* $\gamma = 0.5772\ldots$ *is the Euler constant, holds.*

Milin's theorem captures the sharp growth order in $n$ of logarithmic means. Later it gave rise to his conjecture on logarithmic functionals implying the Bieberbach conjecture

(Section 12). Theorems C3 and C4 together with the celebrated de Branges theorem (proof of Milin's conjecture, Section 13) are the best known results on the logarithmic coefficients of functions in $S$. Other results and references can be found in the books by I.M. Milin [208, Chapter 3] and P.L. Duren [49, Chapters 5, 7, and 8], and in [12,45,50,83,87,90,92–94,107,192,209,210,213,214,223,272] (see Sections 9 and 13). They include the second Milin conjecture on logarithmic coefficients [209] which asserts that *each function $f \in S$ satisfies the inequality*

$$\sum_{n=1}^{\infty} n \left| \left\{ \log \frac{f(z)}{z} \right\}_n \right|^2 r^n \leqslant 2 \log \frac{\max_{|z|=r} |f(z)|}{r}$$

*for every $r \in (0, 1)$*. I.M. Milin verified this interesting conjecture in some cases [209,210].

Finally, we note that although both coefficient formulations of Grunsky's result (inequalities (6) and (7)) contain all the information about a univalent function, this information is by no means easily accessible. The prime example of this is the long but vain attempt by many to prove the Bieberbach conjecture using Grunsky's conditions (related results and references can be found in [49]). Some versions and sequences of (6) and (7) (e.g., Theorems C2, C3 and C4, and inequality (9)) are more suitable for further application than the original inequalities themselves. A number of interesting consequences of the Grunsky conditions, including generalizations of the Bazilevich inequality and Garabedian–Schiffer inequalities (Section 7), are given in the book by N.A. Lebedev [181]. Furthermore, many inequalities of the Grunsky-type and Goluzin-type for systems of functions with nonoverlapping image domains follow from the basic inequalities (7) and (8) (see R. Kühnau [162], N.A. Lebedev [181], and discussion before Theorem D4 in Section 7). An important use of the Grunsky inequalities stems from their operator-theoretic interpretation (Section 9).

## 7. The Lebedev area theorem for nonoverlapping domains

In 1961, Lebedev proved an area theorem for classes $\mathcal{M}(a_0, a_1, \dots, a_n)$. As a parameter $Q(w)$, he used a function with the regular single-valued derivative in the uncovered part of the plane. The Lebedev theorem gives a general multiparameter inequality for functions with nonoverlapping image domains (see [181, Chapter 3] and [106,112,114,116,162,164, 177,179,182] for this and related results of N.A. Lebedev, L.L. Gromova, R. Kühnau, L.L. Gromova and N.A. Lebedev, N.A. Lebedev and L.V. Mamai, and the author and Z.D. Kolomoitseva). We use our exposition in [106].

THEOREM D ([177], [181, Chapter 3]). *Suppose that $\{f_k(z)\}_0^n \in \mathcal{M}(\infty, a_1, \dots, a_n)$, that the function $q(w)$ is regular in the complement of the union of all images $f_k(E)$ ($k = 0, 1, \dots, n$), and that a nonconstant function $Q(w)$ is defined by*

$$Q(w) = q(w) + \sum_{k=1}^{n} \beta^{(k)} \log(w - a_k),$$

*where* $\beta^{(k)}$ $(k = 1, \ldots, n)$ *are arbitrary complex numbers. Furthermore, in an annulus* $r < |z| < 1$, *let*

$$q \circ f_k(z) = \sum_{q=-\infty}^{\infty} \beta_q^{(k)} z^q \quad (k = 0, 1, \ldots),$$

*and in E let*

$$\sum_{q=1}^{\infty} a_q^{(l)} z^q = \sum_{k=1,\, k \neq l}^{n} \beta^{(k)} \log\left[\frac{f_l(z) - a_k}{a_l - a_k}\right] + \beta^{(l)} \log\left[\frac{f_l(z) - a_l}{z f_l'(0)}\right],$$

*if* $l = 1, \ldots, n$, *and*

$$\sum_{q=1}^{\infty} a_q^{(l)} z^q = \sum_{k=1}^{n} \beta^{(k)} \log\big[z f_0'(0)\big(f_0(z) - a_k\big)\big],$$

*if* $l = 0$ $(f_0'(0) = \lim_{z \to 0}(z f_0(z))^{-1})$. *Then*

$$\begin{aligned}
&\sum_{l=0}^{n} \sum_{q=1}^{\infty} q \left|\beta_q^{(l)} + a_q^{(l)}\right|^2 \\
&\quad \leqslant \sum_{l=0}^{n} \sum_{q=1}^{\infty} q \left|\beta_{-q}^{(l)}\right|^2 - 2\Re \sum_{l=0}^{n} \overline{\beta^{(l)}} \beta_0^{(l)} - 2 \sum_{l=0}^{n} \left|\beta^{(l)}\right|^2 \log\left|f_l'(0)\right| \\
&\quad\quad - 4 \sum_{1 \leqslant k < l \leqslant n} \Re\left\{\beta^{(k)} \overline{\beta^{(l)}}\right\} \log|a_l - a_k|,
\end{aligned} \tag{10}$$

*where* $\beta^{(0)} = -\sum_{k=1}^{n} \beta^{(k)}$. *The equality sign holds if and only if* $\{f_k(z)\}_0^n$ *is a full system of mappings.*

The proof is based on the area principle and an effective use of Green's formula. Inequality (10) as inequality (5) in Theorem C is the analytic expression of the fact that the image area of a certain set is nonnegative. This time the set is the complement of the union $\bigcup_{k=1}^{n} f_k(E)$ and the mapping is a single-valued branch of $Q(w)$ (Figure 4). More precisely, we pass to the limit as $r \to 1-$ in the area inequality for the image (under a chosen branch of $Q(w)$) of a simply connected domain which is a suitably cut $(n+1)$-connected complement of $\bigcup_{k=1}^{n} f_k(E_r)$, $E_r = \{z\colon |z| \leqslant r\}$.

An analogous inequality for systems of analytic functions $\{f_k(z)\}_0^n$ in the classes $\mathcal{M}(a_0, a_1, \ldots, a_n)$ follows from Theorem D via a linear-fractional transformation. In this case the area inequality is independent of the way in which the functions $f_k(z)$ are indexed [181, Chapter 3].

One application of Theorem D is the following Lebedev theorem on inner conformal radii of nonoverlapping domains which generalizes the well known result of

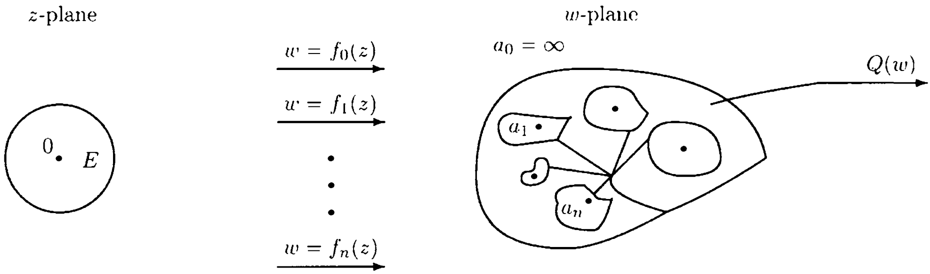


Fig. 4.

M.A. Lavrent'ev on two analytic functions without common values [173], see also G.M. Goluzin [72, Chapter 4].

THEOREM D1 ([177], [181, Chapter 3]). *Let* $\{f_k(z)\}_0^n \in \mathcal{M}(a_0, a_1, \ldots, a_n)$, *where* $a_k \neq \infty$, $k = 0, 1, \ldots, n$. *Then*

$$\prod_{k=0}^{n} |f_k'(0)|^{|x_k|^2} \leqslant \prod_{0 \leqslant k < l \leqslant n} |a_k - a_l|^{-2\Re(\overline{x_l} x_k)}$$

*for all complex numbers* $x_1, \ldots, x_n$ *with* $\sum_{k=0}^{n} x_k = 0$. *The equality holds only if* $\{f_k(z)\}_0^n$ *is a full system of mappings and certain additional conditions are fulfilled* (*see details in* [181, Chapter 3]).

Sharp inequalities for bounded functions with nonoverlapping domains are also contained in Lebedev's area theorem. For, suppose that $\{f_k(z)\}_1^n \in \mathcal{M}(a_1, \ldots, a_n)$, $|f_k(z)| < 1$ in $E$ for $k = 1, \ldots, n$, and that the function $q(w)$ is regular in the complement in $\overline{E}$ of the union of all images $f_k(E)$ $(k = 1, \ldots, n)$. Then one can apply Theorem D with $f_0(z) = 1/z$ and obtain a general area inequality for systems of "nonoverlapping" bounded functions with equality occurring if and only if $\{f_k(z)\}_1^n$ is a full system of mappings in $E$ (Section 2). L.L. Gromova and N.A. Lebedev used another approach in this situation: apply Theorem D to the system $\{f_k(z)\}_1^{2n} \in \mathcal{M}(a_1, \ldots, a_{2n})$, where $f_{n+k}(z) = \overline{f_k(\overline{z})}^{-1}$, $a_{n+k} = \overline{a_k}^{-1}$ for $k = 1, \ldots, n$ [114,116], [181, Chapter 3]. If $n = 1$ this approach implies the following inequalities established by Z. Nehari for bounded univalent functions in 1953.

THEOREM D2 [219]. *Let* $f(z)$ *be analytic and univalent in* $E$ *and let* $|f(z)| < 1$ *there. If* $z_1, \ldots, z_N \in E$ *then*

$$\left| \sum_{n,k=1}^{N} x_n x_k \log \frac{f(z_n) - f(z_k)}{z_n - z_k} \right| \leqslant \sum_{n,k=1}^{N} x_n \overline{x_k} \log \frac{1 - f(z_n)\overline{f(z_k)}}{1 - z_n \overline{z_k}}$$

*for all complex numbers* $x_1, \ldots, x_N$ *with* $\sum_{n=0}^{N} x_n = 0$, $N = 1, 2, \ldots$.

The Nehari inequalities bear some similarity to the Goluzin inequalities for the class $\Sigma$ (Theorem C2, Section 6). They provide necessary and sufficient conditions for a function $f(z)$, analytic about the origin, to be analytic, univalent, and bounded by 1 in $E$. See, e.g., G.G. Shlionskii [252], D.W. DeTemple [43,44], and the books by O. Tammi [255,256] for related results and references.

Several useful consequences of Theorem D were proved independently after 1961. Among those are the Grunsky-type inequalities for Bieberbach–Eilenberg functions and an inequality for logarithmic areas (see Theorems D4 and D3 below; other results and references can be found in [230, Chapter 4] and [181, Introduction and Chapter 3]). In the '60s and '70s, some authors have worked with weaker versions of Theorem D. For instance, one can insist on $\Re\{Q(w)\}$ being single-valued, when specifying a function $Q(w)$ with regular single-valued derivative in the uncovered part of the plane. This assumption, allowing a straightforward use of Green's formula, restricts the coefficients $\beta^{(k)}$ of the singular logarithmic terms of $Q(w)$ to the real line. Since the complex numbers $\beta^{(k)}$ are arbitrary in Theorem D, any inequality obtained under this restriction would follow automatically from (10). Further details and references can be found in [106], where the parameters $\beta^{(k)}$ in Theorem D were chosen in the best possible way to produce optimal area inequalities.

Perhaps, the most common applications of Theorem D and related area theorems are the ones dealing with pairs of univalent functions mapping $E$ conformally onto nonoverlapping domains in the $w$-plane. By virtue of their geometric properties the Bieberbach–Eilenberg functions, Gel'fer's ones, and bounded functions can be studied in this setting (see the basic properties and references in [49, Chapter 8], [76, Chapter 12], [181, Introduction and Chapter 3], [230, Chapter 4]).

The *Bieberbach–Eilenberg* functions are defined as analytic functions $g(z)$, $g(0) = 0$, in $E$ satisfying the condition $g(z)g(\zeta) \neq 1$ for any pair of points $z, \zeta \in E$ (see also Section 15). Clearly this definition includes all analytic functions bounded by 1 in $E$. As observed by D. Aharonov [1], some results for classes $\mathcal{M}(a_0, a_1)$ can be conveniently stated in the "Bieberbach–Eilenberg form", namely, in terms of pairs $\{f, g\}$ of analytic and univalent functions in $E$ which vanish at the origin and satisfy the condition $f(z)g(\zeta) \neq 1$ for any $z, \zeta \in E$. We denote the class of these normalized pairs of functions by $\mathcal{M}_A$. The author proved the following theorem (Theorem D3) in this class by applying the elementary area principle to $p$-symmetric functions. It is easy to see that Theorem D with $Q(w) = \log(w)$ implies Theorem D3 (which is also a consequence of some area theorems less general than Theorem D).

THEOREM D3 ([82], see also [208, Chapter 3]). *Let* $\{f, g\} \in \mathcal{M}_A$. *Then*

$$\sum_{n=1}^{\infty} n\left(|\alpha_n|^2 + |\beta_n|^2\right) \leqslant -2\log\left|f'(0)g'(0)\right|,$$

*where* $\alpha_n$ *and* $\beta_n$ *are the logarithmic coefficients of the functions* $f$ *and* $g$, *i.e.*,

$$\alpha_n = \left\{\log\frac{f(z)}{z}\right\}_n, \qquad \beta_n = \left\{\log\frac{g(z)}{z}\right\}_n \quad (n = 1, 2, \ldots),$$

*and equality holds if and only if* $\{1/f, g\}$ *is a full system of mappings.*

We give a short proof of this inequality in the well known case $f = g$ (Sections 15 and 16). Clearly, a function $g$ with $\{g, g\} \in \mathcal{M}_A$ is a univalent Bieberbach–Eilenberg function. For such a function the conclusion of Theorem D3 reduces to the following inequality:

$$\sum_{n=1}^{\infty} n \left| \left\{ \log \frac{g(z)}{z} \right\}_n \right|^2 \leqslant -2 \log |g'(0)|. \tag{11}$$

PROOF OF INEQUALITY (11) [81]. Let

$$g_p(z) = \left[g\left(z^p\right)\right]^{1/p} = a^{1/p} z + \sum_{n=1}^{\infty} b_{np+1} z^{np+1} \quad \text{and}$$

$$g_{-p}(z) = \left[g\left(z^p\right)\right]^{-1/p} = a^{-1/p}/z + \sum_{n=1}^{\infty} c_{np-1} z^{np-1},$$

where $a = g'(0)$ and $p = 2, 3, \ldots$. Note that these two functions are univalent in $E$ for each $p$, and $g_p(E)$ is contained in the complement of $g_{-p}(E)$ (see Figure 5, where we took $|g(z)| < 1$ in $E$ and $p = 4$).

Comparison of the corresponding areas yields:

$$|a|^{2/p} + \sum_{n=1}^{\infty} (np+1) |b_{np+1}|^2 \leqslant |a|^{-2/p} - \sum_{n=1}^{\infty} (np-1) |c_{np-1}|^2 \ (p = 2, 3, \ldots).$$

Consequently, we have

$$\frac{1}{p+2} \sum_{n=1}^{\infty} n |p b_{np+1}|^2 + \frac{1}{p+2} \sum_{n=1}^{\infty} n |p c_{np-1}|^2 \leqslant |a|^{-2/p} - |a|^{2/p},$$

or

$$\frac{1}{(p+2)\pi} \iint_E \left| \left(p[g(z)/z]^{1/p}\right)' \right|^2 d\sigma + \frac{1}{(p+2)\pi} \iint_E \left| \left(p[z/g(z)]^{1/p}\right)' \right|^2 d\sigma$$
$$\leqslant |a|^{-2/p} - |a|^{2/p}.$$

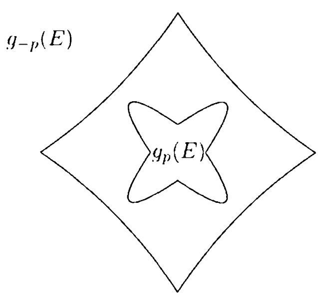


Fig. 5.

Hence

$$\frac{2}{\pi}\iint_E \left|(\log[g(z)/z])'\right|^2 d\sigma$$
$$\leqslant \frac{1}{\pi}\iint_E \left|(\log[g(z)/z])'\right|^2\left(\left|g(z)/z\right|^{2/p} + \left|z/g(z)\right|^{2/p}\right) d\sigma$$
$$\leqslant (p+2)\left(|a|^{-2/p} - |a|^{2/p}\right).$$

Letting $p \to \infty$ and noting the equality

$$\frac{1}{\pi}\iint_E \left|(\log[g(z)/z])'\right|^2 d\sigma = \sum_{n=1}^{\infty} n\left|\left\{\log\frac{g(z)}{z}\right\}_n\right|^2$$

we obtain the desired result. □

For a bounded function $f \in S$, inequality (11) implies that

$$\sum_{n=1}^{\infty} n\left|\left\{\log\frac{f(z)}{z}\right\}_n\right|^2 \leqslant 2\log\sup_{z\in E}\left|f(z)\right|.$$

This was one of the motivations for the second Milin conjecture on logarithmic coefficients (Section 6). Applications of the last inequality are given in [208, Chapter 3], [83,107].

Functions $\varphi(z)$ with $\{\varphi(z), -\varphi(z)\} \in \mathcal{M}(1,-1)$ are known as the univalent *Gel'fer functions*. In general, the Gel'fer functions are defined as the analytic functions $\varphi(z)$, $\varphi(0) = 1$, in $E$ satisfying the condition $\varphi(z) + \varphi(\zeta) \neq 0$ for any pair of points $z, \zeta \in E$ [62] (see Section 16). Elementary transformations relate the class $S$, Bieberbach–Eilenberg functions, and Gel'fer functions. If $f \in S$ and $f(z) \neq a$ in $E$, then $\varphi(z) = [1 - f(z)/a]^{1/2}$ is a univalent Gel'fer function. The converse also holds: if $\varphi$ is a univalent Gel'fer function, then $f(z) = [(\varphi(z))^2 - 1]/(2\varphi'(0))$ belongs to $S$. Furthermore, $\varphi$ is a Gel'fer function if and only if $g = (\varphi - 1)/(\varphi + 1)$ is a Bieberbach–Eilenberg function.

In 1967, P. Garabedian and M. Schiffer obtained multiparameter inequalities for functions in $S$ (or $\Sigma$) in two versions: analytic and geometric. These inequalities can be viewed as both a generalization and a consequence of Grunsky's inequalities (Theorem C1, Section 6). It is not difficult to derive the geometric version from Theorem D directly (Theorem D4 below). For this purpose it is convenient to express the Garabedian–Schiffer inequalities in terms of the Bieberbach– Eilenberg or Gel'fer functions (see [181, Chapters 1 and 3], [208, Chapter 3], and [230, Chapter 4] for details, generalizations, and references). Theorem D4 is a good example of a Grunsky-type result that can be obtained by working either with one univalent function or with a system of functions with nonoverlapping image domains. In fact, as noted by R. Kühnau [162], many inequalities for mappings onto nonoverlapping domains can be obtained, excluding equality case, through limit processes for mappings of only one domain (see Theorem 1 in [162] and remarks on the Grunsky and Goluzin inequalities in Section 6; cf. R. Kühnau [164], Nehari's inequalities (Theorem D2), and related results in Lebedev's book [181, Chapter 3]).

THEOREM D4 [61]. *Let $g$ be a univalent Bieberbach–Eilenberg function and let the coefficients $\beta_{nk}$ be defined by the expansion*

$$\log\left[\frac{g(z)-g(\zeta)}{1-g(z)g(\zeta)}\cdot\frac{1}{z-\zeta}\right]=\sum_{n,k=0}^{\infty}\beta_{nk}z^n\zeta^k,\quad z,\zeta\in E.$$

*Then*

$$\Re\left(\sum_{n,k=0}^{N}\beta_{nk}x_nx_k\right)\leqslant\sum_{n=1}^{N}\frac{|x_n|^2}{n}$$

*for any complex numbers $x_1, x_2, \dots, x_N$, real $x_0$, and $N = 1, 2, \dots$.*

Theorem D4 gives necessary and sufficient conditions for a function $g$, analytic in $E$, to be a univalent Bieberbach–Eilenberg function. An analogous result for the Gel'fer class is obtained by taking $g = (\varphi - 1)/(\varphi + 1)$, where $\varphi$ is a univalent Gel'fer function.

Following W. Rogosinski [238,239] and S.A. Gel'fer [62], one can use the subordination principle to show that some results for pairs of univalent functions with nonoverlapping image domains, as well as for the Bieberbach–Eilenberg functions and Gel'fer functions, are still valid when these functions are not univalent (see also [72, Chapter 8], [181, Introduction and Chapter 3], [49, Chapter 6]). Thus the following consequence of inequality (11) for the univalent Bieberbach–Eilenberg functions is, in fact, true for all Bieberbach–Eilenberg functions. This result, known as *Jenkins' growth theorem for Bieberbach–Eilenberg functions*, was established by J.A. Jenkins in 1954.

THEOREM D5 [147]. *Given a point $z_0$ with $0 < |z_0| = r < 1$, the inequality*

$$|f(z_0)|\leqslant r/\left(1-r^2\right)^{1/2}$$

*holds for all Bieberbach–Eilenberg functions $f(z)$. Equality happens only for functions*

$$f(z)=\pm\frac{r(1-r^2)^{1/2}z}{z_0(1-\bar{z}_0z)}.$$

In a similar way one can use Theorem D3 to prove Alenitsyn's inequality [9], a generalization of Jenkins' theorem for pairs of functions without common values or, equivalently, for pairs of functions in the class $\mathcal{M}_A$ and nonunivalent functions of this kind (see also N.A. Lebedev [176], D. Aharonov [1], and the author [82]). At the same time, Theorem D3 implies a stronger fact, namely the following integral inequality established by N.A. Lebedev in 1961 as a nontrivial consequence of Theorem D. We formulate this Lebedev result in terms of the function pairs in $\mathcal{M}_A$ (the case of nonunivalent functions follows as a direct consequence of it).

THEOREM D6 ([177], [181, Chapter 3]). *Let* $\{f, g\} \in \mathcal{M}_A$. *Then*

$$\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2 \, d\theta \cdot \frac{1}{2\pi}\int_0^{2\pi} |g(e^{i\theta})|^2 \, d\theta \leqslant 1.$$

*Equality holds if and only if* $f(z) = z/(a + bz)$, $g(z) = (|a|^2 - |b|^2)z/(\bar{a}\lambda - \bar{b}z)$, *where* $|a| > |b|$ *and* $|\lambda| = 1$.

In 1949, N.A. Lebedev and I.M. Milin [183,215] proved the following conjecture of W. Rogosinski [238]: for each Bieberbach–Eilenberg function $f(z)$, $|\{f\}_n| \leqslant 1$, $n \geqslant 1$, and equality is attained only for functions of the form $\lambda z^n$, $|\lambda| = 1$. Using Theorem D6 N.A. Lebedev [177] improved this joint result, showing that each Bieberbach–Eilenberg function $f(z)$ satisfies the inequality

$$\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2 \, d\theta \leqslant 1.$$

Later this inequality, Theorem D6 and their generalizations were proved by direct exponentiating inequality (11) and the inequality in Theorem D3 (see Section 11; the papers of D. Aharonov, Z. Nehari, and the author [2,82,221]; and also [208, Chapter 3]).

In connection with the Bieberbach–Eilenberg functions we note that there is a very similar (in some cases more natural) class of functions introduced by H. Grunsky [119]. These functions, known as the elliptically schlicht mappings, are sometimes called the Grunsky (also Grunsky–Shah, Lebedev–Milin) functions. They are defined as analytic functions $g(z)$, $g(0) = 0$, in $E$ satisfying the condition $g(z)\overline{g(\zeta)} \neq -1$ for any pair of points $z, \zeta \in E$ (geometrically the image of $E$ and the antipodal set of $E$ on the Riemann sphere are nonoverlapping, in the univalent case $\{g(z), -\overline{g(\bar{z})}\} \in \mathcal{M}_A$). A survey on univalent elliptically schlicht mappings (and more general mappings in the multiply-connected case) is given in Kühnau's book [165], see also [183, §3], [149, Chapter 7], [177], and [181, Introduction and Chapter 3]. Out of a few results for these functions we mention Kühnau's analog of Theorem D4 [164], Shah's analog of Theorem D5 [247] (see also [146], [149, Chapter 7], and [165, p. 36]), Lebedev's integral inequality (a consequence of Theorem D6, [177]), and a stronger result, analog of inequality (11), which can be proved as in [81] (see above) if one uses the odd values of $p$.

## 8. Quasiconformal and regularly measurable maps

The Grunsky inequalities, other consequences of area theorems, and the area theorems themselves, can be sharpened for conformal mappings with homeomorphic extension or, more generally, for pairs of mappings with homeomorphic assembling. These extensions and assemblings are usually described in terms of quasiconformal maps or their generalizations.

Let $w = \omega(z)$ be an orientation-preserving homeomorphism of a domain $D \subset \overline{\mathbf{C}}$ into $\overline{\mathbf{C}}$. Let $\omega$ be absolutely continuous on lines in $D$, in the sense that $\omega$ is absolutely continuous on almost every closed horizontal and vertical segment in $D$. It follows that the complex partial derivatives $\omega_z$ and $\omega_{\bar{z}}$ are defined almost everywhere in $D$. Their quotient $\mu = \omega_{\bar{z}}/\omega_z$ is called the *complex dilatation* of $\omega$. The function $\omega(z)$ is said to be *quasiconformal* in $D$, if its dilatation $\mu$ satisfies the condition $\|\mu\|_\infty < 1$ there, $\omega$ is $k$-*quasiconformal* if $\|\mu\|_\infty \leqslant k < 1$. If $k = 0$ then $\omega$ is a conformal map. Also the derivatives $\omega_z$ and $\omega_{\bar{z}}$ are locally in $L^2(D)$. See, e.g., the books by L. Ahlfors [4], S.L. Krushkal and R. Kühnau [158], and O. Lehto and K.I. Virtanen [187] for alternative definitions, details, and references.

We now turn to a generalization of quasiconformality, namely to regular measurability [91, §2]. Let $D$ be a domain in the extended $z$-plane, conformally equivalent to a bounded domain, and let $W$ be the class of measurable functions $u(z)$ in $D$ (multivalent in general), whose generalized derivatives $u_z$ and $u_{\bar{z}} \in L^2(D)$, satisfy the inequality $|u_{\bar{z}}| \leqslant |u_z|$ almost everywhere in $D$. Thus, $W$ is a subspace of the corresponding Sobolev space $W_2^1$ [253]. For $u \in W$, denote by $\mathbf{D}(u)$ the generalized Dirichlet integral

$$\iint_D \left(|u_z|^2 + |u_{\bar{z}}|^2\right) dx\, dy, \quad z = x + iy,$$

and set

$$\rho(u) = \begin{cases} \left[\iint_D |u_{\bar{z}}|^2\, dx\, dy \Big/ \iint_D |u_z|^2\, dx\, dy\right]^{1/2}, & \text{if } \mathbf{D}(u) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Given an orientation-preserving homeomorphism $w = \omega(z)$ in $D$ a nonconstant function $Q(w)$ in $\omega(D)$ with regular single-valued derivative there is called *admissible* for $\omega$ if $u = Q \circ \omega \in W$. The homeomorphism $\omega$ is called *regularly measurable* in $D$ if the family $\mathcal{F}_\omega$ of its admissible functions is nonempty.

Denote by $\Omega = \Omega(D)$ the set of all regularly measurable homeomorphisms $\omega(z)$ of the domain $D$. It is easy to show that the set $\Omega$ contains all quasiconformal homeomorphisms of the domain $D$ and all orientation-preserving homeomorphisms from the class $BL = BL(D)$ (maps with a finite Dirichlet integral over $D$ [254]). For $\omega \in \Omega$ the *homeomorphicity coefficient* of $\omega$ is defined by $\varrho(\omega) = \sup_{Q \in \mathcal{F}_\omega} \rho(Q \circ \omega)$. It follows that $\varrho(\omega) \in [0, 1]$ and that $\varrho(\omega)$ remains unchanged under conformal transformations of both $z$ and $w$ planes. If for $\omega \in \Omega$, $\varrho(\omega) = 0$, then $\omega$ is a conformal map. Let $\Omega_k = \Omega_k(D)$ be the set of those maps $\omega \in \Omega$ for which $\varrho(\omega) \leqslant k < 1$. Then all the $k$-quasiconformal (in $D$) maps $\omega$ belong to $\Omega_k$. At the same time $\Omega_k$ for any $k \in (0, 1)$ contains quasiconformal mappings $\omega$ with the complex characteristic $\|\omega_{\bar{z}}/\omega_z\|_\infty$ arbitrarily close to 1 [91, §2].

All of the above equally applies to the case of a closed domain $\overline{D}$, with understanding that if for $\omega \in \Omega(\overline{D})$, $Q$ is admissible, then the derivative $Q'(w)$ is regular and single-valued in some open neighborhood of $\omega(\overline{D})$.

A map $w = \omega(z) \in \Omega$ is *completely regularly measurable* in a domain of definition if every function $Q(w)$ with a regular single-valued derivative in the image of this domain under $\omega$ is admissible for $\omega$. Note that maps which are $k$-quasiconformal in a closed domain $\overline{D}$ are completely regularly measurable and belong to $\Omega_k(\overline{D})$.

## 9. The Grunsky operator and quasiconformally extendible functions

The Grunsky inequalities have an operator-theoretic interpretation often stated in terms of functions in $S$. For $f \in S$ the expansion

$$\log \frac{z-\zeta}{f(z)-f(\zeta)} = \sum_{n,\ell=0}^{\infty} \alpha_{n,\ell} z^n \zeta^\ell \quad (z, \zeta \in E)$$

gives rise to the *Grunsky operator*

$$G_f = \{\sqrt{n\ell}\, \alpha_{n,\ell}\}_{n,\ell=1}^{\infty} : \ell^2 \to \ell^2,$$

where $\ell^2$ is the Hilbert space of all square-summable complex sequences $x = (x_1, x_2, \ldots)$ with the norm $\|x\| = (\sum_{n=1}^{\infty} |x_n|^2)^{1/2}$. Note that a function $F(z) = [f(1/z)]^{-1} +$ constant $\in \Sigma$ $(z \in U)$ generates the same Grunsky operator as $f$ does. According to the Grunsky inequalities (see (3), (4), (6), and (7) in Section 6), $G_f$ is a contraction, that is the operator norm of $G_f$, called the *Grunsky norm* of $f$, is at most 1: $\|G_f\| \leqslant 1$ (see, e.g., [49, Chapter 4] or [230, Chapter 9]).

Denote by $S(k)$, $k \in [0, 1]$, the subclass of functions $f \in S$ which satisfy the condition $\|G_f\| \leqslant k$ [91,93]. It follows from this definition that $S = S(1)$ and $S(k_1) \subset S(k_2)$ for $0 \leqslant k_1 < k_2 \leqslant 1$. Furthermore, the set $S(0)$ consists just of the Möbius transformations of the form $z/(1-\zeta z)$, $\zeta \in \overline{E}$ [93].

The Grunsky norm plays an important role in the theory of conformal mappings with a quasiconformal extension. Let $S_k$, $0 \leqslant k < 1$, denote the subclass of functions in $S$ which have a $k$-quasiconformal extension onto the whole sphere $\overline{\mathbf{C}}$ (Section 8). Note that $S_0$ coincides with $S(0)$. Some geometric reasoning shows that $\|G_f\| \leqslant k$ for any $f \in S_k$, $0 < k < 1$ (see R. Kühnau's paper [163] containing also an improvement of Goluzin's inequalities (8) for this class, and see [230, Chapter 9], [158, Part 2, Chapter 2] for details and references). Therefore $S_k$ lies in $S(k_0)$ if $k \leqslant k_0$. However, for each $k_0 \in (0, 1)$, the class $S(k_0)$ contains not only the $k$-quasiconformally extendible functions with $k \leqslant k_0$, but also some functions which have no $k$-quasiconformal extension for any $k \leqslant k_0$. The first example of this was given by R. Kühnau, in 1981, in terms of functions in $\Sigma$. Summarizing we have

KÜHNAU'S INCLUSION THEOREM ([163,167], [158, Part 2, Chapter 2]). *For each $k \in (0, 1)$, $S_k$ is strictly contained in $S(k)$.*

In fact something stronger is true: class $S(k)$ contains all those functions in $S$ which have a completely regularly measurable continuation onto $\overline{U}$ with the homeomorphicity coefficient not exceeding $k$ (Section 8, [91, §2 and §4]. At the same time, each function in $S(k)$, $k < 1$, has a quasiconformal extension onto the whole sphere $\overline{\mathbf{C}}$:

POMMERENKE'S THEOREM ON QUASICONFORMAL EXTENSION [230, Chapter 9]. *Let $f \in S(k), k < 1$. Then $f \in S_{k_0}$ for some $k_0 < 1$.*

Recently R. Kühnau has found an explicit expression (dilatation bound) for $k_0 = k_0(k)$ in terms of elliptic integrals [169]. Ch. Pommerenke stated the above theorem in terms of the corresponding subclass of functions in $\Sigma$ satisfying the sharpened Grunsky inequalities. He also showed that the sharpened Goluzin inequalities hold in this subclass (see Theorem E1 below; a proof of Theorem E1 is outlined in Kühnau's work [163] as well) and used this fact to obtain a difference inequality (Theorem E2) and other important results [230, Chapter 9]. The following representation of the Grunsky norm allows us to strengthen Milin's inequality (5) and thus any inequality for the class $S$ (or $\Sigma$), stemming from Theorem C (Section 6).

THEOREM E [91,93]. *Let $w = f(z) \in S$ and let $N(f)$ be the set of all nonconstant functions $Q(w)$ which are analytic in the complement of $f(E)$. Let $Q \in N(f)$ and for some $r \in (0, 1)$, let*

$$Q \circ f(z) = \sum_{n=-\infty}^{\infty} \alpha_n z^n, \quad r < |z| < 1,$$

*and*

$$\|f\|_Q = \left[ \sum_{n=1}^{\infty} n|\alpha_n|^2 \bigg/ \sum_{n=1}^{\infty} n|\alpha_{-n}|^2 \right]^{1/2}.$$

*Then*

$$\|G_f\| = \sup_{Q \in N(f)} \|f\|_Q.$$

Theorem E shows that the inequality $\|f\|_Q \leqslant k < 1$ is valid for any function $f \in S(k)$ if $Q \in N(f)$. The case when $f \in S_k$ was earlier established by V.Ya. Gutlyanskii in an equivalent form [122], see also V.G. Sheretov [249]. Actually, this inequality holds for any function $f \in S$ which has a regularly measurable continuation onto $\overline{U}$ with the homeomorphicity coefficient not exceeding $k$ and $Q$ as an admissible function (Section 8 and [91, §2 and §4]). We now restate the two results of Ch. Pommerenke in terms of functions in $S(k)$.

THEOREM E1 ([230, Chapter 9], [163, p. 100]). *Let $f \in S(k)$, $k \leqslant 1$, and let $z_n \in E$, $n = 1, \ldots, N$. Then*

$$\left| \sum_{n,m=1}^{N} x_n x_m \log \frac{[f(z_n)]^{-1} - [f(z_m)]^{-1}}{z_n^{-1} - z_m^{-1}} \right| \leqslant -k \sum_{n,m=1}^{N} x_n \overline{x_m} \log(1 - z_n \overline{z_m}) \tag{12}$$

*for all complex numbers $x_1, x_2, \ldots, x_N$ and $N = 1, 2, \ldots$.*

THEOREM E2 [230, Chapter 9]. *Let $f \in S(k)$, $k < 1$, and let $|z| = |\zeta| = 1$. Then*

$$\eta|z-\zeta|^{1+k} \leqslant \left|[f(z)]^{-1} - [f(\zeta)]^{-1}\right| \leqslant 4|z-\zeta|^{1-k},$$

*where $\eta > 0$ depends only on $k$. If $k = 1$ the left inequality is false and the right one is trivial.*

Let $f(z) \in S(k)$, $k \leqslant 1$, and $F(z) = [f(1/z)]^{-1}$, $z \in U$. Theorem E with $Q(w) = \log[w - F(\zeta)]$, $\zeta \in U$, defined in the complement of $F(U)$ (cf. case (iii) in Section 6), implies that

$$\sum_{n=1}^{\infty} n\left|A_n(\zeta)\right|^2 \leqslant k^2 \log\frac{1}{1-|\zeta|^{-2}}, \quad \zeta \in U, \tag{13}$$

where $A_n(\zeta)$ are given by (3). This inequality was first proved for the class $S_k$ by R. Kühnau [163] and then by V.Ya. Gutlyanskii [122]. Inequality (13) was used by the author [93] (see also [87]) to prove the following theorems on logarithmic coefficients of functions in $S(k)$ improving upon Milin's Theorem C4 and Bazilevich's Theorem C3 respectively.

THEOREM E3 [93]. *Let $f(z) \in S(k)$, $k \in [0, 1]$. Then*

$$\sum_{m=1}^{n} m\left|\left\{\log\frac{f(z)}{z}\right\}_m\right|^2 \leqslant (1+k)^2 \sum_{m=1}^{n} \frac{1}{m} + k(1+k)\log\frac{\Delta}{k}, \quad n \geqslant 1,$$

*where $\Delta = e^{2\delta_m} = 1.8656\ldots$, $\delta_m$ is the best known upper bound for Milin's constant (Section 6).*

*The inequality is asymptotically sharp for each class $S(k)$. For $k = 0$ the equality takes place if and only if $f = z/(1-\lambda z)$, $|\lambda| = 1$.*

The asymptotic sharpness follows, since the function

$$f_k(z) = z/(1-z)^{1+k}$$

belongs to class $S(k)$ for $k \in [0, 1]$. In fact, $f_k \in S_k$ if $k < 1$, its $k$-quasiconformal continuation to $\overline{U}$ is defined by $z(1-z)^{-1}(1-\bar{z}^{-1})^{-k}$ [90,93].

Functions of *maximal* (*largest*) *growth* in $S(k)$ are defined by the condition [93,109]

$$\alpha_f(k) = \limsup_{r\to 1-0} \max_{|z|=r} \left|f(z)\right|(1-r)^{1+k} > 0.$$

If $k = 1$ this formula gives the functions of maximal growth in $S$, i.e., functions with nonzero Hayman's index (Section 6).

THEOREM E4 [93]. *Let $f$ be a function of maximal growth in $S(k)$, $k \in [0, 1]$. Then*

$$\sum_{n=1}^{\infty} n \left| \left\{ \log \frac{f(z)}{z} \right\}_n - \frac{1+k}{n} e^{-i\theta_0 n} \right|^2 \leqslant 2k \log \frac{2^{1-k}}{\alpha_f(k)},$$

*where $\theta_0 \in [0, 2\pi)$ is defined by the condition $f(e^{i\theta_0}) = \infty$.*

Many coefficient and other results in the class $S(k)$ are based on Theorems E, E1, E2, E3, and E4 (cf. [230, Chapter 9], [91,93,108,109], and Section 17). For instance, Theorem E was used by the author to prove that the Grunsky norm does not decrease with a $p$th root transformation. One can use Krushkal's example [156] to show the sharpness of this result.

THEOREM E5 [101]. *Given a function $f \in S$ and an integer $p \geqslant 2$ define its pth root transformation $g$ by $g(z) = \sqrt[p]{f(z^p)}$, $z \in E$. Then*

$$\|G_f\| \leqslant \|G_g\|.$$

*This inequality is sharp for each subclass $\{f \in S\colon \|G_f\| = k\}$, $0 \leqslant k \leqslant 1$.*

In general, $\|G_g\|$ in Theorem E5 does not admit an upper bound smaller than 1 as can be seen by considering any unbounded function $f \in S$ with $\|G_f\| = k$, $0 \leqslant k < 1$. By way of illustration, take $f(z) = f_k(z)$. Then $f_k \in S_k$ and Theorem E3 imply that $\|G_{f_k}\| = k$ for each $k \in [0, 1]$ [93]. Since the function $g(z) = \sqrt[p]{f_k(z^p)}$ has at least two logarithmic poles on the unit circle, it cannot have a quasiconformal extension onto $\overline{\mathbf{C}}$. By Pommerenke's theorem on quasiconformal extension, $\|G_g\| = 1$.

Given $f \in S$ the sequence $\nu_p = \|G_{f_p}\|$ $(p = 2, 3, \ldots)$, where $f_p(z) = \sqrt[p]{f(z^p)}$, is not necessarily nondecreasing (see [101] and the earlier works by R. Kühnau [168] and S.L. Krushkal [156]). But if we set $\nu = \nu(f) = \limsup_{p\to\infty} \nu_p$, Theorem E5 still implies that

$$\|G_f\| \leqslant \nu_p \leqslant \nu \quad (p = 2, 3, \ldots).$$

Furthermore, if $f \in S$, with $\|G_f\| < 1$, is a bounded function which has a $k$-quasiconformal extension $\tilde{f}$ onto $\overline{\mathbf{C}}$ with $\tilde{f}(\infty) = \infty$, then it follows that $\nu(f) \leqslant k < 1$. This observation led the author to the conjecture that *the smallest possible value of $k$ is, in fact, equal to $\nu(f)$ for any function $f$ of the considered type.*

## 10. Pairs of functions with nonoverlapping image domains and the $\tau$-norm

In this section we consider an improvement of an important case of Lebedev's area theorem (Theorem D, Section 7). Let $\mathcal{M}$ be the class of all pairs $\{f, h\}$ of univalent functions mapping $E$ conformally onto nonoverlapping domains of the $w$-plane. Thus, $\mathcal{M}$ is the union of all classes $\mathcal{M}(a_0, a_1)$. For a pair $\{f, h\}$ in $\mathcal{M}$ of functions whose homeomorphic extensions to $\overline{\mathbb{C}}$ exist and are suitably compatible, certain information on this compatibility leads to an area inequality stronger than the one given by Theorem D. This improvement takes the natural form of a restriction of a $[0,1]$-valued functional on $\mathcal{M}$, the $\tau$-norm [93], appearing as a generalization of the Grunsky norm. In fact, the relation between the $\tau$-norm of two functions and information on their compatibility (homeomorphic assembling) is analogous to that one which is between the Grunsky norm of a function and information on its homeomorphic (quasiconformal) extendibility (Section 9). We note that the Kühnau's [166] and Shchepetev's [248] results gave some basis for this development (cf. the author [90]). In particular, R. Kühnau established some Grunsky-type and Goluzin-type inequalities (Section 6) for the class of homeomorphic mappings of the whole plane which are conformal for $|z| < r$ and $|z| > 1/r$ $(0 < r < 1)$, and $k$-quasiconformal for $r \leqslant |z| \leqslant 1/r$. Clearly, one obtains the results for pairs of mappings with nonoverlapping image domains (Section 7) through the obvious limit processes. V.A. Shchepetev proved a related area theorem.

We follow our presentation in [90,93]. Given $\{f, h\} \in \mathcal{M}$ denote by $N(f,h)$ the set of all nonconstant functions $Q(w)$ having a regular single-valued derivative in the complement of the union of $f(E)$ and $h(E)$. For $Q \in N(f,h)$ denote by $T_Q(f,h)$ the class (possibly void) of orientation-preserving homeomorphisms $\omega(z)$ of the extended complex plane onto itself, conformal in the exterior of some annulus $D_\omega = \{z\colon r_1(\omega) < |z| < r_2(\omega)\}$ $(0 < r_1, r_2 < \infty)$ and satisfying the conditions:

(1) there exist $\theta_1(\omega), \theta_2(\omega) \in [0, 2\pi)$ such that

$$f(z) = \omega\big(r_1 e^{i\theta_1} z\big), \qquad h(z) = \omega\big(r_2 e^{i\theta_2}/z\big), \qquad z \in E;$$

(2) $\omega$ is regularly measurable in the closed annulus $\overline{D}_\omega$ and has $Q$ as an admissible function (see Section 8). If for some $Q \in N(f,h)$ $T_Q(f,h) \neq \emptyset$ we say that functions $f$ and $h$ are (*regularly measurably*) *assembled* by the homeomorphism $\omega$.

For $\{f,h\} \in \mathcal{M}$, let $Q \in N(f,h)$. Then in some annulus $r < |z| < 1$ we have the expansions

$$Q \circ f(z) = \sum_{n=-\infty}^{\infty} a_n z^n + \beta \log z, \qquad Q \circ h(z) = \sum_{n=-\infty}^{\infty} b_n z^n - \beta \log z.$$

Let

$$\sigma_Q(f,h) = \sum_{n=1}^{\infty} n\big(|a_{-n}|^2 - |b_n|^2 + |b_{-n}|^2 - |a_n|^2\big) + 2\Re\big[\overline{\beta}(b_0 - a_0)\big],$$

$$\tilde{\sigma}_Q(f,h) = \sum_{n=1}^{\infty} n\big(\big||a_{-n}|^2 - |b_n|^2\big| + \big||b_{-n}|^2 - |a_n|^2\big|\big) + 2\big|\Re\big[\overline{\beta}(b_0 - a_0)\big]\big|.$$

Since the area of the image of the complement to $f(E)$ and $h(E)$ under any branch of $Q(w)$ is $\pi\sigma_Q(f,h)$ [181, Chapter 3] (see also [90]), we conclude that $0 \leqslant \sigma_Q(f,h) \leqslant \tilde{\sigma}_Q(f,h)$. Let $\tau_Q(f,h) = [(\tilde{\sigma}_Q - \sigma_Q)/(\tilde{\sigma}_Q + \sigma_Q)]^{1/2}$ if $\tilde{\sigma}_Q \neq 0$; otherwise set $\tau_Q(f,h) = 0$

if $T_Q(f, h) \neq \emptyset$ (i.e., $f$ and $h$ are Möbius transformations with $f(z) = h(\lambda/z)$, $|\lambda| = 1$) and $\tau_Q(f, h) = 1$ if it occurs that $T_Q(f, h) = \emptyset$ (i.e., $f$ or $h$ is not a Möbius transformation). It turns out that if the class $T_Q(f, h)$ is not empty, we can say more about the functional $\tau_Q$.

THEOREM F [90,93]. *Let* $\{f, h\} \in \mathcal{M}$, $Q \in N(f, h)$, *and* $T_Q(f, h) \neq \emptyset$. *Then for any* $\omega \in T_Q(f, h)$, *the inequality*

$$\tau_Q(f, h) \leqslant \rho(Q \circ \omega)$$

*holds.* (*See Section* 8 *for the definition of* $\rho$.)

Define the *$\tau$-norm* on $\mathcal{M}$ by $\tau(f, h) = \sup_{Q \in N(f,h)} \tau_Q(f, h)$ (cf. Theorem E, Section 9). Thus, $0 \leqslant \tau_Q(f, h) \leqslant \tau(f, h) \leqslant 1$ for all $\{f, h\} \in \mathcal{M}$ and $Q \in N(f, h)$. It follows that for any pair $\{f, h\} \in \mathcal{M}$ such that the intersection $T(f, h)$ of all classes $T_Q(f, h)$, $Q \in N(f, h)$, is nonempty, and so it consists of completely regularly measurable maps, we have (see Section 8):

THEOREM F1 [90,93]. *Let* $\{f, h\} \in \mathcal{M}$ *and* $T(f, h) = \bigcap_{Q \in N(f,h)} T_Q(f, h) \neq \emptyset$. *Then for any* $\omega \in T(f, h)$,

$$\tau(f, h) \leqslant \varrho(\omega).$$

Let $\mathcal{M}^{(k)}$, $k \in [0, 1]$, be the class of all pairs $\{f, h\} \in \mathcal{M}$ for which $\tau(f, h) \leqslant k$. Then (cf. Kühnau's inclusion theorem, Section 6) all pairs $\{f, h\} \in \mathcal{M}$ with a $k$-quasiconformal assembling belong to $\mathcal{M}^{(k)}$ (Section 8) [90,93]. Using Aharonov's normalization (Section 7) we set $\mathcal{M}_A(k) = \{\{f, g\} \in \mathcal{M}_A\colon \{f, g^{-1}\} \in \mathcal{M}^{(k)}\}$, $k \in [0, 1]$. Consequently $\mathcal{M}_A = \mathcal{M}_A(1)$ and $\mathcal{M}_A(k_1)$ is contained in $\mathcal{M}_A(k_2)$ for $0 \leqslant k_1 < k_2 \leqslant 1$. Also we have [93]

$$\mathcal{M}_A(0) = \big\{\{az, bz\}\colon 0 < |ab| \leqslant 1\big\}.$$

Next theorem improves upon Theorem D3 (Section 7).

THEOREM F2 [93]. *Let* $\{f, g\} \in \mathcal{M}_A(k)$, $k \in [0, 1]$. *Then*

$$\sum_{n=1}^{\infty} n\big(|\alpha_n|^2 + |\beta_n|^2\big) \leqslant -2k \log\big|f'(0)g'(0)\big|,$$

*where*

$$\alpha_n = \left\{\log \frac{f(z)}{z}\right\}_n, \qquad \beta_n = \left\{\log \frac{g(z)}{z}\right\}_n \quad (n = 1, 2, \ldots),$$

*and equality holds if and only if* $\tau_{\log}(f, g^{-1}) = k$.

For pairs $\{f, g\} \in \mathcal{M}_A(k)$ Theorem F2 implies, in particular, the following sharpening of Lebedev's integral inequality (Theorem D6):

THEOREM F3 [93]. *Let* $\{f, g\} \in \mathcal{M}_A(k)$, $k \in [0, 1]$. *Then the inequality*

$$\frac{1}{2\pi}\int_0^{2\pi} \left|f\left(e^{i\theta}\right)\right|^2 d\theta \cdot \frac{1}{2\pi}\int_0^{2\pi} \left|g\left(e^{i\theta}\right)\right|^2 d\theta \leqslant \left|f'(0)g'(0)\right|^{2(1-k)}$$

*holds.*

## 11. Exponentiation, Milin's method, and descent technique

Since many important facts in the univalent function theory naturally come in a "logarithmic shell", one often faces an obstacle in using them to solve an extremal problem. The attempts to overcome this difficulty gave rise in the late '60s early '70s to the *exponentiation method* for univalent functions. Figuratively, one can describe exponentiation, as a process of "pushing down" some information on the logarithm of a function resulting in establishing properties of that function:

$$\text{“desired properties of } f\text{”} = e^{\text{“given information about } \log f\text{”}}.$$

We indicate three exponentiation approaches.

The first approach, a very productive one, was introduced and applied to several coefficient problems by I.M. Milin in [201,203–205,207], and [208, Chapters 2 and 3]. Properties of formal power series generated by the exponential function play a crucial role in Milin' theory. His approach and its development incorporate the Grunsky criterion (Section 6), other logarithmic inequalities of geometric nature, and the so-called descent technique for formal power series. The descent technique allows one to estimate the derivatives (usually Taylor coefficients) of an analytic univalent function in terms of certain combinations of the derivatives of its logarithm (logarithmic functionals). To attack an extremal problem *a la* Milin one should first find satisfactory bounds on a suitably constructed logarithmic functional and then try to transform them by means of descent into a desired inequality. The result would be sharp if an extremal function is retained in the process.

A number of longstanding coefficient problems in the class $S$ and related classes (odd functions, starlike functions, Bieberbach–Eilenberg functions, functions with restricted Grunsky norm, quasiconformally extendible functions, Gel'fer functions, and others) were essentially settled by exponentiation of Milin's type, while other methods fell short of being effective (see Milin's book [208, Chapter 3] and the papers of D. Aharonov [2, 3], L. de Branges [32], P.L. Duren [48], A.Z. Grinshpan [81,82,84,86–88,90,93,100], A.Z. Grinshpan and I.M. Milin [107,214], A.Z. Grinshpan and Ch. Pommerenke [108], W.K. Hayman and J.A. Hummel [129], K. Hu [138,140], K. Hu and X.H. Dong [141], R. Kühnau and B. Dittmar [170], Y.J. Leung [188,189], Z. Nehari [221], W.Q. Yang and B.H. Liu [267], Z.Q. Ye [268]; and also the books by P.L. Duren [49, Chapter 5], A.W. Goodman [76, Chapter 12], W.K. Hayman [128], and Ch. Pommerenke [230]).

Recently the author established new sharp inequalities for arbitrary complex vectors and weights generated by the gamma function [102,103]. An alternative approach to Milin's exponentiation for univalent functions is one application of this result. In fact, a limiting case of these new inequalities generalizes the basic exponential inequalities developed by Milin. In particular, it gives the most general form of the Lebedev–Milin exponential inequalities (see Sections 11.2 and 12 below).

Another exponentiation approach, discovered by C. FitzGerald [57], stems from the fact that Goluzin's and similar inequalities (Theorem C2, Section 6) can be viewed as conditions of nonnegativity of certain Hermitian forms. The idea is to use the algebraic properties of Hermitian forms to free the initial inequalities of logarithms and then reduce them to inequalities for the Taylor coefficients of a univalent function. This method yielded the best at the time nonsharp coefficient bound for functions in $S$ as well as some coefficient inequalities not accessible to this day by any other method. The central result is the following:

FITZGERALD'S INEQUALITY [57]. *For every $f \in S$ and $n \geqslant 2$,*

$$\left|\{f\}_n\right|^4 \leqslant \sum_{k=1}^{n} k\left|\{f\}_k\right|^2 + \sum_{k=n+1}^{2n-1} (2n-k)\left|\{f\}_k\right|^2.$$

FitzGerald's approach was refined and used by N.A. Lebedev, Ch. Pommerenke, N.A. Shirokov, and D. Horowitz, and some applications of it were given by D. Bshouty and V.I. Kamotskii. For details see [181, Chapter 2], [230, Chapter 3], [132], [49, Chapter 5], and [152]. A generalized version of this technique in the setting of multiparameter Hermitian forms can be found in [91,93].

The third approach is based on the integral and differential inequalities, the traditional instruments of function theory (see, e.g., the books by G.M. Goluzin [72], W.K. Hayman [128], and Ch. Pommerenke [230,232]). These inequalities can be quite effective for solving coefficient problems (particularly if combined with other tools, e.g., Milin's exponential inequalities, Pommerenke's difference inequality; see [108,109]).

We extend Milin's term *descent technique* to mean the collection of all tools of exponentiation together with auxiliary results. The rest of this section is devoted to discussion of various tools of descent.

**11.1.** *Inequalities for formal power series/Hermitian forms generated by entire functions with nonnegative coefficients.* The first result of this kind for formal power series generated by the exponential function was given by N.A. Lebedev and I.M. Milin in 1965 ([184], see Lemma 2). The underlying properties of Hermitian forms go back to I. Schur [246], and their proofs can be found in, e.g., [181, Appendix] and [49, Chapter 5].

LEMMA 2. *For any sequence of complex numbers* $\{A_n\}_{n=1}^{\infty}$, $\sum_{n=1}^{\infty} n|A_n|^2 < \infty$, *let the coefficients* $D_n$ *be defined by the expansion*

$$\sum_{n=0}^{\infty} D_n z^n = \exp\left(\sum_{n=1}^{\infty} A_n z^n\right).$$

*Then*

$$\sum_{n=0}^{\infty} |D_n|^2 \leqslant \exp\left(\sum_{n=1}^{\infty} n|A_n|^2\right)$$

*with equality if and only if* $A_n = a^n/n$ *for* $n = 1, 2, \ldots$, *where* $|a| < 1$.

Effective applications of Lemma 2 were found by I.M. Milin and other authors (cf. [208, Chapter 3], [2,82,93,107,221]). Some generalizations of this result are given in [208, Chapter 2], [35,42,91,136,242,269] (see also [191,193]).

The following inequalities from [91] hold in the setting of Hilbert spaces of formal power series.

LEMMA 3. *Let* $\phi = \sum_{n=1}^{\infty} \alpha_n z^n$ *be a formal power series with nonnegative coefficients (not all zero), and let* $\mathcal{H}_\phi$ *be the Hilbert space of all formal power series* $\mathbf{a} = \sum_{n=1}^{\infty} A_n z^n$, *such that a coefficient* $A_n = 0$ *if* $\alpha_n = 0$ *and all the nonzero coefficients satisfy the condition*

$$\sum_{n:\, \alpha_n \neq 0} \frac{|A_n|^2}{\alpha_n} < \infty.$$

*Define the inner product on this space by*

$$\langle \mathbf{a}, \mathbf{b} \rangle_{\mathcal{H}_\phi} = \sum_{n:\alpha_n \neq 0} \frac{A_n \overline{B_n}}{\alpha_n}, \quad \mathbf{a} = \sum_{n \geqslant 1} A_n z^n, \quad \mathbf{b} = \sum_{n \geqslant 1} B_n z^n \in \mathcal{H}_\phi.$$

*Let* $\mathbf{a}_n \in \mathcal{H}_\phi$ *and* $x_n \in \mathbf{C}$ $(n = 1, 2, \ldots)$. *Then for any entire function* $g(w)$, $g(0) = 0$, *with nonnegative coefficients*, $g(\mathcal{H}_\phi) \subset \mathcal{H}_{g \circ \phi}$ *and*

$$\left\| \sum_{n=1}^{N} x_n g \circ \mathbf{a}_n \right\|_{\mathcal{H}_{g\circ\phi}}^2 \leqslant \sum_{n,k=1}^{N} x_n \overline{x_k} g\big(\langle \mathbf{a}_n, \mathbf{a}_k \rangle_{\mathcal{H}_\phi}\big), \quad N = 1, 2, \ldots,$$

*with equality if* $\mathbf{a}_n = \phi(\beta_n z)$ *for some* $\beta_n \in \mathbf{C}$ *with* $\phi(|\beta_n|^2) < \infty$ $(n = 1, \ldots, N)$. *Furthermore,*

$$\left\| \sum_{n=1}^{N} x_n \left\{ g\left(\sum_{n=1}^{\infty} \mathbf{a}_n z^n\right)\right\}_n \right\|_{\mathcal{H}_{g\circ\phi}}^2$$

$$\leqslant \sum_{n,k=1}^{N} x_n \overline{x_k} \left\{ g\left( \sum_{n,k=1}^{\infty} \langle \mathbf{a}_n, \mathbf{a}_k \rangle_{\mathcal{H}_\phi} z^n \overline{\zeta^k} \right) \right\}_{n,k}, \quad N = 1, 2, \ldots,$$

*with equality if* $\mathbf{a}_n(\zeta) = \alpha_n \beta^n \zeta^n$, $\beta \in \mathbf{C}$, $n = 1, \ldots, N$.

*Here every possible indeterminate "0/0" is considered to be* 0.

The case $N = 1$ in the first inequality of Lemma 3 contains the Lebedev–Milin inequality of Lemma 2 ($g(w) = e^w - 1$ and $\phi(z) = -\log(1 - z)$) as well as a generalization of it due to I.M. Milin [208, Chapter 2] (see also [185]). Applications of Lemma 3 and its generalization for several complex variables are given in [91]. See also Milin's book [208, Chapter 2] and the papers of I.M. Milin and the author [91,96,107,213] (Section 15) for other inequalities for formal power series and their applications.

**11.2.** *The Milin monotonicity lemma and its consequences.* This very useful lemma [208, Lemma 2.2] (Lemma 4) is stated in the same setting as Lemma 2 but, unlike its predecessor, it touches upon deep interrelated properties of the exponential and gamma functions. Applications of this result are given in Milin's book [208, Chapter 3] and in the papers [2,32,48,58,81–83,87,90,93,100,108,170,188,221] (Sections 12 and 14–17); see also a generalization of Lemma 4 in [136]. The best known application is the proof of Milin's theorem on logarithmic functionals (Section 12).

LEMMA 4. *Let* $\{A_m\}_1^\infty$ *be an arbitrary sequence of complex numbers, and let the sequence* $\{D_m\}_0^\infty$ *be defined by the expansion*

$$\sum_{m=0}^{\infty} D_m z^m = \exp\left( \sum_{m=1}^{\infty} A_m z^m \right). \tag{14}$$

*Let for* $n \geqslant 1$,

$$\Theta_n(a) = \left[ d_{n-1}(a+1) \right]^{-1} \sum_{\nu=0}^{n-1} \frac{|D_\nu|^2}{d_\nu(a)}$$
$$\times \exp\left[ -\left[ a/d_{n-1}(a+1) \right] \sum_{m=1}^{n} d_{n-m-1}(a) \sum_{k=1}^{m} \left( k|A_k/a|^2 - 1/k \right) \right],$$

*where* $a > 0$ *and* $d_n(a) = \{(1-z)^{-a}\}_n = \Gamma(n+a)/[\Gamma(a)n!]$ $(n = 0, 1, \ldots;$ $\Gamma$ *is the gamma function*).

*Then* $\cdots \leqslant \Theta_n(a) \leqslant \cdots \leqslant \Theta_2(a) \leqslant \Theta_1(a) = 1$, *and* $\Theta_n(a) = 1$ *for some* $n > 1$ *if and only if there is some* $\lambda$ *with* $|\lambda| = 1$ *such that* $A_m = a\lambda^m/m$, $m = 1, \ldots, n-1$.

Basing on Milin's argument in [208], we prove this lemma in the important case $a = 1$ (see [100]). By writing $\Theta_n = \Theta_n(1)$ we have

$$\Theta_n = n^{-1} \sum_{\nu=0}^{n-1} |D_\nu|^2 \exp\left[ n^{-1} \sum_{m=1}^{n} (n-m)\left(m^{-1} - m|A_m|^2\right) \right], \quad n \geqslant 1.$$

Thus our task is to show that $\cdots \leqslant \Theta_n \leqslant \cdots \leqslant \Theta_2 \leqslant \Theta_1 = 1$, *and* $\Theta_n = 1$ *for some* $n > 1$ *if and only if there is some* $\lambda$ *with* $|\lambda| = 1$ *such that* $A_m = \lambda^m/m$, $m = 1, \ldots, n-1$.

PROOF OF LEMMA 4 IN THE CASE $a = 1$. Differentiation of (14) and coefficient comparison yield

$$n D_n = \sum_{m=1}^{n} m A_m D_{n-m}, \quad n \geqslant 1.$$

By the Cauchy–Schwarz inequality,

$$|D_n|^2 \leqslant \sigma_n \sum_{m=0}^{n-1} |D_m|^2/n, \quad \text{where } \sigma_n = \sum_{m=1}^{n} |m A_m|^2/n. \tag{15}$$

As

$$\frac{\Theta_{n+1}}{\Theta_n} = \frac{n}{n+1}\left( \sum_{\nu=0}^{n} |D_\nu|^2 \Big/ \sum_{\nu=0}^{n-1} |D_\nu|^2 \right) \exp\left[ \sum_{m=1}^{n} \left(\frac{1}{n} - \frac{1}{n+1}\right)\left(1 - |m A_m|^2\right) \right],$$

(15) and the inequality $(1-x)e^x \leqslant 1$ with $x = (1-\sigma_n)/(n+1)$ give

$$\frac{\Theta_{n+1}}{\Theta_n} \leqslant \frac{n}{n+1}\left(1 + \frac{\sigma_n}{n}\right) \exp\left[\frac{1-\sigma_n}{n+1}\right] \leqslant 1, \quad n \leqslant 1.$$

If $\Theta_{n+1} = 1$, then $|A_1|^2 = \sigma_1 = 1$ and there are constants $\lambda_\nu$ $(\nu = 1, \ldots, n)$ such that $m A_m = \lambda_\nu \overline{D}_{\nu-m}$, $m = 1, \ldots, \nu$ . Since $D_0 = 1$ and $D_1 = A_1$, we get $\nu A_\nu = \lambda_\nu$, $|\lambda_1| = 1$, and $\lambda_\nu = \lambda_{\nu-1}\lambda_1$. It follows that $A_m = \lambda_1^m/m$, $m = 1, \ldots, n$. □

The inequalities $\Theta_n(a) \leqslant 1$ implied by Lemma 4 are known as the *Lebedev–Milin exponential inequalities*. I.M. Milin first presented them without proof in his 1967 report [204] as a joint work with N.A. Lebedev, and later used Lemma 4 to give the proof in [208, Chapter 2]. We conclude this subsection with another consequence of Lemma 4 also known as a Lebedev–Milin exponential inequality. It was first stated in [204] without proof and then proved by I.M. Milin in [208, Chapter 2].

COROLLARY OF LEMMA 4. *For each* $n = 1, 2, \ldots$

$$|D_n| \leqslant \exp\left\{\frac{1}{2}\sum_{m=1}^{n}\left(m|A_m|^2 - 1/m\right)\right\},$$

*with equality if and only if* $A_m = \lambda^m/m$, $m = 1, 2, \ldots, n$, *for some complex constant* $\lambda$ *with* $|\lambda| = 1$.

**11.3.** *quasiexponential inequalities.* We give an example of estimates for the coefficients $B_n$ generated by a sequence of complex numbers $\{A_n\}_{n=1}^{\infty}$ and a bounded function $g(z)$ via the expansion

$$\sum_{n=0}^{\infty} B_n z^n = g(z) \cdot \exp\left\{\sum_{n=1}^{\infty} A_n z^n\right\} \tag{16}$$

(see [84,86,108]).

LEMMA 5. *Let* $g(z) = \sum_{n=0}^{\infty} b_n z^n$ *be analytic and bounded by* 1 *in the closed disk* $E_r = \{z\colon |z| \leqslant r\}$, $r \leqslant 1$, *and let* $\{A_n\}_{n=1}^{\infty}$ *be a sequence of complex numbers. Then for every* $n = 1, 2, \ldots$

$$|B_n| \leqslant B \cdot \exp\left\{\frac{1}{2}\sum_{m=1}^{n}\left(m|A_m|^2 - 1/m\right)\right\},$$

*where*

$$B = \frac{a}{2}\exp\{b/a\}, \quad a = b + \left(b^2 + 4\right)^{\frac{1}{2}}, \quad b = \left(\sum_{m=1}^{n}\frac{|mb_m r^{m-n}|^2}{n}\right)^{1/2},$$

*and the coefficients* $B_n$ *are defined by* (16).

Note that Lemma 5 contains the inequality of Corollary of Lemma 4 as the case $g(z) \equiv 1$. See [84,86,108], and Section 14 for applications of Lemma 5.

**11.4.** *Asymptotic equalities for compositions of exponential type.* A basic asymptotics for exponential compositions is given by the Milin Tauberian theorem [208, Theorem 2.7], motivated by Hayman's earlier work (see [126,128]). We state a slightly modified version of this result (cf. [93, Theorem 4] and also [49, Theorem 5.7]).

MILIN'S TAUBERIAN THEOREM. *Suppose* $\omega(z) = \sum_{n=1}^{\infty} a_n z^n$ *is regular in* $E$, $\sum_{n=1}^{\infty} n|a_n|^2 < \infty$, *and* $\sup_n \Re \sum_{k=1}^{n} a_n < \infty$. *Then for* $\varphi(z) = e^{\omega(z)}$ *and any* $\mu > 1/2$ *we have*

$$\lim_{n\to\infty}\left[\left\{\varphi(z)(1-z)^{-\mu}\right\}_n / \left\{(1-z)^{-\mu}\right\}_n - \varphi(r)\right] = 0,$$

*where* $r = r(n)$ *is such that* $\log[n(1-r)]$ *remains bounded for large* $n$.

Some applications of Milin's Tauberian theorem are given in [208, Chapter 3], [81,93] (Section 17); see also [130,139].

**11.5.** *Integral and differential inequalities.* The development of these techniques has been originally motivated by the integral approach of J.E. Littlewood [194] and H. Prawitz [234] of the '20s (see also [49, Chapter 2] and [72, Chapter 4]). Many applications of integral and differential inequalities can be found in [49,72,108,128,230,232]. The following lemma of Ch. Pommerenke [231], [232, p. 180] and some similar results have useful applications ([40,109] and Section 17).

LEMMA 6. *Let $n \geqslant 2$ and let $a_k(r)$ $(k = 0, \dots, n-1)$ be continuous in $r_0 \leqslant r < 1$. Assume that $a_k(r) \geqslant 0$ for $k = 0, \dots, n-2$. Let $u(r)$ and $v(r)$ be $n$ times differentiable in $[r_0, 1)$ and*

$$u^{(n)}(r) < \sum_{k=0}^{n-1} a_k(r) u^{(k)}(r), \qquad v^{(n)}(r) = \sum_{k=0}^{n-1} a_k(r) v^{(k)}(r).$$

*If $u^{(k)}(r_0) < v^{(k)}(r_0)$ for $k = 0, \dots, n-1$ then*

$$u(r) < v(r), \quad r_0 \leqslant r < 1.$$

**11.6.** *Formal calculus and computers.* Inequalities and identities for formally defined functions have first been put to use in the univalent function theory in the mid '60s (Section 11.1) and methods of computer aid for analysts have even longer history. However the idea of great potential to combine these approaches has found its first implementation in the subject only much later. In 1994, D. Zeilberger obtained an identity for formal functions and a "formal calculus" proof of Lemma 7 (see Example 2 below) in collaboration with a computer (Shalosh B. Ekhad) [55]. Zeilberger's identity leads to a simple proof of Weinstein's integral representation for Milin's logarithmic functionals (see [263] and Sections 12, 13). A short but "human" proof of this identity is given in [99].

ZEILBERGER'S IDENTITY [55]. *For $t \geqslant 0$ and $z \in E$ let*

$$f(z,t) = z \exp\left(t + \sum_{m=1}^{\infty} c_m z^m\right),$$

$$\Phi_m(z,t) = 2\left(1 + \sum_{\nu=1}^{m} \nu c_\nu z^\nu\right) - m c_m z^m, \quad m \geqslant 1,$$

$$V_m(t) = \left\{\Phi_m(z,t)\overline{\Phi_m\left(\overline{z^{-1}}, t\right)} f_t(z,t) / \left[z f_z(z,t)\right]\right\}_0,$$

*and*

$$h(z,t) = \sum_{m=1}^{\infty} \left(m|c_m|^2 - 4/m\right) w^m,$$

*where $c_m = c_m(t)$, $m \geqslant 1$, are formal functions of $t$ and $w = w(z, t)$ is the Pick function ($e^t K(w(z,t)) = K(z)$, see Section 13). Then the following formal identity holds:*

$$h_t = \frac{1-w}{1+w} \sum_{m=1}^{\infty} \operatorname{Re}(V_m) w^m, \quad t \geqslant 0,\ z \in E.$$

**11.7.** *Examples of auxiliary inequalities and identities for estimating logarithmic functionals.* This subsection consists of three special examples. For other material consult, e.g., [208, Chapter 3], [32,87,90,100,263].

EXAMPLE 1. The following inequality was used by I.M. Milin to prove Theorem C4 (Section 6 and [208, Chapter 3]):

$$\sum_{k=1}^{n} \frac{r^{-k}}{k} + \log \frac{1}{1-r} < 2 \sum_{k=1}^{n} \frac{1}{k} + \int_0^x \frac{e^t - 1}{t}\, dt - \log x - \gamma,$$

where $r = \exp\{-2x/(2n+1)\}$, $x > 0$, $n \geqslant 1$, and $\gamma$ is the Euler constant. Other inequalities of this type were used in [87,90].

EXAMPLE 2. The polynomial property given in Lemma 7 was used (in an equivalent form) by L. de Branges in his proof of Milin's conjecture on logarithmic functionals (Sections 12 and 13). This property has been viewed as a case of the Askey–Gasper inequalities for special functions [13,32,58], as a consequence of the addition theorem for Legendre polynomials (1785) [263], or as a "computer fact" [55]. Its elementary and self-contained proof was found by the author and M.E.H. Ismail in 1996 [104]. A simpler version is given by the author in [100].

LEMMA 7. *The polynomials $P_{m,n}(x)$ defined by the formal expansion*

$$\left[1 - \left(2(1-x) + x(\zeta + \zeta^{-1})\right)z + z^2\right]^{-1} = \sum_{n=0}^{\infty} \left[\sum_{m=0}^{n} P_{m,n}(x)\left(\zeta^m + \zeta^{-m}\right)\right] z^n \tag{17}$$

*are nonnegative for $x \in [0, 1]$.*

EXAMPLE 3. The identity below was used by the author in the simplified proof of Milin's conjecture [100].

LEMMA 8. *Given a sequence of complex numbers $\{a_m\}_{m \geqslant 1}$, define a new sequence by*

$$b_m = 2\left(1 + \sum_{\nu=1}^{m} a_\nu\right) - a_m, \quad m = 1, 2, \ldots.$$

*Then*

$$4\Re\left(1+\sum_{\nu=1}^{m}\bar{a}_\nu b_\nu\right)=|a_m+b_m|^2,\quad m\geqslant 1. \tag{18}$$

PROOF. We use induction on $m$. Since $a_k+b_k=b_{k+1}-a_{k+1}$, (18) holds for $m=k+1$ if it is valid for $m=k$, and (18) holds for $m=1$ because $b_1-a_1=2$. □

## 12. The Milin theorem and conjecture on logarithmic functionals

In 1971, I.M. Milin established a far-reaching connection between the Bieberbach conjecture (Section 4) and the logarithmic coefficients of univalent functions. He constructed a sequence of logarithmic functionals on $S$, conjectured that they were nonpositive and gave an elementary argument showing that his conjecture implies Bieberbach's [208, Chapter 3].

MILIN'S THEOREM ON LOGARITHMIC FUNCTIONALS [208, discussion before Theorem 3.2]. *For $f\in S$ and $n\geqslant 1$, define*

$$I_n(f)\equiv\sum_{m=1}^{n}(n+1-m)\left(m\left|\left\{\log\left[f(z)/z\right]\right\}_m\right|^2-4/m\right). \tag{19}$$

*If*

$$I_n(f)\leqslant 0 \tag{20}$$

*for each $f\in S$ and each $n\geqslant 1$, then the Bieberbach conjecture is true.*

The functionals $I_n$ in (19) are called *Milin's functionals* and (20) is known as *Milin's conjecture*. Since $I_n(K)=0$ for all $n\geqslant 1$, (20) suggests an extremal property of the Koebe function deeper than that of Bieberbach's conjecture. Although certain cases of Milin's conjecture were proved in 1972 [83], few seriously believed back then that one could effectively attack the Bieberbach conjecture through (20). However it was this theorem that became a turning point in the long and unsuccessful quest.

Milin's theorem and conjecture are both a motivation for and a result of his exponentiation approach ([208, Chapters 2 and 3] and Section 11). As mentioned before, the monotonicity lemma (Lemma 4, Section 11), a key component of this approach, is used to prove the theorem. In fact, only a case of Lemma 4 is needed to obtain (1) [208, Chapter 3]: given $f\in S$ and $n\geqslant 2$ apply the Cauchy–Schwarz inequality and then the Lebedev–Milin inequality $\Theta_n(a)\leqslant 1$ with $a=1$ and

$$A_m=\left\{\log\left[f(z)/z\right]\right\}_m/2,\quad m=1,\dots,n-1,$$

to

$$\{f\}_n = \sum_{m=0}^{n-1} D_m D_{n-m-1}, \quad D_m = \{[f(z)/z]^{1/2}\}_m.$$

Theorem A (Section 4) allows to settle the case of equality (see also [100]).

## 13. De Branges' theorem and the proof of the Bieberbach conjecture

In 1984, L. de Branges proved a multiparameter inequality for the Löwner chains of bounded univalent functions that implies nonpositivity of Milin's functionals (19) (the final version of his proof was published in [32]). Thus, in particular, de Branges proved the following theorem.

DE BRANGES' THEOREM (Proof of the Milin conjecture [32]). *For each $f \in S$, Milin's functionals* (19) *satisfy*

$$I_n(f) \leqslant 0, \quad n \geqslant 1. \tag{21}$$

This remarkable result coupled with Milin's theorem on logarithmic functionals (Section 12) allowed L. de Branges to confirm the truth of the Bieberbach conjecture.

MAIN THEOREM (Proof of the Bieberbach conjecture [32]). *Let $f \in S$. Then*

$$|\{f\}_n| \leqslant n, \quad n = 2, 3, \ldots.$$

*Equality holds for any given n only for the Koebe function $K(z) = z/(1-z)^2$ and its rotations $\bar{\lambda} K(\lambda z)$, $|\lambda| = 1$.*

De Branges' original proof of Milin's conjecture was based on Löwner's method but involved other fields of mathematics as well. Simplifications made by several authors brought the proof of the Bieberbach and Milin conjectures to the wide readership (see I.M. Milin's comments [212], C. FitzGerald and Ch. Pommerenke [58], L. Weinstein [263], and the author [100]).

It's noteworthy that the essential structure of the proof has never been changed: apply Löwner's method and use the fact that certain functions introduced by de Branges (*de Branges' functions*) are nondecreasing in Löwner's parameter $t$. In general, every de Branges function comes from a Löwner chain starting at some mapping $f$. In the proof of (21) every such function furnishes a delicate link between the value of a Milin functional at $f$ and 0 (see Sections 5 and 12). A proof of de Branges' theorem given in [100] uses a relatively simple coefficient representation of de Branges' functions (22) and some observations from [58,263] on treating Löwner's equation and the auxiliary polynomials. We outline our presentation in [100]:

Since $I_n$ is a continuous functional on $S$, it is sufficient to prove (21) for the dense subclass of $S$ consisting of all single-slit mappings that omit a subray of the negative real axis (Lemma 1, Section 3). Fix $f(z)$ in this subclass and construct the Löwner chain $\{f(z,t)\colon\ t \geqslant 0\}$ as in Theorem B (Section 5). Then there exists some $T = T(f) \geqslant 0$ such that $f(z,t) = e^t K(z)$ for $t \geqslant T$ (discussion after Theorem B). Define the differentiable function

$$\varphi_n(t) = \left\{ K(z) \sum_{m=1}^{n} \left(m\left|c_m(t)\right|^2 - 4/m\right) w^m(z,t) \right\}_{n+1}, \qquad t \in [0,T], \tag{22}$$

where $c_m(t) = \{\log[f(z,t)/z]\}_m$ and $w(z,t)$ is the *Pick function* defined implicitly by the equation

$$e^t K\left(w(z,t)\right) = K(z), \quad z \in E,\ t \geqslant 0$$

(geometrically, for each $t > 0$, $w(z,t) = e^{-t}z + \cdots$ maps $E$ onto $E$ cut along the negative real axis from $-1$ to $1 - 2e^t(1 - \sqrt{1 - e^{-t}})$).

Observe that $w(z,0) = z$ and hence $\varphi_n(0) = I_n(f)$. Also $\varphi_n(T) = 0$ since

$$c_m(T) = \left\{\log\left[e^T K(z)/z\right]\right\}_m = 2/m, \quad m \geqslant 1.$$

Thus the desired conclusion would follow if $\varphi_n' \geqslant 0$. Miraculously, Equations (22), (2) and (17), the definitions of $w = w(z,t)$ and $K(z)$, and Lemma 8 (Section 11) imply that

$$\varphi_n'(t) = \sum_{m=1}^{n} P_{m,n}\left(e^{-t}\right)\left|c_m'(t)\right|^2. \tag{23}$$

So, in view of Lemma 7 (Section 11), $\varphi_n'(t) \geqslant 0$, $t \in [0,T]$. It follows that $I_n(f) \leqslant 0$.

Alternative representations for the derivatives of de Branges' functions, related identities, and further references can be found in [99]. No wonder that each of these representations implies, as does (23), the monotonic (nondecreasing) behavior of Milin's functionals along a Löwner chain (Section 5). This monotony stated by de Branges for more general functionals is a fact of fundamental importance in the theory of univalent functions. At present its depth can only be widely appreciated by use of Milin's exponentiation.

The most general and at the same time simple form of de Branges' discovery, which gives his multiparameter inequality in [32], is just the statement that functions (22) are nondecreasing along any normalized subordination chain of analytic functions. The proof is a version of the above argument involving Weinstein's integral representation [263] (see also [55,99], and Section 11.6) and Pommerenke's Theorem B1 (Section 5).

THEOREM G. *Let $g(z,t)$, $z \in E$, $t \in [0,T]$, be a normalized subordination chain ($g$ may not be univalent for $z \in E$). For each $n \geqslant 1$, the function $\varphi_n(t)$ in* (22), *where $c_m(t) = \{\log[g(z,t)/z]\}_m$, is nondecreasing on* $[0,T]$.

If $g(z,t) = f \exp P(f)$, where $f = f(z,t)$ is a univalent chain and $P$ is a polynomial, this theorem gives de Branges' representation.

Various aspects of de Branges' result and related questions were widely publicized (see, e.g., [7,8,15,55,58,214,258,259,263,264,270,271]). It has been known that the truth of Milin's conjecture implies the truth of Robertson's conjecture for odd functions (Section 14) and Rogosinski's conjecture for subordinate functions in $S$ (see details and references in [49, Chapter 6]). De Branges' theorem also allowed to confirm the Bazilevich conjecture on logarithmic areas (see [214] and Theorem G1) and author's conjecture on the coefficients of powers of univalent functions [86] (Theorem G2):

THEOREM G1 [214]. *For each $f \in S$ and $r \in (0,1)$, the inequality*

$$\sum_{n=1}^{\infty} n \left| \left\{ \log \frac{f(z)}{z} \right\}_n \right|^2 r^n \leqslant 4 \log \frac{1}{1-r}$$

*holds. The equality takes place if and only if $f(z)$ is a rotation of the Koebe function $K(z) = z/(1-z)^2$.*

THEOREM G2 [129,214,267]. *For each $f \in S$ and $a > 1$, the inequalities*

$$\left| \left\{ \left[ \frac{f(z)}{z} \right]^a \right\}_n \right| \leqslant \left\{ (1-z)^{-2a} \right\}_n, \quad n \geqslant 1,$$

*hold. The equality takes place if and only if $f(z)$ is a rotation of $K(z)$.*

To prove Theorem G2 one needs to use Lemma 4 in its full strength. Case $a = 1$ is the assertion of Main Theorem. Case $a = 1/2$, i.e., coefficient bounds for odd functions, is discussed in Section 14. For $a \in (0, 1/2]$, the Koebe function is no longer extremal for $n > 1$ [89]. Given $n \geqslant 1$ let $A_n$ be the least upper bound of all powers $a$ for which the last inequality fails to be true. The author has proved that $A_{2n} = 1$, $n \geqslant 1$, and $A_{2n-1} < 1$, $n \leqslant 2$, and conjectured that $A_{2n-1} < 1$ for all $n > 2$ [89]. The lower bounds for $A_{2n-1}$ found in [34] suggest that the sequence $\{A_{2n-1}\}$ monotonically increases to 1 [94]. This lack of power stability may provide an explanation for the formidable difficulty of the Bieberbach conjecture.

De Branges' theorem and the results just mentioned strongly emphasize the special role that the Koebe function plays for class $S$. In this connection we describe the situation for functionals more general than (19).

A nonzero real vector $X_n = (x_1, \dots, x_n)$ is said to be *admissible* if the Koebe function maximizes the functional of Milin's type

$$I_{X_n}(f) = \sum_{k=1}^{n} x_k \left( k \left| \left\{ \log \frac{f(z)}{z} \right\}_k \right|^2 - \frac{4}{k} \right)$$

on $S$. Thus, for every admissible vector $X_n$ and each $f \in S$, $I_{X_n}(f) \leqslant 0$. In fact, as conjectured by I.M. Milin and proved by L. de Branges, all vectors of the form

$Y_n = (n, n-1, \dots, 1), n \geqslant 1$, are admissible. Additional considerations show that Milin's functionals $I_n = I_{Y_n}$ vanish only at the Koebe function and its rotations [32,58], and that $\lim_{n\to\infty} I_n(f) = -\infty$ for all other functions $f \in S$ [83]. The case when $x_k = 1/k$ $(k = 1, \dots, n)$ corresponds to the Duren–Leung conjecture [50], which remains open for $n \geqslant 3$. According to the Milin and author work [214], every admissible vector $X_n = (x_1, \dots, x_n)$ necessarily satisfies the condition

$$\min_{\theta\in[0,\pi]} \sum_{k=1}^{n} x_k \sin(k\theta) = 0. \tag{24}$$

The diagonal vectors $\mathbf{1}_n = (1, \dots, 1)$, $n \geqslant 2$, do not satisfy (24), and, thus, fail to be admissible. Recall, however, that $I_{\mathbf{1}_n}(f) \leqslant 4\delta$ for any $f \in S$ and $n \geqslant 1$, where $\delta$ is Milin's constant (Theorem C4, Section 6). It is known that for $n < 3$, condition (24) describes all the corresponding admissible vectors, but in general this description of admissible vectors is not complete [92,94]. For related results see [12,50,92,94,96,154,155,192,214].

## 14. Successive coefficients of univalent functions and coefficients of odd functions

For $f \in S$, let

$$\Delta_n = \Delta_n(f) = \bigl|\{f\}_{n+1}\bigr| - \bigl|\{f\}_n\bigr|, \quad n \geqslant 1.$$

Thus for the Koebe function $\Delta_n(K) = 1$ for all $n$. The Bieberbach theorem (Theorem A, Section 4) implies that $|\Delta_1(f)| \leqslant 1$ for all $f \in S$. However, one could not dream of proving the Bieberbach conjecture from the inequality $|\Delta_n| \leqslant 1$, since $\sup_S |\Delta_n| > 1$ for each $n \geqslant 2$. This "anomaly" was discovered thanks to several mathematicians through a deep study of odd univalent functions.

Denote the subclass of all odd functions $g \in S$ by $S^{(2)}$ (one can think of these functions as square-root transformations of functions in $S$). Clearly all the even coefficients of an odd function are zero. In 1932, J.E. Littlewood and R. Paley proved that the odd coefficients of every function $g \in S^{(2)}$ satisfy $|\{g\}_n| \leqslant B$, $n = 3, 5, 7, \dots$, where $B$ is an absolute constant ($B < 14$). Inspired by the Bieberbach conjecture and having Theorem A (which shows that $\sup_{S^{(2)}} |\{g\}_3| = 1$) under their belts, they conjectured that $B = 1$ [195]. A year later M. Fekete and G. Szegö [56] disproved (using Löwner's method, Section 5) the Littlewood–Paley conjecture:

$$\sup_{S^{(2)}} \bigl|\{g\}_5\bigr| = 1/2 + \exp(-2/3) = 1.013\dots.$$

Then A.C. Schaeffer and D.C. Spencer showed that $\sup_{S^{(2)}} |\{g\}_n| > 1$ for each odd $n \geqslant 5$ [244]. Consequently the square-root transformation of the Koebe function $K_2(z) = z/(1-z^2)$ does not have the largest coefficients in $S^{(2)}$. Later W.K. Hayman [126], [128, Chapter 5] proved that $\lim_{n\to\infty} |\{g\}_{2n-1}| \leqslant 1$ for every $g \in S^{(2)}$ and that equality holds only for $K_2(z)$ and its rotations.

For the coefficient differences the Schaeffer–Spencer result shows that $\sup_S |\Delta_n| > 1$ for $n \geqslant 4$. Their approach still yields $\sup_S |\Delta_3| > 1$. The case $n = 2$ was handled by G.M. Goluzin [68,70] (with a minor algebraic error) and also by J.A. Jenkins [148]: $\sup_S |\Delta_2| = 1.029\ldots$.

In 1946, G.M. Goluzin proved that the order of growth of successive coefficient differences of functions in $S$ is lower than the order of the coefficient growth itself: $\sup_S |\Delta_n(f)| = o(\sup_S |\{f\}_n|)$ as $n \to \infty$ [68] (see details and references in [208, Chapter 3] or [49, Chapter 3]).

In 1963, W.K. Hayman [127] established a general result implying that $\sup_{S,n} |\Delta_n|$ is finite. The numerical bounds for $|\Delta_n|$ on $S$ were gradually lowered by I.M. Milin and other authors from 14 (accidentally coinciding with the Littlewood–Paley bound for odd functions) to 3.26 by means of Milin's exponentiation method and its modification (see [205], [208, Chapter 3], [143], [230, p. 81], [84, 268, 140]). An approach yielding the best known bound uses inequality (9) (Section 6) and Lemma 5 (Section 11) (cf. the author [84], Z.Q. Ye [268], and K. Hu [140]). Despite the fact that $|\Delta_n(f)|$ can be greater than 1 for every $n \geqslant 2$, the individual behavior for large $n$ is proper: the results of W.K. Hayman [126], B.G. Eke [53,54], and D.H. Hamilton [125] show that for each function $f \in S$, $\limsup_{n\to\infty} |\Delta_n(f)| \leqslant 1$ (see also [208, Chapter 3], [49, Chapter 5], and [141]). Interestingly enough the inequalities $|\Delta_n(f)| \leqslant 1 (n \geqslant 2)$ do hold for all starlike functions $f \in S$. This was conjectured by Ch. Pommerenke [229] and proved by Y.J. Leung via Milin's exponentiation [188] (see also [49, Chapter 5]). Furthermore, for every $k < 1$, $\sup_{S(k)} |\Delta_n| \to 0$ as $n \to \infty$ (see Sections 9 and 17, and [93,108]). For the class $S_k$, this was first shown by R. Kühnau and B. Dittmar [170] (see also [87]).

Going back to the uniform bounds for the coefficients of odd functions we note that the early Littlewood–Paley result has been dramatically improved (see [49, Chapter 5]). In 1967, I.M. Milin used his Theorem C4 (Section 6) and Corollary of Lemma 4 (Section 11) to show that

$$\sup_{S^{(2)}} \left|\{g\}_n\right| \leqslant \exp(\delta/2) < 1.17, \quad n \geqslant 5,$$

where $\delta$ is Milin's constant [204], [208, Chapter 3]. This coefficient bound was lowered by V.I. Milin [216] to 1.14 and then by K. Hu to 1.1305 [138].

The following conjecture for the coefficients of odd functions is stronger than Bieberbach's but weaker than Milin's and it should justly be regarded as a link between them. In 1936, M.S. Robertson conjectured that *each function* $g \in S^{(2)}$ *satisfies the inequality*

$$\sum_{k=1}^{n} \left|\{g\}_{2k-1}\right|^2 \leqslant n$$

*for every* $n \geqslant 2$ *with equality if and only if* $g$ *is a rotation of* $K_2$ [237] (see also [59,83]). L. de Branges confirmed the Robertson conjecture in 1984 by proving Milin's ([32] and Section 13).

An application of the new exponential inequalities [102,103] to (21) allows us to show that for any $f \in S$, real numbers $a$ and $b$ ($a > 0,\ b \geqslant 0,\ 2a + b \geqslant 2$), and $n = 1, 2, \ldots$, the following inequality holds

$$\sum_{k=0}^{n} \frac{d_{n-k}(b)}{d_k(2a)} \left| \left\{ \left[ \frac{f(z)}{z} \right]^a \right\}_k \right|^2 \leqslant d_n(2a+b),$$

where $d_k(b) = \{(1-z)^{-b}\}_k$ ($k = 0, 1, \ldots$). The equality takes place if and only if $f$ is a rotation of the Koebe function. This inequality contains the Bieberbach ($a = 1,\ b = 0$), Robertson ($a = 1/2,\ b = 1$), and Milin ($a \to 0+,\ b = 2$) cases, as well as the statement of Theorem G2 ($a > 1,\ b = 0$).

For more general coefficient results, i.e., bounds for coefficients and coefficient differences of $p$-valent functions and powers of univalent functions, see Section 19 (Goodman's conjecture), Hayman's book [128] and its bibliography, and also [33,34,86, 88,89,94,129,137,192,211,214,240,267].

## 15. Estimates for functions whose range has a finite logarithmic area

The study of some traditional classes of analytic and univalent functions in $E$, such as bounded functions, functions with a finite range area, and Bieberbach–Eilenberg functions (Section 7), has led to a consideration of more general family defined by a condition of logarithmic type [93,107]. This new family of functions retains a number of old features but is more natural for studying the interplay between geometric and analytic function properties. The corresponding class $\mathcal{A}_S$ of the suitably normalized functions is discussed below.

Let $\pi\sigma(B)$ be the area of a measurable plane set $B$. Similarly, for $g(z)$ analytic in $E$, let $\pi\sigma(g)$ be the area of its image domain $g(E)$ on the corresponding Riemann surface. The *logarithmic area* of the simply connected domain $B = f(E)$ containing zero, where $f$, $f(0) = 0$, is analytic and univalent in $E$, is defined to be $\pi\sigma(\log[f(z)/z])$ [107]. For such a domain $B$ (or Riemann mapping $f$), the $\mathcal{A}$-measure of $B$, $\mathcal{A}(B)$ (or $\mathcal{A}(f)$) is defined by $\mathcal{A}(B) = 2\log d + \sigma(\log[f(z)/z])$, where $d$ is the inner conformal radius of $B$ with respect to the origin [82]. Note that for simply connected domains, the Teichmüller reduced logarithmic area [257] coincides with the $\mathcal{A}$-measure (see [82,107]). A number of nontrivial properties of the $\mathcal{A}$-measure was established in [82]. In particular, a sharp inequality

$$\mathcal{A}(B) = \mathcal{A}(f) \leqslant \log\sigma(B)$$

was proved there and it was shown that the $\mathcal{A}$-measure (unlike the logarithmic area) is monotonic as a function of domain. One consequence of this inequality is the following result of P.L. Duren and M.M. Schiffer (see [52]): *for $f \in S$ with a finite transfinite diameter $R$,*

$$\sum_{n=1}^{\infty} n \left| \left\{ \log \frac{f(z)}{z} \right\}_n \right|^2 \leqslant 2 \log R.$$

Clearly, for a bounded function $f \in S$ the last inequality for $\mathcal{A}$-measure is stronger than $\mathcal{A}(f) \leqslant 2 \log \sup_{z \in E} |f(z)|$ (Section 7), however its applications are not known in abundance.

An example of a function in $S$ having an infinite logarithmic area of its range is the Koebe function. Any bounded function in $S$ or, more generally, any function in $S$ with a finite range area supplies an example of the opposite. The function $f(z) = -\log(1-z) \in S$ shows that an unbounded function whose image domain has an infinite area can have a finite logarithmic area [93].

Let $\mathcal{A}_S$ consist of all functions $f(z)$, $f(0) = 0$, that are analytic and univalent in $E$ and satisfy the condition $\mathcal{A}(f) \leqslant 0$ [107]. It follows that functions in this class have a finite logarithmic area of the range. Moreover, every function $g \in S$ whose range has a finite logarithmic area can be written as $g = \lambda f$ where $f \in \mathcal{A}_S$ and $\lambda$ is a constant.

A number of sharp inequalities for functions $f(z) \in \mathcal{A}_S$ was obtained by exponentiation in [107]. We list some of them:

$$\frac{1}{2\pi} \int_{|z|=1} |f(z)||dz| \leqslant |f'(0)|^{1/2}, \tag{25}$$

$$\frac{1}{2\pi} \int_{|z|=1} |f(z)|^2 |dz| \leqslant 1, \tag{26}$$

$$\frac{1}{\pi} \iint_E |f(z)/z|^4 |dz| \leqslant 1, \tag{27}$$

$$|f(z)| \leqslant |z| \left(1 - |z|^2\right)^{-1/2}, \quad z \in E. \tag{28}$$

The proofs of these and similar inequalities use various generalizations of Lemma 2 (Section 11; [107,213]). Equality in (25)–(28) occurs only for functions of the form

$$f(z) = f'(0) z (1 - \zeta z)^{-\varepsilon}, \quad |f'(0)| = \left(1 - |\zeta|^2\right)^{\varepsilon^2/2}, \quad \zeta \in E, \tag{29}$$

for $\varepsilon = 1$ in (26)–(28) and for $\varepsilon = 2$ in (25). The inequalities (26) and (28) are extensions to the class $\mathcal{A}_S$ of the Lebedev (Corollary of Theorem D6) and Jenkins (Theorem D5) inequalities for Bieberbach–Eilenberg functions (Section 7).

Consider now the problem of estimating the Taylor coefficients of functions in $\mathcal{A}_S$. First note that every function $f(z)$ which is analytic, univalent, and bounded by 1 in $E$ satisfies the area inequality

$$\sigma(f) = \sum_{n=1}^{\infty} n |\{f\}_n|^2 \leqslant 1.$$

Hence we have

$$\left|\{f\}_n\right| \leqslant n^{-1/2} \quad (n = 1, 2, \ldots). \tag{30}$$

It turns out that inequality (30) (actually the one with a smaller constant) holds for the whole class $\mathcal{A}_S$ and is sharp as far as the order of growth in $n$ is concerned. This result for Bieberbach–Eilenberg functions was published independently by D. Aharonov [2] and Z. Nehari [221] in 1970 and by the author [82]. The approach used allows one to prove (30) for all functions in $\mathcal{A}_S$. This is accomplished by combining the inequality $\mathcal{A} \leqslant 0$ with the Corollary of Lemma 4 (Section 11). The sequence of functions

$$f_n(z) = z\left[n^{1/2} - (n-1)^{1/2} z\right]^{-1} \in \mathcal{A}_S \quad (n = 1, 2, \ldots)$$

shows that the growth order in (30) cannot be improved, since $|\{f_n\}_n| = n^{-1/2}(1 - n^{-1})^{(n-1)/2}$ is asymptotically equivalent to $(en)^{-1/2}$ $(n \to \infty)$. Further details can be found in the author's work [93] where it was proved for $n \leqslant 2$ and conjectured for $n \geqslant 3$ that $\sup |\{f\}_n|$ *over the class* $\mathcal{A}_S$ *is attained by a function of the form* (29) *with some* $\zeta \in E$ *and* $\varepsilon$, $0 < \varepsilon \leqslant 1 + |\zeta|^{-1}$, *both depending on* $n$.

## 16. Coefficient properties of Gel'fer functions

In 1946, S.A. Gel'fer extended the well known class $\mathcal{P}$ of the Carathéodory functions, i.e., functions $\varphi(z)$, $\varphi(0) = 1$, that are analytic in $E$ and have positive real part there, to the class of analytic functions satisfying the condition $\varphi(z) + \varphi(\zeta) \neq 0$ for all $z, \zeta \in E$ ([62] and Section 7). Gel'fer's functions have many curious and useful properties (cf. [62,81, 105,182], [181, Introduction and Chapter 3], [16,41,64,85,90,93,142,190,236,266,273]). We discuss some of them related to their Taylor coefficients.

S.A. Gel'fer showed that the half-plane mapping $\varphi_0(z) = (1+z)/(1-z) = 1 + 2z + 2z^2 + \cdots$, which is known to be extremal for a number of basic functionals on $\mathcal{P}$ (such as $|\{\varphi\}_n|$: $|\{\varphi\}_n| \leqslant 2$, $n \geqslant 1$, for $\varphi \in \mathcal{P}$), remains extremal for some of them on the extended (Gel'fer's) class [62]. In particular, he proved that $|\{\varphi\}_1| \leqslant 2$ for his functions.

An interesting conjecture raised by Gel'fer is that the function $\varphi_0$ is still extremal for the estimate of $|\{\varphi\}_n|$, $n \geqslant 2$, on the whole Gel'fer class. To this end, we note that for every univalent Gel'fer function $\varphi$ the sequence $|\{\varphi\}_n|$ converges to a limit not exceeding 2, and 2 is attained only by the function $\varphi_0(z)$ and its rotations $\varphi_0(\lambda z)$, $|\lambda| = 1$ (compare with Hayman's asymptotics for coefficients of odd functions, Section 14). The proof of this result given in [81] is based on the Bazilevich logarithmic inequality (Theorem C3, Section 6) and Milin's Tauberian theorem (Section 11). Furthermore, every Gel'fer function is subordinate to some univalent Gel'fer function [62] (see also [181, Introduction]). However, the inequality $|\{\varphi\}_2| \leqslant 2$, was disproved by J.A. Hummel [142], and it was consequently shown by the author [85] that for univalent Gel'fer functions $\sup |\{\varphi\}_n|$ is greater than 2 for any even value of $n$.

On the other hand, S.A. Gel'fer proved that the coefficients of univalent functions in his class are uniformly bounded: $|\{\varphi\}_n| \leqslant 13.56\ldots$ [62] (see the correction in [200, p. 358]).

Later the author improved this bound using a double exponentiation of Milin's type (see [81,90]):

$$\left|\{\varphi\}_n\right| < 2.3, \quad n = 2, 3, \ldots.$$

The proof involves a generalization of Theorem C4 (Section 6), inequality (11) (Section 7), and Lemma 4 (Section 11).

The sharp growth order in $n$ of the Taylor coefficients of nonunivalent Gel'fer functions is still unknown. This question seems to remind the following result of W. Rogosinski [239]: the best possible coefficient estimate for analytic in $E$ functions which are subordinate to a function with bounded coefficients is $O(n^{1/2})$ as $n \to \infty$. But a version of the Littlewood–Prawitz integral approach (Section 11.5) involving subordination principle (Section 7) allows us to obtain the uniform coefficient estimate $O(\log n)$ $(n \to \infty)$ for all Gel'fer functions.

## 17. Coefficient estimates depending on Grunsky norm

The coefficient properties of univalent functions are in close connection with the size of their Grunsky operators and issues of quasiconformal extendibility (Section 9). It turns out that the condition of logarithmic type

$$\|G_f\| \leqslant k < 1$$

fits as an input into the exponentiation machine. As a result we obtain the best known estimates for the Taylor coefficients (see author's papers [87,90,93]) and coefficient differences (see the author and Ch. Pommerenke [108]) of functions in the class $S(k)$. As a consequence, the same results are valid for all functions $f$ in $S$ which have a $k$-quasiconformal extension onto the whole sphere $\overline{\mathbf{C}}$ (class $S_k$, Section 9). For instance, the following estimates of the sharp growth order in $n = 2, 3, \ldots$ hold in $S(k)$:

$$\left|\{f\}_n\right| < 1.6n^k \quad (0 < k < 1),$$

$$\left|\left|\{f\}_{n+1}\right| - \left|\{f\}_n\right|\right| < \vartheta n^{k-1} \quad (k > 1/2),$$

where $\vartheta$ depends only on $k$. The proofs are based on Theorems E3 and E2 (Section 9) and Lemma 4 (Section 11). The function $f_k(z) = z/(1-z)^{1+k} \in S_k$ (Section 9 and [93]) shows that the growth order in $n$ cannot be improved in either case (see related conjectures in [94]). When $k \to 1-$, the second estimate yields Hayman's result for successive coefficients of univalent functions (Section 14, [127], [128, Chapter 6]). Another exponentiation procedure involving Milin's Tauberian theorem (Section 11) and Theorem E4 (Section 9) led the author [93] to the following asymptotic equality for functions $f(z) \in S(k)$, $k \in [0, 1]$:

$$\lim_{n\to\infty}\left[\frac{\{f\}_n}{\{f_k\}_n} - \frac{e^{-i\theta_0 n} f(re^{i\theta_0})}{f_k(r)}\right] = 0,$$

where $\theta_0 \in [0, 2\pi)$ is defined by the condition $f(e^{i\theta_0}) = \infty$ (if $f$ is bounded in $E$, $\theta_0$ can be any real number) and $r$ satisfies $\log[n(1-r)] = O(1)$ $(n \to \infty)$. This result improves upon Hayman's coefficient asymptotics for univalent functions [126] (cf. Hayman's regularity theorem, Section 6). It has been also shown in [93] that functions of maximal growth in $S(k)$ (Section 9) satisfy, for $k > 1/2$, the condition

$$\lim_{n\to\infty}\left[\frac{\{f\}_n - e^{-i\theta_0 n}\{f\}_{n-1}}{\{f_k\}_n - \{f_k\}_{n-1}} - \frac{e^{-i\theta_0 n} f(re^{i\theta_0})}{f_k(r)}\right] = 0.$$

The case $k = 1$ corresponds to Hayman's theorem on the asymptotic behavior of the coefficient differences in the class $S$ [126] (see also [208, Chapter 3] and Section 14).

The coefficients of bounded functions with a restricted Grunsky norm is a rather more technical matter. The corresponding theorem of the author and Ch. Pommerenke [109] gives both the sharpened coefficient estimates for bounded functions in $S$ and the estimates of the best known growth order in $n$ for functions $f$ in $S_k$, $k < 1$, having a $k$-quasiconformal extension $\tilde{f}$ with $\tilde{f}(\infty) = \infty$ (class $S_k(\infty)$). Namely, for a bounded function $f(z)$ in $S(k)$, $k \in (0, 1]$, we have:

$$\left|\{f\}_n\right| = o\left(n^{\varsigma(k)-1}\right) \quad (n \to \infty),$$

$$\text{where } \varsigma(k) = \begin{cases} 3k^2/2 + 5k^3, & 0 < k \leqslant 1/4; \\ k - 1/19, & 1/4 < k \leqslant 1/3; \\ k\sqrt{14}/4, & 1/3 < k \leqslant 0.517; \\ k/100 + 0.4784, & 0.517 < k \leqslant 1. \end{cases}$$

The exponentiation procedure involves Lemma 6 (Section 11) as well as various properties of bounded functions in $S(k)$. An example of M. Chuaqui, B. Osgood and Ch. Pommerenke [39] shows that for small values of $k$, the best possible estimate for $\varsigma(k)$ must be greater than $ck^2$, where $c > 0$ is an absolute constant. See also Krushkal's result in $S_k(\infty)$ for fixed $n$ and small $k$ [157]. The value $\varsigma(1)$ corresponds to the best known estimate for uniformly bounded univalent functions, which is due to N.G. Makarov and Ch. Pommerenke (circulated in 1995, published in [199]). This case ($k = 1$) has a long history (see [232] for details and references) and it has been conjectured by L. Carleson and P.W. Jones [37] that the best possible value for $\varsigma(1)$ is $1/4$.

We refer to the book by S.L. Krushkal and R. Kühnau [158] for the wide applications of quasiconformally extendible univalent functions from the classes $\Sigma$ and $S$ (in particular, of those in $S_k(\infty)$, $k < 1$). The latest development on the connection between the Grunsky norm and quasiconformal extendibility is discussed in [101] (Section 9).

Furthermore, we note that exponentiation also applies to a similar logarithmic condition for $\tau$-norm, namely $\tau(f, g^{-1}) \leqslant k \leqslant 1$ for $\{f, g\} \in \mathcal{M}_A$. For example, Theorem F2 (Section 10) and Corollary of Lemma 4 (Section 11) imply that under this condition,

$$n\left|\{f\}_n \cdot \{g\}_n\right| < \left|f'(0) \cdot g'(0)\right|^{1-k} \quad (n = 2, 3, \ldots).$$

Further details and examples can be found in [93].

## 18. The Goodman conjecture and polynomial compositions

Once the Bieberbach conjecture has been settled (Sections 4, 12, 13) one looks for the next problem in the natural sequence. This is the coefficient problem for $p$-valent functions with $p > 1$. As a rule, one considers the coefficient problem for analytic $p$-valent functions in $E$ (although the problem for meromorphic functions is worth studying). The efforts of the early workers in the field culminated in the 1936 theorem of M. Biernacki.

BIERNACKI'S THEOREM ([30], [31, Chapter 1]). *If $F(z)$, $F(0) = 0$, is analytic and p-valent in $E$ (in the sense that given $w$, the equation $w = F(z)$ has at most p solutions in $E$), then for all $n > p$,*

$$|\{F\}_n| \leqslant C(p) b n^{2p-1},$$

*where $b = \max\{|\{F\}_1|, \ldots, |\{F\}_p|\}$ and $C(p)$ is an absolute constant that depends only on $p$.*

This result was quite good at the time, although it was not sharp (see also G.M. Goluzin [67]). In 1948, in his thesis, A.W. Goodman [73] proposed the following sharp conjecture.

*If $F(z)$, $F(0) = 0$, is analytic and p-valent in $E$, then for all $p \geqslant 2$ and all $n > p$*

$$|\{F\}_n| \leqslant \sum_{k=1}^{p} \frac{2k(n+p)!}{(p+k)!(p-k)!(n-p-1)!(n^2-k^2)} |\{F\}_k|. \tag{31}$$

Goodman showed that, if true, this conjecture is sharp for every nonzero set $\{|\{F\}_1|, \ldots, |\{F\}_p|\}$. It turns out that the Goodman conjecture is a very hard problem. Even the simplest case $p = 2, n = 3$ is still open (see M. Watson [261] and also [97,196]). However, this conjecture has been proved in some special cases. The best known result in its favor is due to A.W. Goodman and M.S. Robertson who proved in 1951 [78] that the inequality (31) holds for the class of typically-real functions of order $p$, a very large class of multivalent functions. The Goodman and Robertson work depends on an interesting algebraic identity which was proved in [78] by a function theoretic argument (see also [77]). An algebraic proof of this identity was found by the author [95].

Another reasonable type of test functions for Goodman's conjecture is the class of polynomials in univalent functions:

$$F(z) = \sum_{m=1}^{p} b_m f^m(z) \quad (b_p \neq 0,\ f \in S). \tag{32}$$

For such compositions, A. Lyzzaik and D. Styer, who first studied this important case [198], showed that inequality (31) is equivalent to a collection of determinant inequalities in terms of the coefficients of powers of functions from the class $S$. The author used this result to prove that the following two assertions are equivalent [97,98].

(i) *On the class of all functions*

$$\Psi(z_1, \dots, z_p) = \prod_{m=1}^{p} f^k(z_m) \prod_{1 \leqslant \nu < \mu \leqslant p} [f(z_\mu) - f(z_\nu)]$$

$$(z_m \in E;\ m = 1, \dots, p),$$

*where* $p, k \geqslant 1$ *and* $f(z)$ *is in* $S$, *the coefficients of all terms of the form*

$$z_p^n \prod_{m=1}^{p-1} z_m^{k+m} \quad (p + k \leqslant n,\ n \geqslant 1)$$

*in the Taylor series expansion of* $\Psi$ *about* $(0, \dots, 0)$ *become maximal in absolute value under the choice* $f(z) = K(z)$ (*the Koebe function*).

(ii) *The Goodman conjecture is true for all polynomial compositions* (32) $(p \geqslant 1)$.

Thus, the polynomial case of the Goodman conjecture is equivalent to the coefficient conjecture (i) for functions of several complex variables, which seems to be a very deep generalization of the Bieberbach conjecture. This case might be reduced to a "logarithmic problem" for exponentiation [98], if one discovers a suitable version of Milin's approach (Section 11) in the case of $p \geqslant 2$ variables.

It is interesting to note that the Bieberbach conjecture can be generalized for $p$-valent functions in two distinctly different ways, which coalesce when $p = 1$. In the first generalization (31), the sharp upper bound for $|\{F\}_n|$ depends on the first $p$ coefficients. In the second generalization, this bound depends on the location of the zeros of $F(z)$, see A.W. Goodman [74,75].

## 19. A remark on noncoefficient problems

No doubt that the exponentiation machinery was primarily designed to deal with the difficult coefficient problems for univalent functions. However it may be quite effective for problems of other kind. Some examples of it are given in Sections 7 and 15. We conclude our survey with two more examples.

In 1951, N.A. Lebedev and I.M. Milin [183] conjectured that for each function $f \in S$ with a finite image area, $\pi\sigma(f)$, the following inequality holds

$$\frac{1}{2\pi} \lim_{r \to 1-} \int_{|z|=r} |f(z)||dz| \leqslant \sigma^{1/4}(f),$$

where equality occurs if and only if $f(z) = z$.

This inequality happened to be a consequence of more general result proved by exponentiation. In 1991, I.M. Milin and the author confirmed the Lebedev–Milin conjecture and showed that the exponent $1/4$ is the best possible [107].

Two inequalities on growth and distortion for functions $f(z) \in S(k)$ (functions in $S$ whose Grunsky norm does not exceed $k$), $k \in [0, 1)$, follow from estimates established by exponentiation ([93], Sections 9 and 17). Namely, for every $z \in E\backslash\{0\}$, we have:

$$|f(z)| < \sqrt{2} \cdot f_k(|z|) \quad \text{and} \quad |f'(z)| < \sqrt{2} \cdot f_k'(|z|),$$

where $f_k(z) = z/(1-z)^{1+k} \in S(k)$. Clearly, the growth order as $|z| \to 1$ is sharp in both cases and although $|f(z)|$ and $|f'(z)|$ are not maximized by $f_k$ (for $z \neq 0$) even on a subclass $S_k$ (functions with a $k$-quasiconformal extension) of $S(k)$, $0 < k < 1$, these simple bounds are of practical interest. The sharp estimates are harder to establish and much harder to apply, see, e.g., the growth theorem for $S_k$, $k < 1$, proved by V.Ya. Gutlyanskii and V.A. Shchepetev [124].

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CHAPTER 11

# Circle Packing and Discrete Analytic Function Theory

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*Contents*



**Abstract**

Circle packings – configurations of circles with specified patterns of tangency – came to prominence with analysts in 1985 when Thurston conjectured that maps between such configurations would approximate conformal maps. The proof by Rodin and Sullivan launched

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a topic which has grown steadily as ever more connections with analytic functions and conformal structures have emerged. Indeed, the core ideas have matured to the point that one can fairly claim that circle packing provides a *discrete analytic function theory*.

There are two related but distinct aspects *vis-a-vis* the classical model – analogy and approximation. This survey concentrates on the analogies, with a largely pictorial tour intended for the reader familiar with classical conformal geometry. The companion survey, K. Stephenson, Proceedings of the Third CMFT Conference, Vol. 11, treats approximation.

Section 1 of the chapter covers circle packing basics and introduces discrete analytic functions as maps between circle packings. Representatives of the standard classes of analytic functions on the sphere, plane, and unit disc are illustrated and the discrete parallels to basic theorems in complex analysis are reviewed. Section 2 visits several geometric facets of conformal geometry in their discrete encarnations: extremal length, harmonic measure, conformal welding, and conformal structures, among others. The final topic there, conformal tiling, shows both the potential for synergy between the discrete and classical theories and the value of the experimental capabilities available with circle packing. Appendix A comments on computational aspects of circle packing and Appendix B summarizes the circle packing literature.

## Introduction

A *circle packing* is a configuration of circles with a specified pattern of tangencies. Early examples may be helpful, so the reader is invited to the small menagerie collected in Figure 1; of course, our packings will involve more features than we are prepared to illustrate at this early stage.

Scattered examples of circle packings can be traced back to antiquity, but our notion was formulated by William Thurston in his famous *Notes* [96] for use in constructing certain orbifolds. Recognizing a rigidity reminiscent of that associated with conformal structures, he was led to a conjecture, announced at the 1985 Bieberbach conference [97], on the approximation of conformal mappings using circle packings. This conjecture, to which we return in §7, was proven shortly thereafter by Rodin and Sullivan [83], marking the beginning of the association between circle packings and analytic function theory. (Note that circle packing has essentially *no contact* with the more familiar "sphere packing" studies – how many ping pong balls fit in a boxcar. Here it is the *pattern of tangencies* which is central.)

After seeing the basic notions of circle packing in a moment, I think you will find these configurations a fascinating blend of local geometric rigidity and global geometric flexibility. A mapping between one configuration of circles realizing a certain tangency pattern and another configuration realizing that same pattern will be termed a "discrete" analytic function. These maps are the main objects of study here, and the claim is, quite simply, that *discrete analytic functions behave geometrically like classical analytic functions*. Incidentally, I'm using the adjective "classical" simply to distinguish the standard continuous theory from our discrete version; for the classical theory, see, e.g., [6,5].

The emerging discrete theory offers two related but distinct lines of inquiry: namely, *analogy* and *approximation*. The deep and surprisingly comprehensive parallels with the classical theory are the focus in this article, and my central task is to guide the reader in *discretizing* the intuition he/she has developed in the classical setting. In addition, however, the objects of this discrete theory invariably converge under appropriate refinement to their classical counterparts. Such approximations, direct descendents of Thurston's original conjecture, appear here in a supporting role, and the interested reader is encouraged to see the survey article [95] for more details.

With space at a premium, our survey takes a decidedly graphical approach. The paper has two parts. The first provides the bare minimum of circle packing definitions, notation, and mechanics, with images intended to evoke the central geometric intuition. Motivated by those images we define discrete analytic functions, formulate discrete versions of several fundamental classical results, and illustrate familiar classes of analytic functions in their discrete forms. Parallels with the classical theory remain at the forefront.

With the basic theory in place, the second part of the paper formulates discrete encarnations of various geometric companions of analyticity – type problems, harmonic measure, extremal length, conformal structures, conformal welding, and so forth. In examples and selected applications the reader will see how the discrete and classical theories complement one another, with sometimes one and sometimes the other taking the lead. Our final topic, conformal tiling, shows in particular the powerful synergy which can develop.

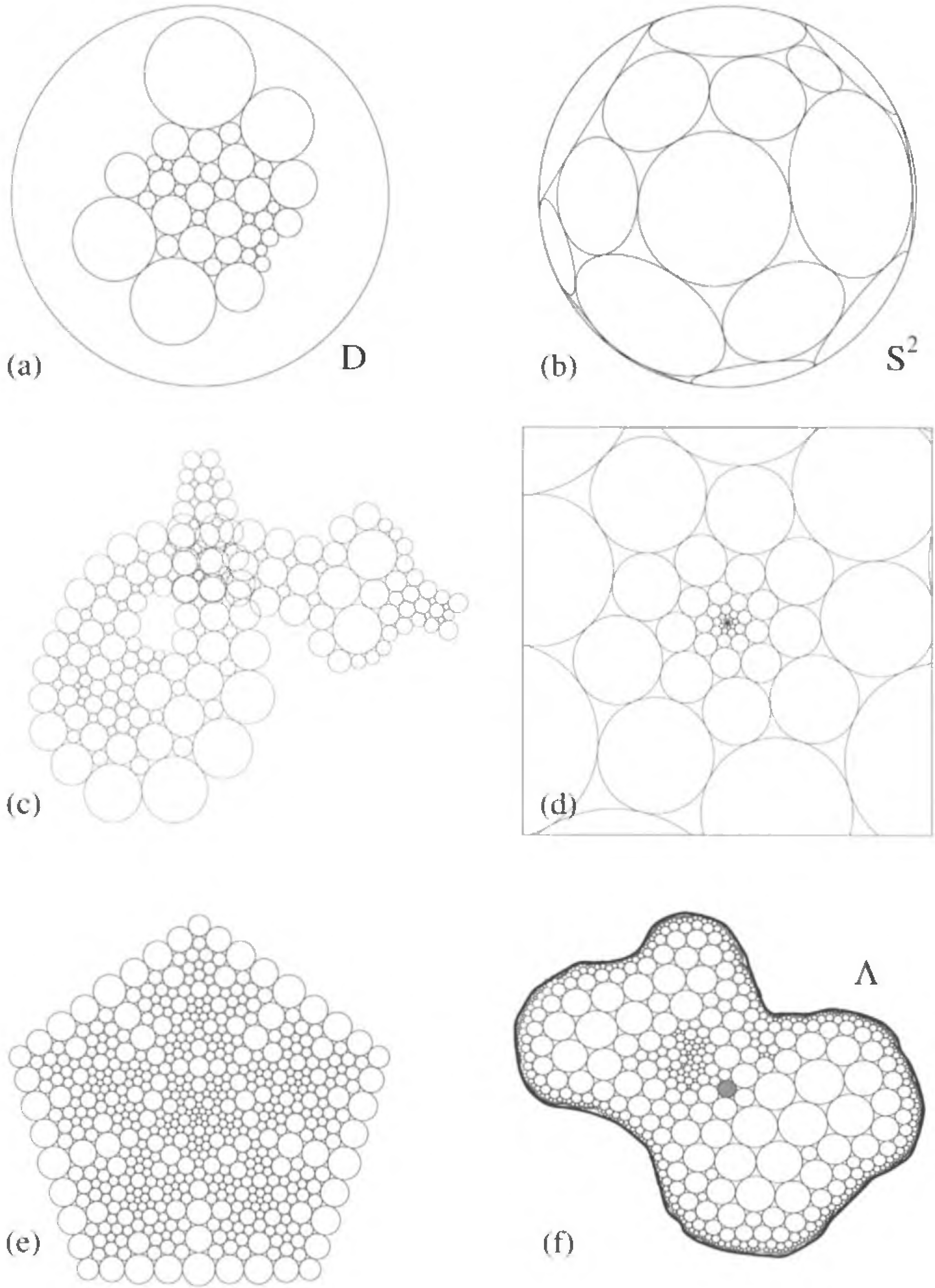



Fig. 1. A small menagerie of circle packings.

A third aspect of circle packing is implicit throughout the paper: namely, circle packings are effectively computable, so the discrete objects can be created and manipulated experimentally. Our illustrations were generated by the software package `CirclePack` (see Appendix A), which implements all the topics discussed here. One of the joys of circle packing is seeing results emerge in live action; we're restricted here to static images, but our approach will emphasize the dynamical nature of the subject and the visual geometric intuition.

This survey is based on talks I gave at the March 1996 Oberwolfach conference *Funktionentheorie: konforme und quasikonforme Abbildungen*. My thanks to the organizers, F.W. Gehring, R. Kühnau, and St. Ruscheweyh, and to the many colleagues in function theory who have encouraged the pursuit of this topic in recent years.

## 1. Discrete analytic function theory

The theory of circle packing began on the sphere with this theorem, proven successively by P. Koebe, E.M. Andreev, and W. Thurston [64,7,8,96]:

KOEBE–ANDREEV–THURSTON THEOREM. *Given an abstract triangulation $\mathcal{K}$ of a topological sphere, there exists a circle packing $P$ on the Riemann sphere $\mathbb{S}^2$ having the combinatorics of $\mathcal{K}$. $P$ is unique up to Möbius transformations and inversions of $\mathbb{S}^2$.*

The conditions on $P$ mean that each vertex $v$ of $\mathcal{K}$ is represented by a circle $c_v \in P$, with $c_v, c_u$ tangent whenever vertices $v$ and $u$ are connected by an edge in $\mathcal{K}$. The richness of circle packings may be traced to their dual natures: *combinatoric* in the pattern of prescribed tangencies and *geometric* in the realization by actual circles.

### 1.1. *Circle packing basics*

We focus attention mainly on circle packings in the familiar metric spaces of constant curvature, namely the plane $\mathbb{C}$, the sphere $\mathbb{S}^2$, and the hyperbolic plane $\mathbb{D}$ (the unit disc with the Poincaré metric), though we have occasion to treat packings on more general Riemann surfaces, accommodated using standard covering arguments. Circle packing requires a surprisingly modest system of bookkeeping. The main objects and terms are as follows:

**Complex:** Combinatorics of a packing are encoded in an abstract simplicial 2-complex $\mathcal{K}$. The conditions are most easily described by requiring that $\mathcal{K}$ be a triangulation of an oriented topological surface.

**Label:** $R$ is a collection $\{R(v)\}$ of positive numbers associated with vertices $v$ of $\mathcal{K}$. If we are working in a metric space $\mathcal{D}$, these represent putative radii, and the label is described as *euclidean*, *spherical*, or *hyperbolic* depending on whether the metric is curvature 0, 1, or $-1$, respectively.

**Angle sum:** A label $R$ determines an angle sum $\theta_R(v)$ at each vertex of $\mathcal{K}$. In particular, for each face $\langle v, u, w\rangle \in \mathcal{K}$, let $\alpha_R(v; u, w)$ denote the angle at $c_v$ in a triangle formed by the centers of a mutually tangent triple $\langle c_v, c_u, c_w\rangle$ of circles of radii $\langle R(v), R(u), R(w)\rangle$ in $\mathbb{S}^2$, $\mathbb{C}$, or $\mathbb{D}$, depending on curvature; this angle can be computed using the appropriate law of cosines. The angle sum is then $\theta_R(v) = \sum_{\langle v,u,w\rangle} \alpha_R(v; u, w)$, summing over all faces containing $v$.

**Packing:** A circle packing $P$ for $\mathcal{K}$ is a configuration of circles in the metric space $\mathcal{D}$ satisfying the tangency pattern of $\mathcal{K}$. That is, $P$ is a collection $\{c_v\}$ of circles, one for each vertex $v$ of $\mathcal{K}$, so that $c_v$ is tangent to $c_u$ whenever $\langle v, u\rangle$ is an edge of $\mathcal{K}$ and so that a triple $\langle c_v, c_u, c_w\rangle$ is positively oriented if $\langle v, u, w\rangle$ is an oriented face of $\mathcal{K}$. Were the radii of $P$ listed in a label $R$, we would write $P \leftrightarrow \mathcal{K}(R)$.

**Flower:** The flower of vertex $v$ refers to the "central" circle $c_v$ and the chain of "petal" circles intended to be tangent to it. Combinatorially, the flower is identified with the "star" of $v$ in $\mathcal{K}$.

**Carrier:** carr$(P)$ denotes the concrete geometric complex in $\mathcal{D}$ formed by connecting the centers of tangent circles of a packing $P$ with geodesic segments; this is simplicially equivalent to the abstract complex $\mathcal{K}$.

Figure 2 illustrates these basics with a simple finite complex labeled $\mathcal{K}$; (b) is a rather generic hyperbolic packing for $\mathcal{K}$ in $\mathbb{D}$, (c) is a euclidean packing in $\mathbb{C}$ arranged so that its carrier (shaded in the picture) forms a rectangle, and (d) is the maximal packing in $\mathbb{D}$, to be discussed shortly. Note that from one packing to another the circles' *radii* and even

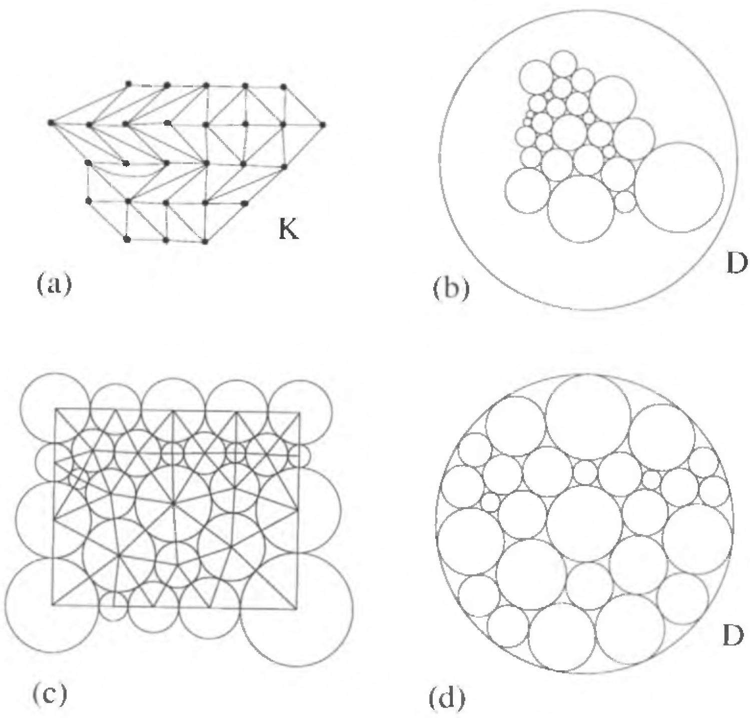



Fig. 2. Circle packings for a simple complex $\mathcal{K}$.

the ambient geometry are subject to change – it is the *combinatorics* which these packings share.

The surprise, perhaps, is that for a given complex $\mathcal{K}$ there exist *any* packings whatsoever, much less a potentially huge variety of packings. The local compatibility of the labels is key.

DEFINITION. A label $R$ is a *packing label* for $\mathcal{K}$ if the angle sum $\theta_R(v)$ is a positive integral multiple of $2\pi$ for every interior vertex $v$.

This is clearly a necessary condition for $R$ to represent the radii of a packing $P$, since $\theta_R(v) = 2\pi n$, $n \geqslant 1$, simply reflects the fact that the petal circles of the flower of $v$ must reach precisely $n$ times around $c_v$. It is also a sufficient condition if there are no topological obstructions:

LEMMA 1. *If $\mathcal{K}$ triangulates a simply connected surface, then a label $R$ for $\mathcal{K}$ represents the radii for a circle packing of $\mathcal{K}$ if and only if $R$ is a packing label.*

We say little about circle centers because they play a secondary role: once a packing label $R$ is in hand, the circles can be laid out in $\mathcal{D}$ ($\mathbb{S}^2$, $\mathbb{C}$, or $\mathbb{D}$, as appropriate) using a process akin to analytic continuation. The resulting circle packing $P \leftrightarrow \mathcal{K}(R)$ is *essentially unique* – that is, it is unique up to a conformal automorphism of $\mathcal{D}$.

A bit of terminology: A packing is *univalent* if its circles have mutually disjoint interiors. Univalence fails locally at a *branch* circle (or vertex), that is, at an interior circle whose neighbors wrap two or more times around it (angle sum $4\pi, 6\pi, \ldots$). In Figure 3, the same nine circles which wrap once about the central circle in Figure 3(a) wrap exactly twice around the smaller central circle of Figure 3(b), illustrating branching of order 1. The *branch set* for a packing $P$, denoted $br(P)$, refers to the indices of its branch circles, repeated according to multiplicities. If all interior angle sums are $2\pi$, then $br(P) = \emptyset$ and the packing is said to be *locally univalent*. Even with local univalence at every circle, global univalence can fail when circles from one part of a packing overlap those from another, as in Figure 1(c).

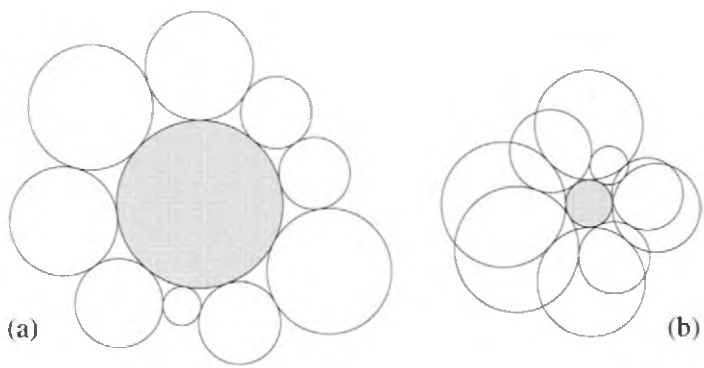



Fig. 3. Local univalence *versus* branching.

### 1.2. *Maximal packings*

Given a complex $\mathcal{K}$, there are numerous results on the existence and variety of packings for $\mathcal{K}$; we need two main types. Let us begin with the *extreme rigidity* displayed by maximal packings.

CIRCLE PACKING THEOREM. *Given a complex $\mathcal{K}$, there exists a unique Riemann surface $\mathcal{X}$ and a circle packing $P_{\mathcal{K}}$ for $\mathcal{K}$ in the intrinsic metric of constant curvature on $\mathcal{X}$ so that $P_{\mathcal{K}}$ is univalent and fills $\mathcal{X}$. The packing $P_{\mathcal{K}}$ is unique up to conformal automorphisms of $\mathcal{X}$ and is called the* maximal packing *for $\mathcal{K}$.*

This is just the Koebe–Andreev–Thurston Theorem when $\mathcal{K}$ triangulates a sphere; it was extended by Beardon and Stephenson to arbitrary $\mathcal{K}$ of bounded degree [10] and then by He and Schramm to cases of unbounded degree [55].

Figure 4 illustrates maximal packings for several complexes. Note that each $\mathcal{K}$ "chooses" its own geometry: that is, $\mathcal{K}$, which begins as a topological surface, is endowed with a

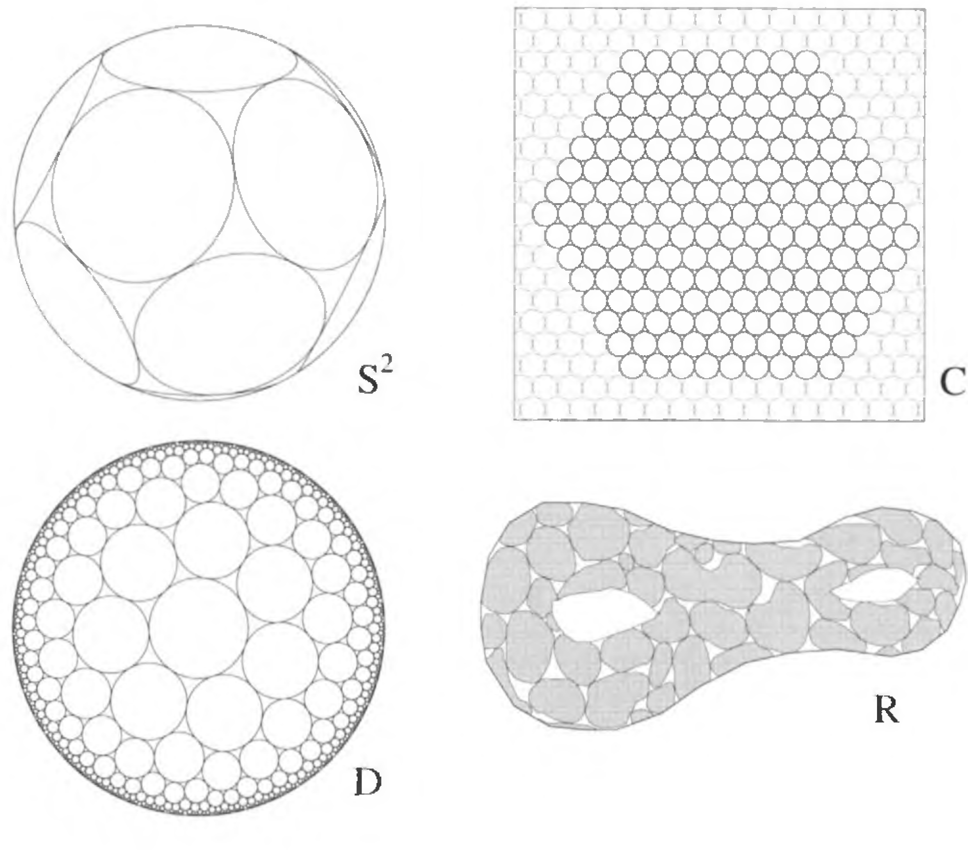



Fig. 4. Maximal packings.

unique conformal structure and associated metric which support $P_{\mathcal{K}}$. The complex $\mathcal{K}$ is said to be *spherical*, *parabolic*, or *hyperbolic* depending on the curvature of that geometry.

We cannot go into the details of the proof, but it may be helpful to highlight some of the familiar geometric notions which are pivotal. The first case is the key, both in theory and practice.

***Case I:*** *$\mathcal{K}$ triangulates a closed topological disc.*

Here $\mathcal{K}$ is finite, simply connected, and has a boundary. Figure 2(a) is an elementary example, and the extremal nature of its maximal packing, Figure 2(d), is visually evident – the boundary circles are horocycles, circles internally tangent to the unit circle. Indeed, horocycles are naturally interpreted as circles of infinite hyperbolic radius, so our task reduces to finding a hyperbolic packing label for $\mathcal{K}$ whose boundary labels are infinite.

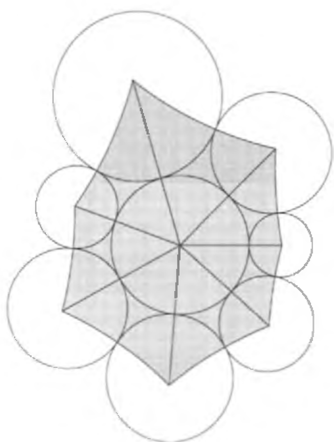
Formally, the proof relies on monotonicity results and a "Perron" method. One defines $\Phi$ as the collection of *subpacking* labels for $\mathcal{K}$, that is, (hyperbolic) labels $R$ with the property that the associated angle sums for interior vertices are greater than or equal to $2\pi$. One shows that $\Phi$ is non-empty, closed under maxima, and uniformly bounded for interior $v$. The upper envelope, the label $\widehat{R}$ defined by

$$\widehat{R}(v) = \sup_{R\in\Phi}\big\{R(v)\big\},$$

can be shown to be a packing label with infinite boundary radii. With these radii in hand, successively laying out the circles gives $P_{\mathcal{K}}$.

It is more informative (and practical) to approach the packing problem algorithmically. The geometric ingredients are surprisingly elementary. Consider a univalent flower in $\mathbb{D}$ as shown below, with central (hyperbolic) radius $r$ and $m$ petal radii $r_1, r_2, \ldots, r_m$; the (hyperbolic) faces are shown for reference. Then:

**Monotonicity:** If $r$ is increased, then the angle sum at the center circle goes *down* while the angle sum at each of the petal circles goes *up*.



**Rodin/Sullivan Ring Lemma:** There's a constant $C = C(m)$ so that $r_j/r \geqslant C$; that is, in a (univalent) flower, no petal circle can be too much smaller than the center circle.

**Bound:** $r \leqslant -\log(\sin(\pi/m))$. In the hyperbolic plane, a circle cannot be too large if $m$ petal circles can wrap at least once around it.

To get some taste for the reasoning, suppose we were to grant the existence of *some* initial hyperbolic packing $P_0$ for $\mathcal{K}$. I ask the reader to imagine the effect of increasing one of its boundary circles. It's an interesting exercise depending solely on *monotonicity*; deduce an *upward* pressure reverberating through *all* of the interior radii as they adjust in order to keep their angles sums at $2\pi$. The adjustments ultimately lead to a new packing for $\mathcal{K}$ which accommodates the increased boundary circle. Iterating this *increment/repack* cycle allows one to push the boundary radii to $\infty$; monotonicity and our bound force the interior radii to converge to finite limiting values. The result is the maximal packing label $R_{\mathcal{K}}$. The argument is quite striking when implemented live on a computer screen. Figure 5(a) is an initial packing $P_0$ for the complex of Figure 2. Incrementally increasing the boundary radii generates a succession of intermediate packings, several of which

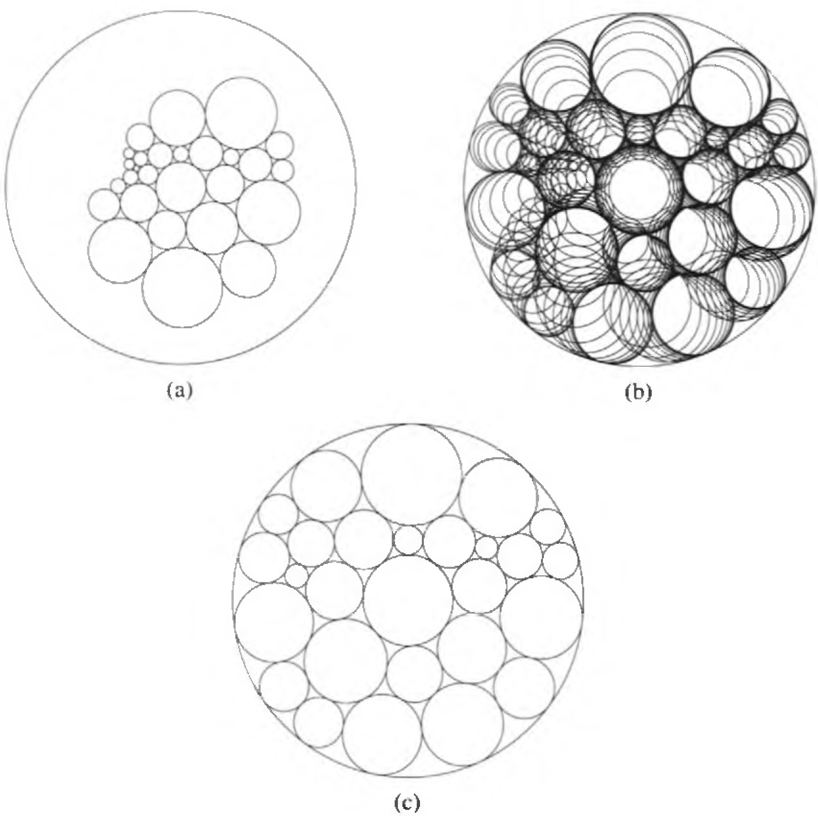


Fig. 5. Growing to the max.

are superimposed in Figure 5(b). One can see the monotonicity at work as the maximal packing, isolated in Figure 5(c), emerges.

The bound on radii for interior circles, which fails in euclidean geometry, along with the availability of circles (horocycles) having infinite radius are what make the proof click in the hyperbolic setting. The uniqueness of $P_{\mathcal{K}}$ (up to automorphisms of $\mathbb{D}$) can be established using monotonicity along with the association in hyperbolic geometry between "angles" and "area". (It also follows from fixed point arguments due to He and Schramm, which we'll comment on later.) The fact that the circles of the maximal packing have mutually disjoint interiors is a consequence of local univalence and a standard topological argument principle.

Let's register an important observation about $R_{\mathcal{K}}$ which can be spun off from the proof.

LEMMA 2. *Suppose $\mathcal{K}$ triangulates a closed topological disc. If $R$ is any hyperbolic packing label for $\mathcal{K}$, then $R \leqslant R_{\mathcal{K}}$, meaning that $R(v) \leqslant R_{\mathcal{K}}(v)$ for every vertex $v \in \mathcal{K}$. Moreover, equality holds at a single interior vertex if and only if $R = R_{\mathcal{K}}$.*

This justifies the adjective "maximal" for $P_{\mathcal{K}}$ and will shortly become the key ingredient for handling infinite complexes.

***Case II:*** *$\mathcal{K}$ triangulates a sphere.*

The spherical case is an easy consequence of Case I. Choosing a vertex $v_\infty$ of $\mathcal{K}$, form a reduced complex $\mathcal{K}'$ by removing $v_\infty$ and all edges and faces containing $v_\infty$. Then $\mathcal{K}'$ triangulates a closed disc, so Case I yields a maximal packing $P' = P_{\mathcal{K}'}$ in $\mathbb{D}$. Stereographic projection to $\mathbb{S}^2$ carries $P'$ to a circle packing in the southern hemisphere. All its boundary circles, as projections of horocycles in $\mathbb{D}$, will be tangent to the northern hemisphere, so by including that hemisphere as the circle $c_\infty$ for $v_\infty$, we have a spherical packing in $\mathbb{S}^2$ for $\mathcal{K}$ itself. A Möbius transformation can be applied for any desired normalization.

This stereographic projection argument is Thurston's and was originally used to prove Case I *from* Case II. However, as you will see in the next case, the hyperbolic arguments give added value.

***Case III:*** *$\mathcal{K}$ triangulates an open topological disc.*

Here $\mathcal{K}$ is infinite, simply connected, and without boundary. Designate some interior vertex $v_0$ and exhaust $\mathcal{K}$ by a nested sequence of finite simply connected complexes $v_0 \in \mathcal{K}_1 \subset \mathcal{K}_2 \subset \cdots \to \mathcal{K}$. For each $j$, write $P_j$ for the maximal packing of $\mathcal{K}_j$, $R_j$ for the maximal packing label, and $r_0^{(j)}$ for the label at $v_0$. Note that within the maximal packing $P_{j+1}$ lies a packing for $\mathcal{K}_j$. By Lemma 2, therefore, $r_0^{(j+1)} < r_0^{(j)}$; that is, the labels at $v_0$ are strictly decreasing,

$$r_0^{(1)} > r_0^{(2)} > r_0^{(3)} > \cdots .$$

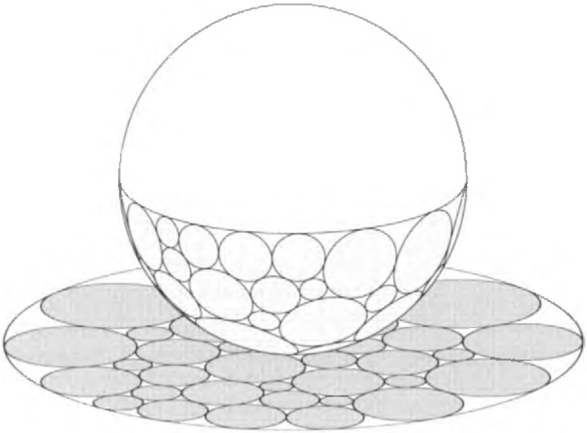


Fig. 6. Projection to the sphere.

In other words, as more circles of $\mathcal{K}$ are incorporated, the circle at the origin gets smaller. That brings us to a crucial

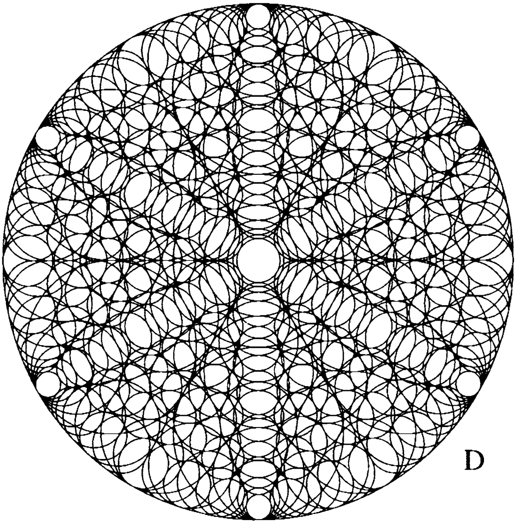
$$\text{Dichotomy:} \quad \begin{cases} \text{(P)}: & r_0^{(j)} \to 0, \\ \text{(H)}: & r_0^{(j)} \to r_0^{(\infty)} \text{ for some } r_0^{(\infty)} > 0. \end{cases}$$

We will find that alternatives (P) and (H) correspond to *parabolic* and *hyperbolic*, respectively. It might be best to illustrate with a pair of particularly clean examples. Let $\mathcal{K}^6$ denote the familiar hexagonal (constant 6-degree) complex and $\mathcal{K}^7$, the heptagonal (constant 7-degree) complex. In each instance choose $\mathcal{K}_j$ to consist of vertices within $j$-generations of $v_0$ and normalize $P_j$ so that $v_0$ is at the origin and some designated neighbor $v_1$ is centered on the positive imaginary axis.

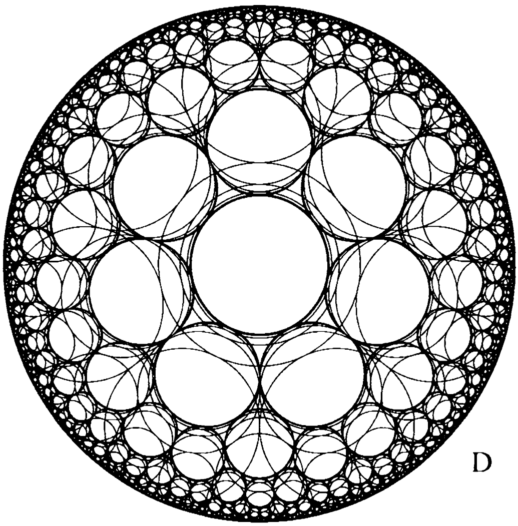
In Figure 7(a) I have superimposed the first few packings $P_j$ associated with $\mathcal{K}^6$. Note that the circle for $v_0$ at the origin is shrinking rather rapidly – this is alternative (P). In contrast, in Figure 7(b) I have superimposed the first few packings associated with $\mathcal{K}^7$; the circle at the origin again gets smaller, but very quickly stabilizes at a positive value – this is alternative (H).

The maximal packing $P_{\mathcal{K}}$ in each case results from a "geometric" limiting process. In the heptagonal case, Figure 7(b), one can almost see the limit packing emerging. Formally, one uses the Ring Lemma to prove existence of positive limit radii for *every* vertex $v$ and diagonalization to deduce limits for circle centers. It is relatively easy to show that the limit packing $P$ is univalent and fills $\mathbb{D}$. In other words, $\mathcal{K}^7$ is *hyperbolic* and the packings $P_j$ converge to $P_{\mathcal{K}^7}$.

On the other hand, it is certainly difficult to see a penny packing emerging in Figure 7(a). Indeed, in alternative (P), the Ring Lemma implies that *all* circle radii decrease to zero. Consequently, we shift perspective, treating each $P_j$ as a euclidean packing and scaling it so the circle for $v_0$ is the unit circle. Now the Ring Lemma yields positive and finite limits for all the (euclidean) radii and again diagonalization provides us with a univalent




6-degree




7-degree

Fig. 7. The hyperbolic/parabolic dichotomy.

limit packing $P$. Thus $\mathcal{K}^6$ is *parabolic* and $P = P_{\mathcal{K}^6}$, the familiar "penny" packing. The proof that carr $P_{\mathcal{K}^6} = \mathbb{C}$ in the parabolic case is more difficult than it might seem. It was confirmed for $\mathcal{K}$ having bounded degree in [10] using quasiconformal arguments. The proof in the general case is even more subtle and was provided by He and Schramm in [55]. Their key observation? *Distinct circles can intersect in at most* 2 *points*! I can't give details, but their arguments deserve mention not only for their elegance but for the powerful tools they bring to the discrete setting – versions of the winding number arguments so central in classical complex analysis.

If $\mathcal{K}$ is infinite, simply connected, but has boundary, then it falls under the *hyperbolic* alternative (H). With this observation, we find that we have taken care of all simply connected complexes, Case I being *hyperbolic*, Case II *spherical*, and Case III either *hyperbolic* or *parabolic* depending on combinatorics. Before going on, it is important to note the essential uniqueness of all the extremal packings obtained so far: *when $\mathcal{K}$ is simply connected, its maximal packing $P_{\mathcal{K}}$ is unique up to Möbius transformations of the sphere, plane, or disc, as appropriate.*

***Case IV:*** *$\mathcal{K}$ triangulates a surface $S$.*

We assume that $S$ is an oriented topological surface and, in view of our earlier cases, that $S$ is not simply connected. It is well known that a triangulation $\mathcal{K}$ of $S$ can be lifted to a complex $\widetilde{\mathcal{K}}$ triangulating the universal covering surface $\widetilde{S}$ of $S$. There is an associated simplicial projection $p: \widetilde{\mathcal{K}} \to \mathcal{K}$ and a group $G$ of simplicial automorphisms $g: \widetilde{\mathcal{K}} \to \widetilde{\mathcal{K}}$ satisfying $p \circ g \equiv p$.

$\widetilde{\mathcal{K}}$ is simply connected, and since the sphere covers only itself, $\widetilde{\mathcal{K}}$ must be an infinite triangulation of a topological disc, Case III. Let $\mathcal{D}$ denote the plane or disc, depending on whether $\widetilde{\mathcal{K}}$ is parabolic or hyperbolic, and let $\widetilde{P} = P_{\widetilde{\mathcal{K}}}$ denote the maximal packing in $\mathcal{D}$ for $\widetilde{\mathcal{K}}$. The situation is illustrated in Figure 8 for a hyperbolic case. The circle packing shown in $\mathbb{D}$ is just the part of the infinite packing $\widetilde{P}$ associated with a fundamental domain for the covering $p$.

The essential uniqueness of $\widetilde{P}$ in $\mathcal{D}$ becomes the key ingredient as we deploy a standard arrow-chasing argument. Briefly, each simplicial automorphism $g$ of $\widetilde{\mathcal{K}}$ must induce a Möbius transformation $M_g$ of $\mathcal{D}$ which maps the packing $\widetilde{P}$ to itself. $\Gamma = \{M_g\colon\ g \in G\}$ is a discrete group of Möbius transformations of $\mathcal{D}$ isomorphic to $G$. Let $\mathcal{X}$ denote the Riemann surface $\mathcal{D}/\Gamma$ obtained in the classical manner from $\mathcal{D}$ by identifying all points equivalent modulo $\Gamma$, and write $\pi: \mathcal{D} \to \mathcal{X}$ for the analytic covering projection. As topological surfaces, $\mathcal{X}$ and $S$ are homeomorphic, however $\mathcal{X}$ inherits a conformal structure and conformal metric from $\mathcal{D}$ under $\pi$. This "intrinsic" metric is either hyperbolic or euclidean, depending on $\mathcal{D}$, and each circle in $\mathcal{D}$ projects, *a fortiori*, to a "circle" in the intrinsic metric on $\mathcal{X}$. Clearly, the projected circles $\pi(\widetilde{P})$ in $\mathcal{X}$ provide an *in situ* packing for $\mathcal{K}$. This is precisely the maximal packing $P_{\mathcal{K}}$ we have been looking for.

This concludes our overview of maximal packings. Even in this last case, note that $\mathcal{X}$ is uniquely determined among all Riemann surfaces homeomorphic to $S$ based purely on the combinatorics of $\mathcal{K}$, so the take-home message is that $\mathcal{K}$ again "chooses" the appropriate geometry for its maximal packing.

### 1.3. *Packing variety*

In stark contrast to the *rigidity* of maximal packings, one finds tremendous *variety* among general packings. In Figure 9, for example, all packings share one complex $\mathcal{K}$, the lower right image being its maximal packing. The variety comes not from combinatorics, but

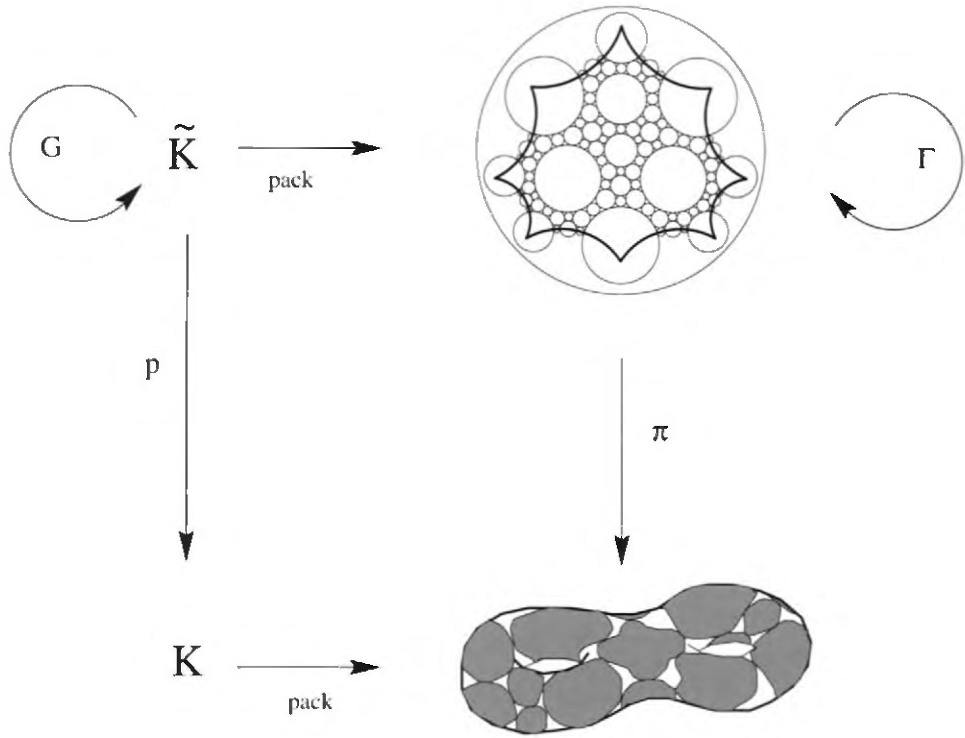



Fig. 8. Discrete covering map.

from solving packing problems with various prescribed boundary value and branching conditions.

Assume for a moment that $\mathcal{K}$ triangulates a closed topological disc and that $\beta = \{b_1, \ldots, b_k\}$ is a list (perhaps empty, perhaps with repetitions) of interior vertices of $\mathcal{K}$ satisfying the following property:

> *If $\gamma$ is a simple closed edge path in $\mathcal{K}$ with $m$ edges, then $\gamma$ encloses no more than $(m-1)/2$ points of $\beta$.*

Dubejko [38] and Bowers [15] have shown that this purely combinatoric condition is necessary and sufficient for $\beta$ to be the branch set of a packing for $\mathcal{K}$. Writing $\partial\mathcal{K}$ for the boundary vertices of $\mathcal{K}$, we have

THEOREM 3. *Let $\mathcal{K}$, $\beta$, and a boundary label $g : \partial\mathcal{K} \to (0, \infty)$ be given. Then there exists a unique euclidean packing label $R$ for $\mathcal{K}$ with branch set $\beta$ and satisfying $R(w) = g(w)$, $w \in \partial\mathcal{K}$. The same result holds in hyperbolic geometry, where $g$ is also allowed to assume the value $+\infty$.*

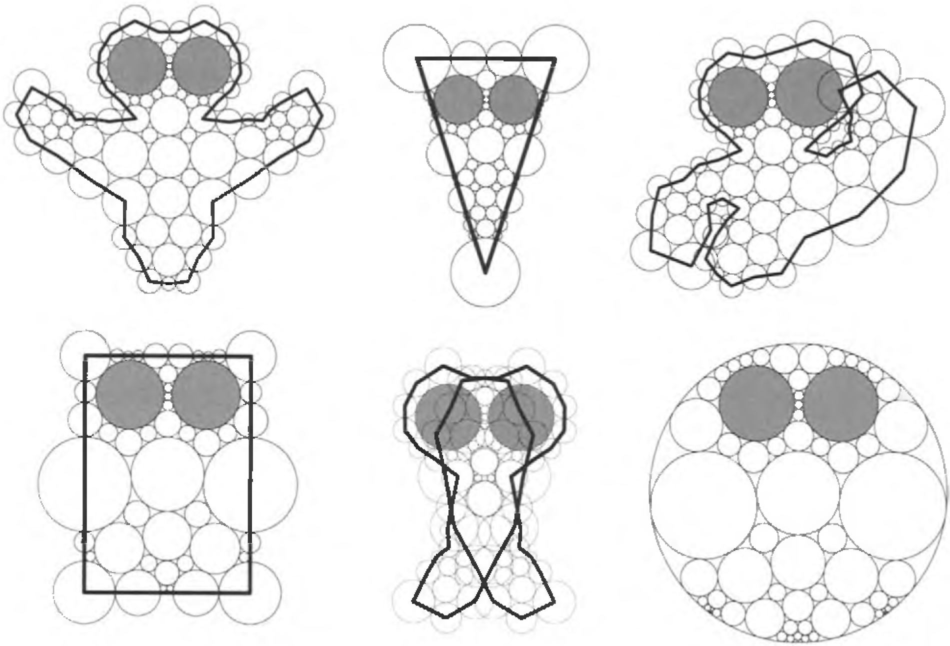


Fig. 9. Owl abuse.

This means that in principle one can find all euclidean and hyperbolic packings for $\mathcal{K}$. For instance, the maximal packing for $\mathcal{K}$ is simply the hyperbolic packing having empty branch set and prescribed boundary radii $g(w) = \infty$, $w \in \partial\mathcal{K}$. Moreover, in practice, these finite circle packings are computable by methods we discuss in Appendix A, so Theorem 3 is the basis for the practical implementation of circle packing. (One can also solve *boundary angle sum* problems, used, for example, in obtaining Figure 2(c).)

The global malleability of circle packings for $\mathcal{K}$ should not obscure the *local rigidity* implied by the packing condition at each vertex. The local-to-global linkage is moderated by the combinatorics of $\mathcal{K}$, and though we will shortly see a few additional methods for generating circle packings, many fascinating existence and computational issues remain open: existence and computation of branched spherical packings, computation of packings in prescribed regions, univalence criteria for packing labels, packings for infinite complexes, with issues of "ideal" boundary values, infinite branch sets, and so forth.

### 1.4. *Discrete analytic functions*

How can one impose some useful order on the potentially huge variety of packings for a given complex $\mathcal{K}$? I propose a *function theory* paradigm, wherein one associates each packing with the range of a mapping from the maximal packing $P_{\mathcal{K}}$. Let me begin with the broadest definition, which we will narrow and then justify through examples.

DEFINITION. A *discrete analytic function* is a map $f : Q \to P$ between circle packings which preserves tangency and orientation. The corresponding *ratio function* $f^{\#}$ is defined by $f^{\#}(c) = \text{radius}(f(c))/\text{radius}(c)$, $c \in Q$.

There is no loss in restricting to cases where $Q$ and $P$ are packings for the same complex $\mathcal{K}$ and where for each vertex $v \in \mathcal{K}$, the circle for $v$ in $Q$ is mapped by $f$ to the circle for $v$ in $P$. Specifying further that $Q$ be the maximal packing $P_{\mathcal{K}}$, gives us our

ORGANIZING PRINCIPLE. *Associate each packing $P$ for $\mathcal{K}$ with the discrete analytic function $f : P_{\mathcal{K}} \to P$.*

Instead of studying packings *per se*, we are now studying functions, with the richness of analytic function theory as a guide. Moreover, since $Q$ packs $\mathbb{S}^2$, $\mathbb{C}$, or $\mathbb{D}$, the standard classifications and terminology apply: rational, meromorphic, entire, univalent/multivalent, bounded/unbounded, branched, and so forth, have their usual meanings. Thus, if $\mathcal{K}$ is hyperbolic and $P$ is a packing in $\mathbb{S}^2$, then $f : P_{\mathcal{K}} \to P$ would be a *discrete meromorphic* function on the disc.

### 1.5. *Examples of discrete functions*

Figure 10 is a cartoon comparing the univalent, locally univalent, and branched discrete analytic functions to their classical models. Let us consider a range of more explicit examples.

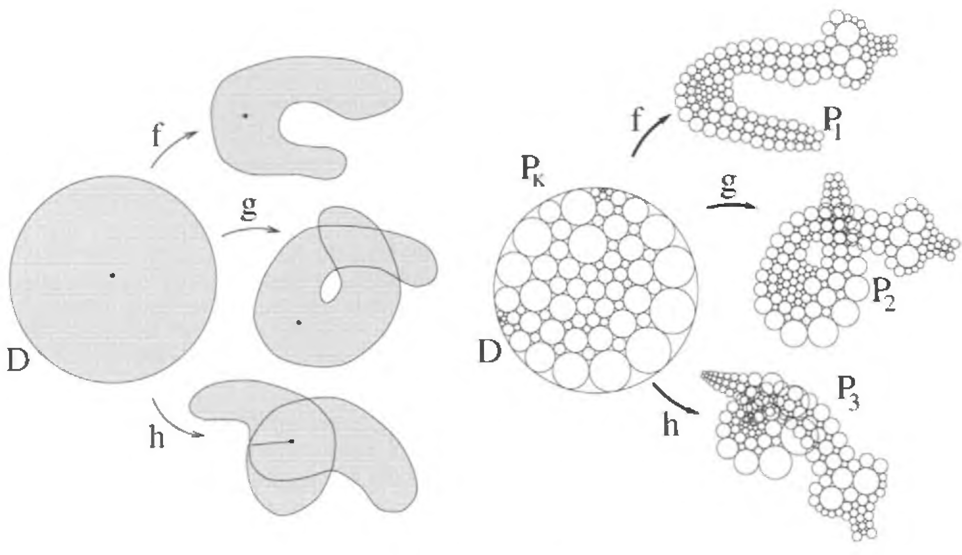



Fig. 10. Discrete analytic functions on the disc.

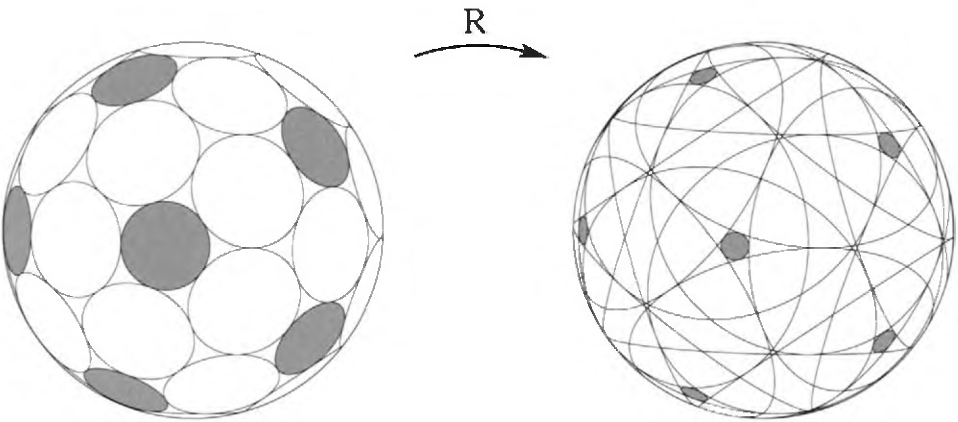



Fig. 11. A discrete rational function.

*Discrete rational functions:* When $\mathcal{K}$ triangulates a sphere it is easy to show that any circle packing $P$ for $\mathcal{K}$ must lie in $\mathbb{S}^2$. $f : P_{\mathcal{K}} \to P$ clearly represents a *discrete rational function*.

With no boundary to consider, packing issues revolve around the branching. An $n$-degree discrete rational map will, as usual, have $2n - 2$ branch points (counting multiplicities). Packings with "polynomial" branching (half their branching at a single vertex) are easily generated, since they can be projected from the hyperbolic setting. However, existence and uniqueness for packings with general branching remain a challenging open questions. Figure 11 displays a 7-fold branched covering of $\mathbb{S}^2$ with 12 simple branch circles (shaded); construction relied heavily on "Schwarz triangles" (see [20]).

*Discrete functions on* $\mathbb{D}$*:* We say $f : P_{\mathcal{K}} \to P$ is *defined on the disc* if $P_{\mathcal{K}}$ packs $\mathbb{D}$, that is, if $\mathcal{K}$ is simply connected and hyperbolic. We will describe two of the many important classes of functions included.

When $P$ is a univalent packing for $\mathcal{K}$ in the plane, as in Figure 12, $f$ is termed a *discrete conformal mapping*. Thurston's 1985 conjecture revolved around such mappings, comparing them to the classical Riemann mapping $F : \mathbb{D} \to \Omega$ (as we will see in Section 1.7).

The range packing in Figure 12 is simply cut from a regular hexagonal packing. In general, however, constructions of univalent packings are much more problematic, and practice falls far short of theory. Here, for example, is a remarkable result due to He and Schramm [58]. Suppose $\mathcal{K}$ triangulates an open topological disc and is hyperbolic. Assume $P_{\mathcal{K}}$ has been normalized to center the circle for $v$ at the origin and the circle for $u$ on the positive $x$-axis.

THEOREM 4. *Let $\Omega$ be any proper, open, simply connected subset of the plane, $0 \in \Omega$. Then there exists a unique univalent circle packing $P$ for $\mathcal{K}$ so that $c_v$ is centered at the origin and $c_u$ is centered on the positive $x$-axis.*

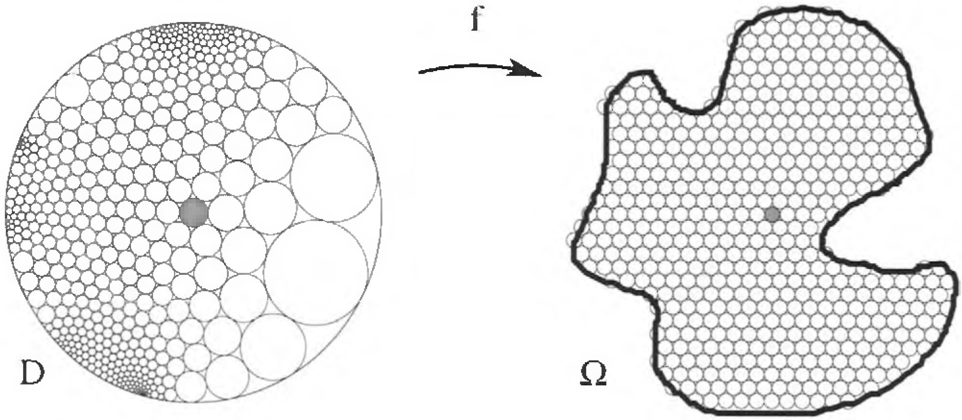



Fig. 12. A discrete conformal mapping.

In other words, there is a unique discrete conformal mapping $f_{\mathcal{K}}:\mathbb{D}\to\Omega$ with the standard normalization $f(0)=0$ and "$f'(0)>0$". This is quite amazing: for example, given *any* $\Omega$, the 7-degree packing of Figure 4 can in theory be repacked to fill $\Omega$! Unfortunately, I can not show you any nontrivial examples, since to my knowledge there is no algorithm for *computing* such packings. Even for finite complexes $\mathcal{K}$, there are as yet no *univalence criteria* to tell whether a given packing label will lead, when the circles are laid out, to a univalent packing $P$.

Foregoing univalence and allowing branching, Theorem 3 provides a huge and computationally accessible variety of discrete analytic functions on the disc. The extreme opposite, in some sense, to the univalent mappings are the proper self-mappings of $\mathbb{D}$, classically the finite Blaschke products. Figure 13 illustrates a *discrete finite Blaschke product* obtained by setting boundary labels to infinity and prescribing 3 simple branch circles (shaded in the domain of Figure 13). The image packing in Figure 13 covers the disc with multiplicity 4, so I have shown only the chain of boundary circles (wrapping 4 times around $\partial\mathbb{D}$), along with three extremely small dots near the origin, the branched circles.

*Discrete entire functions:* When $\mathcal{K}$ is simply connected and parabolic, $P_{\mathcal{K}}$ fills $\mathbb{C}$; a discrete analytic function $f: P_{\mathcal{K}}\to P$ would be called *discrete entire* if $P$ lies in $\mathbb{C}$ or *discrete meromorphic* if $P$ lies in $\mathbb{S}^2$. Here are some examples.

Among the most pleasing infinite packings are the *Doyle spirals*, Figure 1(d) being but one of a two-parameter family of such spirals. Based on the hexagonal complex $H$, these packings exist by virtue of the symmetries of $H$ and a special scaling noted by Peter Doyle. A discrete entire function from $P_H$ to the Doyle spiral $P$ of Figure 1(d) should clearly be considered as a *discrete exponential*, since one can observe familiar properties – periodicity (note: every circle of $P$ has infinitely many preimages in $P_H$), nonvanishing, local univalence, growth of $f^{\#}$, and so forth. Creating infinite packings $P$ – or even proving existence – is a significant challenge, even for the hexagonal complex $H$. Open question: *do there exist any locally univalent packings of $H$ other than $P_H$ and the Doyle spirals?*

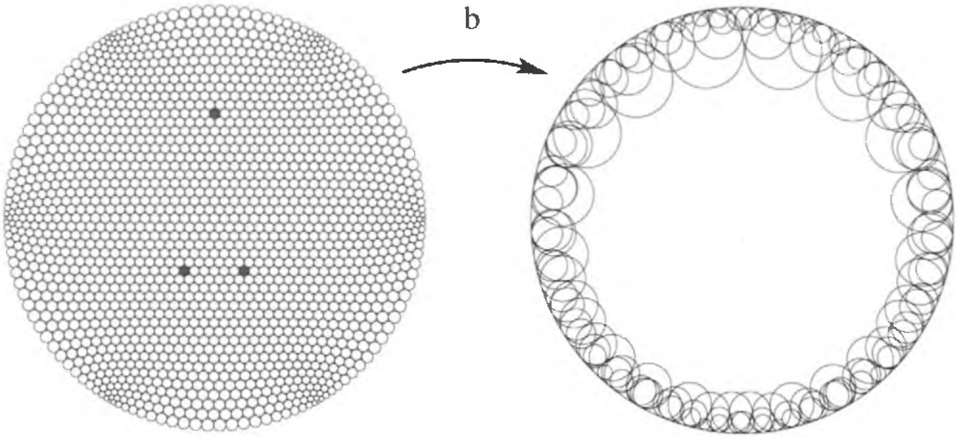



Fig. 13. A 4-fold discrete Blaschke product.

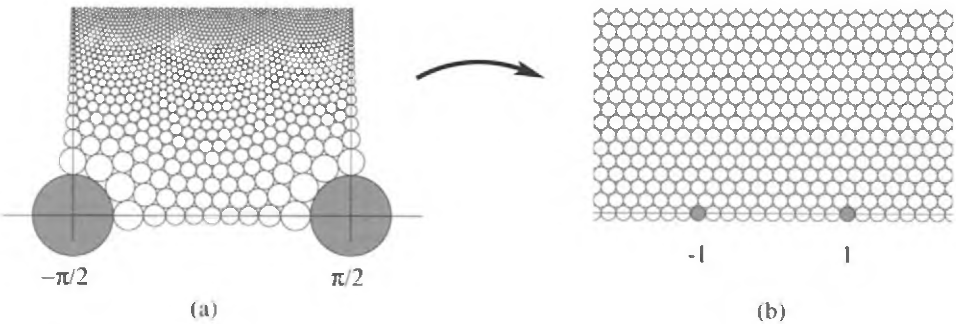



Fig. 14. A discrete sine function.

Exploiting symmetry in a different way leads to a *discrete sine* function. A regular hexagonal packing of a half plane, Figure 14(b), has been repacked to fill a half-infinite strip in Figure 14(a). Reflections in and identifications along the edges in (a) lead to a univalent packing of the plane, *a fortiori* a maximal packing $P_{\mathcal{K}}$. The natural identification of (a) with (b) can be extended *a la* "Schwarz reflection" to a discrete entire function with domain $P_{\mathcal{K}}$. This map will have precisely the geometric mapping properties of $z \mapsto \sin(z)$; in particular, note that the construction gives $2\pi$ periodicity, with the circles at points $\pi/2 + n\pi$ in (a) becoming simple branch circles with image circles at $\pm 1$.

Only a few more general construction techniques are known; Figure 15 illustrates two approaches to the construction of *discrete polynomials*. Dubejko [43] specifies branch points for packings of the hexagonal complex $H$. A finite stage in constructing the analogue of $P : z \mapsto z(z^2 - 1)$ is shown in the top of Figure 15; note the shaded branch circles on the left. Valence considerations are key to proving that the finite stage packings converge to a packing for $H$ itself.

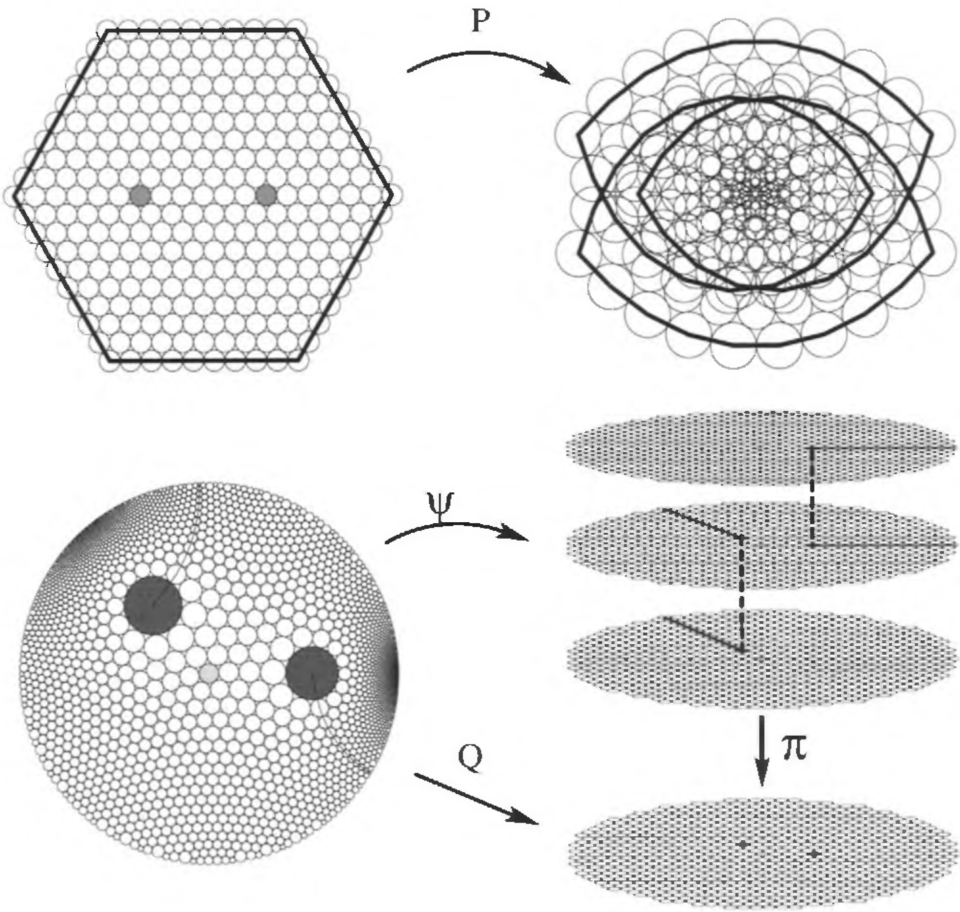



Fig. 15. Discrete polynomials.

Alternately, one can construct discrete polynomials with prescribed branch *values* by directly constructing their image packings, a capability not available in the classical setting. Figure 15 shows a finite stage in constructing a polynomial $Q$ of degree 3. The image surface is built from three (truncated) copies of the regular hexagonal packing which have been slit and cross-connected along curves from circles over the desired branch values. With the branched image packing comes a new simply connected complex whose maximal packing (appropriately scaled) becomes the domain. $Q$ is the composition of $\psi$ with the projection $\pi$.

*Discrete conformal structures:* By the Circle Packing Theorem, combinatorics determines conformal structure. For instance, Figure 16(a), with the indicated side pairings, represents a genus 2 surface $S$. The additional markings define what is known as a "dessin" and lead in a canonical way to a triangulation $\mathcal{K}$ of $S$. There exists a unique conformal

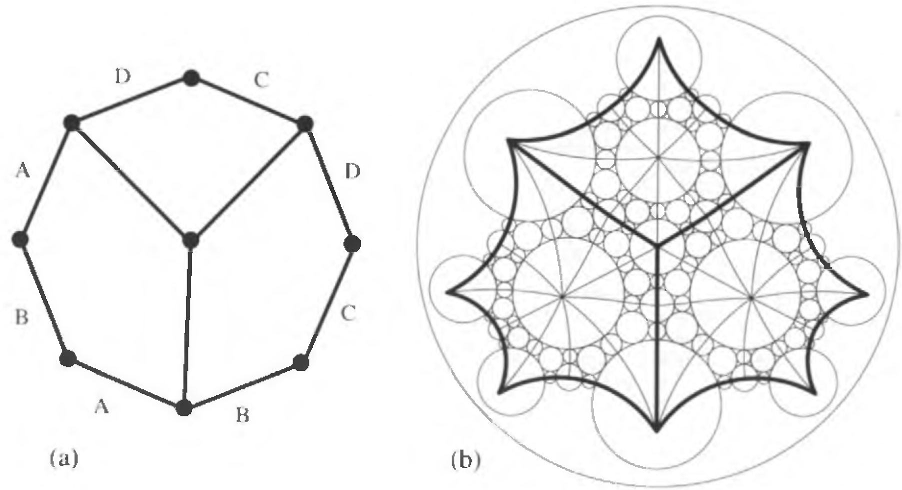



Fig. 16. A discrete conformal structure.

structure on $S$ which supports (in its intrinsic metric) a circle packing $P_{\mathcal{K}}$ for $\mathcal{K}$; that Riemann surface is laid out as a fundamental domain in $\mathbb{D}$ in Figure 16(b), with the carrier of the packing included for reference. Note that this represents a classical conformal structure – a unique point in Teichmüller space $\mathbf{T}_2$; we use the adjective "discrete" simply to emphasis that it is determined by the abstract combinatorics of $\mathcal{K}$.

### 1.6. *Examples of discrete theory*

Were this topic mere mimicry, it would not deserve our long term attention. To understand the deeper connections, let us be a little more explicit about discrete analyticity. First, as to the intrinsic structure on domains implied by the Circle Packing Theorem, we might paraphrase the fundamental analogy:

"A Riemann surface $S$ has a *conformal structure* which determines an *infinitesimal metric* of constant curvature, while a complex $\mathcal{K}$ has a *combinatorial structure* which determines a *discrete metric* of constant curvature."

Likewise, the notion of discrete analyticity of mappings is not so very far from our experience: a familiar saying has it that an analytic function is one which "maps infinitesimal circles to infinitesimal circles". The discrete versions simply operate instead on *real* circles; where $|f'|$ measures the stretching or shrinking of infinitesimal circles, the ratio function $f^{\#}$ measures the stretching/shrinking of *real* circles. The reader can not go far wrong by using this "infinitesimal *versus* real" analogy to transfer classical geometric intuition to the discrete setting.

In this spirit, the Circle Packing Theorem of Section 3 applied to simply connected complexes clearly qualifies as the *Discrete Uniformization Theorem*, while for non-simply connected $\mathcal{K}$, it serves as the *Discrete Covering Theorem*. Within the proof, one can isolate two particularly fundamental results.

The Schwarz–Pick Lemma lies behind a huge portion of the classical theory of functions, and one might anticipate an equally important role for its discrete version.

DISCRETE SCHWARZ–PICK LEMMA. *If $f:\mathbb{D}\to\mathbb{D}$ is a discrete analytic function, then $f$ is a contraction in the hyperbolic metric. In particular, if $f(0)=0$, then $f^{\#}(0)\leqslant 1$, with equality if and only if $f$ is a Möbius transformation.*

Let me interpret: For some simply connected hyperbolic complex $\mathcal{K}$, $f$ maps $P_{\mathcal{K}}\subset\mathbb{D}$ to a packing $P\subset\mathbb{D}$ for $\mathcal{K}$. $f(0)=0$ indicates that a circle $c_v$ is centered at the origin in $P_{\mathcal{K}}$ and $f(c_v)$ is centered at the origin in $P$. Assuming $P_{\mathcal{K}}\leftrightarrow\mathcal{K}(R_{\mathcal{K}})$ and $P\leftrightarrow\mathcal{K}(R)$, we observed earlier that $R\leqslant R_{\mathcal{K}}$. Applied to $v$, that immediately gives $f^{\#}(c_v)\leqslant 1$ (i.e., $f^{\#}(0)\leqslant 1$), the *Schwarz* portion of the result. More subtly, if $d(\cdot,\cdot)$ denotes the distance between circle centers, then it is the case that for any circles $c_u, c_{u'}$ of $P_{\mathcal{K}}$, $d(f(c_u), f(c_{u'}))\leqslant d(c_u, c_{u'})$. This is the *Pick*, or *hyperbolic contraction* portion of the result.

When $\mathcal{K}$ is simply connected and parabolic, the criterion for the hyperbolic/parabolic dichotomy shows that there can be no non-trivial hyperbolic packing label for $\mathcal{K}$ – that is, there can be no circle packing for $\mathcal{K}$ lying in $\mathbb{D}$. This gives

DISCRETE LIOUVILLE THEOREM. *There are no bounded discrete entire functions.*

(In case the reader anticipated the adjective "nonconstant" in the statement, note that there seems to be no natural notion for constant function in the discrete theory.) An alternate statement, proven by Dubejko [44] with probabilistic methods under the assumption that $\mathcal{K}$ has bounded degree, is that if $f:P_{\mathcal{K}}\to P$ is entire and $f^{\#}$ is bounded, then $f^{\#}$ is constant; that is, a discrete entire function with bounded derivative is linear.

There are several other geometric results, standard in classical analytic function theory, which have substantially parallel discrete versions. Formal statements involve more details than we have room for here, so I'll simply mention the *Discrete Picard Theorem* [26], a *Discrete Distortion Lemma* [45], companion to the Schwarz–Pick result, *Discrete Maximum Principles* for $f$ and $f^{\#}$, and a *Discrete Löwner Theorem* for self-maps of the disc. There are also valuable random walk methods, the analogue of Brownian motion, which we discuss later, and the powerful discrete winding number techniques of He and Schramm [55].

### 1.7. *Comment on approximation*

If anything could lend more substance to the analogies we have been discussing, it is the companion approximation results. Under appropriate refinement of the circle packings involved, virtually all of our discrete analytic objects converge to their classical models. Though we are concentrating on analogy, I would be remiss if I did not give at least the flavor of *approximation*, so let me illustrate Thurston's original conjecture.

Return to the situation pictured in Figure 12, where we used a Jordan region $\Omega$ to cut out a finite piece of a regular hexagonal packing of circles. Supposing these circles

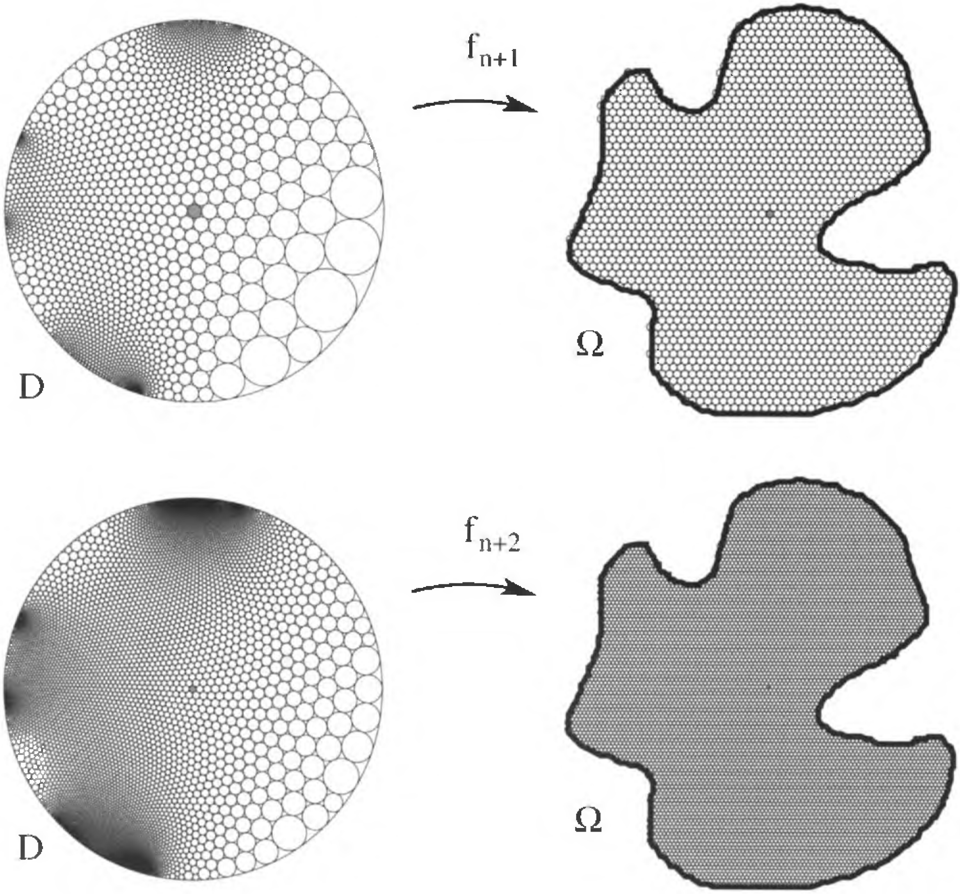



Fig. 17. Illustrating Thurston's Conjecture.

have radii $2^{-n}$, we will write $P_n$ for this finite packing in $\Omega$, $\mathcal{K}_n$ for its complex, and $f_n : P_{\mathcal{K}_n} \to P_n$ for the associated discrete conformal mapping from $\mathbb{D}$ to $\Omega$ (with appropriate normalizations). Figure 17 repeats the construction for two successively larger values of $n$ – a typical "refinement" process.

Thurston conjectured and Rodin and Sullivan proved that the discrete conformal maps $f_n$ converge uniformly on compact subsets of $\mathbb{D}$ to the classical conformal mapping $F : \mathbb{D} \to \Omega$ as $n \to \infty$. In recent work, He and Schramm have removed quasiconformal notions in certain key arguments, so the proof of Thurston's Conjecture now provides an independent *proof* of the Riemann Mapping Theorem!

Approximation results have been extended far beyond the conformal mapping setting, including approximation not only of analytic functions but also of conformal structures. See Appendix A for brief remarks on computational aspects of circle packing, and the survey [95] for more details on approximation.

## 2. Discrete conformal geometry

At this juncture, I hope the reader has successfully transferred some geometric intuition from the classical setting to our new discrete world. I will rely on that now as we investigate various standard geometric topics and see how they are manifest in the setting of circle packings.

### 2.1. *The "type" problem*

The classical "type" problem presents us with an open and simply connected Riemann surface, defined in some fairly explicit form, and asks whether it is conformally equivalent to (is the *type* of) the plane or the unit disc.

DISCRETE TYPE PROBLEM. *Given a specific triangulation $\mathcal{K}$ of an open topological disc, is $\mathcal{K}$ parabolic or hyperbolic?*

The constant $m$-degree complexes $\mathcal{K}^m$ are instructive here. Cases $m = 5, 6$, and 7 are shown in Figure 4. The reader can confirm the local geometry on a flat table by contemplating construction of an $m$-flower from $m + 1$ equal sized discs. The familiar 6-petal flower will lie perfectly flat, whereas the petals of a 5-petal flower must be drawn up if they are to form a closed chain, leading to positive curvature, while the petals of a 7-petal flower must be drawn alternately up and down, giving the saddle shape characteristic of negative curvature. As a rule of thumb, *degree* 6 *vertices tend to be flat* (*euclidean*), *degree* 5 *and less tend to be spherical, and degree* 7 *and more, hyperbolic.*

Global effects when the degrees are mixed are much more subtle. Various type conditions have been established, many reminiscent of those encountered elsewhere, e.g., in discrete potential and graph theory. Perhaps one of the cleanest is due to Gareth McCaughan [71]:

THEOREM 5. *Let $\mathcal{K}$ be an infinite simply connected complex with a uniform upper bound $d$ on the degree of its vertices and consider the simple random walk on the* 1*-skeleton of $\mathcal{K}$. Then $\mathcal{K}$ is parabolic if the random walk is recurrent and hyperbolic if the random walk is transient.*

This is a discrete version of the classical Kakutani type condition: *A simply connected Riemann surface $R$ is parabolic if Brownian motion on $R$ is recurrent, and hyperbolic if Brownian motion is transient.* In general terms, a complex tends to be hyperbolic if it has lots of "room" at infinity, for instance if the number of vertices $n$ generations from some fixed vertex $v_0$ grows rapidly with $n$ or if the average degree of the vertices exceeds six or if areas grow slowly compared to their perimeters.

Motivated by the dichotomy between packings of constant 6- and 7-degree complexes, let us arrange a mixture. Attach a half-plane of 6-degree combinatorics to a half-plane of 7-degree combinatorics to get the complex $\mathcal{K}$ suggested by Figure 18(a). The maximal packing lies in $\mathbb{D}$; the circles on the dividing line between the 6- and 7-degree regions are

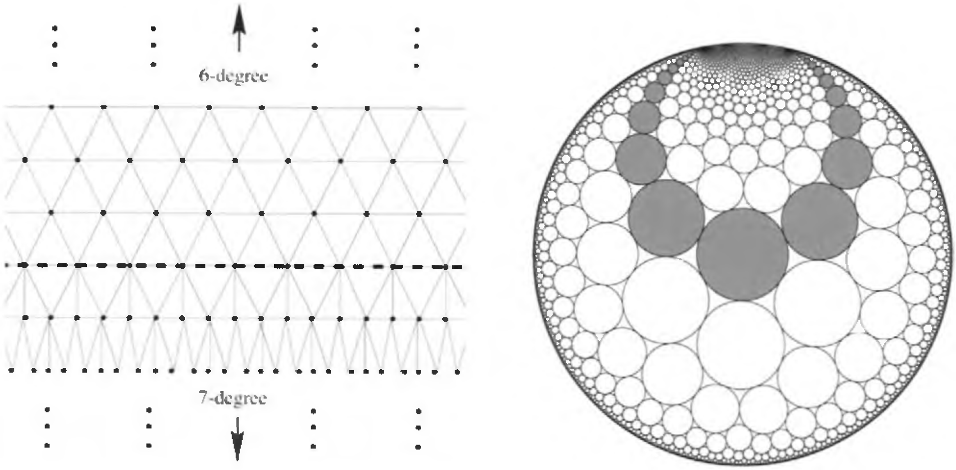



Fig. 18. A split 6/7-degree complex.

shaded in the figure, and one sees that the 6-degree circles are forced into a horocyclic region. Applying McCaughan's criterion, transience is clearly winning out: a random walker stumbling into the degree 7 side risks getting lost and never returning to its starting point, so the walk is transient.

## 2.2. *Harmonic measure*

Of the last example, one might well observe that the "flat half" of $\mathcal{K}$ has a null *harmonic boundary* – full harmonic measure on the "ideal boundary" of $\mathcal{K}$ resides on the boundary of the 7-degree portion, as suggested by the maximal packing. Boundary influences, mediated by combinatorics, are rife in this topic, and though not yet quantified, they strongly resemble the classical influence of "harmonic measure". Careful examination of Figure 12, for example, would show that circles "closer" to the origin in $\Omega$ become (euclideanly) larger in the maximal packing in $\mathbb{D}$ compared to circles which in $\Omega$ are further from or obscured from the origin. Probability theory holds some hope of quantifying these effects.

Given a Jordan region $\Lambda$, classical harmonic measure $\mu$ (with respect to the $0 \in \Lambda$) is defined for Borel subsets $E$ of $\partial \Lambda$ as the probability that a Brownian particle starting at the origin will first exit $\Lambda$ at a point of $E$. To illustrate a discrete version, look to $\Lambda$ in Figure 1(f), with packing $P$, complex $\mathcal{K}$, and interior circle $c_v$ centered at the origin. Given a random walk $\mathcal{W}$ on the circles of $P$ (e.g., on the embedding of the 1-skeleton of $\mathcal{K}$), there is an associated hitting probability $\mu_{\mathcal{W}}$ on the boundary circles analogous to $\mu$. The general questions: *How closely are $\mu_{\mathcal{W}}$ and $\mu$ related? Can $\mu_{\mathcal{W}}$ play the roles in discrete function theory that $\mu$ plays in the classical?*

A first step, of course, is to decide *which* random walk $\mathcal{W}$ to consider – a walk is determined by local "transition probabilities", yet we would like it to reflect global behavior. In fact, there are walks closely attuned to packing geometry.

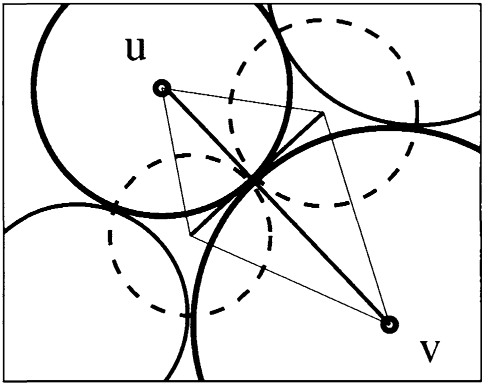



Fig. 19. Geometry of conductances.

In a euclidean packing $P$, consider an edge $\langle u, v\rangle$, as pictured in Figure 19. Each of the triples of circles to which $c_u$ and $c_v$ belong determines a mutually orthogonal "dual" circle. Let $r_u, r_v$ be the radii of the circles $c_u, c_v$ and let $a, b$ be the radii of the dual circles. Define the *conductance* of the edge $\langle u, v\rangle$ as

$$C_{u,v} = \frac{a+b}{r_u + r_v}.$$

Conductances on all the edges determine, in a standard way, a random walk on $P$, and we will refer to the walk resulting from these particular conductances as the *tailored random walk* $\mathcal{W}_P$ on $P$.

I think tailored random walks may encode probabilistically the local-to-global transition that packings themselves seem to accomplish. In fact, these walks were inspired by observations of the packing process itself. Developed first in the hyperbolic setting, they model the flow of hyperbolic angles and area due to perturbations of boundary labels and were used in a probabilistic proof of Thurston's Conjecture [94]. The euclidean tailored walks were developed and applied by Dubejko [40], who discovered the formulation given here.

### 2.3. *Potential theory*

There is a considerable literature on discrete harmonic functions, and in the presence of these harmonic influences, one might anticipate some notion of discrete harmonic function. For example, every random walk is associated with a *discrete Laplacian* and corresponding *discrete harmonic functions*. To show that $\mathcal{W}_P$ has some claim to being the geometrically "correct" random walk for $P$, consider the "center function" $Z_P : \mathcal{K}^{(0)} \to \mathbb{C}$. That is, for each vertex $v$ of $\mathcal{K}$, $Z_P(v)$ is the center of the circle $c_v$ of $P$.

THEOREM 6. *Let $P$ be a euclidean circle packing, $Z_P$ its center function, and $\mathcal{W}_P$ its tailored random walk. Then $Z_P$ is a harmonic function for $\mathcal{W}_P$.*

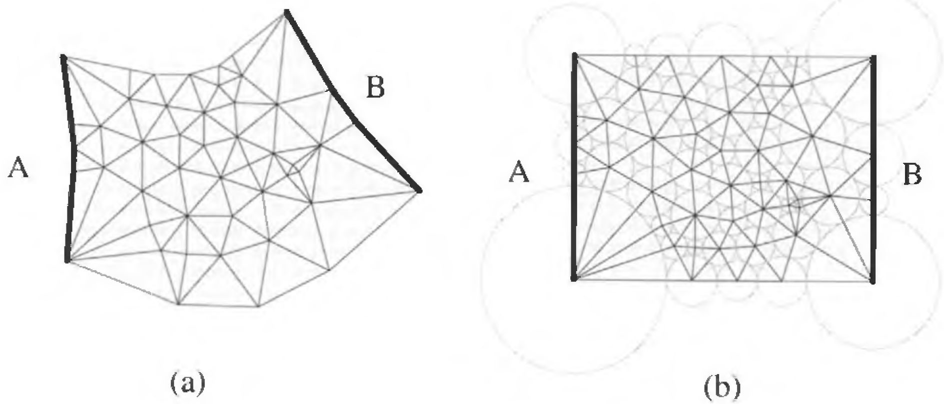



Fig. 20. A combinatorial and a packed quadrilateral.

This is a very modest beginning. Blending harmonic and analytic notions in this discrete setting seems a highly challenging business. Is there, for example, some sense in which the real and imaginary parts of discrete analytic functions are "harmonic"?

## 2.4. *Extremal length*

A conformal quadrilateral $Q$ is a conformal disc with four distinguished boundary points; extremal length $\mathrm{EL}(A, B; Q)$ measures in a conformally invariant way the distance between opposite ends of $Q$. In analogy, a combinatorial quadrilateral $\mathcal{Q} = \{\mathcal{K}; a, b, c, d\}$ is a triangulation $\mathcal{K}$ of a closed disc with four distinguished boundary vertices; an example is shown in Figure 20(a), with ends $A, B$. The reader will recognize the familiar quantities going into extremal length in this combinatorial version due to Duffin:

DEFINITION. Let $\Gamma$ denote the edge-paths $\gamma$ of $\mathcal{Q}$ connecting $A$ to $B$. For any set $W = \{w_j\}$ of nonnegative "weights" assigned to the vertices $v_j$ of $\mathcal{K}$, let $L_W(\Gamma)$ denote the length of $\Gamma$, $L_W = \inf_\gamma \sum_{v_j \in \gamma} w_j$, and let $\mathrm{Area}(W) = \sum_j w_j^2$. Then the *vertex extremal length* of $\Gamma$ is defined by

$$\mathrm{VEL}(A, B; \mathcal{Q}) = \sup_W \left( \frac{L_W^2(\Gamma)}{\mathrm{Area}(W)} \right).$$

It is pleasing to discover concrete geometry here. For instance, $\mathcal{Q}$ can be represented as a "squared rectangle", a rectangle packed with squares having the extremal weights $w_j$ as edge lengths (see [28,88]). Of course, our inclination would be to pack $\mathcal{Q}$ instead. Prescribing angle sums $\pi/2$ at the vertices $a, b, c, d$ and $\pi$ at the remaining boundary vertices leads to a euclidean circle packing $P$ for $\mathcal{K}$ having a rectangular carrier $R_{\mathcal{Q}}$ with corners $a, b, c, d$, as in Figure 20(b). The rectangle $R_Q$ is determined up to similarity by combinatorics, allowing us to define the *packing extremal length* of $\mathcal{Q}$ as the aspect ratio

$$\mathrm{PEL}(A, B; \mathcal{Q}) = \frac{\mathrm{length}(R_Q)}{\mathrm{height}(R_Q)}.$$

Typically $\mathrm{VEL}(A, B; \mathcal{Q}) \sim \mathrm{PEL}(A, B; \mathcal{Q})$ (meaning that these discrete extremal lengths are "comparable" within constants depending only on the degree of $\mathcal{Q}$.) However, PEL, besides being more computationally accessible, also parallels the classical geometry associated with EL.

To see what I mean, let us return to Figure 20. Let $Q$ denote the conformal quadrilateral on the left; in fact, $Q$ is the carrier of a circle packing $P_{\mathrm{l}}$ (not shown). Classical considerations begin with a conformal mapping $F : Q \to R_Q$, where $R_Q$ is a euclidean quadrilateral. Likewise, we have the discrete conformal mapping $f : P_{\mathrm{l}} \to P$, treated as a map of $Q = \operatorname{carr} P_{\mathrm{l}}$ to the euclidean quadrilateral $R_Q = \operatorname{carr} P$. Classically, $\mathrm{EL}(A, B; \Omega)$ is the aspect ratio of $R_Q$ and the extremal density on $Q$ is $|F'|$. Discretely, $\mathrm{PEL}(A, B; \mathcal{Q})$ is the aspect ratio of $R_Q$ and the discrete extremal density (what you multiply radii in $P_{\mathrm{l}}$ by to get the radii of $P$) is precisely $f^{\#}$. Furthermore, due to the Rodin/Sullivan Ring Lemma, $\mathrm{EL}(A, B; \Omega) \sim \mathrm{PEL}(A, B; \mathcal{Q})$. In other words, PEL brings with it discrete "conformal invariance", the true hallmark of extremal length. Similar considerations apply to discrete extremal length in the settings of annuli, tori, and so forth.

### 2.5. *"Packable" surfaces and dessins*

According to the Circle Packing Theorem, each complex $\mathcal{K}$ "chooses" its geometry; that is, there is a unique conformal structure on $\mathcal{K}$ so that the resulting Riemann surface $\mathcal{X}$ supports $P_{\mathcal{K}}$. We'll say in this case that $\mathcal{X}$ is *packable*.

Since the number of complexes $\mathcal{K}$ is countable, there exist at most countably many packable surfaces. Which ones are they? This is largely a mystery, with a few results but mostly intriguing open questions. By results of Brooks [25] and Bowers and Stephenson [18,19], we have

THEOREM 7. *The packable Riemann surfaces are dense in Teichmüller space for surfaces of finite topological type* $T_{g,n,m}$.

Let us illustrate two classical surfaces of genus 3, the Klein and Picard surfaces. These have intrinsic triangulations which can be circle packed to give the fundamental domains shown in Figure 21 on the left and right, respectively (side-pairings are not shown). The ubiquitous symmetries of the Klein surface imply that this discrete version is in fact exact and the Klein surface is packable – indeed, our picture duplicates a century-old image from Fricke and Klein [47]. Our discrete Picard surface, with the circles shown for reference, is most likely only an approximation and I don't know if the Picard surface itself is packable.

These issues are of more than casual interest. Conformal structures are central to Grothendieck's theory of *dessins d'enfants*, which establishes connections among

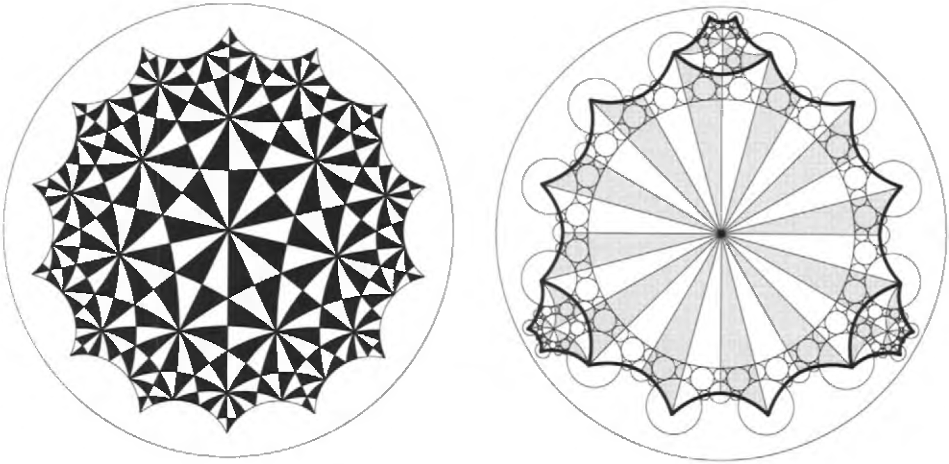


Fig. 21. Discrete Klein and Picard surfaces.

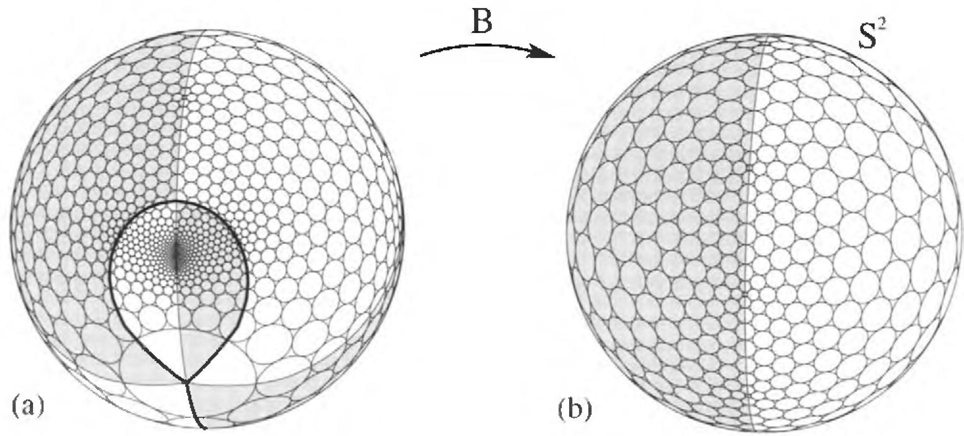



Fig. 22. The Belyĭ map for a genus 0 dessin.

"drawings", triangulations, conformal structures, Belyĭ maps (meromorphic functions branching over at most three points), and number fields. The parallel discrete theory based on circle packing is developed in [17]. Like the packable surfaces, the "dessin" surfaces are dense in Teichmüller space, but the latter have an algebraic characterization. Is there an analogous characterization for packable surfaces? McCaughan [70] has shown that a packable 1-torus will have an algebraic modulus, while a packable $n$-torus, $n \geqslant 2$, will have a Möbius covering group which is conjugate to one having (complex) algebraic entries. The converse statements remain open.

Figure 22 illustrates a genus 0 dessin (the dark graph), the associated triangulation (shaded/unshaded), and the discrete Belyĭ map (i.e., discrete rational function). The triangulation would converge to the conformally correct one if we were to refine the circle packings.

## 2.6. *Conformal welding*

Let $\phi : \partial\mathbb{D} \to \partial\mathbb{D}$ be an orientation preserving homeomorphism. Attach two copies of $\mathbb{D}$, say $\mathbb{D}_1$ and $\mathbb{D}_2$, along their boundaries by identifying $z \in \partial\mathbb{D}_1$ with $w = \phi(z) \in \partial\mathbb{D}_2$. The result is a topological sphere. Pump it full of air to form a sphere and see where the seam $\Gamma$ between the two discs settles. $\phi$ is called a "welding homeomorphism", and the Jordan curve $\Gamma$ is its "welding curve". The well-studied correspondence $\phi \leftrightarrow \Gamma$ is (essentially) a bijection between the class of quasisymmetries $\phi$ of $\partial\mathbb{D}$ and quasicircles $\Gamma$.

We can see the analogous theory, as formulated by Williams [101], in the pictures of Figure 23. In (a), a triangulation $\mathcal{K}$ of a disc has been doubled across its boundary and the result packed in the sphere (the Koebe–Andreev–Thurston Theorem). This symmetric packing is now broken at the equator in (b), giving the two triangulations $\mathcal{K}_1$ and $\mathcal{K}_2$, both isomorphic to $\mathcal{K}$. $\mathcal{K}_1$ and $\mathcal{K}_2$ are reidentified along their boundaries in (c), but now using (a discretization of) $\phi$, interpolating extra boundary vertices as necessary. The resulting triangulation of a topological sphere is packed in $\mathbb{S}^2$ (Koebe–Andreev–Thurston again), giving (d). The common boundary $\Gamma$ of the two triangulations in their new locations – the *discrete welding curve* – is the heavy sinuous line in (d). Maps $f$ and $g$ carrying circles in the northern and southern hemispheres of (a) to the regions above and below $\Gamma$ in (d), respectively, are discrete conformal mappings. The reader familiar with classical conformal welding will find precise parallels here: If $\phi$ is a $k$-quasisymmetry, then $\Gamma$ is a $\hat{k}$-quasicircle ($\hat{k}$ depending on $k$) and under refinement such discrete welding curves converge to the classical welding curve for $\phi$, while the discrete conformal maps $f$ and $g$ of the hemispheres converge to the classical conformal maps $F$ and $G$ satisfying $F \equiv G \circ \phi$ on $\mathbb{D}$.

## 2.7. *Classical analysis issues*

We have seen several instances now of the close parallels between the discrete objects of circle packing and the familiar objects of classical analysis and conformal geometry. I would like to conclude this survey by emphasizing that the connections are a two-way street.

Circle packing brings to the classical theory a significant experimental capability, new methods of approximation, and a flexible visualization tool. It also has the potential to suggest new ideas, as it apparently did for He and Schramm, who in [55] took a major step towards solving Koebe's Uniformization Conjecture using key ideas from their circle packing work.

In the other direction, circle packing theory has obviously benefited by following the rich classical model. That in turn has allowed it to carry notions from complex analysis into new topics. Graph embedding is a good example: any locally planar graph can be endowed with an intrinsic geometry *via* circle packing. In addition to ease of computation and visual

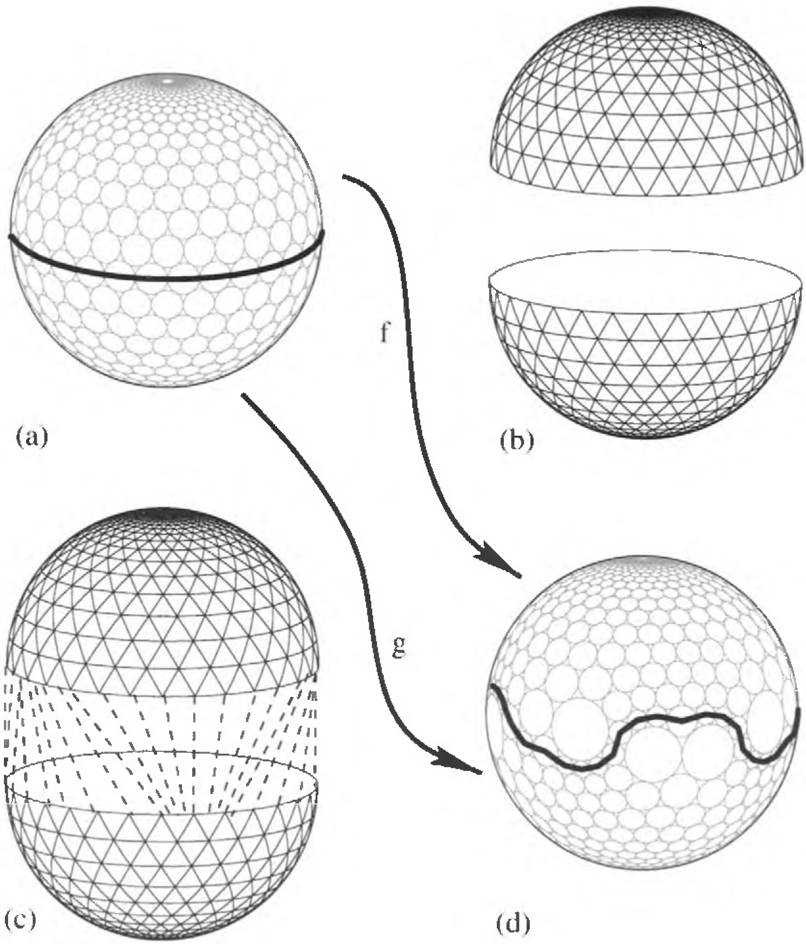
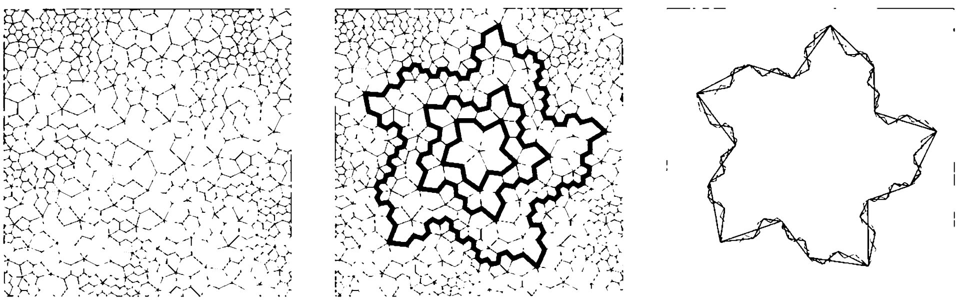



Fig. 23. A discrete conformal welding.

appeal, these embeddings are often useful precisely because of their discrete conformal natures, for example in obtaining estimates for graph "separators" in [73], in estimating graph "resolution" in [65,4], and in proving existence of finite Dirichlet functions in [13].

Let us wrap up with a final example illustrating these mutual influences. The topic is "conformal tiling", a notion introduced by Bowers and me in [21]. The original motivation came from the abstract combinatorial patterns of Jim Cannon, Bill Floyd, and Walter Parry (see [27]) arising in their investigation of Thurston's Geometrization Conjecture. The pentagonal tiling of Figure 24(a) is one example.

Circle packing began simply as a convenient embedding device for intricate graphs. Once concrete images were in hand, however, notions of discrete conformality quickly pointed to a classical formulation. The requisite theory was in fact available a century ago, but now the visual and experimental capabilities of circle packing kicked in – only



(a) (b) (c)

Fig. 24. Structure in a conformal tiling.

with circle packing could one study specific examples in depth and develop essential insights. Unexpected structural features were observed and conjectures made; some have been confirmed using classical theory, others remain tantalizingly open.

Consider, for instance, the tiling of Figure 24(a). This is associated with a "twisted pentagonal" subdivision rule and though the circles themselves are not shown, this embedding is obtained from a circle packing. Relevant "aggregate" tiles arising in the combinatorics have been outlined in (b). Close observation suggests an underlying self-scaling, which seems to be confirmed when the outlines are rotated, scaled by $\lambda \approx 1.82$, and overlayed in (c) – the corners of each aggregate tile fall directly on corners at the next aggregation level. Motivated by these experimental images, Cannon, Floyd, and Parry were lead first to a proof of scaling and then, with Rick Kenyon, to connections with the dynamics of rational functions leading to the exact scaling factor

$$\lambda = \left|25/16(5+3\sqrt{-15})\right|^{1/5} \approx 1.8162516.$$

This example concludes our survey and perhaps shows best the potential for synergy between the familiar classical notions of analytic function theory and this new discrete realization in terms of circle packing.

## Appendix A: Computational notes

The effective computability of circle packings and the availability of software for creating, manipulating, and displaying them have been of central importance both in the theory and the application of circle packing. Here are some brief remarks on computational issues.

THE ALGORITHM. The computational task in circle packing involves approximation of packing labels. Thurston suggested an iterative scheme: start with an initial (perhaps even random) label $R_0$, repeatedly revisit individual vertices of $\mathcal{K}$, on each visit adjusting the label of that vertex so that it is compatible with its neighbors – that is, so its angle sum takes the intended value (typically $2\pi$ or a multiple of $2\pi$ at branch points). This scheme has been implemented and refined and works remarkably well in the euclidean and hyperbolic settings (monotonicity is absent in the spherical setting, and no one yet has a spherical algorithm). A generic boundary value problem involving, say, 5000 circles, will typically "pack" in a few seconds on a personal computer, while packings of over 500 000 circles may take a few hours, but are routine. See [31] for a discussion of the algorithm and its implementation. Packings of multiply connected complexes $\mathcal{K}$, e.g., Figures 16, 21, are done intrinsically and then automatically laid out as fundamental domains in $\mathbb{D}$ or $\mathbb{C}$, as appropriate. These computations are often remarkably fast.

SOFTWARE. The packing algorithm is implemented in my comprehensive software package called `CirclePack`. This package is freely available over the web, but currently operates only under X-Windows on Unix machines (e.g., under *Linux*). It allows for the creation, storage, manipulation, analysis, display, and printing of circle packings. Many parts of the theory have come directly from live experiments, from observing unanticipated phenomena and searching for explanations, often motivated by the emerging connections with classical analytic function theory and conformal geometry.

CIRCLE PACKING MAPS. The definition of a discrete analytic function $f: Q \to P$ as a map between collections of circles is rather abstract. When more concrete point mappings are required, e.g., with approximations, one can formulate $f: \operatorname{carr} Q \to \operatorname{carr} P$ as a point mapping. Define $f$ by mapping the center of each circle $c_v$ to the center of the circle $f(c_v)$ and extending to the edges and faces. From a practical standpoint, piecewise-affine or barycentric coordinates are easiest, but some of the deepest work in the topic depends on extensions introduced by He [51], based on Möbius transformations between packing interstices.

APPROXIMATION. In this survey I have alluded to various "approximation" aspects of circle packing, both in Thurston's original conjecture on conformal mapping (*a la* Figure 17) and in more general settings. There are two key ingredients which reappear in one guise or another in all the approximation results: namely, *refinement* and *distortion control*. "Refinement" refers to the process of conveniently generating sequences of successively finer circle packings (that is, more numerous and smaller circles) appropriate to a setting – hexagonal packings with successively smaller radii serve that purpose in Figure 17. "Distortion" concerns the behavior of discrete analytic functions (interpreted as point mappings); distortion is normally quantified in terms of *quasiconformal dilatation*, so dilatation converging to 1 means the mappings converge to conformal ones. It seems that the objects of the discrete theory invariably approximate their classical counterparts, often with surprising precision, though errors and rates of convergence are as yet poorly quantified. The interested reader should consult [95] for details.

## Appendix B: The literature

William Thurston coined the term "circle packing" in [96], where he introduced circle packings for the construction of certain orbifolds. He found that his proof reinterpreted results on reflection groups due to E.M. Andreev [7,8]. More recently it was pointed out by R. Kühnau that both these authors were preceded by P. Koebe [64]. Hence the name Koebe–Andreev–Thurston Theorem for our fundamental existence and uniqueness result. Brooks [24,25] made early use of circle packings to parameterize Schottky groups. However, circle packing came to prominence with analysts only after Thurston's conjecture [97] on their use in approximating conformal maps and its proof by Rodin and Sullivan [83]. As this survey has suggested, circle packing has developed in several different directions. For the classical models of many of these notions, one cannot go wrong with the books of Ahlfors, particularly [5,6]. Here is an attempt to place the circle packing literature in rough categories.

*Existence/Uniqueness*: [7,8,10,12,15,20,38,43,53,58,62,64,66–68,76,79,82,86–89,96, 99]. *Rodin/Sullivan and Extensions*: [2,29,34–37,51,54,60,69,83,89,93,94,97]. (Note: Early work centered on hexagonal combinatorics, later work successively removed combinatorial restrictions.) *Function Theory*: [1,10,12,17,26,30,38,40–46,50,55,56,58,59, 74,80,81,90,98,100,101]. *Type Problems*: [11,16,57,70,71,91]. *Riemann Surfaces*: [17–19,24,25,48,74,77]. *Probability*: [13,39,44,69,71,93,94]. *Approximation*: [17,21,34,45,46, 61,95,100,101]. *Graph embedding*: [13,16,21,52,63,65,72,73,75,78,84]. *Packing Algorithms*: [22,31–33,75,97]. (Caution: authors are not always clear in distinguishing between convergence of "packing algorithms" and convergence of packings to analytic functions.) *Geometry*: [1,3,4,9,14,21,23,24,49,79,85,92,101].

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CHAPTER 12

# Extreme Points and Support Points

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## 1. Introduction

Let $D = \{z\colon\ |z| < 1\}$ denote the open unit disk in the complex plane $\mathbf{C}$ and let $\Gamma = \{z\colon\ |z| = 1\}$ denote the unit circle. Let $\mathcal{A} = \{f\colon\ f$ is analytic in $D\}$ with the topology of uniform convergence on compact subsets of $D$. $\mathcal{A}$ is a complete, metrizable, locally convex linear topological space, that is, a Fréchet space.

Subclasses of functions from $\mathcal{A}$ have been studied throughout the twentieth century. However the systematic application of linear methods to study the extreme points and support points of subclasses is more recent. In the field of geometric function theory it began to play an important role starting in the 1970s. In this survey we hope to convey some of the substance and flavor of the use of linear methods in this field. References and additional comments are given at the end of our exposition in a section labeled Notes.

Let $\mathcal{F} \subset \mathcal{A}$. We use $co\,\mathcal{F}$ to denote the *convex hull* of $\mathcal{F}$ and $\overline{co}\,\mathcal{F}$ to denote the *closed convex hull* of $\mathcal{F}$. $co\,\mathcal{F}$ is the minimal convex family containing $\mathcal{F}$, which is the same as the set of all *finite* convex combinations of functions in $\mathcal{F}$. Also $\overline{co}\,\mathcal{F}$ is the minimal closed, convex family containing $\mathcal{F}$. A function $f$ is an *extreme point* of $\mathcal{F}$ if $f \in \mathcal{F}$ and $f$ is not a proper convex combination of two distinct functions in $\mathcal{F}$. A function $f$ is a *support point* of $\mathcal{F}$ if $f \in \mathcal{F}$ and there is a continuous linear functional $L$ on $\mathcal{A}$, that is $L \in \mathcal{A}^*$, with $\operatorname{Re} L$ nonconstant on $\mathcal{F}$ such that $\operatorname{Re} L(f) \geqslant \operatorname{Re} L(g)$ for all $g \in \mathcal{F}$.

Let $\mathcal{E}\mathcal{F}$ and $\sigma\mathcal{F}$ denote, respectively, the set of extreme points and the set of support points of $\mathcal{F}$. A *linear extremal problem* over $\mathcal{F}$ is simply: maximize $\operatorname{Re} L$ over $\mathcal{F}$ for some $L \in \mathcal{A}^*$. Two useful representations for continuous linear functionals on $\mathcal{A}$ are given in the next statement.

THEOREM 1. *Let $L \in \mathcal{A}^*$. Then*

(1) *there is a sequence of complex numbers $\{b_n\}_{n=0}^{\infty}$ satisfying $\overline{\lim\limits_{n\to\infty}} |b_n|^{1/n} < 1$ and such that $L(f) = \sum_{n=0}^{\infty} b_n a_n$, where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < 1$;*

(2) *there is a complex valued regular Borel measure $\lambda$ supported on a compact subset of $D$ such that*

$$L(f) = \int_D f(z)\,d\lambda(z) \quad \text{for } f \in \mathcal{A}.$$

*Conversely any such sequence or any such measure defines a continuous linear functional on $\mathcal{A}$ according to the formulas in* (1) *and in* (2).

Examples of continuous linear functionals on $\mathcal{A}$ include point evaluation and coefficient functionals. Now let $\mathcal{F}$ be compact. Standard arguments using convexity and the Krein–Milman Theorem yield:

$$\begin{aligned}\max_{\mathcal{F}} \operatorname{Re} L &= \max_{\overline{co}\,\mathcal{F}} \operatorname{Re} L = \max_{\mathcal{E}\mathcal{F}} \operatorname{Re} L = \max_{\mathcal{E}\,\overline{co}\,\mathcal{F}} \operatorname{Re} L = \max_{\sigma\mathcal{F}} \operatorname{Re} L = \max_{\sigma\,\overline{co}\,\mathcal{F}} \operatorname{Re} L \\ &= \max_{\mathcal{E}\,\overline{co}\,\mathcal{F} \cap \sigma\mathcal{F}} \operatorname{Re} L\end{aligned}$$

for each $L \in \mathcal{A}^*$. Since $\mathcal{E}\,\overline{co}\,\mathcal{F} \subset \mathcal{E}\mathcal{F}$ and $\sigma\mathcal{F} \subset \sigma\,\overline{co}\,\mathcal{F}$, linear extremal problems are solved over the two distinguished subsets $\mathcal{E}\,\overline{co}\,\mathcal{F}$ and $\sigma\mathcal{F}$.

For a given compact family $\mathcal{F}$ two basic problems in the application of linear methods are:

PROBLEM 1. Determine $\overline{co}\,\mathcal{F}$, $\mathcal{E}\,\overline{co}\,\mathcal{F}$ and $\sigma\mathcal{F}$.

PROBLEM 2. Identify geometric–analytic properties of the functions in $\mathcal{E}\,\overline{co}\,\mathcal{F}$ and in $\sigma\mathcal{F}$.

Linear methods also are applicable to investigate certain nonlinear optimization problems such as integral means, convex and Fréchet differentiable functionals and quotients of continuous linear functionals as well as for solving linear problems over compact families related to a family whose extreme points are known.

In many cases the subordination family associated with an individual function or a family plays an important role. For $f, g \in \mathcal{A}$ we say $g$ is *subordinate* to $f$ if there is a Schwarz function $\varphi$, that is $\varphi \in \mathcal{A}$, $\varphi(0) = 0$ and $|\varphi(z)| < 1$ for $|z| < 1$, such that $g(z) = f(\varphi(z))$ for $|z| < 1$. We write $g \prec f$. When $f$ is univalent, $g \prec f$ if and only if $g(D) \subset f(D)$ and $g(0) = f(0)$. For a given function $F$ or a given family $\mathcal{F}$, let $s(F)$ and $s(\mathcal{F})$ denote the collections of functions, respectively, subordinate to $F$ or to some function in $\mathcal{F}$. For $\mathcal{F}$ compact $s(\mathcal{F})$ is also compact.

## 2. Integral representations

Among the many interesting subfamilies of $\mathcal{A}$ perhaps the most famous and widely utilized in geometric function theory is the compact, convex family $\mathcal{P}$ of normalized functions of positive real part. Specifically, $f \in \mathcal{P}$ if and only if $f \in \mathcal{A}$, $f(0) = 1$ and $\operatorname{Re} f(z) > 0$ for $|z| < 1$. The *Riesz–Herglotz Representation Theorem* is given in the next statement.

THEOREM 2. *A function $f$ is in $\mathcal{P}$ if and only if there is a probability measure $\mu$ on $\Gamma$ such that*

$$f(z) = \int_{\Gamma} \frac{1+xz}{1-xz}\, d\mu(x) \quad \text{for } |z| < 1.$$

*Moreover*

$$\mathcal{E}\mathcal{P} = \left\{ \frac{1+xz}{1-xz} \colon |x| = 1 \right\} \quad \text{and} \quad \sigma\mathcal{P} = co\,\mathcal{E}\mathcal{P}.$$

The family $\mathcal{P}$ is an example of a subordination class. In fact if $p(z) = (1+z)/(1-z)$ then $\mathcal{P} = s(p) = \overline{co}\, s(p)$. Many other subfamilies of $\mathcal{A}$ give rise to integral representations. Basic properties of such integral representations can be summarized as follows.

THEOREM 3. *Let $X$ be a compact Hausdorff space and let the function $k\colon D \times X \to \mathbf{C}$ satisfy*:

(i) *for each $x \in X$, the map $z \mapsto k(z,x)$ is analytic in $D$.*
(ii) *For each $z \in D$, the map $x \mapsto k(z,x)$ is continuous in $X$.*
(iii) *The family $\{k(\cdot,x)\colon x \in X\}$ is locally bounded in $D$.*
*Let $\mathcal{F} = \{f_\mu(z) = \int_X k(z,x)\,d\mu(x)\colon \mu$ is a probability measure on $X\}$. Then*
(1) *$\mathcal{F}$ is a compact subset of $\mathcal{A}$.*
(2) *$\mathcal{F} = \overline{co}\{k(\cdot,x)\colon x \in X\}$, and*
(3) *$\mathcal{E}\mathcal{F} \subset \{k(\cdot,x)\colon x \in X\}$.*
(4) *If the map $\mu \mapsto f_\mu$ is one-to-one, then $\mathcal{E}\mathcal{F} = \{k(\cdot,x)\colon x \in X\}$.*

For the family $\mathcal{P}$ we have $X = \Gamma$ and $k(z,x) = p(xz) = (1+xz)/(1-xz)$, $x \in \Gamma$. In this case the map $\mu \mapsto f_\mu$ is one-to-one and this determines $\mathcal{E}\mathcal{P}$.

Many compact, convex subclasses of $\mathcal{A}$ are included as special cases of, or are linearly isomorphic to, one of the following general integral families. Let $\mathbf{P}(X)$ denote the set of probability measures on $X$. Let

$$\mathcal{F}_\alpha = \left\{ \int_\Gamma \frac{1}{(1-xz)^\alpha}\, d\mu(x)\colon \mu \in \mathbf{P}(\Gamma) \right\}, \quad \alpha \geqslant 0;$$

$$\mathcal{F}_{\alpha,c} = \left\{ \int_\Gamma \left( \frac{1+cxz}{1-xz} \right)^\alpha d\mu(x)\colon \mu \in \mathbf{P}(\Gamma) \right\}, \quad \alpha \geqslant 0,\ |c| \leqslant 1;$$

$$\mathcal{G}_{\alpha,\beta} = \left\{ \int_{\Gamma\times\Gamma} \frac{(1-xz)^\alpha}{(1-yz)^\beta}\, d\mu(x,y)\colon \mu \in \mathbf{P}(\Gamma\times\Gamma) \right\}, \quad \alpha,\beta \geqslant 0.$$

We note that $\mathcal{F}_{\alpha,0} = \mathcal{F}_\alpha$ and, less obviously, that $\mathcal{F}_{\alpha,c} \subset \mathcal{G}_{\alpha,\alpha}$. The families $\mathcal{F}_\alpha$ and $\mathcal{F}_{\alpha,c}$ are well understood. The extreme points and the support points of $\mathcal{G}_{\alpha,\beta}$ have only been determined for certain values of $\alpha$ and $\beta$. A useful property of the families $\mathcal{F}_\alpha$ is described in the following theorem.

THEOREM 4 (Product Lemma). *$\mathcal{F}_\alpha\mathcal{F}_\beta \subset \mathcal{F}_{\alpha+\beta}$. That is, if $f \in \mathcal{F}_\alpha$ and $g \in \mathcal{F}_\beta$ then $fg \in \mathcal{F}_{\alpha+\beta}$.*

In many cases the functions of interest are related to the family $S$ of univalent (one-to-one) functions in $\mathcal{A}$ normalized by $f(0) = 0$ and $f'(0) = 1$. Families defined by geometric–analytic conditions frequently lead to integral representations which in turn yield a complete answer to Problem 1 posed in the introduction: determine $\overline{co}\,\mathcal{F}$, $\mathcal{E}\,\overline{co}\,\mathcal{F}$ and $\sigma\mathcal{F}$. The family $S$ itself has not yielded to such a tractable description. However some success has been achieved in answering Problem 2 for $S$. The extreme points of the normalized family of meromorphic univalent functions is completely known. In section 6 we elaborate on these remarks.

## 3. Subclasses of $S$

We begin with four illustrative special classes of functions in $S$: The family $K$ of functions in $S$ with convex range; the family $S^*$ of functions in $S$ with range starlike with respect to 0;

the family $C$ of functions in $S$ with close-to-convex range; the family $S_{\mathbf{R}}$ of functions in $S$ such that $f(z)$ is real if $z$ is real $(-1 < z < 1)$. Also let $T$ denote the set of typically real functions, that is, the functions in $\mathcal{A}$ normalized by $f(0) = 0$ and $f'(0) = 1$ and satisfying $f(z)$ is real if and only if $z$ is real.

THEOREM 5. *Assume that* $f \in \mathcal{A}$, $f(0) = 0$ *and* $f'(0) = 1$.

(1) $f \in K$ *if and only if* $\operatorname{Re}\{zf''(z)/f'(z) + 1\} > 0$ *for* $|z| < 1$.
(2) $f \in S^*$ *if and only if* $\operatorname{Re}\{zf'(z)/f(z)\} > 0$ *for* $|z| < 1$.
(3) $f \in C$ *if and only if* $\operatorname{Re}\{zf'(z)/(e^{i\gamma}g(z))\} > 0$ *for* $|z| < 1$, *for some* $g \in S^*$ *and* $\gamma$ *real.*
(4) $f \in T$ *if and only if the analytic function* $q$ *defined by* $q(z) = (1 - z^2)f(z)/z$ $(q(0) = 1)$ *is real when* $z$ *is real and satisfies* $\operatorname{Re} q(z) > 0$.

These analytic characterizations of geometric properties lead to the following integral representations and to the determination of extreme points and support points. Let $\Gamma^+ = \{x \in \Gamma\colon \operatorname{Im} x \geqslant 0\}$.

THEOREM 6.

(1)
$$\overline{co}\, K = \left\{ \int_\Gamma \frac{z}{1 - xz}\, d\mu(x)\colon \mu \in \mathbf{P}(\Gamma) \right\} \quad \textit{and}$$
$$\mathcal{E}\, \overline{co}\, K = \sigma K = \left\{ \frac{z}{1 - xz}\colon |x| = 1 \right\}.$$

(2)
$$\overline{co}\, S^* = \left\{ \int_\Gamma \frac{z}{(1 - xz)^2}\, d\mu(x)\colon \mu \in \mathbf{P}(\Gamma) \right\} \quad \textit{and}$$
$$\mathcal{E}\, \overline{co}\, S^* = \sigma S^* = \left\{ \frac{z}{(1 - xz)^2}\colon |x| = 1 \right\}.$$

(3)
$$\overline{co}\, C = \left\{ \int_{\Gamma \times \Gamma} \frac{z - \frac{1}{2}(x + y)z^2}{(1 - yz)^2}\, d\mu(x, y)\colon \mu \in \mathbf{P}(\Gamma \times \Gamma) \right\} \quad \textit{and}$$
$$\mathcal{E}\, \overline{co}\, C = \sigma C = \left\{ \frac{z - \frac{1}{2}(x + y)z^2}{(1 - yz)^2}\colon |x| = |y| = 1 \textit{ and } x \neq y \right\}.$$

(4)
$$\overline{co}\, S_{\mathbf{R}} = T = \left\{ \int_{\Gamma^+} \frac{z}{(1 - xz)(1 - \bar{x}z)}\, d\mu(x)\colon \mu \in \mathbf{P}(\Gamma^+) \right\} \quad \textit{and}$$
$$\mathcal{E}\, \overline{co}\, S_{\mathbf{R}} = \sigma S_{\mathbf{R}} = \left\{ \frac{z}{(1 - xz)(1 - \bar{x}z)}\colon x \in \Gamma^+ \right\}.$$

For $\overline{co}\, K$, $\overline{co}\, S^*$ and $\overline{co}\, S_{\mathbf{R}}$ the maps $\mu \mapsto f_\mu$ described by Theorem 6 are one-to-one so that the statements about extreme points follow from Theorem 3. For $\overline{co}\, C$ the map $\mu \mapsto f_\mu$ is not one-to-one. We note that by differentiation we obtain

$$[\overline{co}\, C]' \equiv \{f'\colon f \in \overline{co}\, C\} = \left\{ \int_{\Gamma \times \Gamma} \frac{1 - xz}{(1 - yz)^3}\, d\mu(x, y) \right\}.$$

Thus $[\overline{co}\,C]' = \mathcal{G}_{1,3}$. Also $\frac{1}{z}\overline{co}\,K = \mathcal{F}_1$ and $\frac{1}{z}\overline{co}\,S^* = \mathcal{F}_2$.

As an illustration of the arguments yielding Theorem 6 we sketch a proof of (2). For $f \in S^*$ we have

$$\frac{zf'(z)}{f(z)} = \int_\Gamma \frac{1+xz}{1-xz}\,d\mu(x)$$

by using Theorems 5 and 2. Since an analytic branch of $\log \frac{f(z)}{z}$ is well-defined in $D$, differentiation yields

$$\frac{d}{dz}\log\frac{f(z)}{z} = \frac{1}{z}\left[\frac{zf'(z)}{f(z)} - 1\right] = \int_\Gamma \frac{2x}{1-xz}\,d\mu(x).$$

Then

$$\log\frac{f(z)}{z} = -2\int_\Gamma \log(1-xz)\,d\mu(x) \quad \text{and}$$

$$f(z) = z\exp\left[-2\int_\Gamma \log(1-xz)\,d\mu(x)\right].$$

Any $\mu \in \mathbf{P}(\Gamma)$ is the limit of a sequence of measures each of which is a finite convex combination of unit point masses. For such a measure

$$\nu = \sum_{k=1}^{m} t_k\delta_{x_k},\ \exp\left[-2\int_\Gamma \log(1-xz)\,d\nu(x)\right] = \exp\left[-2\sum_{k=1}^{m} t_k\log(1-x_k z)\right]$$
$$= \frac{1}{\prod_{k=1}^{m}(1-x_k z)^{2t_k}}.$$

If we apply Theorem 4 and then take the limit we obtain

$$f(z) = \int_\Gamma \frac{z}{(1-xz)^2}\,d\mu(x) \quad \text{for some } \mu \in \mathbf{P}(\Gamma).$$

Since each kernel function in this integral is an element of $S^*$ the description of $\overline{co}\,S^*$ given in Theorem 6 follows.

To obtain the support points of $S^*$, let $L$ be a continuous linear functional with $\operatorname{Re} L$ not constant on $S^*$ and let $\{b_n\}$ be the sequence associated with $L$ given by Theorem 1. Then

$$\operatorname{Re} L\left(\int_\Gamma \frac{z}{(1-xz)^2}\,d\mu(x)\right) = \int_\Gamma \operatorname{Re} L\left(\frac{z}{(1-xz)^2}\right)d\mu(x)$$
$$\leqslant \max_{|x|=1} \operatorname{Re}\sum_{n=1}^{\infty} nb_n x^{n-1}.$$

The condition $\overline{\lim}_{n\to\infty}|b_n|^{1/n} < 1$ implies that the function $h(z) = \sum_{n=1}^{\infty} nb_n z^{n-1}$ is analytic on $\overline{D}$. Since $\operatorname{Re} L$ is nonconstant on $S^*$, $\operatorname{Re} h$ is not constant on $\Gamma$. Hence, $\max_{|x|=1} \operatorname{Re} h(x)$ is achieved at only finitely many points of $\Gamma$, say $x_1, x_2, \ldots, x_k$. It follows that $f \in \overline{co}\, S^*$ maximizes $\operatorname{Re} L$ if and only if

$$f(z) = \sum_{j=1}^{k} t_j \frac{z}{(1 - x_j z)^2}, \quad \text{where } t_j \geqslant 0 \text{ for } j = 1, 2, \ldots, k \text{ and } \sum_{j=1}^{k} t_j = 1.$$

Also given any finite set $\{x_1, x_2, \ldots, x_k\} \subset \Gamma$ it is not difficult to construct a continuous linear functional $L$ such that $\operatorname{Re} L$ peaks over $\overline{co}\, S^*$ precisely on the set $\{\sum_{j=1}^{k} t_j \frac{z}{(1-x_j z)^2}\}$. Therefore $\sigma\, \overline{co}\, S^* = co(\mathcal{E} S^*)$. Now $\sigma S^* = S^* \cap \sigma\, \overline{co}\, S^*$. If

$$f(z) = \sum_{j=1}^{k} t_j \frac{z}{(1 - x_j z)^2} \in S^*,$$

the univalence of $f$ forces $k = 1$ and hence we obtain $\sigma S^* = \mathcal{E} S^*$.

There are a number of generalizations of the families $K$, $S^*$ and $C$ and the closed convex hulls and the extreme points of several of them have been determined. We shall describe a few such examples. The family $S^*(\alpha)$ of starlike functions of order $\alpha$ ($\alpha < 1$) consists of the normalized functions in $\mathcal{A}$ satisfying $\operatorname{Re} zf'(z)/f(z) > \alpha$ for $|z| < 1$. An argument analogous to that given above for $S^*$ shows that $\overline{co}\, S^*(\alpha) = \frac{1}{z}\mathcal{F}_{2-2\alpha}$. Similarly the class $K(\alpha)$ of convex functions of order $\alpha$ can be treated. Let $C(\beta)$ denote the class of close-to-convex functions of order $\beta$ ($\beta > 0$). A normalized function in $\mathcal{A}$ belongs to $C(\beta)$ provided that

$$\left|\arg \frac{zf'(z)}{e^{i\alpha} g(z)}\right| < \beta \cdot \frac{\pi}{2} \quad \text{for some } g \in S^* \text{ and some real } \alpha.$$

An important related class denoted $V_k$ consists of the normalized functions in $\mathcal{A}$ with boundary rotation at most $k\pi$ ($k \geqslant 2$). For $k > 2$ it can be shown that $V_k \subset C(k/2 - 1)$. Also for $k \geqslant 4$, $\overline{co}\, V_k = \overline{co}\, C(k/2 - 1)$ and $\overline{co}\, V_k' = \mathcal{G}_{k/2-1,k/2+1}$. The arguments for some of these relations are based on results described in the next section. At this point, we simply note that if $h \in \mathcal{A}$, $h(0) = 1$ and $|\arg h(z)| < \beta\pi/2$ then

$$h(z) = \left(\int_\Gamma \frac{1 + xz}{1 - xz}\, d\mu(x)\right)^\beta \quad \text{for some } \mu \in \mathbf{P}(\Gamma).$$

## 4. Subordination classes

Much of the success in addressing Problem 1 for the classes in Section 3 relies on subordination techniques. Questions about extreme points and support points for subordination families $s(\mathcal{F})$, where $\mathcal{F}$ is a family of functions, can be difficult to answer.

However if we consider a given function $F$ which maps $D$ conformally onto a domain $\Omega$, then the interplay between the geometry of $\Omega$ and the description of the extreme points and support points of the subordination class $s(F)$ can become extremely interesting. If, in addition, $\Omega$ is convex, then $\overline{co}\,s(F) = s(F)$. The starting point again is the family $\mathcal{P} = s(p) = \overline{co}\,s(p)$, where $p(z) = (1+z)/(1-z)$ maps $D$ onto a half-plane $\Omega$, and the results are displayed in Theorem 2. For a function $F \in \mathcal{A}$, let $F_x(z) = F(xz)$, where $|x| = 1$. Then it is not difficult to show it is always the case that $\{F_x\colon |x| = 1\} \subset \mathcal{E}\,\overline{co}\,s(F)$, which thus determines the minimal possible set of extreme points. Observe that the minimal set is achieved for $\mathcal{P}$. A key argument in generalizing this result is based on the following fact.

THEOREM 7 (Subordination Lemma). *Suppose that* $|c| \leqslant 1$ *and* $c \neq -1$. *Let* $F(z) = (1+cz)/(1-z)$. *If* $\alpha, \beta > 0$, $f \in s(F^\alpha)$ *and* $g \in s(F^\beta)$ *then* $fg \in s(F^{\alpha+\beta})$.

The proof of Theorem 7 depends upon the fact that $\log F$ is univalent and convex in $D$.

THEOREM 8. *Let $c$ and $F$ be described as in Theorem 7. If* $\alpha \geqslant 1$ *then*

$$\overline{co}\,s(F^\alpha) = \left\{ \int_\Gamma F_x^\alpha \, d\mu(x)\colon \mu \in \mathbf{P}(\Gamma) \right\} \quad \textit{and} \quad \mathcal{E}\,\overline{co}\,s(F^\alpha) = \{F_x^\alpha\colon |x| = 1\}.$$

*If* $\alpha = 1$ *then* $\sigma s(F) = co[\mathcal{E}s(F)]$ *and if* $\alpha > 1$ *then* $\sigma s(F^\alpha) = \mathcal{E}\,\overline{co}\,s(F^\alpha)$.

We note that $\overline{co}\,s(F^\alpha) = \mathcal{F}_{\alpha,c}$. For $\alpha = 1$ Theorem 8 is contained in Theorem 2 and $\overline{co}\,s(F) = s(F)$. For $\alpha > 1$ Theorem 8 essentially states that

$$\left[ \int_\Gamma \frac{1+cxz}{1-xz} \, d\mu(x) \right]^\alpha = \int_\Gamma \left( \frac{1+cxz}{1-xz} \right)^\alpha d\nu(x)$$

for some $\nu \in \mathbf{P}(\Gamma)$. The proof of this is an application of Theorem 7 as we now show.

Suppose that $f \in s(\mathcal{F}^\alpha)$ so that $f = g^\alpha$ where $g \in s(F)$. Assume that $g \notin \mathcal{E}s(F)$. Then $g = tg_1 + (1-t)g_2$ with $g_1, g_2 \in s(F)$, $g_1 \neq g_2$ and $0 < t < 1$. Hence,

$$f = g^\alpha = g^{\alpha-1} g = tg^{\alpha-1} g_1 + (1-t) g^{\alpha-1} g_2 = tf_1 + (1-t) f_2$$
$$\text{where } f_1, f_2 \in s(F^\alpha)$$

by Theorem 7. Clearly $f_1 \neq f_2$ and thus $f \notin \mathcal{E}s(F^\alpha)$. Therefore the only candidates for membership in $\mathcal{E}s(F^\alpha)$ are the functions $F_x^\alpha$ with $|x| = 1$. Hence,

$$\mathcal{E}\,\overline{co}\,s(F^\alpha) = \mathcal{E}s(F^\alpha) = \{F_x^\alpha\colon |x| = 1\} \quad \text{and} \quad \overline{co}\,s(F^\alpha)$$

is as described.

A corollary of Theorems 7 and 8 is a product theorem for $\mathcal{F}_{\alpha,c}$: If $\alpha + \beta \geqslant 1$, then $\mathcal{F}_{\alpha,c}\mathcal{F}_{\beta,c} \subset \mathcal{F}_{\alpha+\beta,c}$.

A remarkable further generalization of these results is contained in the following theorem.

THEOREM 9. *Let $F$ be analytic, univalent and nonzero on $D$. Suppose that $\mathbf{C}\backslash F(D)$ is convex. If $\alpha \geqslant 1$, then*

$$\mathcal{E}\,\overline{co}\,s(F^\alpha) = \{F_x^\alpha\colon |x| = 1\} \quad \text{and} \quad \overline{co}\,s(F^\alpha) = \left\{\int_\Gamma F_x^\alpha\, d\mu(x)\colon \mu \in \mathbf{P}(\Gamma)\right\}.$$

*When $\alpha = 1$ the hypothesis that $F$ is nonzero is not necessary.*

Contrasting greatly with the results above, in which the minimal set of extreme points is achieved, are examples where the function $F$ belongs to the Hardy space $H^1$ or where $F$ maps $D$ conformally onto a convex domain which is not a half-plane.

THEOREM 10. *If $F \in H^1$ and $\varphi$ is an inner function satisfying $\varphi(0) = 0$, then the composition $F \circ \varphi \in \mathcal{E}\,\overline{co}\,s(F)$.*

Observe that if $F \in H^p$ for some $p$, $0 < p < 1$, then the set of extreme points can be much smaller. For example this occurs when $F(z) = 1/(1-z)^\alpha$ and $\alpha > 1$ as described in Theorem 8. An inequality of Littlewood implies that if $f \in s(F)$ and $F \in H^p$ for some $p > 0$ then $f \in H^p$ and $\|f\|_{H^p} \leqslant \|F\|_{H^p}$. Moreover, for nonconstant $F$, $\|f\|_{H^p} = \|F\|_{H^p}$ if and only if $f = F \circ \varphi$ where $\varphi$ is a inner function and $\varphi(0) = 0$. Initially the last fact was used to prove the statement in Theorem 10 where $F \in H^p$ for some $p > 1$.

Now let $F$ map $D$ conformally onto a convex domain $\Omega$. If $\Omega$ is not a half-plane, then Theorem 10 is applicable since $F \in H^p$ for some $p > 1$. The next theorem presents results in this setting, where $\partial\Omega$ denotes the boundary of $\Omega$.

THEOREM 11. *Let $\mathcal{F} = s(F)$ and let $\Omega = F(D)$.*

(1) *If $\Omega$ is a strip or a wedge, then $f \in \mathcal{E}\mathcal{F}$ if and only if $f(e^{i\theta}) \in \partial\Omega$ for almost all $\theta$.*

(2) *If $\Omega$ is convex and not a half-plane, a strip or a wedge, then there exists $f \in \mathcal{E}\mathcal{F}$ such that $f(e^{i\theta}) \notin \partial\Omega$ for almost all $\theta$.*

The existence of $f(e^{i\theta}) \equiv \lim_{r\to 1^-} f(re^{i\theta})$ almost everywhere follows from facts about $H^p$ spaces. Examples of part (1) of Theorem 11 are given by the functions

$$F(z) = \log\left(\frac{1+z}{1-z}\right) \quad \text{and} \quad F(z) = \left(\frac{1+z}{1-z}\right)^\alpha \quad \text{where } 0 < \alpha < 1.$$

In the first case each $f \in \mathcal{F}$ belongs to $H^p$ for every $p > 0$. In the second case each $f \in \mathcal{F}$ belongs to $H^p$ for $0 < p < 1/\alpha$. The proof of part (1) of Theorem 11 uses Theorem 10 and a construction involving harmonic measure.

Facts about extreme points of subordination classes have also been related to the quantity $\rho(\theta) =$ distance between $f(e^{i\theta})$ and $\partial\Omega$. The classical example of this is the family of Schwarz functions $\mathcal{B}_0 = s(F)$ where $F(z) = z$ or, equivalently, the family $\mathcal{B}$ of functions $\varphi$ in $\mathcal{A}$ such that $|\varphi(z)| \leqslant 1$ for $|z| < 1$. Then $\omega \in \mathcal{B}_0$ if and only if $\omega(z) = z\varphi(z)$ for some $\varphi \in \mathcal{B}$. The relationship between the extreme points is the same.

THEOREM 12. *$\varphi \in \mathcal{E}\mathcal{B}$ if and only if*

$$\int_0^{2\pi} \log(1 - |\varphi(e^{i\theta})|)\, d\theta = -\infty.$$

*$\sigma\mathcal{B}$ consists of all finite Blaschke products.*

Obviously in this case, for a given $\varphi$, $\rho(\theta) = 1 - |\varphi(e^{i\theta})|$. This result has been generalized in several ways.

THEOREM 13. *Suppose that $\Omega$ is a bounded convex domain such that $\partial\Omega$ can be parametrized by $w = w(t)$, $a \leqslant t \leqslant b$, where $w \in C^2[a,b]$ and the curvature of $\partial\Omega$ is always positive. Then $f \in \mathcal{E}s(F)$ if and only if $f \in s(F)$ and $\int_0^{2\pi} \log \rho(\theta)\, d\theta = -\infty$.*

Certain properties of $\varphi$ and $f$ in the relation $f = F \circ \varphi$ can be connected, especially when $F$ is univalent. Not only do $F(e^{i\theta})$, $f(e^{i\theta})$ and $\varphi(e^{i\theta})$ exist almost everywhere but also $f(e^{i\theta}) = F(\varphi(e^{i\theta}))$ holds almost everywhere. The next theorem is a general result in this setting.

THEOREM 14. *Suppose that $F$ is analytic and univalent in $D$ and $\Omega = F(D)$ is a Jordan domain. If $f \in \mathcal{E}s(F)$ then*
(1) $\int_0^{2\pi} \log \frac{\rho(\theta)}{\rho(\theta)+1}\, d\theta = -\infty$, *and*
(2) $f = F \circ \varphi$ *where* $\varphi \in \mathcal{E}\mathcal{B}_0$.
*If $F$ is bounded the condition* (1) *is equivalent to* $\int_0^{2\pi} \log \rho(\theta)\, d\theta = -\infty$.

If $F$ is non-constant then, as with extreme points, it is not difficult to show that $\{F_x\colon |x| = 1\} \subset \sigma s(F)$. According to Theorem 8, for $F(z) = ((1+cz)/(1-z))^\alpha$ where $|c| \leqslant 1$, $c \neq -1$ and $\alpha > 1$ there is equality in this inclusion.

THEOREM 15. *Suppose that $F$ is analytic and $F'(z) \neq 0$ in $D$. Then $\sigma s(F) \subset \{F \circ \varphi\colon \varphi$ is a finite Blaschke product and $\varphi(0) = 0\}$. If $F$ is a convex mapping then we have equality in this inclusion.*

We close this section with a brief look at the question of determining the extreme points and the support points of $\overline{co}\, s(\mathcal{F})$ where $\mathcal{F}$ is a family of functions. Even for our four illustrative examples $K$, $S^*$, $C$ and $S_{\mathbf{R}}$ the results are incomplete. Notice below that the result for the support points of $K$, which utilizes Theorem 15, is distinctly different from the analogous result for $S^*$, which relies on Theorem 8.

THEOREM 16.

(1) $$\overline{co}\, s(K) = \left\{ \int_{\Gamma\times\Gamma} \frac{xz}{1-yz}\, d\mu(x,y)\colon \mu \in P(\Gamma \times \Gamma) \right\},$$

$$\mathcal{E}\,\overline{co}\, s(K) = \left\{ \frac{xz}{1-yz}\colon |x| = |y| = 1 \right\} \quad \textit{and}$$

$$\sigma s(K) = \{f \circ \varphi\colon\ f \in K,\ \varphi \in \sigma\mathcal{B}_0\}.$$

(2) $$\overline{co}\, s(S^*) = \left\{ \int_{\Gamma\times\Gamma} \frac{xz}{(1-yz)^2}\, d\mu(x,y)\colon\ \mu \in \mathbf{P}(\Gamma\times\Gamma) \right\},$$

$$\mathcal{E}\,\overline{co}\, s(S^*) = \left\{ \frac{xz}{(1-yz)^2}\colon |x| = |y| = 1 \right\} \quad \textit{and} \quad \sigma s(S^*) = \{ f_x\colon f \in S^*, |x| = 1 \}.$$

(3) $$\overline{co}\, s(C) = \left\{ \int_{\Gamma\times\Gamma\times\Gamma} w \frac{z - \frac{1}{2}(x+y)z^2}{(1-yz)^2}\, d\mu(x,y,w)\colon\ \mu \in \mathbf{P}(\Gamma\times\Gamma\times\Gamma) \right\} \quad \textit{and}$$

$$\mathcal{E}\,\overline{co}\, s(C) = \left\{ w \frac{z - \frac{1}{2}(x+y)z^2}{(1-yz)^2}\colon\ |w| = |x| = |y| = 1,\ x \neq y \right\}.$$

To sample the techniques utilized here we sketch the proof for the convex hull and extreme points for $s(S^*)$. If $f \in \mathcal{E}\,\overline{co}\, s(S^*)$ then necessarily $f \prec g$ where $g \in \mathcal{E}\,\overline{co}(S^*)$. Thus $f \prec \frac{z}{(1-xz)^2}$ for some $x$ with $|x| = 1$ and then $xf \prec \frac{z}{(1-z)^2}$. Since

$$\frac{z}{(1-z)^2} = \frac{1}{4}\left[\left(\frac{1+z}{1-z}\right)^2 - 1\right],$$

Theorem 8 yields

$$xf(z) = \frac{1}{4}\int_\Gamma \left[\left(\frac{1+yz}{1-yz}\right)^2 - 1\right] d\mu(y).$$

Hence,

$$f(z) = \int_\Gamma \frac{\bar{x}yz}{(1-yz)^2}\, d\mu(y).$$

Therefore

$$\overline{co}\, s(S^*) \subset \left\{ \int_{\Gamma\times\Gamma} \frac{xz}{(1-yz)^2}\, d\mu(x,y)\colon\ \mu \in \mathbf{P}(\Gamma\times\Gamma) \right\}.$$

Clearly $xz/(1-yz)^2 \in s(S^*)$ and hence the reverse containment also holds.

## 5. Applications

Suppose that $\mathcal{F}$ is a compact subset of $\mathcal{A}$ and $L$ is a continuous linear functional on $\mathcal{A}$. Then, as mentioned in the introduction, to solve the extremal problem $\max_{f\in\overline{co}\,\mathcal{F}} \operatorname{Re} L(f)$, it suffices to solve it only over $\mathcal{E}\,\overline{co}\,\mathcal{F}$. In many of the examples described earlier $\mathcal{E}\,\overline{co}\,\mathcal{F}$ is given in such a simple analytic way that linear extremal problems are readily solvable by this approach.

For a simple illustration suppose that $f \in \mathcal{A}$ and $f$ is subordinate to some function in $S^*$. Let $a_n$ denote the $n$th coefficient of $f$. Then Theorem 16 and the remarks above imply that

$$\operatorname{Re} a_n \leqslant \max_{|x|=1,\ |y|=1} \operatorname{Re}\{nxy^{n-1}\} = n.$$

For other families or functionals the reduction to extreme points frequently requires further analysis. As an example the use of extreme point theory in the coefficient problem for the family $V_k$ ($k > 2$) helped reduce the problem to comparing the $n$th coefficient of $((1+xz)/(1-z))^\alpha$ with that of $((1+z)/(1-z))^\alpha$, where $|x| = 1$ and $\alpha \geqslant 1$.

Let $L$ be identified by the sequence $\{b_n\}$ according to Theorem 1. Then, for $\mathcal{F}$ compact,

$$\max_{f\in\mathcal{F}} \operatorname{Re} L(f) = \max_{f\in\mathcal{E}\,\overline{co}\,\mathcal{F}} \operatorname{Re}\left\{\sum_{n=0}^{\infty} b_n a_n\right\} \quad \text{where } f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

For example, if $\mathcal{F} = s(S^*)$, then

$$\max_{f\in s(S^*)} \operatorname{Re} L(f) = \max_{|x|=1,\ |y|=1} \operatorname{Re} F(x, y) = \max_{|y|=1} |G(y)|$$

$$\text{where } F(x, y) = x\sum_{n=1}^{\infty} nb_n y^{n-1} = xG(y)$$

and $G$ is analytic in $\overline{D}$. For several of the families described earlier such reductions occur where $F$ is realized as an analytic function in one or more variables.

Extreme point methods are also applicable to other types of functionals. If $\mathcal{F}$ is a convex subset of $\mathcal{A}$, then $J : \mathcal{A} \to \mathbf{R}$ is a called a *convex functional* on $\mathcal{F}$ provided that

$$J[tf + (1-t)g] \leqslant tJ(f) + (1-t)J(g) \quad \text{for } f, g \in \mathcal{F} \text{ and } 0 < t < 1.$$

THEOREM 17. *Suppose that $\mathcal{F}$ is a compact subset of $\mathcal{A}$ and $J$ is a continuous, convex functional on $\overline{co}\,\mathcal{F}$. Then*

$$\max_{f\in\overline{co}\,\mathcal{F}} J(f) = \max_{f\in\mathcal{F}} J(f) = \max_{f\in\mathcal{E}\,\overline{co}\,\mathcal{F}} J(f).$$

Theorem 17 has a variety of applications. For example, if $L$ is a continuous linear functional on $\mathcal{A}$, then $J = |L|$ is a continuous, convex functional. Such examples include the absolute value of the $n$th coefficient or, more generally, $|f^{(n)}(z)|$ when $z$ is fixed, $|z| < 1$ and $n = 0, 1, 2, \ldots$. If $p \geqslant 1$ then the Minkowski inequality implies that the functional

$$J(f) = \left[\int_0^{2\pi} |f^{(n)}(re^{i\theta})|^p\, d\theta\right]^{1/p}$$

is convex on $\mathcal{A}$ for $0 < r < 1$ and $n = 0, 1, 2, \ldots$ so that Theorem 17 can be applied.

To illustrate suppose that $f \in \mathcal{A}$ and $f$ is subordinate to some function in the family $C$. Then Theorem 16 implies that

$$\int_0^{2\pi} |f^{(n)}(re^{i\theta})|^p \, d\theta \leqslant \max_{|x|=1} \int_0^{2\pi} |F^{(n)}(re^{i\theta})|^p \, d\theta$$

$$\text{where } F(z) = \frac{z - \frac{1}{2}(x+1)z^2}{(1-z)^2} \text{ and } |x| = 1.$$

An appeal to facts about symmetric rearrangements shows that the maximum occurs when $x = -1$. In other words,

$$\max_{f \in s(C)} \int_0^{2\pi} |f^{(n)}(re^{i\theta})|^p \, d\theta$$

takes place for the Koebe function $k(z) = z/(1-z)^2$.

Extreme point methods also apply to quotients of continuous linear functionals. As a special case of a more general result we illustrate with the simple and elegant proof of the following result.

THEOREM 18. *Let $L$, $M \in \mathcal{A}^*$ with $M(f) \neq 0$ for $f \in \mathcal{P}$.*
*Let*

$$\mathcal{P}_0 = \left\{ t\frac{1+xz}{1-xz} + (1-t)\frac{1+yz}{1-yz} \colon 0 \leqslant t \leqslant 1,\ |x| = |y| = 1 \right\}.$$

*Then*

$$\max_{f \in \mathcal{P}} \operatorname{Re}\left\{ \frac{L(f)}{M(f)} \right\} = \max_{f \in \mathcal{P}_0} \operatorname{Re}\left\{ \frac{L(f)}{M(f)} \right\}.$$

PROOF. Suppose that $w = L(f)/M(f)$ for some $f \in \mathcal{P}$. Theorem 2 implies that 0 belongs to the closed convex hull of

$$\Lambda = \{F(x) \colon |x| = 1\} \quad \text{where } F(x) = L\left(\frac{1+xz}{1-xz}\right) - wM\left(\frac{1+xz}{1-xz}\right).$$

Since $\Lambda$ is compact and connected, $0 = tF(x) + (1-t)F(y)$ for some $x, y$ with $|x| = |y| = 1$ and $0 \leqslant t \leqslant 1$. Equivalently,

$$\frac{L(g)}{M(g)} = w \quad \text{where } g(z) = t\frac{1+xz}{1-xz} + (1-t)\frac{1+yz}{1-yz}. \qquad \square$$

Suppose that $\mathcal{L}\colon \mathcal{A} \to \mathcal{A}$ is a continuous linear operator. Then $\mathcal{L}$ is called of *order zero* if $\mathcal{L}(f(w))|_{w=xz} = \mathcal{L}(f(xz))$ for $f \in \mathcal{A}$ and $0 < |x| \leqslant 1$. Examples are given by

$(\mathcal{L}f)(z) = zf'(z)$, $(\mathcal{L}f)(z) = \frac{1}{z}\int_0^z f(w)\,dw$, $(\mathcal{L}f)(z) = [zf(z)]'$ and the $n$th partial sum of a power series. If $f, g \in \mathcal{A}$ then $f$ is called *hull subordinate* to $g$ if

$$\{w\colon\ w = f(z),\ |z| \leqslant r\} \subset \overline{co}\{w\colon\ w = g(z),\ |z| \leqslant r\} \quad \text{for each } r\,(0 < r < 1).$$

THEOREM 19. *Suppose that $\mathcal{F}$ is a compact subset of $\mathcal{A}$ and $\mathcal{E}\,\overline{co}\,\mathcal{F} = \{F_x\colon\ |x| = 1\}$ for some $F \in \mathcal{A}$. If $\mathcal{L}$ is a continuous linear operator of order zero then $\mathcal{L}(f)$ is hull subordinate to $\mathcal{L}(F)$ for each $f \in \mathcal{F}$.*

Theorem 9 describes a large class of subordination families for which the condition in the theorem on $\mathcal{E}\,\overline{co}\,\mathcal{F}$ occurs. For example if

$$f \in \mathcal{P} = s\left(\frac{1+z}{1-z}\right), \quad f(z) = 1 + \sum_{k=1}^{\infty} a_k z^k \quad \text{and} \quad s_n(z) = 1 + \sum_{k=1}^{n} a_k z^k,$$

then

$$\{s_n(z)\colon\ |z| \leqslant r\} \subset \overline{co}\{S_n(z)\colon\ |z| \leqslant r\} \quad \text{where } S_n(z) = 1 + \sum_{k=1}^{n} 2z^k.$$

In particular $\operatorname{Re} s_n(z) \geqslant 0$ for $|z| \leqslant \frac{1}{2}$ and $n = 1, 2, \ldots$, since $S_n(z)$ has that property. Indeed

$$\min_{f \in \mathcal{P}} \min_{|z| \leqslant r} \operatorname{Re} s_n(z) = \min_{|z| \leqslant r} \operatorname{Re} S_n(z).$$

Another area of application concerns modulus majorization. If $f, g \in \mathcal{A}$ we say that $f$ is *majorized* by $g$ provided that $|f(z)| \leqslant |g(z)|$ for $|z| < 1$. Equivalently, $f(z) = \varphi(z)g(z)$ for some $\varphi \in \mathcal{B}$. For $\mathcal{F} \subset \mathcal{A}$, let $m(\mathcal{F}) = \{f\colon\ f$ is majorized by some function in $\mathcal{F}\}$.

THEOREM 20. *Suppose that $J : \mathcal{A} \to \mathbf{R}$ is a continuous convex functional and $\mathcal{F}$ is a compact subset of $\mathcal{A}$. Let $\mathcal{G} = \{f(0)\colon\ f \in \mathcal{F}\} \cup m(\mathcal{E}\,\overline{co}\,\mathcal{F})$. Then $\max_{f \in m(\mathcal{F})} J(f) = \max_{f \in \mathcal{G}} J(f)$.*

Theorem 20 can be combined with results such as Theorem 6 to yield an effective tool to solve extremal problems about majorization families in geometric function theory. Such problems reduce to problems about the family $\mathcal{B}$.

The last area of application we mention concerns the form of solutions to more general extremal problems and we illustrate with a result about $S^*$.

THEOREM 21. *For $|\xi| < 1$, let $V_n = V_n(S^*, \xi)$ denote the set of points $(w_0, w_1, \ldots, w_{n-1})$ in $\mathbf{C}^n$ such that $w_k = f^{(k)}(\xi)$ $(k = 0, 1, \ldots, n-1)$ for some $f \in S^*$. Suppose that $\Phi$ is a complex-valued non-constant function which is defined and continuous on $V_n$ and analytic on the interior of $V_n$. If $f \in S^*$ and*

$$\operatorname{Re}\Phi\left[f(\xi), f'(\xi), \ldots, f^{(n-1)}(\xi)\right] = \max_{g \in S^*} \operatorname{Re}\Phi\left[g(\xi), g'(\xi), \ldots, g^{(n-1)}(\xi)\right]$$

*for some* $\xi\,(0 < |\xi| < 1)$, *then* $f$ *has the form*

$$f(z) = \frac{z}{\prod_{k=1}^{n}(1 - x_k z)^{\lambda_k}} \quad \textit{where } |x_k| = 1,\ \lambda_k \geqslant 0 \textit{ for } k = 1, 2, \ldots, n$$

$$\textit{and } \sum_{k=1}^{n} \lambda_k = 2.$$

The proof requires a lengthy analysis to show that each point on the boundary of $V_n$ corresponds uniquely to a function in $S^*$ of the form given in the theorem. Ultimately, however, the proof depends on the fact that

$$\mathcal{E}\mathcal{F} = \left\{ \log \frac{1}{1 - xz} \colon |x| = 1 \right\}$$

$$\text{where } \mathcal{F} = \left\{ \int_{\Gamma} \log\left(\frac{1}{1 - xz}\right) d\mu(x) \colon \mu \in P(\Gamma) \right\}.$$

Application of extreme point methods in geometric function theory has given a general framework for deriving many classical results about extremal problems over special families and has been used to resolve a number of unsolved problems.

## 6. Univalent functions

At the end of Section 3 we introduced the class $S$ of normalized univalent functions in $\mathcal{A}$. A closely related family, which we denote $\Sigma$, is the set of normalized univalent functions which are analytic in $E = \{z\colon |z| > 1\}$. Specifically, $f \in \Sigma$ if and only if $f$ is analytic and univalent in $E$ and has a Laurent series at $\infty$ of the form

$$f(z) = z + \sum_{n=1}^{\infty} \frac{a_n}{z^n}.$$

Equipped with the topology of uniform convergence on compact subsets of $E$, the Fréchet space of all analytic functions on $E$ contains $\Sigma$ as a compact subset.

Considerable effort has been devoted to the study of extreme points and support points of $S$. In general many factors play a role in the determination of extreme points and support points of a given class of functions. In a family with some underlying linear structure, like $K$, or with nice analytic or geometric mapping properties, like $B$, the extreme points and support points are easily identified. This is not the case of the class $S$ which is very rich in functions (Riemann Mapping Theorem), which has no linear structure (sums of univalent functions are almost never univalent) and which is defined by the nonanalytic property of univalence. Consequently the focus for $S$ has shifted to obtaining descriptive information about the extreme points and support points.

A variety of techniques and variational methods have been exploited to identify geometric–analytic properties of these extreme points and support points. For example, if $f \in S$ and $f(D)$ omits an open set, it is easy to show, for $w$ in an open subset omitted by $f$, that

$$f_\varepsilon(z) = f(z) + \varepsilon \frac{f^2(z)}{w - f(z)} \in S$$

for all sufficiently small complex numbers $\varepsilon$. This exhibits an elementary variation of $f$ which constructs functions in $S$ close to $f$ and is the initial step in the proof we present of the following theorem.

THEOREM 22. *If $f \in \sigma S$, then $f(D)$ is dense in* $\mathbf{C}$.

PROOF. Let $f$ yield the maximum of $\operatorname{Re} L$ over $S$ for the continuous linear functional $L$, whose real part is nonconstant on $S$. Suppose $f(D)$ is not dense in $\mathbf{C}$ and choose $w$ in an open subset of $\mathbf{C}\backslash f(D)$. Then, with $f_\varepsilon$ as given above, $\operatorname{Re} L(f_\varepsilon) \leqslant \operatorname{Re} L(f)$ so that $\operatorname{Re} L(\varepsilon f^2/(w-f)) \leqslant 0$ for all sufficiently small complex numbers $\varepsilon$. It follows that $L(f^2/(w-f)) = 0$. According to Theorem 1 there is a compact set $Q \subset D$ and a measure $\lambda$ supported on $Q$ with

$$L(h) = \int_Q h(z)\, d\lambda(z) \quad \text{for } h \in A.$$

The univalence of $f$ assures that any $w \notin F(D)$ is connected to $\infty$ so that $w$ lies in an open, connected neighborhood of $\infty$ disjoint from $f(Q)$. The function

$$H(w) = \int_Q \frac{f^2(z)}{w - f(z)}\, d\lambda(z)$$

is analytic on this neighborhood and vanishes on an open subset by the above argument, hence is identically zero. Expanding $H(w)$ in a series at $\infty$ yields

$$L(f^n) = \int_Q f^n(z)\, d\lambda(z) = 0 \quad \text{for every integer } n \geqslant 2.$$

By applying Runge's Theorem to functions analytic on the range of $f$ and transferring this information back to $D$ we obtain $L(h) = L(1)h(0) + L(f)h'(0)$ for every $h \in A$. Thus $L$ is constant on $S$, contrary to hypothesis. □

A second elementary argument allows one to decompose a function in $S$ into a convex combination of two distinct functions in $S$, each of whose range omits an open set, whenever $f$ omits two values of equal modulus. One obtains the following result.

THEOREM 23. *If $f$ is an extreme point or a support point of $S$ then $f$ maps the unit disk onto the complement of a continuous arc tending to $\infty$ with increasing modulus.*

Somewhat surprisingly one can obtain a complete description of the extreme points of $\overline{co}\,\Sigma$, as follows.

THEOREM 24. *Suppose that $f \in \Sigma$. Then $f \in \mathcal{E}\,\overline{co}\,\Sigma$ if and only if the two-dimensional Lebesgue measure of $\mathbf{C}\backslash f(E)$ is zero.*

For $f \in \Sigma$ by an application of the Area Theorem the coefficients must satisfy $\sum_{n=1}^{\infty} n|a_n|^2 \leqslant 1$. Also the measure of $\mathbf{C}\backslash f(E)$ is given by $\pi(1 - \sum_{n=1}^{\infty} n|a_n|^2)$. One direction of the proof of Theorem 24 follows from the fact that the extreme points of the space of functions $g(z) = \sum_{n=1}^{\infty} b_n z^n$ in $\mathcal{A}$ which satisfy $\sum_{n=1}^{\infty} n|b_n|^2 \leqslant 1$ consists precisely of those functions for which $\sum_{n=1}^{\infty} n|b_n|^2 = 1$. To establish the other direction suppose that $f \in \Sigma$ and the measure of the compact set $J = \mathbf{C}\backslash f(E)$ is positive. The proof then relies on the significant result that there exists a nonconstant bounded analytic function $F$ on $(\mathbf{C} \cup \{\infty\})\backslash J$ which satisfies a Lipschitz condition. It follows that $F_1(w) = w + \alpha F(w)$ and $F_2(w) = w - \alpha F(w)$ are both univalent in $E$ for all small $\alpha > 0$. Thus $f = \frac{1}{2} f_1 + \frac{1}{2} f_2$, where $f_n = F_n \circ f \in \Sigma$ for $n = 1, 2$, and $f_1 \neq f_2$. Consequently $f \notin \mathcal{E}\,\overline{co}\,\Sigma$.

The powerful Schiffer boundary variation allows one to obtain considerably more information about the omitted set of a support point of $S$ or $\Sigma$ and to deduce that the omitted set consists of finitely many analytic arcs. Of course for $S$ it is a single arc by Theorem 23. By constructing two variations, which in the case of an omitted arc essentially amounts to modifying the arc to produce neighboring functions, one shows that the omitted set $J$ of a support point of $S$ or $\Sigma$ lies on the trajectories of a quadratic differential.

We summarize some of the many consequences of this much exploited result in the following two theorems.

THEOREM 25. *Let $f$ be a support point of $S$ corresponding to the continuous functional $L$ and let $J = \mathbf{C}\backslash f(D)$. Then*

(1) *$J$ is an analytic arc which satisfies the differential equation*

$$\frac{1}{w^2} L\left(\frac{f^2}{f - w}\right)\left(\frac{dw}{dt}\right)^2 > 0,$$

*where $w = w(t)$ is a parametrization of $J$.*

(2) *At each point $w \in J$, except perhaps the finite tip, the tangent line makes an angle less than $\frac{\pi}{4}$ with the radial line from $0$ to $w$. The value $\frac{\pi}{4}$ can be attained at the tip.*

(3) *The arc $J$ is asymptotic to a line at $\infty$.*

(4) *The function $f$ is analytic in the closed unit disk except for a pole or order two at one point on the unit circle.*

THEOREM 26. (1) *If $f$ is a support point of $\Sigma$ corresponding to the continuous linear functional $L$, then $J = \mathbf{C}\backslash f(E)$ consists of finitely many analytic arcs which satisfy the differential equation*

$$L\left(\frac{1}{f - w}\right)\left(\frac{dw}{dt}\right)^2 > 0, \quad \text{where } w = w(t) \text{ is a parametrization of } J.$$

(2) *If $f \in \Sigma$ and $\mathbf{C}\backslash f(E)$ is a single analytic arc, then $f$ is a support point of $\Sigma$.*

The second statement in Theorem 26 is established by a direct construction of a suitable functional.

Considerable additional information is known for support points of $S$ associated with special types of functionals. For example, the famous Bieberbach Conjecture–de Branges' Theorem states that the $n$th coefficient functional is maximized by the Koebe function $f(z) = z/(1-z)^2$. For point evaluation functionals the omitted arc of an extremal function has a monotonic argument and a monotonic radial angle. Similar results hold for derivative point evaluations.

One other category of interesting extremal functions and functionals arises in the process of truncation. If one starts with the omitted arc of a support point, removes a piece starting at its base point and constructs and normalizes the map to the new region created, one obtains a new support point maximizing a new functional associated with the original function and functional. One can think of the process in reverse, elongation of the arc.

For the class $\Sigma$ the omitted arc of an analytic slit mapping can always be elongated analytically and then renormalized to produce a new support point. For the class $S$ this is not always the case and there exist support points, called terminal support points, which do not allow an elongation to produce another support point. Since properties like monotone argument and monotone radial angle are preserved under truncation, existence of and information about terminal support points of $S$ is especially useful. At this point, however, little is known about terminal support points and a number of important and interesting conjectures are associated with them.

Finally, Theorem 23 above suggests that extreme points and support points of $S$ share important geometric properties. One might think that the two sets coincide. However examples exist of extreme points which are not support points. It remains open whether or not support points exist which are not extreme points.

## 7. Notes

Many of the results mentioned in this survey appear in the three books by Duren, Hallenbeck and MacGregor, and Schober listed at the top of our references as [Du], [HM], and [Sch]. Rather than make an extensive list of references we will merely refer to these texts in which a comprehensive list of references is given. For example, Theorem 1 appears in Duren, pp. 278–80, Theorems 9.1 and 9.3. We will abbreviate this as [Du] (pp. 278–80, Thms. 9.1 and 9.3). [HM] has references listed sequentially in alphabetical order and Du has references listed alphabetically, then sequentially under a name. So, for example, for general results on compact, convex sets and the Krein–Milman Theorem one can turn to *Linear Operators* by Dunford and Schwartz, [Du] (Dunford [1]) or [HM] ([26]). Additional references not included in these books will be listed below sequentially.

The fact that, for $\mathcal{F}$ a compact subset of $A$, it is also true that $s(\mathcal{F})$ and $m(\mathcal{F})$ are compact is in [HM] (p. 64, Lemma 5.19). For the original paper on the Riesz–Herglotz Representation Theorem see [HM] ([47]). For a linear methods proof see [Sch] (pp. 3–4, Thms. 1.5 and 1.6). Theorem 3 is in [HM] ([18]) and Theorem 4 is in [HM] ([16]). These two references were fundamental in initiating many of the ideas relating linear methods to geometric function theory. Theorem 5 can be found in [Du] (Sections 2.5, 2.6 and 2.8) or in [HM] (Chapter 2). See also [Sch] (pp. 6–15). With the exception of the support points of $C$, Theorem 6 is in [HM] ([18]). For the supports points of $C$ see [HM] ([37]) and ([13]).

The family $\mathcal{F}_\alpha$ was introduced in [HM] ([18]). The family $\mathcal{F}_{\alpha,c}$ and the related Theorems 7 and 8 can be found in [HM] (pp. 50–52, Lemma 5.6 and Thm. 5.7 and p. 104, Thm. 7.15). For more about these families plus results about the families $S^*(\alpha)$, $K(\alpha)$, $C(\beta)$ and $V_k$ see [HM] ([16]) and [Sch] (Chapter 2). A summary of results about $\mathcal{G}_{\alpha,\beta}$ and its relationship to the above families is in [12]. Theorem 9 arose from a conjecture by Sheil-Small ([5], problem 5.59, p. 554) and evolved through the series of papers [2,3,10, 11]. Theorem 10 is in [1]. The discussion following Theorem 10 is expanded upon in [HM] (pp. 117–122 and Thm. 8.1) and Littlewood's inequality is in [HM] (p. 25, Thm. 3.3). Theorems 11–14 appear as part of a more general discussion of subordination and extreme point theory in [HM] (Chapter 8). Specifically, see [HM] (pp. 131–138, Thms. 8.14, 8.15, 8.18 and 8.20). The second statement in Theorem 11 is in [9]. Also note that the statement about support points in Theorem 12 as well as in Theorem 15 can be found as part of a discussion of support points in [HM] (pp. 101–107, Thms. 7.12, 7.16 and 7.17). Theorem 16 is a synthesis from [HM] (pp. 65–69, Thms. 5.21, 5.22 and 5.23). A general presentation of applications to extremal problems is available in [HM] (Chapter 6). Theorem 17 is in [HM] (p. 45, Thm. 4.6). For our specific Theorems 18 and 19 see [HM] (pp. 85–89, Thms. 6.13 and 6.15). Theorem 20 can be found in [HM] (pp. 64–5, Lemma 5.20) and Theorem 21 in [HM] (p. 163, Thm. 9.13).

For a general introduction to the elementary theory of univalent functions and the families $S$ and $\Sigma$ see [Du] (Chapter 2) and for a discussion of general extremal problems over $S$ and $\Sigma$ see [Du] (Chapter 9). In particular Theorems 22–24 are in [Du] (pp. 284–290, Thm. 9.4 and Corollary, Thm. 9.5 and Corollaries 1 and 2, Thms. 9.6 and 9.8). The Schiffer boundary variation is given a thorough treatment both in [Du] (Chapter 10) and [Sch] (Appendix C, p. 181–190). With regard to Theorem 25, parts (1) and (2) are essentially [Du] (p. 306, Thm. 10.3) although one should also consult [Du] (Pearce [1]). One can find parts (3) and (4) in [Du] (Brickman and Wilken [1]). Theorem 26, part (1) is a consequence of what is called Schiffer's Theorem. See [Du] (p. 297, Thm. 10.1). Part (2) can be found in ([4]).

The celebrated de Branges proof of the Bieberbach conjecture can be found in the authors' own words in [6]. For a general survey of truncation and terminal support points which includes both results and open questions see [7]. Also included there is an excellent list of references. In the context of what are called generalized support points of $S$ one can find an example of an extreme point which is not a support point of $S$ in [8].

We have separated our references into two categories, the second of which is labeled Further References. This second list provides selected references about extreme points and support points which have appeared after the publication of [Du], [HM] or [Sch] and whose results are not described in the text or the notes above. In this list [21,27,31,33] are survey articles. The two reference lists combined with the comprehensive lists given in [Du], [HM] and [Sch] should give the interested reader access to most of the literature in this field.

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CHAPTER 13

# The Method of the Extremal Metric


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*Contents*




**Abstract**

This work provides an account of the method of the extremal metric with special focus on its applications in geometric function theory. Almost exclusively reference is made only to publications which use this method in an essential manner or to those with close connections to such publications. Not every item in the bibliography has a corresponding reference in the text.



HANDBOOK OF COMPLEX ANALYSIS: GEOMETRIC FUNCTION THEORY, VOLUME 1

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## 1. History, definitions and standard results

**1.** The method of the extremal metric has its origin in the simple remark that for a conformal mapping area distortion is the square of length distortion. As a result if, for a family of curves in a domain under a conformal mapping there is a lower estimate for the lengths of the images of the curves and an upper estimate for the area of the image of the domain, an application of Schwarz's inequality or something similar may provide a useful inequality. This is called the length-area method.

The first application of this method appears to be in a paper of Courant [24] dealing with boundary correspondence. This was reproduced in the book of Hurwitz and Courant [74]. Other early applications were made by Bohr [19] and Gross [64]. A somewhat different technique was employed by Faber [33] in his work on boundary correspondence.

The basis for the current form of the method of the extremal metric was laid in the years 1928–1946 by the work of Herbert Grötzsch, Arne Beurling, Lars Ahlfors and Oswald Teichmüller.

Grötzsch developed the technique of Faber to obtain what he called the method of strips. With it he obtained [44,45,47–49,52–56,58,60–62] the solution of numerous extremal problems for conformal mappings. He also did the pioneering work for quasiconformal mappings (although he did not use this term).

In his thesis Beurling [12] introduced the concept of extremal distance between two sets in the closure of a domain. His chief interest there was to obtain inequalities between these quantities and harmonic measures.

In his thesis Ahlfors [1] considered strip mappings and using length-area arguments obtained two results which he called the first and second fundamental inequalities. The first of these is now habitually called the Ahlfors Distortion Theorem.

Teichmüller's contributions are contained in three papers. In [228] he manifested the fundamental importance of quadratic differentials for extremal problems in geometric function theory. In [227] he gave a coefficient result for univalent functions associated with quadratic differentials. In [226] he studied with great precision certain questions for the characteristic conformal invariants of doubly-connected domains.

At the Scandinavian Mathematical Congress in Copenhagen in 1946 Beurling and Ahlfors gave companion lectures introducing the abstract form of the method of the extremal metric. Beurling's lecture provided the theoretical background. Ahlfors' lecture dealt with more explicit problems including some for multiple curve families although not in the context considered in Section 3. It was published in the proceedings of the conference [8]. It was said that Beurling's lecture would be published in Acta Mathematica. Unfortunately it did not appear. It seems impossible at this distance of time to recover its exact content. It seems unlikely that the obscure fragment which was included by the editors in Beurling's collected works [14] represents the substance of this lecture.

In the spring of 1947 Ahlfors, in his graduate course at Harvard, gave the first detailed account of the method of the extremal metric as constituted at that time. Shortly afterwards in lectures [3] at Oklahoma State University he gave an abbreviated account of this material for which mimeographed notes were prepared.

In the academic year 1948–1949 Beurling was visiting professor at Harvard and he and Ahlfors prepared a rough draft of a monograph giving a somewhat expanded version of the same material. Nothing was ever published from this. Some of the new results which were given later appeared in Ahlfors' book [6] in particular Sections 2.5 and 4.6 although to some extent in a modified form. They did give a brief account of the method in several publications [9,10].

**2.** The fundamental entity in the method of the extremal metric is most effectively defined on a Riemann surface.

DEFINITION. Let $\mathcal{R}$ be a Riemann surface. By a conformally invariant metric $\rho(z)|dz|$ defined on $\mathcal{R}$ we mean an entity which associates with every local uniformizing parameter $z$ of $\mathcal{R}$ a real-valued non-negative measurable function $\rho(z)$ satisfying the conditions:

(i) If $\gamma$ is a rectifiable curve in the parameter plane neighborhood for $z$, $\int_\gamma \rho(z)|dz|$ exists (as a Lebesgue–Stieltjes integral) possibly having the value $+\infty$.

(ii) If the neighborhood on $\mathcal{R}$ for the local uniformizing parameter $z^*$ overlaps that for $z$ and with $z^*$ is associated the function $\rho^*(z^*)$ then at every common point of the neighborhoods for $z$ and $z^*$ we have

$$\rho^*(z^*) = \rho(z)\left|\frac{dz}{dz^*}\right|.$$

DEFINITION. A curve on a Riemann surface is called locally rectifiable if, for every compact subcurve lying in the neighborhood on $\mathcal{R}$ for a local uniformizing parameter $z$, the corresponding curve in the $z$-plane is rectifiable.

DEFINITION. If $\Gamma$ is a family of locally rectifiable curves on a Riemann surface $\mathcal{R}$ and $P$ is the class of conformally invariant metrics $\rho(z)|dz|$ on $\mathcal{R}$ for which $\rho(z)$ is locally of integrable square in the $z$-plane for each local uniformizing parameter $z$ while

$$A_\rho(\mathcal{R}) = \iint_{\mathcal{R}} \rho^2\, dA$$

and

$$L_\rho(\Gamma) = \underset{\gamma\in\Gamma}{\text{g.l.b.}} \int_\gamma \rho|dz|$$

are not simultaneously 0 or $\infty$ we call

$$m(\Gamma) = \underset{\rho\in P}{\text{g.l.b.}} \frac{A_\rho(\mathcal{R})}{(L_\rho(\Gamma))^2} \tag{1}$$

the module of $\Gamma$. This quantity admits the values 0 and $\infty$.

This definition is designed to provide a conformal invariant in an evident sense.

The definition of module can be normalized in several manners. Most important is the $L$-normalization.

If $P_L$ is the subclass of $P$ such that for $\rho \in P_L$ and $\gamma \in \Gamma$

$$\int_\gamma \rho|dz| \geqslant 1 \tag{2}$$

then

$$m(\Gamma) = \underset{\rho \in P_L}{\text{g.l.b.}}\, A_\rho(\mathcal{R}), \tag{3}$$

where we understand that if $P_L$ is void $m(\Gamma) = \infty$. The metrics in $P_L$ are called admissible metrics.

Also if $P_A$ is the subclass of $P$ such that for $\rho \in P_A$

$$A_\rho(\mathcal{R}) \leqslant 1$$

then

$$m(\Gamma) = \underset{\rho \in P_A}{\text{g.l.b.}} \left|L_\rho(T)\right|^{-2}.$$

This definition is closest to Beurling's original formulation.

Ahlfors and Beurling usually featured the reciprocal of the module which they called extremal length in analogy to Beurling's earlier terminology. This term is not intuitively natural as compared with Beurling's term extremal distance [12] where distance was an actual Euclidean distance and extremal referred to competing conformal mappings. In applications it is usually the module which is useful. In any case for multiple curve families there is no feasible analogue for the reciprocal.

If a metric in $P$ exists for which the greatest lower bound is attained it is called an extremal metric. Since it is immediately seen to be uniquely determined up to sets of two-dimensional measure zero we may speak of the extremal metric. In the most general context there is no existence theorem for extremal metrics. The best result in this direction is due to Suita [221,222].

There have been various modifications of the above definition, some using weaker conditions, some stronger. With weaker conditions the problem is that the solutions may not have desired properties. For example, Fuglede [38] pointed out that Hersch's definition [73] leads to a value of the module in general larger than the standard value and it is not countably sub-additive. With stronger conditions it may be more difficult to verify their validity. The definition given here is intuitively natural and effective for most applications.

Modules can be defined in a formal manner in analogy to the above definition, particularly the $L$-normalization for higher dimensional spaces. A good account is found in [38]. Some of the same techniques can be applied but the most profound and elegant results do not obtain. In particular there is nothing analogous to quadratic differentials.

**3.** Some properties of modules follow directly from the definition

(1) If $\Gamma_1 \subset \Gamma_2$, $m(\Gamma_1) \geqslant m(\Gamma_2)$.

(2) If every element of $\Gamma_1$ contains an element of $\Gamma_2$, $m(\Gamma_1) \leqslant m(\Gamma_2)$.

(3) If $\Gamma_1$, $\Gamma_2$ lie in disjoint open sets and every element of $\Gamma$ contains an element of $\Gamma_1$ and an element of $\Gamma_2$, $m(\Gamma)^{-1} \geqslant m(\Gamma_1)^{-1} + m(\Gamma_2)^{-1}$.

(4) If $\Gamma_1$, $\Gamma_2$ lie in disjoint open sets and every element of $\Gamma_1$ and every element of $\Gamma_2$ contains an element of $\Gamma$, $m(\Gamma) \geqslant m(\Gamma_1) + m(\Gamma_2)$.

There are two special cases of modules which are of great importance. First let $D$ be a non-degenerate doubly-connected domain on the complex sphere. (Non-degenerate means that neither complementary continuum reduces to a point.) $D$ can be mapped conformally on a circular ring $\Delta$: $r_1 < |z| < r_2$, $0 < r_1 < r_2 < \infty$. To the concentric circles $\gamma_r$: $|z| = r$, $r_1 < r < r_2$, correspond Jordan curves in $D$ called the level sets of $D$. They determine an unsensed homotopy class $\mathcal{H}$ in $D$. $m(\mathcal{H})$ is equal to the module of the corresponding class $\widehat{\mathcal{H}}$ in $\Delta$. Let $\rho(z)|dz|$ be an admissible metric in the $L$-normalization for $m(\widehat{\mathcal{H}})$. Then

$$\int_{\gamma_r} \rho r \, d\theta \geqslant 1, \quad r_1 < r < r_2,$$

and

$$\int_{\gamma_r} \rho \, d\theta \geqslant \frac{1}{r}.$$

Integrating

$$\iint_\Delta \rho \, dr \, d\theta \geqslant \log \frac{r_2}{r_1}.$$

Now $(1/(2\pi|z|))|dz|$ is an admissible metric for $m(\widehat{\mathcal{H}})$ with

$$\iint_\Delta \frac{1}{4\pi^2 r^2} r \, dr \, d\theta = \frac{1}{2\pi} \log \frac{r_2}{r_1}$$

while

$$\iint_\Delta \left(\rho - \frac{1}{2\pi r}\right)^2 r \, dr \, d\theta \geqslant 0$$

and

$$\iint_\Delta \rho^2 r \, dr \, d\theta - \frac{1}{\pi} \iint_\Delta \rho \, dr \, d\theta + \frac{1}{2\pi} \log \frac{r_2}{r_1} \geqslant 0,$$

i.e.,

$$\iint_\Delta \rho^2 r \, dr \, d\theta \geqslant \frac{1}{2\pi} \log \frac{r_2}{r_1}.$$

Thus

$$m(\widehat{\mathcal{H}}) = \frac{1}{2\pi} \log \frac{r_2}{r_1}$$

and

$$\frac{1}{2\pi |z|} |dz|$$

is the extremal metric.

One can also consider the module problem for the class of locally rectifiable open curves in $D$ tending respectively to the two complementary continua of $D$ in the respective senses. Its module is readily seen to be

$$\left(\frac{1}{2\pi} \log \frac{r_2}{r_1}\right)^{-1}.$$

When we speak of the module of a doubly-connected domain without qualification we mean the former.

Second let $D$ be a quadrangle, i.e., a simply-connected domain of hyperbolic type with four assigned (distinct) border elements called vertices. These divide the cycle of border elements of $D$ into subsets called sides. $D$ can be mapped conformally on a rectangle $R$: $0 < x < a, 0 < y < b$, $z = x + iy$, so that the vertices of $D$ correspond to the corners of $R$. Let $\Gamma_1$ be the class of locally rectifiable open curves in $D$ which tend respectively to the sides of $D$ corresponding to the vertical sides of $R$ in the respective senses. It is readily seen that $m(\Gamma_1) = b/a$. If $\Gamma_2$ is the corresponding class for the horizontal sides $m(\Gamma_2) = a/b$ in each case with the evident extremal metric.

We can derive at once the two results known as Grötzsch's Lemmas on which he based his method of strips.

(I) Let $D_j$ be non-overlapping doubly-connected domains of modules $M_j$ (finite or countable in number) lying in the doubly-connected domain $D$ of module $M$ and having the same topological situation. Then

$$\sum_j M_j \leqslant M$$

with equality only if the $D_j$ are obtained by splitting $D$ along level sets so that the sum of the areas of the $D_j$ in the extremal metric for $D$ is equal to the area of $D$.

(II) Let $D$ be a doubly-connected domain of module $M$ for the class of curves tending respectively to the complementary continua of $D$ and $Q_j$ (finite or countably infinite in number) quadrangles each with a pair of opposite sides on the respective border continua of $D$ with modules $M_j$ for the class of curves tending respectively to these sides. Then

$$\sum_j M_j \leqslant M$$

with equality only if the $Q_j$ are obtained by splitting $D$ along orthogonal trajectories of the level sets so that the sum of the areas of the $Q_j$ in the extremal metric for $D$ is equal to the area of $D$.

Kühnau [169] has given a formula for the module of a parametric family of parametrically defined curves. Juve [164] had earlier given essentially the same formula as a lower bound for the module.

Teichmüller [226] introduced the concept of reduced module. Let $D$ be a simply-connected domain of hyperbolic type on the complex $z$-sphere and $z_0$ a (finite) point of $D$. If $r > 0$ is sufficiently small the domain

$$D \cap \{z\colon |z - z_0| > r\}$$

is doubly-connected. Let its module be $M(r)$. For $0 < r' < r$

$$M(r) + \frac{1}{2\pi}\log\frac{r}{r'} \leqslant M(r')$$

thus

$$M(r) + \frac{1}{2\pi}\log r \leqslant M(r') + \frac{1}{2\pi}\log r'$$

and $M(r) + (1/(2\pi))\log r$ has a limit as $R$ tends to zero which is seen at once to be finite. It is called the reduced module of $D$ with respect to $z_0$.

Reduced module is not a conformal invariant but rather a covariant. For this reason if we extend the definition to a Riemann surface we must specify a fixed local uniformizing parameter at the distinguished point. Several authors [6,200] have modified the definition to obtain a conformal invariant. Unfortunately the entities considered are of little interest for applications. Even more unfortunately these authors have used the same term for them.

**4.** In Ahlfors' lectures considerable emphasis was placed on a relation between modules and harmonic functions. The same considerations appear in his book [6] and in Beurling's Mittag-Leffler's lectures in 1977–1978 [14].

Specifically let $D$ be a domain on the complex sphere bounded by a finite number of Jordan curves, $E_0$ and $E_1$ disjoint sets on the boundary each composed of a finite number of arcs and/or complete boundary curves. Let $u$ be the solution of the mixed Dirichlet–Neumann problem taking the value 0 on the boundary interior of $E_0$ and the value 1 on the boundary interior of $E_1$. Let $\Gamma$ be the family of curves joining $E_0$ and $E_1$ in $D$. Then the module of $\Gamma$ is given by the Dirichlet integral

$$\iint_D \left(u_x^2 + u_y^2\right) dA.$$

## 2. Quadratic differentials

**1.** Quadratic differentials have long been known as formal generalizations of linear differentials (see for example the book of Hensel and Landsberg [72]). In his variational method Schiffer [215] obtained an equation which can be interpreted as stating the equality of two quadratic differentials. He did not, however, pursue this development. It was the remarkable achievement of Teichmüller to recognize the essential role which quadratic differentials play in many extremal problems of geometric function theory.

A meromorphic quadratic differential is an entity defined on a Riemann surface $\mathcal{R}$ which assigns to every local uniformizing parameter $z$ of $\mathcal{R}$ a meromorphic function $Q(z)$ such that if $z^*$ is a second local uniformizing parameter and $Q^*(z^*)$ the corresponding meromorphic function then for corresponding points of overlapping parameter neighborhoods

$$Q^*(z^*) = Q(z)\left(\frac{dz}{dz^*}\right)^2.$$

A quadratic differential is denoted generically by $Q(z)\,dz^2$. It is thus possible to speak of a quadratic differential having a zero or pole of a given order at a point of $\mathcal{R}$. These points are called critical points of $Q(z)\,dz^2$. Zeros and simple poles are called finite critical points, their totality denoted by $C$. The totality of higher order critical points is denoted by $H$.

An essential role is played by the trajectories of a quadratic differential. For a quadratic differential $Q(z)\,dz^2$ defined on a Riemann surface a maximal open arc or Jordan curve which has a smooth parameterization such that for every local uniformizing parameter

$$Q\big(z(t)\big)\left(\frac{dz(t)}{dt}\right)^2 > 0$$

is called a trajectory of $Q(z)\,dz^2$. This is expressed by writing $Q(z)\,dz^2 > 0$. The trajectories of $-Q(z)\,dz^2$ are called the orthogonal trajectories of $Q(z)\,dz^2$. Teichmüller described the local structure of the trajectories at ordinary and critical points with little indication of proof and identified certain domains which can occur in the global trajectory structure. He did not raise the question of whether these were the only type of structure domains which could occur even for relatively simple Riemann surfaces.

Schaeffer and Spencer [214] also had a variational method which led to a differential equation related to quadratic differentials. They were aware of Teichmüller's work and tried to fill in some of the gaps. They stated results only for hyperelliptic quadratic differentials, i.e., meromorphic quadratic differentials on the sphere. They gave a detailed but quite complicated treatment of the local structure of trajectories (where the hyperelliptic restriction is evidently inessential) and studied the global structure in some special cases. They were concerned as to whether there could be recurrent trajectories. An open arc $\gamma(t)$, $0 < t < 1$, is called recurrent if there are parameter values $\{t_n\}_1^\infty$, with $\lim_{n\to\infty} t_n = 0$ or 1 and $\lim_{n\to\infty} \gamma(t_n) = \gamma(t_0)$, $0 < t_0 < 1$. Schaeffer and Spencer proved that for a hyperelliptic quadratic differential with one or two poles there could be no recurrent trajectory and obtained the same result for a special case when there were three poles. They believed that this would be true in general but the author early demonstrated that there could be recurrent trajectories in relatively simple cases.

**2.** The author [87,101] gave a simpler treatment for the local structure of the trajectories. The essential description is as follows.

(I) Every point $P \in \mathcal{R} - \{C \cup H\}$ has a neighborhood $N$ homeomorphic by a mapping $\phi$ to the square $-1 < x, y < 1$ so that $\phi(P) = 0$ and the image of the intersection of a trajectory with $N$ is an open segment $-1 < x < 1$, $y$ constant.

(II) If $P$ is a point of $C$ of order $\mu$ ($\mu > 0$ for a zero, $\mu = -1$ for a simple pole) there exist a neighborhood $N$ of $P$ and a homeomorphic mapping of $N$ onto the disc $|w| < 1$ under which a maximal open arc on a trajectory in $N$ is mapped onto an open arc on which $\Im w^{(n+2)/2}$ is constant. There exist $\mu + 2$ trajectories with limiting end points at $P$ spaced at equal angles equal to $2\pi/(\mu + 2)$.

(III) If $P$ is in $H$ and is a pole of order $\nu > 2$ there exists a neighborhood $N$ of $P$ such that

(i) every trajectory through a point of $N$ in each sense either leaves $N$ or tends to $P$ in one of $\nu - 2$ directions equally spaced at angles of $2\pi/(\nu - 2)$,

(ii) $P$ has a subneighborhood $N^*$ of $N$ such that every trajectory which meets $N^*$ tends in at least one sense to $P$ remaining in $N^*$,

(iii) if a trajectory lies entirely in $N$ and thus tends in each sense to $P$ it does so for its respective senses in two adjacent limiting directions; the domain $D$ in $N$ enclosed by the Jordan curve obtained by adjoining $P$ to this trajectory is such that a trajectory through any of its points tends to $P$ respectively in its two senses in these adjacent limiting directions; $D$ is mapped by a suitable branch of $\zeta = \int (Q(z))^{1/2}\, dz$ onto a half-plane $\Im\zeta > c$ ($c$ real).

(IV) Let $P$ be in $H$ and be a pole of order 2. Let $z$ be a local uniformizing parameter in terms of which $P$ is represented by $z = 0$. Let $(Q(z))^{1/2}$ have (for one choice of the root) the expansion about $z = 0$

$$\left(Q(z)\right)^{1/2} = (a + ib)z^{-1}\left(1 + b_1 z + b_2 z^2 + \cdots\right)$$

for suitable real constants $a, b$ and complex $b_1, b_2, \ldots$.

(IV A) $a \neq 0$, $b \neq 0$. For $\alpha > 0$ sufficiently small every trajectory image which meets $|z| < \alpha$ tends in one sense to $z = 0$ and in the other sense leaves $|z| < \alpha$. Both modulus and argument of $z$ vary monotonically on a trajectory image in $|z| < \alpha$. Every trajectory image spirals about $z = 0$ behaving asymptotically like a logarithmic spiral.

(IV B) $a \neq 0$, $b = 0$. For $\alpha > 0$ sufficiently small every trajectory image which meets $|z| < \alpha$ tends in the one sense to $z = 0$ and in the other sense leaves $|z| < \alpha$. The modulus of $z$ varies monotonically on each trajectory image in $|z| < \alpha$. Distinct trajectory images have distinct limiting directions at $z = 0$.

(IV C) $a = 0, b \neq 0$. Given $\varepsilon > 0$ there exists $\alpha(\varepsilon) > 0$ such that for $0 < \alpha < \alpha(\varepsilon)$ a trajectory image which meets $|z| = \alpha$ is a Jordan curve which lies in the circular ring

$$\alpha\,(1 + \varepsilon)^{-1} < |z| < \alpha\,(1 + \varepsilon).$$

**3.** The author has analyzed completely the global trajectory structure for a positive quadratic differential on a finite Riemann surface.

A finite open Riemann surface $S$ is a Riemann surface on which there exists a finite number of mutually disjoint Jordan curves $\gamma_j$, $j = 1, \ldots, n$, each of which separates $S$ with one component $D_j$ of $S - \gamma_j$ not containing any $\gamma_k$, $k \neq j$, and conformally equivalent by a mapping $\chi_j$ to a (non-degenerate) circular ring $1 < |w| < R_j$ such that $S - \bigcup_{j=1}^{n} D_j$ is compact. A Riemann surface is said to be finite if it is either a finite open Riemann surface or a closed Riemann surface.

A finite open Riemann surface $S$ can be homeomorphically imbedded in a compact topological space $\widetilde{S}$ so that

$$\widetilde{S} - S = \bigcup_{j=1}^{n} C_j,$$

where each $C_j$ is the homeomorphic image of a circumference contained in the closure of $D_j$ relative to $\widetilde{S}$ and the mapping $\chi_j$ can be extended to a homeomorphism of $D_j \cup C_j$ onto $1 < |w| \leqslant R_j$.

The set

$$\bigcup_{j=1}^{n} C_j = C$$

is called the border of $S$. A homeomorphic mapping $\psi$ of a neighborhood $N$ of $P \in C$ with $\psi$ regular in $N \cap S$, $\psi(N \cap S)$ an open semicircular disc in the upper half-plane and $\psi(N \cap C)$ the diameter of this disc on the real axis is called a border uniformizer for $\widetilde{S}$ (or $S$) at $P$. A complex-valued function $f$ defined in $\widetilde{S}$ is said to be regular at $P \in C$ if, for a boundary uniformizer $\psi$, $f\psi^{-1}$ is regular at $\psi(P)$.

The following are the basic entities in the description of the global structure of the trajectories of a positive quadratic differential on a finite Riemann surface. By a $Q$-set $K$ we mean a set such that every trajectory of $Q(z)\,dz^2$ which meets $K$ lies in $K$.

An end domain $\mathcal{E}$ is a maximal connected open $Q$-set with the following properties.

(i) $\mathcal{E}$ contains no critical point of $Q(z)\,dz^2$,

(ii) $\mathcal{E}$ is swept out by trajectories of $Q(z)\,dz^2$ each of which has a limiting end point in its two senses at a given point $A$ of $H$,

(iii) $\mathcal{E}$ is mapped by $\zeta = \int (Q(z))^{1/2}\,dz$ conformally onto an upper or lower half-plane in the $\zeta$-plane (depending on the choice of determination).

(It is seen at once that $A$ must be a pole of $Q(z)\,dz^2$ of order at least three.)

A strip domain $\mathcal{S}$ is maximal connected open $Q$-set with the following properties.

(i) $\mathcal{S}$ contains no critical point of $Q(z)\,dz^2$,

(ii) $\mathcal{S}$ is swept out by trajectories of $Q(z)\,dz^2$ each of which has in one sense a limiting end point at a given point $A$ of $H$ and in the other sense a limiting end point at a given point $B$ of $H$ (possibly coincident with $A$),

(iii) $\mathcal{S}$ is mapped by $\zeta = \int (Q(z))^{1/2}\,dz$ conformally onto a strip $a < \Im\zeta < b$, $a, b$ finite real numbers, $a < b$.

A circle domain $\mathcal{C}$ is a maximal connected open $Q$-set with the following properties.

(i) $\mathcal{C}$ contains a single double pole $A$ of $Q(z)\,dz^2$,

(ii) $\mathcal{C} - A$ is swept out by trajectories of $Q(z)\,dz^2$ each of which is a Jordan curve separating $A$ from the boundary of $\mathcal{C}$,

(iii) for a suitably chosen purely imaginary constant $c$

$$w = \exp\left\{c \int \left(Q(z)\right)^{1/2} dz\right\}$$

extended to have the value 0 at $A$ maps $\mathcal{C}$ conformally onto a circular disc.

A ring domain $\mathcal{D}$ is a maximal connected open $Q$-set with the following properties.

(i) $\mathcal{D}$ contains no critical point of $Q(z)\,dz^2$,

(ii) $\mathcal{D}$ is swept out by trajectories of $Q(z)\,dz^2$ each of which is a Jordan curve separating the boundary components of $\mathcal{D}$,

(iii) for a suitably chosen purely imaginary constant $c$

$$w = \exp\left\{c \int \left(Q(z)\right)^{1/2} dz\right\}$$

maps $\mathcal{D}$ onto a circular ring

$$r_1 < |w| < r_2, \quad 0 < r_1 < r_2 < \infty.$$

A density domain $\mathcal{F}$ is a maximal connected open $Q$-set with the following properties.

(i) The closure of $\mathcal{F}$ contains no point of $H$,

(ii) $\mathcal{F} - C$ us swept out by trajectories of $Q(z)\,dz^2$ each of which is everywhere dense in $\mathcal{F}$.

The proof of the following complete description of the global structure of the trajectories of a positive quadratic differential on a finite Riemann surface is given in [107].

A quadratic differential $Q(z)\,dz^2$ on a finite open Riemann surface is said to be positive if for any boundary uniformizer $Q(z)$ is positive on the relevant segment of the real axis apart from possible zeros of $Q(z)$. On a closed Riemann surface any meromorphic quadratic differential is understood to be positive.

BASIC STRUCTURE THEOREM. *Let $\mathcal{R}$ be a finite Riemann surface and $Q(z)\,dz^2$ a positive quadratic differential on $\mathcal{R}$ where we exclude the following possibilities and all configurations obtained from them by conformal equivalence.*

*$\mathcal{R}$ the $z$-sphere, $Q(z)\,dz^2 = dz^2$,*

*$\mathcal{R}$ the $z$-sphere, $Q(z)\,dz^2 = e^{i\theta}\,dz^2/z^2$, $\theta$ real,*

*$\mathcal{R}$ a torus, $Q(z)\,dz^2$ regular on $\mathcal{R}$.*

*Let $\Lambda$ denote the union of the closures of all trajectories of $Q(z)\,dz^2$ which have one limiting end point at a point of $C$ and a second limiting end point at a point of $C \cup H$. Then*

(i) *$\mathcal{R} - \Lambda$ consists of a finite number of end, strip, circle, ring and density domains,*

(ii) *each such domain is bounded by a finite number of trajectories together with their limiting end points; every boundary component of such a domain contains a point of $C$ except that a boundary component of a circle or ring domain may coincide* *with a border component of $\mathcal{R}$; for a strip domain the two boundary elements arising from points of $H$ divide the boundary into two parts on each of which is a point of $C$,*

(iii) *every pole of $Q(z)\,dz^2$ of order $m$ greater than two has a neighborhood covered by the inner closure of the union of $m-2$ end domains and a finite number (possibly zero) of strip domains,*

(iv) *every pole of $Q(z)\,dz^2$ of order 2 either has a neighborhood covered by the inner closure of a finite number of strip domains or has a neighborhood contained in a circle domain.*

**4.** The method used by Schaeffer and Spencer required the absence of density domains. The proof of the General Coefficient Theorem eliminated this difficulty. However it is of interest to know what are the general circumstances in which one can affirm the absence of recurrent trajectories. The exact conditions are that the finite Riemann surface must be schlichtartig and that the total number of poles (actual points not order) plus the number of border components is at most three. This is called the Three Pole Theorem and the author has give two function-theoretic proofs [101,158]. Simple examples show that in other cases recurrent trajectories can occur.

Actually though, this is a purely topological result [124]. For this we introduce the concept of a trajectoire curve family.

A trajectoire curve family $F$ defined on a Riemann surface $\mathcal{R}$ on which is assigned a set $S$ of isolated points consists of disjoint open arcs and Jordan curves such that every point of $\mathcal{R} - S$ lies on one. For every point $P \in \mathcal{R}$ there is to be a neighborhood $N$ and a homeomorphism $\phi$ such that in $\phi(N)$ the images of the elements of $F|N$ coincide with trajectories of a meromorphic quadratic differential. Thus we can define sets $C$ and $H$ and orders of their points as in the function theoretic case.

TOPOLOGICAL THREE POLE THEOREM. *Let $F$ be a trajectoire curve family on a domain $D$ obtained from the sphere by deleting at most three points. Let $H$ be empty and $C$ contain no point of order $-1$. Then there is no recurrent element in $F$.*

It should be noted that no assumption is made concerning the behavior of the elements of $F$ in the neighborhood of a deleted point.

This result is stronger than the corresponding function theoretic result since it is easy to give examples of a trajectoire curve family which is not globally topologically equivalent to the trajectories of a quadratic differential.

## 3. Modules of multiple curve families

**1.** The definition (1) of modules admits various generalizations. In particular the $L$-normalization is susceptible of numerous such. One can even replace (2) by the condition

$$\int_\gamma \rho|dz| \geqslant c(\gamma), \quad \gamma \in \Gamma,$$

where $c(\gamma)$ is a non-negative function. With this degree of generality only very superficial results are possible. However there is one case where extremely profound and elegant results can be obtained.

By a free family of homotopy classes $\mathcal{H}_j$, $j = 1, \ldots, L$, on a finite Riemann surface $\mathcal{R}$ we mean a family of distinct free unsensed homotopy classes of closed curves which can be represented by disjoint Jordan curves $C_j$, $j = 1, \ldots, L$. A family of disjoint doubly-connected domains $D_j$, $j = 1, \ldots, L$, associated with the free family $\mathcal{H}_j$ will be said to be an admissible family if the level curves of $D_j$ lie in $\mathcal{H}_j$, $j = 1, \ldots, L$. We allow certain domains to be missing and speak of them as degenerate.

In this context we consider two types of extremal problems.

PROBLEM $P(a_1, \ldots, a_L)$. Let $a_j$, $j = 1, \ldots, L$, be non-negative real numbers not all zero. Let $P(a_1, \ldots, a_L)$ denote the class of conformally invariant metrics on $\mathcal{R}$ for which $\rho(z)$ is locally of integrable square in the $z$-plane for each local uniformizing parameter $z$ and such that for $\gamma_j$ rectifiable in $\mathcal{H}_j$

$$\iint_{\gamma_j} \rho|dz| \geqslant a_j, \quad j = 1, \ldots, L.$$

Find the greatest lower bound $M(a_1, \ldots, a_L)$ of

$$\iint_R \rho^2 \, dA_z$$

for $\rho \in P(a_1, \ldots, a_L)$.

This is called the module of the multiple curve family.

PROBLEM $\mathcal{P}(a_1, \ldots, a_L)$. Let $a_j$, $j = 1, \ldots, L$, be non-negative real numbers not all zero. For an admissible family of domains $D_j$ of module $M_j$, $j = 1, \ldots, L$, find the least upper bound of $\sum_{j=1}^{n} a_j^2 M_j$.

**2.** We will state the fundamental theorem in the basic case of a finite Riemann surface and a free family of homotopy classes later indicating various extensions. Also we tacitly exclude the cases where $\mathcal{R}$ is a disc, a doubly-connected domain or a torus. In the first case the theorem is vacuous, in the second trivial and in the third it is generally true apart from certain uniqueness statements.

FUNDAMENTAL THEOREM. *Let $\mathcal{R}$ be a finite Riemann surface, $\mathcal{H}_j$, $j = 1, \ldots, L$, a free family of homotopy classes on $\mathcal{R}$. Then the solution of Problem $P(a_1, \ldots, a_L)$ is given by an (essentially) unique extremal metric $|Q(z)|^{1/2}|dz|$ where $Q(z)\,dz^2$ is a regular positive quadratic differential on $\mathcal{R}$ all of whose structure domains are ring domains. Enumerating these appropriately as $D_j(a_1, \ldots, a_L)$, $j = 1, \ldots, L$, they form an admissible family for $\mathcal{H}_j$. If $D_j(a_1, \ldots, a_L)$ is non-degenerate its level curves all have length $a_j$ in the metric $|Q(z)|^{1/2}|dz|$. If it is degenerate there is a geodesic in this metric belonging to*

*$\mathcal{H}_j$ composed of trajectories joining zeros of $Q(z)\,dz^2$ plus their end points and having length at least $a_j$ in this metric. If $D_j(a_1, \ldots, a_L)$ has module $M_j(a_1, \ldots, a_L)$*

$$M(a_1, \ldots, a_L) = \sum_{j=1}^{L} a_j^2 M_j(a_1, \ldots, a_L).$$

*The solution of Problem $\mathcal{P}(a_1, \ldots, a_L)$ is given by the domains $D_j(a_1, \ldots, a_L)$, $j = 1, \ldots, L$, the least upper bound being $M(a_1, \ldots, a_L)$ and is a maximum attained uniquely for these domains.*

We give also a more detailed statement of the uniqueness properties.

UNIQUENESS THEOREM. *Let $\mathcal{R}$ be a finite Riemann surface $\mathcal{H}_j$, $j = 1, \ldots, L$, a free family of homotopy classes on $\mathcal{R}$. Let $Q(z)\,dz^2$ be a regular quadratic differential on $\mathcal{R}$ all of whose structure domains are ring domains and such that suitably enumerated and allowing for degenerate domains they form an admissible family $D_j^*$, $j = 1, \ldots, L$, for $\mathcal{H}_j$. Let the module of $D_j^*$ be $M_j^*$. Suppose that for a non-degenerate domain $D_j^*$ all trajectories in $D_j^*$ have length $a_j$ while for a degenerate domain $D_j^*$ there is geodesic in the $Q$-metric belonging to $\mathcal{H}_j$ composed of trajectories of $Q(z)\,dz^2$ joining zeros of $Q(z)\,dz^2$ plus their end points of length $a_j^*$. Then for non-negative numbers not all zero with $a_j \leqslant a_j^*$ in the case of degenerate domains $|Q(z)|^{1/2}|dz|$ provides the (essentially) unique solution of Problem $P(a_1, \ldots, a_L)$ while the domains $D_j^*$, $j = 1, \ldots, L$, provide the unique solution of Problem $\mathcal{P}(a_1, \ldots, a_L)$.*

These results were first proved by the author [99] by a variational method. More recently the author [142] has given a simpler and more illuminating proof using only the techniques of the method of the extremal metric.

The Fundamental Theorem as stated above admits immediate extensions as indicated in [99]. We can consider distinguished points $P_j$, $j = 1, \ldots, J$, on $\mathcal{R}$ and distinguished points $Q_k$, $k = 1, \ldots, K$, on $\beta(\mathcal{R})$ the border of $\mathcal{R}$ and then homotopy classes of closed paths on $\mathcal{R} - \bigcup_{j=1}^{J} P_j$ (where now we must require that no class consists of curves homotopic to one of the $P_j$) and homotopy classes of paths on

$$\mathcal{R} \cup \beta(\mathcal{R}) - \left\{ \bigcup_{j=1}^{J} P_j \cup \bigcup_{k=1}^{K} Q_k \right\}$$

which joins pairs (not necessarily distinct) of border components. With the former we associate as before doubly-connected domains and with the latter quadrangles which have one pair of opposite sides in $\beta(\mathcal{R}) - \bigcup_{k=1}^{K} Q_k$ on the appropriate components of $\beta(\mathcal{R})$. For these we use the module of the quadrangle for paths joining these sides. The concept of an admissible family of domains extends at once and the Fundamental Theorem restated in this context can be derived immediately from the previous form. Now $Q(z)\,dz^2$ may have simple poles at the points $P_j$, $j = 1, \ldots, J$, $Q_k$, $k = 1, \ldots, K$.

**3.** The Fundamental Theorem extends almost immediately to the case of an open Riemann surface of finite genus [160]. The definitions of free family of homotopy classes and admissible family of doubly-connected domains are as before. The result is as follows.

Let $\mathcal{R}$ be a Riemann surface and $\mathcal{H}_j$, $j = 1, \dots, L$, a free family of homotopy classes on $\mathcal{R}$. Then for Problem $P(a_1, \dots, a_L)$ there exists an extremal metric of the form $|Q(z)|^{1/2}|dz|$ where $Q(z)\,dz^2$ is a regular quadratic differential on $\mathcal{R}$. Provided that $\mathcal{R}$ is not a closed surface of genus one or a doubly-connected domain trajectories of $Q(z)\,dz^2$ with limiting end points at the zeros of $Q(z)\,dz^2$ together possibly with certain trajectories running from the ideal boundary of $\mathcal{R}$ back to the same divide $\mathcal{R}$ into an admissible family of doubly-connected domains $D_j^*$ of module $M_j^*$, $j = 1, \dots, L$, associated with the free family of homotopy classes and

$$M(a_1, \dots, a_L) = \sum_{j=1}^{L} a_j^2 M_j^*.$$

For an admissible family of domains $D_j$ with modules $M_j$, $j = 1, \dots, L$, associated with the free family of homotopy classes

$$\sum_{j=1}^{L} a_j^2 M_j \leqslant M(a_1, \dots, a_L).$$

Subject to the previous exclusions equality can occur only for the domains $D_j^*$.

The Fundamental Theorem can also be extended to infinite free families of homotopy classes on an open Riemann surface but only under certain subsidiary conditions [220].

**4.** Problem $\mathcal{P}(a_1, \dots, a_L)$ can be considered independently of its relationship with Problem $P(a_1, \dots, a_L)$. It then becomes a problem of extremal decompositions. Its solution is of the form that there is a quadratic differential whose structure domains provide the extremal decomposition.

In this context one can consider further types of domains competing in the problem, in the first instance simply-connected domains with a distinguished interior point (and a local uniformizing parameter assigned at that point in the case of Riemann surfaces). Such a domain contributes a term of the form $a_j^2 \mu_j$, where $\mu_j$ is the reduced module of the entity considered. The results in this case are easily derived from those for doubly-connected domains. A similar concept can be introduced with distinguished points on the border.

Emel'yanov [32] introduced the concept of reduced module for a biangle (simply-connected domain of hyperbolic type with two distinguished boundary elements) and gave a statement for extremal decompositions involving these entities in the case of domains on the sphere analogous to the extremal decomposition part of the Fundamental Theorem. One can proceed similarly for a triangle (simply-connected domain of hyperbolic type with three distinguished boundary elements one of which is designated for the limiting process) [218].

Other papers in this context are [181,182,230].

**5.** Emel'yanov [31] and Solynin [216] gave variational formulas for the modules of multiple curve families. If, in the notation of the Fundamental Theorem,

$$M(a_1,\dots,a_L)=\sum_{j=1}^{L}a_j^2M_j(a_1,\dots,a_L)$$

it is easily seen that

$$\frac{\partial}{\partial a_j}M(a_1,\dots,a_L)=2a_jM_j(a_1,\dots,a_L),\quad j=1,\dots,L.$$

Also if on a plane domain $P$ is a distinguished point and $M$ is the module for a corresponding Problem $P(a_1,\dots,a_L)$ with quadratic differential $Q(z)\,dz^2$ having a simply pole at $P$ then

$$\frac{\partial}{\partial P}M=\lim_{z\to P}(z-P)Q(z).$$

**6.** In the case of simple explicit situations the problem of extremal decompositions can become a region of values problem. Teichmüller [226] early considered such a problem for reduced modules. Other examples are found in [37,84].

**7.** Renelt [209] introduced another kind of extremal decomposition in the case of plane domains which he treated by a different method. Stated analogously to Problem $\mathcal{P}(a_1,\dots,a_L)$ we have the following.

PROBLEM $R(b_1,\dots,b_L)$. Let $b_j$, $j=1,\dots,L$, be positive numbers. For an admissible family of domains $D_j$ of module $M_j$, $j=1,\dots,L$, find the greatest lower bound of $\sum_{j=1}^{L}b_j^2M_j^{-1}$.

The author [142] showed that for domains on a finite Riemann surface the solution of Problem $R(b_1,\dots,b_L)$ can easily be derived from that of Problem $\mathcal{P}(a_1,\dots,a_L)$. Recently the author [150] has shown that both problems are special cases of the following problem.

PROBLEM $X(a_1,\dots,a_N,b_{N+1},\dots,b_L)$. Let $a_j$, $j=1,\dots,N$, and $b_j$, $j=N+1,\dots,L$, be non-negative numbers not all zero with any $b_j$ which actually occurs being positive, $0\leqslant N\leqslant L$. For an admissible family of domains $D_j$ of module $M_j$, $j=1,\dots,L$, find the least upper bound of

$$\sum_{j=1}^{N}a_j^2M_j-\sum_{j=N+1}^{L}b_jM_j^{-1}.$$

The corresponding result is the following theorem.

DECOMPOSITION THEOREM. *Let $\mathcal{R}$ be a finite Riemann surface, $\mathcal{H}_j$, $j = 1, \ldots, L$, a free family of unsensed homotopy classes on $\mathcal{R}$. Let $a_j$, $j = 1, \ldots, N$, $b_j$, $j = N + 1, \ldots, L$, be non-negative numbers not all zero such that any $b_j$ actually present (i.e., $N < L$) is positive. Then apart from the usual exceptional cases the solution of Problem $X(a_1, \ldots, a_N, b_{N+1}, \ldots, b_L)$ is given by a positive quadratic differential $Q(z)\,dz^2$ on $\mathcal{R}$ all of whose structure domains are ring domains $D_j(a_1, \ldots, a_N, b_{N+1}, \ldots, b_L)$, $j = 1, \ldots, L$, admissible for $\mathcal{H}_j$, $j = 1, \ldots, L$, with modules $M_j(a_1, \ldots, a_N, b_{N+1}, \ldots, b_L)$. The extremal value is*

$$\sum_{j=1}^{N} a_j^2 M_j(a_1, \ldots, a_N, b_{N+1}, \ldots, b_L)$$

$$- \sum_{j=N+1}^{L} b_j^2 M_j^{-1}(a_1, \ldots, a_N, b_{N+1}, \ldots, b_L).$$

*$Q(z)\,dz^2$ is uniquely determined up to positive multipliers.*

The Decomposition Theorem can be extended in manners analogous to the Fundamental Theorem.

## 4. The General Coefficient Theorem

**1.** As mentioned in Section 1 Teichmüller was the first to recognize the essential role played by quadratic differentials in numerous extremal problems in geometric function theory. In particular he enunciated the following principle for extremal problems for univalent regular and meromorphic functions. If in such a problem a point is to be fixed without further requirement the quadratic differential will have a simple pole there. If in addition the functions treated in the problem are required at such a point to have in terms of a suitable assigned local uniformizing parameter fixed values for their first $n$ derivatives the quadratic differential will have a pole of order $n + 1$ there. More generally the highest derivative occurring may not be required to be fixed but some condition on its region of values may be desired. He was led to this principle by abstraction from the numerous results of Grötzsch. Teichmüller never gave anything in the nature of an explicit general result embodying this principle.

He did, however, prove a coefficient theorem which provides an explicit realization of the principle in one special case [227]. It can be given various slightly modified forms, one of which is the following theorem.

TEICHMÜLLER'S COEFFICIENT THEOREM. *Let the meromorphic quadratic differential $Q(z)\,dz^2$ be regular on the $w$-sphere apart from a pole of order $n + 4$ $(n > -1)$ at the point at infinity at which it has the expansion*

$$Q(w)\,dw^2 = (\alpha w^n + \textit{decreasing powers of } w)\,dw^2.$$

*Let $D$ be a domain on the sphere containing the point at infinity whose boundary consists of trajectories and arcs of trajectories of $Q(w)\,dw^2$ together with their end points which is the image of $|z| > 1$ under a mapping $w = f^*(z)$ with $f^* \in \Sigma$ where $f^*$ has the expansion at the point at infinity*

$$f^*(z) = z + b_0 + b_1 z^{-1} + \cdots + b_k z^{-k} + \cdots.$$

*Let $f \in \Sigma$ have expansion at the point at infinity*

$$f(z) = z + c_0 + c_1 z^{-1} + \cdots + c_k z^{-k} + \cdots$$

*with $c_j = b_j$, $j = 0, 1, \ldots, n$. Then*

$$\mathcal{R}(\alpha c_{n+1}) \leqslant \mathcal{R}(\alpha b_{n+1})$$

*equality occurring only if $f(z) = f^*(z)$.*

**2.** The General Coefficient Theorem is a result which applies on general finite Riemann surfaces and embodies a very general representation of Teichmüller's principle. It has passed through a number of extensions and generalizations [88,101,106,115,116]. We will now give several definitions preliminary to stating it in its current form. $\mathcal{R}$ will be a finite Riemann surface, $Q(z)\,dz^2$ a positive meromorphic quadratic differential defined on $\mathcal{R}$.

By an admissible family $\{\Delta\}$ of domains $\Delta_j$, $j = 1, \ldots, k$, on $\mathcal{R}$ with respect to $Q(z)\,dz^2$ we mean the complement on $\mathcal{R}$ of a finite number of trajectories of $Q(z)\,dz^2$ each of which is either closed or has a limiting end point in each sense at a point of $C$, possible end points of these trajectories and a finite number of arcs in $\mathcal{R} - H$ on closures of trajectories.

Let $\Delta_j$ be a domain in an admissible family, $f_j$ a conformal mapping of $\Delta_j$ into $\mathcal{R}$. $f_j$ is said to admit an admissible homotopy into the identity if there exists a function $F(P, t)$, $P \in \Delta_j$, $0 \leqslant t \leqslant 1$, with values in $\mathcal{R}$ continuous in both variables together satisfying the following conditions

(a) $F(P, 0) = f_j(P)$, $P \in \Delta_j$,
(b) $F(P, 1) = P$, $P \in \Delta_j$,
(c) $F(P, t) = P$, $P$ a pole of $Q(z)\,dz^2$ in $\Delta_j$, $0 \leqslant t \leqslant 1$,
(d) $F(P, t) \neq Q$, $Q$ a pole of $Q(z)\,dz^2$ in $\mathcal{R}$, $P \neq Q$, $0 \leqslant t \leqslant 1$.

If $f_j$ admits an admissible homotopy into the identity it is readily seen that if $A \in H$ lies in $\Delta_j$ and $N$ is a neighborhood of a local uniformizing parameter $z$ at $A$ for which $A$ is represented by $z = 0$, for $P$ sufficiently close to $A$, $F(P, t)$, $0 \leqslant t \leqslant 1$, will lie in $N - A$ and if $\delta(F, P)$ denotes the change of argument as we describe the image of $F(P, t)$ from the image of $F(P, 1)$ to that of $F(P, 0)$ in the $z$-plane the limit

$$\lim_{P \to A} \delta(F, P)$$

exists and is independent of the choice of the local uniformizing parameter. This is called the deformation degree for $f_j$ and $F$ at $A$ and is denoted by $d(F, A)$.

Let $\{\Delta\}$ be an admissible family of domains $\Delta_j$, $j = 1, \ldots, k$, on $\mathcal{R}$ with respect to $Q(z)\,dz^2$. By an admissible family $\{f\}$ of functions $f_j$, $j = 1, \ldots, k$, associated with $\{\Delta\}$ we mean a family with the following properties:

(i) $f_j$ maps $\Delta_j$ conformally into $\mathcal{R}$,

(ii) if a pole $A$ of $Q(z)\,dz^2$ lies in $\Delta_j$, $f_j(A) = A$,

(iii) $f_j(\Delta_j) \cap f_l(\Delta_l) = 0$, $j \neq l$, $j, l = 1, \ldots, k$,

(iv) if $A$ is a pole of order $m$ greater than two of $Q(z)\,dz^2$ in $\Delta_j$ in terms of a local uniformizing parameter $z$ at $A$ representing $A$ as the point at infinity $f_j(z)$ admits locally the representation

$$f_j(z) = z + \sum_{i=s}^{\infty} a_i z^{-i},$$

where $m - 3 \geqslant s \geqslant \frac{1}{2}m - 2$,

(v) each function $f_j$, $j = 1, \ldots, k$, admits an admissible homotopy $F$ into the identity,

(vi) the homotopy $F$ can be chosen so that if $A$ is a pole of $Q(z)\,dz^2$ of order greater than two on the boundary of a strip domain then $d(F, A) = 0$.

**3.** We are now ready to state the General Coefficient Theorem in its current form.

GENERAL COEFFICIENT THEOREM. *Let $\mathcal{R}$ be a finite Riemann surface, $Q(z)\,dz^2$ a positive quadratic differential on $\mathcal{R}$, $\{\Delta\}$ an admissible family of domains $\Delta_j$, $j = 1, \ldots, k$, on $\mathcal{R}$ relative to $Q(z)\,dz^2$ and $\{f\}$ an admissible family of functions $f_j$, $j = 1, \ldots, k$, associated with $\{\Delta\}$. Let $Q(z)\,dz^2$ have double poles $P_1, \ldots, P_r$ and poles $P_{r+1}, \ldots, P_n$ of order greater than two. We allow either of these sets to be void but not both. Let $P_j$, $j \leqslant r$, lie in the domain $\Delta_l$ and in terms of a local uniformizing parameter $z$ representing $P_j$ as the point at infinity let $f_l$ have the expansion*

$$f_l(z) = a^{(j)} z + a_0^{(j)} + \textit{negative powers of } z$$

*and $Q$ the expansion*

$$Q(z) = \alpha^{(j)} z^{-2} + \textit{higher powers of } z^{-1}.$$

*Let $P_j$, $j > r$, a pole of order $m_j$ lie in the domain $\Delta_l$ and in terms of a local uniformizing parameter $z$ representing $P_j$ as the point at infinity let $f_l$ have the expansion*

$$f_l(z) = z + \sum_{i=t_j}^{\infty} a_i^{(j)} z^{-i},$$

*where $t_j$ is the smallest integer greater than or equal to $\frac{1}{2}m_j - 2$ and $Q$ the expansion*

$$Q(z) = \alpha^{(j)} \left[ z^{m_j - 4} + \sum_{i=1}^{\infty} \beta_i^{(j)} z^{m_j - i - 4} \right].$$

*Let*

$$T_j = \alpha^{(j)} \left[ a^{(j)}_{m_j - 3} + \sum_{i=1}^{t_j} \beta^{(j)}_i a^{(j)}_{m_j - i - 3} \right]$$

*if $m_j$ is odd and*

$$T_j = \alpha^{(j)} \left[ a^{(j)}_{m_j - 3} + \sum_{i=1}^{t_j+1} \beta^{(j)}_i a^{(j)}_{m_j - i - 3} + \frac{1}{2}\left(\frac{1}{2} m_j - 2\right)\left(a^{(j)}_{t_j}\right)^2 \right]$$

*if $m_j$ is even.*

*Then*

$$\mathcal{R}\left\{ \sum_{j=1}^{r} \alpha^{(j)} \log a^{(j)} + \sum_{j=r+1}^{n} \alpha^{(j)} T_j \right\} \leqslant 0, \tag{4}$$

*where* $\log a^{(j)} = \log |a^{(j)}| + i d(F, P_j)$, $j \leqslant r$.

If equality occurs in (4) each $f_j$, $j = 1, \ldots, k$, must be an isometry in the $Q$-metric, $|Q(z)|^{1/2}|dz|$, each trajectory of $Q(z)\,dz^2$ in $\bigcup_{j=1}^{k} \Delta_j$ must go into another such and the set $\bigcup_{j=1}^{k} f_j(\Delta_j)$ must be dense in $\mathcal{R}$.

If equality occurs in (4) $f_l$ reduces to the identity in a domain $\Delta_l$ for which any of the following conditions holds.

(i) There is in $\Delta_l$ a pole $P_j$, $j \leqslant r$, of order $m_j$ such that $a^{(j)}_i = 0$ for $i < \min(t_j + 1, m_j - 3)$.

(ii) There is in $\Delta_l$ a pole $P_j$, $j \leqslant r$, with the corresponding coefficient $a^{(j)}$ equal to one.

(iii) There is in $\Delta_l$ a simple pole of $Q(z)\,dz^2$ or a point on a trajectory of $Q(z)\,dz^2$ with an end point at a simple pole.

Equality can occur in (4) when there is a double pole $P_j$, $j \leqslant r$, such that for the corresponding coefficient $|a^{(j)}| \neq 1$ only when $\mathcal{R}$ is conformally equivalent to the sphere and $Q(z)\,dz^2$ is a quadratic differential whose only critical points are two double poles. If further $\{\Delta\}$ consists of a single domain the corresponding admissible function is conformally equivalent to a linear transformation with the points corresponding to these poles as fixed points.

**4.** Most of the subsidiary conditions in the General Coefficient Theorem are of no hindrance in explicit applications of that result. However any relaxation of the coefficient normalization would have important consequences.

One result in this direction is due to Obrock [197]. This is given only in the original Teichmüller context. It proceeds by introducing an auxiliary quadratic differential and using it to calculate the length and area evaluations used in the proof analogously to those used in the proof of the General Coefficient Theorem. While no coefficient normalization is applied to the competing functions there is a normalization required for this auxiliary quadratic differential. This, together with the complicated form of the coefficient inequality obtained, has limited the possibility of application for this result.

The author [117] has given a result without coefficient normalization by restricting consideration to special admissible families of functions. For these, conditions (c) and (d) in the definition of an admissible homotopy are to apply also to zeros of odd order as well as to poles. The same is to be true in condition (ii) in the definition of an admissible family while condition (iv) there is replaced by requiring that each branch of $\int (Q(z))^{1/2}dz$ is single valued in a deleted neighborhood of a pole of $Q(z)\,dz^2$ of order greater than two and condition (vi) is deleted. The result is an inequality involving line and area integrals. It is called the Special Coefficient Theorem.

## 5. Symmetrization

**1.** Symmetrization is a geometric process. Its importance in geometric function theory derives chiefly from the effect it has on modules particularly the module of a proper doubly-connected domain and the reduced module of a simply-connected domain of hyperbolic type with respect to an assigned interior point.

Teichmüller [226] early proved a particular result of this sort. Let $D$ be a doubly-connected domain of module $M(D)$, one complementary continuum of which contains the origin and a point of modulus $r_1$, while the other complementary continuum contains the point at infinity and a point of modulus $r_2$, $0 < r_1, r_2 < \infty$. Let $D^*$ be the domain of module $M(D^*)$ obtained by splitting the plane rectilinearly from 0 to $r_1$ and from $-r_2$ to the point at infinity along the negative real axis. Then

$$M(D) \leqslant M(D^*)$$

with equality only if $D$ is obtained from $D^*$ by rotation about the origin.

The type of symmetrization most important in geometric function theory is circular symmetrization. This was studied by Pólya and Szegö [205,206] both in the plane and three-dimensional Euclidean space. In Pólya's paper [205] it was shown that with reasonable smoothness conditions the area of the boundary of an open set in three-dimensional Euclidean space was not increased by circular symmetrization. This was applied, using a limiting process to show that the Dirichlet integral of functions defined in plane domains was not increased by an appropriate circular symmetrization. This leads then to a result for modules. Because of the limiting process this treatment does not provide an equality statement. The first proof of uniqueness results was given by the author [89].

The relevant results are the following.

Let $D$ be a doubly-connected domain of module $M(D)$ in the $w$-plane, $P$ a point of the plane and $\lambda$ a ray with end point at $P$. Let $K_1$ and $K_2$ be the complementary continua of $D$, both of which we assume to be non-degenerate. Let the respective intersections of $K_1$ and $K_2$ with the circumference of centre $P$ and radius $R$ have angular Lebesgue measures

$l_1(R)$ and $l_2(R)$. Let $R$, $\Phi$ be polar coordinates with pole $P$ and initial ray $\lambda$. Let $K_1^*$ be the set

$$-\frac{1}{2}l_1(R) \leqslant \Phi \leqslant \frac{1}{2}l_1(R)$$

for those values of $R$ for which the circumference $|w - P| = R$ meets $K_1$ plus either of $P$ or the point at infinity which is in $K_1$. Let $K_2^*$ be the set

$$\pi - \frac{1}{2}l_2(R) \leqslant \Phi \leqslant \pi + \frac{1}{2}l_2(R)$$

for those values of $R$ for which the circumference $|w - P| = R$ meets $K_2$ plus either of $P$ or the point at infinity which is in $K_2$. The complement of $K_1^* \cup K_2^*$ is a doubly-connected domain $D^*$ of module $M(D^*)$ which is called the circular symmetrization of $D$ determined by $P$ and $\lambda$. Then

$$M(D) \leqslant M\big(D^*\big).$$

Equality occurs only if $D^*$ is obtained from $D$ by a rotation about $P$.

Let $D$ be a simply-connected domain of hyperbolic type in the $w$-plane, $Q$ an interior point of $D$ and $m$ the reduced module of $D$ with respect to $Q$. Let $P$ be a point of the plane and $\lambda$ a ray with end point at $P$. Let $K$ be the complement of $D$, $K^*$ obtained from $K$ as $K_2^*$ was obtained from $K_2$. Let $D^*$ be the complement of $K^*$. Let $Q^*$ be a point of $\lambda$ with $|Q^* - P| = |Q - P|$. Let $m^*$ be the reduced module of $D^*$ with respect to $Q^*$. Then

$$m \leqslant m^*.$$

Equality occurs only if $D^*$ is obtained from $D$ by a rotation about $P$.

**2.** These results are readily extended to the following context [113]. By a condenser is meant a domain on the $w$-sphere whose complement consists of two closed disjoint sets $K_0$ and $K_1$ such that the Dirichlet problem given by assigning the value 0 at the boundary points of $D$ in $K_0$ and the value at the boundary points of $D$ in $K$, has a solution $u$, called the potential function. The Dirichlet integral of $u$ is called the capacity of the condenser and its reciprocal is called the module $M(D)$ of the condenser. It is equal to the module of a curve family (in an extended sense) each member of which consists of a finite number of rectifiable Jordan curves in $D$ which together separate $K_0$ and $K_1$.

For a point $P$ in the plane and ray $\lambda$ with end point at $P$ let the respective intersections of $K_0$ and $K_1$ with the circumference $|w - P| = R$ have angular Lebesgue measures $l_0(R)$ and $l_1(R)$. Let $R$, $\Phi$ be polar coordinates with pole $P$ and initial ray $\lambda$. Let $K_0^*$ be the set

$$-\frac{1}{2}l_0(R) \leqslant \Phi \leqslant \frac{1}{2}l_0(R)$$

for those values of $R$ for which the circumference $|w - P| = R$ meets $K_0$ plus either of $P$ or the point at infinity which is in $K_0$. Let $K_1^*$ be the set

$$\pi - \frac{1}{2}l_1(R) \leqslant \Phi \leqslant \pi + \frac{1}{2}l_1(R)$$

for those values of $R$ for which the circumference $|w - P| = R$ meets $K_1$ plus either of $P$ or the point at infinity which is in $K_1$. The complement of $K_0^* \cup K_1^*$ is a domain $D^*$. Provided that $D^*$ is again a condenser having module $M(D^*)$ we have

$$M(D) \leqslant M\big(D^*\big).$$

Equality occurs if and only if $D$ is invariant under circular symmetrization with respect to some ray.

**3.** Symmetrization methods can be combined with the results of Section 3 [82]. Let $D$ be a triply-connected domain given by the complement in $|z| < R$ of disjoint continua $K_1$ and $K_2$ contained therein. With pole 0 and the positive real axis as ray let $K_1^*$ and $K_2^*$ be constructed as above. Let $D^*$ be

$$\big\{|z| < R\big\} - \big\{K_1^* \cup K_2^*\big\}.$$

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the (unsensed) homotopy classes determined by Jordan curves in $D$ respectively separating $K_1$ from $K_2$ and $|z| = R$ and $K_2$ from $K_1$ and $|z| = R$. Let for $a_1$, $a_2$ not both zero $M(a_1, a_2)$ be the corresponding module for $D$. Let $\mathcal{H}_1^*$ and $\mathcal{H}_2^*$ be the (unsensed) homotopy classes determined by Jordan curves in $D$ respectively separating $K_1^*$ from $K_2^*$ and $|z| = R$ and $K_2^*$ from $K_1^*$ and $|z| = R$. Let $M^*(a_1, a_2)$ be the corresponding module for $D^*$. Then

$$M(a_1, a_2) \leqslant M^*(a_1, a_2)$$

with equality occurring only if $D^*$ is obtained from $D$ by a rotation about the origin.

**4.** The author [80] obtained the following result by a symmetrization argument.

For $f \in S$ let $L(f, r)$ denote the Lebesgue length measure of the set of values on $|w| = r$ not covered by the image of $|z| < 1$ under the mapping $w = f(z)$. For $\frac{1}{4} < r < 1$ we have the sharp bound

$$L(f, r) \leqslant 2r \cos^{-1}\big(8r^{1/2} - 8r - 1\big)$$

with equality only for functions given explicitly in [80].

Lewandowski [185] applied the method of the paper [80] to starlike functions.

**5.** The author [146] has recently proved the following result which generalizes the result of Teichmüller mentioned at the beginning of this section for the case $r_1 = r_2$.

THEOREM OF $n$-FOLD SYMMETRIZATION. *Let $P_j$, $j = 1, \ldots, 2n$, $n > 1$, be points $e^{i\theta_j}$ on $|z| = 1$ with $\theta_1 < \theta_2 < \cdots < \theta_{2n} < \theta_1 + 2\pi$. Let $D$ be a doubly-connected domain on the $z$-sphere with complementary continua $K_1$ and $K_2$ such that $K_1$ contains $P_{2l+1}$, $l = 0, \ldots, n-1$, and $K_2$ contains $P_{2l}$, $l = 1, \ldots, n$. Then the module $M$ of $D$ satisfies $M \leqslant 1/(2n)$. Equality occurs for the domain $D^*$ where $\theta_j = ((j-1)/n)\pi$, $j = 1, \ldots, 2n$, and $K_1$ is the continuum $K_1^*$ composed of segments from the origin to $P_{2l+1}$, $l = 0, \ldots, n-1$, and $K_2$ is the continuum $K_2^*$ composed of rays of constant argument from $P_{2l}$, $l = 0, \ldots, n$, to the point at infinity and only for domains obtained from $D^*$ by linear transformations which preserve the unit circumference. In particular if $K_1$ contains the origin and $K_2$ contains the point at infinity equality occurs only for $D^*$ and for domains obtained from it by rotation about the origin.*

## 6. Boundary correspondence. Boundary distortion

**1.** A simply-connected domain $\Delta$ of hyperbolic type on the sphere has two types of closure, the ordinary point set closure and the closure obtained by adjoining its border given by regarding $\Delta$ as a finite Riemann surface. The theory of boundary correspondence establishes relations between these entities.

Let $C$ be the point set boundary of $\Delta$. By a crosscut of $\Delta$ we mean an open arc in $\Delta$ which tends to $C$ in each sense. We take a continuum $E \subset \Delta$.

By a fundamental sequence associated with $C$ and $\Delta$ we mean a non-increasing sequence $\{E_n\}$ of non-void sets $E_n \subset \Delta - E$ tending to $C$ and such that if $\Gamma_n$ denotes the family of locally rectifiable crosscuts of $\Delta$ separating $E$ and $E_n$ for the module $m(\Gamma_n)$ of $\Gamma_n$

$$\lim_{n \to \infty} m(\Gamma_n) = \infty.$$

By a $C$-fundamental sequence associated with $C$ and $\Delta$ we mean a non-increasing sequence $\{E_n\}$ of non-void sets $E_n \subset \Delta - E$ tending to $C$ and such that given $\varepsilon > 0$ for $n \geqslant N(\varepsilon)$ there exists a crosscut of $\Delta$ in $\Delta - E$ separating $E$ and $E_n$ of spherical diameter less than $\varepsilon$.

This is essentially Carathéodory's definition.

The first step is to show that a sequence $\{E_n\}$ is $C$-fundamental if and only if it is fundamental.

Then two fundamental sequences $\{E_n\}$ and $\{E'_n\}$ are said to be equivalent if $\{E_n \cup E'_n\}$ is a fundamental sequence.

Equivalence classes of fundamental sequences are called prime ends or boundary elements associated with $C$ and $\Delta$. This definition is essentially independent of the choice of $E$ and clearly conformally invariant.

This is substantially the exposition given in Ahlfors' lectures and repeated in the preliminary draft mentioned in Section 1. However the proof of transitivity in the above equivalence relation contained a serious gap. In his book [6] Ahlfors tried to remedy this by artificial means. However in his Zentralblatt review of [6] the author pointed out that no such artifice is necessary. A detailed exposition appears in [136].

**2.** This treatment leads to a very simple proof of the result that a conformal mapping between Jordan domains can be extended to a homeomorphism of the closures. More important it can give under suitable conditions homeomorphic or continuous extensions to subarcs of the boundary.

The above considerations extend in an obvious manner to domains of finite connectivity. Suita [223] developed similar concepts for a domain of arbitrary connectivity.

Nobuyuhi Suita and the author [162] used the results of Section 3 to obtain results on the representation and compactification of Riemann surfaces.

**3.** A sequence of points of $\Delta$ is said to converge to a prime end if its terminal sequences form a fundamental sequence in the equivalence class. The impression of a prime end consists of all accumulation points of sequences converging to the prime end. There is an extensive literature categorizing prime ends in terms of their impressions. A survey of such results can be found in [200].

**4.** The prototype of theorems on boundary distortion is an old result of Löwner [187] which states the following.

Let $\phi(z)$ be a function regular for $|z| < 1$ such that $\phi(0) = 0$, $|\phi(z)| < 1$ for $|z| < 1$ and that $\phi(z)$ assumes continuously values of modulus 1 on an arc $A$ on $|z| = 1$. Then the image of $A$ has length at least equal to that of $A$. Equality of these lengths occurs only for $\phi(z) = \mathrm{e}^{\mathrm{i}\theta} z$, $\theta$ real.

By confining attention to univalent functions and using the results of Sections 3 and 5 more sophisticated results can be obtained [95]. A typical result is as follows.

Let $D$ be a plane domain of connectivity $N$, $n \geqslant 2$, with border components $C_1, C_2, \ldots, C_n$. Let $A$ be a border arc on $C_1$, $A^*$ its complement on $C_1$. Then there exist values $R_0$, $R_1$ with $0 < R_1 < R_0$ such that for $R_1 \leqslant R \leqslant R_0$ there exists a unique mapping $f(z, R)$ of $D$ into $1 < |w| < R$ with the following properties. $f(z, R_0)$ admits continuous extensions to the border components with $A$ mapped onto $|w| = R_0$, $A^*$ onto the rectilinear slit $-R_0 < w \leqslant -p$, $1 < p < R_0$, $C_2$ onto $|w| = 1$ and $C_j$, $j = 3, \ldots, n$, onto radial slits. $f(z, R)$, $R_1 \leqslant R < R_0$, admits a continuous extension to the border components with $A$ mapped onto an arc given by $R\,\mathrm{e}^{\mathrm{i}\theta}$, $-\theta(R) \leqslant \theta \leqslant \theta(R)$, $0 < \theta(R) < \pi$, $A^*$ onto the open arc given by $R\,\mathrm{e}^{\mathrm{i}\theta}$, $\theta(R) < \theta < 2\pi - \theta(R)$, plus a rectilinear slit $-R < w \leqslant -p(R)$, $1 < p(R) < R$, $C_2$ onto $|w| = 1$, $C_j$, $j = 3, \ldots, n$, onto trajectory arcs for a quadratic differential regular on $1 < |w| < R$, positive on the open arc given by $R\,\mathrm{e}^{\mathrm{i}\theta}$, $-\theta(R) < \theta < \theta(R)$, negative on the open arc given by $R\,\mathrm{e}^{\mathrm{i}\theta}$, $-\theta(R) < \theta < 2\pi - \theta(R)$, with simple poles at $R\,\mathrm{e}^{\mathrm{i}\theta(R)}$, $R\,\mathrm{e}^{-\mathrm{i}\theta(R)}$ and a double zero at $-R$.

If $f(z)$ is regular and univalent in $D$ with continuous extension to $C_2$ and $A$ mapping them onto $|w| = 1$ and an arc on $|w| = R$, $R_1 \leqslant R \leqslant R_0$, of length $l$ respectively and satisfies $1 < |f(z)| < R$, $z \in D$, then $l \geqslant 2R\theta(R)$ with equality only if $f(z) = \mathrm{e}^{\mathrm{i}\chi} f(z, R)$, $\chi$ real.

Kühnau [170] considered some similar problems.

## 7. Univalent regular and meromorphic functions and other function families

**1.** In [101] the form of the General Coefficient Theorem given there was used to derive numerous results including all the basic results for the families $S$ and $\Sigma$ as well as many region of values results due primarily to Grötzsch and Grunsky. For the most part we will not reproduce these developments here as well as applications to canonical conformal mappings.

The present improved form of the General Coefficient Theorem makes possible many other applications.

**2.** In [104] the author worked out in highly explicit detail a number of low order versions of the General Coefficient Theorem and proved a number of new results for the families $S$ and $\Sigma$. We will use the generic representations

$$f(z) = z + A_2 z^2 + \cdots + A_n z^n + \cdots$$

for $f \in S$ and

$$f(z) = z + c_0 + c_1 z^{-1} + \cdots + c_n z^{-n} + \cdots$$

for $f \in \Sigma$.

A central role is played by the function $f(z, \tau, \phi) \in \Sigma$ which maps $|z| > 1$ conformally onto an admissible domain for the quadratic differential

$$Q_\phi(w)\, dw^2 = \mathrm{e}^{-2\mathrm{i}\phi}\bigl(\tau\,\mathrm{e}^{\mathrm{i}\phi} - w\bigr) w^{-1}\, dw^2$$

$\phi$ real, $0 < \tau \leqslant 4$, bounded by the rectilinear segment from 0 to $\tau\,\mathrm{e}^{\mathrm{i}\phi}$ and equal arcs on the other trajectories of $Q_\phi(w)\, dw^2$ with end points at $\tau\,\mathrm{e}^{\mathrm{i}\phi}$. Its expansion about the point at infinity is

$$z + \frac{1}{2}\tau\left[1 - \log\frac{\tau}{4}\right]\mathrm{e}^{\mathrm{i}\phi} + \left[\frac{1}{8}\tau^2\left(1 - 2\log\frac{\tau}{4} - 1\right)\right]\mathrm{e}^{-2\mathrm{i}\phi} z^{-1} + \cdots.$$

Applying the General Coefficient Theorem we get the following results.

Let $f \in \Sigma$ and not take the value 0 in $|z| > 1$. The for real $\phi$

$$\Re\bigl\{\mathrm{e}^{-2\mathrm{i}\phi}\bigl(c_1 - \tau\,\mathrm{e}^{\mathrm{i}\phi} c_0\bigr)\bigr\} \geqslant \begin{cases} -1 - \frac{3}{8}\tau^2 + \frac{1}{4}\tau^2 \log\frac{\tau}{4}, & 0 < \tau \leqslant 4, \\ -1, & \tau = 0. \end{cases}$$

For $\tau = 0$ equality occurs only for the functions

$$z + \mathrm{i}a\,\mathrm{e}^{\mathrm{i}\phi} - \mathrm{e}^{2\mathrm{i}\phi} z^{-1}, \quad -2 \leqslant a \leqslant 2.$$

For $0 < \tau < 4$ equality occurs only for the function $f(z, \tau, \phi)$ and functions obtained from it by translations along trajectories. For $\tau = 4$ equality occurs only for

$$f(z, 4, \phi) \equiv z + 2\mathrm{e}^{\mathrm{i}\phi} + \mathrm{e}^{2\mathrm{i}\phi} z^{-1}.$$

Let $f \in S$. Then for real $\phi$

$$\Re\{e^{-2i\phi}A_3 + e^{-2i\phi}A_2^2 + \tau e^{i\phi}A_2\} \geqslant \begin{cases} -1 + \frac{3}{8}\tau^2 + \frac{1}{4}\tau^2 \log\frac{\tau}{4}, & 0 < \tau \leqslant 4, \\ -1, & \tau = 0. \end{cases}$$

For $\tau = 0$, equally occurs only for the functions

$$z\left(1 + ia\,e^{i\phi} - z^{2i\phi}\right)^{-1}, \quad -2 \leqslant a \leqslant 2.$$

For $0 < \tau < 4$, equality occurs only for the function $1/f(1/z, \tau, \phi)$ and functions obtained from it by translation along trajectories. For $\tau = 0$, equality is attained only for the function

$$\frac{1}{f(1/z, 4, \phi)} \equiv z\left(1 + e^{i\phi}z\right)^{-2}.$$

From this can be derived numerous bounds involving $A_2$ and $A_3$, in particular the bound $|A_3| \leqslant 3$ and the corresponding equality result.

One can similarly obtain inequalities involving $c_1$ and $c_2$ for functions in $\Sigma$ in particular the precise bound $|c_2| \leqslant 2/3$ with appropriate equality statement.

For inequalities involving $c_1$, $c_2$ and $c_3$ we have the following result.

Let $f \in \Sigma$. For $\psi$ real

$$\Re\left\{e^{-i4\psi}\left(c_3 + \frac{1}{2}c_1^2 - \sigma^2 e^{2i\psi}c_1\right)\right\} \geqslant \begin{cases} -\frac{1}{2} - \frac{3}{16}\sigma^4 + \frac{1}{8}\sigma^4 \log\frac{\sigma^2}{4}, & 0 < \sigma \leqslant 2, \\ -\frac{1}{2}, & \sigma = 0. \end{cases}$$

For $0 \leqslant \sigma < 2$ equality occurs only for the functions

$$\left[f\left(z^2, \sigma^2, 2\psi\right)\right]^{1/2} + k$$

with the proper choice of root and $k$ constant and for functions obtained from them by translation along the trajectories of the quadratic differential

$$e^{-4\psi i}\left(\sigma^2 e^{2i\psi} - w^2\right)dw^2.$$

For $\sigma = 2$ equality occurs only for the functions

$$\left[f\left(z^2, 4, 2\psi\right)\right]^{1/2} + k$$

for proper choice of root and $k$ constant.

This result has various consequences, in particular the precise bound

$$|c_3| \leqslant \frac{1}{2} + e^{-4}$$

with the appropriate equality statement.

**3.** In [108] these results are extended to coefficients of general index. In particular for $f \in \Sigma$ with real $\psi$ and $c_j = 0$ for $j < n$, $n \geqslant 2$,

$$\Re\left\{-e^{-2(n+1)\psi i}c_{2n+1} + \delta\, e^{-(n+1)\psi i}c_n - \frac{1}{2}n\, e^{-2(n+1)\psi i}c_n^2\right\}$$

$$\leqslant \frac{3}{8(n+1)}\delta^2 - \frac{1}{4(n+1)}\delta^2 \log\frac{\delta}{4} + \frac{1}{n+1}, \quad 0 < \delta \leqslant 4, \tag{5}$$

$$\Re\left\{-e^{-2(n+1)\psi i}c_{2n+1} - \frac{1}{2}n\, e^{-2(n+1)\psi i}c_n^2\right\} \leqslant \frac{1}{n+1}. \tag{6}$$

Equality occurs only for the functions

$$\left[f\left(z^n, \delta, n\psi\right)\right]^{1/n}$$

(with the proper choice of root) and for functions obtained from them by translation along the trajectories of the quadratic differential

$$e^{-2n\psi i}w^{n-2}\left(\delta\, e^{n\psi i} - w^n\right)dw^2$$

for inequality (5). Equality occurs in (6) only for the functions $[f(z^n, 0, n\psi]^{1/n}$.

More specifically we have the following bounds.

If $c_j = 0$ for $j \leqslant (n-2)/2$, $n > 0$,

$$|c_n| \leqslant \frac{2}{n+1}.$$

If $c_j = 0$ for $j < n$, $n > 0$,

$$|c_{n+1}| \leqslant (n+1)^{-1}\left[1 + 2\exp\left(-2\frac{n+2}{n}\right)\right].$$

In each case there is an appropriate equality statement.

Golusin [42] stated similar results but his proof is defective as is his proof for the result $|c_2| \leqslant 2/3$ attributed to him in [101].

**4.** These results can be reformulated for functions in $S$. In particular we have the following result.

If $f(z)$ is univalent and meromorphic in $|z| < 1$ with expansion about the origin

$$f(z) = z + A_2z^2 + A_3z^3 + \cdots + A_nz^n + \cdots$$

and $A_2 = 0$ then

$$|A_3| \leqslant \frac{1}{2} + e^{-2/3}$$

with an appropriate equality statement. This improves a result of Fekete and Szegö [34] who required that the function be regular and odd.

**5.** The family of Bieberbach–Eilenberg functions $C$ consists of functions regular for $|z| < 1$ with $f(0) = 0$ such that $f(z_1)f(z_2) \neq 1$, $|z_1|, |z_2| < 1$. Using the method of the extremal metric and symmetrization the author [85] found the best upper bound for $|f(z)|$, $|z| = r$, $0 < r < 1$, for $f \in C$ and the best upper and lower bounds for $|f(z)|$, $|z| = r$, $0 < r < 1$, for $f \in C$ and univalent with given $|f'(0)|$ (necessarily $|f'(0)| \leqslant 1$).

The results are the following.

If $f \in C$, $|z| = r$, $0 < r < 1$, $|f(z)| \leqslant r(1 - r^2)^{-1/2}$, equality occurring only for the functions $\pm f_r(z\,e^{i\theta})$ at the point $z = ir\,e^{-i\theta}$ where

$$f_r(z) = \left(1 - r^2\right)^{1/2} z(1 + irz)^{-1}.$$

Given $0 < r < 1$ and $0 < c < 1$, there exists a unique function $F(z; r, c) \in C$ which maps $|z| < 1$ conformally onto an admissible domain with respect to a quadratic differential

$$Q(w)\,dw^2 = \frac{(w - im)(w + im^{-1})}{w^2(w - il)(w + il^{-1})}\,dw^2,$$

where $m > 0$, $0 < l < m$, bounded by two trajectories of $Q(w)\,dw^2$ joining its zeros and a possible segment on the imaginary axis with end point at $-im^{-1}$ of length less than $m^{-1}$ and such that $F(ir; r, c) = il$ and $|F'(0; r, c)| = c$.

If $f \in C$, $|f'(0)| = c$, is univalent, $|z| = r$, $0 < r < 1$, $|f(z)| \leqslant \Im F(ir; r, c)$ with equality only if $f(z\,e^{i\theta}) = F(z; r, c)$, $\theta$ real.

Given $0 < r < 1$ and $0 < c < 1$, there exists a unique function $H(z; r, c) \in C$ which maps $|z| < 1$ conformally onto an admissible domain with respect to a quadratic differential

$$\widetilde{Q}(w)\,dw^2 = -\frac{(w - im)(w + im^{-1})}{w^2(w - il)(w + il^{-1})}\,dw^2,$$

where $m > 0$, $-m^{-1} < l < 0$, bounded by two trajectories of $\widetilde{Q}(w)\,dw^2$ joining its zeros and a possible segment on the imaginary axis with end point at $-im^{-1}$ of length less than $m^{-1} + l$ and such that $H(-ir; r, c) = il$ and $|H'(0; r, c)| = c$.

If $f \in C$, $|f'(0)| = c$, is univalent, $|z| = r$, $0 < r < 1$, $|f(z)| \geqslant -\Im H(-ir; r, c)$ with equality only if $f(ze^{i\theta}) = H(z; r, c)$, $\theta$ real.

In [90] the following result is obtained.

If $0 < \lambda < 1$, there is a unique function $G(z, \lambda) \in C$ which maps $|z| < 1$ conformally onto an admissible domain with respect to a quadratic differential

$$\widehat{Q}(w)\,dw^2 = -\frac{(w - im)(w + im^{-1})}{w^2(w - il)(w + il^{-1})}\,dw^2,$$

where $m > 0$, $-m^{-1} < l < 0$, bounded by two trajectories of $\widetilde{Q}(w)\,dw^2$ joining its zeros and a segment on the imaginary axis joining $il$ and $-im^{-1}$ and such that $|G'(0, \lambda)| = \lambda$.

If $f \in C$, $|f'(0)| = \lambda$, is univalent the smallest modulus of a boundary point of the image of $|z| < 1$ under $f$ is at least $|l| = \mu(\lambda)$. Equality occurs only if $f(ze^{i\theta}) = G(z, \lambda)$, $\theta$ real.

If $C(\lambda)$ is the subset of functions $f$ in $C$ with $|f'(0)| = \lambda$ for $0 < \lambda < 1$, the best upper bound of $|f(z)|$ for $f \in C(\lambda)$ is $(\mu(\lambda))^{-1}$. There is no function in $C(\lambda)$ for which this bound is attained.

**6.** Grunsky [66] introduced a family of functions, usually denoted by $K$, which have many properties similar to those of the Bieberbach–Eilenberg functions. This family consists of functions $f$ regular for $|z| < 1$ and such that $f(0) = 0$, $f(z_1)\overline{f(z_2)} \neq -1$, $z_1, z_2 < 1$. The proofs are frequently somewhat simpler because they do not require the use of symmetrization. In [101] the author mistakenly ascribed the introduction of this family to Shah.

Kühnau [168] proved that if a univalent function in $K$ has expansion about the origin $a_1 z + a_2 z^2 + \cdots$ then $|a_2| \leqslant a_M$ where $a_M$ is given by certain identities between elliptic integrals, $a_M = 0.58\ldots$. Combined with the author's result [118] that for certain extremal problems the extremal functions for $C$ and $K$ coincide it follows that the same bound is true for univalent functions in $C$.

**7.** The author [103] used the General Coefficient Theorem to study the subclass $S_R$ of $S$ consisting of functions with real coefficients. In particular he determined the exact domain covered by the image of the unit disc under every such function, the region of values of such functions at a point of the unit disc and new bounds involving the derivative of these functions.

For the family $S$ Phelps [204] gave explicitly the coefficient body for $(A_2, A_3, A_4)$ in the case where these coefficients are real.

**8.** The author has applied the method of the extremal metric and the General Coefficient Theorem to study the Schlicht Bloch Constant. He gave a criterion [140] which eliminated certain functions which had been conjectured to be extremal in this problem and qualitative properties [148] of a possible extremal function. He also [111] gave the lower bound 0.5705 for the Schlicht Bloch Constant and more recently [148] by an improved method the lower bound 0.57088.

**9.** The General Coefficient Theorem is particularly effective in dealing with problems involving functions whose values are conditioned in terms of complementary domains. The author has proved the following results [101,106].

(I) Let $Q_\mu(w)\,dw^2$ denote the quadratic differential $w^{-2}(w - \mu)\,dw^2$, $\mu > 0$. Let $T_\mu$ be the trajectory of $Q_\mu(w)\,dw^2$ which runs from $w = \mu$ back to that point. Then for $\mu \leqslant 16\pi^{-2}$ there exists a function $f_\mu \in \Sigma$ such that the mapping $w = f_\mu(z)$ carries $|z| > 1$ onto a domain bounded by $T_\mu$ and a possible slit on the real axis to the right of $\mu$. For $\mu > 16\pi^{-2}$ there exists a function $f_\mu \in \Sigma$ such that the mapping $w = f_\mu(z)$ carries $|z| > 1$ onto a domain bounded by a closed trajectory of $Q_\mu(z)\,dz^2$ which separates $T_\mu$ from the origin.

Let $f \in \Sigma$ map $|z| > 1$ onto a domain whose complement contains a domain with inner conform radius with respect to the origin at least $r$ $(0 < r < 1)$. Let $f(z)$ have expansion about the point at infinity

$$f(z) = z + a_0 + \text{terms in } z^{-1}.$$

Then the region of values of $a_0$ is given by

$$|a_0| \leqslant P_r, \quad \text{where } P_r = 2 - \frac{1}{4} e^2 r$$

for $r \leqslant 64/(\pi^2 e^2)$ and

$$P_r = 4k^{-2} - 2 - \mu$$

with

$$\mu = 16\pi^{-2}k^{-2}E^2(k), \qquad r = 4\mu \exp\left(-2 - \frac{\pi K'(k)}{K(k)} + \frac{\pi^2}{2K(k)E(k)}\right)$$

for $r > 64\pi^{-2}e^{-2}$.

The value $P_r e^{i\theta}$, $\theta$ real, is attained by $a_0$ only for the function $e^{i\theta} f_\mu(e^{-i\theta} z)$ where $\mu = \frac{1}{4} e^2 r$ in the first case, $\mu$ and $R$ are related as above in the second case.

(II) Let $Q_\nu(w)\,dw^2$ denote the quadratic differential $w^{-2}(w^2 - \nu^2)\,dw^2$ where $\nu > 0$. Then $Q_\nu(w)\,dw^2$ has two trajectories $T_\nu$, $T'_\nu$ each of which joins the points $\pm\nu$. For $\nu \leqslant 4\pi^{-1}$ there exists a function $f_\nu \in \Sigma$ such that the mapping $w = f_\nu(z)$ carries $|z| > 1$ onto a domain bounded by $T_\nu$, $T'_\nu$ and possible slits of equal length on the real axis to the right of $\nu$ and to the left of $-\nu$. The expansion of $f_\nu(z)$ about the point at infinity is

$$f_\nu(z) = z + \frac{1}{2}(2 - \nu^2)z^{-1} + \text{higher powers of } z^{-1}.$$

The interior of the complement of $f_\nu(\{|z| > 1\})$ has inner conform radius with respect to the origin equal to $2\nu e^{-1}$. For $\nu > 4\pi^{-1}$ there exists a function $f_\nu \in \Sigma$ such that the mapping $w = f(z)$ carries $|z| > 1$ onto a domain bounded by a closed trajectory of $Q_\nu(w)\,dw^2$ which separates $T_\nu$, $T'_\nu$ from the origin. The expansion of $f(z)$ about the point at infinity is

$$f_\nu(z) = z + \left(2k^{-2} - 1 - \frac{1}{2}\nu^2\right)z^{-1} + \text{higher powers of } z^{-1}$$

when $2k^{-1}E(k) = \frac{1}{2}\pi\nu$. The interior of the complement of $f_\nu(\{|z| > 1\})$ has inner conform radius with respect to the origin equal to

$$2\nu e^{-1} \exp\{-2k^{-1}\nu^{-1}[K'(k) - E'(k)]\}.$$

Let $f \in \Sigma$ map $|z| > 1$ onto a domain whose complement contains a domain with inner conform radius with respect to the origin at least $r$ $(0 < r < 1)$ and have expansion about the point at infinity

$$f(z) = z + a_0 + a_1 z^{-1} + \text{higher powers of } z^{-1}.$$

Then the region of values of $a_1$ is given by

$$|a_1| \leqslant Q_r, \quad \text{where } Q_r = 1 - \frac{1}{8} e^2 r$$

when $r \leqslant 8\pi^{-1} e^{-1}$ and

$$Q_r = 2k^{-1} - 1 - \frac{1}{2} \nu^2 \quad \text{with } \nu = 4\pi^{-1} k^{-1} E(k),\ r = 2\nu e^{-1},$$

when $r > 8\pi^{-1} e^{-1}$. For $r < 8\pi^{-1} e^{-1}$ the value $Q_r e^{2i\theta}$ is attained only for the function $e^{i\theta} f_\nu(e^{-i\theta} z)$, $\theta$ real, $0 \leqslant \theta < \pi$, where $\nu = \frac{1}{2} er$ and functions obtained from it by translation along trajectories. For $r \geqslant 8\pi^{-1} e^{-1}$ the value $Q_r e^{2i\theta}$, $\theta$ real, $0 \leqslant \theta < \pi$, is attained only for the function $e^{i\theta} f_\nu(e^{-i\theta} z)$ where $\nu$ is determined as above.

Further results in this direction were given by Kühnau [174].

Duren and Schiffer [28] published a result which is an immediate consequence of (I).

**10.** Obrock [198], using the General Coefficient Theorem in combination with the continuity method, studied coefficient bodies for the functions of the preceding section and some other classes of functions obtaining results analogous to those obtained in [214] for $S$ by the variational method.

**11.** By weighted distortion is meant an expression for a regular function $f(z)$ involving $|f'(z)|$ and other entities. In [105] among other results the author determined, using the General Coefficient Theorem, the exact region of values for the pair $(|f(z)|, |f'(z)|)$, $|z| < 1$, for $f \in S$ and used it to solve some problems on weighted distortion.

Recently the author proved the following results improving earlier work by Blatter [17] and Kim and Minda [166].

(I) If $f$ is regular and univalent in $D$: $|z| < 1$ and $z_1, z_2$ are distinct points in $D$ we have for $p \geqslant 1$:

$$|f(z_1) - f(z_2)| \geqslant \frac{\sinh 2\rho}{2(\cosh 2p\rho)^{1/p}} \left(|D_1 f(z_1)|^p + |D_1 f(z_2)|^p\right)^{1/p}$$

when $D_1 f(z) = f'(z)(1 - |z|^2)$ and $\rho$ is the hyperbolic distance between $z_1$ and $z_2$. Equality occurs if and only if $f$ maps $D$ onto the plane slit along a ray on the line determined by $f(z_1)$ and $f(z_2)$.

The inequality does not obtain for $0 < p < 1$.

(II) If $f$ is regular and univalent in $D$ and $z_1$, $z_2$ are distinct points of $D$ we have for $p > 0$

$$|f(z_1) - f(z_2)| \leqslant \frac{1}{2^{1+p}} \sinh 2\rho\big(|D_1 f(z_1)|^p + |D_1 f(z_2)|^p\big)^{1/p}$$

with $D$, $D_1 f(z)$ and $\rho$ as in (I). Equality occurs if and only if $f$ maps $D$ onto the plane slit symmetrically through the point at infinity on the line determined by $f(z_1)$ and $f(z_2)$.

**12.** In [120] the author considered relations between the General Coefficient Theorem and the variational method. In particular it was shown that for functions in $S$ if we restrict the variations to leave fixed those coefficients which in the General Coefficient Theorem are required to vanish the necessary variational condition given by a quadratic differential equation is also sufficient. This gives a very simple (but not elementary) proof of inequality $|A_3| \leqslant 3$ for $f \in S$ and the corresponding equality statement.

## 8. Strip domains. Angular derivatives

**1.** Ahlfors' thesis [1] consists of three parts. In the first he considered mappings of strip domains and proved two theorems which he called the First and Second Fundamental Inequalities.

We understand the term strip domain in a general sense, that is, a simply-connected domain $\mathcal{S}$ in the plane which has two point boundary elements $P_1$ and $P_2$ of abscissae $A$ and $B$, $A < B$. For $A < x < B$ there exists on the line $\Re z = x$ a segment $\sigma(x)$ which separates $P_1$ from $P_2$ in $\mathcal{S}$. Let $\sigma(x)$ have length $\theta(x)$ (the possibility of infinite length is not excluded). It is easily seen that we can assume $\theta(x)$ to be measurable and that $\sigma(x_1)$ separates $\sigma(x_2)$ from $P_1$ for $x_1 < x_2$. Let $\mathcal{S}$ be mapped conformally on the rectilinear strip $S$: $0 < \Im\zeta < a$, so that $P_1$ and $P_2$ correspond to the boundary elements of $S$ determined by the point at infinity with neighborhoods in $\Re\zeta < 0$ and $\Re\zeta > 0$ respectively. Let $\tau(x)$ denote the image of $\sigma(x)$ in $S$. Let

$$\xi_1(x) = \underset{\zeta\in\tau(x)}{\text{g.l.b.}}\, \Re\zeta, \qquad \xi_2(x) = \underset{\zeta\in\tau(x)}{\text{l.u.b.}}\, \Re\zeta.$$

Ahlfors' results consist in giving upper and lower bounds for expressions involving these quantities in terms of $\int dx/\theta(x)$. His first result is now habitually called the Ahlfors Distortion Theorem. We state it as follows.

AHLFORS DISTORTION THEOREM. *For $A < x_1 < x_2 < B$*

$$\int_{x_1}^{x_2} \frac{dx}{\theta(x)} \leqslant \begin{cases} \frac{1}{a}\big(\xi_1(x_2) - \xi_2(x_1)\big) + 2, & \xi_2(x_1) - \xi_1(x_2) < a, \\ 1, & \xi_2(x_1) - \xi_1(x_2) \geqslant a. \end{cases}$$

This result has a very simple proof by the method of the extremal metric.

Let $\Gamma$ be the family of the segments $\sigma(x)$, $x_1 < x < x_2$. One component of

$$\mathcal{S} - \big(\sigma(x_1) \cup \sigma(x_2)\big)$$

becomes a quadrangle $Q$ (possibly degenerate) of module $M$ for the pair of sides complementary to $\sigma(x_1)$ and $\sigma(x_2)$. The module of $\Gamma$ is $\int_{x_1}^{x_2} dx/\theta(x)$. Thus

$$M \geqslant \int_{x_1}^{x_2} \frac{dx}{\theta(x)}.$$

Now $\tau(x_1)$, $\tau(x_2)$ are a pair of opposite sides of a quadrangle $\mathcal{Q}$ conformally equivalent to $Q$ whose complementary sides are respectively on $\Im f = 0$ and $\Im f = a$ with module $M$ for the family of curves joining these latter sides. Using the $L$-normalization we get an admissible metric in the case $\xi_2(x_1) - \xi_1(x_2) < a$ setting

$$\rho(\zeta) = \begin{cases} \frac{1}{a}, & \zeta \in \mathcal{Q}, \xi_2(x_1) - a < \Re\zeta < \xi_1(x_2) + a, \\ 0 & \text{elsewhere in } \mathcal{Q}. \end{cases}$$

The above estimate is immediate.

Ahlfors' proof was a complicated application of the length-area method. Teichmüller [226] also gave a proof and a functional inequality involving elliptic functions.

Some authors have applied the Ahlfors Distortion Theorem as a technique particularly in certain extremal problems. Examples can be found in the book of Nevanlinna [196]. These considerations seem usually to be rather artificial and simpler proofs of better results usually can be given by a direct application of the method of the extremal metric.

**2.** Ahlfors' second result gave an upper bound for $\xi_2(x_2) - \xi_1(x_1)$ in terms of $\int_{x_1}^{x_2} dx/\theta(x)$ under rather strong subsidiary conditions. Kôtaro Oikawa and the author [152] showed that certain of these restrictions were unnecessary and proved the following result using the previous terminology.

Let $\sigma(x)$ be given by $-\theta_1(x) < y < \theta_2(x)$, $0 < \theta_j(x) \leqslant L$, $A < x < B$, $j = 1, 2$. Let

$$\min_{x' < x < x''} \{\theta_1(x), \theta_2(x)\} = \theta^{(m)}(x', x'').$$

Then for $A + 2L < x_1 < x_2 < B - 2L$

$$\begin{aligned} \frac{1}{a}\big(\xi_2(x_2) - \xi_1(x_1)\big) \leqslant \frac{1}{a}\int_{x_1}^{x_2} &\frac{dx}{\theta_1(x) + \theta_2(x)} \\ &+ \frac{1}{2} L\big(\theta^{(m)}(x_1, x_2)\big)^{-2}\big(V_1(x_1, x_2) + V_2(x_1, x_2)\big) \\ &+ 2L\big[\theta^{(m)}(x_1 - 2L, x_1 + 2L)\big]^{-1} \\ &+ 2L\big[\theta^{(m)}(x_2 - 2L, x_2 + 2L)\big]^{-1}, \end{aligned}$$

where $V_j(x', x'')$ denotes the total variation of $\theta_j$ on $[x', x'']$, $j = 1, 2$.

The authors [154] also gave a generalization employing a weaker form of variation.

**3.** One normally speaks of a complex-valued function having a derivative at a point when it is defined in a (plane) neighborhood of the point. Quite early this concept was extended to a function having a derivative at a boundary point of a domain. In particular if $f(z)$ is regular in a disc, say $|z| < 1$, and continuously extendable to the boundary point $e^{i\theta}$, $f(z)$ was said to have an angular derivative at $e^{i\theta}$ if

$$\lim_{z\to e^{i\theta}} \frac{f(z) - f(e^{i\theta})}{z - e^{i\theta}}$$

exists as $z$ approached $e^{i\theta}$ in a Stolz (angular) domain. The problem can be normalized in various manners. In particular one can take the domain to be the right-hand half-plane, the distinguished point and its image both to be the point at infinity. The most significant results have been obtained under the further assumption that the function is univalent.

In the second part of his thesis Ahlfors considered the problem of angular derivatives. His significant contribution was to reformulate the problem in terms of strip domains. We will use that formulation in the results to follow. The corresponding formulation for the half-plane normalization is presented in detail in [155].

**4.** We consider therefore a regular univalent function $f$ defined in the strip

$$S = \left\{ z \,\middle|\, |\Im z| < \frac{\pi}{2} \right\}.$$

We denote by $S_\delta$ the set $\{z \mid |\Im z| < \pi/2 - \delta\}$, $0 < \delta < \pi/2$ and by $S_\delta(a)$ the set $S_\delta \cap \{z \mid \Re z > a\}$. The image of $S$ by $f$ is a simply-connected domain $D$ in the $w$-plane. We assume that for every $\varepsilon > 0$ there exists $a$ such that $S_\varepsilon(a) \subset D$ and that

$$\lim_{S_\delta \ni z \to \infty} f(z) = +\infty$$

for every $\delta$, $0 < \delta < \pi/2$.

If $\lim(z - f(z)) = \alpha$ as $S_\delta \ni z \to \infty$, $\alpha \neq \pm\infty$ ($\alpha$ is necessarily real) $f$ is said to be conformal at $+\infty$ with angular derivative $\alpha$.

If this holds for $S \ni z \to \infty$, $f$ is said to have unrestricted derivative $\alpha$.

If $\lim \Im(z - f(z)) = 0$ as $S_\delta \ni z \to \infty$, $f$ is said to be semi-conformal at $+\infty$.

If this holds for $S \ni z \to \infty$, $f$ is said to have unrestrictedly semi-conformal at $+\infty$.

For a sufficiently large $a \geqslant a_0$, let $\sigma_a$ denote the crosscut of $D$ on the line $\Re z = a$ which meets the real axis. For $a, b \geqslant a_0$ let $D^*(a, b)$ be the component of $D - (\sigma_a \cup \sigma(b))$ which contains the segment $(a, b)$. Let $\Gamma(a, b)$ be the family of locally rectifiable curves in $D^*(a, b)$ separating $\sigma_a$ from $\sigma_b$. Let $m(a, b)$ be its module.

In [155] the following results are proved.

(A) A necessary and sufficient condition that $f$ have a finite angular derivative is that

$$m(a, b) = \frac{1}{\pi}(b - a) + o(1)$$

as $b > a \to +\infty$.

(B) A necessary and sufficient condition that $f$ be semi-conformal at $+\infty$ is that given $\varepsilon > 0$ there exists $a$ such that for $b > a$

$$\left\{w \,\middle|\, \left|w - \left(b + \mathrm{i}\frac{\pi}{2}\right)\right| < \varepsilon\right\} \not\subseteq D,$$

$$\left\{w \,\middle|\, \left|w - \left(b - \mathrm{i}\frac{\pi}{2}\right)\right| < \varepsilon\right\} \not\subseteq D.$$

For $a$ sufficiently large let $D(a)$ denote the component of $D \cap \{w \mid \Re w > a\}$ containing the ray $(a, \infty)$ and $D^*(a)$ denote the component of $D - \sigma_a$ containing the ray $(a, \infty)$.

(E) A necessary and sufficient condition that $f$ have a finite unrestricted derivative is that the following conditions hold:
 (i) $m(a, b) = \frac{1}{\pi}(b - a) + \mathrm{o}(1)$, $b > a \to +\infty$,
 (ii) for any $d > 0$ there exists $a$ such that every component of

$$D^*(a) - \bigcup_{u > a} \sigma_u$$

has orthogonal projection on the real axis of length less than $D$,
 (iii) for any $\varepsilon > 0$ there exists $b$ such that

$$D(b) \subset \left\{w \,\middle|\, |\Im w| < \frac{\pi}{2} + \varepsilon\right\}.$$

(F) A necessary and sufficient condition that $f$ be unrestrictedly semi-conformal at $+\infty$ is that given $\varepsilon > 0$, $0 < \varepsilon < \pi/2$, and $c$ there exists $a$ sufficiently large such that

$$\partial D^*(a) - \sigma_a \subset \left\{w \,\middle|\, c \leqslant \Re w,\ \frac{\pi}{2} - \varepsilon < |\Im w| < \frac{\pi}{2} + \varepsilon\right\},$$

The authors [157] also gave an alternative proof of a result of Ostrowski [201].
(A) was also proved by Rodin and Warschawski [210].
(B) is due to Ostrowski [201].
The analogous results in the half-plane normalization are easily derived from (A), (B), (E), (F). Actually the latter are stronger because there are functions in the strip normalization case which do not derive from such in the half-plane normalization case.

Result (A) must almost certainly be regarded as the ultimate solution of the angular derivative problem. Some authors have held out the hope of a solution in purely Euclidean terms but the description of a general strip domain in such terms is so complicated that there seems to be little hope. Strong evidence is provided by the fact that since the appearance of [155] no significant result in this direction has been obtained.

**5.** Burdzy [23] introduced the concept of Lipschitz majorant in the case of the half-plane normalization and in terms of it gave a conditional necessary and sufficient condition for the existence of an angular derivative, his method employing Brownian motion.

Transferred to the strip normalization this becomes the concept of Lipschitz approximations. Let $D$ be a strip domain in the $w$-plane containing the real axis and such that $\partial D$ has non-empty intersection with the first and fourth quadrants. Let $B^+$ be the family of Lipschitz-1 functions $g$ defined for $u > 0$ such that

$$\partial D \cap \{w \mid \Re w > 0,\ \Im w > 0\}$$

lies above the graph of $1/2 + g(u)$. Let $B^-$ be the family of Lipschitz-1 functions $g$ defined for $u > 0$ such that

$$\partial D \cap \{w \mid \Re w > 0,\ \Im w < 0\}$$

lies below the graph of $-1/2 + g(u)$. Let

$$h_+(u) = \operatorname*{l.u.b.}_{g \in B^+} g(u), \qquad h_-(u) = \operatorname*{g.l.b.}_{g \in B^-} g(u).$$

These are again Lipschitz-1 functions. Rodin and Warschawski [211] formulated the following statement.

RW. *For a strip domain D as above under the assumption that*

$$\int_0^\infty \min\bigl(h_+(u), 0\bigr)\, du > -\infty, \qquad \int_0^\infty \max\bigl(h_-(u), 0\bigr)\, du < +\infty$$

*a necessary and sufficient condition that D be conformal at infinity is that*

$$\int_0^\infty \max\bigl(h_+(u), 0\bigr)\, du < +\infty, \quad \int_0^\infty \min\bigl(h_-(u), 0\bigr)\, du > -\infty.$$

Burdzy believed that this followed from his results but this was based on an unjustified and almost certainly unjustifiable assertion [23, p.106]. Rodin and Warschawski [211] proved by the method of the extremal metric the sufficiency part of RW and thought this to be a new proof of that part of Burdzy's result but actually it was a new result. They were unable to prove the necessity part.

Swati Sastry [213] has proved the necessity part using the method of the extremal metric.

**6.** By a comb domain we will mean a strip domain $D$ whose boundary in the right-hand half-plane consists of vertical slits

$$a_n + iv, \quad v \geqslant \frac{1}{2} - \theta_n, \quad n = 1, 2, \ldots, \quad a_n \nearrow +\infty, \quad \theta_n < \frac{1}{2},$$
$$a_m + iv, \quad v \leqslant -\frac{1}{2} + \tilde{\theta}_m, \quad m = 1, 2, \ldots, \quad a_m \nearrow +\infty, \quad \tilde{\theta}_m < \frac{1}{2}.$$

Using the results of the previous section the author [145] proved the following result.

Let

$$\sum_{n=1}^{\infty} \theta_n^2 < \infty, \qquad \sum_{m=1}^{\infty} \tilde{\theta}_m^2 < \infty.$$

Then for $f$ related to $D$ as in Section 4 to be conformal at $+\infty$ it is necessary and sufficient that

$$\sum_{n=1}^{\infty} (a_{n+1} - a_n)^2 < \infty, \qquad \sum_{m=1}^{\infty} (a_{m+1} - a_m)^2 < \infty.$$

Rodin and Warschawski [212] proved the sufficiency part of a weaker result but were unable to prove necessity.

## 9. Harmonic measures and triad modules

**1.** We have already observed several connections between harmonic measures and modules. Most important is the relationship between harmonic measures of a certain type and triad modules. We consider a simply connected domain $D$ of hyperbolic type, an open border arc $\alpha$ of $D$ and a point $P$ in $D$. The combination $(P, \alpha, D)$ is called a triad. We denote the harmonic measure of $\alpha$ taken at $P$ with respect to $D$ by $\omega(P, \alpha, D)$. Let $\alpha^*$ be the complementary (closed) border arc of $\alpha$. Let $\Gamma$ be the family of locally rectifiable open arcs in $D - P$ running from $\alpha$ back to $\alpha$ and separating $P$ from $\alpha^*$. Its module is denoted by $m(P, \alpha, D)$ and is called a triad module. Evidently there is a strictly monotone increasing function from the interval $[0, 1)$ to the ray $[0, \infty)$ which relates $\omega(P, \alpha, D)$ to $m(P, \alpha, D)$. The concept of triad module was introduced by Beurling and Ahlfors and presented in Ahlfors' 1947 lectures. The term triad module first appeared in [102].

Triad modules satisfy a two-constant theorem as follows.

Let $(P, \alpha, D)$ be a triad, $u(z)$ a subharmonic function on $D$ which satisfies

$$\overline{\lim_{z \to \zeta}}\, u(z) \leqslant a, \quad \zeta \in \alpha,$$

$$\overline{\lim_{z \to \zeta}}\, u(z) \leqslant b, \quad \zeta \in \alpha^*,$$

with $a < b$ and $a < u(P) < b$. Let $S(a, b)$ be the strip in the $(u, v)$-plane defined by $a < u < b$ and $g(a)$ its boundary arc $u = a$. Then

$$m(P, \alpha, D) \leqslant m\big(u(P), g(a), S(a, b)\big).$$

**2.** Gaier [39] raised the following problem. Let $D$ be the unit disc $|z| < 1$, $\alpha$ a half-open arc in $D$ with end points $\zeta \in D$, $\zeta \neq 0$, and 1. Let $G = D - \{\alpha\}$. Find the greatest lower bound of $\omega(0, \alpha, D)$. The author [125] pointed out that the problem is better formulated as follows. Let $F$ be the closed unit disc $|z| \leqslant 1$, $\alpha$ an arc in $F$ with end points $\zeta \in D$ and 1 not containing 0. Let $G_\alpha$ be the component of $D - \{\alpha\}$ containing 0. Find the minimum of $\omega(0, \alpha, G_\alpha)$. He solved the problem completely using triad modules and the Fundamental Theorem for modules. He first solved the problem when the change in argument on $\alpha$ sensed from 1 to $\zeta$ has a given value. The result is as follows.

Let $D$ denote the open unit disc in the $z$-plane, $F$ the closed unit disc. Let $\alpha$ be an arc in $F - \{0\}$ with end points $\zeta \in D$ and 1, sensed from 1 to $\zeta$. Let the change of argument $\Delta_\alpha(\arg z)$ have an assigned value. Let $G_\alpha$ be the component of $D - \{\alpha\}$ containing 0, $\tilde{\alpha}$ the border entity of $G_\alpha$ arising from $\alpha$. Then

$$\omega(0, \tilde{\alpha}, G_\alpha) \geqslant \omega(0, \alpha^*, G_{\alpha^*}),$$

where $\alpha^*$ is a competing arc uniquely determined as follows. There is an unique point $e^{i\chi}$, $\chi$ real, such that the quadratic differential

$$Q(z)\,dz^2 = c\left(z - e^{i\chi}\right)\left[z(z-\zeta)\left(z - \bar{\zeta}^{-1}\right)(z-1)\right]^{-1} dz^2$$

with constant $c$ ($\neq 0$) is real on the unit circumference and $\alpha^*$ consists of a trajectory of $Q(z)\,dz^2$ on $|z| = 1$ from 1 to $e^{i\chi}$ and a trajectory in $D$ from $e^{i\chi}$ to $\zeta$ together with their end points. Equality can occur only if $\alpha$ coincides with $\alpha^*$. The point $e^{i\chi}$ can coincide with 1 only if $\zeta$ is real and positive and $c$ is negative.

The author then proved that in the solution of the original problem $|\Delta_\alpha(\arg z)|$ must be minimal. This is uniquely determined unless $\zeta$ is real and negative when there are two extremal configurations related by conjugation.

**3.** Fuchs [16] raised the problem of finding the greatest lower bound of the harmonic measure at the origin of a set in $|z| \leqslant 1$ which meets every radius. If the set is restricted to be a continuum the result of the proceeding section readily gives a characterization of the extremal [135].

Let $D$ be the open unit disc in the $z$-plane, $F$ the closed unit disc and $C$ a continuum in $f$ not containing the origin which meets every radius of $f$. Let $G$ be the component of $D - C$ containing the origin, $\alpha$ the border entity of $G$ determined by $C$. Then $\omega(0, \alpha, G)$ attains its minimal value in the following context: $\widetilde{C}$ consists of an arc $\lambda$ on $|z| = 1$ of angular measure at least $\pi$ with end points 1 and $e^{i\chi}$, $\pi \leqslant \chi < 2\pi$, plus a trajectory of the quadratic differential

$$Q(z)\,dz^2 = c\left(z - e^{i\chi}\right)\left[z(z-r)\left(z - r^{-1}\right)(z-1)\right]^{-1} dz^2,$$

$0 < r < 1$, together with its end points at $e^{i\chi}$ and $r$ where $c$ is such that $Q(z)\,dz^2 > 0$ on the open arc $\operatorname{int}\lambda$. The minimum occurs only for continua obtained from the preceding by rotation about the origin and reflection in the real axis.

Marshall and Sundberg [190] proved that $e^{i\chi} = -1$ and $r = 7 - 4\sqrt{3}$. The author [143, 146] has given two much simpler proofs.

Solynin [218] extended the previous results considering a sequence of similar problems where the competing continua are to have change of argument $\pi n$ ($n$ integer $\neq 0$). His solutions were of the previous form but depended on constants which were determined implicitly from certain equations. Thus he obtained an analytically implicit solution.

The author [149] has given a simpler treatment by the method of [146] and obtained a geometrically explicit solution.

**4.** Gaier [39] formulated a second problem as follows. Let $\zeta_1$, $\zeta_2$ be points of the open unit disc $D$, $C$ a continuum containing them and not containing the origin. Find the greatest lower bound of the harmonic measure of $C$ at 0. Once again the problem is better formulated in terms of the closed unit disc.

Liao [186] treated this problem by considering a continuum of problems where the doubly-connected domain obtained by deleting $C$ from $D$ is to have a given module. In this context there is also a corresponding maximum problem.

Solynin [215] considered a similar continuum of problems for Gaier's first problem parametrized by the inner conform radius of $G_\alpha$ with respect to the origin. Once again there is also a maximum problem.

**5.** FitzGerald, Rodin and Warschawski [36] proved by elementary means that if a continuum in $|z| \leqslant 1$ subtends an angle $\phi \leqslant \pi$ at the origin then its harmonic measure as above at the origin is at least $(1/(2\pi))\phi$, i.e., at least equal to the harmonic measure of an arc of angle $\phi$ on the circumference. This result is easily proved by the method of the extremal metric [135].

The result is still true for some angles greater than $\pi$ but not for angles very near to $2\pi$. It is readily seen that there is a value $\phi_0$, $\pi < \phi_0 < 2\pi$, such that the above result holds for $\phi \leqslant \phi_0$ but fails for $\phi > \phi_0$. Solynin [218] gave some numerical estimates for $\phi_0$ and showed that for $\phi < 2\pi$ a boundary arc of angle $\phi$ gives a local minimum for the harmonic measure.

Marshall and Sundberg [191] have given very detailed numerical computations concerning the dividing value $\phi_0$.

**6.** The following result is due to Hall [67].

Let $E$ be a set consisting of a finite number of arcs in the right-hand half-plane $H$ which bound with imaginary axis a domain $D$. Let $\omega(z, E, D)$ be the harmonic measure of $E$ at $z$ with respect to $D$. Let $E^*$ denote the circular projection (centre the origin) of $E$ onto the positive imaginary axis. Let $\omega(z, E^*, H)$ denote the corresponding harmonic measure. Then there exists an absolute constant $k$, $2/3 < k \leqslant 1$, such that

$$\omega(x + \mathrm{i}y, E, D) \geqslant k\omega(x + \mathrm{i}y, E^*, H).$$

It has been shown [70] that actually $k < 1$. However Gaier [40] has shown that if $E$ consists of a single arc with end point at 0 the result is valid with $k = 1$. Using triad modules and symmetrization the author [127] has obtained a number of improvements of this result. Solynin [218] has shown that the requirement that such $E$ have an end point at 0 is unnecessary.

## 10. Application to non-univalent functions

**1.** Most methods developed to treat the theory of univalent regular functions apply only in that context or in slightly modified situations. The method of the extremal metric can be applied in a wide variety of problems in geometric function theory.

The simplest situation occurs when one can define an appropriate metric on the image Riemann surface of a regular or meromorphic function, often by elevating a metric from the corresponding base surface.

In the following result due to Kôtaro Oikawa and the author [156] we use that type of definition.

Let $S$ denote the half-strip $a < x < b$, $y > 0$. Let $T$ be a subset of $S$, $T_\lambda$ its subset with $y \geqslant \lambda$. For $f$ defined on $S$ the cluster set $\mathcal{C}(f, T, \sigma)$ of $f$ on $T$ at $\sigma$, the boundary point of $S$ at infinity, is defined to be $\bigcap_{x>0} \mathrm{Cl}\, f(T_\lambda)$ where Cl denotes closure on the sphere. $\mathcal{F}(S)$ denotes the family of functions meromorphic on $S$ for which the Riemann image has finite spherical area.

Let $f \in \mathcal{S}$. Let $T$ be a closed subset of $S$ such that for fixed $L$ ($> 0$) every rectangle $a < x < b$, $Y \leqslant y \leqslant Y + L$, $Y > 0$, contains a subcontinuum of $T$ of diameter at least $\delta$ for a certain positive $\delta$. Let $U$ be a subset of $a' \leqslant x \leqslant b'$, $y > 0$ with $a < a' < b' < b$. Then

$$\mathcal{C}(f, U, \sigma) \subset \mathcal{C}(f, T, \sigma).$$

For $x$ in $[a', b']$ the spherical distance of $f(x + \mathrm{i}y)$ from $\mathcal{C}(f, T, \sigma)$ tends uniformly to zero as $y$ tends to infinity.

**2.** The third part of Ahlfors' thesis [1] contains the proof of the Denjoy Conjecture. By an asymptotic value of an integral function is meant a finite value $a$ such that, for a path $z(t)$, $0 \leqslant t \leqslant 1$, with $z(1) = \infty$, $\lim_{t\to 1} f(z(t)) = a$. The conjecture states that an integral function of order $k$ has at most $2k$ distinct asymptotic values. Ahlfors' proof transferred the problem to the context of strip domains and used his distortion theorem.

Actually the result applies to subharmonic functions and the function needs only to be bounded on the paths not necessarily having a limit. In this context the result takes the following form.

DENJOY CONJECTURE. *Let $A_j$, $j = 1, \ldots, n$, be arcs on the sphere joining the origin and the point at infinity pairwise disjoint apart from these points with $A_j$, $A_{j+1}$ ($A_{n+1} = A_1$), $j = 1, \ldots, n$, bounding a domain $D_j$ which meets no $A_k$. Let $u$ be subharmonic in the plane, bounded on the $A_j$, $j = 1, \ldots, n$, and unbounded in each domain $D_j$. Let $\sigma$ be the greatest lower bound of numbers $s$ such that $u(z) = \mathrm{O}(r^s)$, $|z| = r$. Then $\sigma \geqslant \frac{1}{2}n$.*

In the set $|z| \geqslant r_0$, $r_0 > 0$, there will be subarcs $A'_j$ of the $A_j$, $j = 1, \ldots, n$, running from $|z| = r_0$ to the point at infinity and dividing $|z| > r_0$ into domains $D'_j \subset D_j$, $j = 1, \ldots, n$. On the boundary of $D'_j$ in the plane we will have $u(z) < a$ for a suitable value $a$. In $D'_j$ there will be a point $P_j$ with $u(P_j) > a$. Let $\beta = \max_{j=1,\ldots,n} |P_j|$. For $r > \beta$ there will be on $|z| = r$ in $D'_j$ a crosscut $\alpha^*_j(r)$ which separates $P_j$ and the boundary arc of $D'_j$ on

$|z| = r_0$ from the point at infinity. Let the component of $D'_j - \alpha^*_j(r)$ containing $P_j$ be denoted by $D_j(r)$. Let the boundary arc of $D_j(r)$ complementary to $\alpha^*_j(r)$ be denoted by $\alpha_j(r)$. Let $\rho_j(z,r)|dz|$ be the extremal metric in the module problem giving the triad module $m(P_j, \alpha_j(r), D_j(r))$, $j = 1, \dots, n$. Let $K(\beta, r)$ be the circular ring $\beta < |z| < r$. In $K(\beta, r)$ we take the metric $\rho(z,r)|dz|$ with

$$\rho(z,r) = \begin{cases} \frac{1}{n}\rho_j(z,r), & z \in K(\beta,r) \cap D_j(r),\ j = 1, \dots, n, \\ 0 & \text{elsewhere in } K(\beta,r). \end{cases}$$

It is admissible for the module problem for $K(\beta, r)$. Thus

$$\frac{1}{2\pi}\log\frac{r}{\beta} \leqslant \frac{1}{n^2}\sum_{j=1}^{n} m\big(P_j, \alpha_j(r), D_j(r)\big)$$

and for $s > \sigma$

$$u(z) \leqslant a \quad \text{on } \alpha_j(r),$$
$$u(z) \leqslant r^s \quad \text{on } \alpha^*_j(r).$$

By the two-constant theorem

$$\sum_{j=1}^{n} m\big(P_j, \alpha_j(r), D_j(r)\big) \leqslant \sum_{j=1}^{n} m\big(u(P_j), g(a), S(a, r^s)\big)$$
$$\leqslant \frac{n}{\pi}\log r^s + \mathrm{O}(1).$$

Passing to the limit we have $s \geqslant n/2$ thus $\sigma \geqslant n/2$.

**3.** Ahlfors [1] gave a generalization of the Denjoy Conjecture in which he imposed a condition on the behavior of the asymptotic paths. In the collected works [7] he commented that Hayman had pointed out to him that his proof was inadequate and that Hayman had given a proof so lengthy and complicated that the validity of the result appeared to be almost some kind of "freak". Not at all. The author [134] using a slightly more sophisticated version of the proof of the preceding section has given a brief and very natural proof. The result is as follows.

If we assume in addition that for the paths $A_j$ parametrized by $z(t)$, $0 \leqslant t \leqslant 1$, $z(t) \to \infty$ at $t \to 1$

$$\overline{\lim_{t\to 1}}\frac{|\arg z(t)|}{\log|z(t)|} = \lambda$$

then $\sigma \geqslant \frac{1}{2}n(1 + \lambda^2)$.

The author [122] also has given a more general form of the Denjoy Conjecture and a similar generalization of the Phragmén–Lindelöf Theorem.

**4.** Kennedy [165] gave an extended form of the previous results. Let $A_j$, $D_j$, $j = 1, \ldots, n$, be as above. On $\{|z| = r\} \cap D_j$ there will be open arcs $\alpha_1^{(j)}(r)$ and $\alpha_2^{(j)}(r)$ each of which divides $D_j$ into two domains with the origin and the point at infinity as respective boundary points where $\alpha_1^{(j)}(r)$ separates the origin from all other such and $\alpha_2^{(j)}(r)$ separates the point at infinity from all other such ($\alpha_1^{(j)}(r)$ and $\alpha_2^{(j)}(r)$ may coincide). For $0 < r < R$ let $Q_{kl}^{(j)}(R, r)$, $k, l = 1, 2$, denote the quadrangle bounded by $\alpha_k^{(j)}(R)$, $\alpha_l^{(j)}(r)$ and a pair of arcs on $A_j$, $A_{j+1}$. Let $M_{kl}^{(j)}(R, r)$ denote the module of $Q_{kl}^{(j)}(R, r)$ for the family of curves joining the latter pair of sides.

The following is a slightly modified form of a result stated by Kennedy.

Let $u_j(z)$, $j = 1, \ldots, n$, be a function subharmonic in $D_j$ such that

$$\overline{\lim_{z \to \zeta}}\, u_j(z) \leqslant 0,$$

$z \in D_j$, $\zeta$ finite on $A_j$ or $A_{j+1}$ while

$$u_j(z) > 0$$

for some $z \in D_j$. Let

$$\sigma_j(r) = \underset{D_j \cap \{|z| = r\}}{\text{l.u.b.}}\, u_j(z).$$

Let

$$\sigma(r) = \max \sigma_j(r), \quad j = 1, \ldots, n.$$

Assume

$$\underline{\lim}_{r \to \infty} \frac{\sigma(r)}{r^{n/2}} < \infty.$$

Then as $r$ tends to infinity, $j = 1, \ldots, n$,

$$\log \sigma_j(r) = \lambda_j + \pi M_{kl}^{(j)}(r, r_0) + \mathrm{o}(1)$$

for $k, l = 1, 2$, fixed $r_0 > 0$ and an appropriate constant $\lambda_j$,

$$\log \sigma_j(r) - \frac{1}{2} n \log r = \mathrm{o}\big((\log r)^{1/2}\big),$$

$$\prod_{j=1}^{n} \sigma_j(r) \sim \gamma r^{n^2/2}$$

for an appropriate positive constant $\gamma$.

For $r$ sufficiently large $\alpha_1^{(j)}(r)$, $\alpha_2^{(j)}(r)$ coincide.

For a well-defined branch of $\arg z$ on $A_j$, $j = 1, \ldots, n$,

$$\arg z = o\big(\log|z|\big)^{1/2}$$

as $z \to \infty$ on $A_j$.

Kennedy was able to prove his corresponding result only under an additional assumption [165]. The author [122] gave the first correct proof of this result.

**5.** The third manner of applying the method of the extremal metric to non-univalent functions is by the method of simple coverings. In suitable circumstances it is possible to obtain an admissible metric on the Riemann image for a function by elevating a metric on the base surface onto only part of the image. A simple but typical example is given by the following proof of a well-known result.

Let $f$ be regular in the ring $\Delta$: $1 < |z| < r$ and satisfy the bounds $1 < |f(z)| < R$ with

$$\left|\frac{1}{2\pi \mathrm{i}}\int_C \frac{f'(z)}{f(z)}\,dz\right| = n$$

(integer $\neq 0$) where $C$ is a Jordan contour separating $|z| = 1$, $|z| = r$. Then $r^n \leqslant R$ with equality only if $f$ maps $\Delta$ onto an $n$-fold unbranched unbordered covering surface of $1 < |w| < R$.

Over every segment $w = \sigma\,\mathrm{e}^{\mathrm{i}\phi}$, $\phi$ fixed, $1 < \sigma < R$, not covered by a branch point there lie in the Riemann covering surface image of $\Delta$, $n$ segments each the image of a curve running between the boundary circumferences of $\Delta$. Taking on them the metric $(1/(2\pi n))|dw|/|w|$ and transferring it back to $\Delta$ this can be done in such a way that the metric so obtained is measurable and so admissible in the module problem for $\Delta$. Thus

$$\frac{1}{2\pi}\log r \leqslant \frac{1}{n}\,\frac{1}{2\pi}\log R \quad \text{and} \quad r^n \leqslant R.$$

The equality statement follows by a standard extremal metric argument.

Nobuyuki Suita and the author [159,161] (see also [126]) have extended this result in a far-reaching manner.

Let $\mathcal{R}$ be an open Riemann surface with a regular partition $(A, B)$ of its ideal boundary, $A$, $B$ not void. Let $\{\mathcal{R}_\nu\}$ be an exhaustion of $\mathcal{R}$ by canonical subdomains with boundaries divided into sets of contours $\alpha_\nu$, $\beta_\nu$ corresponding to $A$, $B$ where the $\alpha_\nu$ are to be sensed positively, the $\beta_\nu$ sensed negatively with respect to $\mathcal{R}_\nu$. Let $f$ be a regular function on $\mathcal{R}$, $m$, $n$ integers, $m > n$. Let $P_m(\alpha_\nu)$ denote the set of points of the sphere for which $f(\alpha_\nu)$ has index at least $m$, $Q_n(\beta_\nu)$ the set for which $f(\beta_\nu)$ has index at most $n$. $\mathrm{Cl}\,P_m(\alpha_\nu) \subset \mathrm{int}\,\mathrm{Cl}\,P_m(\alpha_\mu)$ and $\mathrm{Cl}\,Q_n(\beta_\nu) \subset \mathrm{int}\,\mathrm{Cl}\,Q_n(\beta_\mu)$ for $\mu < \nu$. If all $P_m(\alpha_\nu)$, $Q_n(\beta_\nu)$ are non-void we have disjoint closed sets $P_m(A) = \bigcap_\nu \mathrm{Cl}\,P_m(\alpha_\nu)$ and $Q_n(B) = \bigcap_\nu \mathrm{Cl}\,Q_n(\beta_\nu)$.

Let $m(\mathcal{R})$ denote the module of the family of rectifiable cycles on $\mathcal{R}$ separating $A$ and $B$. Let $f$ be a regular function on $\mathcal{R}$ for which there are non-void sets $P_m(A)$, $Q_n(B)$. Let $\Delta$

denote the union of the (finite number of) domains in the complement of $P_m(A) \cup Q_n(B)$ which have points of both sets on their boundary. Let $m(\Delta)$ denote the module of the family of rectifiable cycles in $\Delta$ separating $P_m(A)$ and $Q_n(B)$. Then

$$(m-n)m(\mathcal{R}) \leqslant m(\Delta).$$

If $m(\mathcal{R})$ is finite equality can occur only if $f$ is a $(m-n, 1)$ map of $\mathcal{R}$ onto $\Delta$ apart possibly from a relatively closed subset of $\Delta$ of logarithmic capacity zero. Further the indices of $f(\alpha_\nu)$, $f(\beta_\nu)$ with respect to each point of $P_m(A)$, $Q_n(B)$ respectively would be equal to $m$ and $n$ for each $\nu$.

**6.** The author [128] has proved the following result by the method of simple coverings, improving earlier results due to R. Nevanlinna [196] and Ahlfors [2].

Let $D$ be a domain in the $z$-plane which has as one boundary component a Jordan curve $J$ composed of two arcs $L_1$, $L_2$ joining the lines $x = x_1$, $x = x_2$ $(x_1 < x_2)$, certain arcs in the strip $x_1 \leqslant \Re z \leqslant x_2$ each running from $x = x_j$, $j = 1, 2$, back to that line, denoted in their totality by $L_3$ and certain arcs outside the strip $x_1 < \Re z < x_2$ each running from $x = x_j$, $j = 1, 2$, back to that line, denoted in their totality by $L_4$ (either class might include in particular a segment on one of these lines). Let the finite domain bounded by $J$ be denoted by $\Delta$ and let $D$ be a subset of $\Delta$. Let $f$ be a function regular in $D$ such that for positive numbers $a, b$ its cluster set on $L_1$ is contained in $\Im w \leqslant -b$, its cluster set on $L_2$ is contained in $\Im w \geqslant b$ and its cluster set on $L_3$ and on each boundary component of $D$ other than $J$ is contained in $\Im w \leqslant -b$ or $\Im w \geqslant b$. Further let the image of $D$ contain no point satisfying $\Re w \leqslant -a$, $-b \leqslant \Im w < b$ or $\Re w \geqslant a$, $-b \leqslant \Im w \leqslant b$. Let the intersection $\mathfrak{S}(x)$ of $D$ with the line of abscissa $x$, $x_1 < x < x_2$, have measure $\Theta(x)$. Then

$$\int_{x_1}^{x_2} \frac{dx}{\Theta(x)} \leqslant \frac{a}{b}.$$

Equality can occur if and only if $D$ is a rectangle

$$x_1 < x < x_2, \qquad c < y < c + \frac{b}{a}(x_2 - x_1)$$

and $w = f(z)$ $(w = u + iv)$ maps $D$ linearly onto the rectangle

$$-a < u < a, \qquad -b < v < b.$$

**7.** There is one further case where the method of the extremal metric can be applied to non-univalent functions, namely in the theory of multivalent functions. This aspect is deferred to the next section.

## 11. Multivalent functions

**1.** A function regular in a domain $D$ is said to be $p$-valent ($p$ a positive integer) if it takes no value more than $p$ times in $D$. One can also consider functions which are $p$-valent in an average sense. There are two particularly important cases.

A function $f$ regular on a domain $D$ is said to be circumferentially mean $p$-valent if on the Riemann image $\mathcal{R}$ of $D$ by $f$ the total angular measure of the open arcs on $\mathcal{R}$ covering $|w| = r$, all $r > 0$, is at most $2\pi p$.

A function $f$ regular on a domain $D$ is said to be areally mean $p$-valent if on the Riemann image $\mathcal{R}$ of $D$ by $f$ the total area of $\mathcal{R}$ covering the disc $|w| < R$ is at most $p\pi R^2$.

**2.** One can consider in particular the class $F_p$ of functions $f$ regular and circumferentially mean $p$-valent in $|z| < 1$ whose expansion about the origin begins

$$f(z) = z^p + A_{p+1}z^{p+1} + \cdots .$$

The symmetrization results of Section 5 are readily extended to Riemann domains [101] and using this it can be shown that the class $F_1$ has all the same basic properties as the class $S$. These results are presented in detail together with consequent results for the class $F_p$ in [92] and [101] and will not be reproduced here.

One important result was not included. Gronwall [63] considered the problem of finding for functions in $S$ the maximum of $|f(r\,\mathrm{e}^{\mathrm{i}\theta})|$, $0 < r < 1$, $\theta$ real, for functions with given $|A_2|$, $0 \leqslant |A_2| < 2$. The corresponding minimum problem is elementary. Historical details of the problem are found in [86].

The author solved this problem in [86], proving the following result.

Given $c$, $r$, $0 \leqslant c < 2$, $0 < r < 1$, there exists a unique function $f(z, r, c)$ in $S$ with second coefficient $c$ mapping $|z| < 1$ onto an admissible domain for a quadratic differential with a triple pole at the origin, a simple pole at $b > 0$, a simple pole at the point at infinity, a zero on the real axis not on $[0, b]$ (exceptionally the last two may cancel out) and positive on $(0, b)$. If $f \in F_1$ has expansion about the origin

$$f(z) = z + A_2z^2 + \cdots$$

with $|A_2| = c$

$$\left|f\left(r\,\mathrm{e}^{\mathrm{i}\theta}\right)\right| \leqslant f(r, r, c),$$

$0 < r < 1$, $\theta$ real.

The result is proved in [86] for functions in $S$ but the proof extends at once to functions in $F_1$.

**3.** As observed above the Koebe $\frac{1}{4}$-theorem holds for functions in $F_1$. Spencer [219] considered the same problem for areally mean 1-valent functions. He was able to prove that the image covered a circle centre the origin of radius $1/7$. The author [98] proved the result with value $1/4$, in fact proved the following stronger result.

Let $f$ be regular in $|z| < 1$ with $f(0) = 0$, $f'(0) = 1$ and map $|z| < 1$ onto a Riemann surface $\mathcal{F}$ covering the $w$-plane. $f$ is called logarithmically areally mean 1-valent if $\mathcal{F}$ covers some disc $|w| < R_0$ ($R_0 > 0$) simply, while the portion of $\mathcal{F}$ over every circular ring $R_0 < |w| < R$ has logarithmic area at most $2\pi \log(R/R_0)$.

The result is the following.

Let $f$ be regular and logarithmically areally mean 1-valent for $|z| < 1$ with $f(0) = 0$, $f'(0) = 1$. Then $f$ takes in $|z| < 1$ every value of modulus less than $1/4$ and omits a value of modulus $1/4$ only if it is one of the functions $z(1 + z\,\mathrm{e}^{\mathrm{i}\theta})^{-2}$, $\theta$ real.

**4.** Hayman [69] discussed multivalent functions utilizing integrals involving the quantity $p(R)$ defined as follows.

Let $f$ be regular in $|z| < 1$ and let $n(w)$ denote the number of roots in $|z| < 1$ of the equation $f(z) = w$. Let

$$p(R) = \frac{1}{2\pi} \int_0^{2\pi} n\big(R\,\mathrm{e}^{\mathrm{i}\phi}\big)\, d\phi.$$

Kôtaro Oikawa and the author [152] proved the following result using the method of the extremal metric.

Let $f$ be regular in $|z| < 1$ with at most $q$ zeros in $|z| < s$, $s \leqslant 1$, and have expansion about the origin

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Let $\mu_q = \max_{\nu \leqslant q} |a_\nu|$, $R_1 = (q+2)\,2^{q-1}\mu_q$, $M(r, f) = \max_{|z|=r} |f(z)|$.

Then

$$\int_{R_1}^{M(r,f)} \frac{dR}{R\,p(R)} < 2\log\frac{1+r}{1-r} + \pi^2 + \log\frac{1}{2}.$$

From this was derived the following result, a major improvement of a result of Hayman. In the preceding notation, if $f(z)$ is areally mean $p$-valent in $|z| < 1$

$$M(r, f) < A(p)\mu_p(1-r)^{-2p}, \quad 0 < r < 1,$$

with

$$A(p) = (p+2)2^{p-1}\exp\left(p\pi^2 + \frac{1}{2}\right).$$

**5.** Eke [29,30] studied the behavior of areally mean $p$-valent functions with regard to directions of maximal growth. His method was to map suitable neighborhoods of the points on $|z| = 1$ corresponding to directions of maximal growth on parallel strips and to use analogues of the Ahlfors Distortion Theorem with integrals of the form $\int dR/(Rp(R))$ replacing the expression in that theorem.

Kôtaro Oikawa and the author [151] have given an extensive generalization of these results by considering the class of slowly increasing unbounded harmonic functions. Confining ourselves to the special case of the domain $\Omega$: $r_0 < |z| < 1$, $0 < r_0 < 1$, we take $u$ non-constant harmonic on $\Omega$ with level sets $l(c)$ where $u(z) = c$ and

$$\Theta(c) = \int_{l(c)} |{*du}|$$

($*du$ is the conjugate differential of $du$). If $l(c) = 0$ we take $\Theta(c) = 0$. For $-\infty \leqslant a < b \leqslant \infty$, $\Gamma(a, b) = \{l(c) \mid a < c < b\}$. If it is non-void its module is denoted by $\mu(a, b)$ which equals $\int_a^b dc/\Theta(c)$. We consider functions for which there exists $a$ such that $\int_a^b dc/\Theta(c) < \infty$ for every $b > a$. The quantity $2\pi \int_a^b dc/\Theta(c)$ plays the role of $\int_{R_1}^{R_2} dR/(R\,p(R))$ for multivalent functions.

Typical results are the following.

If a harmonic function $u$ defined in $\Omega$ satisfies

$$\mu(a, \infty) = \infty, \qquad \overline{\lim_{r\to 1}}\, u\left(r\,\mathrm{e}^{\mathrm{i}\phi}\right) = \infty$$

for some $a \in u(\Omega)$ and $\phi$ real, then as $z \to \mathrm{e}^{\mathrm{i}\phi}$ in a Stolz domain the uniform limit

$$\lim\left(\mu\left(a, u(z)\right) - \frac{1}{\pi}\log\frac{1}{|z - \mathrm{e}^{\mathrm{i}\phi}|}\right) = \alpha, \quad -\infty \leqslant \alpha < \infty, \text{ exists.}$$

If a harmonic function $u$ defined in $\Omega$ satisfies $[a, \infty) \subset u(\Omega)$ and

$$\overline{\lim_{r\to 1}}\left(\mu\left(a, r\,\mathrm{e}^{\mathrm{i}\phi}\right) - \frac{1}{\pi}\log\frac{1}{1 - r}\right) > -\infty$$

for some $a$ and $\phi$

$$\mu\left(a, u(z)\right) = \mathrm{o}\left(\left(\log\frac{1}{1 - |z|}\right)^{1/2}\right)$$

uniformly as $|z| \to 1$, $u(z) > a$, on any sector whose closure relative to $\Omega$ is contained in $\Omega - \{z \mid \arg z = \phi\}$.

If we define $m(r) = \max_{|z|=r} u(z)$, $r_0 < r < 1$, we say $u$ attains maximum growth if

$$\overline{\lim_{r\to 1}}\left(\mu\left(a, u\left(r\,\mathrm{e}^{\mathrm{i}\Phi}\right)\right) - \frac{1}{\pi}\log\frac{1}{1 - r}\right) > -\infty.$$

A principal result is the following.

A harmonic function $u$ on $\Omega$ satisfying $[a,\infty) \subset u(\Omega)$ for some $a$ attains maximum growth if and only if it attains maximum growth in one direction $e^{i\phi}$, $\phi$ real, i.e.,

$$\overline{\lim_{r\to 1}}\left(\mu\left(a, u\left(r\,e^{i\phi}\right)\right) - \frac{1}{\pi}\log\frac{1}{1-r}\right) > -\infty.$$

The direction $e^{i\phi}$ is uniquely determined and finite limiting values

$$\lim_{r\to 1}\left(\mu\left(a, m(r)\right) - \frac{1}{\pi}\log\frac{1}{1-r}\right),$$
$$\lim_{r\to 1}\left(\mu\left(a, u\left(r\,e^{i\phi}\right)\right) - \frac{1}{\pi}\log\frac{1}{1-r}\right)$$

exist and coincide and the growth in directions other than $e^{i\phi}$ is relatively slower in the sense that

$$\mu\left(a, u(z)\right) = o\left(\left(\log\frac{1}{1-|z|}\right)^{1/2}\right)$$

uniformly as $|z| \to 1$, $u(z) > a$, on every sector whose closure does not meet the ray $\arg z = \phi$.

The authors also treated the case of maximal simultaneous growth in more than one direction.

Hamilton [68] gave some further results in this context in the special case that the directions of maximal simultaneous growth are evenly distributed.

Villamor [231] completed these developments by treating the case of general directions.

## 12. Quasiconformal mappings

**1.** In its earliest development the theory of quasiconformal mappings was closely associated with the method of the extremal metric. The first substantial work in this direction is due to Grötzsch [46,50,57]. He did not use the term quasiconformal but spoke of nichtkonforme (non-conformal) or möglichst konforme (as conformal as possible) mappings. He did not confine himself to (1, 1) mappings. His basic assumption was that the mapping was a homeomorphism from a plane domain onto a Riemann covering surface which apart from branch point antecedents was continuously differentiable so that an "infinitesimal" circle was mapped onto an "infinitesimal" ellipse with bounded ratio of the axes. The ratio of major to minor axis was to be bounded by a constant $Q$.

In [46] he introduced basic concepts and proved a generalization of the second Picard theorem.

In [50] he solved a number of extremal problems for univalent functions of this type involving areas and distances, generalizing known results for regular functions.

**2.** A conformal entity such as a quadrangle or a doubly-connected domain which is characterized by a module does not in general admit a conformal mapping onto another such. Grötzsch [57] considered for them mappings which he called "möglichst konforme", that is, mappings for which the constant $Q$ in his definition above was minimal. Today these would be called extremal quasiconformal mappings. He gave explicitly a number of such mappings, taking his entities in canonical form. For example, for two rectangles $0 < x < a, 0 < y < b$; $0 < u < c, 0 < v < d$ the extremal mapping is the linear mapping

$$u = \frac{c}{a} x, \qquad v = \frac{d}{b} y.$$

Here one sees clearly the genesis of Teichmüller's developments.

**3.** Teichmüller used the term quasiconformal which had been introduced by Ahlfors. He considered in the first instance homeomorphisms between plane domains which had continuous partial derivatives except at isolated points in both directions. At non-exceptional points he took the modulus of the quotient of the maximal directional derivative by the minimal such and called it the dilatation quotient. If this is bounded the mapping is called quasiconformal. The greatest lower bound $K$ of such bounds is called the maximal dilatation. Actually Teichmüller sometimes allowed more general estimations for the dilatation coefficient and both sense-preserving and sense-reversing homeomorphisms but now it is generally agreed that quasiconformal mappings should be sense-preserving with bounded dilatation coefficient. These definitions extend in a straightforward manner to Riemann surfaces.

If $\mathcal{R}$ is a closed Riemann surface and $Q(z)\,dz^2$ is a regular quadratic differential on $\mathcal{R}$, at a point $P$ which is not a zero of $Q(z)\,dz^2$ a branch of $\zeta = \int_P (Q(z))^{1/2}\,dz$ will map a suitable neighborhood of $P$ onto a rectangle $-a < \xi < a$, $-b < \eta < b$, $\zeta = \xi + i\eta$, and thus provide a local uniformizing parameter. At zeros of $Q(z)\,dz^2$ a slightly more complicated construction applies. If we apply on the $\zeta$-plane the affine mapping $\sigma = K\xi$, $\tau = \eta$ with a corresponding situation for the zeros of $Q(z)\,dz^2$ we obtain, taking on the point set of $\mathcal{R}$ these images as local uniformizing parameters, a new Riemann surface $\tilde{\mathcal{R}}$ and by point identification a homeomorphism $f$ of $\mathcal{R}$ into $\tilde{\mathcal{R}}$ with maximal dilation $K$. Teichmüller [228] proved by the method of the extremal metric that if $\tilde{f}$ is any other quasiconformal mapping of $\mathcal{R}$ onto $\tilde{\mathcal{R}}$ homotopic to the preceding that its maximal dilation is at least $K$ so that $f$ is an extremal quasiconformal mapping.

Teichmüller's remarkable achievement was to recognize that also the converse is true namely that given homeomorphic Riemann surfaces $\mathcal{R}$ and $\tilde{\mathcal{R}}$ for any homotopy class of quasiconformal mappings of $\mathcal{R}$ onto $\tilde{\mathcal{R}}$ there is an extremal quasiconformal mapping determined in the above manner by a quadratic differential and that this mapping is unique except in the case of genus one, the quadratic differential being determined up to a positive constant. In this way one obtains a space consisting of Riemann surfaces with a topological determination which is called Teichmüller space.

In [228] Teichmüller only conjectured the existence result but later [229] gave a proof of this by the continuity method.

These results are readily extended to finite bordered Riemann surfaces.

Teichmüller's uniqueness proof shows that a self-conformal mapping of a closed Riemann surface (other than a torus) homotopic to the identity is the identity. Tamrazov [224] extended this result to an admissible family of domains for a positive quadratic differential for a finite Riemann surface as a sort of supplement to the General Coefficient Theorem.

**4.** Ahlfors [4] in a generally expository paper gave a definition of quasiconformal mappings independent of any differentiability conditions. A homeomorphism is called quasiconformal if there exists a constant $K$ such that for any quadrangle $Q$ and its image $\tilde{Q}$ for a corresponding pair of modules $\mathcal{M}$ and $\tilde{\mathcal{M}}$ we have $\tilde{\mathcal{M}} \leqslant K\mathcal{M}$. The greatest lower bound of acceptable $K$ is again called the maximal dilatation.

Ahlfors also attempted to give a proof of the existence part of Teichmüller's result but it is defective, as appears to be the case with a subsequent attempt at a correction [5].

**5.** The considerations of Section 6 concerning boundary correspondence apply also to quasiconformal mappings. In particular a quasiconformal mapping of the upper half-plane onto itself can be extended to a homeomorphism of the (completed) real axis onto itself. Beurling and Ahlfors [15] gave a necessary and sufficient condition for such a homeomorphism to be the extension of a quasiconformal mapping. However it is not a necessary and sufficient condition for it to be the extension of a quasiconformal mapping of maximal dilatation $K$. The author [137] has given such a condition.

A (sense-preserving) homeomorphism $\phi$ of a Jordan curve $J$ bounding a domain $D$ onto a Jordan curve $J^*$ bounding a domain $D^*$ can be extended to a $K$-quasiconformal mapping of $D$ onto $D^*$ if and only if for every polygon problem $P(a_1, \ldots, a_L)$ for $D$ and for the associated polygon problem $P^*(a_1, \ldots, a_L)$ for $D^*$, where as vertices for the latter we take the images under $\phi$ of the vertices for the former we have for the corresponding modules $M(a_1, \ldots, a_L)$ and $M^*(a_1, \ldots, a_L)$

$$K^{-1}M(a_1, \ldots, a_L) \leqslant M^*(a_1, \ldots, a_L) \leqslant K\, M(a_1, \ldots, a_L).$$

**6.** Beurling [13] proved that for a regular univalent mapping of the unit disc radial limits exist except at most for radii with end points at a set of outer capacity zero (the same being true for a finitely multivalent function) and that the limit can have a given value at most for such a set. The author [94] extended these results to quasiconformal mappings by the use of the method of the extremal metric (Lohwater [188] gave another quite different proof). In this context the author introduced for the first time the interpretation for sets of capacity zero in terms of modules. Pfluger [203] gave an estimate of capacity of sets for non-zero capacity.

In the paper [94] the author considered also quasiconformal mappings of the unit disc onto a Riemann surface and showed that the analogue of Fatou's Theorem (existence of radial limits for bounded functions except at a set of boundary points of measure zero) and the non-constancy of boundary values for non-constant functions except at a set of boundary points of measure zero are equivalent to the (Lebesgue) measurability of the homeomorphic extension of a quasiconformal mapping of the unit disc onto itself. Since Beurling and Ahlfors [15] proved that this extension need not be measurable these results do not hold for quasiconformal mappings.

**7.** A number of authors have studied by other methods functions in $S$ or $\Sigma$ which have quasiconformal extensions.

Deiermann [25,26] treated several such problems by the method of the extremal metric.

**8.** Recently considerable interest has attached to (sense-preserving) homeomorphisms which have some similarity to quasiconformal mappings but satisfy weaker conditions. These conditions are usually expressed in terms of the complex dilatation. If the homeomorphic mapping $f$ has partial derivatives almost everywhere the complex dilatation $\mu$ is defined at points $z$ where $f_z(z) \neq 0$ by

$$\mu(z) = \frac{f_{\bar{z}}(z)}{f_z(z)}.$$

The definition can be extended to points $z$ where $f_z(z) = 0$ by setting $\mu(z) = 0$.

It is well known that the essential least upper bound $\|\mu\|_\infty \leqslant 1$ and that if $\|\mu\|_\infty \leqslant q < 1$ the homeomorphism is a quasiconformal mapping with maximal dilatation $\leqslant (1+q)/(1-q)$.

In the more general case where $\|\mu\|_\infty = 1$ important use is made of inequalities for the modules of the images of circular rings and rectangles involving the complex dilatation. For an ACL mapping of complex dilatation $\mu$ of the circular ring $r_1 < |z| < r_2$, $0 < r_1 < r_2 < \infty$, the inequalities for the module $M(r_1, r_2)$ of the image are, $r = e^{i\phi}$,

$$\int_{r_1}^{r_2} \left( \int_0^{2\pi} \frac{|1 - e^{-2i\phi}\mu(r\,e^{i\phi})|^2}{1 - |\mu(r\,e^{i\phi})|^2}\, d\phi \right)^{-1} \frac{dr}{r} \leqslant M(r_1, r_2)$$
$$\leqslant \left[ \int_0^{2\pi} \left( \int_{r_1}^{r_2} \frac{|1 + e^{-2i\phi}\mu(r\,e^{i\phi})|^2}{1 - |\mu(r\,e^{i\phi})|^2}\, \frac{dr}{r} \right)^{-1} d\phi \right]^{-1}$$

provided the relevant integrals exist. This result is stated in [208] for quasiconformal mappings. For the rectangle $0 < x < a$, $0 < y < b$ the corresponding inequalities for the module $M(a, b)$ of the image for the family of curves joining the images of the vertical sides are, $z = x + iy$,

$$\int_0^b \left( \int_0^a \frac{|1 + \mu(z)|^2}{1 - |\mu(z)|^2}\, dx \right)^{-1} dy \leqslant M(a, b)$$
$$\leqslant \left[ \int_0^a \left( \int_0^b \frac{|1 - \mu(z)|^2}{1 - |\mu(z)|^2}\, dy \right)^{-1} dx \right]^{-1}.$$

In the paper [20] Melkana Brakalova and the author proved, in addition to other results, the following generalization of the Teichmüller–Wittich–Belinskii Theorem.

If $f$ is a sense-preserving ACL homeomorphism of the punctured unit disc $U_0$ onto itself with complex dilatation $\mu$ and

$$\iint_{U_0} \frac{|\mu(z)|}{1 - |\mu(z)|} \frac{dA}{|z|^2} < \infty$$

then $f$ is conformal at $z = 0$.

In the paper [21] the preceding authors gave extensions of these results.

Conversely it is well known that if $\mu(z)$ is a (complex-valued) measurable function defined in the plane with $\|\mu\|_\infty \leqslant q < 1$ the Beltrami equation

$$f_{\bar{z}} = \mu f_z$$

has a solution uniquely determined up to normalization which is a quasiconformal mapping of maximal dilatation $\leqslant (1+q)/(1-q)$. A proof of this result using the method of the extremal metric is found in the book [184]. Melkana Brakalova and the author [22] have generalized this result to the context of functions $\mu$ satisfying weaker conditions, improving results obtained earlier by David by a quite different approach. In particular the following results have been proved.

Let $\Delta$ be a domain in the $z$-plane, $\mu$ a measurable function defined a.e. on $\Delta$ with $\|\mu\|_\infty \leqslant 1$. Suppose that for every bounded measurable set $B \subset \Delta$ there exists a positive constant $\Phi_B$ such that

$$\iint_B \exp \frac{1/(1-|\mu|)}{1+\log(1/(1-|\mu|))}\, dA < \Phi_B$$

and

$$\iint_{(|z|<R)\cap\Delta} \frac{1}{1-|\mu|}\, dA = \mathrm{O}\big(R^2\big), \quad R \to \infty.$$

Then there exists a homeomorphism $f$ of $\Delta$ into the plane which is ACL and whose partial derivatives $f_z$ and $f_{\bar{z}}$ are in $\mathbb{L}^q$ on every compact subset $\Delta$ for every $q < 2$ and which satisfies the Beltrami equation a.e. The partials $f_z$ and $f_{\bar{z}}$ are distributional derivatives.

With $\mu$ and $f$ as above and $\Delta$ the plane let $\hat{f}$ be a homeomorphism of the plane onto itself which has a.e. partial derivatives $\hat{f}_z$, $\hat{f}_{\bar{z}}$ locally in $\mathbb{L}^2$. If $\hat{f}$ satisfies the Beltrami equation a.e. then

$$\hat{f}(z) = af(z) + b,$$

where $a$ and $b$ are constants, $a \neq 0$.

If $\hat{f}$ is a homeomorphism of a domain $\Delta$ onto a domain $\Theta$ with the above properties

$$\hat{f}(z) = \xi\big(f(z)\big),$$

where $\xi$ is a conformal mapping of $f(\Delta)$ onto $\Theta$.

## 13. Various results

In this section we collect a variety of topics which, while individually important do not fall under the purview of the preceding sections.

**1.** Grötzsch [51] treated the following problem.

Given distinct points $z_j$, $j = 1, \ldots, n$, in the plane find a continuum of minimal capacity containing them.

Grötzsch's solution is readily manifested in terms of the ideas of Section 3. The unique extremal configuration consists of trajectories (plus their end points) of a quadratic differential with a double pole at infinity and at most simple poles at the $z_j$.

The author [121] used similar ideas to obtain the solution of the problem of maximizing the sum of pairwise distances for three points in a continuum of capacity one.

The author [119] used similar ideas to consider certain problems of minimal capacity.

Kuz'mina [176–178] considered a related problem for functions regular and univalent in the unit disc which omit two finite values and obtained more detailed results in this context.

**2.** Grötzsch [59] solved the problem of four point interpolation for meromorphic univalent functions in a general schlichtartig domain. Kühnau [167] has given an explicit analytic formula for the domain $|z| > 1$. Avci and Zlotkiewicz [11] rediscovered Grötzsch's result in the case of simply-connected domains. It should be observed that the normalization at infinity for functions in $\Sigma$ does not affect cross ratio.

The author [131] used a method related to but not identical with that of Grötzsch to treat a problem for meromorphic univalent functions in a general schlichtartig domain which omit two values.

**3.** Dugué [27] asserted that if a function is meromorphic in the sphere apart from $n$ essential singularities there can be at most $n + 1$ local exceptional values.

The author [93] proved by the method of the extremal metric that there can be actually $2n$.

Matsumoto [192] used the method of [93] to prove that if $K$ is a set of capacity zero which is the union of a countable number of compact sets there exists a compact set $E$ of capacity zero on the sphere and a function meromorphic on its complement which has an essential singularity at every point of $E$ with $K$ as its set of exceptional values at each singularity.

**4.** In the paper [158] the author showed how a positive quadratic differential on a finite bordered Riemann surface of genus zero can be imbedded in a meromorphic quadratic differential on the sphere. This is closely associated with well known results of de la Vallée Poussin and Julia.

Later Walsh [232] gave by other methods a more explicit result in this direction.

In the paper [100] the author showed how the earlier considerations gave such a theorem and in particular proved the following result.

Let $D$ be a domain on the $z$-sphere with boundary components $B_1, B_2, \ldots, B_\mu$; $C_1, C_2, \ldots, C_\nu$, $\mu\nu \neq 0$. Then there exists a conformal mapping of $D$ onto a domain $\Delta$ in the $Z$-sphere where $\Delta$ is defined by

$$1 < \left|T(Z)\right| < e^{1/\tau},$$

$$T(Z) = \frac{A(Z - a_1)^{M_1}(Z - a_2)^{M_2} \cdots (Z - a_\mu)^{M_\mu}}{(Z - b_1)^{N_1}(Z - b_2)^{N_2} \cdots (Z - b_\nu)^{N_\nu}},$$

$$\sum_{j=1}^{\mu} M_j = \sum_{j=1}^{\nu} N_j = 1, \quad \tau > 0.$$

The exponents $M_j$ and $N_j$ are positive but need not be rational. The locus $|T(Z)| = 1$ consists of $\mu$ mutually disjoint Jordan curves, respective images of the $B_j$ which separate $\Delta$ from the $a_j$; the locus $|T(Z)| = \mathrm{e}^{1/\tau}$ consists of $\nu$ mutually disjoint Jordan curves, respective images of the $C_j$, which separate $\Delta$ from the $b_j$.

**5.** In the paper [114] the author used a modified form of the method of the extremal metric to obtain generalizations of the usual span theorems as well as results for functions defined in a multiply-connected domain for which one boundary component is a rectangle.

**6.** Heins [71] proved that a Riemann surface which can be imbedded in every closed Riemann surface of a fixed positive genus is schlichtartig. The author [129] proved this result by the method of the extremal metric. Other proofs of Heins' theorem have been given but the method of [129] was the only one which succeeded in the work of Jussila [163] on extension of Riemann surfaces.

**7.** Gol'dberg [41] considered the problem of functions meromorphic, regular, rational or polynomial in the unit disc for which the multiplicities with which the values $0, 1, \infty$ are taken are finite and distinct and the maximum modulus for a point at which one of these values is assumed and the greatest lower bound $A_1$, $A_2$, $A_3$, $A_4$ of those quantities for the above classes. He gave in particular an explicit numerical upper bound for $A_1$ and numerical upper and lower bounds for $A_2$. The author [132] used the method of the extremal metric to give an explicit lower bound for $A_1$ (better than Gol'dberg's lower bound for $A_2$) and an upper bound for $A_1$ (better than Gol'dberg's).

**8.** In the context of Section 7.6, Beurling [13] showed that if a function meromorphic in the unit disc $D$ mapped $D$ onto a Riemann surface $S$ of finite spherical area and had a given radial limit $\alpha$ at a set $E$ of boundary points of $D$ and if the surface had a certain restrictive behavior in a neighborhood of $\alpha$ then $E$ would have capacity zero. The author [141] showed that the restriction of finite spherical area is unnecessary and that various weaker conditions on $S$ would lead to the same conclusion. A typical result is as follows.

If $f(z)$ is a non-constant meromorphic function in the unit disc $D$ mapping $D$ onto a Riemann surface $S$ and $\alpha$ is a value such that, for every point $P$ in a closed set $E \subset \partial D$, $\alpha$ is a cluster value of $f$ along every path tending to $P$ in $D$ while for $\zeta$ representing $\alpha$ as the origin and if $S(r)$ denotes the subset of $S$ covering the set with $|\zeta| < r$ and

$$A(R) = \iint_{S(R)} dA$$

with for a constant $A^* > 0$

$$\overline{\lim_{R\to 0}}\, A(R)\bigl(R^2 \log R^{-1}\bigr)^{-1} < A^*$$

then $E$ has capacity zero.

**9.** Jacqueline Ferrand [35] gave a method for defining metrics on a domain $G$ in $\mathbb{R}^n$ by the use of modules of curve families obtaining a quantity denoted by $\lambda_\omega(x, y)$, $x, y \in G$. It is readily seen that $\lambda_\omega(x, y)^{-1/n}$ is a metric on $G$. Vuorinen raised the question whether $\lambda_\omega(x, y)^{-1/(n-1)}$ is itself a metric. In 1987 Vuorinen asked the author whether this would be true even in the case $n = 2$ with $G$ the sphere punctured at two points and not long afterward the author communicated to him a very simple proof by the method of the extremal metric. In the paper [144] the author published this proof as well as an extension to a general domain of finite connectivity on the sphere.

Solynin [217] also gave a proof in the case of the punctured sphere which is much more complicated, even using elliptic functions.

**10.** In the paper [171] Kühnau gave explicit analytic representations for the solutions of numerous extremal problems especially those connected with quadratic differentials.

**11.** Pu [207] gave the value for the module of the basic homotopy class of a Möbius strip but his attempt to extend these considerations to a more general context was unsuccessful as was pointed out by Blatter [18]. A more complete solution was given by Tamrazov [225].

**12.** Marden and Rodin [189] extended the results mentioned in Section 1.4, to open Riemann surfaces. Further results in this direction were given by Minda [193] and Wiener [233].

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CHAPTER 14

# Universal Teichmüller Space


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**Abstract**

We present an outline of the theory of universal Teichmüller space, viewed as part of the theory of $QS$, the space of quasisymmetric homeomorphisms of a circle. Although elements of $QS$ act in one dimension, most results about $QS$ depend on a two-dimensional proof. $QS$ has a manifold structure modelled on a Banach space, and after factorization by $PSL(2, \mathbb{R})$ it becomes a complex manifold. In applications, $QS$ is seen to contain many deformation spaces for dynamical systems acting in one, two and three dimensions; it also contains deformation spaces of every hyperbolic Riemann surface, and in this naive sense it is universal. The deformation spaces are complex submanifolds and often have certain universal properties themselves, but those properties are not the object of this article. Instead we focus on the analytic foundations of the theory necessary for applications to dynamical systems and rigidity.

## Introduction

The origins of this theory lie in the study of deformations of complex structure in spaces of real dimension 2 and the moduli problem for Riemann surfaces. It seems appropriate, therefore, to begin with a brief sketch of how the notion of a Teichmüller space first arose, within this problem of variation of complex structure on a topologically fixed compact Riemann surface. For brevity we shall restrict attention to *hyperbolic* Riemann surfaces, which have as universal covering space the unit disc; the terminology refers to the fact that, via projection form Poincaré's Riemannian metric on the disc, all the surfaces are endowed with a structure of hyperbolic geometry.

The definition of Teichmüller space stands out clearly as a key stage in the struggle to justify, and to make precise, the famous assertion of Riemann [*Theorie der Abel'schen Functionen*, Crelle J., B. 54 (1857)] that the number of (complex) parameters (or 'moduli') needed to describe all surfaces of genus $g \geqslant 2$ up to conformal equivalence is $3g-3$. After preliminary work over many years by a substantial number of eminent mathematicians, including F. Schottky, A. Hurwitz, F. Klein, R. Fricke and H. Poincaré, the crucial new idea was introduced by O. Teichmüller around 1938, following earlier work of H. Grötzsch. One specifies a *topological marking* of the base surface and then considers all homeomorphisms to a target Riemann surface which have the property that they distort the conformal structure near each point by at most a bounded amount, using a precise analytic measure of the distortion to be defined below. Grötzsch (see [37,38]) had used this approach to resolve similar problems in estimating distortion for smooth mappings between plane domains; the term *quasiconformal* was coined by L.V. Ahlfors around 1930 for the class of homeomorphisms to be employed. The method was strengthened, generalized and applied to the case of closed Riemann surfaces with striking effect by Teichmüller, as we indicate below.

A fundamental relationship exists between the quasiconformal homeomorphisms of the hyperbolic disc and the induced boundary maps of the circle, and this lies at the heart of the viewpoint on Teichmüller theory to be presented here: for a general Riemann surface, one must consider not only deformations of complex structure in the interior but also the ways in which the conformal structure may change relative to the boundary. It turns out that both aspects are best studied on the universal covering surface, the unit disc $\Delta = \{|z| < 1\}$: quasiconformal mappings of the disc extend to homeomorphisms of the closed disc and many (but not all) of the properties of a quasiconformal homeomorphism can be expressed solely in terms of the boundary homeomorphism of the circle induced by it.

Let $QS$ be the space of sense-preserving, quasisymmetric self-maps of the unit circle; such maps turn out to be precisely those occurring as the boundary value of some quasiconformal self map of the disc $\Delta$. A map $H : \Delta \to \Delta$ is called *quasiconformal* (sometimes abbreviated to q-c) if $K(H) < \infty$, where $K(H)$ is the essential supremum, for $z \in \Delta$, of the local dilatations $K_z(H)$, and the *local dilatation* $K_z(H)$ at $z$ is defined as

$$K_z(H) = \limsup_{\varepsilon \to 0} \frac{\max_\theta\{|H(z+\varepsilon e^{i\theta}) - H(z)|\}}{\min_\theta\{|H(z+\varepsilon e^{i\theta}) - H(z)|\}},$$

which may be interpreted as the upper bound of local distortion as measured on circles centered at $z$; compare with the definition (18) in Section 2. The set of all possible quasiconformal extensions $H : \Delta \to \Delta$ of a given quasisymmetric map $h$ may be regarded as the *mapping class* of $h$ in the disc, and a mapping $H_0$ is called *extremal* for its class if $K(H_0) \leqslant K(H)$ for every extension $H$ of $h$. This notion of extremality for a mapping (within a homotopy class of quasiconformal maps between two plane regions) was also introduced by Grötzsch (op. cit.), but it was Teichmüller who recognized the significance of extremal maps in the study of deformations of complex structures. He applied them decisively in [71,72], to establish a measure of distance between two marked surfaces: here the upper bound for the local distortion of the mapping over the base surface is to be minimized. A base (hyperbolic) Riemann surface is given as the quotient space $X_0 = \Delta/\Gamma$ of the unit disc under the action of a Fuchsian group $\Gamma$, which is by definition a discrete group of Möbius transformations which are conformal automorphisms of the disc; topologically, $\Gamma$ represents the group of deck transformations of the covering projection from $\Delta$ to $X_0$. Suppose now that we are given a quasisymmetric map $h$ of the circle with the property that the conjugate group $\Gamma_1 = h \circ \Gamma \circ h^{-1}$ is also Fuchsian: the two orbit spaces $\Delta/\Gamma$ and $\Delta/\Gamma_1$ can be viewed as the same topological surface but with different complex structures. The *mapping class* of $h$ for the group $\Gamma$ is the subset of its mapping class (viewed as all q-c self-mappings of the disc extending the map $h$) consisting of those q-c extensions $H$ of $h$ with the property that every element $H \circ \gamma \circ H^{-1}$ of $\Gamma_1$ acts as a Möbius transformation of the disc $\Delta$ to itself. For a Fuchsian group $\Gamma$ that covers a compact Riemann surface, Teichmüller's theorem establishes a profound link between an extremal representative $H$ for a given class and a holomorphic quadratic differential for $\Gamma$ – a complete proof is given in [6]. As a consequence, one may infer that the space of marked deformations of the compact surface $\Delta/\Gamma$ is a complete metric space homeomorphic to a $(6g-6)$-dimensional real cell. The metric is called Teichmüller's metric and the distance between the base surface $X_0 = \Delta/\Gamma$ and the marked surface $X_1 = \Delta/\Gamma_1$, with $\Gamma_1 = H_0 \circ \Gamma \circ H_0^{-1}$, is $\log K(H_0)$, where $H_0$ is extremal in its class. This type of extremal mapping is a feature of continuing interest, partly because of the connection with Thurston's theory of measured laminations on hyperbolic surfaces [10,18,30,40,41,68].

The final ingredient, which makes it possible to construct these holomorphic parameter spaces for all types of Riemann surface, is the relationship between the quasiconformal property and the solutions of a certain partial differential equation. By a fundamental observation of Lipman Bers (see [3]), if $H$ is a quasiconformal self-map of the disc, it satisfies the Beltrami equation

$$H_{\bar{z}}(z) = \mu(z) H_z, \tag{1}$$

where $\mu$, with $\|\mu\|_\infty < 1$, is a measurable complex-valued function on the disc, which represents the complex dilatation at each point of $\Delta$; $\mu$ is often called the *Beltrami coefficient* of $H$. Conversely, by virtue of solvability properties of this equation, $\mu$ determines $H$ uniquely up to postcomposition by a Möbius transformation. By using the analytic dependence of $H = H_\mu$ on its Beltrami coefficient $\mu$, and deploying a construction known as the Bers embedding, each $T(\Gamma)$ is embedded as a closed subspace of the complex

Banach space $B$ of univalent functions on $\Delta$ which have quasiconformal extensions to the sphere – more details are given in Section 2.5. It then follows that $T(\Gamma)$ has a natural structure of complex manifold for *any* Fuchsian group $\Gamma$. Furthermore, each inclusion $\Gamma' \subset \Gamma$ of Fuchsian groups induces a contravariant inclusion of these Teichmüller spaces $T(\Gamma) \subset T(\Gamma')$, which implies that the Banach space $T(1) = B$, which corresponds to the trivial Fuchsian group $\Gamma' = 1 = \langle \mathrm{Id} \rangle$ is *universal* in the sense that it contains the Teichmüller spaces of every hyperbolic Riemann surface $\Delta/\Gamma$.

In the period after World War II, the verification of Teichmüller's ideas and the subsequent rigorous development of the foundational complex analytic deformation theory outlined above by L.V. Ahlfors, L. Bers, H.E. Rauch and their students occupied more than 20 years. The circumstances of Teichmüller's life and particularly his political activities caused much controversy and, coupled with the relative inaccessibility of his publications, this perhaps contributed to some early reluctance to pursue a theory based on his claims; for commentary on mathematical life in Germany under the Third Reich, the reader might consult [74] and the review of Teichmüller's Collected Works [73]. Detailed expositions of this foundational work on moduli are given in [17,30,42,53,57].

In a landmark study of the local complex analytic geometry of Teichmüller space, H.L. Royden [64] showed that when $T(\Gamma)$ is finite dimensional, the complex structure of the space determines its Teichmüller metric. In fact, he proved that Teichmüller's metric coincides with the Kobayashi metric [45], which is defined purely in terms of the set of all holomorphic maps from the unit disc into $T(\Gamma)$. Royden also showed that every biholomorphic automorphism of $T(\Gamma)$ is induced geometrically by an element of the mapping class group, a result which extends to many infinite dimensional Teichmüller spaces; we examine this important rigidity theorem more carefully in Sections 1.8 and 2.8.

The case of compact Riemann surfaces and their deformation spaces calls for techniques involving aspects of surface topology and geometry which will not be considered in this article. Instead, we present a formulation which focusses on the real analytic foundations of the theory, important for applications to real and complex dynamical systems and matters which relate to rigidity. It was observed by S.P. Kerckhoff (see for instance [75]) and later, independently, by S. Nag and A. Verjovsky [58] that the almost complex structure on each $T(\Gamma)$ corresponding to its complex structure is given by the Hilbert transform acting on the relevant space of vector fields defined on the unit circle. This fact indicates that deep results concerning the complex structure of Teichmüller space can be viewed purely as theorems of real analysis. With this principle in mind, we divide the exposition into two parts. The first part concentrates on the real theory of $QS$ and we present the theorems in real terms as far as possible; the basic properties are stated mostly without proof, except in certain cases where an easy real-variable proof is available. The second part of the paper follows closely the outline of the first but brings in the complex analysis: in our view, despite their very real nature, properties of quasisymmetric maps are most easily understood by consideration of their possible two-dimensional quasiconformal extensions.

## 1. Real analysis

### 1.1. *Quasisymmetry*

A *quasisymmetric* map $h$ of an interval $I$ to an interval $J$ is an increasing homeomorphism $h$ for which there exists a constant $M$ such that

$$\frac{1}{M} \leqslant \frac{h(x+t)-h(x)}{h(x)-h(x-t)} \leqslant M \tag{2}$$

for every $x$ and $t>0$ with $x-t$, $x$ and $x+t$ in $I$. It is not hard to prove the quasisymmetric maps form a pseudo-group. That is, if $h$ is quasisymmetric from $I$ to $J$ with constant $M$, then $h^{-1}$ from $J$ to $I$ is quasisymmetric with constant $M_1$ depending only on $M$. Moreover, if $g$ is quasisymmetric from $I_1$ to $I_2$ with constant $M_g$ and $h$ is quasisymmetric from $I_2$ to $I_3$ with constant $M_h$, then $h \circ g$ is quasisymmetric from $I_1$ to $I_3$ with constant $M_{h\circ g}$ depending only on $M_g$ and $M_g$. Also, $h$ is Hölder continuous with Hölder exponent $\alpha$ depending only on $M$. For purposes of illustration, we prove this fact here.

LEMMA 1. *A quasisymmetric map of an interval $I$ to an interval $J$ satisfying condition* (2) *is Hölder continuous.*

PROOF (*See Ahlfors* [1], *pp.* 65 *and* 66). Pre- and post-composition of $h$ by affine maps yields a map $\tilde{h} = A \circ h \circ B$ with the same constant $M$ of quasisymmetry. To show $h$ is Hölder continuous at a point $p$, it suffices to show $\tilde{h}$ is Hölder continuous at 0 and to assume the intervals $I$ and $J$ contain $[0,1]$, and that $\tilde{h}(0)=0$ and $\tilde{h}(1)=1$. By plugging in $x-t=0$, $x=1/2$, $x+t=1$ inequality (2) yields

$$1/(M+1) \leqslant \tilde{h}(1/2) \leqslant M/(M+1).$$

Repeated applications of (2) with $x=1/2^n$ and $t=1/2^n$ yield

$$1/(M+1)^n \leqslant \tilde{h}\big(1/2^n\big) \leqslant \big(M/(M+1)\big)^n.$$

Since $\tilde{h}$ is increasing, this implies that for $1/2^n \leqslant x \leqslant 1/2^{n-1}$,

$$\tilde{h}(x) \leqslant \tilde{h}\big(1/2^{n-1}\big) \leqslant \big(M/(M+1)\big)^{n-1} = \big((M+1)/M\big)\big(M/(M+1)\big)^n.$$

But since $x \geqslant 1/2^n$, the previous inequality implies

$$\tilde{h}(x) \leqslant \big((M+1)/M\big)x^\alpha, \quad \text{where } \alpha = \frac{\log((M+1)/M))}{\log 2}.$$

To finish the proof we note that $h$ and $\tilde{h}$ have the same Hölder exponent $\alpha$. □

A quasisymmetric homeomorphism of the unit circle $S^1 = \{e^{i\theta}: \theta \text{ real}\}$ is an orientation preserving homeomorphism of the circle for which there exists a constant $M$ such that

$$\frac{1}{M} \leqslant \left| \frac{f(e^{i(x+t)}) - f(e^{ix})}{f(e^{ix}) - f(e^{i(x-t)})} \right| \leqslant M, \tag{3}$$

for all $x$ and all $|t| < \frac{\pi}{2}$. Obviously the restriction of any Möbius transformation preserving the unit disc to the unit circle is quasisymmetric. Let a finite number of smooth real-valued charts $\varphi_j$ that cover the circle be given and assume the maps from intervals to intervals defined by $\varphi_j \circ h \circ (\varphi_k)^{-1}$ are quasisymmetric with constants $M_{jk}$ on the intervals where they are defined. Then $h$ will be quasisymmetric with a constant $M$ depending on the $M_{jk}$ and the coordinate charts $\varphi_j$ and $\varphi_k$. Conversely, suppose $h$ is quasisymmetric and $\varphi_j$ are a finite system of smooth charts whose domains of definition cover the circle. Then each $\varphi_j \circ f \circ (\varphi_k)^{-1}$ is quasisymmetric on the interval where it is defined. Thus, if when expressed in terms of a finite number of smooth charts that cover the circle $h$ is quasisymmetric on the intervals for some constant $M$, then $h$ is quasisymmetric on the circle for some possibly different constant $M_1$.

### 1.2. *The quasisymmetric topology*

Now we introduce a topology on the group of orientation preserving homeomorphisms $h$ of a circle that satisfy inequality (3) by specifying a neighborhood basis $V(\varepsilon)$ of the identity. By definition $h$ is in $V(\varepsilon)$, for given $\varepsilon > 0$, if two conditions are satisfied:

(a) $\sup\{|h(e^{ix}) - e^{ix}|, |h^{-1}(e^{ix}) - e^{ix}|\} < \varepsilon$, and

(b) inequality (3) is satisfied with $M = 1 + \varepsilon$.

This system of neighborhoods has the following properties:

(i) $\bigcap_{n=1}^{\infty} V(1/n) = \{\mathit{identity}\}$,

(ii) for every $\varepsilon > 0$, there exists $\delta > 0$, such that $V(\delta) \circ V(\delta) \subset V(\varepsilon)$, and

(iii) for every $\varepsilon > 0$, there exists $\delta > 0$, such that $(V(\delta))^{-1} \subset V(\varepsilon)$.

The system of neighborhoods induces a right and a left topology on $QS$ by right and left translation. That is, $V \circ h$ is a right neighborhood of $h$ when $V$ is a neighborhood of the identity. These neighborhoods are precisely those that make right translation maps $h \mapsto h \circ g$ continuous. Similarly, there is the system of left neighborhoods $h \circ V$ of the $h$, and these make left translation maps $h \mapsto g \circ h$ continuous. However, these properties constitute only part of the structure necessary to make $QS$ a topological group. In the next section we examine this discrepancy in more detail.

### 1.3. *The symmetric subgroup*

There is a brief theory of groups that are also Hausdorff topological spaces satisfying axioms (i), (ii), and (iii) above, developed by Gardiner and Sullivan in [35]. We summarize this theory and its application to $QS$ in this section.

DEFINITION. A *topological group* is a group $G$ that is also a Hausdorff topological space and such that the map $(f, g) \mapsto f \circ g^{-1}$ from $G \times G$ to $G$ is continuous.

It turns out that $QS$ is not a topological group because taking inverses is not continuous. However, it does satisfy the axioms for what we call a partial topological group.

DEFINITION. A *partial topological group* is a group with a Hausdorff system of neighborhoods of the identity satisfying (i), (ii) and (iii) above.

As we have seen in Section 1.2, at a general point $h$ of the group there are two neighborhood systems. If $U$ runs through the neighborhood system at the identity, then $h \circ U$ and $U \circ h$ run through systems of left and right neighborhoods of $h$, respectively. The following three theorems are proved in [35].

THEOREM 1. *The following conditions on a partial topological group are equivalent*:

(i) *it is a topological group with the given neighborhood system of the identity,*
(ii) *the left and the right neighborhood systems agree at every point,*
(iii) *the adjoint map $f \mapsto h \circ f \circ h^{-1}$ is continuous at the identity for every $h$ in the group.*

In a general partial topological group the properties of Theorem 1 will not hold. One of the two topologies in a partial topological group will be left translation invariant and the other right translation invariant. The inverse operation interchanges these two topologies.

One can consider those elements $h$ of a partial topological group for which the two neighborhood systems at $h$ agree, that is, for which conjugation by $h$ maps the neighborhood system at the identity isomorphically onto itself. These elements form a closed subgroup: the two topologies agree on this subgroup and give it the structure of a topological group. We call this subgroup the *characteristic topological subgroup*.

If a subset of a partial topological group is invariant under the inverse operation, then it is closed for one topology if, and only if, it is closed for the other. In particular, one may speak without ambiguity of a closed subgroup of a partial topological group.

The next result is elementary.

THEOREM 2. *The characteristic topological subgroup of a partial topological group is a closed topological subgroup.*

DEFINITION. A quasisymmetric map $h$ has *vanishing ratio distortion* if there is a function $\varepsilon(t)$ with $\varepsilon(t)$ converging to zero as $t$ converges to zero, such that inequality (3) is satisfied with $M$ replaced by $1 + \varepsilon(t)$.

It turns out that the characteristic topological subgroup of $QS$ comprises precisely those homeomorphisms that have vanishing ratio distortion. We shall call this subgroup the *symmetric subgroup* $S$. A direct proof that $S$ is a topological group is elementary. Here we prove only the following fact.

THEOREM 3. *S is a closed subgroup of QS.*

PROOF. We shall use the following notation. $I$ and $J$ are contiguous intervals, $I = [a, b]$, $J = [b, c]$, and $|I| = b - a$ is the length of $I$. Let a constant $C > 1$ be given. One first shows that if $I$ and $J$ are contiguous with

$$1/C \leqslant |I|/|J| \leqslant C$$

and if $g$ is sufficiently near the identity in the quasisymmetric topology, then

$$\frac{1}{1+\varepsilon} \leqslant \frac{|g(I)|}{|g(J)|} \cdot \frac{|J|}{|I|} < 1+\varepsilon.$$

Assume $h_n$ is a sequence of elements of $S$ converging in the right $QS$-topology to $h$. This means that for sufficiently large $n$,

$$\frac{1}{1+\varepsilon} \leqslant \frac{|h \circ h_n^{-1}(I)|}{|h \circ h_n^{-1}(J)|} \cdot \frac{|J|}{|I|} < 1+\varepsilon.$$

Also, assume that for each $h_n$ there is a function $\varepsilon_n(t)$ approaching zero as $t$ approaches 0 such that for all contiguous intervals $K$ and $L$ with $1/C \leqslant |K|/|L| \leqslant C$,

$$\frac{1}{1+\varepsilon_n(|K|)} < \frac{|h_n(K)|}{|h_n(L)|} \cdot \frac{|L|}{|K|} < 1+\varepsilon_n\big(|K|\big).$$

Taking the product, we obtain

$$\begin{aligned}\frac{1}{(1+\varepsilon)(1+\varepsilon_n(|K|))} &\leqslant \frac{|h \circ h_n^{-1}(I)|}{|h \circ h_n^{-1}(J)|} \cdot \frac{|J|}{|I|} \cdot \frac{|h_n(K)|}{|h_n(L)|} \cdot \frac{|L|}{|K|} \\ &< (1+\varepsilon)\big(1+\varepsilon_n\big(|K|\big)\big).\end{aligned}$$

Since there is a uniform bound on the quasisymmetric norm of $h_n$, we may assume $1/C \leqslant |h_n(K)|/|h_n(L)| \leqslant C$, and thus we may substitute in $I = h_n(K)$ and $J = h_n(L)$. We obtain

$$\frac{1}{(1+\varepsilon)(1+\varepsilon_n(|K|))} \leqslant \frac{|h(K)|}{|h(L)|} \cdot \frac{|L|}{|K|} < (1+\varepsilon)\big(1+\varepsilon_n\big(|K|\big)\big).$$

For given $\delta > 0$, we can pick $n_0$ large enough so that $\varepsilon_{n_0}(|K|) < \varepsilon$ whenever $|K| < \delta$ and $1/C < |K|/|L| < C$. Then

$$\frac{1}{(1+\varepsilon)^2} < \frac{|h(K)|}{|h(L)|} \cdot \frac{|L|}{|K|} < (1+\varepsilon)^2,$$

and this implies $h$ has vanishing ratio distortion. □

### 1.4. *Dynamical systems and deformations*

By definition, *universal Teichmüller space* $T$ is $QS$ factored by the closed subgroup of Möbius transformations that preserve the unit disc. It is universal in the naive sense that it contains the deformation spaces of nearly all one-dimensional dynamical systems $F$ that act on the unit circle. When we say this, we have in mind the following two types of dynamical systems. $F$ is either a Fuchsian group acting on the unit circle or a $C^2$-homeomorphism acting on the unit circle. In the second situation, it is often useful to assume $F$ has irrational rotation number. The theory is already complicated when $F$ is a diffeomorphism and becomes even more so if $F$ is allowed to have one critical point. Under smooth changes of coordinate, we may assume $F$ maps an interval on the real axis to another interval on the real axis and maps the origin to a point $c$. We also assume that in suitable smooth coordinates $F$ takes the form of a power law:

$$F(x) = |x|^{\alpha}\,\mathrm{sign}(x) + c,$$

for some constant $\alpha > 1$.

The deformation space $T(F)$ (sometimes called the *Teichmüller space of* $F$), is defined to be the space of equivalence classes of quasisymmetric maps $h \in QS$ such that $h \circ F \circ h^{-1}$ is a dynamical system of the same type. Two maps $h_0$ and $h_1$ are equivalent if there is a Möbius transformation $A$ such that $A \circ h_0 = h_1$. Thus, it is the set of quasisymmetric conjugacies to dynamical systems of the same type factored by this equivalence relation.

In the case that $F$ is a Fuchsian group with generators $\gamma_j$ this means that, for each $j$, the conjugate $h \circ \gamma_j \circ h^{-1}$ is also a Möbius transformation preserving the unit circle. In the case $F$ is a $C^2$ homeomorphism, possibly with a power law, this means $h \circ F \circ h^{-1}$ is also a $C^2$-homeomorphism. If $h$ is itself a Möbius transformation, we consider the dynamical systems generated by $F$ and by $h \circ F \circ h^{-1}$ as not differing in any essential way. For this reason, we view $T(F)$ as a subspace of $QS$ mod $PSL(2, \mathbb{R})$. That is, two elements $h_1$ and $h_2$ are considered equivalent if there is a Möbius transformation $A$ such that $A \circ h_1 = h_2$.

It turns out that the factor space $T = QS$ mod $PSL(2, \mathbb{R})$ carries in a natural way the structure of a complex manifold, as do the subspaces $T(F)$ for many dynamical systems $F$. Even the statement that $T(F)$ is connected is already significant, and knowledge of geometrical properties of curves which join pairs of points in $T(F)$ can have dynamical consequences.

Here we explain why conjugacies that allow distortion of eigenvalues cannot be smooth: therefore, to obtain interesting conjugacies one must expand into the quasisymmetric realm.

LEMMA 2. *Let $F_0$ and $F_1$ be two discrete dynamical systems acting on the real axis, generated by $x \mapsto \gamma_0(x) = \lambda_0 x$ and by $x \mapsto \gamma_1(x) = \lambda_1 x$, respectively, and assume $1 < \lambda_0 < \lambda_1$. Let $h$ be a conjugacy, so that $h \circ \gamma_0 \circ h^{-1} = \gamma_1$. Then $h$ can be at most Hölder continuous with exponent $\alpha = \log\lambda_0 / \log\lambda_1$.*

PROOF. Because $h(\lambda_0^n x) = \lambda_1^n h(x)$, by plugging in $x = 1$ and letting $n$ approach $-\infty$ and $\infty$, one sees that $h$ must fix 0 and $\infty$. By postcomposition of $h$ with a real dilation, we may assume $h(1) = 1$ and this implies $h(\lambda_0^n) = \lambda_1^n$. But any such map taking these values for arbitrarily large negative values of $n$ cannot satisfy an inequality of the form $|h(x)| \leqslant C|x|^\alpha$ unless $\alpha \leqslant \log\lambda_0/\log\lambda_1$. □

## 1.5. *Tangent spaces to QS and S*

In this section we identify the circle with the extended real line $\overline{\mathbb{R}}$. By postcomposing with a Möbius transformation we may assume any homeomorphism representing an element of $T$ fixes infinity. Consider a smooth curve $h_s$ of homeomorphisms in $QS$ parameterized by $s$ and passing through the identity at $s = 0$. We may assume each homeomorphism $h_s$ fixes infinity. By smooth we shall mean that

$$h_s(x) = x + sV(x) + \mathrm{o}(s), \tag{4}$$

where the distance measured in the quasisymmetric norm from the identity to $h_s$ is less than or equal to a constant times $s$. In particular,

$$\frac{1}{1+Cs} \leqslant \frac{h_s(x+t) - h_s(x)}{h_s(x) - h_s(x-t)} \leqslant 1 + Cs.$$

By substituting (4) into this formula we arrive at the following condition on the continuous function $V$:

$$\left|V(x+t) - 2V(x) + V(x-t)\right| = \mathrm{O}(t). \tag{5}$$

If $h_s$ is a smooth curve in the symmetric subspace $S$, then

$$\left|V(x+t) - 2V(x) + V(x-t)\right| = \mathrm{o}(t). \tag{6}$$

We will call (5) and (6), respectively, the big and little *Zygmund conditions*. Since $V$ is to be regarded as the tangent vector to the one-parameter family of homeomorphisms $h_s$, $V(x)\frac{\partial}{\partial x}$ is a vector field.

If instead we consider the mappings $h_s$ as acting on the unit circle $|z| = 1$ then the condition that the vector field $W$ point in a direction tangent to the unit circle is that

$$\widetilde{W}(x) = W(\mathrm{e}^{\mathrm{i}x})/\mathrm{i}\mathrm{e}^{\mathrm{i}x} \tag{7}$$

be real-valued. The boundedness conditions on $QS$ and $S$ correspond to the conditions that the continuous, periodic function $\widetilde{W}$ satisfy (5) and (6). We denote the spaces of continuous vector fields satisfying these conditions by $Z$ and $Z_0$, respectively.

A simple example of a tangent vector in $Z_0$ is generated by a curve of Möbius transformations preserving the unit circle and passing through the identity. Such a curve has a tangent vector of the form

$$W(z)\frac{\partial}{\partial z} = \left(\alpha z^2 + \beta z + \gamma\right)\frac{\partial}{\partial z},$$

where $\alpha$, $\beta$ and $\gamma$ are constants which make $W(z)$ real-valued along $z\bar{z}=1$. We call such tangent vectors *trivial*. Thus the quadratic polynomials which satisfy this reality condition define the tangent vectors to trivial curves of homeomorphisms.

We will show that any tangent vector satisfying the big Zygmund condition is the tangent vector to a smooth curve in $QS$ passing through the identity, and correspondingly any tangent vector satisfying the little Zygmund condition is the tangent vector to a smooth curve in $S$.

DEFINITION. Let $\mathcal{Z}$ and $\mathcal{Z}_0$ be the spaces $Z$ and $Z_0$ factored by the quadratic polynomials.

Eventually in Section 2.3 we shall identify a Banach space $\mathcal{A}$ such that the Banach dual of $\mathcal{Z}_0$ is isomorphic to $\mathcal{A}$ and the Banach dual of $\mathcal{A}$ is isomorphic to $\mathcal{Z}$. In particular, $Z_0^{**} \cong Z$.

Let $Q$ be any quadruple of points $a, b, c$ and $d$ arranged in counter-clockwise order on the unit circle or in increasing order on the real axis and define the *cross ratio* $cr(Q)$ by

$$cr(Q) = \frac{(d-c)(b-a)}{(c-b)(a-d)}. \tag{8}$$

Recall that $cr(Q)$ is Möbius invariant in the sense that $cr(A(q)) = cr(Q)$ for any Möbius transformation $A$. In consequence we may define a norm $\| \ \|_{cr}$ on vector fields which is Möbius invariant in the sense that

$$\|W\|_{cr} = \left\| \frac{W \circ A}{A'} \right\|_{cr},$$

for every Möbius transformation $A$.

Define $W[a,b,c,d]$ to be the alternating sum

$$\frac{W(d)-W(c)}{d-c} - \frac{W(c)-W(b)}{c-b} + \frac{W(b)-W(a)}{b-a} - \frac{W(a)-W(d)}{a-d}.$$

For a given quadruple $Q$ the term $cr(Q)\rho(cr(Q))W[a,b,c,d]$ measures the velocity of the cross-ratio (8) with respect to the Poincaré metric $\rho(z)|dz|$ on the sphere punctured at $0, 1$ and $\infty$ when each of the points $a, b, c$ and $d$ move with complex velocities $W(a)$, $W(b)$, $W(c)$ and $W(d)$, respectively. The *infinitesimal cross-ratio norm* is defined for the space $\mathcal{Z}$ by

$$\|W\|_{cr} = \sup_{Q} \left| cr(Q)\rho\big(cr(Q)\big) W[a,b,c,d] \right|. \tag{9}$$

Note that $\|W\|_{cr} = 0$ if, and only if, $W$ is a quadratic polynomial. Furthermore, if $Q$ has the form $Q = (-\infty, x-t, x, x+t)$ then $cr(Q) = -1$. If in addition we assume $|W(z)| = o(|z|^2)$, which is tantamount to the assumption that $W(z)\frac{\partial}{\partial z}$ vanishes at infinity, then the alternating sum $W[a,b,c,d]$ is equal to

$$-\frac{W(x+t) - 2W(x) + W(x-t)}{t}.$$

### 1.6. *The Hilbert transform and almost complex structure*

If the vector field $\widetilde{V}(z)\frac{\partial}{\partial z}$ is continuous and real-valued on the circle, then the function $V(x) = \widetilde{V}(e^{ix})/ie^{ix}$ is continuous, real-valued and periodic on the real axis. Define a function $W(x)$ by the formula

$$W(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|y-x| \geqslant \varepsilon} V(y) \cot\left(\frac{y-x}{2}\right) dy, \tag{10}$$

where the integral is taken over values of $y$ for which $-\pi \leqslant y \leqslant \pi$ and $|y - x \pm 2\pi n| \geqslant \varepsilon$ for all integers $n$. Transporting $W$ back to the unit circle by the formula

$$\widetilde{W}(e^{ix}) = W(x)ie^{ix},$$

one obtains a complex-valued function $\widetilde{W}$ defined on the circle for which

$$\widetilde{W}(z)\frac{\partial}{\partial z}$$

is again real-valued. By this process $\widetilde{V}$ is transformed to $J\widetilde{V} = \widetilde{W}$, another field of vectors on the unit circle whose directions are tangent to the circle.

This rule defines an operator $J$ called the *Hilbert transform*; it extends to a bounded operator for many different smoothness classes. For example,

$$\|W\|_p \leqslant C_p \|V\|_p$$

for $p \geqslant 2$, where

$$\|V\|_p^p = \frac{1}{2\pi} \int_{-\pi}^{\pi} |V(x)|^p \, dx.$$

More important to us are the following properties of $J$:
(1) the Zygmund classes $Z$ and $Z_0$ are preserved by $J$,
(2) $J$ is anti-involutory in the sense that $J^2 = -I$, and
(3) $J(\sin kx) = \cos kx$ and $J(\cos kx) = -\sin kx$.

Proofs of all of these statements are greatly simplified by considering different possible extensions of $V$ to the complex plane, as we shall see in Section 2.

The anti-involutory property of $J$ yields an almost complex structure on $\mathcal{Z}$, the tangent space to universal Teichmüller space. In fact, whenever an anti-involutory automorphism $J$ of a vector space $X$ over $\mathbb{R}$ is given, $X$ becomes a vector space over $\mathbb{C}$ by defining, for every $v$ in $X$,

$$(a + ib)v = av + bJ(v).$$

The reader should check that

$$\big((a_1 + ib_1)(a_2 + ib_2)\big)v = (a_1 + ib_1)\big((a_2 + ib_2)v\big).$$

Instead of using the (inverse) exponential map $z = e^{ix} \mapsto x$ as a real-valued chart for the unit circle, one can use the stereographic map $z \mapsto u$ from the circle to $\overline{\mathbb{R}}$ where

$$u = U(z) = \frac{z + i}{iz + 1}. \tag{11}$$

This map sends the four points $1,\ i,\ -1,\ -i$ on the unit circle to the four points $1,\ \infty,\ -1,\ 0$, respectively, on the real axis. The real-valued vector field $\widetilde{V}(z)\frac{\partial}{\partial z}$ on the circle $\{z\colon\ |z| = 1\}$ is related to the vector field $\widehat{V}(u)$ defined for $u$ on the real axis by

$$\widetilde{V}(z) = \widehat{V}(u)\left(\frac{-2}{(u+i)^2}\right).$$

Since we assume $\widetilde{V}$ is continuous, and in particular bounded on the circle, that implies at most quadratic growth of $\widehat{V}$ near $\infty$. That is

$$\widehat{V}(u) = O\big(|u|^2\big)$$

as $|u| \to \infty$.

In the special case when $\widetilde{V}(z)\frac{\partial}{\partial z} = (c_0 + c_1 z + c_2 z^2)\frac{\partial}{\partial z}$ is a real-valued Möbius vector field, then $c_1$ is pure imaginary, $c_2 = -\overline{c_0}$,

$$\widehat{V}(u) = \left(c_0 + c_1\left(\frac{i(u-i)}{(u+i)}\right) + c_2\left(\frac{i(u-i)}{(u+i)}\right)^2\right)\left(\frac{-2}{(u+i)^2}\right)$$

is a quadratic polynomial in $u$, and

$$V(x) = \frac{(c_0 + c_1 e^{ix} - \overline{c_0}e^{2ix})}{ie^{ix}} = \frac{1}{2}a_0 + a_1 \cos x + b_1 \sin x.$$

Here, $a_0$, $a_1$, and $b_1$ are real and $a_0 = 2\,\mathrm{Im}\, c_1$, $a_1 = 2\,\mathrm{Im}\, c_2$, $b_1 = 2\,\mathrm{Re}\, c_2$.

It will turn out that the quadratic polynomials are preserved by the Hilbert transform and so $J$ is well-defined on the quotient space

$$\mathcal{Z} = \big\{\widetilde{V} \in Z\big\}\big/\{\textit{quadratic polynomials}\}. \tag{12}$$

In Section 2.6, when we use complex methods to deal with Hilbert transforms, we will find it useful to map the interior of the circle to the upper half-plane by the stereographic map $u = U(z)$ given in (11) and then compute the Hilbert transform in the upper half-plane. When this is done, it must be remembered that the function $\widehat{V}$ is permitted to have at most quadratic growth near infinity.

### 1.7. *Scales and trigonometric approximation*

If a vector field $\widetilde{V}(z)\frac{\partial}{\partial z}$, on the unit circle is given by a finite sum of the form

$$\widetilde{V}_n(z) = \sum_{k=-n}^{n} c_k z^k, \tag{13}$$

and is real-valued, then $c_{n+2} = -\overline{c_{-n}}$. The corresponding function $V_n(x) = \widetilde{V}_n(e^{ix})/ie^{ix}$, is the *trigonometric polynomial*

$$a_0/2 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)$$

of degree $n$, where $a_k = 2\operatorname{Im} c_{k+1}$ and $b_k = 2\operatorname{Re} c_{k+1}$. We may think of the trigonometric polynomial $V_n$ with

$$\|V_n\|_\infty = M$$

as a typical vector field having a definite oscillation down to intervals whose length is as small as $\frac{1}{Mn}$. That is, if $V(x) = 1$ and $0 < t < \frac{1}{Mn}$, then $V(x+t) > 0$. This is because of the mean value theorem and the following lemma due to Bernstein [5].

LEMMA 3. *If $V_n(x)$ is a trigonometric polynomial of degree $n$, then*

$$\left\| \frac{d}{dx} V_n(x) \right\|_\infty \leqslant n \left\| V_n(x) \right\|_\infty .$$

PROOF. We follow the proof given in [55, p. 39]. To begin, assume there is a trigonometric polynomial $V_n$ with $\|V_n'\| = nL$ and $L > \|V_n\|$. Thus at some point $x_0$, $|V_n'(x_0)| = nL$, and we can assume that $V_n'(x_0) = nL$. Since $V_n'$ is a maximum at $x_0$, $V''(x_0) = 0$.

Consider the trigonometric polynomial

$$T_n(x) = L \sin n(x - x_0) - V_n(x)$$

of degree $n$. In the interval $[x_0, x_0 + 2\pi)$ there are $2n$ points where $\sin n(x - x_0)$ takes the values $\pm 1$, and between any two of these points the polynomial $T_n$ takes values of opposite sign. Hence, $T_n$ has $2n$ different zeros in this interval, and so

$$T_n'(x) = nL \cos n(x - x_0) - V_n'(x)$$

also has $2n$ different zeros. One of these zeros is $x_0$, since

$$T_n'(x_0) = nL - V_n'(x_0).$$

Also,

$$T_n''(x) = -n^2 L \sin n(x - x_0) - V_n''(x)$$

vanishes at $x = x_0$. Moreover, $T_n''$ has $2n$ zeroes between the zeros of $T_n'$. Thus $T_n''$ has at least $2n + 1$ zeros in this interval, and since it is a trigonometric polynomial of degree $n$ it must be identically zero. Thus $T_n'$ is constant, but since $T_n'(x_0) = 0$, this implies $T_n$ is constant. But this contradicts the statement that $T_n$ changes sign and we conclude that the original assumption could not be correct, that is, we have $L \leqslant \|V_n\|_\infty$, which means $\|V_n'\|_\infty \leqslant n\|V_n\|_\infty$. □

Assume $V$ and $W$ are continuous functions of period $2\pi$. An inequality of the form $\|V(x) - W(x)\|_\infty < 1/2^n$ for large $n$ implies that the graph of $V(x)$ closely resembles the graph of $W(x)$. Now consider $M_{k,I}V(x) = \frac{1}{2^k}V(2^k x)$, where $x$ lies in some interval $I$ of length $2\pi/2^k$. Then $M_{k,I}$ is a magnification operator of degree $k$, magnifying the graph of $V$ over the interval $I$ by the same factor in both the domain and range.

In fractal geometry, one considers graphs that have roughly the same shape no matter how much they are magnified. Thus, suppose that we go to some fine scale $M_{k,I}V$. Then the picture of the graph seen at this scale should roughly resemble the picture of the graph of $M_{n,J}$ if $n$ is any number larger than $k$ and $J$ is any interval of size $2\pi/2^n$.

A general trigonometric polynomial does not possess this property. Suppose a polynomial $V_{2^n}$ of degree $2^n$ is magnified by the operator $M_{k,I}$ of degree $k$. If $k$ is larger than $n$, then because of Bernstein's inequality, one does not see any oscillation in the graph of $M_{k,I}V_n$. This observation motivates the following theorem due to Zygmund and Jackson, [43,55,76].

THEOREM 4. *Suppose $V(x)$ is a continuous, periodic function defined on the real axis. Then $V$ is in the Zygmund class $Z$ defined by*

$$\left|\frac{V(x+t) + V(x-t) - 2V(x)}{t}\right| \leqslant C$$

*if, and only if, there exists a constant $C'$ such that for every positive integer $n$ there is a trigonometric polynomial $V_n$ of degree at most $n$, such that*

$$\|V - V_n\|_\infty \leqslant \frac{C'}{n}.$$

*Moreover, the number $C'$ can be estimated purely in terms of $C$ and vice versa.*

PROOF. We begin by proving that if such trigonometric approximations are possible for every $n$, then $V$ is in the Zygmund class. For each integer of the form $2^k$, let $V_{2^k}$ be a trigonometric polynomial of degree $2^k$ such that

$$\|V - V_{2^k}\|_\infty \leqslant \frac{C}{2^k}. \tag{14}$$

Now select $n$ so that $\frac{1}{2^{n+1}} \leqslant t \leqslant \frac{1}{2^n}$, and write $V$ in the form $V = W_1 + W_2$ where $W_1 = V - V_{2^n}$ and $W_2 = V_{2^n}$. In general, define the difference operator $\Delta_t$ by

$$\Delta_t G(x) = G(x+t) - G(x).$$

Then

$$\Delta_t^2 G(x) = \Delta_t\big(\Delta_t(G)\big)(x) = G(x+2t) - 2G(x+t) + G(x).$$

From the hypothesis,

$$\left|\Delta_t^2 W_1(x)\right| \leqslant \frac{8C}{2^{n+1}} \leqslant 8Ct. \tag{15}$$

Putting $V_0 = 0$, we may rewrite $W_2$ as a sum over scales:

$$W_2(x) = V_1 - V_0 + \sum_{k=1}^{n}(V_{2^k} - V_{2^{k-1}}). \tag{16}$$

Each term $V_{2^k} - V_{2^{k-1}}$ has norm bounded by

$$|V_{2^k} - V| + |V - V_{2k-1}| \leqslant \frac{3C}{2^k},$$

and is a trigonometric polynomial of degree less than or equal to $2^k$. So by Lemma 3, the second derivative of $V_{2^k} - V_{2^{k-1}}$ is at most $3C2^k$. Thus, by the second mean value theorem

$$\left|\Delta_t^2(V_{2^k} - V_{2^{k-1}})\right| \leqslant 3C2^k t^2.$$

By using Equation (16), we obtain

$$\left|\Delta_t^2 W_2\right| \leqslant \sum_{k=1}^{n} 3C2^k t^2 \leqslant \sum_{k=1}^{n} \frac{3C2^k}{2^{2n}} = \frac{3C2^{n+1}}{2^{2n}} = \frac{6C}{2^n} \leqslant 12Ct. \tag{17}$$

Putting inequalities (15) and (17) together, we obtain $|\Delta_t^2 V| \leqslant 20Ct$ and this proves the first half of the theorem.

To prove the other half, for every $n$ we must construct a trigonometric polynomial $V_n$ of degree $n$ that approximates $V$ in the sup-norm to within $C/n$. Let $K_n$ be the Jackson kernel defined by $K_n = \sigma_{2n-1} - 2\sigma_n$, where

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_{n-1}}{n} = \frac{1}{2\pi n}\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^2$$

is the Féyér kernel and

$$s_n = \frac{\sin(2n+1)\frac{t}{2}}{2\sin\frac{t}{2}}.$$

By convolution of $V$ with the Jackson kernel $K_n$, one gets a trigonometric polynomial of degree $2n-1$ that approximates $V$ to within $O(1/n)$ in the sup norm. For details of this proof we refer to [55, p. 55–56], or to [76]. □

THEOREM 5. *The Zygmund spaces $Z$ and $Z_0$ are invariant under the Hilbert transform $J$.*

PROOF. Since a much easier proof of the same result is given in Section 2.6, here we only outline the argument given by Zygmund in [76]. Begin by using a result of Favard [25]: if $|g'| \leqslant M$ then $|Jg - J\sigma_n(g)| \leqslant A/n$. Then employ the Zygmund–Jackson theorem. Let $V$ be in $Z$ and $V_n$ be a trigonometric polynomial of degree $n$ with $|V - V_n| \leqslant \frac{C}{n}$. Let $G' = V$ and $T_n' = V_n$. Then $|(G - T_n)'| \leqslant \frac{C}{n}$ and therefore

$$\left|J(G - T_n) - J\sigma_n(G - T_n)\right| \leqslant \frac{AC}{n^2}.$$

Thus $JG$ is approximable in the sup norm to within $\frac{AC}{n^2}$ by a trigonometric polynomial of degree $n$. This implies that $JV$ is approximable to within $\frac{AC}{n}$ by a trigonometric polynomial of degree $n$, therefore $JV$ is in the Zygmund class.

The proof for the class $Z_0$ is similar. □

### 1.8. *Automorphisms of Teichmüller space*

Given any quasisymmetric homeomorphism $f$ of $S^1$, the map $\rho_f([h]) = [h \circ f^{-1}]$ is a bicontinuous self-map of $T = QS \bmod PSL(2, \mathbb{R})$. Moreover, $\rho_f$ preserves the almost complex structure. We call biholomorphic automorphisms of $T$ of this form *geometric* automorphisms.

An *almost complex structure* on a real Banach manifold $M$ is a smoothly varying family of automorphisms $J_x$, $x \in M$, of each fiber of the tangent bundle such that $J_x^2 = -I$. A diffeomorphism $F$ of $M$ is *almost complex* if

$$J_{F(x)}\bigl(F_x'(v)\bigr) = F_x'\bigl(J_x(v)\bigr).$$

THEOREM 6. *Any almost complex automorphism of $T$ is geometric. That is, given a diffeomorphism $F$ of*

$$QS \bmod PSL(2, \mathbb{R})$$

*whose derivative commutes with the almost complex structure, there exists a quasisymmetric map $f$ such that $F = \rho_f$.*

An outline of the proof of this theorem is given in the last section of this chapter.

## 2. Complex analysis

### 2.1. *Quasiconformal extensions*

Roughly speaking, a homeomorphism of $\mathbb{R}^n$ is quasiconformal if it distorts standard shapes by a bounded amount, see [33,54,63]. When $n \geqslant 2$, it turns out that quasiconformal maps are differentiable almost everywhere and the distortion of shape can be measured infinitesimally. An observation of central importance for the deformation theory of one-dimensional dynamical systems is that this statement is not true when $n = 1$. That is, quasisymmetric maps may not be differentiable anywhere.

In any case, the measurement of quasiconformal distortion at a point $z$ for a mapping $f$ when $n = 2$ is by means of a quantity called the *local dilatation* $K_z(f)$:

$$K_z(f) = \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|}. \tag{18}$$

Any quasisymmetric homeomorphism $h$ of the real axis extends to a quasiconformal self-mapping of the upper half-plane. This pivotal result was first proved by Ahlfors and Beurling [4]. The formula given in [1] for such an extension of $h$ is $H_1(z) = F(z) + \mathrm{i}G_1(z)$, where

$$F(x+\mathrm{i}y) = \frac{1}{2y}\int_{x-y}^{x+y} h(t)\,dt,$$

$$G_1(x+\mathrm{i}y) = \frac{1}{2y}\left\{\int_{x}^{x+y} h(t)\,dt - \int_{x-y}^{y} h(t)\,dt\right\}. \tag{19}$$

This formula does not extend the identity by the identity. In particular, for $h(x) = x$, the extension $H_1(z) = x + \frac{1}{2}\mathrm{i}y$. It is therefore convenient to multiply the expression for $G_1$ by a factor two. That is, we put

$$G(x+\mathrm{i}y) = \frac{1}{y}\left\{\int_{x}^{x+y} h(t)\,dt - \int_{x-y}^{y} h(t)\,dt\right\}. \tag{20}$$

Although $H = F + \mathrm{i}G$ differs from $H_1$, it still yields a quasiconformal extension $ex(h)$ of $h$, but with the additional property that the identity is extended by the identity. It is useful to view $H$ as an extension to the whole plane by stipulating that $H(\bar{z}) = \overline{H}(z)$. The reader should check that this extension process is natural for real affine transformations in the sense that if $A(z) = c_1 z + c_2$ and $B(z) = c_3 z + c_4$ where $c_1, \ldots, c_4$ are real, then

$$ex(A \circ h \circ B) = A \circ ex(h) \circ B.$$

Hence, if we assume

$$h(x) + 1 = h(x+1), \tag{21}$$

then $H(z+1) = H(z) + 1$.

It is important to note that a lift of a self-homeomorphism of the circle by the universal covering $x \mapsto e^{2\pi ix}$ yields a homeomorphism $h$ satisfying (21), and conversely, if a homeomorphism of the real-axis satisfies (21) then it projects to a homeomorphism of the circle. Moreover, the covering $x \mapsto e^{2\pi ix}$ extends to the covering $z \mapsto e^{2\pi iz}$ of the punctured unit disc $\mathbb{D}^* = \mathbb{D} - \{0\}$ by the upper half-plane. The extension of $h$ to the disc punctured at 0 is a quasiconformal map preserving 0, and therefore if we stipulate that the extension preserves 0, it becomes a quasiconformal extension to the entire disc. This method of extension also has the following *asymptotic property*.

Assume that $h(0) = 0$, $h(x+1) = h(x) + 1$, $h$ is quasisymmetric and

$$\frac{1}{1+\varepsilon} \leqslant \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \leqslant 1 + \varepsilon$$

for $|t| < \varepsilon$, with $\delta$ sufficiently small. Then if $|\operatorname{Im} z| < \delta$, the dilatation $K_z$ of $ex(h)$ at $z$ satisfies $K_z < 1 + \varepsilon'$, where $\varepsilon'$ converges to zero as $\varepsilon$ converges to zero.

## 2.2. *Teichmüller's metric*

The *Teichmüller distance* between two points $[h_1]$ and $[h_2]$ in $QS$ mod $PSL(2, \mathbb{R})$ is defined to be

$$d([h_1], [h_2]) = \frac{1}{2} \log K_0\big(h_2 \circ (h_1)^{-1}\big), \tag{22}$$

where

$$K_0(h) = \inf\{K(\tilde{h}): \text{ where } \tilde{h} \text{ is any quasiconformal extension of } h\}.$$

As a consequence of basic properties of quasiconformal mappings, $d([id], [h])$ is always realized by an extremal mapping $\tilde{h}$ which is an extension of $h$. To see this one can assume that $h$ fixes three points on the real axis, say 0, 1 and $\infty$, and then select extensions $\tilde{h}_n$ of $h$ such that

$$\frac{1}{2} \log K(\tilde{h}_n) < \frac{1}{2} \log K_0([h]) + \frac{1}{n}.$$

Since the mappings $\tilde{h}_n$ are normalized and have uniformly bounded dilatation, they are equicontinuous. Therefore there is a subsequence of $\tilde{h}_n$ that converges uniformly in the spherical metric to some self-mapping of the upper half-plane $\tilde{h}_0$. Since each $\tilde{h}_n$ coincides with $h$ at every point of the real axis, so does $\tilde{h}_0$. Moreover, the maximal dilatation of $\tilde{h}_0$ must be less than $K([h]) + \frac{1}{n}$ for every positive integer $n$. On the other hand, $K(\tilde{h}_0)$ cannot be less than $K_0([h])$ because by definition $K_0([h])$ is the infimum of the dilatations of all possible extensions of $h$. We conclude that every mapping $h$ of the real axis has an *extremal* quasiconformal extension $\tilde{h}_0$ to the upper half-plane, that is, an extension for which

$$K_0([h]) = K(\tilde{h}_0).$$

It will turn out that certain quasisymmetric mappings $h$ have many extremal extensions and thus we do not expect to find a general formula that yields an extremal extension. In particular, the Beurling–Ahlfors extension formula given in the previous section will almost never yield an extremal extension.

In formula (22) the Teichmüller distance is seen as the solution to an infimum problem. It turns out that it is also the solution to a supremum problem. Consider the vector space $\mathcal{R}(\mathbb{H})$ of integrable holomorphic quadratic differentials $\varphi(z)\,dz^2$ in the upper half-plane $\mathbb{H}$ with only a finite number of poles on the real axis and for which $\varphi(z)\,dz^2$ is real-valued on the real axis. Any element $\varphi(z)\,dz^2$ of $\mathcal{R}(\mathbb{H})$ has the form

$$\varphi(z)\,dz^2 = \frac{p(z)\,dz^2}{(z-x_1)\cdots(z-x_n)}, \tag{23}$$

where $x_1,\ldots,x_n$ are distinct points on the real axis and $p(z)$ is a polynomial of degree less than or equal to $n-3$ with real coefficients. Such a quadratic differential determines a decomposition of $\mathbb{H}$ into a finite number of strips, $S_1,\ldots,S_k$, where $k \leqslant n-3$. The interior of each strip is swept out by horizontal trajectories of this quadratic differential, that is, parameterized curves $\alpha(t)$ along which $\varphi(\alpha(t))\alpha'(t)^2 > 0$. A choice of coordinate

$$\zeta = \pm\int\sqrt{\varphi(z)}\,dz + (\text{const})$$

can be made so that $z \mapsto \zeta$ maps the $j$th strip to a rectangle $R_j$ and takes the horizontal trajectories $\alpha(t)$ to horizontal line segments that join the left side of the rectangle to its right side. Let $a_j$ and $b_j$ be the width and height of the rectangle $R_j$ measured in the parameter $\zeta$. Then

$$\|\varphi\| = \iint_{\mathbb{H}} |\varphi(z)|\,dx\,dy = \sum_{j=1}^{k} a_j b_j.$$

That is, $\|\varphi\|$ is equal to the sum of the areas of these rectangles. Moreover, if $\beta$ is any arc in $\mathbb{H}$ with endpoints on $\partial\mathbb{H}$ transversal to the horizontal trajectories of $\varphi$, then we can assign to it a *height* $ht_\varphi(\beta)$ given by

$$ht_\varphi(\beta) = \int_\beta \operatorname{Im}\big(\sqrt{\varphi(z)}\,dz\big),$$

equal to the sum of the heights of the rectangles corresponding to the strips $S_j$ crossed by $\beta$.

Let $I_j$ be the intervals on $\partial\mathbb{H}$ whose endpoints are successive pairs from the (ordered) sequence of points $x_1,\ldots,x_k$ and assume that the endpoints of the arc $\beta \subset \mathbb{H}$ lie on the intervals $I_{j_1}$ and $I_{j_2}$. If $\beta'$ is another arc transverse to the horizontal trajectories of $\varphi(z)\,dz^2$ with endpoints on the same intervals $I_{j_1}$ and $I_{j_2}$, then $ht_\varphi(\beta) = ht_\varphi(\beta')$, and so the height function $ht_\varphi$ is a nonnegative function defined on all possible pairs of intervals $I_{j_1}$ and $I_{j_2}$ taken from the set $I_1,\ldots,I_k$.

A sense-preserving selfmap $h$ of $\partial\mathbb{H}$ takes the points $x_j$ to points $x'_j = h(x_j)$ and thus determines a new height function defined on pairs of intervals $I'_{j_1} = h(I_{j_1})$ and $I'_{j_2} = h(I_{j_2})$. By a theorem of Hubbard and Masur in [41] (see also [28]), there is a unique quadratic differential of the form

$$\psi(z)\,dz^2 = \frac{q(z)\,dz^2}{(z-x'_1)\cdots(z-x'_n)},$$

such that $q(z)$ has real coefficients and degree $\leqslant n-3$ and the heights of $\psi$ for the interval pairs $I'_{j_1}$ and $I'_{j_2}$ are equal to the heights of $\varphi$ for the corresponding interval pairs $I_{j_1}$ and $I_{j_2}$. Moreover, $\psi$ is unique among all continuous integrable quadratic differentials on $\overline{\mathbb{C}} \setminus \{x'_1, \ldots, x'_k\}$, real-valued on the real axis, with heights between pairs of intervals $I'_{j_1}$ and $I'_{j_2}$ greater than or equal to the corresponding heights of $\varphi$ on intervals $I_{j_1}$ and $I_{j_2}$ and with $\|\psi\|$ as small as possible.

If $h$ has a quasiconformal extension $\tilde{h}$ with dilatation $K_0$, then $\|\psi\| \leqslant K_0\|\varphi\|$, and one obtains the following expression for $K_0$:

$$K_0 = \sup \frac{\|\psi\|}{\|\varphi\|}, \tag{24}$$

where the supremum is taken over all non-zero quadratic differentials $\varphi$ of the form (23) and $\psi$ is the quadratic differential with simple poles at the points $x'_j = h(x_j)$, $1 \leqslant j \leqslant n$, and with the same corresponding heights with respect to these points that $\varphi$ has with respect to the points $x_j$, $1 \leqslant j \leqslant n$.

### 2.3. *Quadratic differentials*

Let $\mathcal{A} = \mathcal{A}(\Omega)$ be the Banach space of integrable functions $\varphi(z)$ holomorphic in $\Omega$ where $\Omega = \mathbb{H}$ or $\Omega = \Delta = \{z\colon |z| < 1\}$ with norm

$$\|\varphi\| = \iint_\Omega |\varphi(z)|\,dx\,dy < \infty.$$

In this section we introduce a pairing between $\mathcal{A}$ and $\mathcal{Z}$ and show that $\mathcal{A}^* \cong \mathcal{Z}$ and that $(\mathcal{Z}_0)^* \cong \mathcal{A}$. By $\mathcal{Z}$ we mean the vector fields $V$ defined on $\partial\Omega$ such that $V(z)\frac{\partial}{\partial z}$ is real-valued on $\partial\Omega$, and such that $\|V\|_{\mathcal{Z}} < \infty$. Since there is a Möbius transformation transforming $\mathbb{H}$ onto $\Delta$ and since the statements we prove will be invariant under pull-back by Möbius transformations, we can work interchangeably with either $\mathbb{H}$ or with $\Delta$.

Our first step is to prove a special case of *Bers' approximation theorem* [2,6]. Let $\mathcal{R}(\mathbb{H})$ be the space of finite linear combinations of the form

$$\lambda_1\varphi_{x_1}(z) + \cdots + \lambda_n\varphi_{x_n}(z), \tag{25}$$

where $x_1, \ldots, x_n$ and $\lambda_1, \ldots, \lambda_n$ are real numbers and

$$\varphi_x(z) = \frac{x(x-1)}{z(z-1)(z-x)}.$$

THEOREM 7. *$\mathcal{R}$ is dense in $\mathcal{A}$.*

PROOF. A similar and much deeper result is true if $\mathbb{H}$ is replaced by any plane domain; see [2,6].

Let $L$ be any linear functional on the Banach space $\mathcal{A}$ that annihilates $\mathcal{R}$. To show that $\mathcal{R}$ is dense in $\mathcal{A}$ it is sufficient to show that $L$ annihilates $\mathcal{A}$. By the Hahn–Banach and Riesz representation theorems, there exists a bounded measurable function $\mu$ defined in $\mathbb{H}$ so that

$$L(\varphi) = \text{real part of} \left(\iint_{\mathbb{H}} \mu(z)\varphi(z)\,dx\,dy\right).$$

If we extend $\varphi(z)$ and $\mu(z)$ to the lower half-plane by the rules $\overline{\varphi(z)} = \varphi(\bar{z})$ and $\overline{\mu(z)} = \mu(\bar{z})$, then we can write the formula for $L$ as

$$L(\varphi) = \frac{1}{2} \iint_{\mathbb{C}} \mu(z)\varphi(z)\,dx\,dy.$$

The assumption that $L$ annihilates $\mathcal{R}$ implies that

$$V(z) = -\frac{z(z-1)}{\pi} \iint_{\mathbb{C}} \frac{\mu(\zeta)}{\zeta(\zeta-1)(\zeta-z)}\,d\xi\,d\eta = 0 \tag{26}$$

whenever $z$ is a real number. One shows that $V$ has the following properties:

(1) $\bar{\partial} V = \mu$ in the sense of distributions,

(2) $|V(z)| = O(|z|\log|z|)$ as $z \to \infty$, and

(3) $V(z)$ has an $|\varepsilon \log \varepsilon|$-modulus of continuity, that is to say, given $R > 0$, there exists a $C$ such that for every $z_1$ and $z_2$ with $|z_1|$ and $|z_2| < R$ and with $|z_1 - z_2| < 1/2$,

$$\left|V(z_1) - V(z_2)\right| \leqslant C|z_1 - z_2| \log\left(1/|z_1 - z_2|\right).$$

Let $D_{\varepsilon,R}$ be a semi-disc in the upper half-plane with diameter of length $2R$ along the line $y = \varepsilon$ and with midpoint on the $y$-axis. The curved part of the boundary of $D_\varepsilon$ is parameterized by the curve $z = \mathrm{i}\varepsilon + R\mathrm{e}^{\mathrm{i}\theta}$, $0 \leqslant \theta \leqslant \pi$. Assume further that $\varphi$ is continuous on the real axis and $\varphi(z) = O(|z|^{-4})$ as $z \to \infty$. Then the subspace of $A(\mathbb{H})$ comprising those $\varphi$ with these properties is dense in $A(\mathbb{H})$. Since $\varphi$ is integrable and $\mu$ is bounded,

$$\iint_{\mathbb{H}} \mu\varphi = \lim \iint_{D_{\varepsilon,R}} \mu\,d\xi\,d\eta,$$

where the limit is taken both as $\varepsilon \to 0$ and as $R \to \infty$. On the other hand, from Green's formula,

$$\iint_{D_{\varepsilon,R}} \mu \, d\xi \, d\eta = \int_{\partial D_{\varepsilon,R}} V(\zeta)\varphi(\zeta)\, d\zeta.$$

Because $V(z)$ is identically zero when $z \in \mathbb{R}$, if we first take the limit in this line integral as $\varepsilon \to 0$ we obtain

$$\int_0^{\pi} V\big(R\mathrm{e}^{\mathrm{i}\theta}\big)\varphi\big(R\mathrm{e}^{\mathrm{i}\theta}\big) R\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\, d\theta = 0.$$

Because of the vanishing condition on $\varphi$,

$$\int_0^{\pi} V\big(R\mathrm{e}^{\mathrm{i}\theta}\big)\varphi\big(R\mathrm{e}^{\mathrm{i}\theta}\big) R\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\, d\theta$$

is dominated by a constant times $(\log R)/R$ and thus vanishes as $R \to \infty$. □

We are now ready to introduce the pairing between an element $V$ in $\mathcal{Z}$ and $\varphi$ in $\mathcal{A}$. Given $V$ in $\mathcal{Z}$ we select any extension $\widetilde{V}$ of $V$ to the upper half-plane with the properties that $\bar{\partial}\widetilde{V} = \mu$ is essentially bounded and that $|V(z)| = \mathrm{O}(|z|^2)$. Then we define

$$(V, \varphi) = \mathrm{Re}\left(\iint_{\mathbb{H}} \mu\varphi\right). \tag{27}$$

We must show first that any $V$ in $\mathcal{Z}$ has such an extension and second that if a different extension is taken, the integration (27) yields the same result. $\widetilde{V}$ can be defined by the Beurling–Ahlfors formula (19) applied to the vector field $V$:

$$\mathrm{Re}\big(\widetilde{V}(x+\mathrm{i}y)\big) = \frac{1}{2y}\int_{x-y}^{x+y} V(t)\, dt \quad \text{and}$$

$$\mathrm{Im}\big(\widetilde{V}(x+\mathrm{i}y)\big) = \frac{1}{y}\left(\int_{x}^{x+y} V(t)\, dt - \int_{x-y}^{x} V(t)\, dt\right).$$

We leave it to the reader to verify that $\widetilde{V}$ has the appropriate growth rate and that the Zygmund condition implies

$$\bar{\partial}\widetilde{V} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right)\widetilde{V}$$

is bounded. To show that the right-hand side of (27) depends only on the values of $V$ on the real axis we first note that from Theorem 7 it suffices to show the right-hand side of (27) depends only on the values of $V$ on the real axis when $\varphi$ has the special form (25). In that case, if we assume $V$ vanishes at 0 and 1 and has growth rate $o(|z|^2)$, then by Green's formula,

$$(V, \varphi) = \frac{\pi}{2} \sum_j \lambda_j V(x_j). \tag{28}$$

THEOREM 8. *The pairing* (27) *induces an isomorphism between* $\mathcal{Z}$ *and* $\mathcal{A}^*$.

PROOF. From the preceding discussion we have seen how an element $L$ of $\mathcal{A}^*$ determines by the correspondence an element $V$ in $\mathcal{Z}$. Conversely, because of the residue formula (28), the extension formula and because $\mathcal{R}$ is dense in $\mathcal{A}$, any element $V$ of $\mathcal{Z}$ determines by this correspondence an element of $\mathcal{A}^*$. Note that $\|V\|_{cr} = 0$ is equivalent to the condition that $V(z) = a_0 + a_1 z + a_2 z^2$. Since by definition elements

$$\varphi(z) = \sum_{j=1}^{n} \frac{\lambda_j}{z - x_j}$$

of $\mathcal{R}$ satisfy $|\varphi(x)| = O(|z|^{-4})$ as $z \to \infty$, they also satisfy $\sum \lambda_j = 0$, $\sum_j \lambda_j x_j = 0$ and $\sum_j \lambda_j x_j^2 = 0$. One therefore sees that any quadratic polynomial vector field $V(x)\frac{\partial}{\partial x}$ annihilates all elements of $\mathcal{R}$, and so also annihilates $\mathcal{A}$ since $\mathcal{R}$ is dense in $\mathcal{A}$. □

Because $\mathcal{A}^*$ is isomorphic to $\mathcal{Z}$, the norm on $\mathcal{Z}$ dual to the norm on $\mathcal{A}$ is equivalent to $\| \; \|_{cr}$. We call this norm the *infinitesimal Teichmüller norm* and denote it by $\| \; \|_{\mathcal{T}}$. It is given by either of the following formulas:

$$\begin{aligned} \|V\|_T &= \sup\left\{ \left| \iint_{\mathbb{H}} \varphi \bar\partial \widetilde{V} \, dx \, dy \right| : \; \varphi \in \mathcal{A} \text{ with } \|\varphi\| = 1 \right\} \\ &= \inf\left\{ \|\bar\partial \widetilde{V}\|_\infty : \text{ where } \widetilde{V} \text{ is any extension of } V \right\}. \end{aligned}$$

DEFINITION. We say a sequence $\varphi_n$ in $\mathcal{A}$ is *degenerating* if there is a constant $C > 1$ such that $C^{-1} \leqslant \|\varphi_n\| \leqslant C$ and $\varphi_n(z) \to 0$ for every $z \in \mathbb{H}$.

We now wish to focus attention on the closed subspace $\mathcal{Z}_0$ of $\mathcal{Z}$ defined in Section 1.6, the tangent vector fields to the symmetric circle maps.

THEOREM 9. *The following conditions on an element* $V$ *of* $\mathcal{Z}$ *are equivalent*:

(1) *with respect to any smooth local coordinate* $x$ *on the boundary of* $\Omega$,

$$\left| \frac{V(x+t) - 2V(x) + V(x-t)}{t} \right| \leqslant c(t),$$

*where* $c(t)$ *approaches* 0 *as* $t \to 0$,

(2) *$V$ has a continuous extension $\widetilde{V}$ for which $\bar\partial \widetilde{V} = \mu$ is vanishing in the sense that for every $\varepsilon > 0$ there exists a compact subset of $\mathbb{H}$ such that if $z$ lies outside the compact set then $|\mu(z)| < \varepsilon$,*

(3) *$V$ annihilates every degenerating sequence in $\mathcal{A}$.*

PROOF. Given $V$ satisfying condition (1), the Beurling–Ahlfors formula (19) yields a vector field $\widetilde{V}$ with the property that $\bar\partial \widetilde{V}$ is vanishing. Thus (1) implies (2). It is easy to see that if $|\mu(z)| < \varepsilon$ for $z$ outside a sufficiently large compact set and if $\varphi_n$ is degenerating, then

$$\lim_{n\to\infty} \iint \mu\varphi_n \, dx\, dy \to \infty,$$

and so (2) implies (3). To see that (3) implies (1), consider the following sequence of quadratic differentials $\varphi_n$, where $t_n \to 0$ and $x_n$ is arbitrary:

$$\varphi_n(z) = \frac{1}{t_n}\left\{\frac{1}{z-(x_n-t_n)} - \frac{2}{z-x_n} + \frac{1}{z-(x_n+t_n)}\right\}$$
$$= \frac{2t_n}{(z-(x_n-t_n))(z-x_n)(z-(x_n+t_n))}.$$

Note that

$$\iint_{\mathbb{H}} |\varphi_n| = \iint_{\mathbb{H}} \left|\frac{2t_n}{(z-t_n)z(z+t_n)}\right| dx\, dy = \iint_{\mathbb{H}} \left|\frac{1}{(z-1)z(z+1)}\right| dx\, dy,$$

which is a positive constant not depending on $n$ and, for fixed $z$ in the upper half-plane, $\varphi_n(z) \to 0$ as $t_n \to 0$.

By formula (28)

$$(V, \varphi_n) = \frac{\pi}{2}\left\{\frac{V(x_n - t_n) - 2V(x_n) + V(x_n + t_n)}{t_n}\right\}$$

and we know that this quantity approaches zero as $t_n \to 0$, no matter which sequence $\{x_n\}$ is selected. Thus (3) implies (1). □

THEOREM 10. *The pairing $(V, \varphi)$ defined in* (27) *induces an isomorphism from $\mathcal{A}$ onto $\mathcal{Z}_0^*$.*

PROOF. We first observe that the pairing defined in (27) is non-degenerate between $\mathcal{Z}_0$ and $\mathcal{A}$. Since it is a non-degenerate pairing between $A$ and $\mathcal{Z}$ and since $\mathcal{Z}_0 \subset \mathcal{Z}$, whenever $V \in \mathcal{Z}_0$, $(V, \varphi) = 0$ for all $\varphi \in \mathcal{A}$ implies $V = 0$. Moreover, for $|z_0| < 1$ and $\varepsilon < 1 - |z_0|$, by the mean value property

$$\varphi(z_0) = \frac{1}{\pi\varepsilon^2}\iint_{|z-z_0|<\varepsilon} \varphi(z)\, dx\, dy = \iint \mu_0\varphi,$$

where

$$\mu_0(z) = \begin{cases} \frac{1}{\pi\varepsilon^2} & \text{for } |z - z_0| \leqslant \varepsilon, \\ 0 & \text{for } z \text{ elsewhere.} \end{cases}$$

Because the pairing is non-degenerate, the mapping from $\mathcal{A}$ to $\mathcal{Z}_0^*$ given by $\varphi \mapsto \{V \mapsto (V, \varphi)\}$ is well-defined and injective. In order to show it is surjective it suffices to show the unit ball of $\mathcal{A}$ is compact with respect to the weak topology. To this end, assume $\varphi_n$ is a sequence in $\mathcal{A}$ with $\|\varphi_n\| = 1$ and $L$ is a linear functional of the form

$$L(\varphi) = \iint_{\mathbb{H}} \mu\varphi,$$

where $|\mu(z)| < \varepsilon$ for $z$ outside sufficiently large compact subsets of $\mathbb{H}$. By normal convergence, $\varphi_n$ has a subsequence that converges uniformly on compact subsets to some $\varphi$, which (in order to avoid cumbersome notation) we denote also by $\varphi_n$. Note that by Lebesgue dominated convergence

$$\lim_{n\to\infty} \iint \bigl(|\varphi_n - \varphi| - |\varphi_n|\bigr) = \lim_{n\to\infty} \bigl(|\varphi_n - \varphi| - |\varphi_n|\bigr) = -\|\varphi\|,$$

and, since $\|\varphi_n\| = 1$, $\|\varphi_n - \varphi\| \to 1 - \|\varphi\|$.

We divide the argument into three cases: either $\|\varphi\| = 0$ or $0 < \|\varphi\| < 1$ or $\|\varphi\| = 1$. In the first case, $\varphi_n$ is degenerating and $L(\varphi_n) \to 0$. In the second case, since $\|\varphi\| < 1$, if we put

$$\tilde{\varphi}_n = \frac{\varphi_n - \varphi}{\|\varphi_n - \varphi\|},$$

then the denominator is bounded away from zero. Thus $\tilde{\varphi}_n$ is a degenerating sequence and $L(\tilde{\varphi}_n)$ converges to zero, which implies $L(\varphi_n)$ converges to $L(\varphi)$. In the third case $\|\varphi_n - \varphi\| \to \infty$, and so $L(\varphi_n)$ converges to $L(\varphi)$. Thus, in all cases $L(\varphi_n)$ converges to $L(\varphi)$. □

DEFINITION. We say a sequence $V_n$ in $\mathcal{Z}$ is *vanishing* if there is a constant $C > 1$ such that $C^{-1} \leqslant \|V_n\|_T \leqslant C$ and $V_n(x)$ approaches zero for every $x$ on the boundary.

The following theorem enables one to deduce that an automorphism of $\mathcal{Z}$ that is an isometry for the infinitesimal Teichmüller norm necessarily preserves the closed subspace $\mathcal{Z}_0$. This is a key step in the proof of the automorphism theorem, Theorem 6 of Section 1.8.

THEOREM 11. *An element $V$ of $\mathcal{Z}$ with $\|V\|_T = 1$ is in $\mathcal{Z}_0$ if, and only if, for every vanishing sequence of elements $W_n$ in $\mathcal{Z}$ with $\|W_n\|_T = 1$,*

$$\limsup_{n\to\infty} \|V + W_n\|_T \leqslant \|V\|_T.$$

For the proof we refer to [19,33].

### 2.4. *Reich–Strebel inequalities*

Let $\mu$ be the Beltrami coefficient of a quasiconformal extension $\tilde{h}$ of a quasisymmetric mapping $h$, and let $K_0 = K_0(h)$ be the smallest possible dilatation of a quasiconformal extension of $h$. For any holomorphic quadratic differential $\varphi$ in $\mathcal{A}(\mathbb{H})$ with

$$\|\varphi\| = \iint_{\mathbb{H}} |\varphi(z)|\, dx\, dy = 1,$$

one has the following bounds on $K_0$:

$$\frac{1}{K_0} \leqslant \iint_{\mathbb{H}} \frac{|1 - \mu \frac{\varphi}{|\varphi|}|^2}{1 - |\mu|^2} |\varphi(z)|\, dx\, dy, \tag{29}$$

and

$$K_0 \leqslant \sup_{\|\varphi\|=1} \iint_{\mathbb{H}} \frac{|1 + \mu \frac{\varphi}{|\varphi|}|^2}{1 - |\mu|^2} |\varphi(z)|\, dx\, dy. \tag{30}$$

These inequalities were proved by Reich and Strebel [61,62], who also observed that they yield the infinitesimal form of Teichmüller's metric:

$$d_T\big([0], [t\mu]\big) = \frac{1}{2} \log K_0(t\mu) = t \sup_{\|\varphi\|=1} \left| \iint_{|z|<1} \mu(z)\varphi(z)\, dx\, dy \right| + \mathrm{o}(t), \tag{31}$$

for $t > 0$. The inequality

$$d_T\big([0], [t\mu]\big) \geqslant t\, \mathrm{Re}\left\{ \iint_{|z|<1} \mu\varphi \right\} + \mathrm{o}(t),$$

for $\|\varphi\| = 1$, follows by replacing $\mu$ by $t\mu$ and calculating the first variation in (29). Similarly, the inequality

$$d_T\big([0], [t\mu]\big) \leqslant t \sup_{\|\varphi\|=1} \mathrm{Re}\left\{ \iint_{|z|<1} \mu\varphi \right\} + \mathrm{o}(t),$$

follows on replacing $\mu$ by $t\mu$ and calculating the first variation in (30).

### 2.5. *Tangent spaces revisited*

Let $M$ denote the open unit ball in $L_\infty(\mathbb{H})$ and suppose $\mu_t$ is a smooth curve in $M$ such that each $f^{\mu_t}$ is equal to the identity on the boundary of $\mathbb{H}$. Then from inequality (29) we see that for every holomorphic quadratic differential $\varphi(\zeta)d\zeta^2$ on $\mathbb{H}$,

$$1 \leqslant \iint_{\mathbb{H}} \frac{|1 - \mu_t \frac{\varphi}{|\varphi|}|^2}{1 - |\mu_t|^2} |\varphi|\, d\xi\, d\eta.$$

By putting

$$\|\mu_t - t\nu\|_\infty = \mathrm{o}(t) \tag{32}$$

and computing the first variation in this inequality, one obtains

$$\iint_{\mathbb{H}} \varphi(\zeta)\nu(\zeta)\, d\xi\, d\eta = 0, \tag{33}$$

for every such $\varphi$.

Conversely, suppose (33) holds for every holomorphic quadratic differential $\varphi$ on the upper half-plane. Restricting $\varphi(\zeta)\, d\zeta^2 = d\zeta^2/(\zeta - z)^4$ to the lower half-plane, we conclude that

$$\iint_{\mathbb{H}} \frac{\nu(\zeta)}{(\zeta - z)^4}\, d\xi\, d\eta = 0, \tag{34}$$

where $\zeta = \xi + i\eta$ is in the upper half-plane.

A key existence theorem (see [30], p. 107) says that (34) implies there exists a curve $\mu_t$ such that

$$\|\mu_t - t\nu\|_\infty = \mathrm{o}(t)$$

and $f^{\mu_t}(z) = z$ for every $z$ in the closure of the lower half-plane. Here, $f^{\mu_t}$ is the unique quasiconformal self-map of the whole plane that fixes 0, 1 and $\infty$, and that has Beltrami coefficient equal to $\mu_t$ in the upper half-plane and identically equal to 0 in the lower half-plane. In particular, if we let $M_0$ be the closed subspace of those $\mu$ in $M$ for which $f^\mu(x) = x$ for all $x \in \mathbb{R}$, then the tangent space $N$ to $M_0$ consists of those $\nu$ for which

$$\iint_{\mathbb{H}} \nu(\zeta)\varphi(\zeta)\, d\xi\, d\eta = 0$$

for every quadratic differential $\varphi$ holomorphic in $\mathbb{H}$. Moreover, the tangent space to Teichmüller space $T$ is isomorphic to the factor space $L_\infty(\mathbb{H})/N$; see [30,33].

For $\mu$ in $L_\infty(\mathbb{H})$, define

$$\hat{\mu} = \begin{cases} \mu(\zeta) & \text{for } \zeta \text{ in } \mathbb{H}, \\ \overline{\mu(\bar{\zeta})} & \text{for } \zeta \text{ in } \mathbb{H}^*. \end{cases}$$

Let $\alpha : L^\infty(\mathbb{H}) \to Z$ be the map $\alpha : \mu \mapsto V_\mu(x)$, where $x$ is in $\mathbb{R}$ and

$$V_\mu(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\hat{\mu}(\zeta)}{\zeta(\zeta - 1)(\zeta - z)}\, d\xi\, d\eta. \tag{35}$$

Also define

$$\tilde{\mu}(\zeta) = \begin{cases} \mu(\zeta) & \text{for } \zeta \text{ in } \mathbb{H}, \\ 0 & \text{for } \zeta \text{ in } \mathbb{H}^*, \end{cases}$$

and let $\beta$ be the Bers map $\beta : \mu \mapsto \left(W_\mu\right)'''(z)$, where

$$W_\mu(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\tilde{\mu}(\zeta)}{\zeta(\zeta-1)(\zeta-z)} \, d\xi \, d\eta. \tag{36}$$

THEOREM 12. *The maps $\alpha$ and $\beta$ defined above induce isomorphisms of Banach spaces from $L_\infty(\mathbb{H})/N$ onto $\mathcal{Z}$ and from $L_\infty(\mathbb{H})/N$ onto $B$, where $B$ is the Banach space of holomorphic functions $\psi(z)$ defined in the lower half-plane $\mathbb{H}^*$ for which*

$$\|\psi\|_B = \sup_{z \in \mathbb{H}^*} \left|\psi(z) y^2\right| < \infty.$$

PROOF. Note that

$$\alpha(\mu) = \mathrm{Re}\left(-\frac{2}{\pi} \iint_{\mathbb{H}} \frac{\mu(\zeta)}{\zeta(\zeta-1)(\zeta-x)} \, d\xi \, d\eta\right),$$

and therefore the condition that $\alpha(\mu)(x) = 0$ for all $x$ in $\mathbb{R}$ implies

$$\iint_{\mathbb{H}} \frac{\mu(\zeta)}{\zeta(\zeta-1)(\zeta-z)} \, d\xi \, d\eta = 0$$

for all $z$ in the lower half-plane. On taking the third derivative with respect to $z$, we find that

$$\iint_{\mathbb{H}} \frac{\mu(\zeta)}{(\zeta-z)^4} \, d\xi \, d\eta = 0,$$

for all $z$ in the lower half-plane. Since finite linear combinations of the form

$$\varphi(\zeta) = \sum_j c_j \frac{1}{(\zeta - z_j)^4},$$

where $z_j$ are points in the lower half-plane, are dense in the space of integrable holomorphic quadratic differentials in the upper half-plane (see [2,6]), we see that $\alpha(\mu)(x) = 0$ for all $x$ in $\mathbb{R}$ implies $\iint_{\mathbb{H}} \varphi(\zeta)\mu(\zeta) \, d\xi \, d\eta = 0$ for all $\varphi$, which implies $\mu$ is in $N$.

Conversely, if $\mu$ is orthogonal to every $\varphi$, then

$$\iint_{\mathbb{H}} \frac{\mu(\zeta)}{(\zeta-z)^4} \, d\xi \, d\eta = 0,$$

for every $z$ in the lower half-plane and, by integrating three times and normalizing so that $V_\mu(z)$ vanishes at 0, 1 and $\infty$, we find that

$$V_\mu(x) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\hat{\mu}(\zeta)}{\zeta(\zeta-1)(\zeta-x)} \, d\xi \, d\eta = 0,$$

for all $x$ in $\mathbb{R}$.

To see that $\alpha$ is surjective, we apply the extension formula (20) to the real vector field $V(x)\frac{\partial}{\partial x}$. That is, for given $V(x)\frac{\partial}{\partial x}$ representing an element in $\mathcal{Z}$, we put $V(z) = W_1(z) + \mathrm{i}W_2(z)$, where

$$W_1(x+\mathrm{i}y) = \frac{1}{2y}\int_{x-y}^{x+y} V(t)\,dt, \tag{37}$$

and

$$W_2(x+\mathrm{i}y) = \frac{1}{y}\left\{\int_{x}^{x+y} V(t)\,dt - \int_{x-y}^{y} h(t)\,dt\right\}. \tag{38}$$

Then it is a routine calculation (see [35]) to show that $\|\frac{\partial}{\partial \bar{z}}V(z)\|_\infty < \infty$.

We leave it to the reader to show that the Bers map $\beta$ is an isomorphism, and in particular that $\beta(\mu) = 0$ if, and only if, $V_\mu(x) = 0$ for every $x$ in $\mathbb{R}$. A detailed proof may be found in [33, p. 134], or in [48, pp. 111–114]. □

## 2.6. *Hilbert transform and almost complex structure*

For a smooth real-valued function $f(x)$ defined on the real axis with compact support, the Hilbert transform $Jf$ is normally defined as the principal part of a divergent integral. That is,

$$(Jf)(x) = -\frac{1}{\pi}\lim_{\varepsilon\to 0}\left\{\int_{-\infty}^{x-\varepsilon} f(t)\,dt + \int_{x+\varepsilon}^{\infty} f(t)\,dt\right\}. \tag{39}$$

This formula hides a description of the transform in terms of harmonic conjugates which is invariant under conformal changes of coordinate. This description has three steps. The first step, if it is possible, is to form the unique harmonic extension $\tilde{f}(z)$ to the upper half-plane, characterized by the properties that $\tilde{f}(z)$ is harmonic and $\tilde{f}(x)$ coincides with $f(x)$ for $x$ real. Then one forms $\tilde{g}(z)$, which is unique up to an additive constant and such that $\tilde{f}(z) + \mathrm{i}\tilde{g}(z)$ is holomorphic in the upper half-plane. Finally, $Jf(x)$ is defined to be the restriction of $\tilde{g}(z)$ to the real axis.

Of course, the definition of $Jf$ in given this way is determined only up to additive constant, but in order for $J$ to be well-defined on $\mathcal{Z}$, we need only to define $JV$ up to the addition of a quadratic polynomial $p(z) = az^2 + bz + c$.

We wish to give an alternative description of the Hilbert transform on the space $\mathcal{Z}$. Since $\alpha$ is surjective, we can assume $V$ is of the form

$$V(x) = -\frac{1}{\pi}\iint_{\mathbb{C}} \frac{\mu(\zeta)}{\zeta(\zeta-1)(\zeta-x)}\,d\xi\,d\eta,$$

where $\mu(z) = \overline{\mu(\overline{z})}$. We shall say that $\mu$ satisfying this equation is *symmetric*. For symmetric $\mu$, we define $\hat{\mu}$ to be the Beltrami coefficient given by the formula

$$\hat{\mu}(\zeta) = \begin{cases} \mathrm{i}\mu(\zeta) & \text{for } \zeta \text{ in } \mathbb{H}, \\ -\mathrm{i}\mu(\zeta) & \text{for } \zeta \text{ in } \mathbb{H}^*. \end{cases} \tag{40}$$

Then $\hat{\mu}$ is also symmetric and

$$(V_\mu + \mathrm{i}V_{\hat{\mu}})(z) = \frac{-2}{\pi} \iint_{\mathbb{H}^*} \frac{\mu(\zeta)}{\zeta(\zeta-1)(\zeta-z)}\, d\xi\, d\eta,$$

where the integration is over the lower half-plane $\mathbb{H}^*$. It is obvious that this function is holomorphic in the upper half-plane; therefore, up to the addition of a quadratic polynomial, $JV_\mu$ is the restriction to the real axis of $V_{\hat{\mu}}(z)$.

Note that $\|\hat{\mu}\|_\infty = \|\mu\|_\infty$. This reformulation shows that $\mathcal{Z}$ is invariant under $J$, and in fact $J$ is an isometry for the infinitesimal Teichmüller norm on the tangent space to Teichmüller space. The argument is easily modified to show that $\mathcal{Z}_0$ is also invariant under $J$, see [35]. It also shows that the Hilbert transform applied to the vector field $V(x)\frac{\partial}{\partial x}$ corresponds to the mapping $\mu \mapsto \mathrm{i}\mu$ for Beltrami coefficients given in the upper half-plane. Since multiplication by i on Beltrami coefficients determines the standard almost complex structure on Teichmüller space, the Hilbert transform gives the same almost complex structure. This observation is due to Steven Kerckhoff (unpublished, but see [75]).

## 2.7. *Complex structures on quasi-Fuchsian space*

The view of the almost complex structures summarized in the previous section has been exploited by Giannis Platis [60] to yield *three* inter-related but distinct almost complex structures on the *quasi-Fuchsian spaces* $QF = QF(\Gamma)$. These are complex deformation spaces, whose points are given by arbitrary quasiconformal conjugates of a given (cofinite volume) Fuchsian group $\Gamma$, acting discretely on the union of $\mathbb{H}$ and $\mathbb{H}^*$, the complement of the circle $\overline{\mathbb{R}}$ inside the Riemann sphere. Such a group $H\Gamma H^{-1}$ is called a *quasi-Fuchsian group*. It operates discretely, but not necessarily symmetrically, on the complement of the quasicircle $H(\overline{\mathbb{R}})$; our earlier definition of the Teichmüller spaces implies that $QF(\Gamma) \supset T(\Gamma)$ as a diagonal subset, corresponding to q-c conjugates where the mapping $H$ is given by a symmetric Beltrami coefficient. Together with a certain hermitian 2-form $\Omega$ defined on the space $QF(\Gamma)$, the three anti-involutions determine a hyper-Kählerian structure. One views the tangent space to $QF(\Gamma)$ as a space of (complex) vector fields $V(x)\frac{\partial}{\partial x}$ which can be expressed in the form

$$V(x) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\mu(\zeta)}{\zeta(\zeta-1)(\zeta-x)}\, d\xi\, d\eta.$$

In this formula, $V(x)$ is usually complex-valued because there is no assumption about symmetry for $\mu$.

We define $I$ to be the map of vector fields induced by $\mu \mapsto \mathrm{i}\mu$. Writing

$$\mu = \begin{cases} \mu_1(\zeta) & \text{for } \zeta \text{ in } \mathbb{H}, \\ \mu_2(\zeta) & \text{for } \zeta \text{ in } \mathbb{H}^*, \end{cases}$$

we define $V \mapsto J(V)$ to be the map induced by

$$\mu \mapsto \begin{cases} \overline{\mu_2(\bar{\zeta})} & \text{for } \zeta \text{ in } \mathbb{H}, \\ -\overline{\mu_1(\bar{\zeta})} & \text{for } \zeta \text{ in } \mathbb{H}^*. \end{cases}$$

A simple calculation shows that $IJ = -JI$, so that if we write $K = IJ$, then $I^2$, $J^2$ and $K^2$ are all equal to minus the identity and $IJ = K$, $JK = I$ and $KI = J$. Moreover, $K$ restricted to symmetric Beltrami coefficients coincides with the almost complex structure defined (via the Hilbert transform) in the preceding section on Teichmüller space.

In [60], Platis shows that for finite co-volume Fuchsian groups, $I$, $J$, and $K$ together with the hermitian form $\Omega$ yield a hyper-Kählerian structure on $QF(\Gamma)$. The form $\Omega$ is constructed using derivatives of a finite spanning set of complex length functions (see for instance [44]); it is compatible with the almost complex structure induced on $QF$ as a product space $T(\Gamma) \times \overline{T(\Gamma)}$ by the anti-involution $J$, satisfies a complex analogue of Wolpert's reciprocity formula for hyperbolic length functions on Teichmüller space, and restricts on the diagonal subspace to give the Weil–Petersson metric on Teichmüller space.

## 2.8. *Automorphisms are geometric*

To close this article we return to the rigidity theorem, Theorem 6, formulated in Section 1.8. This result is the analogue for universal Teichmüller space of the classical result of H. Royden [64] and of Earle and Kra [24] that says that any automorphism of the Teichmüller space of a surface of genus greater than 3 and possibly with a finite number of punctures is induced by an element of the mapping class group. That the parallel result holds for any surface of finite genus with a finite number of holes removed was proved in [19] and for any open surface of finite genus by Lakic in [50].

Suppose we are given an almost complex diffeomorphism $F$ of universal Teichmüller space, $T$. Since Kobayashi's metric coincides with Teichmüller's metric on $T$ [29], the automorphism is an isometry in Teichmüller's metric, and since Teichmüller's metric is the integral of its infinitesimal form [59], this means that if $F([0]) = \tau$, then $F' = dF$ defines an isometry from the tangent space at [0] to the tangent space at $\tau$. Since we may select a geometric isomorphism $\rho_h$ such that $\rho_h \circ F([0]) = [0]$ and since geometric isomorphisms are isometries, we obtain an automorphism $\rho_h \circ F$ which preserves the basepoint [0] and induces an isometry on the tangent space $\mathcal{Z}$ at [0] to Teichmüller space. One shows that this isometry is necessarily equal to the identity and thus $F = \rho_{h^{-1}}$, that is, every automorphism of Teichmüller space is induced by the action of a quasisymmetric mapping on the boundary of the hyperbolic plane.

We outline the key steps in the proof. One first shows that any isometry $\sigma$ of $\mathcal{Z}$ with Teichmüller's infinitesimal metric must be induced by an isometry of the predual space

$A$. This result follows from the results of Section 2.3, and in particular from Theorem 11. Such an isometry must preserve the closed subspace $\mathcal{Z}_0$ and therefore it is equal to the second dual of its restriction to $\mathcal{Z}_0$. Thus there is an isometry $\hat{\sigma}$ of $\mathcal{A}$ such that $\sigma$ is the dual of $\hat{\sigma}$ under the natural pairing between $A$ and $\mathcal{Z}$. Then one shows that $\hat{\sigma}$ is induced by multiplication by a complex constant $c$ and by composition with a conformal map. That is,

$$\hat{\sigma}(\varphi) = c\varphi\big(f(z)\big)f'(z)^2,$$

where $f$ is a conformal self-map of the base Riemann surface. For the case of universal Teichmüller space, the base Riemann surface is the upper half-plane and $f$ is a real Möbius transformation. Finally, one shows that the constant $c$ is equal to 1: for this step, see [19, 33].

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CHAPTER 15

# Application of Conformal and Quasiconformal Mappings and Their Properties in Approximation Theory

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**Abstract**

This is a survey of some results in the constructive theory of functions of a complex variable, obtained by the author and his collaborators during the last 15–20 years by an application of methods and results from modern geometric function theory and the theory of quasiconformal mappings.

HANDBOOK OF COMPLEX ANALYSIS: GEOMETRIC FUNCTION THEORY, VOLUME 1
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The study of polynomial approximation of a function $f \in A(K)$, continuous on a compact set $K \subset \mathbb{C}$ and analytic at its interior points, has a rather long history, in the course of which approximation theory was reshaped several times in response to the challenges posed by a series of radically new problems. The main aim of this survey is to acquaint the reader with some results in constructive theory of functions of a complex variable obtained by the author and his collaborators during the last 15–20 years by an application of methods and results from modern geometric function theory and the theory of quasiconformal mappings.

There is an extensive bibliography dealing with this field. We are compelled to cite here only papers directly related to the presented results. For more detail, see bibliography in cited papers and in monographs [26,36,27,33,16] as well as survey papers [25,22].

As a starting point of our paper we choose Mergelian's theorem (see Section 1), which gives the rate of uniform polynomial approximation of a function $f \in A(K)$ on an arbitrary continuum $K$ with connected complement (with respect to the extended complex plane $\overline{\mathbb{C}}$). We discuss in detail different refinements of this result (see Section 3).

Section 2 includes an auxiliary material about geometry of continua in the complex plane.

We often use the notions of direct and inverse theorems which can be described as follows. A *direct theorem* is one in which the properties of $f$ are in the hypothesis and the rate of convergence of polynomials to $f$ is the conclusion. An *inverse theorem* is in the converse direction, that is, the rate of convergence of polynomials to $f$ is in the hypothesis and the properties of $f$ form the conclusion.

The next crucial achievement was due to V.K. Dzjadyk. In the late fifties – early sixties he showed that a constructive description of Hölder classes (subclasses of $A(K)$ which are quite natural to consider) requires a "nonuniform scale" of approximation on the boundary $\partial K$. In what follows this approximation will be called *Dzjadyk type approximation*. In Section 4 we give a practically complete description of continua on which the functions under consideration admit a Dzjadyk type approximation.

Section 5 is devoted to approximation theory for functions (not necessarily analytic) on a quasidisk. The theory of quasiconformal mappings (see [1,28]) and the local theory of distance distortion under conformal mappings which is due to V.I. Belyi (see [21,16]) work here in an extremely effective way.

Section 6 is devoted to the approximation of harmonic functions by harmonic polynomials. We are interested in this problem because of the following circumstance: In spite of the fact that the overwhelming majority of problems of analytic and harmonic approximation are almost alike and the formulations of the results are practically the same, there are some unexpected exceptions from this rule. We are going to clarify the essence of this effect.

## 1. Polynomial approximation on general continua

Let $K \subset \mathbb{C}$ be a compact set with connected complement $\Omega := \overline{\mathbb{C}} \setminus K$, where $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. We assume that $K$ is an infinite set of points (the nontrivial case for the questions discussed below).

For $f$ given on $K$, set

$$\|f\|_K := \sup\{|f(z)|: z \in K\}.$$

Denote by $A(K)$ the class of all functions $f$ continuous on $K$ and analytic in its interior $K^\circ$. Let $\mathbb{P}_n$, $n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$, be the set of all complex polynomials of degree at most $n$.

Set

$$E_n(f, K) := \inf\{\|f - p\|_K: p \in \mathbb{P}_n\}, \quad f \in A(K), \ n \in \mathbb{N}_0.$$

By the well-known Mergelian theorem (see [35]),

$$\lim_{n\to\infty} E_n(f, K) = 0$$

for any $f \in A(K)$.

The next result, which is also due to Mergelian, can be considered a starting point for the material presented in this paper.

For $f$ given on $K$ we introduce its *modulus of continuity*

$$\omega_{f,K}(\delta) := \sup\{|f(z_1) - f(z_2)|: z_1, z_2 \in K, \ |z_1 - z_2| \leqslant \delta\}, \quad \delta > 0.$$

In what follows we denote by $c, c_1, \ldots$ positive constants and by $\varepsilon, \varepsilon_1, \ldots$ sufficiently small positive constants (each time different in general) that either are absolute or depend on parameters not essential for the arguments; otherwise, such a dependence will be indicated. We will use the symbol $a \preceq b$ to mean $a \leqslant cb$ and $a \asymp b$ to mean that both $a \preceq b$ and $b \preceq a$.

Let $\omega(\delta)$, $\delta > 0$, be a function of the *modulus of continuity type*, i.e., positive, nondecreasing (with $\omega(+0) = 0$) and satisfying

$$\omega(t\delta) \leqslant ct\omega(\delta), \quad \delta > 0, \ t > 1,$$

for some $c \geqslant 1$.

Denote by $A^\omega(K)$ the class of all functions $f \in A(K)$ satisfying

$$\omega_{f,K}(\delta) \leqslant c\omega(\delta), \quad \delta > 0.$$

Let $B = K$ be a continuum, i.e., $\Omega$ is simply connected. Denote by $\Phi$ the conformal mapping of $\Omega$ onto the exterior $\Delta := \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ of the unit disc $\mathbb{D} := \{z: |z| < 1\}$, normalized by the conditions $\Phi(\infty) = \infty$, $\Phi'(\infty) > 0$.

For $u > 0$ set

$$L_u := \{\zeta: |\Phi(\zeta)| = 1 + u\}, \quad L := \partial B,$$

$$d(u) := \sup\{\operatorname{dist}(z, L): z \in (\operatorname{int} L_u) \setminus B\},$$

where by $\operatorname{int}\Gamma$ we understand the bounded domain whose boundary coincides with $\Gamma$ (it is assumed that $\Gamma$ is a bounded Jordan curve),

$$\operatorname{dist}(A', A'') := \inf\{|z - \zeta|: z \in A', \zeta \in A''\}, \quad A', A'' \subset \mathbb{C}.$$

THEOREM (Mergelian, see [35]). *Let $B$ be a continuum with connected complement and let $f \in A^{\omega}(B)$. Then*

$$E_n(f, B) \leqslant c\omega\left(d\left(\frac{\log n}{n}\right)\right), \quad n = 2, 3, \ldots, \tag{1.1}$$

*where the constant $c$ is independent of $n$.*

The main line of this paper is to impose certain conditions upon $B$ in order to improve (1.1) and even to present "more subtle" estimates (instead of (1.1)) for polynomial approximation which are unimprovable for the classes $A^{\omega}(B)$ and in many cases even characterize them.

## 2. Classes of sets

Below we introduce different classes of planar sets which play a significant role in our constructions. Most of them as, e.g., quasiconformal curves and arcs (this concept was introduced by L. Ahlfors and appears in many different contexts), John domains, the class $S$ (i.e., arcs and curves satisfying the so-called Carleson's condition) are considered in different branches of mathematics. In the next sections we explain how they work in Approximation Theory. The class of continua $H^*$ is especially typical for the Dzjadyk type approximation where it has a necessary and sufficient character for the corresponding direct theorems.

**2.1.** An important role in our consideration will be played by domains bounded by a quasiconformal curve or quasiconformal arcs. We recall that, by definition, a $K$-quasiconformal ($K \geqslant 1$) or briefly *quasiconformal curve* is the image of the unit circle under some $K$-quasiconformal mapping $F: \overline{\mathbb{C}} \to \overline{\mathbb{C}}$. Any subarc of a $K$-quasiconformal curve is called a $K$-quasiconformal or briefly a *quasiconformal arc*.

There exists a geometric characterization of quasiconformal curves and arcs. For example, for curves it is the well-known Ahlfors' theorem which can be formulated as follows (cf. [28]): $L$ is a quasiconformal curve if and only if there exists a constant $c > 0$, depending only on $L$, such that for $z_1, z_2 \in L$,

$$\min\{\operatorname{diam} L', \operatorname{diam} L''\} \leqslant c\,|z_1 - z_2|, \tag{2.1}$$

where $L'$ and $L''$ denote the two arcs of which $L \setminus \{z_1, z_2\}$ consists. Moreover, the constant $c$ in (2.1) and the coefficient of quasiconformality $K$ of $L$ are mutually dependent.

The corresponding result for arcs (see [31,16]) repeats practically word for word Ahlfors' theorem.

A domain bounded by a quasiconformal curve is called a *quasidisk*.

If in (2.1) we use the length (of $L'$ and $L''$) instead of the diameter, we obtain the notion of so-called *quasismooth curves* (in the sense of Lavrentiev), see [30]. Any subarc of quasismooth curve is called a *quasismooth arc*.

**2.2.** Next, we introduce the notion of John domains playing an important role in Geometric Function Theory (cf. [30]).

A domain $\Omega \subset \overline{\mathbb{C}}$, $\infty \in \Omega$, is called a *John domain*, if any point $\zeta \in \Omega \setminus \{\infty\}$ can be joined to infinity by a Jordan arc $\gamma = \gamma(\zeta, \infty) \subset \Omega$ such that

$$\operatorname{dist}(z, \partial\Omega) \geqslant c \operatorname{length} \gamma(\zeta, z)$$

for each point $z \in \gamma$, where $\gamma(\zeta, z)$ denotes the subarc of $\gamma$ lying between the corresponding points.

Sometimes, however it is more convenient to use another definition of a John domain, which we are going to introduce now.

A bounded Jordan domain $G$ is called a *$k$-quasidisk*, $0 \leqslant k < 1$, if any conformal mapping of the unit disk $\mathbb{D}$ onto $G$ can be extended to a $K$-quasiconformal homeomorphism of $\overline{\mathbb{C}}$ onto itself, $K := (1+k)/(1-k)$.

It is easy to verify that, for example, the domain $G = G(k, \delta)$, $0 \leqslant k < 1$, $\delta > 0$, which is symmetric with respect to the real and imaginary axes and bounded by two circular arcs which meet in an inner angle of $\pi(1-k)$ at the vertices $\pm\delta$, is a $k$-quasidisk.

We say that $\Omega$ satisfies a *$k$-quasidisk condition*, $0 \leqslant k < 1$, if for each point $\zeta \in \Omega$, there exists a $k$-quasidisk $D_\zeta \subset \Omega$ such that $\zeta \in \partial D_\zeta$ and $\operatorname{diam} D_\zeta \geqslant c$.

THEOREM 1 [14]. *A domain $\Omega$ is a John domain if and only if for some $0 \leqslant k < 1$ it satisfies a $k$-quasidisk condition.*

In particular, the complement of each closed quasidisk or quasiconformal arc is a John domain.

**2.3.** Next, we introduce the classes $H$ and $H^*$ of continua and discuss their properties in detail.

Let $B \subset \mathbb{C}$ be a continuum with connected complement $\Omega := \overline{\mathbb{C}} \setminus B$.

We say that $B \in H$, if every pair of points $z, \zeta \in B$ can be joined by an arc $\gamma(z, \zeta) \subset B$ whose length satisfies

$$\operatorname{length} \gamma(z, \zeta) \leqslant c\,|z - \zeta|. \tag{2.2}$$

By virtue of their definition continua of the class $H$ can be called continua without external zero angles. We will denote by the symbol $\Phi$ also the homeomorphism between a compactification $\widetilde{\Omega}$ of the domain $\Omega$ by prime ends (see [30]) and $\overline{\Delta}$ which in $\Omega$ coincides with $\Phi(z)$. Let $\Psi := \Phi^{-1}$, $\tilde{L} := \widetilde{\Omega} \setminus \Omega$.

THEOREM 2 [8]. *Suppose that $B \in H$. In this case*

(i) *all prime ends $Z \in \tilde{L}$ are of the first kind, i.e., they have one-point impressions $|Z| \in L$,*

(ii) *each point $z \in L$ is of a multiplicity not greater than a certain number $k = k(B) \geqslant 1$, i.e., $z$ can be an impression of at most $k$ prime ends.*

We say that an arc or curve $\gamma \subset \Omega$ separates the prime ends $Z_1, Z_2, \ldots \in \widetilde{\Omega}$ from $\mathcal{Z}_1, \mathcal{Z}_2, \ldots \in \widetilde{\Omega}$, if $\Omega \setminus \gamma$ consists of two connected components such that for one of them $Z_1, Z_2, \ldots$ are adjacent and for the other $\mathcal{Z}_1, \mathcal{Z}_2, \ldots$ are adjacent. The term *adjacent* means for $Z \in \widetilde{\Omega}$ that in the domain and subdomain the prime end can be defined by the same null-chain of crosscuts.

We denote by $\gamma_Z(r)$, $Z \in \tilde{L}$, a subarc of the circle $\{\zeta\colon\ |\zeta - |Z|| = r\}$ that separates $Z$ from $\infty$ (if there are several such arcs, we take for $\gamma_Z(r)$ the arc that separates $Z$ from all the others). For $0 < r < R < d/2$, $d := \operatorname{diam} B$, the arcs $\gamma_Z(r)$ and $\gamma_Z(R)$ are the sides of a certain quadrilateral $Q_Z(r, R) \subset \Omega$ such that its other two sides are parts of the boundary $L$. We denote by $m_Z(r, R)$ its module, i.e., the module of the family of arcs which separate in $Q_Z(r, R)$ the sides $\gamma_Z(r)$ and $\gamma_Z(R)$ (see [1,28]).

THEOREM 3 [8]. *Suppose that $B \in H$, $Z \in \tilde{L}$, $0 < r_1 < r_2 < r_3 < d/2$. Then*

$$0 \leqslant m_Z(r_1, r_3) - \bigl(m_Z(r_1, r_2) + m_Z(r_2, r_3)\bigr) \leqslant c_1,$$

$$\frac{1}{2\pi}\log\frac{r_2}{r_1} \leqslant m_Z(r_1, r_2) \leqslant c_2 \log\frac{r_2}{r_1} + c_3.$$

Following [2] we say that $B \in H^*$, if $B \in H$ and there exist positive constants $c$ and $\varepsilon$ such that for any prime ends $Z \in \tilde{L}$ and $\mathcal{Z} \in \tilde{L}$ with the property $|z - \zeta| < \varepsilon$, $z := |Z|$, $\zeta := |\mathcal{Z}|$, the inequality

$$\bigl|m_Z\bigl(|z - \zeta|, \varepsilon\bigr) - m_{\mathcal{Z}}\bigl(|z - \zeta|, \varepsilon\bigr)\bigr| \leqslant c \tag{2.3}$$

holds.

Assume that $Z \in \tilde{L}$, $\mathcal{Z} \in \widetilde{\Omega}$. We denote by $r_Z(\mathcal{Z})$ the supremum of those $r > 0$ for which the arc $\gamma_Z(r)$ separates $Z$ from $\mathcal{Z}$.

If $B \in H$, then directly from the definition, we have

$$|z - \zeta| \leqslant c\, r_Z(\mathcal{Z}), \quad z := |Z|,\ \zeta := |\mathcal{Z}|.$$

The following result is useful for verifying condition (2.3).

THEOREM 4 [8]. *Assume that $B \in H$, $Z \in \tilde{L}$, $\mathcal{Z} \in \tilde{L}$, $z := |Z|$, $\zeta := |\mathcal{Z}|$, $0 < |z - \zeta| < \varepsilon < d/2$. Then*

(i) *if $r_Z(\mathcal{Z}) \leqslant c|z - \zeta|$, then relation* (2.3) *holds,*

(ii) *if $r_Z(\mathcal{Z}) \geqslant |z - \zeta|$, then relation* (2.3) *is equivalent to the condition*

$$\bigl|m_Z\bigl(|z - \zeta|, r_Z(\mathcal{Z})\bigr) - m_{\mathcal{Z}}\bigl(|z - \zeta|, r_Z(\mathcal{Z})\bigr)\bigr| \leqslant c. \tag{2.4}$$

The connection between the moduli of quadrilaterals and their configurations has been extensively investigated. Consequently, the conditions $B \in H$, (2.3) and (2.4) yield a rather complete description of the geometry of the continuum $B \in H^*$. In particular, the closure of a domain with quasiconformal boundary and sets of $B_k^*$ type studied by Dzjadyk belong to $H^*$ (see [2]).

The arcs of the class $H$ are refered to as quasismooth. Each point $z \in L$, except for its ends, is the impression of two different prime ends $Z_1 \in \tilde{L}$ and $Z_2 \in \tilde{L}$.

THEOREM 5 [8]. *Let $L$ be a quasismooth arc with endpoints $\zeta_1$ and $\zeta_2$. The following conditions are equivalent*:

(i) $L \in H^*$;

(ii) *For* $z = |Z_1| = |Z_2| \in L \setminus \{\zeta_1, \zeta_2\}$ *and* $0 < \delta < \varepsilon_z := \min\{|z - \zeta_1|, |z - \zeta_2|\}$ *the inequality*

$$\left|m_{Z_1}(\delta, \varepsilon_z) - m_{Z_2}(\delta, \varepsilon_z)\right| \leqslant c \tag{2.5}$$

*holds.*

Theorem 5 provides a description of arcs belonging to $H^*$ convenient for geometric verification. Condition (2.5) in a certain sense reflects the "symmetry" of the arc $L \in H^*$ and is associated with the smoothness of $L$. However, examples show that (2.5) and the smoothness of $L$ have essentially different nature.

**2.4.** Let $B \subset \mathbb{C}$ be a continuum with simply connected complement $\Omega$. For $z \in \mathbb{C}$ and $\delta > 0$, we set

$$D(z, \delta) := \{\zeta\colon |\zeta - z| < \delta\}, \qquad d(z, B) := \operatorname{dist}(z, B),$$

$$B_\delta := \bigcup_{z \in B} D(z, \delta) = \{\zeta\colon d(\zeta, B) < \delta\}.$$

We say that $B \in R$ if there exists a constant $c = c(B) > 1$ such that for any point $z \in \partial B$ and $\delta > 0$ we have

$$D(z, c\delta) \cap (\mathbb{C} \setminus B_\delta) \neq \emptyset.$$

THEOREM 6 [5]. $H \subset R$.

We associate the arc or curve $L$ with the class $S$, if for $z \in L$ and $\delta > 0$,

$$\operatorname{length} L \cap D(z, \delta) \leqslant c\delta \tag{2.6}$$

with some $c = c(L) > 1$. Inequality (2.6) is sometimes called the *Carleson condition*.

THEOREM 7 [4]. *Each arc $L \in S$ belongs to $R$.*

## 3. Uniform approximation. Approximation characteristic

**3.1.** We set

$$\delta_n := \sup\{\operatorname{dist}(\zeta, B)\colon\ \zeta \in \operatorname{int} L_{1/n}\}, \quad n \in \mathbb{N}.$$

THEOREM 8 [5]. *Let $B \in R$ and $f \in A^{\omega}(B)$. Then there exists a sequence of polynomials $p_n \in \mathbb{P}_n$, $n \in \mathbb{N}$, such that for $z \in B$ and $n \in \mathbb{N}$,*

$$|f(z) - p_n(z)| \leqslant c_1 \omega(\delta_n), \tag{3.1}$$

$$|p_n'(z)| \leqslant c_2 \frac{\omega(\delta_n)}{\delta_n}, \tag{3.2}$$

*where the constants $c_j$, $j = 1, 2$, are independent of $z$ and $n$.*

The existence of a sequence of polynomials satisfying inequality (3.1) is in some sense the standard assertion of direct theorems in Constructive Function Theory. It turns out that it is not possible to improve this result as shown by N.A. Shirokov [34].

In some cases we can invert Theorem 8, obtaining in this way an *approximation characterization* of the classes $A^{\omega}(B)$. Therefore, the missing condition for a description of the classes $A^{\omega}(B)$ is the restriction (3.2) on the growth of the derivatives of the polynomials approximating the function.

THEOREM 9 [5]. *Let $B \in H$. Then the relation $f \in A^{\omega}(B)$ is equivalent to the existence of a sequence of polynomials satisfying the relations* (3.1) *and* (3.2).

Next, we state some results about polynomial approximation on compact sets whose complement is a John domain (not necessarily simply connected).

THEOREM 10 (cf. [14]). *Let $K \subset \mathbb{C}$ be a compact set whose complement $\Omega := \overline{\mathbb{C}} \setminus K$ satisfies a $k$-quasidisk condition, $0 \leqslant k < 1$. Then, for any $f \in A^{\omega}(K)$ and $n \in \mathbb{N}$,*

$$E_n(f, K) \leqslant c\omega(n^{k-1}), \tag{3.3}$$

*where $c > 0$ is independent of $n$.*

It turns out that for each Hölder class

$$\Lambda^{\alpha}(K) := A^{\omega}(K), \quad \omega(\delta) = \delta^{\alpha},\ 0 < \alpha \leqslant 1,$$

and $0 \leqslant k < 1$, it is not possible to improve estimate (3.3).

THEOREM 11 [14]. *For any $0 \leqslant k < 1$ and $0 < \alpha < 1$, there exists a closed Jordan domain $\overline{G} = \overline{G(k)}$, whose complement satisfies a $k$-quasidisk condition, and a function $f \in \Lambda^{\alpha}(\overline{G})$ such that*

$$E_n(f, \overline{G}) \geqslant cn^{\alpha(k-1)}, \quad n \in \mathbb{N},$$

*where $c > 0$ is independent of $n$.*

**3.2.** In this section we consider the case when $\partial B$ has interior zero angles with respect to $\Omega$.

We begin with the case when $B = L$ is an arc. We say that $L \in U(\theta)$, where $\theta(\delta),\ \delta > 0$, is a positive nondecreasing function, $\theta(+0) = 0$, if

(i) $L = l^+ \cup l^-,\ l^+ \cap l^- = \{z_0\},\ l^\pm$ are quasismooth;

(ii) for the points $z \in l^\pm,\ 0 < |z - z_0| < (\operatorname{diam} l^\pm)/2$, we have the relations

$$\operatorname{dist}(z, l^\mp) \asymp |z - z_0| \theta\big(|z - z_0|\big).$$

If $\theta(\delta) = \delta^\beta,\ \beta > 0$, then the corresponding class of arcs is denoted by $U_\beta$.

The simplest example of an arc $L \in U(\theta)$ is a piecewise smooth arc, whose smooth parts make a cusp at the point $z_0$ (the fuction $\theta(\delta)$ characterizes the order of their tangency).

THEOREM 12 [4]. *Let $L \in U(\theta)$. Then $f \in A^\omega(L)$ if and only if there exists a sequence of polynomials satisfying for $z \in L$ the relations* (3.1) *and* (3.2).

The following assertion shows that the estimate (3.1) cannot be improved for each arc $L \in U_\beta$ and for each Hölder class $\Lambda^\alpha(L)$. On the other hand, it is simple to see that (3.1) is not sufficient for a characterization of the classes $\Lambda^\alpha(L)$. Hence, Theorem 12 shows that the missing condition which will allow this is given by additional information about the approximating polynomials in the form of relation (3.2).

THEOREM 13 [4]. *Let $L \in U_\beta,\ \beta > 0,\ 0 < \alpha \leqslant 1$. Then there exists $f_\alpha \in \Lambda^\alpha(L)$ such that*

$$E_n(f_\alpha, L) \geqslant c\delta_n^\alpha, \quad n \in \mathbb{N}.$$

In conclusion, we note that Theorems 12 and 13 can be extended to regions with exterior zero angles on the boundary.

**3.3.** Next, we consider the question concerning the description of classes of functions with given rate of decrease of their uniform polynomial approximations.

By a *normal majorant* we shall mean any function $\mu(\delta)$, defined, positive and nondecreasing on the half-line $\delta > 0$, for which $\mu(+0) = 0$ and so that for some constant $c > 0$ one has

$$\mu(2\delta) \leqslant c\, \mu(\delta), \quad \delta > 0.$$

For example, for any $\alpha > 0$ the function $\mu(\delta) = \delta^\alpha$ is a normal majorant.

By $E^\mu(B)$ we denote the class of functions $f \in A(B)$, for which

$$E_n(f, B) = \mathrm{O}\big(\mu(1/n)\big), \quad n \to \infty.$$

For $z \in L = \partial B$ and $\delta > 0$ we introduce $r(z, \delta)$ by the relation

$$\rho_{r(z,\delta)}(z) = \delta,$$

where

$$\rho_u(z) := \operatorname{dist}(z, L_u), \quad u > 0.$$

By the *local modulus of smoothness* of order $k \in \mathbb{N}$ we mean

$$\omega_{k,B}(f, z, \delta) := E_{k-1}\big(f, B \cap \overline{D(z, \delta)}\big), \quad \text{where } z \in B.$$

THEOREM 14 [3]. *Let $B \in H^*$, $\mu(\delta)$ be an arbitrary normal majorant. In order that $f \in E^{\mu}(B)$, it is necessary and sufficient that for some $k \in \mathbb{N}$ and $c > 0$ as well as for all $z \in L$ and $\delta > 0$ the inequality*

$$\omega_{k,B}(f, z, \delta) \leqslant c\, \mu\big(r(z, \delta)\big)$$

*holds.*

The last result can be written in a more convenient form for closed quasidisks $B = \overline{G}$. In this case we introduce the following characteristic of the smoothness properties of a function $f \in A(\overline{G})$ on $L$ or, more precisely, of the function $\tilde{f}(w) := f(\Psi(w))$ on $\mathbb{T} := \{w\colon |w| = 1\}$:

$$\overline{\omega}_k(\delta) := \sup\big\{E_{k-1}\big(f, \Psi(\gamma)\big)\colon \gamma \text{ subarc of } \mathbb{T}, \text{ length } \gamma \leqslant \delta\big\}, \quad \delta > 0,$$

where $k \in \mathbb{N}$.

When $G = \mathbb{D}$, the $\overline{\omega}_k(\delta)$ is equivalent to the $k$th modulus of continuity of the function $f$ on $\partial G$ (cf. [36]).

We note also that $\overline{\omega}_1(\delta)$ is simply equivalent to the usual modulus of continuity of the function $\tilde{f}$ on $\mathbb{T}$.

THEOREM 15 [12]. *Let $G$ be a quasidisk, $\mu(\delta)$ a normal majorant, $f \in A(\overline{G})$. In order that*

$$E_n\big(f, \overline{G}\big) = \mathrm{O}\big(\mu(1/n)\big), \quad n \to \infty, \tag{3.4}$$

*it is necessary that for all sufficiently large $k \geqslant k_0(\mu, G)$ and sufficient that for some $k \in \mathbb{N}$,*

$$\overline{\omega}_k(\delta) = \mathrm{O}\big(\mu(\delta)\big), \quad \delta \to 0.$$

Acording to Theorem 15, the inequality

$$E_n\big(f, \overline{G}\big) \leqslant c\overline{\omega}_k(1/n), \quad n \in \mathbb{N}, \tag{3.5}$$

holds for all integers $k$, where $c = c(k, G) > 0$.

In the majority of the known results of this kind the particular case of (3.5) for $k = 1$ is most popular. Unfortunately, simple examples show that even the condition $E_n(f, \overline{G}) = 0$,

$n \geqslant 1$, is not sufficient in order to assert that $\overline{\omega}_1(\delta) = O(\delta^\alpha)$ as $\delta \to 0$. This fact, in particular, explains the role of the quantity $\overline{\omega}_k(\delta)$, because the transition from $k = 1$ to an arbitrary $k \in \mathbb{N}$ gives us the possibility to obtain the description of the class of functions with property (3.4).

We complete this section with the following result.

THEOREM 16 [12]. *Let $G$ be a quasidisk, $\mu(\delta)$ a normal majorant, $f \in A(\overline{G})$. In order that*

$$c_1\mu(1/n) \leqslant E_n\big(f, \overline{G}\big) \leqslant c_2\mu(1/n), \quad n \in \mathbb{N}, \tag{3.6}$$

*holds it is necessary and sufficient that*

$$c_3\mu(\delta) \leqslant \overline{\omega}_k(\delta) \leqslant c_4\mu(\delta), \quad 0 < \delta < 1,$$

*holds for all sufficiently large $k \geqslant k_0(G, \mu)$.*

## 4. Dzjadyk type approximation

In 1959–1963 Dzjadyk obtained for certain domains $G \subset \mathbb{C}$ with piecewise smooth boundary a constructive characterization of functions that are analytic in $G$ and satisfy a Hölder condition in $\overline{G}$. The central role in the description is played by the distance $\rho_{1/n}(z)$ from the boundary point $z \in \partial G$ to the $(1 + 1/n)$-level line of the function $\Phi(z)$ which maps the domain $\overline{\mathbb{C}} \setminus \overline{G}$ conformally and univalently onto the exterior of the unit circle with standard normalization at infinity. Obviously, the distance $\rho_{1/n}(z)$ depends on the properties of the function $\Phi$ or, which is the same, on the geometric structure of the domain $G$. These results were later extended to more general sets by Dzjadyk, Lebedev, Shirokov, Tamrazov, Mikljukov, Belyi etc. (for more detail, see [26,36,33,16,25,22]).

Our aim is to present geometric conditions for the continuum $B$, which are sufficient and, under certain additional assumptions, necessary for the validity of a Dzjadyk type theorem on $B$ (see Section 4.1). The statement of these conditions makes use of the concept of a module of a quadrilateral (see [1,28]), which, although depending on the configuration of the quadrilateral, is not to the same extent a geometric characteristic as, for example, diameter or area.

This fact calls for a more detailed investigation of the geometric conditions mentioned above and their relationship with the metric properties of the mapping function $\Phi$ which was presented in Section 2.

**4.1.** We shall study functions belonging to the classes $W^rCA^\omega = W^rCA^\omega(B)$, $C =$ const $> 0$, $r \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$, $\omega(\delta)$ is a function of the modulus of continuity type; that is to say, functions analytic in $B^\circ$ and continuous in $B$, together with their derivatives up to order $r$, which satisfy

$$\big|f^{(r)}(z_1) - f^{(r)}(z_2)\big| \leqslant C\omega\big(|z_1 - z_2|\big), \quad z_1, z_2 \in B.$$

For arbitrary $z \in \mathbb{C}$ and $u > 0$, we set

$$L_u := \{\zeta\colon |\Phi(\zeta)| = 1 + u\}, \qquad \rho_u(z) := \operatorname{dist}(z, L_u).$$

THEOREM 17 (cf. [6]). *Let $B \in H^*$ and $f \in W^r CA^\omega$. Then for any integer $n \geqslant r + 1$ there is a polynomial $p_n \in \mathbb{P}_n$ such that simultaneously for all $\nu = 0, \dots, r$ and for all $z \in \partial B$, the inequality*

$$|f^{(\nu)}(z) - p_n^{(\nu)}(z)| \leqslant c\rho_{1/n}^{r-\nu}(z)\omega(\rho_{1/n}(z))$$

*holds, where $c = c(B, C, r, \omega) > 0$.*

In [20] this result is extended to classes of functions defined by means of the $k$th modulus of continuity, $k > 1$ (for particular cases, see [26,32,23]).

We say that a continuum $B$ has the *D-property* if, for every $f \in W^r CA^\omega$, where $r \in \mathbb{N}_0$, $C = \text{const} > 0$, $\omega(\delta)$ a function of the modulus of continuity type, there is a sequence of polynomials $\{p_n\}_{n \geqslant r+1}$, $p_n \in \mathbb{P}_n$, for which the inequality

$$|f(z) - p_n(z)| \leqslant c\rho_{1/n}^r(z)\omega(\rho_{1/n}(z)), \quad z \in L,$$

holds with some $c = c(B, C, r, \omega) > 0$.

COROLLARY 1. *A continuum $B \in H^*$ has the D-property.*

The $D$-property is of interest because, together with the inverse theorems (cf. [36,26]) it makes it possible, under some standard restriction on $\omega(\delta)$, to obtain a constructive characterization of the classes $W^r A^\omega := \bigcup_{C>0} W^r CA^\omega$.

THEOREM 18 [36,26,16]. *Let $B$ have the D-property, $\omega(t)$ be a function of the modulus of continuity type which satisfies the condition*

$$\delta \int_\delta^1 \frac{\omega(u)}{u^2}\, du \leqslant c_1\omega(\delta), \quad 0 < \delta < 1,$$

*and, if we consider the classes $W^r A^\omega$, $r = 1, 2, \dots$, satisfies the additional condition*

$$\int_0^\delta \frac{\omega(u)}{u}\, du \leqslant c_2\omega(\delta).$$

*Then in order that, for some $r \geqslant 0$, the function $f \in A(B)$ belongs to the class $W^r A^\omega$, it is necessary and sufficient that for this function for any $n = r + 1,\ r + 2,\ \dots$ there is a polynomial $p_n \in \mathbb{P}_n$ such that the inequality*

$$|f(z) - p_n(z)| \leqslant c_3\rho_{1/n}^r(z)\omega(\rho_{1/n}(z)), \quad z \in \partial B,$$

*holds, where the constant $c_3$ is independent of $z$ and $n$.*

We now formulate a geometric necessary condition on the structure of a continuum $B$ to have the $D$-property.

Let $B = \overline{G}$ be the closure of a Jordan domain $G$. For $z \in L$ and $0 < r < \frac{1}{2} \operatorname{diam} G$ we introduce, along with the arc $\gamma_z(r)$, the arc $s_z(r)$ defined as the shortest subarc of the circle $\{\zeta\colon |\zeta - z| = r\}$ containing $\gamma_z(r)$ and all the intersections of this circle with $L$. We denote by $\gamma_z'(r) \subset L$ the arc consisting of the points of $L$ which $\gamma_z(r)$ separates from $\infty$.

We begin with the assumption that in a neighborhood of a point $z_0 \in L$ the condition (2.2) that $\overline{G} \in H$ is violated.

THEOREM 19 [6]. *Let $\varepsilon > 0$ be sufficiently small. By $z_0$ the arc $\gamma_{z_0}'(\varepsilon)$ is divided into two parts $l^+$ and $l^-$. Suppose that*

(i) $G = G^+ \cup G^-$, $\overline{G}^\pm \in H$, $L \setminus l^\pm \subset \partial G^\mp$;

(ii) *there is a normal majorant $\theta(\delta)$ that satisfies*

$$\lim_{\delta \to 0} \theta(\delta) \log \delta = 0,$$

*and has the property that the following relations hold for all $z \in l^\pm$:*

$$\operatorname{dist}(z, l^\mp) \asymp \operatorname{length} \gamma_{z_0}\big(|z - z_0|\big) \asymp \operatorname{length} s_{z_0}\big(|z - z_0|\big) \asymp |z - z_0| \theta\big(|z - z_0|\big).$$

*Then $\overline{G}$ does not possess the $D$-property.*

Consequently, Theorem 19 shows that the presence of exterior cusps on the boundary of $G$, under some mild supplementary restrictions, yields that $\overline{G}$ fails to have the $D$-property. For instance, the closed domain

$$\overline{G}^* := \{z\colon |z - i| \leqslant 1\} \cup \{z\colon |z + i| \leqslant 1\} \cup \{z\colon |z| \leqslant 2, \operatorname{Re} z \leqslant 0\}$$

does not have the $D$-property.

THEOREM 20 (cf. [6]). *If $B \in H$ has the $D$-property, then $B \in H^*$.*

The last theorem shows that condition (2.3), which is in the base for the definition of the class $H^*$, has both a necessary character for the $D$-property (Theorem 20) and a sufficient character (Corollary 1).

**4.2.** In this subsection we are going to discuss the question of approximation of functions on quasismooth arcs. From one side such arcs are elements of the class $H$ and, consequently, the results of the previous section can be applied in this situation. From the other side the set of quasismooth arcs with the $D$-property does not include, for example, such a simple arc as the angle $[0, 1] \cup [0, i]$. Our aim is to derive the concept of pointwise estimates for the rate of polynomial approximation on arcs.

We begin with the following problem (which is an analogue of the classical de la Vallée Poussin problem of finding the exact order of $E_n(|x|, [-1, 1])$. Let $B = L$ be an arc and let $z_0 \in L$ be fixed. Let $L'$ and $L''$ denote the components of $L \setminus \{z_0\}$.

Consider the function

$$f_0(z) := \begin{cases} z - z_0, & z \in \overline{L'}, \\ z_0 - z, & z \in L''. \end{cases}$$

The classical case corresponds to $L = [-1, 1]$, $z_0 = 0$, $L' = (0, 1]$, $L'' = [-1, 0)$, $f_0(z) = f_0(x) = |x|$.

The problem is to find the rate of $E_n(f_0, L)$.

Denote the endpoins of $L$ by $z_1$ and $z_2$ and set

$$w_j := \Phi(z_j), \quad j = 1, 2,$$
$$\Delta_1 := \{w\colon |w| > 1,\ \arg w_1 < \arg w < \arg w_2\}, \qquad \Delta_2 := \Delta \setminus \overline{\Delta_1},$$
$$\rho^*_{1/n}(z) := \max_{j=1,2} \operatorname{dist}\bigl(z, L_{1/n} \cap \Omega^j\bigr), \qquad \Omega^j := \Psi(\Delta_j), \quad j = 1, 2.$$

THEOREM 21 [16]. *Let $L$ be a quasismooth arc. Then*

$$E_n(f_0, L) \asymp \rho^*_{1/n}(z_0).$$

Since under the conditions of Theorem 21, $f_0 \in \Lambda^1(L)$, it is intuitively clear what has to be the analogue of the Dzjadyk type theorem for arcs.

THEOREM 22 [16]. *Let $L$ be a quasiconformal arc, $f \in A^{\omega}(L)$. Then for each $n \in \mathbb{N}$, there exists $p_n \in \mathbb{P}_n$ such that*

$$\bigl|f(z) - p_n(z)\bigr| \leqslant c\omega\bigl(\rho^*_{1/n}(z)\bigr), \quad z \in L, \tag{4.1}$$

*where the constant $c > 0$ is independent of $n$ and $z$.*

According to Theorem 21, the estimate in Theorem 22 cannot be improved.

The quasiconformality of $L$ in Theorem 22 is also essential in the following sense. Consider the Jordan arc

$$L = \{\zeta\colon |\zeta - \mathrm{i}| = 1\colon -\pi/2 \leqslant \arg(\zeta - \mathrm{i}) \leqslant 0\}$$
$$\cup \{\zeta\colon |\zeta + \mathrm{i}| = 1\colon 0 \leqslant \arg(\zeta + \mathrm{i}) \leqslant \pi/2\}.$$

An easy calculation leads for $n \geqslant 2$ to the relations

$$\delta_n \asymp \log^{-2} n, \qquad \rho_n := \sup\{\rho^*_{1/n}(z)\colon z \in L\} \asymp \log^{-1} n.$$

Thus, applying Theorem 8 to $L$ ($\in R$) gives an essentially better estimate for the quantity $E_n(f, L)$ than one could have expected from weakening inequality (4.1). This fact shows that the violation of quasiconformality of the arc under consideration leads to the nonoptimality of the estimate (4.1) for such arcs.

Next, we are going to compare the following three subclasses of $A(L)$:

$$\Lambda^{\alpha}(L) := \{f \in A^{\omega}(L)\colon \omega(\delta) = \delta^{\alpha}\}, \quad 0 < \alpha \leqslant 1,$$
$$E^{\alpha}(L) := \{f \in A(L)\colon E_n(f, L) = O(n^{-\alpha})\}, \quad \alpha > 0,$$
$$B^{\alpha}(L) := \{f \in A(L)\colon \exists\{p_n\}_{n\in\mathbb{N}}\colon |f(z) - p_n(z)| \leqslant c_f(\rho^*_{1/n}(z))^{\alpha},\ z \in L\},$$
$$\alpha > 0.$$

We recall some well-known facts.

By Theorem 22, for any quasiconformal arc $L$,

$$\Lambda^{\alpha}(L) \subset B^{\alpha}(L), \quad 0 < \alpha \leqslant 1.$$

By the Nikolski–Timan–Dzjadyk theorem,

$$\Lambda^{\alpha}([-1,1]) = B^{\alpha}([-1,1]), \quad 0 < \alpha < 1.$$

At the same time for $L_0 := [0, 1] \cup [0, i]$ it can be proved that

$$\Lambda^{\alpha}(L_0) \neq B^{\alpha}(L_0), \quad 0 < \alpha < 1.$$

By the classical Bernstein and Jackson theorems for functions of a real variable,

$$E^{2\alpha}([-1,1]) \subset \Lambda^{\alpha}([-1,1]) \subset E^{\alpha}([-1,1]), \quad 0 < \alpha < 1.$$

In order to compare the classes $\Lambda^{\alpha}(L)$, $E^{\alpha}(L)$ and $B^{\alpha}(L)$ it would be useful to find their description from a "uniform" point of view, i.e., using the same notions.

Denote by $\Lambda(L)$ the class of functions $\lambda(\zeta)$, defined and continuous on $\Omega$ and such that

$$\lambda(\zeta) = 0, \quad |\zeta| > c = c(\lambda).$$

By

$$F_{\lambda}(z) := -\frac{1}{\pi}\iint_{\Omega} \frac{\lambda(\zeta)}{\zeta - z}\, dm(\zeta), \quad z \in L,$$

we denote the *Cauchy transform* of $\lambda \in \Lambda(L)$, where $dm(\zeta)$ means integration with respect to the two-dimensional Lebesgue measure (area).

THEOREM 23 [19]. *Let $L$ be quasismooth, $0 < \alpha < 1$. The following conditions are equivalent*:

(i) $f \in \Lambda^{\alpha}(L)$;

(ii) $f = F_{\lambda}$ *for some* $\lambda \in \Lambda(L)$ *with*

$$|\lambda(\zeta)| \leqslant c[\operatorname{dist}(\zeta, L)]^{\alpha-1}, \quad \zeta \in \Omega.$$

THEOREM 24 [19]. *Let $L$ be quasismooth, $\alpha > 0$. The following conditions are equivalent*:

(i) $f \in E^\alpha(L)$;
(ii) $f = F_\lambda$ *for some* $\lambda \in \Lambda(L)$ *with*

$$|\lambda(\zeta)| \leqslant c\frac{(|\Phi(\zeta)|-1)^\alpha}{\operatorname{dist}(\zeta, L)}, \quad \zeta \in \Omega.$$

Next, for $\zeta \in \Omega \setminus \{\infty\}$ we introduce the following transformations:

$$\begin{aligned}\zeta \in \Omega &\to w := \Phi(\zeta) \to w_L := w/|w| \to \zeta_L := \Psi(w_L)\\ &\to w_L^* := \Phi(\zeta_L) \neq w_L \,(\text{unless } \zeta_L \text{ endpoint})\\ &\to w^* := w_L^*|w| \to \zeta^* := \Psi(w^*).\end{aligned}$$

Thus, the correspondence $\zeta \to \zeta^*$ defines a certain reflection in the arc $L$.

THEOREM 25 [19]. *Let $L$ be quasismooth, $\alpha > 0$. The following conditions are equivalent*:

(i) $f \in B^\alpha(L)$;
(ii) $f = F_\lambda$ *for some* $\lambda \in \Lambda(L)$ *with*

$$|\lambda(\zeta)| \leqslant c\frac{(\operatorname{dist}(\zeta, L) + \operatorname{dist}(\zeta^*, L))^\alpha}{\operatorname{dist}(\zeta, L)}, \quad \zeta \in \Omega.$$

We complete this section with the following remarks.

If $L = [-1, 1]$, then $\operatorname{dist}(\zeta, L) = \operatorname{dist}(\zeta^*, L)$ and by Theorems 23 and 25,

$$\Lambda^\alpha\big([-1,1]\big) = B^\alpha\big([-1,1]\big)$$

(cf. the Nikolski–Timan–Dzjadyk theorem).

If $L = [0, 1] \cup [0, i]$, then $\operatorname{dist}(\zeta, L)$ and $\operatorname{dist}(\zeta, L)+ \operatorname{dist}(\zeta^*, L)$ are not equivalent and the previous remark is not valid.

## 5. Approximation of polyanalytic fuctions on a quasidisk

We consider different possible generalizations and extentions of results from the previous sections to approximation by aggregates which are also "polynomials" but not analytic as before. For details, see [18,17].

One of the natural generalizations of analytic functions comprises the solutions of the equation

$$\bar{\partial}^j f = 0, \tag{5.1}$$

where $j \geqslant 1$ and $\overline{\partial} := \frac{1}{2}(\partial/\partial x + i\partial/\partial y)$ is the *Cauchy–Riemann operator* in the complex plane (of the variable $z = x + iy$).

The main purpose of this section is to present direct and inverse theorems of approximation theory that describe the quantitative connection between the rate of approximation of a solution of (5.1) by polynomial modules and the smoothness properties of the functions being approximated, in particular, a theorem that gives a constructive characterization of Hölder classes of such functions on quasidisks.

For an arbitrary compact set $K \subset \mathbb{C}$ let $A_j(K)$ denote the class of functions that are solutions of (5.1) in $K^\circ$ and are continuous on $K$. It turns out that an arbitrary function $f \in A_j(K)$ can be (uniquely) represented in $K^\circ$ in the form

$$f(z) = \sum_{n=0}^{j-1} \overline{z}^n f_n(z),$$

where the $f_n(z)$ are functions analytic in $K^\circ$.

As the example of $K = \{\zeta\colon |\zeta - 1| \leqslant 1\}$ and $f(z) = \overline{z}z^{-\alpha}$, $0 < \alpha < 1$, shows, the coefficients $f_n(z)$ in this expansion can have a discontinuity on the boundary of $K$. Consequently, the problem of uniform approximation of a function of the class $A_j(K)$ does not reduce in general to the approximation of the coefficients $f_n(z)$ by analytic polynomials. The situation is further complicated by the fact that such fundamental properties of analytic functions as the maximum principle, the uniqueness theorem, etc., do not hold in general for functions of the class $A_j(K)$.

As approximating functions we use the "polynomial modules" of the form

$$P(z) = \sum_{k=0}^{j-1} \overline{z}^k p_k(z),$$

where the $p_n(z)$ are analytic polynomials. This form of approximating aggregates is natural and is dictated by the structure of the solutions of (5.1).

Next, we describe the main results. Let $G$ be a quasidisk and let $y(\zeta)$ be a quasiconformal reflection with respect to $L := \partial G$, i.e., an orientation-changing quasiconformal mapping of $\overline{\mathbb{C}}$ onto itself that carries $G$ into $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$ and vice versa, and that leaves the points of the curve $L$ fixed (see [1,28]). The function $y(\zeta)$ can be chosen so that it is continuously differentiable everywhere exept at the points of $L$ and the point $0 \in G$, with

$$|y(\zeta)| \asymp |\zeta|^{-1}, \qquad |\operatorname{grad} y(\zeta)| \asymp |\zeta|^{-2} \quad \text{as } \zeta \to 0.$$

Let $\Phi : \Omega \to \Delta$ denote the Riemann mapping function (with standard normalization at $\infty$). This function can be naturally extended to a homeomorphism between the closed domains $\overline{\Omega}$ and $\overline{\Delta}$, and we keep the previous notation for this extension.

We further extend $\Phi(z)$ to a quasiconformal mapping of $\overline{\mathbb{C}}$ onto itself by setting

$$\Phi(z) := \begin{cases} 1/\overline{\Phi(y(z))}, & z \in G \setminus \{0\}, \\ 0, & z = 0. \end{cases}$$

It is not hard to verify that the mapping $\Phi(z)$ so constructed and its inverse $\Psi(w)$ belong to the class Lip 1 on compact subsets of the interior of $G$ and interior of the unit disk, respectively.

For $u > -1$, let

$$L_u := \{\zeta\colon |\Phi(\zeta)| = 1+u\}, \qquad G_u := \operatorname{int} L_u, \qquad \Omega_u := \mathbb{C}\setminus \overline{G}_u.$$

For $z \in \mathbb{C}$ and $u > 0$ we define

$$d_u(z) := \max\{|z - \Psi(\Phi(z) + u e^{i\theta})|\colon 0 \leqslant \theta < 2\pi\}.$$

As follows from the properties of quasiconformal mappings, $d_{1/n}(z)$ has for $z \in G_{1+\varepsilon/n} \setminus G_{1-\varepsilon/n}$, $0 < \varepsilon < 1$, the same order as the quantity $\rho_{1/n}(z)$ occuring in Dzjadyk type approximation (cf. Section 4). Clearly, $d_u(z) \asymp u$ on compact sets interior to $G$.

Next, for functions of class $A_j(\overline{G})$ we introduce the concept of the modulus of smoothness of order $k \geqslant 1$. Denote by $\mathbb{P}_{n,j}$, $n \in \mathbb{N}_0$, the class of polynomial modules of the form

$$P(z) = \sum_{\nu=0}^{j-1}\sum_{\mu=0}^{n-\nu} a_{\mu,\nu} z^{\mu}\bar{z}^{\nu}, \quad a_{\mu,\nu} \in \mathbb{C}.$$

For $z \in \overline{G}$ and $\delta > 0$ we define the *local kth-order modulus of smoothness* of a function $f$ at the point $z$ with respect to the set $\overline{G}$ to be the quantity

$$\omega_{j,f,k,z,\overline{G}}(\delta) := \inf_{P\in\mathbb{P}_{k-1,j}} \sup_{\zeta\in\overline{G}\cap\overline{D(z,\delta)}} |f(\zeta) - P(\zeta)|. \tag{5.2}$$

Let $P_{j,f,k,z,\overline{G},\delta}(\zeta)$ denote the polynomial on which the infimum in (5.2) is attained. In what follows we omit some of the indices and write $\omega_{k,z}(\delta)$ and $P_{k,z,\delta}(\zeta)$ in the notation for the moduli of smoothness and the corresponding polynomials of best local approximation whenever no confusion may arise.

We distinguish two global *kth-order moduli of smoothness* for a function $f(z)$:

$$\widetilde{\omega}_{k,\overline{G}}(\delta) := \sup\{\omega_{k,z}(\delta)\colon z \in \partial G\}, \qquad \omega_{k,\overline{G}}(\delta) := \sup\{\omega_{k,z}(\delta)\colon z \in \overline{G}\}.$$

THEOREM 26 [18]. *Let $G$ be a quasidisk. Suppose that $f \in A_j(\overline{G})$ and for some $k \in \mathbb{N}$,*

$$\widetilde{\omega}_{k,\overline{G}}(\delta) \leqslant \mu(\delta), \quad \delta > 0,$$

*where $\mu(\delta)$ is a normal majorant. Then for an arbitrary (but fixed) $m \in \mathbb{N}_0$ there exists a sequence of polynomials $P_{n,m}(z) \in \mathbb{P}_{n,j}$, $n \in \mathbb{N}$, such that for $z \in \overline{G}$,*

$$|f(z) - P_{n,m}(z)| \leqslant c\mu(d_{1/n}(z))\left(\frac{d_{1/n}(z)}{|z - z^*| + d_{1/n}(z)}\right)^m,$$

*where $z^*$ denotes a point of $L := \partial G$ closest to $z$ and $c > 0$ is independent of $z$, $n$ and $f$.*

To obtain direct theorems admitting inversion in terms of uniform polynomial approximation of functions it is necessary to take into account the behaviour of the local modulus of smoothness $\omega_{k,z}(\delta)$. For $z \in \overline{G}$ let the function $r(z,\delta)$ be determined from the equality

$$d_{r(z,\delta)}(z) = \delta.$$

Denote by $E_{n,j}(f,\overline{G})$ the best uniform approximation of a function $f \in A_j(\overline{G})$ by polynomials $P_n \in \mathbb{P}_{n,j}$, $n \in \mathbb{N}_0$.

THEOREM 27 [18]. *Let $G$ be a quasidisk. Suppose that for a function $f \in A_j(\overline{G})$,*

$$\omega_{k,z}(\delta) \leqslant \mu\big(r(z,\delta)\big), \quad \delta > 0,$$

*for some $k \in \mathbb{N}$ and all $z \in L$, where $\mu(\delta)$ is a normal majorant. Then*

$$E_{n,j}\big(f,\overline{G}\big) \leqslant c\mu(1/n),$$

*where $c > 0$ which is independent of $n$ and $f$.*

THEOREM 28 [18]. *Let $G$ be a quasidisk and suppose that for a function $f(z) \in A_j(\overline{G})$ there exists a sequence of polynomials $P_n \in \mathbb{P}_{n,j}$, $n \in \mathbb{N}$, such that*

$$\big|f(z) - P_n(z)\big| \leqslant \mu\big(d_{1/n}(z)\big), \quad z \in \overline{G},$$

*where $\mu(\delta)$ is a normal majorant. Then for $k \in \mathbb{N}$,*

$$\omega_{k,\overline{G}}(\delta) \leqslant c\delta^k \int_\delta^1 \frac{\mu(t)}{t^{k+1}}\,dt, \quad 0 < \delta < 1/2.$$

Theorems 26 and 28 permit us to write

COROLLARY 2. *For a function $f \in A_j(\overline{G})$ to have a sequence of polynomials $P_n(z) \in \mathbb{P}_{n,j}$, $n \in \mathbb{N}$, such that*

$$\big|f(z) - P_n(z)\big| \preceq d_{1/n}^{k+\beta}(z), \quad z \in \overline{G},$$

*where $k \in \mathbb{N}_0$ and $0 < \beta \leqslant 1$, it is necessary and sufficient that*

$$\omega_{k+1,\overline{G}}(\delta) \preceq \delta^{k+\beta}, \quad \text{if } 0 < \beta < 1,$$

$$\omega_{k+2,\overline{G}}(\delta) \preceq \delta^{k+1}, \quad \text{if } \beta = 1.$$

Thus, Corollary 2 gives a constructive description of the solutions of (5.1) belonging to generalized Hölder classes on quasidisks.

THEOREM 29 [18]. *Let $G$ be a quasidisk and suppose that for a function $f(z) \in A_j(\overline{G})$,*

$$E_{n,j}\big(f, \overline{G}\big) \leqslant \mu(1/n),$$

*where $\mu(\delta)$ is a normal majorant. Then for some $k \in \mathbb{N}$,*

$$\omega_{k,z}(\delta) \leqslant c\mu\big(r(z,\delta)\big), \quad z \in \overline{G},\ \delta > 0.$$

Combining Theorems 27 and 29, we obtain

COROLLARY 3. *Let $G$ be a quasidisk. A function $f(z) \in A_j(\overline{G})$ satisfies the relation*

$$E_{n,j}\big(f, \overline{G}\big) \preceq \mu(1/n)$$

*if and only if for some positive integer $k$,*

$$\omega_{k,z}(\delta) \preceq \mu\big(r(z,\delta)\big), \quad z \in \overline{G},\ \delta > 0.$$

## 6. Approximation by harmonic polynomials

In this section we discuss analogues of previous results concerning approximation by analytic polynomials in the case of approximation by harmonic polynomials.

**6.1.** Denote by $\mathrm{Har}(B)$ the class of all real-valued functions $u(z)$ continuous on a continuum $B \subset \mathbb{C}$ and harmonic in $B^\circ$. Let $\omega(\delta), \delta > 0$, be a function of modulus of continuity type and let $\mathrm{Har}^\omega(B)$ denote the set of all $u \in \mathrm{Har}(B)$ such that

$$\big|u(z_1) - u(z_2)\big| \leqslant c\omega\big(|z_1 - z_2|\big), \quad z_1, z_2 \in B.$$

An expression of the form

$$t_n(z) = \mathrm{Re}\, p_n(z), \quad p_n \in \mathbb{P}_n,\ n \in \mathbb{N},$$

is called a harmonic polynomial of degree at most $n$ (briefly, $t_n \in \mathbb{T}_n$).

THEOREM 30 [7]. *Suppose that $B \in H^*$ and $u \in \mathrm{Har}^\omega(B)$. Then for each $n \in \mathbb{N}$ there exists a harmonic polynomial $t_n \in \mathbb{T}_n$ such that*

$$\big|u(z) - t_n(z)\big| \leqslant c\omega\big(\rho_{1/n}(z)\big), \quad z \in \partial B, \tag{6.1}$$

*where the constant $c > 0$ is independent of $z$ and $n$.*

This assertion is an analogue of the direct theorems on Dzjadyk-type polynomial approximation (see Section 4). For a quasidisk and $\omega(\delta)$ satisfying

$$\int_0^\delta \frac{\omega(t)}{t}\,dt \asymp \omega(\delta), \quad 0<\delta<1, \tag{6.2}$$

it can be derived by passing to completion and by an application of results from Section 4 (see [24]).

For quasidisks the direct Theorem 30 can be inverted.

THEOREM 31 [9]. *Let $G$ be a quasidisk. Then in order for $u \in \operatorname{Har}^\omega(\overline{G})$, where $\omega(\delta)$ satisfies the condition*

$$\delta\int_\delta^1 \frac{\omega(t)}{t^2}\,dt \preceq \omega(\delta), \quad 0<\delta<1,$$

*it is necessary and sufficient that there exists a sequence of harmonic polynomials $t_n \in \mathbb{T}_n$, $n \in \mathbb{N}$, for which*

$$\left|u(\zeta) - t_n(\zeta)\right| \preceq \omega\big(\rho_{1/n}(z)\big)$$

*for $z \in L$ and $\zeta \in \overline{G} \cap D(z, \rho_{1/n}(z))$.*

Superficially, the last theorem almost completely coincides with the corresponding results for analytic functions. However, the harmonic case has a number of essential peculiarities. We mention only one of them.

A central role in the proof of sufficiency in Theorem 31, as in the analytic case, is played by the following analogue of the Markov–Bernstein theorem.

LEMMA 1. *Let $G$ be a quasidisk and let the harmonic polynomial $t_n \in \mathbb{T}_n$ satisfy*

$$\left|t_n(\zeta)\right| \leqslant \omega\big(\rho_{1/n}(z)\big)$$

*for $z \in L$ and $\zeta \in \overline{G} \cap D(z, \rho_{1/n}(z))$. Then the inequality*

$$\left|\operatorname{grad} t_n(\zeta)\right| \leqslant c\,\frac{\omega(\rho_{1/n}(z))}{\rho_{1/n}(z)}$$

*holds for $z \in L$ and $\zeta \in D(z, \rho_{1/n}(z))$.*

This proposition in essence mimics the corresponding result for the analytic case (cf. [36]). At the same time, the requirement of quasiconformality in the hypotheses of Lemma 1 is essential, in contrast to the analytic case.

**6.2.** For problems on the constructive description of functions that belong to generalized Hölder classes and are analytic in $G$, by Tamrazov's results (see [36]) it makes no difference which modulus of continuity (on $\partial G$ or $\overline{G}$) is used in the definition of these classes. Simple examples show that this is not the case for harmonic functions. We discuss this phenomenon below.

We formulate the appropriate results only for the case of a quasidisk $G$ (the general theory can also be constructed, see [10]). Let $z_0 \in G$ be a given point. On the circle $\{\zeta\colon |\zeta - z| = r\}$, $z \in L := \partial G$, $0 < r < \varepsilon$, there are one or a finite number of arcs that separate $z$ from $z_0$, i.e., divide $G$ into two subdomains, one of which contains $z_0$ and the closure of the second of which contains $z$. We denote by $\gamma_z'(r)$ the arc for which the second subdomain (we denote it by $g_z(r)$) is smallest (in the sense of set-theoretical inclusion).

We denote by $m_z'(r, R)$, $0 < r < R < \varepsilon$, the module of the quadrilateral $Q_z'(r, R) := g_z(R) \setminus \overline{g_z(r)}$, namely the module of the family of arcs that separate the sides $\gamma_z'(R)$ and $\gamma_z'(r)$ in $Q_z'(r, R)$. Set

$$\alpha^*(G) := \pi \liminf_{\substack{R\to 0\\ t\to\infty}} \frac{m_z'(R/t, R)}{\log t}.$$

The parameter $\alpha^*$ describes, in a certain sense, the size of the possible angles on $L$.

Since, for any $G$,

$$m_z'(r, R) \geqslant \frac{1}{2\pi}\log\frac{R}{r}, \qquad 0 < r < R,$$

we have $\alpha^*(G) \geqslant 1/2$.

It is easily verified that if $L$ is a smooth curve (i.e., has a continuously turning tangent), then $\alpha^*(G) = 1$.

In addition, if $L$ consists of a finite number of smooth arcs which make, at their junction points, angles inside $G$ of opening $\alpha_j\pi$, $j = 1, \dots, k$, $0 < \alpha_j < 2$, then, by the properties of the moduli of quadrilaterals,

$$\alpha^*(G) = \min\Big\{1, \Big(\max_{1\leqslant j\leqslant k} \alpha_j\Big)^{-1}\Big\}.$$

For $u \in \mathrm{Har}(\overline{G})$, $E \subset \overline{G}$ and $\delta > 0$ we set

$$\omega_{u,E}(\delta) := \sup\big\{|u(z) - u(\zeta)|\colon z, \zeta \in E,\ |z - \zeta| \leqslant \delta\big\}.$$

THEOREM 32 [9]. *Let $G$ be a quasidisk and let $u \in \mathrm{Har}(\overline{G})$. Then for any sufficiently small $\varepsilon > 0$ and $0 < \delta < \delta_0 = \delta_0(\varepsilon)$,*

$$\omega_{u,\overline{G}}(\delta) \leqslant c\,\delta^{\alpha^*-\varepsilon}\left(\|u\|_{\overline{G}} + \int_\delta^{\delta_0} \frac{\omega_{u,\partial G}(t)}{t^{1+\alpha^*-\varepsilon}}\,dt\right), \tag{6.3}$$

*where $\alpha^* := \alpha^*(G)$, $c = c(\varepsilon) > 0$.*

COROLLARY 4. *If* $0 < \alpha < \alpha^*(G)$, *then, taking* $\varepsilon = (\alpha^*(G) - \alpha)/2$, *one may conclude that, for a function* $u \in \mathrm{Har}(\overline{G})$,

$$u|_L \in \Lambda^\alpha(L) \quad \Rightarrow \quad u \in \Lambda^\alpha_\Delta(\overline{G}),$$

*where* $\Lambda^\alpha(L)$ *and* $\Lambda^\alpha_\Delta(\overline{G}) \subset \mathrm{Har}(\overline{G})$ *denote the corresponding Hölder classes.*

This result (concerning the interval for the exponent $\alpha$) ist best possible.

It can also be shown that it is impossible to replace the factor $\delta^{-\varepsilon}$, $\varepsilon > 0$, on the right-hand side of (6.3) by any other function that grows more slowly as $\delta \to 0$.

Finally, we say some words about approximation of analytic functions and their real part. We begin with the following classical assertion.

THEOREM A (I.I. Privalov). *Let* $f$ *be analytic in the unit disk* $\mathbb{D}$ *and* $u := \mathrm{Re}\, f \in \mathrm{Har}(\overline{\mathbb{D}})$. *If* $u$ *satisfies a Hölder condition on* $\partial\mathbb{D}$ *with exponent* $0 < \alpha < 1$, *then* $f \in A(\overline{\mathbb{D}})$ *and* $f$ *satisfies a Hölder condition on* $\overline{\mathbb{D}}$ *with the same* $\alpha$.

Privalov's theorem can be written in another form. For $u \in \mathrm{Har}(\overline{G})$ and $n \in \mathbb{N}_0$, set

$$E_{n,\Delta}(u, \overline{G}) := \inf\{\|u - t\|_{\overline{G}}\colon t \in \mathbb{T}_n\}.$$

Theorem A is equivalent to the following assertion.

THEOREM B. *Let* $f$ *be analytic in* $G = \mathbb{D}$, $0 < \alpha < 1$. *If* $u := \mathrm{Re}\, f$ *satisfies*

$$E_{n,\Delta}(u, \overline{G}) \leqslant \frac{c_1}{n^\alpha}, \quad n \in \mathbb{N},$$

*then* $f \in A(\overline{G})$ *and*

$$E_n(f, \overline{G}) \leqslant \frac{c_2}{n^\alpha}, \quad n \in \mathbb{N},$$

*where* $c_2 = c_2(c_1, \alpha)$.

THEOREM 33 [11]. *The statement of Theorem B is also true for any* $\alpha > 0$ *and any quasidisk* $G$.

Consider the following example. Let

$$G = G(\beta) := \{re^{i\theta\pi}\colon 0 < r < 1,\ -1 + \beta/2 < \theta < 1 + \beta/2\}, \quad 0 < \beta < 1,$$
$$f(z) = z^{1/(2-\beta)}.$$

It is clear that $u \in \Lambda^1(\partial G)$, but $f \notin \Lambda^\alpha(\overline{G})$ for $\frac{1}{2-\beta} < \alpha < 1$.

This shows that even for simple domains with corners (such as in the last example) there is no direct analogue of Theorem A.

**6.3.** Let $L_0$ denote the segment $[-1, 1]$ and let $f(x)$ be a function continuous on $L_0$. The famous theorem of Jackson states that for any integer $n \in \mathbb{N}$ there exists a polynomial $p_n(x)$ of degree at most $n$ such that for any $x \in L_0$

$$\left|f(x) - p_n(x)\right| \leqslant c\omega_f(1/n), \tag{6.4}$$

where $\omega_f$ is the modulus of continuity of $f$.

Many papers are devoted to generalizations of this statement by means of considering $L_0$ as a part of the complex plane $\mathbb{C}$ rather than merely part of the real line.

In particular, D.J. Newman [29] raised the question, whether (6.4) remains true when $L_0$ and $p_n(x)$ are replaced by some other Jordan arc $L \subset \mathbb{C}$ and algebraic polynomial $p_n(z)$ of the complex variable $z \in \mathbb{C}$. He proposed to say that a Jordan arc has the Jackson property (briefly, has $(J)$), if an analogy of (6.4) written for $L$ remains true.

The problem of determining if $L$ has $(J)$ or not turned out to be difficult. However, on the whole, thanks to the efforts of D.J. Newman, J.M. Anderson, F.D. Lesley, J.I. Mamedhanov, V.V. Maimeskul and A. Hinkkanen, it seems to be solved.

Let $\omega(\delta)$, $\delta > 0$, be a function of the modulus of continuity type. We shall say that a Jordan arc $L \subset \mathbb{C}$ has $(JH)$, if for any $\omega$, any real-valued function $f \in A^{\omega}(L)$ and each $n \in \mathbb{N}$ the inequality

$$E_{n,\Delta}(f, L) \leqslant c\omega(1/n)$$

holds with some constant $c > 0$ depending only on $L$ and $\omega$.

Arcs possessing $(J)$ and $(JH)$ have a few similar properties. For example, repeating word for word a proof suggested by D.J. Newman [29, Theorem 1], one can show that if $L$ has infinite length, then $L$ does not have $(JH)$.

The following statement shows that the properties $(J)$ and $(JH)$ are essentially different.

As usual, the Jordan arc $L$ is of class $C^{2+}$ (briefly, $C^{2+}$-smooth), if it has a parametrization $L = \{w(t)\colon\ 0 \leqslant t \leqslant 1\}$, where $w$ is two times continuously differentiable and satisfies $w'(t) \neq 0$,

$$\left|w''(t_1) - w''(t_2)\right| \leqslant c\,|t_1 - t_2|^{\alpha} \quad (0 \leqslant t_1 < t_2 \leqslant 1)$$

with some constants $c > 0$ and $0 < \alpha < 1$.

THEOREM 34 [15]. *Any Jordan arc $L$ consisting of a finite number of $C^{2+}$-smooth arcs without cusps has $(JH)$.*

At the same time, it is well known that piecewise smooth arcs need not have $(J)$ (cf. Theorem 21).

**6.4.** Harmonic functions can be considered as a bridge from Complex Analysis to higher-dimensional Real Analysis. Some results emphasizing this point of view are presented below.

Let $K$ be a compact set of the $k$-dimensional Euclidean space $\mathbb{R}^k$, $k \geqslant 3$, with connected complement $\Omega := \mathbb{R}^k \setminus K$. Denote by $\mathrm{Har}(K)$ the class of all functions continuous on $K$ and harmonic at its interior points, and let $\mathbb{H}_n$, $n \in \mathbb{N}_0$, be the class of all harmonic polynomials of degree at most $n$.

For a function $f$ given on $K$ and $n \in \mathbb{N}_0$, we set

$$\|f\|_K := \sup\{|f(\mathbf{x})|: \mathbf{x} \in K\},$$
$$E_{n,\Delta}(f, K) := \inf\{\|f - h\|_K: h \in \mathbb{H}_n\}.$$

The main purpose of this subsection is to describe estimates for the quantity $E_{n,\Delta}(f, K)$ depending – like to the previous results for analytic and harmonic functions in the complex plane – on the smoothness properties of the function $f$ and the geometric structure of the compact set $K$.

For $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$, $\mathbf{y} = (y_1, \ldots, y_k) \in \mathbb{R}^k$, and $\delta > 0$, we set

$$|\mathbf{x} - \mathbf{y}|^2 := \sum_{j=1}^{k} (x_j - y_j)^2, \qquad B(\mathbf{x}, \delta) := \{\mathbf{y}: |\mathbf{x} - \mathbf{y}| < \delta\},$$
$$d(\mathbf{x}, K) := \inf\{|\mathbf{x} - \mathbf{y}|: \mathbf{y} \in K\}.$$

The domain $\Omega$ is called a *John domain* if each point $\mathbf{x} \in \Omega$ can be joined to infinity by a Jordan curve $\gamma = \gamma(\mathbf{x}) \subset \Omega$ having the following property. If $\gamma$ is defined by $\mathbf{y} = \mathbf{y}(s)$, $0 \leqslant s \leqslant \infty$, $\mathbf{y}(0) = \mathbf{x}$, $\mathbf{y}(\infty) = \infty$, where $s$ is the arc length parameter, then for every $s > 0$ we require

$$d(\mathbf{y}(s), K) \geqslant cs,$$

where $c > 0$ is independent of $\mathbf{x}$ and $s$.

Let $\omega(\delta)$ be a function of the modulus of continuity type. We denote by $\mathrm{Har}^{\omega}(K)$ the class of functions $f \in \mathrm{Har}(K)$ for which

$$|f(\mathbf{x}) - f(\mathbf{y})| \leqslant c\omega(|\mathbf{x} - \mathbf{y}|), \quad \mathbf{x}, \mathbf{y} \in K.$$

THEOREM 35 [13]. *Let $K \subset \mathbb{R}^k$ be a compact set whose complement $\Omega$ is a John domain. Then for $f \in \mathrm{Har}^{\omega}(K)$ the following estimate holds*:

$$E_{n,\Delta}(f, K) \leqslant c_1\, \omega(n^{-c}), \quad n \in \mathbb{N},$$

*where the constants $c, c_1 > 0$ are independent of $n$.*

The proof of Theorem 35 is based on the procedure of "removal of the poles" suggested in the case of approximation of analytic functions by M.V. Keldysh. Moreover, the same reasoning can be applied to the proof of the harmonic analogue of the classical Bernstein–Walsh theorem.

THEOREM 36 [13]. *Let $K \subset \mathbb{R}^k$ be an arbitrary compact set with simply connected complement, and let the function $f$ be harmonic in some neighbourhood of $K$. Then for some $0 < q < 1$, the estimate*

$$E_{n,\Delta}(f, K) \leqslant cq^n, \quad n \in \mathbb{N}_0,$$

*holds, where the constant $c > 0$ is independent of $n$.*

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