CHAPTER 4

Conformal Welding

D.H. Hamilton

Department of Mathematics, University of Maryland, College Park, MD 20742, USA
E-mail: dhh@math.umd.edu

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1. Introduction

In conformal welding (or sewing or glueing) one uses conformal mappings of the inside and outside of the unit disk $\mathbf{U} = \{|z| < 1\}$ to represent homeomorphisms ϕ of the unit circle $\mathbf{T} = \{e^{it}\colon\ 0 \leqslant t \leqslant 2\pi\}$. These homeomorphisms need not be differentiable, let alone analytic. The theory has important applications including Teichmüller space (i.e., the space of all Riemann surfaces), first imagined by Riemann in the nineteenth century. Its importance carries forward into physics as String Theory (i.e., the grand unified theory) which is based on Teichmüller space, see Witten's 1986 address to the ICM [34].

The following provides a simple example of the conformal welding. We assume $\phi : \mathbf{T} \to \mathbf{T}$ is analytic on some neighbourhood of $\mathbf{T}$. Construct the abstract sphere Ω by joining the unit disk $\mathbf{U}$ to its exterior $\mathbf{L} = \{|z| > 1\} \cup \{\infty\}$ along the circle $\mathbf{T}$ by the correspondence

$$z \to 1/\overline{\phi(1/\bar{z})}.$$

However the abstract Riemann surface Ω is conformally equivalent to the ordinary Riemann sphere. Translating: this means there are conformal maps $f : \mathbf{U} \to A$ and $g : \mathbf{L} \to B$ so that

$$f \circ \phi = g, \quad z \in \mathbf{T},$$

where A and B are disjoint domains with common boundary γ which is an analytic curve. (In particular, f, g are conformal on $\mathbf{T}$.) This observation is an immediate consequence of Koebe's (1905) Uniformisation Theorem. It is one of a host of basic ideas of conformal pasting developed early in the century which go back to Schwarz's conformal representation of polygonal domains, see Carathéodory [7] or Kühnau [19]. Koebe [18] even gave conformal welding for multiple domains. As an application Courant (in the late 30s) used a variational method based on conformal welding in his solution of the Plateau–Douglas problem of minimal surfaces, see his 1950 book [8] for an account (without attribution). In the late 40s the welding theorem was proved by Schaeffer and Spencer [29] by a variational technique (another variational problem which yields f, g was given by Grunsky [11]), in both cases ϕ is analytic. Some early results may also be found in the 1952 book of Goluzin [10]. Actually in the early 30s Lavrentieff [20] went beyond analytic ϕ and introduced quasiconformal mapping (in the same paper) although it was not until 1946 that Volkovskykii [33] used quasiconformal mapping to obtain conformal welding. (One notes that conformal welding seems analogous to the classical Riemann–Hilbert problem but is different, and solved by different means. However the Hilbert–Haseman problem with its Carleman shifts generalizes both (classical) problems, see [21].)

Consider an arbitrary Jordan curve γ with complementary domains A, B and corresponding conformal mappings $f : \mathbf{U} \to A$ and $g : \mathbf{L} \to B$. Now from Carathéodory (1912) the mappings extend to homeomorphisms of $\mathbf{T}$ so that

$$\phi = f^{-1} \circ g$$

is a homeomorphism of $\mathbf{T}$. Thus it is natural to conjecture that any homeomorphism $\phi : \mathbf{T} \to \mathbf{T}$ can be so represented. In fact this is not true, indeed there are counterexamples for which ϕ is analytic except at one point (e.g., $|\phi(e^{it}) - 1| = t^3$ $(t \to 0+)$, t^2 $(t \to 0-)$). There are several ways to understand this. One way is to realize that the relationship between f and g implies estimates on the harmonic measures of adjacent subarcs of γ and consequently bounds on what ϕ can do to adjacent subarcs of $\mathbf{T}$, see [23].

The connection with quasiconformal mappings has been a central theme of conformal welding. A homeomorphism $\Phi : \mathbf{C} \to \mathbf{C}$ is quasiconformal, see [22], if it has generalized L^2 derivatives which satisfy the Beltrami equation

$$\overline{\partial}\Phi = \mu \partial \Phi$$

for some measurable μ satisfying $\|\mu\|_\infty \leqslant k < 1$. Geometrically this means that Φ maps small disks to small ellipses of bounded eccentricity.

It is easier to understand the main result of conformal welding if we transform $\mathbf{T}$ to the line $\mathbf{R} \cup \{\infty\}$ so that $\mathbf{U}$ is now the upper half plane and $\mathbf{L}$ the lower half plane. A famous result of Beurling and Ahlfors [2] characterizes $\phi : \mathbf{R} \to \mathbf{R}$ which extend to quasiconformal mappings Φ by the property that ϕ is quasisymmetric, i.e., there exists a constant c such that for any real x and h

$$c^{-1} \leqslant \frac{\phi(x+h) - \phi(x)}{\phi(x) - \phi(x-h)} \leqslant c.$$

This condition just means that the family of rescalings and translations of ϕ is equicontinuous, so for example any bilipschitz ϕ is quasisymmetric, but so is $\phi(x) = x^{1/3}$ (as are more exotic singular examples). *The fundamental theorem of conformal welding is that conformal welding is possible for arbitrary quasisymmetric functions.* This was proved by Pfluger [27] in 1960. Lehto and Virtanen [22] shortly afterwards gave a different proof.

These are statements of the classical results of the field. In the rest of the article we shall discuss the general problem of the existence and uniqueness of conformal welding. Next we mention applications to Teichmüller space where there is Bers' theorem of Simultaneous Uniformisation, one of the major achievements of twentieth century mathematics. Finally we consider the problem of how the regularity ϕ of determines the regularity of f, g.

2. Existence

We begin with the proof that any quasisymmetric mapping $\phi : \mathbf{R} \to \mathbf{R}$ has conformal welding, by conformal mappings f, g which extend to quasiconformal mappings of the plane. As mentioned before, the central ingredient is the Ahlfors and Beurling extension of ϕ to a quasiconformal mapping Φ of the whole plane with complex dilatation μ. The other ingredient is the solution of the Beltrami equation by quasiconformal mappings for any measurable μ. (In the final form due to Bojarski, see [24], using the Calderón–Zygmund L^p estimates on the Beurling transform, but earlier authors had less general cases including Gauss who did the real analytic case.) In any case one now solves the Beltrami equation

$$\bar{\partial} g = \begin{cases} \mu \partial g, & z \in \mathbf{U}, \\ 0, & z \in \mathbf{L}. \end{cases}$$

The quasiconformal solution g is therefore analytic on $\mathbf{L}$ while $f = g \circ \Phi^{-1}$ is analytic on $\mathbf{U}$. In particular, as $\Phi = \phi$ on $\mathbf{R}$

$$g = f \circ \phi, \quad z \in \mathbf{R}.$$

Also one can now characterize the Jordan boundary γ as a "quasicircle": a Jordan curve through ∞ satisfying the Ahlfors "3-point" condition that there is a constant c with

$$\frac{|z_1 - z_2| + |z_2 - z_3|}{|z_1 - z_3|} \leqslant c$$

for any ordered points z_1, z_2, z_3 on γ, see [24].

One might try to give general necessary and sufficient conditions for ϕ to admit conformal welding. There are conditions (weaker than quasisymmetry) due to Lehto [23], see also [32], which are known to be sufficient:

$$c^{-1}(h) \leqslant \frac{\phi(x+h) - \phi(x)}{\phi(x) - \phi(x-h)} \leqslant c(h),$$

where $c(h) = \mathrm{O}(\log(1/h))$ as $h \to 0$. On the other hand, there are counterexamples with $c(t) = \mathrm{O}(h^{\varepsilon})$. The sharp result is not known.

At the core of existence is obtaining conditions when ϕ is analytic except at one point ζ. The problem is that the topological plane $\Omega - \{\zeta\}$ we first constructed may be hyperbolic rather than parabolic. In this case A and B are still disjoint domains but their common boundary is an open Jordan arc clustering at a nontrivial continuum. The theory of modulus provides the simplest test for when an isolated point of a Riemann surface is removable. An annullus $\{r < |z| < 1\}$ has capacity $m = 1/\log(1/r)$ which is equal to the infimum of the Dirichlet integrals

$$\iint_{r<|z|<1} |\nabla u|^2 \, dx \, dy$$

where $u = 0$ on $|z| = 1$ and $u = 1$ on $|z| = r$. In particular, if the capacity is zero then $r = 0$. Thus $\Omega - \{\zeta\}$ is parabolic if and only if the modulus of annulli surrounding ζ on the abstract sphere can be made small. So for every $\varepsilon > 0$ we consider continuous functions ψ on $\mathbf{R}$ with compact support so that $\psi(\zeta) = 1$. Then consider the harmonic function u on $\mathbf{U}$ with boundary value ψ on $\mathbf{R}$ and the harmonic function v on $\mathbf{L}$ with boundary value $\psi(\phi)$ on $\mathbf{R}$. The condition we obtain is that the Dirichlet integrals satisfy

$$(\mathcal{D}) \qquad \iint_{\mathbf{U}} |\nabla u|^2 \, dx \, dy + \iint_{\mathbf{L}} |\nabla v|^2 \, dx \, dy \leqslant \varepsilon.$$

These conditions are not as impossible as one might think. Now for any Dirichlet integrals on some domain D and quasiconformal mapping Φ of D:

$$\iint_{\Phi(D)} \left|\nabla\big(u(\Phi)\big)\right|^2 dx\,dy \leqslant K \iint_D |\nabla v|^2\,dx\,dy.$$

One immediately sees that quasisymmetric mappings satisfy condition ($\mathcal{D}$). On the other hand quasisymmetry is too strong as we only need certain Dirichlet integrals to be bounded, in fact any test functions will give a corresponding test for parabolic. The $\mathcal{D}$ test shows that we have conformal welding in this case. This approach, doing it as a type problem is seen in Volkovyskii [33], Oikawa [26] (although Courant is already clear about the problem).

Of course one wishes to generalize the above results away from the special case that ϕ be analytic except at one point. Now we switch back to the unit circle $\mathbf{T}$. The general condition will be a uniform version of condition $\mathcal{D}$ namely: for every $\varepsilon > 0$ there exits a $\delta > 0$ that for every annullus $\{r < |z-\zeta| < 1\}$, $\zeta \in \mathbf{T}$ of small capacity $m < \delta$ the corresponding Dirichlet integrals, i.e., the capacity of the abstract annullus has capacity $m' < \varepsilon$. To obtain conformal welding in this case one simply approximates ϕ by the piecewise-linear homeomorphism ϕ_n, ensuring that the $\mathcal{D}$ condition holds uniformly for the ϕ_n. The corresponding conformal weldings f_n, g_n are normalized so that the capacity condition ensures that small rings map to small rings, uniformly. Thus we have an equicontinuous family f_n, g_n on the unit circle from which we extract a subsequence which converges to a pair f, g which is a welding for ϕ. It would seem that our uniform $\mathcal{D}$ condition is also necessary. However in the next section we show that this is not true. This is because of the various types of nonuniqueness associated with conformal welding.

Thus we almost have necessary and sufficient conditions for conformal welding. In other situations (see [12,13]) one requires a generalized form of conformal welding where the boundary between A and B need no longer be a Jordan curve but nevertheless the conformal maps f, g represent the homeomorphism ϕ. One way to do this is to use the angular limits $f(e^{it})$, $g(e^{it})$ (keeping with the unit disk again) which for conformal mappings are not only defined almost everywhere but in fact everywhere except for a set of zero (log) capacity, a result of Beurling (1940). In [12] one uses the Hausdorff dimension dim and defines ϕ to be *regular* if

$$\dim(E) > 0 \quad \Leftrightarrow \quad \dim\big(\phi(E)\big) > 0, \quad \forall E \subset \mathbf{T}.$$

Then it is shown that for *regular* ϕ there exist conformal mappings f, g so that

$$f\big(\phi\big(e^{it}\big)\big) = g\big(e^{it}\big),$$

except for a set of e^{it} of zero arc length. To prove this one takes approximate conformal weldings and ensures convergence. Once again one obtains compactness of the family of approximations but this time not in the space of continuous functions but instead in the Banach space of boundary functions of Dirichlet functions. There are fairly simple ϕ which have no conformal welding in the classical sense but do in the generalized sense.

3. Uniqueness

For many applications it is important that the conformal weldings f, g of ϕ be essentially unique (up to a bilinear transformation). Clearly there is no uniqueness if conformal welding fails in the classical sense, for example, if γ clusters on some continuum K say. For then any conformal mapping h on $\mathbf{C} - K$ gives another conformal welding $h \circ f$, $h \circ g$ of ϕ. However even if ϕ has classical conformal welding with a Jordan curve there need not be uniqueness. The easiest case is when γ has positive area, then one defines a nontrivial quasiconformal mapping Φ with dilatation supported on γ so that $\Phi = h$ is conformal off γ and once again we get another conformal welding, see [4,12].

To understand Jordan curves γ for which there are nontrivial homeomorphisms of $\mathbf{C}$ which are analytic off γ we need concepts from the theory of null sets developed by Ahlfors and Beurling [1]. A compact set E belongs to $\mathcal{N}(\mathcal{D})$ if every function h analytic and with finite Dirichlet integral on $\mathbf{C} - E$ has analytic extension to E. It is a main result of this theory that this is equivalent to there being NO nontrivial conformal maps on $\mathbf{C} - E$. Another related result is that if E is NOT $\mathcal{N}(\mathcal{D})$ there exists a conformal mapping h on $\mathbf{C} - E$ so that $\mathbf{C} - h(\mathbf{C} - E)$ has positive area. However such an h need not be continuous (yes indeed point components of E can be stretched to continua and vice versa).

A parallel concept is for bounded conformal mappings. The requirement that all conformal h on $\mathbf{C} - E$ preserve point components is denoted by $\mathcal{N}(\mathcal{BS})$ (BS meaning bounded schlicht). For example a totally disconnected closed set $E \in \mathbf{R}$ belongs to $\mathcal{N}(\mathcal{D})$ if and only if it belongs to $\mathcal{N}(\mathcal{BS})$. (In particular, no such h can be constructed for $E \subset \mathbf{R}$.) However there are more general sets E which are in $\mathcal{N}(\mathcal{BS})$ but not in $\mathcal{N}(\mathcal{D})$. This means that there are nontrivial functions h conformal on $\mathbf{C} - E$ which necessarily extend to homeomorphisms of $\mathbf{C}$. Then given such a set E it is easy to construct a Jordan curve which contains E.

Consequently if a curve γ contains a set E in $\mathcal{N}(\mathcal{BS})$ but not in $\mathcal{N}(\mathcal{D})$ there exists a nontrivial homeomorphism h which is analytic off γ. There can be no unique conformal mapping for $\phi = f^{-1} \circ g$. Here we constructed examples by the theory of null sets, another approach is given by Bishop [4].

In other examples if γ contains a totally disconnected compact set E which is not on $\mathcal{N}(\mathcal{BS})$ even, then there exist h conformal on $\mathbf{C} - E$ so that at least one point component is stretched to a continuum. Thus ϕ cannot satisfy the uniform $\mathcal{D}$ criterion which ensures that this does not happen, although we have conformal welding. But ϕ has classical conformal welding by f, g and generalized conformal welding by $h \circ f$, $h \circ g$ of ϕ. Therefore the uniform $\mathcal{D}$ cannot be a necessary condition of conformal welding. One might ask if the converse is true, that is, if γ is a Jordan curve and there exists a (nonlinear) homeomorphism h which are conformal off γ then does γ contain a set E in $\mathcal{N}(\mathcal{BS})$ but not in $\mathcal{N}(\mathcal{D})$.

On the other hand if ϕ is quasisymmetric, even though the γ need not be rectifiable, one can prove their are no (nonbilinear) homeomorphisms which are conformal off γ. *Thus we have the very important result that conformal welding is unique for quasisymmetric functions.*

These nonuniqueness results bespoke a certain kind of nonstability of the problem. Conformal welding is obviously unstable in the uniform norm on ϕ. However in the

c-quasisymmetric category, as K-quasiconformal mappings form a compact family, there is stability with respect to the uniform norm, see Huber and Kühnau [17] (in which they even have an explicit formula for the conformal welding functions in the category of diffeomorphisms).

4. Fuchsian groups

By the Uniformisation Theorem any (hyperbolic) Riemann surface R is conformally equivalent to the unit disk $\mathbf{U}$ modulo a discontinuous group G of bilinear mappings $\beta : \mathbf{U} \to \mathbf{U}$. Therefore any homeomorphism Θ of R onto another Riemann surface R' is equivalent to a homeomorphism $\theta : \mathbf{U} \to \mathbf{U}$ so that $\theta \circ G \circ \theta^{-1}$ is a Fuchsian group G' uniformizing R'. If G and G' are of the first kind (i.e., the Limit set of orbits of 0 is dense in T) then θ extends to a homeomorphism $\phi : \mathbf{T} \to \mathbf{T}$ which is equivariant with respect to G, i.e., $\phi \circ \beta \circ \phi^{-1} = \beta' \in G'$ for all $\beta \in G$. In the case of a finitely generated group of the first kind (e.g., any compact Riemann surface) the map ϕ is quasisymmetric. We now apply conformal welding and obtain conformal mappings f, g onto domains A, B bounded by a quasicircle γ. Uniqueness means that both f, g are equivariant. Consequently $\mathcal{G} = f \circ G \circ f^{-1}$ is a discontinuous group acting on A (which has limit set γ). This is conformally equivalent to G, i.e., $A/\mathcal{G}$ is another uniformization of R. On the other hand $\mathcal{G} = g \circ G' \circ g^{-1}$ is a discontinuous group acting on B which is conformally equivalent to G' acting on L. Therefore the two Fuchsian groups G, G' have been simultaneously uniformized by $\mathcal{G}$ acting on A, B. This is Bers' theorem on simultaneous uniformization. The group $\mathcal{G}$ is said to be quasi-Fuchsian and it has limit set $f(\mathbf{T})$ which is a quasicircle.

In general any G equivariant homeomorphism $\phi : \mathbf{T} \to \mathbf{T}$ can be extended to a quasiconformal mapping Φ. Here the problem is that Φ should also be equivariant, a property not given by the original Ahlfors Beurling extension but obtained by Tukia and later by Earle and Hubbard, see [25]. Thus the space of Riemann surfaces (quasiconformal images of a fixed surface R) is realized as the space of G equivariant quasisymmetrics ϕ. To each of these conformal welding assigns an equivariant conformal mapping f on U. This is used to construct the Universal Teichmüller Space $\mathcal{T}$, i.e., those f arising from conformal welding of a quasisymmetric ϕ. These are results of Ahlfors. (The same results hold if one restricts oneself to a fixed Fuchsian group G.) Any further discussion is properly the subject of Teichmüller space, the whole point is to show that conformal welding lies at the basis for its construction. A fine exposition of this theory is Lehto's 1986 book [25].

Until now we restricted our attention to quasisymmetric ϕ. However for infinitely generated groups the ϕ need not be quasisymmetric, indeed nonhomeomorphisms are possible (say if a group of the first kind is transformed to a group of the second). Nevertheless it is possible to obtain a theory of simultaneous uniformization for arbitrary topological transformations of Riemann surfaces, see [13], a theory that depends on generalized conformal welding. The latter depends on special properties of the ϕ associated with a group. A general theory of conformal welding for monotone ϕ which may be nonhomeomorphic has yet to be written down.

In the opposite direction other Teichmüller Spaces based on conformal welding have been considered. There is the model due to Gardiner and Sullivan [9] based on "asymptotically conformal" quasimappings (introduced by Strebel [31], see also Pommerenke [28]) in which the dilatation is continuous. This has been of interest in Dynamics. An even smoother class was considered by Semmes [30] who used "chord arc" curves, i.e., uniformly rectifiable at all scales.

5. Regularity

It is a result going back to Privalov (1919) that for any rectifiable closed Jordan curve γ the harmonic measure taken from the A-side of γ is absolutely continuous with respect to the harmonic measure taken from the B-side, i.e., ϕ is absolutely continuous. By Cauchy's representation theorem for rectifiable γ it is easy to see that there are no (nonbilinear) homeomorphisms which are conformal off γ, so we have uniqueness (up to bilinear mappings). However nothing like the converse is true. In particular, an absolutely continuous ϕ need not have conformal welding. Indeed there are no good necessary and sufficient conditions on ϕ for γ to be rectifiable. For sufficient conditions on the complex dilatation μ for $\Phi(\mathbf{R})$ for γ to be rectifiable see Carleson [5] and also [14] (where a meromorphic function with a rectifiable Julia set is constructed). The requirement that ϕ is absolutely continuous does not suffice, even if ϕ is already quasisymmetric, see Huber [15,16]. Semmes [30] and Bishop [3] showed that even A_p conditions do not suffice. In general there is a loss of regularity between ϕ and the f, g. So if ϕ has continuous k-derivatives (and nonzero first derivative), then f, g have $k-1$ derivatives, which are α Holder continuous for $\alpha < 1$.

It is interesting that the examples of ϕ arising in Teichmüller Theory are often highly irregular. In the case of a finitely generated group of the first kind Tukia proved that the map ϕ has the important property of being either bilinear or totally singular (i.e., zero derivative a.e.) but nevertheless quasisymmetric, see [25]. Furthermore Bowen proved that the limit set γ of a quasi-Fuchsian group is either a circle/line or a Jordan curve with fractal dimension $\mathrm{Dim}(\gamma) > 1$. The analogous result was proved for the Julia set of a rational function, as conformal welding can be used in Complex Dynamics, see [14]. These results are a large part of the interest in fractals at the end of the century. All of this means that the natural applications of conformal welding are for ϕ which are not absolutely continuous even and thus very far removed from the initial observations of which started the subject early in the century.

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