Conformal Maps and Geometry

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Chapter 1

Preliminaries

Here is a short list of the results that we are going to use a lot. You should be familiar with most of them, if you don't know any of them then you can find them in virtually any book with "Complex Analysis" in the title. Standard choices are [1, 16]

- Uniform limit is analytic functions is analytic
- Liouville theorem: the only bounded entire (i.e analytic in the entire complex plane) functions are constants
- Maximum modulus principle: the maximum modulus of a non-constant analytic function is achieved on the boundary of the domain
- Argument principle
- Rouche theorem

In this course we are mostly interested in one-to-one analytic functions. Since we think of them as about mappings from one domain to another we call them maps. You probably should know from course of complex analysis that an analytic function f is locally one-to-one if and only if its derivative never vanishes. Such maps are called *conformal*. Slightly abusing notations we will use this term for globally bijective maps. It is easy to see that the condition that f' never vanishes does not imply global injectivity. Indeed $f(z) = z^2$ is analytic in the complement of the unit disc and its derivative does not vanishe there, but it is two-to-one map.

There are two other terms for analytic one-to-one maps: *univalent* and *schlicht*. We will use them interchangeably.

Chapter 2

Riemann Uniformisation Theorem

In this chapter we are going to discuss a class of uniformizing results. We are mostly interested in the following question: given a domain in the complex plane, can we find a conformal map from this domain onto some simple domain. The first result in this direction is the famous Riemann Uniformization or Riemann Mapping theorem which states that any simply connected domain can be conformally mapped onto the complex sphere $\widehat{\mathbb{C}}$, the complex plane $\mathbb{C}$, or the unit disc $\mathbb{D}$.

We will present the classical Koebe's proof of the uniformization theorem in the simply connected case and will give a complete proof for doubly connected domains. We will briefly mention some other approaches to the construction of the uniformizing maps and proofs for the domains of higher connectivity,

2.1 Möbius transformations and Schwarz lemma

As usual, there are two related questions of uniqueness and existence. In this section we are going to discuss the uniqueness assuming the existence of the uniformazing maps.

Let Ω be a domain in the complex sphere and let us assume that there are two conformal maps f and g from Ω onto some uniformizing domain Ω'. Then the map $\mu = g \circ f^{-1}$ is a conformal automorphism of Ω'. Conversely, if μ is an automorphism of Ω', then $\mu \circ f$ is also a conformal map from Ω onto Ω'. This means that the non-uniqueness of f is given my the group of conformal automorphisms of Ω'.

In this section we are going to describe all conformal automorphisms of $\widehat{\mathbb{C}}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{D}$. It is a well known fact that there are Möbius transformations preserving these domains. Any Möbius transformation is a conformal automorphism of $\widehat{\mathbb{C}}$. For the other domains they are described by the following proposition

Proposition 2.1.1. *The only Möbius transformations that map $\mathbb{D}$, $\mathbb{C}$ or $\mathbb{H}$ to themselves are of the form*

$$\begin{aligned} e^{i\theta}\frac{z-a}{1-\bar{a}z}, &\quad a\in\mathbb{D},\ \theta\in\mathbb{R} \\ az+b, &\quad a,b\in\mathbb{C} \\ \frac{az+b}{cz+d}, &\quad a,b,c,d\in\mathbb{R},\ ad-bc>0. \end{aligned}$$

Exercise 1. *Prove Proposition 2.1.1.*

It turns out that these Möbius transformations are *the only* conformal automorphisms. To prove this we will need a classical result, known as Schwarz lemma.

Theorem 2.1.2 (Schwarz Lemma)**.** *Let f be an analytic function in the unit disc $\mathbb{D}$ normalised to have $f(0)=0$ and $|f(z)|\le 1$, then $|f(z)|\le|z|$ for all $z\in\mathbb{D}$ and $|f'(0)|\le 1$. Moreover, if $|f(z)|=|z|$ for some $z\neq 0$ or $|f'(0)|=1$, then $f(z)=e^{i\theta}z$ for some $\theta\in\mathbb{R}$.*

Proof. Let us define $g(z)=f(z)/z$ for $z\neq 0$. It is easy to see that $z=0$ is a removable singularity, if we define $g(0)=f'(0)$, then g is analytic in $\mathbb{D}$. Next, let us fix some $0<r<1$. On the circle $|z|=r$ we have $|g(z)|<1/r$ and hence, by the maximum modulus principle, the same is true for $|z|<r$. Passing to the limit as $r\to 1$ we show that $|g|\le 1$ in $\mathbb{D}$ which is equivalent to $|f(z)|\le|z|$ and $|f'(0)|\le 1$.

Now assume that there is a point inside $\mathbb{D}$ where $|g(z)|=1$. By the maximum modulus principle, $g(z)=e^{i\theta}$ for some real θ. This proves the second part of the theorem. □

Note that the normalisation that we use is not restrictive: by rescaling and adding a constant, any bounded function in $\mathbb{D}$ could be reduced to this form.

Proposition 2.1.3. *All conformal automorphisms of $\widehat{\mathbb{C}}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{D}$ are Möbius transformations.*

Proof. We are going to prove the unit disc case, the other cases are left as exercises.

Let $f:\mathbb{D}\to\mathbb{D}$ be a conformal automorphism. We define the Möbius transformation $\mu=(z-w)/(1-\bar{w}z)$ where $w=f(0)$. Obviously $g=\mu(f)$ is an analytic map in $\mathbb{D}$ with $g(0)=0$ and $|g(z)|\le 1$. By Schwarz lemma we have $|g(z)|\le|z|$. On the other hand we can also apply Schwarz lemma to the inverse map g^{-1} and obtain $|g^{-1}(z)|\le|z|$. This means that $|g(z)|=|z|$ and hence $g(z)=e^{i\theta}z$ for some θ. This proves that f is inverse of the Möbius transformation $e^{-i\theta}\mu(z)$, hence it is also a Möbius transformation of the same form. □

Exercise 2. *Complete the proof of the Proposition 2.1.3.*

2.2 Normal Families

In this section we discuss some results about convergence of conformal maps that we will need for the proof of Riemann Mapping Theorem.

Definition 2.2.1. Let $\mathcal{F}$ be a family of analytic functions on Ω. We say that $\mathcal{F}$ is a *normal family* if for every sequence f_n of functions from $\mathcal{F}$ there is a subsequence which converges uniformly on all compact subsets of Ω.

The term "normal family" is somewhat old fashioned, in more modern terms it should be called "precompact". The standard way to prove precompactness is to use Arzela-Ascoli theorem, and this is exactly what we will do. Before stating the theorems we need two more definitions.

Definition 2.2.2. We say that a family of functions $\mathcal{F}$ defined on Ω is *equicontinuous* on $A \subset \Omega$ if for every $\epsilon > 0$ there is $\delta > 0$ such that $|f(x) - f(y)| < \delta$ for every $f \in \mathcal{F}$ and all $x, y \in A$ such that $|x - y| < \epsilon$.

Definition 2.2.3. We say that a family of functions $\mathcal{F}$ defined on Ω is *uniformly bounded* on $A \subset \Omega$ if there is M such that $|f(x)| < M$ for all $x \in A$ and every $f \in \mathcal{F}$.

Now we can state the Arzela-Ascoli theorem which we present here without a proof. Interested readers could find it in many books including [16, Theorem 11.28]

Theorem 2.2.4 (Arzela-Ascoli)**.** *Let $\mathcal{F}$ be a family of pointwise bounded equicontinuous functions from a separable metric space X to $\mathbb{C}$. Then every sequence f_n of functions from $\mathcal{F}$ contains a subsequence that converges uniformly on all compact subsets of X.*

Now we are ready to state and prove Montel's theorem which gives a simple sufficient condition for normality of a family of analytic functions.

Theorem 2.2.5 (Montel)**.** *Let $\mathcal{F}$ be a family of analytic functions on a domain Ω that is uniformly bounded on every compact subset of Ω. Then $\mathcal{F}$ is a normal family.*

Proof. First we construct a family of compacts that exhaust Ω. We define K_n to be $\{z \in \Omega \text{ such that } |z| \le n \text{ and } \operatorname{dist}(z, \mathbb{C}\setminus\Omega) \le 1/n\}$. (We assume that all $K_n \neq \emptyset$, otherwise we change indexes so that K_1 is the first non-empty set.) It is easy to see that for every compact $K \subset \Omega$ there is n such that $K \subset K_n$. This also implies that $\cup K_n = \Omega$. Moreover, K_n are increasing and separated, namely $K_n \subset K_{n+1}$ and there are $\delta_n > 0$ such that for all $z \in K_n$ we have $B(z, \delta_n) \subset K_{n+1}$.

Let z and w be two points from K_n with $|z - w| < \delta_n/2$ and f be any function from $\mathcal{F}$. We can use Cauchy formula to write

$$f(z) - f(w) = \frac{1}{2\pi i} \int_\gamma \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - w} \right) f(\zeta) \mathrm{d}\zeta,$$

where γ is a circle of radius δ_n centred at z. Note that $\gamma \subset K_{n+1}$ and since $\mathcal{F}$ is uniformly bounded, there is a constant M_{n+1} independent of f such that $|f(\zeta)| \leq M_{n+1}$. This allows us to estimate

$$|f(z) - f(w)| \leq \frac{2M_{n+1}}{\delta_n}|z - w|$$

which implies that $\mathcal{F}$ is equicontinuous on K_n and hence on every compact subset of Ω.

By Arzela-Ascoli theorem 2.2.4 from each sequence of functions f_n from $\mathcal{F}$ we can choose a subsequence converging uniformly on K_n. Let $f_{n,1}$ be a subsequence converging on K_1, by the same argument is has a subsequence converging on K_2, we denote it by $f_{n,2}$. Continuing like that we construct a family of sequences $f_{n,k}$. By the standard diagonal argument, the sequence $f_{n,n}$ converges uniformly on every K_n and hence on every compact subset of Ω.

□

It is important to mention that Montel's theorem tells us very little about the limit of the subsequence. From uniform convergence we know that the limit is also analytic in Ω, but we don't know whether it belongs to $\mathcal{F}$ or not. We are mostly interested in the case when all functions from $\mathcal{F}$ are univalent, in this case we have the following dichotomy:

Theorem 2.2.6 (Hurwitz). *Let f_n be a sequence of univalent functions in some domain Ω that converge to f uniformly on every compact subset of Ω. Then f is a univalent or a constant function*

Remark 2.2.7. This is a typical example of a dichotomy in complex analysis where we can say that our object is as good as possible or as bad as possible, but not something in between. Another example is classification of isolated singularities.

Proof. Let us assume that the limiting function f is not univalent, i.e. there are distinct points z_1 and z_2 in Ω such that $f(z_1) = f(z_2)$. The sequence of functions $g_n(z) = f_n(z) - f_n(z_2)$ converges to $g(z) = f(z) - f(z_2)$. If us assume that f is not a constant function, then the roots of g are isolated and there is a small circle γ around z_1 such that $\gamma \subset \Omega$, g does not vanish on γ and z_2 is not inside γ. Since g does not vanish on γ, there is $c > 0$ such that $|g| > c$ on γ. By uniform convergence $|g - g_n| < c$ on γ for sufficiently large n. By Rouche's theorem the numbers of the roots of g and g_n inside of γ are the same for sufficiently large n. On the other hand functions g_n are univalent and $g(z_2) = 0$, hence there are no roots inside γ, but $g(z_1) = 0$. This proves that if f is not univalent, hence our assumption that it is non constant must be false. □

2.3 Koebe's proof of Riemann Mapping theorem

Now we are ready to prove the Riemann Uniformisation or Riemann Mapping theorem. It was originally stated by Riemann, but his proof contained a gap. Here we present a proof based on the ideas of Koebe.

Theorem 2.3.1. *Let Ω be a simply-connected domain in the complex sphere $\widehat{\mathbb{C}}$. Then Ω is conformally equivalent to one of three domains: $\widehat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{D}$. To be more precise if $\widehat{\mathbb{C}} \setminus \Omega$ contains at least two points, then Ω is equivalent to $\mathbb{D}$, if it contains one point, then it is equivalent to $\mathbb{C}$ and if it is empty, then $\Omega = \widehat{\mathbb{C}}$.*

Moreover, if Ω is equivalent to $\mathbb{D}$ and z_0 is any point in Ω, then there is a unique conformal map $f : \Omega \to \mathbb{D}$ such that $f(z_0) = 0$ and $f'(z_0) > 0$.

Three uniformising domains $\widehat{\mathbb{C}}$, $\mathbb{C}$, and $\mathbb{D}$ are not conformally equivalent.

Proof. We start from the last part of the theorem. It is easy to see that $\widehat{\mathbb{C}}$ can not be equivalent to $\mathbb{C}$ or $\mathbb{D}$ since they are not even homeomorphic. To show that $\mathbb{C}$ and $\mathbb{D}$ are not equivalent we assume the contrary, that there is a univalent map from $\mathbb{C}$ onto $\mathbb{D}$. This function is a bounded entire function, by Liouville's theorem this function must be constant which contradicts our assumption that it is univalent.

There is nothing to prove when $\Omega = \widehat{C}$. When $\Omega = \widehat{\mathbb{C}} \setminus \{w_0\}$ we can apply Möbius transformation $\mu = 1/(z - w_0)$ which maps Ω onto $\mathbb{C}$.

The only interesting case is when the complement of Ω contains at least two points. To analyse this case we consider the family $\mathcal{F}$ of univalent maps f on Ω such that $|f(z)| \le 1$ and $f(z_0) = 0$, $f'(z_0) > 0$ for some fixed $z_0 \in \Omega$.

We will have to make the following steps to complete the proof:

1. Show that the family $\mathcal{F}$ is non empty.

2. Show that the family $\mathcal{F}$ is normal.

3. Consider a continuous functional on $\mathcal{F}$: $f \mapsto f'(0)$. Let f_n be a sequence of functions maximizing the functional. By the previous step there is a sequence converging to a maximizer. Show that the maximizer is in $\mathcal{F}$.

4. Show that the maximizer is onto $\mathbb{D}$.

Step 1. We know that there are two points outside of Ω, by applying a Möbius transformation we can assume that one of these points is infinity. So, our domain is a proper simply connected sub-domain of $\mathbb{C}$. By assumption there is $w \in \mathbb{C} \setminus \Omega$. Since Ω is simply connected, there is a continuum connecting w to infinity that lies outside of Ω. Using this continuum as a branch-cut we can define a single-valued branch of $\phi(z) = (z - w)^{1/2}$. Notice that this function is univalent. Indeed, if $\phi(z_1) = \phi(z_2)$, then $z_1 - w = z_2 - w$ and $z_1 = z_2$. By the same argument it does not take the opposite values i.e. we can not have $\phi(z_1) = -\phi(z_2)$. Since ϕ maps a small neighbourhood of z_0 onto an open neighbourhood of $w_0 = \phi(z_0)$, there is $r > 0$ such that $B(w_0, r) \subset \phi(\Omega)$ and $B(-w_0, r) \cap \phi(\Omega) = \emptyset$. Composing ϕ with $r/(z + w_0)$ we find a map from $\mathcal{F}$.

Note that for domains with non-empty interior of the complement we only need the last step. The trick with the square root is needed only for domains that are dense in $\mathbb{C}$.

Step 2. Since all functions in $\mathcal{F}$ are bounded by 1, normality follows immediately from the Montel's Theorem 2.2.5.

Step 3. It is a standard corollary of Cauchy formula that if analytic functions f_n converge uniformly to f, then $f_n'(z) \to f'(z)$ for every z. This proves that the functional $f \mapsto f'(z_0)$ is continuous with respect to the uniform convergence on compact sets.

Let M be the supremum of $f'(z_0)$ over all functions from $\mathcal{F}$. There is a sequence f_n such that $f_n'(z_0) \to M$ (note that we do not assume that M is finite). By normality of $\mathcal{F}$ there is a subsequence which converges on all compact subsets of Ω. Abusing notations we denote this subsequence by f_n and its limit by f. Uniform convergence implies that f is analytic in Ω and $f'(z_0) = M$. In particular M is finite.

By Hurwitz Theorem 2.2.6 the limit f is either univalent or constant. Since $M > 0$, f can not be constant.

Step 4. The main idea of this step is rather simple. In some sense the derivative at z_0 pushes the images of other points away from $f(z_0)$. If there is a point w in $\mathbb{D} \setminus f(\Omega)$, then we can construct a function that will push w to the boundary of $\mathbb{D}$. Explicit computation will show that composition of f with this function has larger derivative.

First we compose f with a Möbius transformation $\mu(z) = (z - w)/(1 - \bar{w}z)$. This will map w to the origin. Now, by the same argument as in the first step we can define a single-valued branch of

$$F(z) = (\mu(f(z)))^{1/2} = \sqrt{\frac{f(z) - w}{1 - \bar{w}f(z)}}.$$

Finally we have to compose with another Möbius transformation that will send $F(z_0)$ back to the origin. This is done by

$$G(z) = \frac{|F'(z_0)|}{F'(z_0)|} \frac{F(z) - F(z_0)}{1 - F(z)\overline{F(z_0)}}.$$

The first factor is needed to ensure that the derivative at z_0 is positive (in other words its argument is zero).

Explicit computation shows that

$$G'(z_0) = \frac{|F'(z_0)|}{1 - |F(z_0)|^2} = \frac{1 + |w|}{2\sqrt{|w|}} f'(z_0) > f'(z_0).$$

Since G is a composition of univalent maps and $|G| < 1$ this contradicts the assumption that f maximizes the derivative at z_0.

To complete the proof of the theorem we have to show that the map f is unique. Let us assume that there is another function g which maps Ω onto $\mathbb{D}$ and has the right normalisation. The map $f \circ g^{-1}$ is a conformal automorphism of the unit disc. By Proposition 2.1.1 it has the form

$$e^{i\theta} \frac{z-a}{1-\bar{a}z}.$$

Since 0 is mapped to itself and the derivative at 0 is 1 we must have $e^{i\theta} = 1$ and $a = 0$. This means that $g^{-1} = f^{-1}$ and $f = g$.

□

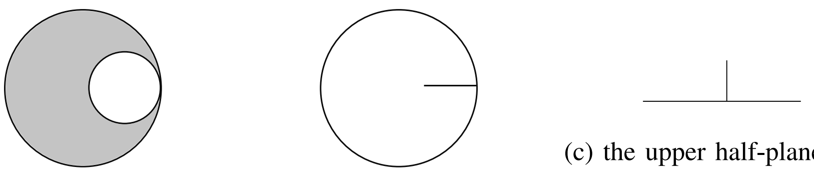
We can see from the proof that the univalent map onto the disc maximizes derivative at the point which is mapped to the origin. There is an alternative extremal formulation. Let us assume that Ω is a simply connected domain such that $\widehat{\mathbb{C}} \setminus \Omega$ contains at least two points. By composing with an appropriate Möbius transformation we can assume that $0 \in \Omega$. We denote by $\mathcal{F}$ the family of all univalent maps on Ω with $f(z_0) = 0$ and $f'(z_0) = 1$. The functional $f \mapsto \sup |f(z)|$ is minimized by the unique univalent map onto the disc of radius $R = \min_f \sup_z |f(z)|$. This radius is called *conformal radius of the domain* Ω *at* z_0.

There is one more statement claiming that the derivative at the fixed point is related to the size of the domain. This result is known as Lindelöf's principle. Let f_1 and f_2 be two univalent functions mapping $\mathbb{D}$ onto Ω_1 and Ω_2 respectively. We also assume that $f_i(0) = 0$ and that $\Omega_1 \subset \Omega_2$. Then $|f_1'(0)| \leq |f_2'(0)|$ with equality holding if and only if $f_2(z) = f_1(e^{i\theta} z)$ for some real θ.

Minimization of the maximum modulus and Lindelöf's principle follow immediately from the proof of the Riemann Mapping theorem. Lindelöf's principle also implies that the conformal radius increases when the domain increases.

Exercise 3. *Unifomise the following domains:*

1. *Domain bounded by two touching circles, see Figure 2.1a.*
2. *Infinite strip,* $0 < \Im z < 1$.
3. *The unit disc with a slit* $\mathbb{D} \setminus [x, 1]$ *with* $-1 < x < 1$, *see Figure 2.1b.*
4. *The upper half-plane with a slit* $\mathbb{H} \setminus [0, it]$ *with* $t > 0$, *see Figure 2.1c.*



(a) Two circles

(b) A disc with a slit

(c) the upper half-plane with a slit

Figure 2.1: Three domains where uniformizing maps could be found explicitly.

2.4 Other normalizations

The Riemann Mapping theorem 2.3.1 tells us that all simply connected domains whose complement contains at leas two points are conformally equivalent. In the proof of this theorem we used the unit disc as the standard uniformizing domain. Obviously, this choice is completely arbitrary. In this small section we are going to discuss other uniformzing domains and normalizations.

First of all we know that the map from a simply connected domain Ω onto $\mathbb{D}$ is not unique, it can be composed with any Möbius transformation preserving the unit disc. The family of these transformations is described by three real parameters: real and imaginary parts of the point which is mapped to the origin and angle of rotation. This means that in general we should be able to fix uniquely any three real parameters by the proper choice of Möbius transformation.

In the standard formulation of the Riemann's theorem we normalize map by requiring that a fixed point z_0 is mapped to the origin and that the argument of the derivative at this point is zero. This corresponds exactly to fixing three real parameters, so it should not be a surprise that such a map is unique. We would like to point out that the argument with the number of parameters is just a rule of thumb, although a very good one, and each separate case requires a rigorous proof.

Other standard ways to choose normalization are: fix one interior and one boundary point, fix three boundary points, fix two boundary points and and derivative at one of them. For some of these normalizations other domains are natural uniformizing domains. Finally, we would like to mention that independently of normalization, the upper half-plane is another standard choice for the uniformizing domain.

One interior and one boundary point. Let Ω be a domain conformally equivalent to $\mathbb{D}$ and let f be a conformal map from Ω onto $\mathbb{D}$. We chose an interior point $z_0 \in \Omega$ and a boundary point $\zeta \in \partial\Omega$. We assume that f can be defined continuously at ζ. Then there is a unique univalent function $g : \Omega \to \mathbb{D}$ such that $g(z_0) = 0$ and $g(\zeta) = 1$. There is a unique univalent function $h : \Omega \to \mathbb{H}$ with $h(z_0) = i$ and $h(\zeta) = 0$.

By Riemann theorem we can assume that $f(z_0) = 0$ and we know that all maps onto $\mathbb{D}$ differ by composition with a Möbius transformation. By Schwarz lemma 2.1.2 the only Möbius automorphisms of $\mathbb{D}$ are rotations. This means that $f(z)/f(\zeta)$ is the only map with desired properties.

The second part is straightforward. We know that there is a unique Möbius transformation $\mu : \mathbb{D} \to \mathbb{H}$ such that $\mu(0) = i$ and $\mu(1) = 0$. The map h is equal to $\mu \circ g$.

Exercise 4. *Find this map μ.*

Three boundary points. As before we assume that there is a map $f : \Omega \to \mathbb{D}$ which can be continuously defined at boundary points ζ_i, $i = 1, 2, 3$. Let z_i be three points on the boundary of $\mathbb{D}$ that have the same order as ζ_i[1]. We know that there is a unique Möbius transformation μ mapping $f(\zeta_i)$ to z_i. Notice that μ will also map the unit disc into itself. Since z_i and $f(\zeta_i)$ have the same order, map μ will send the unit disc to itself. This means that $\mu \circ f$ will send ζ_i to z_i.

Sometimes the unit disc is not the most convenient domain for this type of normalization. It is a bit more useful to map Ω onto the upper half-plane and to send three given points to 0, 1 and ∞.

Two boundary points and derivative. First of all we have to assume that function $f : \Omega \to \mathbb{D}$ is continuous at two boundary points ζ_1 and ζ_2. We assume that the boundary of Ω is analytic near ζ_1, this allows to extend f analytically into a neighbourhood of ζ_1 and to to make sense of the derivative at the boundary point.

It might seem that we want to fix too many parameters: two boundary points give us two real parameters and a derivative is a complex number, hence it also gives two parameters. But we can notice that near ζ_1 the function f maps the smooth boundary of Ω onto smooth boundary of the unit disc. This determines the argument of the derivative and we are left with only one parameter: modulus of the derivative.

The best unifomizing domain for this problem is the half-plane. As before, by composing with a Möbius transformation we can construct a map $g : \Omega \to \mathbb{D}$ with $g(\zeta_1) = 0$ and $g(\zeta_2) = \infty$. It is easy to see that $g(z)/|g'(\zeta_1)|$ maps ζ_1 and ζ_2 to 0 and ∞ and has derivative 1 at ζ_1. It is easy to check that we can choose any two points and the value of the modulus of derivative, but this particular normalization is probably the most useful one.

Exercise 5. *Prove that this map f is unique.*

2.5 Constructive proofs

In this section we briefly discuss some constructive proofs of the theorem. We will present constructions, but will not give complete proofs. It is important to note, that constructive proof give only approximate solutions to the uniformization problem. On the other hand, there are very few domains where Riemann map can be written explicitly in terms of simple functions.

Composition of elementary maps. We assume that $\Omega \subset \mathbb{D}$ and that $0 \in \Omega$, otherwise we can repeat the explicit construction from the first step of the Riemann Mapping theorem's proof. To construct the uniformization map we are going to use the last step from the this proof.

We are going to construct a sequence of domains $\Omega_1 = \Omega, \Omega_2, \Omega_2, \dots$ and conformal maps f_n from Ω_n onto Ω_{n+1}. We will show that $\Omega_n \to \mathbb{D}$ (in some

[1] We say that points on the boundary of Ω are in the counter clockwise order if their images under the Riemann map are in this order

sense) and composition of f_n will converge to a conformal map from Ω onto $\mathbb{D}$. Define $r_n = \inf\{|z|, z \in \mathbb{D} \setminus \Omega_n\}$ and let w_n be some point in $\mathbb{D} \setminus \Omega_n$ with $|w_n| = r_n$. As in the Step 4 we define

$$\psi_n = \sqrt{\frac{z - w_n}{1 - \bar{w_n} z}}$$

and

$$f_n = \frac{|\psi'(0)|}{\psi'(0)|} \frac{\psi(z) - \psi(0)}{1 - \psi(z)\overline{\psi(0)}}.$$

As before we have that $f_n(0) = 0$ and $f_n'(0) = (1 + r_n)/2\sqrt{n} > 1$. It is obvious that $F_n = f_n \circ f_{n-1} \circ \cdots \circ f_1$ is a univalent map from Ω onto $\Omega_{n+1} \subset \mathbb{D}$ with $F_n(0) = 0$ and

$$F_n'(0) = \prod f_i'(0) = \prod \frac{1 + r_i}{2\sqrt{r_i}}.$$

From Schwarz lemma 2.1.2 we know that $|F_n'(0)|$ is bounded by some constant which depends on Ω and z_0 only, but not on other Ω_n. This implies that the product must converge, and hence $r_n \to 1$ as $n \to \infty$. This means that Ω_n is squeezed between $r_n\mathbb{D}$ and $\mathbb{D}$ and hence converges to $\mathbb{D}$ (in Hausdorff topology). It is also possible to show that the sequence of maps F_n converges uniformly on all compact subsets of Ω and that the limiting function is a univalent map from Ω onto $\mathbb{D}$.

This construction follows the same idea that the uniformizing map should maximize the derivative at the point which should be mapped to the origin, but instead of abstract compactness argument we use explicit construction. Another advantage of this approach is that all functions f_n are elementary and easy to compute: they are compositions of Möbius transformations and square root function. From pure practical point of view it might be difficult to compute r_n, but it is easy to see that we don't really need r_n to be optimal, we just need it to be comparable to the optimal.

Christoffel-Schwarz mapping The next method uses domain approximations. The main idea is that any domain can be approximated by a polygonal domains and for a polygonal domain there is a nice expression for a conformal map from the unit disc onto these domains that is given by Christoffel-Schwarz formula. Detailed discussion of Christoffel-Schwarz maps could be found in a boom by Driscoll and Trefethen [9]. Here we just provide a brief description.

Theorem 2.5.1 (Christoffel-Schwarz). *Let Ω be a polygonal domain with n vertices where angles between adjacent edges are equal to $\pi\alpha_k$. Then there is a conformal map from $\mathbb{D}$ onto Ω which has the form*

$$F(w) = C \int_0^w \prod_{k=1}^{n} (w - w_k)^{-\beta_k} \mathrm{d}w + C'$$

where $\beta_k = 1 - \alpha_k$, w_k are some points on the unit circle, and C and C' are complex-valued constants.

There is an alternative version for a map from the upper half-plane. In this case the mapping is given by

$$F(w) = C \int_0^w \prod_{k=1}^{n-1} (w - x_k)^{-\beta_k} \mathrm{d}w + C'$$

where x_i are real numbers. Note that β_n does not appear in this formula explicitly, but it is not an independent parameter, from elementary geometry we know that sum of all β_i is equal to 2.

Using this theorem one can find *explicit* formulas for several simple domains such as triangles and rectangles.

The main disadvantage of this formula is that it is not as explicit as it looks: in practice it is very difficult to compute the points w_k. Even when the points w_k are known, the map is given by an integral which has integrable singularities, which make it not very amenable to straightforward computations. Banjai and Tefethen [3] adopted other techniques to Christoffel-Schwarz algorithm and significantly increased the speed of the computations.

Zipper algorithm Probably the best method for numerical computations is given by zipper algorithm that was discovered by R. Kühnau and D. Marshall[2]. Given points $z_1, \dots z_n$, the algorithm computes a conformal map onto a domain bounded by a curve passing through these points. The conformal map is presented as a composition of simple "slit" maps which are easy to compute. The algorithm is fast and accurate, its complexity depends only on the number of data points, but not on the shape of the domain. In 2007 Marshall and Rohde [14] showed the convergence of the zipper algorithm for the Jordan domains. Finally, we refer readers to the paper of Binder, Braverman, and Yampolsky [5] for a discussion of the computational complexity of the Riemann uniformization problem.

Exercise 6. *Find explicit formulas (that might involve special functions) for conformal maps between Ω and one of the standard uniformizing domains where Ω is*

1. *Semi-infinite strip* $\{z : -\pi/2 < \Re z < \pi/2,\ \Im z > 0\}$
2. *Equilateral triangle*
3. *Rectangle*

Exercise 7. *Let R_1 and R_2 be two rectangles and let $\lambda_i > 1$ be the ratio of side lengths of R_i. Assume that there is a conformal maps $f : R_1 \to R_2$ which is continuous up to the boundary and maps vertices to vertices. Show that $\lambda_1 = \lambda_2$*

[2]Software is available from D. Marshall's page `www.math.washington.edu/~marshall/zipper.html`

2.6 Boundary correspondence

In the previous sections we discussed the existence of univalent maps from general domains onto simple uniformizing domains. These maps are analytic inside the corresponding domains, but a priori we have no information about their boundary behaviour. In this section we will investigate this question and will obtain a simple geometrical answer.

First we notice that by means of elementary maps that are obviously continuous on the boundary we can map any domain onto a bounded domain. This means that without loss of generality we can always assume that all domains in this section are bounded.

Next we make a very simple observation which is purely topological and does not use analyticity: boundaries are mapped onto each other. The precise meaning is given by the following proposition:

Proposition 2.6.1. *Let f be a univalent map from Ω onto Ω' and let $z_n \in \Omega$ be a sequence which tends to the boundary of Ω, which means that all accumulation points are on the boundary of Ω. Then $f(z_n)$ tends to the boundary of Ω'. Alternatively, f is a continuous function between one-point compactifications od Ω and Ω'.*

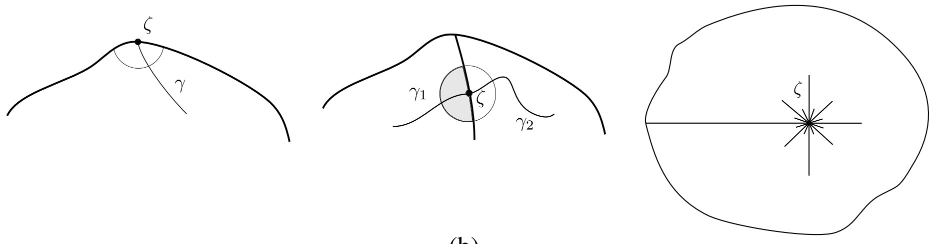
Proof. It is easy to see that the condition that z_n tends to the boundary is equivalent to the fact that for every compact $K \subset \Omega$ there is N such that z_n is outside of K for $n > N$. Let K' be a compact set in Ω', by continuity $K = f^{-1}(K')$ is also a compact set. Since z_n will eventually leave K, $f(z_n)$ will leave K'. □

The previous proposition tells us that the boundary as a whole set is mapped to the boundary, but it does not tell us anything about the continuity. The boundary behavious of analytic functions is a rich and well developed subject but it is beyond the scope of this course. Here we will use only some rather elementary considerations which a surprisingly sufficient since we work with a rather small class of univalent functions. We start by considering boundary behaviour near "regular" boundary points.

Definition 2.6.2. An *accessible boundary point* ζ of a domain Ω is an equivalence class of continuous curves $\gamma : [0, 1] \to \bar{\Omega}$ which join a given point $\zeta \in \partial\Omega$ with an arbitrary interior point. We assume that γ lies completely inside Ω except $\gamma(1) = \zeta$. Two curves are equivalent if for arbitrary neighbourhood U of ζ, parts of the curves that are inside of $\Omega \cap U$ could be joined by a continuous curve.

Notice that accessible points that correspond to different boundary points are always different, but the same boundary point could carry several accessible points. If accessible points are different, then for sufficiently small r_0 there are disjoint components of $B(\zeta_i, r_0) \cap \Omega$ such that the tails of the curves defining accessible points lie in the corresponding components. We denote these components by $B(\zeta, r_0)$.

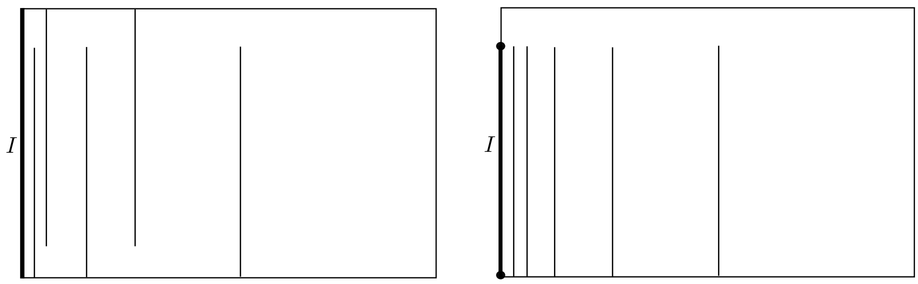
If the boundary of Ω is nice, say, Jordan curve, then each boundary point corresponds to exactly one accessible point. In this case we identify them. It also could be that one boundary point corresponds to more than one accessible points, for examples see Figure 2.2. In the Figure 2.2a ζ is a point on a boundary such that $B(\zeta, r) \cap \Omega$ has only one component for sufficiently small r. This point must correspond to one accessible point. In the Figure 2.2b ζ is a point on a slit and $B(\zeta, r) \cap \Omega$ has two components, each of them gives rise to an accessible point. The last example in the Figure 2.2c is a bit more involved. Let $\zeta = 0$ and for reach dyadic direction $\theta = 2\pi k/2^n$ we remove an interval $[0, e^{i\theta}/2^n]$. For each irrational (mod 2π) angle θ we can consider $\gamma_\theta(t) = (1-t)e^{i\theta}$. It is not very difficult to see that each γ_θ defines an admissible point and that all these admissible points are different.



(a) (b) (c)

Figure 2.2: A single boundary point could correspond to one 2.2a, two 2.2b, or even uncountably many 2.2c accessible points.

It could also be that there are no continuous curves γ approaching a boundary point, in this case the boundary point is not accessible, see Figure 2.3.



(a) (b)

Figure 2.3: In both examples all points of the interval I are non-accessible.

Theorem 2.6.3. *Let Ω be a simply connected bounded domain in the plane and let f be a univalent map from Ω onto $\mathbb{D}$. Then for every accessible point ζ the map f can be continuously extended to ζ and $|f(\zeta)| = 1$. Moreover, for distinct accessible points their images are distinct.*

There are several ways to prove this theorem, one of the standard modern ways is to consider the inverse function and use some powerful results about the existence of the radial limits for functions from the Hardy class H^∞. Here we prefer to give rather elementary geometrical proof. We start with two technical lemmas due to Koebe and Lindelöf.

Lemma 2.6.4 (Koebe). *Let z_n and z'_n be two sequences in the unit disc $\mathbb{D}$ converging to two distinct points ζ and ζ' on the boundary of the unit disc. Let γ_n be Jordan arcs connecting z_n and z'_n inside $\mathbb{D}$ but outside some fixed neighbourhood of the origin. Finally, we assume that a function f is analytic and bounded in $\mathbb{D}$ and that f converges uniformly to 0 on γ_n, that is the sequence $\epsilon_n = \sup_{\gamma_n} |f|$ converges to 0. Then f is identically equal to 0 in $\mathbb{D}$.*

Proof. Let us suppose that f is not identically zero. Without loss of generality we assume that $f(0) \neq 0$, otherwise f has zero of order n at $z = 0$ and we can replace f by $f(z)/z^n$ which satisfies all assumptions of the lemma.

For sufficiently large m there is a sector S of angle $2\pi/m$ such that the radii towards ζ and ζ' lie outside of this sector and infinitely many of γ_n cross this sector. We discard all other curves, as well as their endpoints. Abusing notations we call the remaining curves γ_n. By rotating the unit disc, i.e by considering $f(e^{i\alpha}z)$ instead of $f(z)$, we can assume that the positive real line is the bissectrice of S.

For each curve γ_n we can find its part γ' which is also a simple curve that crosses S, its end points lie on two different sides of S, and no other point lies on the boundary of S. Finally, let γ''_n be the part of γ'_n connecting one of the end points to the first intersection with the real line and $\bar{\gamma}''_n$ be symmetric to γ''_n about the real axis (see the Figure 2.4).

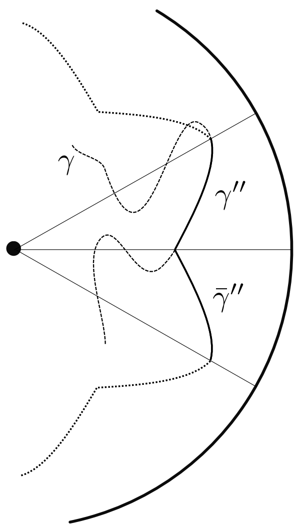


Figure 2.4: The dashed line is the original curve γ, solid line is its part γ'' and $\bar{\gamma}''$. The dotted line is made of rotations of γ'' and $\bar{\gamma}''$

By reflection principle the function $\bar{f}(\bar{z})$ is also analytic in $\mathbb{D}$ and it is bounded by ϵ_n on $\bar{\gamma}_n''$. This means that the function $\phi(z) = f(z)\bar{f}(\bar{z})$ is analytic and bounded on the union of γ_n'' and $\bar{\gamma}_n''$ by $M\epsilon_n$, where $M = \sup_{\mathbb{D}} |f|$.

Let F be the product of rotations of ϕ by $2\pi/m$, namely

$$F(z) = \phi(z)\phi(e^{2\pi i/m}z)\ldots\phi(e^{2\pi i(m-1)/m}z).$$

This function is analytic in $\mathbb{D}$ and bounded by $\epsilon_n M^{2m-1}$ on a closed curve formed by the union of rotations of γ_n'' and $\bar{\gamma}_n''$. By maximum principle this implies that $|f(0)|^{2m} = |F(0)| \leq \epsilon_n M^{2m-1}$, since $\epsilon_n \to 0$ this implies that $f(0) = 0$, which contradicts our initial assumption. $\square$

The second technical result that we take from [15] is the following theorem which we present without proof. (The proof could be found in Section 11.21 of [15].)

Theorem 2.6.5 (Lindelöf)**.** *Let Ω be a simply connected domain bounded by a Jordan curve Γ and let f be a function analytic in Ω satisfying the following conditions*

1. *f is bounded in Ω;*
2. *f is continuous everywhere on the boundary with the exception of a single point ζ_0;*
3. *let ζ_1 be some other point on Γ, two points ζ_0 and ζ_1 separate Γ into two Jordan curves: Γ_1 and Γ_2. The following limits along these two arcs exist*

$$\lim_{\Gamma_1} f(\zeta) = a, \qquad \lim_{\Gamma_2} f(\zeta) = b.$$

Then $a = b$ and f is continuous at ζ_0.

Proof of Theorem 2.6.3. Let $\gamma(t)$ be a curve defining the accessible point ζ, we want to show that $\tilde{\gamma}(t) = f(\gamma(t))$ converges to a point on the unit circle. Let us assume the contrary, then $\tilde{\gamma}$ contains a sequence of arcs with endpoint converging to two distinct boundary points, see the Figure 2.5. Moreover, these arcs converge to the boundary and hence stay away from the origin. The inverse function f^{-1} converges uniformly to ζ on these arcs. Applying the previous lemma to $f^{-1} - \zeta$ we see that f^{-1} must be identically equal to ζ, which is obviously impossible. This proves that as we move along $\tilde{\gamma}$ we must approach a definite point on the unit circle. We define $f(\zeta)$ to be this point.

Next we have to show that this definition is consistent, that is independent of our choice of γ. Let $\gamma'(t)$ be another curve describing the same accessible point ζ. As before we know that $\tilde{\gamma}' = f(\gamma')$ approaches a single point on the unit circle. We assume that $\tilde{\gamma}$ and $\tilde{\gamma}'$ approach two distinct points. By the definition of accessible point, curves γ and γ' can be connected by a Jordan arc within any

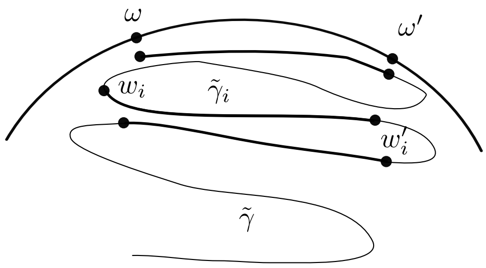


Figure 2.5: Thick parts of the curve γ form arcs whose end-points converge to two distinct points ω and ω'.

neighbourhood of ζ. As these neighbourhood contract to ζ, their images become arcs whose endpoints converge to two distinct points on the unit circle, see the Figure 2.6. On these arcs f^{-1} converges uniformly to ζ and, as before, this implies that f^{-1} is constant.

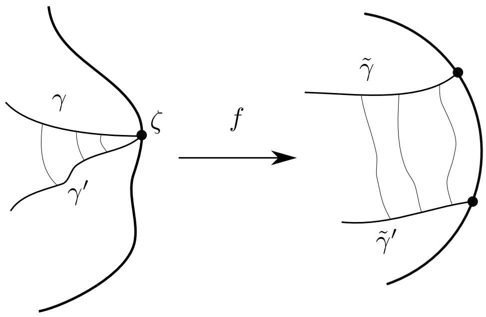


Figure 2.6: Images of arcs connecting curves γ and γ' form a sequence of arcs in $\mathbb{D}$ whose end-points converge to two distinct points on the unit circle.

Let ζ and ζ' be two different accessible points, γ and γ' be the corresponding curves, and $B(\zeta, r_0)$ and $B(\zeta', r_0)$ be the disjoint components of $B(\zeta, r) \cap \Omega$ as in the definition of accessible points. We know that $f(z)$ approaches definite points on the unit circle as z approaches ζ or ζ' along γ or γ'. We assume that they approach the same point $\omega \in \partial\mathbb{D}$ and will show that it leads to a contradiction.

Without loss of generality we can assume that the curves γ and γ' defining these these two accessible points start at the same point and that they have no other common points. The images of these two curves form a Jordan curve that have the only common point with $\partial\mathbb{D}$ and bound a sub-domain in $\mathbb{D}$ that we denote by D.

In this domain we can apply Lindelöf's theorem 2.6.5 to $g = f^{-1}$ and prove that g is continuous at ω. This immediately implies that both accessible points correspond to the same boundary point. Moreover for every $\epsilon > 0$ there is δ such that if $w \in \mathbb{D}$ such that $|w - \omega| < \delta$ then $|z - \zeta| < \epsilon$ where $z = g(w)$.

Since we have two different accessible points, there is r_0 such that the tails of γ and γ' are in the different connected component of $\Omega \cap B(\zeta, r)$ for $r < r_0$. We assume that $\epsilon < r_0$ and chose δ as in the previous paragraph. There is an arc of the circle $|\zeta - w| = \delta$ which connects two points on the images of the arcs γ and γ'. Let γ'' be the image of this arc. It connects two points in the disjoint components of $\Omega \cap B(\zeta, r_0)$, hence there should be a point on γ'' which is outside of $B(\zeta, r_0)$ which contradicts our assumption that γ'' should be inside $B(\zeta, \epsilon)$. This proves that ζ and ζ' should be the same accessible points.

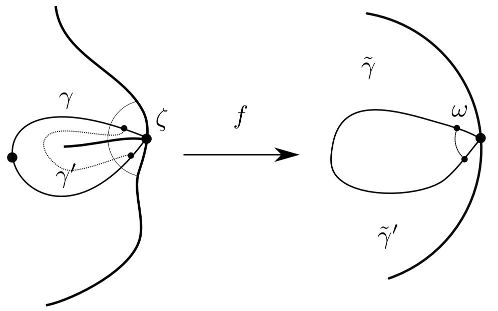


Figure 2.7: Pre-image of an arc connecting $\tilde{\gamma}$ and $\tilde{\gamma}'$ can not stay in a small neighbourhood of ζ.

□

Since all points on a Jordan curve correspond to exactly one accessible point one can easily prove

Theorem 2.6.6 (Caratheodory). *Let Ω be a simply connected domain bounded by a closed Jordan curve Γ and let f be a conformal map from Ω onto $\mathbb{D}$. Then f could be continuously extended to a bijection from Γ onto the unit circle.*

Proof. Existence of the extension and that it is a bijection follows directly from Theorem 2.6.3. The continuity follows from monotonicity of the argument. The details are left to the reader. □

It is not surprising that for analytic boundaries the result is even stronger (but the proof is beyond the scope of this course).

Theorem 2.6.7. *Let Ω be a domain bounded by an analytic Jordan curve, then a conformal map f from Ω onto $\mathbb{D}$ can be extended to a function analytic on the boundary.*

Surprisingly the inverse boundary correspondence holds:

Theorem 2.6.8. *Let f be a continuous function in $\bar{\Omega}$ which is analytic in Ω, we also assume that the boundary of Ω is a positively oriented Jordan curve Γ. If f is a continuous orientation preserving bijection from Γ onto another Jordan curve Γ', then f is a univalent map from Ω onto the domain Ω' bounded by Γ'.*

Proof. Let w_0 be some point in Ω'. Since f maps Γ onto Γ', we have that $f \neq w_0$ on Γ. By continuity, there is a neighbourhood $U \subset \Omega$ of Γ where $f \neq w_0$ as well.

For any closed curve $\gamma \subset \Omega$ we can consider the quantity

$$\frac{1}{2\pi}\Delta_\gamma \arg(f(z) - w_0)$$

the normalized increment of the argument along γ. It is easy to see that when we continuously deform γ, this quantity changes continuously. Since this quantity is integer-valued for any closed curve, it must be constant for all curves that are continuous deformations of each other inside U.

By theorem's assumptions

$$\frac{1}{2\pi}\Delta_\Gamma \arg(f(z) - w_0) = \frac{1}{2\pi}\Delta_{\Gamma'} \arg(w - w_0) = 1.$$

Let $\gamma \subset \Omega$ be a simple curve homotopic to Γ inside U. The argument above implies that the same is true for γ. Let D be the domain bounded by γ. For this domain we can apply argument principle and get that the equation $f = w_0$ has exactly one solution inside D. On the other hand $\Omega \setminus D \subset U$ and by construction $f \neq w_0$ there. This proves that there is a unique point $z_0 \in \Omega$ such that $f(z_0) = w_0$.

By the same argument $f \neq w$ for every w in the interior of complement of Ω'. Finally, no point of Ω is mapped onto a point of Γ', otherwise its neighbourhood would be mapped onto a neighbourhood of a point on the boundary of Ω' and there will be points outside, which contradicts the argument above. $\square$

Note that in the previous theorem can assume that Ω is a domain with Jordan boundary in $\widehat{\mathbb{C}}$. But the domain Ω' should be bounded which can be seen from the following simple example.

Let $\Omega = \Omega'$ be the upper half-plane, the boundary $\Gamma = \Gamma' = \mathbb{R}$. Function $f(z) = z^3$ is a continuous bijection from Γ onto Γ', but it does not map Ω onto Ω'.

Considering simple examples of slit domains where the uniformizing maps are known explicitly we can see that these maps are not continuous if one uses the ordinary Euclidean topology. Maps obviously behave differently on different sides of the slits. On the other hand, from the internal geometry point of view, two points on the different sides of the slit are far away. The notion of an accessible point formalizes this intuition and allows to treat two sides of a slit as two different sets. This allows us to study boundary behaviour for all domains with relatively simple boundary. To complete the study of boundary correspondence we have to study what happens at non-accessible points. For this we will introduce the notion of prime ends that was introduced by Caratheodory [6].

Definition 2.6.9. *A cross-section* or *a cross-cut* in a simply-connected domain Ω is a Jordan arc $\gamma : (0, 1) \to \Omega$ such that the limits $\gamma(t)$ as t approaches 0 and 1 exist and lie on the boundary of Ω. Curve γ separates Ω into two connected domains. We assume that boundaries of both domains contain boundary points of Ω other than the end-points of γ.

It is easy to see that the end-points of a cross-cut must be different accessible points.

Definition 2.6.10. A *chain* is a sequence of cross-cuts γ_n such that $\bar{\gamma}_n \cap \bar{\gamma}{=}\emptyset$, diameter of γ_n tends to zero, and any Jordan curve in Ω connecting a point on γ_n with a given point $z_0 \in \Omega$ must intersect all γ_m for $m < n$.

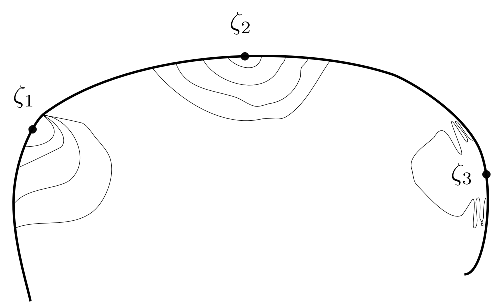


Figure 2.8: Curve near point ζ_1 do not form a chain. A curve near point ζ_3 is not a cross-cut since it is not continuous at the end-points. Curves near ζ_2 form a chain that defines a prime end which corresponds to an accessible point.

Definition 2.6.11. We say that two chains γ_n and γ_n' are equivalent if for every n the arc γ_n separates almost all γ_m' from γ_{n-1} and γ_n' separates almost all γ_m from γ_{n-1}'. A *prime end* is an equivalence class of chains.

There is an alternative way to define the equivalence of the chains. Let D_n be the connected component of $\Omega \setminus \gamma_n$ which does not contain γ_{n-1}. It contains all γ_m with $m > n$. It is easy to see that $D_n \subset D_{n+1}$. Let D_n and D_n' be two collections of sub-domains corresponding to two chains. Then the chains are equivalent if and only if each domain from one collection contain all but finitely many domains from the other collection. Using this notion we can define a prime end by the condition that diameters of $f(D_n)$ tend to zero instead of the diameters of γ_n.

Definition 2.6.12. The *support* of a prime end is defined as $\cap_n D_n$ where D_n are the domains as above.

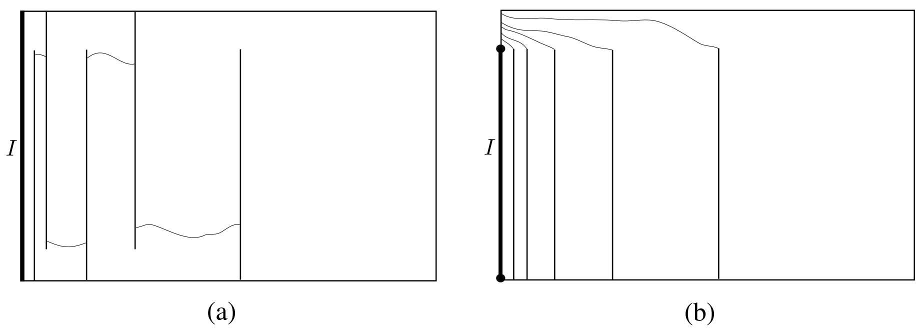


Figure 2.9: In both examples the interval I is the support of a prime end.

It is easy to see that the support is a subset of $\partial\Omega$ which is independent of the choice of a chain. Another simple observation is that each accessible point can be associated with a prime end. Indeed, let us consider an accessible point ζ which is defined by a curve γ. We can define γ_n to be arcs of the circles $|\zeta - z| = 1/n$ that intersect with γ. These arcs form a chain and the support of the corresponding prime end is ζ. Clearly, for different accessible points these prime end are different.

Now we can formulate (without a proof) the most general result about the boundary correspondence.

Theorem 2.6.13 (Caratheodory)**.** *Let Ω be a simply connected domain and let f be a conformal map from f onto $\mathbb{D}$, then f could be continuously extended to a bijection between prime ends and points on the unit circle.*

2.7 Multiply connected domains

2.7.1 Conformal annuli

In the previous section we have shown that all non-trivial simply connected domains are conformally equivalent to the unit disc, hence they all are conformally equivalent to each other. For multiply connected domains this is not true any more. The simplest example is given by the following theorem

Theorem 2.7.1. *Let $A(r, R) = \{z : r < |z| < R\}$ be an annulus with internal radius r and external radius R. There is a conformal map from $A_1 = A(r_1, R_2)$ onto $A_2 = A(r_2, R_2)$ if and only if $R_1/r_1 = R_2/r_2$.*

Proof. As usual, one direction is easy, if the ratios of the radii are the same then $f(z) = zR_2/R_1$ maps $A(r_1, R_2)$ onto $A(r_2, R_2)$. This map is linear and hence conformal. This also means that without loss of generality we can assume in the sequel that $r_i = 1$.

The main part of the theorem is the statement that the ratio of radii is a conformal invariant. Let us assume that there is a map f from one annulus onto another. We are going to show that this implies that the ratios of radii are equal.

First we want to show that f maps boundary circles onto boundary circles. Note that this is much weaker than continuity up to the boundary, and this is why we can show this without use of sophisticated techniques.

Let $S = r\mathbb{T}$ be a circle in A_2 with radius $1 < r < R_2$. Its pre-image under f is a compact set, hence it is bounded away from both boundary circles of A_1. In particular, $K = f(A(1, 1 + \epsilon))$ does not intersect S for sufficiently small ϵ. Since S separates A_2 into two disjoint parts, this means that K is completely inside S or completely outside S. Let us assume for a while that it is inside (). If we consider a sequence $\{z_n\}$ inside $A(1, 1 + \epsilon)$ with $|z_n| \to 1$ then the sequence $\{f(z_n)\}$ does not have points of accumulation inside A_2, hence $|f(z_n)|$ must converge to 1. In the same way we show that $|f(z_n)| \to R_2$ for $|z_n| \to R_1$. The purpose of the trick with S excludes the possibility that $f(z_n)$ oscillates between two boundary circles.

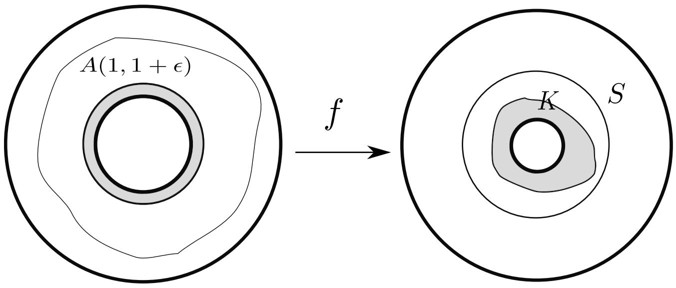


Figure 2.10: Circle S and its pre-image split each annulus into two doubly connected domains. Shaded areas are $A(1, 1+\epsilon)$ and its image. We assume that both of them lie inside S and its pre-image.

In the case when K is outside S we get that $|f(z_n)| \to 1$ as $|z_n| \to R_1$ and $|f(z_n)| \to R_2$ as $|z_n| \to 1$. In this case we change $f(z)$ to $R_2/f(z)$ which also conformally maps A_1 onto A_2 but have the same boundary behaviour as in the first case.

Let us consider the function

$$u(z) = \log|f(z)| = \Re \log(f(z)).$$

This is a real part of an analytic function and hence it is harmonic in A_1. The previous discussion shows that u can be extended continuously to the closure of A_1 by defining $u(z) = 0$ on $|z| = 1$ and $u(z) = \log(R_2)$. There is another harmonic function in A_1 which has the same boundary values:

$$\frac{\log(R_2)}{\log(R_1)} \log|z|.$$

By the maximum modulus principle, these two functions are the same.

The basic idea of the rest is very simple. The equality of the harmonic functions gives $|f| = |z|^\alpha$ where $\alpha = \log(R_2)/\log(R_1)$. This suggests that $f = cz^\alpha$ for some c with $|c| = 1$. On the other hand z^α is one-to-one if and only if $\alpha = 1$ or, equivalently, $R_1 = R_2$. The rigorous justification of this argument is slightly more involved.

Let us consider a harmonic function

$$h(z) = \log|f| - \alpha \log|z|.$$

This function looks like the real part of $\log(f) - \alpha \log(z)$, but we don't know whether it could be defined as a single-valued function.

The argument above shows that h vanishes on the boundary, hence, by maximum principle, it vanishes everywhere in A_1. Equivalently, $\log|f| = \alpha \log|z|$ or $\log(f\bar{f}) = \alpha \log(z\bar{z})$. Applying the Cauchy-Riemann differential operator $\partial = (\partial_x - i\partial_y)/2$ to both functions we get

$$\frac{f'}{f} = \alpha \frac{1}{z}.$$

Take any simple curve γ which goes counter clockwise around the origin inside A_1 and integrate this identity along γ. Dividing by $2\pi i$ we have

$$\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}\mathrm{d}z = \alpha$$

By argument principle the left hand side is the index of $f(\gamma)$ and hence α must be an integer. Since $R_i > 1$, it must be a positive integer. This allows us to define a single valued function $z^{-\alpha}$. Now consider function $z^{-\alpha}f(z)$. Modulus of this function is identically equal to 1 in A_1. By standard corollary of Cauchy-Riemann equations this implies that it must be a constant function. This proves that $f(z) = e^{i\theta}z^\alpha$ for some real θ. On the other hand the only integer power which is univalent in the annulus is z, hence $\alpha = 1$ and $R_1 = R_2$. $\square$

This is a very important theorem and as such it has more than one proof. Here we give one more proof and we will give another one after discussion of extremal lengths. The second proof is based on the following proposition.

Proposition 2.7.2. *Let A_1 and A_2 be two annuli as before. If there is a univalent map $f : A_1 \to A_2$, then $R_2/r_2 \geq R_1/r_1$.*

Proof. As before, we can assume without loss of generality that $r_1 = r_2 = 1$ and that the inner circle is mapped to the inner circle, so that the outer circle is mapped to the outer circle. Since function f is analytic in an annulus it can be written as Laurent series

$$f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$$

We denote by $A(r)$ the area of a domain bounded by a Jordan curve $f(re^{i\theta})$ where θ goes from 0 to 2π. By Green's formula for the area we have

$$\begin{aligned}
A(r) &= \frac{1}{2i}\int \bar{f}(z)\mathrm{d}f(z) = \frac{1}{2i}\int_{|z|=r}\bar{f}(z)f'(z)\mathrm{d}z \\
&= \frac{1}{2i}\int_0^{2\pi}\left(\sum \bar{a}_n r^n e^{-i\theta n}\right)\left(\sum n a_n r^{n-1}e^{i\theta(n-1)}\right) rie^{i\theta}\mathrm{d}\theta \\
&= \pi\sum_{n\in\mathbb{Z}} n|a_n|^2 r^{2n}.
\end{aligned}$$

The last identity holds since $\int e^{i\theta n} = 0$ unless $n = 0$.

Passing to the limit as $r \to 1$ we have

$$\pi = \pi\sum n|a_n|^2.$$

Using this identity we can write

$$A(r) - \pi r^2 = \pi r^2 \sum_{n\in\mathbb{Z}} n|a_n|^2(r^{2n-2} - 1) \geq 0$$

where the last inequality holds term-wise. Passing to the limit as $r \to R_1$ we obtain that $R_2 \geq R_1$. $\square$

To complete the second proof of Theorem 2.7.1 we just use the Proposition for f and f^{-1}.

Exercise 8. *Use reflection principle to give another proof of this theorem.*

Theorem 2.7.1 tells us that not all doubly connected domains are conformally equivalent, and it is easy to believe that the same is true for domains of higher connectivity. This means that we can not use the same uniformizing domain for all domains, instead, we should use sufficiently large family of standard domains. In the doubly connected case the standard choice is the family of all annuli with outer radius 1 (some people prefer annuli with inner radius 1). For higher connectivity there is no unique family, but there are several preferred families. One of the most frequent families is the family of circle domains: domains such that each boundary component is either a circle or a single point. We will discuss other standard families of domains at the end of this section.

Here we will present a rather elementary proof for the doubly connected domains. The proof in the case of finitely connected domains is not extremely difficult, but goes beyond the scope of this course. For infinitely connected domains there is Koebe conjecture stating that every domain can be mapped onto a circle domains. The best result in this direction is due to He and Schramm who proved it in [12] for countably connected domains using circular packing techniques.

We start with a general construction that works for all finitely connected domains. It allows us to assume without loss of generality that all boundary components are analytic Jordan curves.

First of all we can get rid of all single point components. Indeed, if there is a map f from $\Omega \setminus \{z\}$, then z_0 is an isolated singularity and the function is bounded in its neighbourhood, hence it is a removable singularity and f could be extended to the entire domain Ω. On the other hand, if we have a map from domain without a hole at z_0 then we can just restrict it to the domain with a hole.

We use the doubly connected case to illustrate how this works. Let Ω be a double connected domain and one of components of its complement is a single point z_0. Let us consider $\Omega' = \Omega \cup \{z\}$. By Riemann theorem there is a univalent $f : \Omega' \to \mathbb{D}$ with $f(z_0) = 0$. It is obvious that f maps Ω onto the annulus $\{z : 0 < |z| < 1\}$.

To show that we can assume that all boundary components are nice we again use the Riemann uniformization theorem. Let Ω be an n-connected domain and let $E_1, \dots, E_{n+1}$ be the components of its complement. Using the argument above we assume that all E_i are not singletons. Let us consider domain $\Omega \cup E_2 \cup \cdots \cup E_{n+1}$. This is a simply connected domain whose complement is not a single points, hence we can map it to the unit disc. Under this map $\Omega, E_2, \dots, E_{n+1}$ are mapped to some subsets of $\mathbb{D}$ which, abusing the notations, we still call $\Omega, E_2, \dots, E_{n+1}$. By new E_1 we denote the complement of the unit disc. Notice that the boundary of E_1 is now the unit circle which is an analytic Jordan curve. Next we take the union of all domains except E_2, map it to the disc and rename all the sets. After that the boundary of E_2 is the unit circle and the boundary of E_2 is a univalent image of the unit circle, hence it is an analytic Jordan curve. Continuing like that for all components we can map the original domain onto a sub-domain of $\mathbb{D}$ such that one boundary component is the unit circle and the others are analytic Jordan curves.

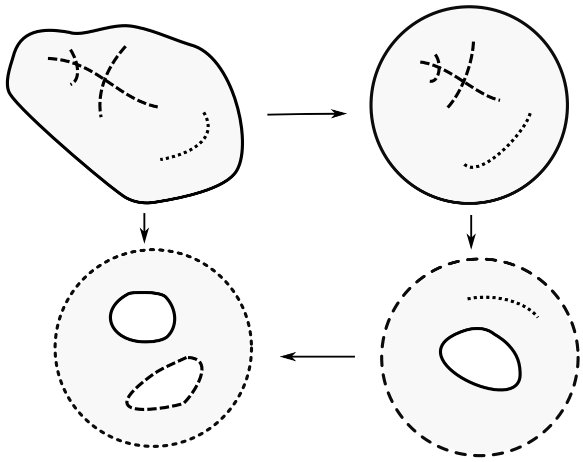


Figure 2.11: Dashed, dotted, and solid lines represent three boundary components and their successive images.

Theorem 2.7.3. *Let Ω be a doubly connected domain, then there is a univalent map f from Ω onto some annulus with outer radius 1. This map is unique up to rotation and inversion of the annulus.*

Proof. We have already proved the uniqueness in the proof of Theorem 2.7.1. To prove existence we first consider two special cases. If both components of the complement are the single points, then we can choose f to be a Möbius transformation sending these two points to 0 and ∞. If only one of them is a single point, then we can map Ω with this point to the unit disc and this point to the origin.

The only interesting case is when both components are non trivial. As we explained before, we can assume that Ω is a doubly connected domain such that one component of its complement is the complement of the unit disc and the other one is bounded by an analytic curve. By composing with one more Möbius transformation we can assume that the origin is inside the second component.

Let us apply the logarithmic function to Ω. Since 0 is not in Ω, the logarithm is analytic but it is not single valued. Each time when we go around the inner boundary component the value of $\log$ changes by $2\pi i$. Logarithm maps Ω onto a vertical strip S such that its right boundary is the imaginary axis and the left boundary is a $2\pi i$-periodic curve. By Riemann theorem there is a univalent map from S onto a vertical strip $S' = \{z : -1 < \Re(z) < 0\}$. Moreover we can assume that $\pm i\infty$ and 0 are mapped to themselves. The point $2\pi i$ is mapped onto some point w_0 on the positive imaginary axis. Rescaling by $2\pi/|w_0|$ we find a map ϕ from S onto $S'' = \{z : -h < \Re(z) < 0\}$ where $h = 2\pi/|w_0|$. This map preserves

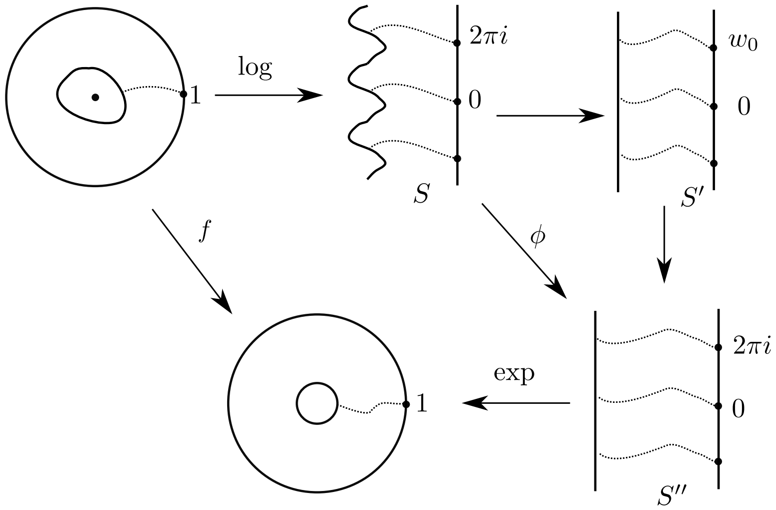


Figure 2.12: The Riemann mapping from a doubly connected domain onto an annulus. Dashed lines are an arbitrary simple curve connecting 1 to the inner boundary component and its images.

$\pm i\infty$, 0 and $2\pi i$. We claim that ϕ satisfies the following equation

$$\phi(z + 2k\pi i) = \phi(z) + 2k\pi i, \tag{2.1}$$

moreover, the same is true for the inverse function. Obviously, it is sufficient to prove this for $k = 1$, the general case follows immediately by induction. Notice that $z \mapsto z+2\pi i$ is a conformal automorphism of S and S'', hence both $f(z)+2\pi i$ and $f(z + 2\pi i)$ map S onto S'' in such a way that three boundary points $\pm i\infty$ and 0 are mapped to $\pm i\infty$ and $2\pi i$. By uniqueness of the Riemann map which sends three given boundary points to three given boundary points, these two maps are the same. The proof for the inverse function is exactly the same.

Finally we compose all these functions

$$f(z) = e^{\phi(\log(z))}.$$

This is an analytic function which maps Ω onto an annulus $A(e^{-h}, 1)$. The problem is that both $\log$ and $\exp$ are not one-to-one, so we can not immediately claim that f is univalent. Despite that, this function is univalent. This function is injective since $\log$ maps z onto a $2\pi i$-periodic sequence. By (2.1), ϕ maps $2\pi i$-periodic sequences to $2\pi i$-periodic sequences, and, finally, $\exp$ maps any $2\pi i$-periodic sequence to a single point. Similar argument for inverse functions gives that f is surjective.

$\square$

Theorem 2.3.1 tells us that all non-trivial simply connected domains are conformally equivalent to each other. Theorems 2.7.1 and 2.7.3 tell us for doubly-connected domains there is a family of equivalence classes. Each doubly connected domain is conformally equivalent to the annulus and the ratio of its radii completely determines the equivalence class. This is a first example of a *conformal invariant*: quantity that does not change under conformal transformation. For various reasons that we will discuss later, the standard conformal invariant of a doubly connected domain Ω which describes the equivalence class is the *conformal modulus* which is defined as

$$\frac{1}{2\pi}\log\frac{R}{r}$$

where R and r are outer and inner radii of an annulus which is conformally equivalent to Ω. By Theorems 2.7.1 and 2.7.3 we know that this quantity is well defined and does not depend on particular choice of an annulus.

Exercise 9. *Find an explicit map from a domain bounded by two non-concentric circles onto an annulus*

Koebe conjecture: every domain is conformally equivalent to a circle domain. Schramm proved this for countable connectivity.

2.7.2 Uniformisation of multiply connected domains

Annuli are the natural "standard" doubly connected domains. For the domains of higher connectivity there is no natural unique choice of uniformizing domains. Instead there are several somewhat standard families of *canonical domains*. In this section we will discuss canonical domains and formulate the corresponding uniformization theorems.

Parallel Slit Domains. These are domains that are the complex sphere $\widehat{\mathbb{C}}$ without a finite union of intervals that are parallel to each other.

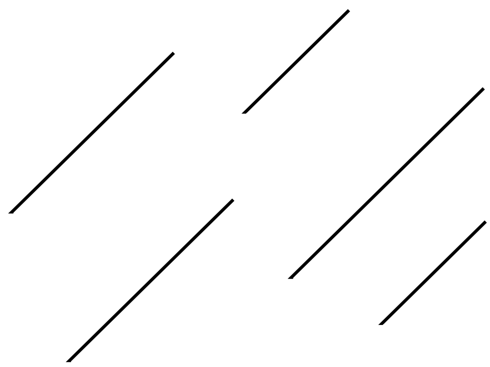


Figure 2.13: A parallel slit domain.

Let Ω be a multiply connected domain, z_0 be some point in Ω and θ be an angle in $[0, 2\pi)$, then there is a unique univalent map $f_{z_0,\theta}$ from Ω onto a parallel slit domain such that the slits form angle θ with the real line, z_0 is mapped to infinity and the Laurent series at z_0 is of the form

$$f_{z_0,\theta}(z) = \frac{1}{z - z_0} + a_1(z - z_0) + a_2(z - z_0)^2 + \dots \tag{2.2}$$

As in the proof of the Riemann Uniformisation theorem the uniformizing map could be described as a function which maximizes a certain functional over a class of admissible functions. For the mapping onto a parallel slit domain the class of admissible functions is the class of all univalent functions in Ω which have expansion as in (2.2) at $z = z_0$. The function $f_{z_0,\theta}$ has the maximal value of

$$\Re\left(e^{-2i\theta}a_1\right)$$

among all admissible functions.

Circular and Radial Slit Domains. These are two similar classes of slit domains consisting of the complex sphere without some slits. In the first case we remove several arcs that lie on concentric circles centred at the origin. In the second case we remove intervals that lie on rays emanating from the origin

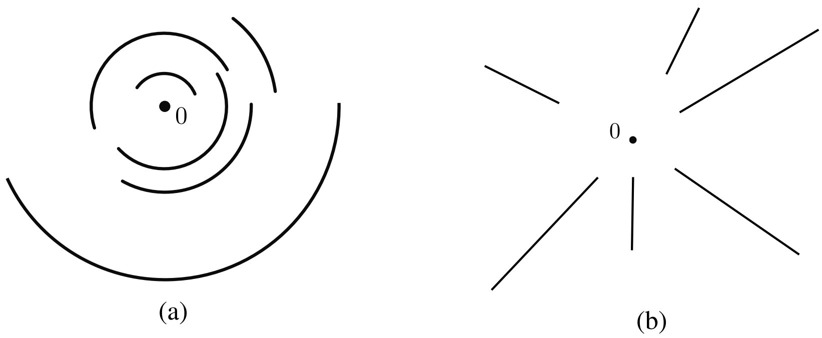


Figure 2.14: Examples of a circular slit domain (a) and a radial slit domain (b).

In both cases we can normalise a map in such a way that two given points z_1 and z_2 from Ω are mapped to the origin and infinity. Let us consider a family of functions f that are univalent in Ω, $f(z_1) = 0$, and there is a simple pole of residue 1 at z_2. The function that maximizes $|f'(z_1)|$ maps the domain onto a circular slit domain and the function that minimizes $|f'(z_1)|$ maps the domain onto a radial slit domain.

Chapter 3

Elementary Theory of Univalent Maps

In this chapter we will discuss some properties of univalent functions, we will be especially interested in their boundary behaviour and connection between geometrical properties of domains and analytical properties of univalent functions on or onto these domains.

3.1 Classes S and Σ

We will be mostly interested in properties of functions from class S (from the German word *schlicht* which is another standard term for univalent functions) consisting of univalent functions in the unit disc normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Alternatively they are given by Taylor series of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots$$

that converge in the unit disc.

For any simply connected domain there is a univalent function from $\mathbb{D}$ onto this domain. By rescaling and shifting the domain the function can be normalized to be from the class S. So up to scaling and translations, functions from S describe all simply connected domains except, of course, $\mathbb{C}$ and $\widehat{C}$.

Another standard class is the family of functions that are univalent in the complement of the unit disc $\mathbb{D}_-$ and have expansion

$$g(z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots$$

We denote this class by Σ. Each $g \in \Sigma$ maps $\mathbb{D}_-$ onto the complement of a compact set E. Sometimes it is more convenient to assume that $0 \in E$. This subclass of Σ is denoted by Σ'. Note that any function from Σ differ from some function from Σ' by subtraction of an appropriate constant, so these two classes are extremely close and share most of the properties.

One of the reasons for introduction of Σ' is that there is a simple bijection between functions from S and Σ'. If f is an arbitrary function from the class S then

$$g(z) = \frac{1}{f\left(1/z\right)}$$

belongs to the class Σ'. Conversely for every $g \in \Sigma'$

$$f(z) = \frac{1}{g\left(1/z\right)} \in S.$$

For functions given by Taylor series it is generally very difficult to check whether they are in S or not. There are some sufficient conditions but they are rather weak and cover only special cases.

One of a very important examples of a function from S is the *Koebe function*

$$K(z) = z + 2z^2 + 3z^3 + 4z^4 + \dots.$$

It is difficult to see that $K \in S$ just by looking at the Taylor series. Fortunately, this series could be written in a closed form as $z/(1-z)^2$. There are two standard ways to show that K is univalent. First one is to observe that it is a rational function of degree 2 and hence it is 2-to-1 on the complex sphere. By explicit computations one can show that the unit circle is mapped onto $[-\infty, -1/4]$ and for all points outside of this ray only one pre-image is inside the unit disc.

Alternative and more intuitive way it to rewrite K as

$$\frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4}.$$

We know that $(1+z)/(1-z)$ is a Möbius map from the unit disc onto the right half-plane $\{z : \Re z > 0\}$. Square function maps it conformally onto the plane with cut along the negative real line, scaling and subtracting $1/4$ maps it onto the plane with cut from $-\infty$ to $-1/4$. Since all maps here are univalent, their composition is also univalent.

Exercise 10. *Show that $K_\alpha \in S$ where*

$$K_\alpha(z) = \frac{1}{2\alpha}\left[\left(\frac{z+1}{1-z}\right)^\alpha - 1\right], \qquad \alpha \in (0, 2].$$

Find $K_\alpha(\mathbb{D})$.

Exercise 11. *Show that Joukowsky function $J = z + 1/z$ belongs to Σ and find $J(\mathbb{D}_-)$. Show that the modifies Joukowsky function $J_k(z) = z + k/z$ is also in Σ for all $-1 < k < 1$. Find the image $J_k(\mathbb{D}_-)$.*

Exercise 12. *Let f be a function from class S. Prove that the following functions are also from S*

1. *Let μ be a Möbius transformation preserving $\mathbb{D}$, then we can define*

$$f_\mu = \frac{f \circ \mu - f \circ \mu(0)}{(f \circ \mu)'(0)}.$$

Important particular case is $f_\theta(z) = e^{-i\theta} f(e^{i\theta} z)$.

2. *Reflection of f defined as $\bar{f}(\bar{z})$.*

3. *Koebe transform $K_n(f)(z) = f^{1/n}(z^n)$ (you also have to show that $K_n f$ could be defined as a single valued function for all positive integer n). The same is true for functions from class Σ'.*

3.2 Bieberbach-Koebe theory

The first example of a theorem relating analytical properties with geometrical is Gronwall's theorem which relates the area of the complementary domain E with coefficients of a function from Σ.

Theorem 3.2.1 (Gronwall's Area Theorem)**.** *Let $g(z) = z + \sum b_n z^{-n}$ be a function from class Σ which maps $\mathbb{D}_-$ onto the complement of a compact set E. The area of E is given by*

$$m(E) = \pi \left(1 - \sum_{n=1}^{\infty} n|b_n|^2 \right).$$

The proof of this theorem uses essentially the same technique as the proof of Proposition 2.7.2.

Proof. To compute the area of E we would like to use Green's theorem for the image of the unit circle. This does not work since the function is not defined on the unit circle and it might be that it could not be even continuously extended to the boundary. Instead we use one of the standard tricks. Take some $r > 1$ and denote by γ_r the image of the circle $|z| = r$ under f. Since f is a univalent map we have that f is a simple closed analytic curve enclosing $E \subset E_r$. By Green's theorem in its complex form the area of E_r is

$$\begin{aligned}
m(E_r) &= \frac{1}{2i} \int_{\gamma_r} \bar{w} \mathrm{d}w = \frac{1}{2i} \int_{|z|=r} \bar{g}(z) g'(z) \mathrm{d}z \\
&= \frac{1}{2i} \int_0^{2\pi} \left(\bar{z} + \sum \bar{b}_b \bar{z}^n \right) \left(1 - \sum n b_n z^{-n-1} \right) r i e^{i\theta} \mathrm{d}\theta \\
&= \pi \left(r^2 - \sum_{n=1}^{\infty} n |b_n|^2 r^{-2n} \right).
\end{aligned}$$

Passing to the limit as $r \to 1$ we complete the proof the theorem. $\square$

Corollary 3.2.2. *Since the measure of E is non-negative we have*

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1,$$

and in particular

$$|b_n| \leq \frac{1}{\sqrt{n}}.$$

This inequality is sharp for $n = 1$ since $J(z) = z + 1/z$ is univalent, but not sharp for $n \geq 2$. Indeed the direct computations show that the derivative of $g(z) = z + b_0 + e^{i\theta}\sqrt{n}z^n$ vanishes at some points in $\mathbb{D}_-$ and hence g is not univalent.

From these inequalities on the coefficients of functions from Σ one can estimate the second coefficient of function from S. This theorem was initially proved by Bieberbach in 1916 [4].

Theorem 3.2.3 (Bieberbach)**.** *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be a function from S, then $|a_2| < 2$. Moreover, $|a_2| = 2$ if and only if f is a rotation of the Koebe function.*

Proof. As we discussed before, the function $1/f(1/z)$ is from class Σ'. Applying the Koebe transform (see Exercise 3) with $n = 2$ we see that

$$g(z) = \frac{1}{\sqrt{f(1/z^2)}}$$

is also from Σ'. From the Taylor series for f we compute

$$\sqrt{f(z^2)} = \sqrt{z^2 + a_2 z^4 + \dots} = z\sqrt{1 + a_2 z^2 + \dots}$$

and

$$g(z) = \frac{z}{\sqrt{1 + a_2 z^{-2} + \dots}} = z + \frac{a_2}{2} z^{-1} + \dots$$

Applying the Corollary to the Gronwall's Area theorem 3.2.2 we get $|a_2|/2 \leq 1$ with equality holding if and only if

$$g(z) = z - e^{i\theta} z^{-1}$$

for some real θ. Rewriting f in terms of g we get

$$g(z) = e^{-i\theta}\left(\frac{e^{i\theta}z}{(e^{i\theta}z - 1)^2}\right) = e^{-i\theta}K(e^{i\theta}z).$$

□

In the same paper [4] Bieberbach used this result as a basis for the following famous conjecture that was probably the main open problem in the geometric function theory for may decades.

Conjecture 3.2.4 (Bieberbach)**.** Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be a function from S, then $|a_n| < n$. Moreover, $|a_n| = n$ for some n if and only if f is a rotation of the Koebe function.

This conjecture motivated a lot of progress in complex analysis until it was finally proved in 1985 by de Branges [8]. Surprisingly the similar question about the coefficients of functions from the class Σ is still open. In fact, even the decay rate (i.e. the best constant γ such that $|b_n|$ is asymptotically bounded by $n^{\gamma-1}$) is not known. To be more precise we define

$$\gamma_g = \limsup_{n\to\infty} \frac{\log |b_n|}{\log n} + 1$$

and

$$\gamma = \gamma_\Sigma = \sup_{g\in\Sigma} \gamma_g.$$

In the similar way for functions in class S we define

$$\gamma_f = \limsup_{n\to\infty} \frac{\log |a_n|}{\log n} + 1$$

and

$$\gamma_S = \sup_{f\in S} \gamma_f,$$

$$\gamma_{S_b} = \sup_{f\in S_b} \gamma_f,$$

where S_b is a family of bounded functions from S. The equality $\gamma_S = 2$ follows immediately from the Bieberbach conjecture, but in fact it can be derived from much simpler estimate $|a_n| \le en$ which was proved by Littlewood in 1925 [13]. Carleson and Jones proved in 1992 [7] that $\gamma_{S_b} = \gamma_\Sigma$ and conjectured that it is equal to $1/4$ (trivial bounds are $0 \le \gamma \le 1/2$). This conjecture is still wide open.

Bieberbach theorem implies a very important corollary about the geometrical properties of functions from S. By analyticity, we know that $\Omega = f(\mathbb{D})$ contains an open neighbourhood of the origin. The lower bound on the distance from the origin to the boundary of Ω is given by

Theorem 3.2.5 (Koebe $1/4$ Theorem)**.** *Let f be a function from S, then $1/4\mathbb{D} \subset \Omega$, where $\Omega = f(\mathbb{D})$. Moreover, if there is $w \notin \Omega$ with $|w| = 1/4$, then f is a rotation of the Koebe function.*

Proof. Let us take any point w which is not in Ω. The function

$$\phi(z) = \frac{wf(z)}{w - f(z)}$$

is obviously analytic in $\mathbb{D}$. To check that it is univalent, we assume that $\phi(z_1) = \phi(z_2)$. Since $w \neq 0$, this implies that $f(z_1) = f(z_2)$ and $z_1 = z_2$. Finally

$$\phi(z) = \frac{wf(z)}{w - f(z)} = \frac{wz + wa_2z^2 + \dots}{w - z - a_2z^2 - \dots} = z + \left(a_2 + \frac{1}{w}\right) z^2 + \dots,$$

which implies that $\phi \in S$ and, by the Bieberbach theorem, $|a_2 + 1/w| \leq 2$. Since we also have $|a_2| \leq 2$ we have $|1/w| \leq 4$ or, equivalently, $|w| \geq 1/4$. This proves that all points with $|w| < 1/4$ must lie in Ω.

To prove the last part we notice that $|w| = 1/4$ implies that $|a_2| = 2$ and the Bieberbach theorem 3.2.3 implies that f must be a rotation of the Koebe function. □

On the other hand, if $\mathbb{D} \subset \Omega$, then the Schwarz lemma 2.1.2 applied to $f^{-1}|_{\mathbb{D}}$ implies that $f(z) = z$. The same argument implies that Ω can not contain a disc centred at the origin of radius larger than 1. Together with the Koebe $1/4$ theorem this proves

Corollary 3.2.6. *Let $f : \mathbb{D} \to \Omega$ be a function from s, then* $\operatorname{dist}(0\partial\Omega) \in [1/4, 1]$.

This could be easily generalized to arbitrary univalent maps:

Theorem 3.2.7 (Koebe Distortion Theorem)**.** *Let $f : \Omega \to \Omega'$ be a univalent map and let z be some point in Ω. Then*

$$\frac{1}{4}\operatorname{dist}(f(z), \partial\Omega') \leq |f'(z)|\operatorname{dist}(z, \partial\Omega) \leq 4\operatorname{dist}(f(z), \partial\Omega')$$

Exercise 13. *Prove the Koebe Distortion Theorem.*

Exercise 14. *Let f be a bounded univalent function in $\mathbb{D}$, prove that $(1-|z|)|f'(z)|$ tends to 0 as $|z| \to 1$. Give an example showing that boundedness is an essential condition.*

We know that locally the distances are distorted by $|f'|$. The Koebe theorem tells us that the same holds globally up to a constant which is between $1/4$ and 4.

We would like to conclude this section with the sharp bounds on the distortion (i.e. on $|f'|$) and on the growth (i.e on $|f|$) near the boundary. Both results will follow from the following inequality which is due to Bieberbach [4] which in its turn follows from the coefficient estimate.

Theorem 3.2.8 (Bieberbach inequality)**.** *Let f be a function from S, z be any point with $r = |z| < 1$, then*

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2r^2}{1-r^2}\right| \leq \frac{4r}{1-r^2}. \tag{3.1}$$

Moreover, this inequality is sharp.

Proof. For $w_0 \in \mathbb{D}$ we can define the function

$$\phi(z) = \frac{f\left(\frac{z+w_0}{1+\bar{w}_0 z}\right) - f(w_0)}{(1-|w_0|^2)f'(w_0)}.$$

This function is a composition of a Möbius automorphism of $\mathbb{D}$, f, and a linear transformation, hence it is univalent in $\mathbb{D}$. Moreover, it is easy to see that $\phi(0) = 0$. Computing the first two derivatives at the origin is a bit more involved, but absolutely straightforward chain rule computation which is left to the reader. Here we just present the result of the computation

$$\phi(z) = z + \left(\frac{1}{2}(1 - |w_0|^2)\frac{f''(w_0)}{f'(w_0)} - \bar{w}_0\right) z^2 + \dots$$

This proves that $\phi \in S$ and by the Bieberbach theorem 3.2.3 the second coefficient is bounded by 2.

$$\left|\frac{1}{2}(1 - |w_0|^2)\frac{f''(w_0)}{f'(w_0)} - \bar{w}_0\right| \leq 2.$$

Changing w_0 to z and multiplying by $2z/(1 - |z|^2)$ we obtain (3.1).

Direct computations for the Koebe function $K(z)$ and $z = r$ show that the inequality is sharp. By rotating the Koebe function we can see that it is sharp for all radial directions. □

In the inequality (3.1) we can change the modulus to the real or imaginary part and obtain

$$\frac{-4r + 2r^2}{1 - r^2} \leq \Re\left(\frac{zf''(z)}{f'(z)}\right) \leq \frac{4r + 2r^2}{1 - r^2} \tag{3.2}$$

and

$$\frac{-4r}{1 - r^2} \leq \Im\left(\frac{zf''(z)}{f'(z)}\right) \leq \frac{4r}{1 - r^2}.$$

On the other hand $zf''/f' = r\partial_r \log f'$ and the inequalities above could be rewritten as

$$\frac{-4 + 2r}{1 - r^2} \leq \partial_r \log |f'(z)| \leq \frac{4 + 2r}{1 - r^2}$$

and

$$\frac{-4}{1 - r^2} \leq \partial_r \arg f'(z) \leq \frac{4}{1 - r^2}.$$

Integrating these inequalities the along straight interval from 0 to z we prove two theorems below.

Theorem 3.2.9 (Distortion Theorem)**.** *For a function $f \in S$ and $r = |z|$ we have*

$$\frac{1 - r}{(1 + r)^3} \leq |f'(z)| \leq \frac{1 + r}{(1 - r)^3}.$$

Moreover, this inequality is sharp and if the equality occurs for some $z \neq 0$, then f must be a rotation of the Koebe function.

Proof. We already proved the main part of the theorem. To prove the last part we notice that for the equality to hold for some $z = re^{i\theta}$, there must be equality in (3.2) for all $z = te^{i\theta}$, $t \in [0, r]$. Dividing by t and passing to the limit $t \to 0$ we have

$$\Re\left(e^{i\theta}\frac{f''(0)}{f'(0)}\right) = \pm 4,$$

which in its turn implies that the second coefficient of f has modulus 4 which happens only for the rotations of the Koebe function. This argument, or the direct computation of the derivative of the Koebe function shows that the inequality is indeed sharp. □

Exercise 15. *Let f be a univalent function in $\mathbb{D}$. Show that for all $z \in \mathbb{D}$*

$$\frac{1}{4}(1-r^2)|f'(z)| \le \operatorname{dist}(f(z), \partial f(\mathbb{D})) \le (1-r^2)|f'(z)|$$

where $r = |z|$.

Theorem 3.2.10 (Rotation Theorem)**.** *For a function $f \in S$ and $r = |z|$ we have*

$$|\arg f'(z)| \le \frac{1+r}{1-r}.$$

The estimate in the Rotation Theorem is not sharp, but the proof of the sharp estimate is beyond the scope of this course.

Finally we prove the universal estimates on the growth of the functions from S

Theorem 3.2.11 (Growth Theorem)**.** *For a function $f \in S$ and $r = |z|$ we have*

$$\frac{r}{(1+r)^2} \le |f(z)| \le \frac{r}{(1-r)^2}.$$

Moreover, if the equality occurs for some $z \neq 0$, then f is a rotation of the Koebe function.

Proof. The upper bound is a simple corollary of the Distortion Theorem 3.2.9. Indeed, for $z = re^{i\theta}$ we have

$$f(z) = \int_0^r f'(se^{i\theta})e^{i\theta}\mathrm{d}s.$$

By triangle inequality and Distortion Theorem

$$|f(z)| \le \int_0^r \frac{1+s}{(1-s)^3}\mathrm{d}s = \frac{r}{(1-r)^2}.$$

To get the lower bound we fix r and observe that it is enough to prove the inequality for z such that $|f(z)|$ is minimal. Let us consider the curve in $\Omega = f(\mathbb{D})$ which is the image of the circle or radius r. This curve is a compact set which does not contain 0. Let w_0 be the point on this curve which minimizes the distance

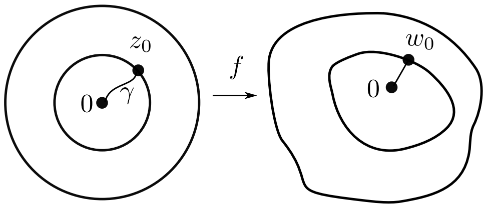


Figure 3.1

to the origin. The interval from 0 to w_0 lies completely inside Ω. We denote its pre-image by γ, which is obviously a simple curve connecting the origin with some point z_0 of modulus r and stays inside the closed disc or radius r (see the Figure 3.1). By construction, $|f(z_0)| = \min_{|z|=r} |f(z)|$. As before, $f(z_0) = \int_\gamma f'(z)\mathrm{d}z$, but in this case our construction implies that the argument of $f'(z)\mathrm{d}z$ is constant along γ so we have

$$|f(z_0)| = \int_\gamma |f'(z)||\mathrm{d}z| \geq \int_0^r \frac{1-r}{(1+r)^3}\mathrm{d}r = \frac{r}{(1+r)^2}.$$

Since both inequalities are obtained by integration of the inequalities from the Distortion Theorem, the equality in any of them implies equality in the Distortion Theorem, which, in its tern, implies that the function is a rotation of the Koebe function.

□

Chapter 4

Conformal Invariants

In this chapter we will discuss various quantities that do not change under conformal transformations. First important example that we have already encountered is the conformal modulus of a doubly connected domain. Another similar example is the modulus of the conformal rectangle: any simply connected domain with four marked points on the boundary could be conformally mapped onto a rectangle. The side ratio of this rectangle is a conformal invariant. Later on we will see that these two invariants are closely related. Other important examples are the harmonic measure, the Green's function and other solutions of boundary problems.

4.1 Green's function and harmonic measure

One of the main applications of conformal mappings is the solution of the boundary problems for Laplacian. This is based on a very simple observation that harmonic functions are invariant under conformal transformations. Indeed, if u is a harmonic function in Ω' and $f : \Omega \to \Omega'$ is a conformal map, then the function $h(z) = u(f(z))$ is harmonic in Ω. This follows from the chain rule and Cauchy-Riemann equations. If the function f is continuous bijection of the boundaries and u is continuous up to the boundary, then h is also continuous up to the boundary and its boundary values are given by that of u. This means that if we want to solve a Dirichlet boundary problem on Ω' then we can solve it in a simpler domain Ω and transfer the result to Ω' by a conformal map from Ω to Ω'. The best choice for the simple domain is $\mathbb{D}$ or $\mathbb{H}$ where explicit formulas for the Poisson kernel are known and solutions to the Dirichlet problems are given by simple integral formulas.

The Green's function plays a fundamental role in the the theory of harmonic functions and in the study of the Dirichlet boundary problems. We define the Green's function $G_\Omega(z_1, z_2)$ in a domain Ω as the only function which is harmonic as a function of z_1 everywhere in $\Omega\backslash\{z_2\}$, near $z_1 = z_2$ it behaves as $-\log|z_1 - z_2|$, and equals to 0 on the boundary of Ω. Conformal invariance of harmonic functions and a simple computation show that if Ω and Ω' are two domains and f is a conformal map from one domain onto another, then $G_\Omega(z_1, z_2) = G_{\Omega'}(f(z_1), f(z_2))$.

This could be interpreted as conformal invariance of the Green's function.

Alternatively, we can consider a simply connected domains Ω with two distinct marked points z_1 and z_2. From the Riemann uniformisation theorem we know that there is a conformal map $f : \Omega \to \mathbb{D}$ such that $f(z_1) = 0$ and $f(z_2) \in (0, 1)$. Moreover, we know that such map f is unique, which means that $f(z_2)$ is uniquely determined by (Ω, z_1, z_2). In other words, this system has a unique conformal parameter $f(z_2)$ which completely determines the conformal type of the configuration. This means that two configurations are conformally invariant if and only if this parameter is the same for both configurations. On the other hand

$$f(z_2) = \exp(\log |f(z_2)|) = \exp(-G_{\mathbb{D}}(0, f(z_2))) = \exp(-G_\Omega(z_1, z_2)).$$

This means that the Green's function is a conformal invariant which completely determines the conformal type of a configuration (Ω, z_1, z_2).

4.2 Harmonic measure

The harmonic measure is one of the fundamental objects in the geometric function theory and plays an important role in many applications. Extensive discussion of the harmonic measure could be found in the book by Garnett and Marshall [10]. There are several ways to define the harmonic measure. Here we will present some of them, but we will not prove that they all are equivalent.

Probably the simplest way to define is via conformal invariance

Definition 4.2.1. For the unit disc we define the harmonic measure $\omega_{\mathbb{D}}(0, A)$ on the boundary of $\mathbb{D}$ as the normalized Lebesgue measure $m(A)/2\pi$. For any simply connected domain Ω and $z \in \Omega$ we define $\omega_\Omega(z_0, A) = \omega_{\mathbb{D}}(0, f(A))$, where f is a conformal map from Ω onto $\mathbb{D}$ with $f(z_0) = 0$. We understand $f(A)$ in terms of prime ends.

Conformal invariance is built into this definition.

Another definition uses the Dirichlet boundary problem

Definition 4.2.2. Let Ω be a simply connected domain and A be a set on it's boundary, the harmonic measure $\omega_\Omega(z, A)$ is defined as $u(z)$ where u is the solution of the Dirichlet boundary problem with the boundary value $u = 1$ on A and $u = 0$ on the rest of the boundary.

It is not difficult to check that these two definitions are equivalent. The main difference it that in the first definition we mainly think of $\omega(z, A)$ as a measure which depends on a parameter z. In the second definition we think that it is a harmonic function of z which depends on a parameter A.

Readers familiar with the Brownian motion might find the following definition more illustrative .

Definition 4.2.3. Let Ω be a domain, A be a set on its boundary and B_t be the standard two-dimensional Brownian motion started from z. The harmonic measure of A at z could be defined as $\omega_\Omega(z, A) = \mathbb{P}(B_\tau \in A)$, where $\tau = \inf\{t > 0 : B_t \notin \Omega\}$ is the first exit time.

One of the main simple properties of harmonic measure is that it is monotone with respect to both Ω and A. The precise statement is given by the following theorem.

Theorem 4.2.4. *Let Ω be a sub-domain of Ω'. Let us assume that $A \subset (\partial\Omega \cap \partial\Omega')$ and that $z \in \Omega$, then $\omega_\Omega(z, A) \le \omega_{\Omega'}(z, A)$. If $A \subset A' \subset \partial\Omega$, then $\omega_\Omega(z, A) \le \omega_\Omega(z, A')$.*

Proof. Both parts of the theorem follow from the maximum principle for harmonic functions. Obviously, $h(z) = \omega_{\Omega'}(z, A)$ is a harmonic function in Ω, moreover, it dominates $\omega_\Omega(z, A)$ on the boundary of Ω. Indeed, the boundary of Ω is made of three parts: A, $(\partial\Omega \cap \partial\Omega') \setminus A$, and $\partial\Omega \cap \Omega'$. On the first two, both harmonic measures are equal to 1 and 0 correspondingly. On the last part, the harmonic measure in Ω is equal to 0 and harmonic measure in Ω' is non-negative.

The second inequality is proved in the similar way. Indeed, considering the boundary values we see that $\omega_\Omega(z, A) + \omega_\Omega(z, A' \setminus A) = \omega_\Omega(z, A')$. As before, $\omega_\Omega(z, A' \setminus A) \ge 0$ and the desired inequality follows immediately.

We can also notice that both inequalities are strict unless $\Omega = \Omega'$ or harmonic measure of $A' \setminus A$ is identically equal to 0. $\square$

4.3 Extremal length

Extremal length is a conformal invariant which has a simple geometric interpretation, this makes it a very powerful tool if one have to estimate some analytical properties like harmonic measure in terms of the geometry of the domain. Here we discuss the main results and applications of the extremal length. More information could be found in [2, 11, 10].

The introduction of extremal lengths is frequently attributed to Ahlfors, in fact, in its modern form, it was introduced by Beurling in early 40's and later developed by Beurling and Ahlfors. Some of the underlying ideas could be traced back to the work of Grötzsch.

4.3.1 Definitions and basic properties

Let Ω be a domain in $\mathbb{C}$. In this sections we are interested in various collections of curves γ in Ω. Abusing notations, by *curve* we call a finite (or countable) union of rectifiable arcs in Ω. A *metric* in Ω is a non-negative Borel measurable function ρ such that the *area* of Ω which is defined as

$$A(\Omega, \rho) = \int_\Omega \rho^2(z) \mathrm{d}m(z)$$

satisfies $0 < A(\Omega, \rho) < \infty$.

Given a metric ρ we can define the *length* of any rectifiable curve γ as

$$L(\gamma, \rho) = \int_\gamma \rho(s)|\mathrm{d}z| = \int_\gamma \rho(s)\mathrm{d}s,$$

where $\mathrm{d}s$ is the usual arc-length. For a family of curves Γ we define the minimal length by

$$L(\Gamma, \rho) = \inf_{\gamma\in\Gamma} L(\gamma, \rho).$$

Definition 4.3.1. The *extremal length* of a curve family Γ in a domain Ω is defined as

$$\lambda_\Omega(\Gamma) = \sup_\rho \frac{L^2(\Gamma, \rho)}{A(\Omega, \rho)},$$

where supremum is over all possible metrics. The *extremal metric* is a metric for which the supremum is achieved.

The expression in the definition of the extremal length is obviously homogeneous with respect to ρ, this means that we can normalize ρ by fixing $L(\Gamma, \rho)$ or $A(\Omega, \rho)$ or any linear relation between them. Indeed, by rescaling ρ one can see that

$$\lambda_\Omega(\Gamma) = \sup_\rho L^2(\Gamma, \rho),$$

where supremum is over all metrics with $A(\Omega, \rho) = 1$. Alternatively

$$\frac{1}{\lambda_\Omega(\Gamma)} = \inf_\rho A(\Omega, \rho),$$

where infimum is over all metrics with $L(\Gamma, \rho) = 1$. The quantity $m_\Omega(\Gamma) = \lambda_\Omega(\Gamma)^{-1}$ is called the *modulus* of Γ. Finally

$$\lambda_\Omega(\Gamma) = \sup_\rho L(\Gamma, \rho) = \sup_\rho A(\Omega, \rho),$$

where supremum is over metrics with $L(\Gamma, \rho) = A(\Omega, \rho)$.

The main property is that the extremal length is conformally invariant

Theorem 4.3.2. *Let $f : \Omega \to \Omega'$ be a conformal map and let Γ' and Γ' be two families of curves in Ω and Ω' such that $\Gamma' = f(\Gamma)$. Then $\lambda_\Omega(\Gamma) = \lambda_{\Omega'}(\Gamma')$.*

Proof. Let ρ' be a metric in Ω', then $\rho(z) = |f'(z)|\rho'(f(z))$ is a metric in Ω and by change of variable formula $A(\Omega, \rho) = A(\Omega', \rho')$. By the same argument, if $\gamma' = f(\gamma)$, then $L(\gamma, \rho) = L(\gamma', \rho')$. This proves that for every metric ρ' there is a metric ρ such that

$$\frac{L^2(\Gamma, \rho)}{A(\Omega, \rho)} = \frac{L^2(\Gamma', \rho')}{A(\Omega', \rho')}.$$

This implies that $\lambda_\Omega(\Gamma) \geq \lambda_{\Omega'}(\Gamma')$. Applying the same argument to f^{-1} we complete the proof of the theorem. $\square$

It is also important to notice that the extremal length depend on Γ but not on Ω. Namely, if we have two domains $\Omega \subset \Omega'$ and Γ is a family of curves in Ω, then $\lambda_\Omega(\Gamma) = \lambda_{\Omega'}(\Gamma)$. This will allow us to write $\lambda(\Gamma)$ instead of $\lambda_\Omega(\Gamma)$. The proof of this independence is quite simple. Let ρ be some metric in Ω, we can extend it to ρ' in Ω' by setting $\rho' = 0$ outside of Ω. Obviously areas and lengths for these two measures are the same and we have $\lambda_\Omega(\Gamma) \le \lambda_{\Omega'}(\Gamma)$. For any ρ' in Ω' we define ρ to be its restriction to Ω. Clearly $L(\Gamma,\rho) = L(\Gamma,\rho')$ and $A(\Omega,\rho) \le A(\Omega',\rho')$, this implies the opposite inequality

$$\lambda_{\Omega'}(\Gamma) = \sup_{\rho'} \frac{L^2(\Gamma,\rho')}{A(\Omega',\rho')} \le \sup_{\rho'} \frac{L^2(\Gamma,\rho)}{A(\Omega,\rho)} \le \lambda_\Omega(\Gamma).$$

4.3.2 Extremal metric

In general, we don't know which families Γ admit an extremal metric, but it is not difficult to show that if it does exist, then it is essentially unique.

Theorem 4.3.3. *Let Γ be a family of curves in Ω and let ρ_1 and ρ_2 be two extremal metrics normalized by $A(\Omega,\rho_i) = 1$, then $\rho_1 = \rho_2$ almost everywhere.*

Proof. For these two metrics we have that $\lambda(\Gamma) = L^2(\Gamma,\rho_i)$. Let us consider a metric $\rho = (\rho_1+\rho_2)/2$, then

$$L(\Gamma,\rho) = \inf_\gamma \int_\gamma \frac{\rho_1(z)+\rho_2(z)}{2}|\mathrm{d}z| \ge \frac{L(\Gamma,\rho_1)+L(\Gamma,\rho_2)}{2} = \lambda^{1/2}(\Gamma). \quad (4.1)$$

By the Cauchy-Schwarz inequality

$$\begin{aligned} A(\Omega,\rho) &= \int_\Omega \frac{(\rho_1+\rho_2)^2}{4} \le \frac{A(\Omega,\rho_1)}{4} + \frac{A(\Omega,\rho_1)}{4} + \int \frac{\rho_1\rho_2}{2} \\ &\le \frac{1}{2} + \frac{1}{2}\left(\int \rho_1^2\right)^{1/2}\left(\int \rho_2^2\right)^{1/2} = 1. \end{aligned} \quad (4.2)$$

Together this implies that

$$\frac{L^2(\Gamma,\rho)}{A(\Omega,\rho)} \ge \lambda(\Gamma).$$

By the definition of the extremal length, this must be an equality and ρ must be an extremal metric and we must have an equality in (4.2). We know that the equality in the Cauchy-Schwarz inequality occurs if and only if ρ_1 and ρ_2 are proportional to each other almost everywhere. Normalization $A(\Omega,\rho_1) = A(\Omega,\rho_2)$ implies that they must be equal almost everywhere. □

As we will see in the next section, computation of the extremal length quite often involves making a good guess for the extremal metric. This could be done in surprisingly many cases, but not always. Sometimes this question could be reversed and we ask: given a metric ρ, is there a family of curves for which ρ is extremal. Beurling in an unpublished work gave a very simple criterion which could also be used to prove that your candidate for the extremal metric is indeed extremal.

Theorem 4.3.4. *A metric ρ_0 is extremal for a curve family Γ in Ω if there is a sub-family Γ_0 such that*

$$\int_\gamma \rho_0(s)\mathrm{d}s = L(\Gamma, \rho_0), \qquad \textit{for all } \gamma \in \Gamma_0$$

and for all real-valued measurable h in Ω we have that $\int_\Omega h\rho_0 \geq 0$ if $\int_\gamma h\mathrm{d}s \geq 0$ for all $\gamma \in \Gamma_0$.

You can think that Γ_0 is a collection of the shortest curves in Γ and they should cover the entire support of ρ_0.

Proof. Let ρ be some other metric normalized by $L(\Gamma, \rho) = L(\Gamma, \rho_0)$. Since all curves from Γ_0 have minimal length with respect to ρ_0 we have that $L(\gamma_0, \rho) \geq L(\gamma_0, \rho_0)$ for any $\gamma_0 \in \Gamma_0$. This implies that for $h = \rho - \rho_0$

$$\int_{\gamma_0} h(s)\mathrm{d}s \geq 0, \qquad \text{for all } \gamma_0 \in \Gamma_0.$$

By assumptions this implies that

$$\int_\Omega (\rho(z) - \rho_0(z))\rho_0(z)\mathrm{d}x\mathrm{d}y = \int_\Omega h(z)\rho_0(z)\mathrm{d}x\mathrm{d}y \geq 0.$$

This inequality together with the Cauchy-Schwarz inequality gives

$$\int_\Omega \rho_0^2 \leq \int_\Omega \rho\rho_0 \leq \left(\int_\Omega \rho^2\right)^{1/2} \left(\int_\Omega \rho_0^2\right)^{1/2}$$

and

$$A(\Omega, \rho_0) = \int_\Omega \rho_0^2 \leq \int_\Omega \rho^2 = A(\Omega, \rho).$$

The last inequality together with normalization of ρ proves that ρ_0 is extremal. $\square$

4.3.3 Composition rules

Proposition 4.3.5 (The comparison rule)**.** *Extremal length is monotone. Namely, let Γ and Γ' be two family of curves such that each curve $\gamma \in \Gamma$ contains a curve $\gamma' \in \Gamma'$, then $\lambda(\Gamma) \geq \lambda(\Gamma')$. In other words, a smaller family of longer curves have larger extremal length (see the Figure 4.1).*

The proof of this statement is really trivial: just by the definition $L(\Gamma, \rho) \geq L(\Gamma', \rho)$ and the admissible metrics are the same.

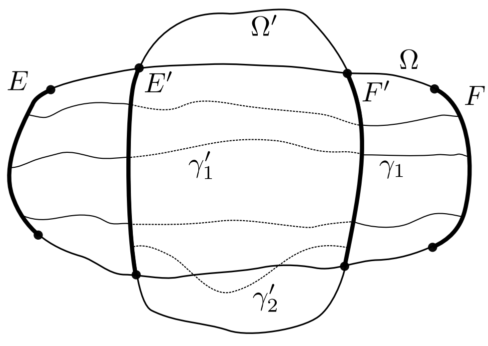


Figure 4.1: Γ and Γ' are the families of curves connecting E with F and E' with F' within Ω and Ω' correspondingly. Each curve from Γ contains a dotted piece which belongs to Γ'. The curve γ_2' is not a part of any curve from Γ.

Proposition 4.3.6 (The serial rule)**.** *Let Ω_1 and Ω_2 be two disjoint domains and Γ_i be two families of curves in these domains. Let Ω be a third domain such that $\Omega_i \subset \Omega$ and Γ be a family of curves in Ω such that each $\gamma \in \Gamma$ contains a curve from each Γ_i (see the Figure 4.2 for a typical example). Then $\lambda(\Gamma) \geq \lambda(\Gamma_1) + \lambda(\Gamma_2)$.*

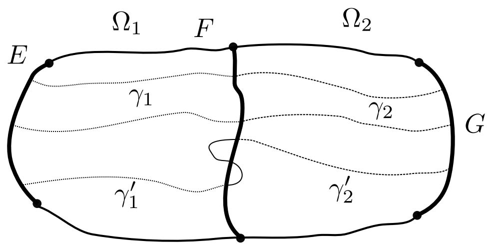


Figure 4.2: Γ is the family of curves connecting E and F in Ω_1, Γ_2 connects F and F in Ω_2, and Γ connects E and G in $\Omega = \Omega_1 \cup \Omega_2$. Each curve from Γ contains a dotted piece from Γ_1 and dashed piece from Γ_2.

Proof. If any of $\lambda(\Gamma_i)$ is trivial i.e equal to 0 or ∞, then the statement follows immediately from comparison rule 4.3.5. From now on assume that both lengths are non-trivial. Let ρ_i be two metrics normalized by $A(\Omega_i, \rho_i) = L(\Gamma_i, \rho_i)$ and define ρ to be ρ_i in Ω_i and 0 everywhere else. For this metric in Ω we have

$$L(\Gamma, \rho) \geq L(\Gamma_1, \rho_1) + L(\Gamma_2, \rho_2)$$

and

$$A(\Omega, \rho) = A(\Omega_1, \rho_1) + A(\Omega_2, \rho_2) = L(\Gamma_1, \rho_1) + L(\Gamma_2, \rho_2).$$

Combining these two we have $\lambda(\Gamma) \geq \lambda(\Gamma_1) + \lambda(\Gamma_2)$. □

Proposition 4.3.7 (The parallel rule)**.** *Let Ω_1 and Ω_2 be two disjoint domains and Γ_i be two families of curves in these domains. Let Γ be a third family of curves* *such that every curve $\gamma_i \in \Gamma_i$ contains a curve $\gamma \in \Gamma$ (see the Figure 4.3 for a typical example). Then*

$$\frac{1}{\lambda(\Gamma)} \geq \frac{1}{\lambda(\Gamma_1)} + \frac{1}{\lambda(\Gamma_2)}.$$

Equivalently

$$m(\Gamma) \geq m(\Gamma_1) + m(\Gamma_2),$$

where m is the conformal modulus.

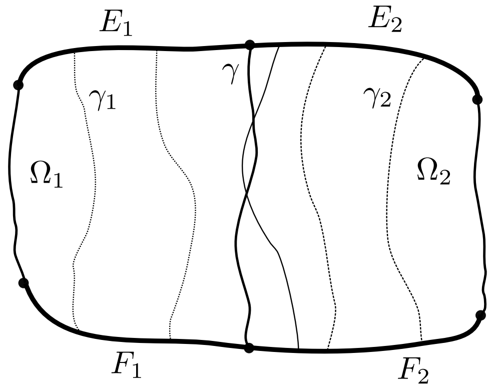


Figure 4.3: Γ_i are the families of curves connecting E_i and F_i inside Ω_i, Γ is the family of curves connecting $E = E_1 \cup E_2$ and $F = F_1 \cup F_2$ inside Ω which is the interior of the closure of $\Omega_1 \cup \Omega_2$. Each curve from Γ_i contains a curve from Γ. In fact, they belong to Γ, but there are curves like $\gamma \in \Gamma$ that are not related to the curves from Γ_i.

Proof. Let Ω be some domain containing Γ. Consider a metric ρ in Ω normalized by $L(\Gamma, \rho) = 1$. Our assumptions immediately imply that $L(\Gamma_i, \rho) \geq L(\Gamma, \rho) = 1$ and

$$A(\Omega, \rho) \geq A(\Omega_1, \rho) + A(\Omega_2, \rho) \geq \frac{1}{\lambda(\Gamma_1)} + \frac{1}{\lambda(\Gamma_2)}$$

where the last inequality follows from $1/A(\Omega_i, \rho) \leq L^2(\Gamma_i, \rho)/A(\Omega_i, \rho) \leq \lambda(\Gamma)$. On the other hand $\inf A(\Omega, \rho) = 1/\lambda(\Gamma)$ where the infimum is over all metrics normalized by $L(\Gamma, \rho) = 1$. □

Proposition 4.3.8 (The symmetry rule)**.** *Let Ω be a domain symmetric with respect to the real line and Γ a symmetric family of curves which means that for every curve $\gamma \in \Gamma$ its symmetric image $\bar{\gamma}$ is also from Γ. Then*

$$\lambda(\Gamma) = \sup_\rho \frac{L^2(\Gamma, \rho)}{A(\Omega, \rho)},$$

where supremum is over all symmetric metrics ρ such that $\rho(z) = \rho(\bar{z})$.

Proof. The proof is almost trivial. Let ρ_1 be some metric and let $\rho_2(z) = \rho_1(\bar{z})$ be its symmetric image. Obviously $L(\Gamma, \rho_1) = L(\Gamma, \rho_2)$ and $A(\Omega, \rho_1) = A(\Omega, rho_2)$. By the same argument as in the proof of Theorem 4.3.3 we have

$$\frac{L^2(\Gamma, \rho)}{A(\Omega, \rho)} \geq \frac{L^2(\Gamma, \rho_1)}{A(\Omega, \rho_1)} = \frac{L^2(\Gamma, \rho_2)}{A(\Omega, \rho_2)},$$

where $\rho = (\rho_1 + \rho_2)/2$. This proves that the supremum over symmetric metrics is equal to the supremum over all admissible metrics. □

Exercise 16. *State and prove a version of the symmetry rule for the symmetry with respect to the unit circle.*

4.3.4 Examples

There are several configurations that are defined by a single conformally invariant parameter: a simply connected domain with four marked points on the boundary (conformal rectangle), a simply connected domain with a marked point inside and two marked points on the boundary, a simply connected domain with two marked interior points, and a doubly connected domain. In these cases we already know conformal invariants that defined the conformal type of configurations. In the first case this is modulus of a rectangle, in the second case this is harmonic measure of an arc between two boundary points evaluated at the interior point, in the third case this is Green's function, and and in the last case this is the conformal modulus of the domain. Here we will discuss how these invariants are related to the extremal length.

One of the most important examples of the extremal length is the *extremal distance*. Let E and F be two subsets of $\bar{\Omega}$, then the extremal distance between them inside Ω is

$$d_\Omega(E, F) = \lambda(\Gamma),$$

where Γ is the family of all rectifiable curves in Ω that connect E and F. The *conjugated extremal distance* is

$$d^*_\Omega(E, F) = \lambda(\Gamma^*),$$

where Γ^* is the family of all (not necessary connected) curves separating E and F inside Ω. Proposition 4.3.5 immediately implies that $d_\Omega(E, F)$ decreases when any of Ω, E, or F increases. The Figures 4.2 and 4.3 give examples how the serial rule 4.3.6 and the parallel rule 4.3.7 could be applied to the extremal distances.

Conformal rectangle. Let Ω be a simply connected domain with four marked (accessible) points on the boundary. They divide boundary into four connected pieces (again in terms of accessible points or prime ends). Let us chose two of them that do not share a common chosen point and call them E and F. We know that there is a map from Ω onto a rectangle such that four marked points are mapped to the vertices. Let us assume that the images of E and F lie on the sides given by $x = 0$ and $x = a$ and two other sides are $y = 0$ and $y = b$. The extremal distance $d_\Omega(E, F) = a/b$ which is the conformal invariant that we have seen before.

By conformal invariance of the extremal length, $d_\Omega(E, F)$ is the same as the extremal distance between vertical sides of the rectangle $R = \{(x, y) : 0 < x < a,\ 0 < y < b\}$. For $\rho = 1$ we have that $A(R, \rho) = ab$ and $L(\Gamma, \rho) = a$, where Γ is the family of all curves connecting two vertical sides. This immediately gives us that $\lambda_R(\Gamma) \geq a^2/ab = a/b$.

We claim that this metric is extremal and $\lambda_R(\Gamma) = a/b$. Let Γ_0 be the family of all horizontal lines connecting two vertical sides. Clearly, these curves have the same length and it is equal to $L(\Gamma, \rho)$. If for some function h we have that

$$\int h(x, y)\mathrm{d}x \geq 0, \quad \forall y,$$

then integrating with respect to y we get

$$\int_R h\mathrm{d}x\mathrm{d}y \geq 0.$$

By the Theorem 4.3.4 this implies that $\rho = 1$ is indeed an extremal metric.

By symmetry we can see that the extremal distance between two other parts of the boundary is given by b/a and is equal to $d^*_\Omega(E, F)$. We can see that

$$d^*_\Omega(E, F)d_\Omega(E, F) = 1.$$

Exercise 17. *Use the symmetry rule (Proposition 4.3.8) to prove the following statement.*

Let Ω_1 be a domain in the upper half plane and let E_1 and F_1 be two sets on $\partial\Omega$. Let Ω_2, E_2, and F_2 be their symmetric images with respect to $\mathbb{R}$. We define $\Omega = \Omega_1 \cup \Omega_2$ (to be completely rigorous we also have to add the real part of the boundary), $E = E_1 \cup E_2$, and $F = F_1 \cup F_2$. Then

$$d_\Omega(E, F) = \frac{1}{2}d_{\Omega_1}(E_1, F_1) = \frac{1}{2}d_{\Omega_2}(E_2, F_2).$$

Conformal annulus. Let Ω be a doubly connected domain and E and F be two boundary components, then the extremal distance $d_\Omega(E, F)$ is equal to the conformal modulus of Ω. This gives a geometrical interpretation of the conformal modulus.

By conformal invariance of extremal distance and conformal modulus, it is sufficient to prove the identity for an annulus $A(r, R)$. We will treat the general case $0 < r < R < \infty$ and leave the cases $r = 0$ and $R = \infty$ to the reader.

As in the case of rectangles, it is easy to guess the extremal metric. Taking $\rho_0 = 1/|z|$ and considering curves along the radial directions we have $L(\Gamma, \rho_0) =$

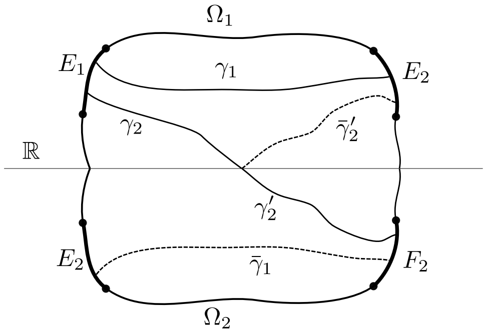


Figure 4.4: Application of symmetry rule

$\int_r^R (1/t)\mathrm{d}t = \log(R/r)$ and $A(\rho_0) = \int_r^R \int_0^{2\pi} 1/t\mathrm{d}t\mathrm{d}\theta = 2\pi\log(R/r)$, hence

$$\lambda(\Gamma) \geq \frac{L(\Gamma,\rho_0)}{A(\rho_0)} = \frac{\log^2(R/r)}{2\pi\log(R/r)} = \frac{1}{2\pi}\log(R/r).$$

To show that this metric is extremal we again use the Theorem 4.3.4 with Γ_0 being the family of straight radial intervals connecting two boundary components.

Since extremal distance is conformally invariant this gives yet another proof of the Theorem 2.7.1.

The same extremal metric and essentially the same argument gives that

$$\lambda(\Gamma^*) = \frac{2\pi}{\log(R/r)},$$

where Γ^* is the conjugated family of the curves that separate two boundary components.

An interior point and a boundary arc. Let Ω be a simply connected domain, z_0 be a point inside and A be a boundary arc. We can consider two families of curves Γ and Γ^*. The first family consists of curves that begin and end on A and go around z_0, the second family consist of all curves that separate A from z_0 (see the Figure 4.5). Both $\lambda(\Gamma)$ and $\lambda(\Gamma^*)$ are conformal invariants of the configuration (Ω, z_0, A). On the other hand, we know that conformal type of such configuration is uniquely determined by harmonic measure $\omega_\Omega(z_0, A)$. This proves that $\lambda(\Gamma)$ and $\lambda(\Gamma^*)$ could be written as functions of harmonic measure. Finding the explicit relation is not easy.

Two interior points. Finally, for the case of two interior points we can consider two families Γ and Γ^* of loops surrounding two points and curves connecting boundary points and separating points from each other (see the Figure 4.6).

By conformal invariance it is sufficient to consider the case when $\Omega = \mathbb{D}$, one of the marked points is 0 and the other one is $x \in (0,1)$. Since our configuration is symmetric with respect to $z \mapsto \bar{z}$ we can apply the symmetry rule 4.3.8 and

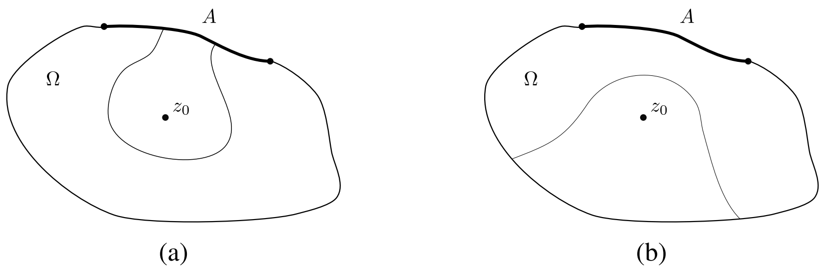


Figure 4.5: Families Γ (a) and Γ^* (b).

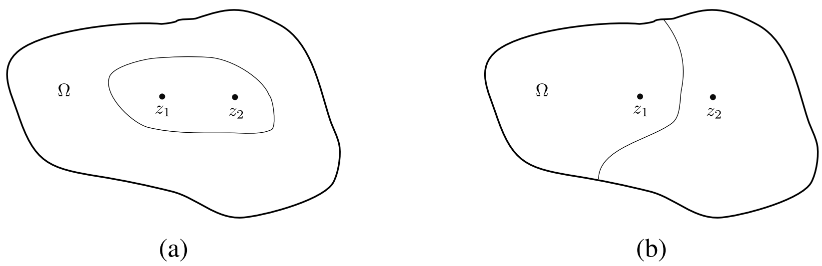


Figure 4.6: Families Γ (a) and Γ^* (b).

consider only symmetric metrics. Using symmetry one can show that it is enough to consider symmetric curves.

Exercise 18. *Complete the argument above and show that* $\lambda^{-1}(\Gamma) = \lambda(\Gamma^*) = d_{\Omega'}(s, \mathbb{T})$*, where* Ω *is a doubly connected domain* $\mathbb{D} \setminus [0, x]$.

This doubly connected domain is conformally equivalent to $\mathbb{D}_- \setminus [1/x, +\infty]$ which is known as Grötzsch annulus and its modulus could be computed in terms of elliptic functions.

4.3.5 Geometric application

In this section we will show two examples how one can use extremal length to obtain purely geometrical inequalities.

Conformal rectangles. Let Ω be a conformal rectangle and denote the four boundary arcs by E, E', F, F', then

$$d(E,F)d(E',F') \le A(\Omega)$$

where d is the Euclidean distance between sets and A is the area of Ω.

The proof of this inequality is almost trivial. Let Γ be the family of curves connecting E and F and Γ' be the family of curves connecting E' and F'. Let us consider $\rho = 1$ in Ω, i.e. the usual Euclidean metric in Ω. For this metric $L(\Gamma, \rho) \ge d(E,F)$ and $L(\Gamma', \rho) \ge d(E', F')$, hence $\lambda(\Gamma) \ge L^2(\Gamma, \rho)/A(\Omega) \ge$

$d^2(E,F)/A(\Omega)$ and the similar inequality holds for $\lambda(\Gamma')$. For conformal rectangles we have that $\lambda(\Gamma)\lambda(\Gamma') = 1$ which together with the previous inequalities imply $1 \geq d^2(E,F)d^2(E',F') \leq A^2(\Omega)$.

Conformal triangles. Let Ω be a conformal triangle, i.e. a simply connected domain with three marked points, then there is a curve γ in Ω such that γ touches all three sides of Ω and

$$\mathrm{length}(\gamma) \leq \sqrt[4]{3}\sqrt{A(\Omega)}.$$

Moreover, the constant $\sqrt[4]{3}$ is sharp.

Let Γ be the family of all curves that touch all three sides of Ω. We would like to compute the extremal length of this family. The situation with conformal triangles is a bit different from all previous examples that we have considered so far: all conformal triangles are conformally equivalent. This means that $\lambda(\Gamma)$ is just an absolute constant.

By conformal invariance, we can can assume that Ω is an equilateral triangle with vertices 0, 1 and $1/2 + i\sqrt{3}/2$.

Exercise 19. *Use Theorem 4.3.4 to show that* $\rho = 1$ *in the triangle and* 0 *outside is the extremal metric for* Γ*. From this deduce that* $\lambda(\Gamma) = \sqrt{3}$*.*

Now let Ω be an arbitrary conformal triangle and consider $\rho = 1$ in Ω, then

$$\sqrt{3} = \lambda(\Gamma) \geq \frac{\min(\mathrm{length}^2(\gamma))}{A(\Omega)}.$$

This is equivalent to the desired inequality. To see that the constant is sharp we consider the equilateral triangle as above and γ is a triangles altitude.

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