MAT 536, Spring 2024, Final Exam, Friday May 10, 11:15am-1:45pm

Name	ID

Part I	Part II	Total

Part I: True/False. Put a "T" or "F" in each box. 2 points each, 20 points total.

(1) [*F*] $(3+i)(2-2i) = 6-4i$

(2) [*T*] $\sum_{n=0}^{\infty} \frac{(\pi i)^n}{n!} = -1$

(3) [*F*] If f is bounded and analytic on $\mathbb{D} = \{|z| < 1\}$ its Taylor series must converge at $z = 1$.

(4) [*F*] If f is analytic and non-zero on a domain Ω there exists an analytic g on Ω so that $g^2 = f$.

(5) [*F*] If f is a 1-1, analytic map from $\mathbb{D}$ to a simply connected domain Ω, then f extends continuously to the boundary.

(6) [*T*] If $\{f_n\}$ are analytic functions on a domain Ω that converge uniformly on Ω to a function f, then f is analytic.

(7) [*T*] If $\mathcal{F}$ is the family of analytic functions f on $\mathbb{D}$ so that $\text{Re}(f)$ is bounded by 1, and $f(0) = 0$, then f is a normal family.

(8) [*F*] If $\{g_n\}$ are non-vanishing analytic maps on a domain Ω, that converge uniformly on compact subsets to g, then g must be non-vanishing.

(9) [*T*] There are only three conformally distinct simply connected Riemann surfaces.

(10) [*T*] $\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1} = \pi e^{-a}$.

Part II: Do three of the following four problems. Mark the boxes next to the problems you want graded. 10 points each, 30 points total.

(1) ☐ Prove the identity $\sin^2 z + \cos^2 z = 1$ holds for all complex z. (You many assume the standard trigonometric identities for z real.)

Sketch: You can calculate using the definitions or simply observe that $\sin^2 z + \cos^2 z - 1 = 0$ is analytic on the plane and zero on the real line, so is zero everywhere in the plane (zeros of a non-constant analytic function must be countable and have no accumulation points).

(2) ☐ Prove that if $u(x, y)$ is a positive harmonic function on $\mathbb{R}^2$, then u is constant.

Sketch: The plane is simply connected so u has a harmonic conjugate v so that $f = u + iv$ is analytic on the whole plane and maps the plane into the right half-plane (since $u > 0$). The half-plane can be mapped to the unit disk by a linear fractional transformation τ. Thus $\tau \circ f$ is constant by Lioville's theorem. Thus f, and hence u, is also constant.

(3) ☐ Suppose that f is analytic in $\mathbb{D}$, that $f(0) = 1$ and that $|f| \leq M$ on $\mathbb{D}$. Show that number of zeros of f inside $\{|z| < r\}$ is at most $\ln M / \ln(1/r)$.

Sketch: Replacing $f(z)$ by $f(tz)$ and taking $t \nearrow 1$, we may assume f is analytic on a neighborhood of the closed unit disk. Let $\{z_k\}_{k=1}^n$ be the zeros of f inside $D(0, r)$, counted with multiplicity. Let B be a finite Blaschke product with these zeros. Then $g = f/B$ is analytic in the unit disk. Hence $\log |g|$ is subharmonic on the unit disk, and $\log |g| \leq \log M$ on the unit circle. Therefore $\log |g(0)| \leq \log M$ and $|B(0)| = \prod_k |z_k|$. Moreover,

$$\log |g(0)| = \log |f(0)| - \log |B(0)| = 0 - \sum_{k=1}^{n} \log |z_k| \geq -n \log r = n \log(1/r),$$

so $n \leq \log M / \log(1/r)$.

(4) ☐ If f is non-constant, non-linear entire function, prove $f(f(z)) = z$ has at least one solution. (Hint: Consider $g(z) = (f(f(z)) - z)/(f(z) - z)$. You may assume g is non-constant if f is neither constant nor linear; this is true but a little trickier to prove.)

Sketch: By the hint we may assume g is not constant. So, by Picard's theorem g can omit at most two values and hence must take on at least one of the values $0, 1, \infty$. If $g(z) = 0$ then $f(f(z)) = z$, as desired. If $g(z) = \infty$ then $f(z) = z$, so $f(f(z)) = f(z) = z$. Finally, if $g(z) = 1$, then $f(f(z)) = f(z)$ so $w = f(z)$ is a fixed point of f, hence also a fixed point of $f(f(z))$.

Proof of the hint (not required for exam): Claim: If f is an entire function, and $g(z) = (f(f(z)) - z)/(f(z) - z)$ is constant, then f must be constant or linear.

Sketch: Suppose g is constant, but f is not constant. If g is the constant 1, then $f(f(z)) = f(z)$ on the image of f. This image is dense in the plane, so f must be the identity, thus linear. If g is the constant 0 then $f(f(z)) = z$ implies f is a 1-1 map, hence linear. Finally, suppose c is neither 0 nor 1, and

$$f(f(z)) - z = c(f(z) - z).$$

Differentiating gives

$$f'(f(z))f'(z) - 1 = c(f'(z) - 1).$$
$$f'(z)(f'(f(z)) - c) = 1 - c.$$

The right side is not zero, so neither factor on the left is ever zero. Thus f' omits zero. It also omits c unless it only takes this value at the points that f omits. Thus f' takes the value c at only finitely many points. By Picard's theorem f' has a pole at ∞, so f' is a polynomial. Since f' never equals zero, it is a constant, so f is linear.

Bonus: For 10 extra points do the following (from the midterm):

Suppose $f_n : \mathbb{D} \to \mathbb{D}$ is analytic for $n = 1, 2, \ldots,$ and suppose that $f_n(z) \to f(z)$ for every $z \in \mathbb{D}$. Prove that f is analytic on $\mathbb{D}$.

Sketch: If R is a compact rectangle $\mathbb{D}$, then since $|f_n| \leq 1$ and $f_n \to f$ pointwise on ∂R, the Lebesgue dominated convergence theorem applies to prove

$$\int_{\partial R} f dz = \lim_n \int_{\partial R} f_n dz = 0.$$

Hence f is analytic on $\mathbb{D}$ by Morera's theorem.