MAT 536, Spring 2024, Sample Final Exam

Name	ID

Part I	Part II	Total

Part I: True/False. Put a "T" or "F" in each box. 2 points each, 20 points total.

(1) T $(1+i)^2 = 2i$

(2) T $\sum_{n=1}^{\infty} \frac{(\pi i)^n}{n!} = -2$

(3) F If f is bounded and analytic on $\mathbb{D} = \{|z| < 1\}$ then its Taylor series converges uniformly to f on $\mathbb{D}$.

(4) T If f is analytic and non-zero on a simply connected domain Ω there exists an analytic g on Ω so that $e^g = f$.

(5) T The conformal map of the unit disk to a bounded polygonal domain must extend continuously to the boundary.

(6) F If $\{f_n\}$ are analytic functions on a domain Ω that converge pointwise on Ω to a function f, then f is analytic.

(7) F If $\mathcal{F}$ is the family of analytic functions f on $\mathbb{D}$ so that $|f'|$ is bounded by 1, then f is a normal family.

(8) F If $\mathcal{F}$ is a Perron family, then $-\mathcal{F} = \{-f : f \in \mathcal{F}\}$ is also a Perron family.

(9) F If u is bounded and harmonic on $\mathbb{D}$, then its harmonic conjugate is also bounded on $\mathbb{D}$.

(10) T $\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = 2\pi/3.$

Part II: Do three of the following four problems. Mark the boxes next to the problems you want graded. 10 points each, 30 points total.

(1) ☐ Prove that if $f = u + iv$ is analytic on the plane and $|u| \leq |v| + 1$, then f is constant.

Sketch: If this condition holds, then f maps into a region on the plane the does not include the disk $\{z : |z - 3| < 1\}$ (many other disks would work too). Thus $1/(f(z) - 3)$ is analytic map of the plane into the unit dis, so it must be constant by Liouville's theorem. Thus f is constant.

(2) ☐ Suppose f is entire (analytic on whole plane). Must f have a fixed point (a solution of $f(z) = z$)? Prove or give a counterexample.

Sketch: No. e^z is never zero, so $f(z) = z + e^z$ is never equal to z.

(3) ☐ Suppose f and g are entire and $f^3 + g^3 = 1$. Prove f and g are constant. (Hint: apply Picard's theorem to f/g).

Sketch: Suppose there are such a non-constant pair f and g. If the equation holds then $(f/g)^3 + 1 = 1/g^3$. Since g is assumed non-constant, this implies f/g is meromorphic and non-constant. Since f/g is meromorphic and non-constant, by Picard's little theorem it can omit at most two values. So there must be some z where f/g is equal one of the three cube roots of -1. But then $1/g^3(z) = 1 - 1 = 0$, so $g(z) = \infty$, contradicting that g is analtyic. The contradiction means there is no such pair f, g.

(4) ☐ Suppose f is analytic on $\mathbb{D} = \{|z| < 1\}$ and that it extends to be continuous and non-vanishing on $\mathbb{T} = \{|z| = 1\}$. Prove there is a analytic g on $\mathbb{D}$ so that $|g| = |f|$ on $\mathbb{T}$ and g is non-vanishing on all of $\overline{\mathbb{D}}$.

Sketch: Since f is continuous on the closure of the disk and non-vanishing on the boundary, all its zeros are bounded away from the circle. Since the zeros can't accumulate inside the disk, there can only be finitely many. Let $B(z) = \prod(z - a_n)/(1 - \overline{a_n} z)$ be the finite Blaschke product with the same zeros as f (counted with multiplicity). Then $|B| = 1$ on the circle, so $g = f/B$ is analytic with no zeros in $\mathbb{D}$ and $|g| = |f|/|B| = |f|$ on the circle.