CHAPTER 3 REVIEW

KEY TERMS

absolute error if B is an estimate of some quantity having an actual value of A, then the absolute error is given by $|A-B|$

computer algebra system (CAS) technology used to perform many mathematical tasks, including integration

improper integral an integral over an infinite interval or an integral of a function containing an infinite discontinuity on the interval; an improper integral is defined in terms of a limit. The improper integral converges if this limit is a finite real number; otherwise, the improper integral diverges

integration by parts a technique of integration that allows the exchange of one integral for another using the formula $\int u\,dv = uv - \int v\,du$

integration table a table that lists integration formulas

midpoint rule a rule that uses a Riemann sum of the form $M_n = \sum_{i=1}^{n} f(m_i)\Delta x$, where m_i is the midpoint of the ith subinterval to approximate $\int_a^b f(x)dx$

numerical integration the variety of numerical methods used to estimate the value of a definite integral, including the midpoint rule, trapezoidal rule, and Simpson's rule

partial fraction decomposition a technique used to break down a rational function into the sum of simple rational functions

power reduction formula a rule that allows an integral of a power of a trigonometric function to be exchanged for an integral involving a lower power

relative error error as a percentage of the absolute value, given by $\left|\frac{A-B}{A}\right| = \left|\frac{A-B}{A}\right| \cdot 100\%$

Simpson's rule a rule that approximates $\int_a^b f(x)dx$ using the integrals of a piecewise quadratic function. The approximation S_n to $\int_a^b f(x)dx$ is given by $S_n = \frac{\Delta x}{3}\begin{pmatrix} f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) \\ + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \end{pmatrix}$

trapezoidal rule a rule that approximates $\int_a^b f(x)dx$ using trapezoids

trigonometric integral an integral involving powers and products of trigonometric functions

trigonometric substitution an integration technique that converts an algebraic integral containing expressions of the form $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, or $\sqrt{x^2-a^2}$ into a trigonometric integral

KEY EQUATIONS

- **Integration by parts formula**
  $\int u\,dv = uv - \int v\,du$
- **Integration by parts for definite integrals**
  $\int_a^b u\,dv = uv|_a^b - \int_a^b v\,du$

To integrate products involving $\sin(ax)$, $\sin(bx)$, $\cos(ax)$, and $\cos(bx)$, use the substitutions.

- **Sine Products**
  $\sin(ax)\sin(bx) = \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x)$
- **Sine and Cosine Products**
  $\sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x)$
- **Cosine Products**
  $\cos(ax)\cos(bx) = \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x)$
- **Power Reduction Formula**
  $\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-1} x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$
- **Power Reduction Formula**
  $\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx$
- **Midpoint rule**
  $M_n = \sum_{i=1}^{n} f(m_i)\Delta x$
- **Trapezoidal rule**
  $T_n = \frac{1}{2}\Delta x\left(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right)$
- **Simpson's rule**
  $S_n = \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right)$
- **Error bound for midpoint rule**
  $\text{Error in } M_n \le \frac{M(b-a)^3}{24n^2}$
- **Error bound for trapezoidal rule**
  $\text{Error in } T_n \le \frac{M(b-a)^3}{12n^2}$
- **Error bound for Simpson's rule**
  $\text{Error in } S_n \le \frac{M(b-a)^5}{180n^4}$
- **Improper integrals**
  $\int_a^{+\infty} f(x)dx = \lim_{t \to +\infty} \int_a^t f(x)dx$
  $\int_{-\infty}^{b} f(x)dx = \lim_{t \to -\infty} \int_t^b f(x)dx$
  $\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_0^{+\infty} f(x)dx$

KEY CONCEPTS

3.1 Integration by Parts

- The integration-by-parts formula allows the exchange of one integral for another, possibly easier, integral.
- Integration by parts applies to both definite and indefinite integrals.

3.2 Trigonometric Integrals

- Integrals of trigonometric functions can be evaluated by the use of various strategies. These strategies include
  1. Applying trigonometric identities to rewrite the integral so that it may be evaluated by u-substitution
  2. Using integration by parts
  3. Applying trigonometric identities to rewrite products of sines and cosines with different arguments as the sum of individual sine and cosine functions
  4. Applying reduction formulas

3.3 Trigonometric Substitution

- For integrals involving $\sqrt{a^2 - x^2}$, use the substitution $x = a\sin\theta$ and $dx = a\cos\theta d\theta$.
- For integrals involving $\sqrt{a^2 + x^2}$, use the substitution $x = a\tan\theta$ and $dx = a\sec^2\theta d\theta$.
- For integrals involving $\sqrt{x^2 - a^2}$, substitute $x = a\sec\theta$ and $dx = a\sec\theta\tan\theta d\theta$.

3.4 Partial Fractions

- Partial fraction decomposition is a technique used to break down a rational function into a sum of simple rational functions that can be integrated using previously learned techniques.
- When applying partial fraction decomposition, we must make sure that the degree of the numerator is less than the degree of the denominator. If not, we need to perform long division before attempting partial fraction decomposition.
- The form the decomposition takes depends on the type of factors in the denominator. The types of factors include nonrepeated linear factors, repeated linear factors, nonrepeated irreducible quadratic factors, and repeated irreducible quadratic factors.

3.5 Other Strategies for Integration

- An integration table may be used to evaluate indefinite integrals.
- A CAS (or computer algebra system) may be used to evaluate indefinite integrals.
- It may require some effort to reconcile equivalent solutions obtained using different methods.

3.6 Numerical Integration

- We can use numerical integration to estimate the values of definite integrals when a closed form of the integral is difficult to find or when an approximate value only of the definite integral is needed.
- The most commonly used techniques for numerical integration are the midpoint rule, trapezoidal rule, and Simpson's rule.
- The midpoint rule approximates the definite integral using rectangular regions whereas the trapezoidal rule approximates the definite integral using trapezoidal approximations.
- Simpson's rule approximates the definite integral by first approximating the original function using piecewise quadratic functions.

3.7 Improper Integrals

- Integrals of functions over infinite intervals are defined in terms of limits.
- Integrals of functions over an interval for which the function has a discontinuity at an endpoint may be defined in terms of limits.

- The convergence or divergence of an improper integral may be determined by comparing it with the value of an improper integral for which the convergence or divergence is known.

CHAPTER 3 REVIEW EXERCISES

For the following exercises, determine whether the statement is true or false. Justify your answer with a proof or a counterexample.

408. $\int e^x \sin(x)dx$ cannot be integrated by parts.

409. $\int \frac{1}{x^4+1}dx$ cannot be integrated using partial fractions.

410. In numerical integration, increasing the number of points decreases the error.

411. Integration by parts can always yield the integral.

For the following exercises, evaluate the integral using the specified method.

412. $\int x^2 \sin(4x)dx$ using integration by parts

413. $\int \frac{1}{x^2\sqrt{x^2+16}}dx$ using trigonometric substitution

414. $\int \sqrt{x}\ln(x)dx$ using integration by parts

415. $\int \frac{3x}{x^3+2x^2-5x-6}dx$ using partial fractions

416. $\int \frac{x^5}{\left(4x^2+4\right)^{5/2}}dx$ using trigonometric substitution

417. $\int \frac{\sqrt{4-\sin^2(x)}}{\sin^2(x)}\cos(x)dx$ using a table of integrals or a CAS

For the following exercises, integrate using whatever method you choose.

418. $\int \sin^2(x)\cos^2(x)dx$

419. $\int x^3\sqrt{x^2+2}\,dx$

420. $\int \frac{3x^2+1}{x^4-2x^3-x^2+2x}dx$

421. $\int \frac{1}{x^4+4}dx$

422. $\int \frac{\sqrt{3+16x^4}}{x^4}dx$

For the following exercises, approximate the integrals using the midpoint rule, trapezoidal rule, and Simpson's rule using four subintervals, rounding to three decimals.

423. **[T]** $\int_1^2 \sqrt{x^5+2}\,dx$

424. **[T]** $\int_0^{\sqrt{\pi}} e^{-\sin(x^2)}dx$

425. **[T]** $\int_1^4 \frac{\ln(1/x)}{x}dx$

For the following exercises, evaluate the integrals, if possible.

426. $\int_1^\infty \frac{1}{x^n}dx$, for what values of n does this integral converge or diverge?

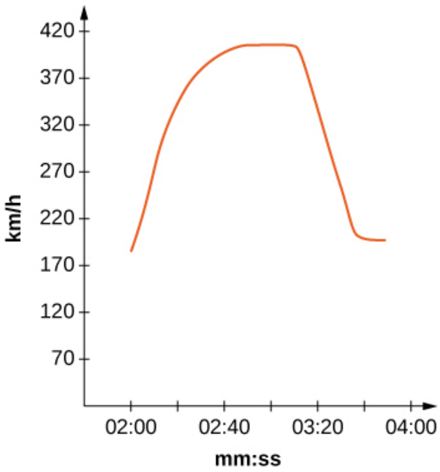
427. $\int_1^\infty \frac{e^{-x}}{x}dx$

For the following exercises, consider the gamma function given by $\Gamma(a) = \int_0^\infty e^{-y}y^{a-1}dy$.

428. Show that $\Gamma(a) = (a-1)\Gamma(a-1)$.

429. Extend to show that $\Gamma(a) = (a-1)!,$ assuming a is a positive integer.

The fastest car in the world, the Bugati Veyron, can reach a top speed of 408 km/h. The graph represents its velocity.



430. **[T]** Use the graph to estimate the velocity every 20 sec and fit to a graph of the form $v(t) = a\exp^{bx}\sin(cx) + d.$ (*Hint:* Consider the time units.)

431. **[T]** Using your function from the previous problem, find exactly how far the Bugati Veyron traveled in the 1 min 40 sec included in the graph.