Intersections on Grassmannians of Two-Planes

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1 Flags and Schubert cycles

A flag $\mathbf{V}$ in $\mathbb{C}^n$ is an increasing sequence of $n+1$ linear subspaces of $\mathbb{C}^n$:

$$\mathbf{V} = (V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq \ldots V_{n-1} \subsetneq V_n). \tag{1}$$

Thus, V_i is a linear subspace of $\mathbb{C}^n$ of dimension i containing V_{i-1}; in particular, $V_0=\{0\}$ and $V_n=\mathbb{C}^n$. The standard flag $\mathbf{V}^{\rm std}$ on $\mathbb{C}^n$ is the sequence as above with $V_i^{\rm std}=\mathbb{C}^i\times\{0\}^{n-i}\equiv\mathbb{C}^i$.

Let $G(2,n)$ denote the Grassmannian of two-dimensional linear subspaces of $\mathbb{C}^n$. For any non-negative integers a,b and a flag $\mathbf{V}$ as in (1), let

$$\sigma_{ab}(\mathbf{V}) = \big\{P\in G(2,n):\ P\subset V_{n-b},\ P\cap V_{n-1-a}\neq\{0\}\big\}. \tag{2}$$

The second condition implies that $n-1-a\ge1$ if $\sigma_{ab}(\mathbf{V})\neq\emptyset$, so that $n-2\ge a$. Since P is a linear subspace of $\mathbb{C}^n$ of dimension 2 and V_{n-1-b} is a linear subspace of V_{n-b}, the first condition in (2) and linear algebra imply that for any $P\in\sigma_{ab}(\mathbf{V})$

$$\dim P\cap V_{n-1-b} \ge \dim P + \dim V_{n-1-b} - \dim V_{n-b} = 1 \qquad\Longrightarrow\qquad P\cap V_{n-1-b}\neq\{0\}.$$

Thus, if $n-1-a > n-1-b$, the second condition in (2) is meaningless since then $V_{n-1-b}\subset V_{n-1-a}$ and $P\cap V_{n-1-a}\neq\{0\}$. Therefore, one always requires that $a\ge b$. Similarly, since V_{n-b} is defined only for $b\ge0$, we assume that $b\ge0$. In summary, $\sigma_{ab}(\mathbf{V})\subset G(2,n)$ is defined by (2) whenever $n-2\ge a\ge b\ge 0$; otherwise, it is defined to be empty.

If $P\in G(2,n)$, then $P\subset V_{n-0}$ and by linear algebra

$$\dim P\cap V_{n-1} \ge \dim P + \dim V_{n-1} - \dim V_n = 1 \qquad\Longrightarrow\qquad P\cap V_{n-1-0}\neq\emptyset.$$

Thus, $\sigma_{00}(\mathbf{V})=G(2,n)$. In general, the integers a and b in (2) indicate how much earlier a typical element P of $\sigma_{ab}(\mathbf{V})$ satisfies the containment and intersection conditions with respect to the given flag. By (2),

$$\sigma_{ab}(\mathbf{V})\supset\sigma_{a'b'}(\mathbf{V}) \qquad \text{if}\quad a\le a',\ b\le b',\ (a,b)\neq(a',b'). \tag{3}$$

The subspace $\sigma_{ab}(\mathbf{V})$ is a compact analytic subvariety of the complex manifold $G(2,n)$ of (complex) codimension $a+b$. The complement $\sigma^*_{ab}(\mathbf{V})$ in $\sigma_{ab}(\mathbf{V})$ of the finitely subspaces $\sigma_{a'b'}(\mathbf{V})$ with (a',b') as in (3) is a smooth complex submanifold of $G(2,n)$, but is generally not compact. Since the dimension of $G(2,n)$ is $2(n-2)$, this means that $\sigma_{ab}(\mathbf{V})$ can be written as a disjoint union of finitely many complex manifolds with the largest dimension equal $2(n-2)-(a+b)$.

If $\mathbf{V}$ and $\mathbf{V}'$ are two different flags in $\mathbb{C}^n$, there is a smooth path of flags $\mathbf{V}^{(t)}$, with $t\in[0,1]$, so that $\mathbf{V}^{(0)}=\mathbf{V}$ and $\mathbf{V}^{(1)}=\mathbf{V}'$. The smooth (generally non-compact) manifold with boundary

$$M=\big\{(t,P)\in[0,1]\times G(2,n):P\in\sigma^*_{ab}(\mathbf{V}^{(t)})\big\}$$

is then a pseudocycle equivalence (even a homotopy) between the cycles $\sigma_{ab}(\mathbf{V})$ and $\sigma_{ab}(\mathbf{V}')$. Thus, the equivalence class of $\sigma_{ab}(\mathbf{V})$ as a cycle in $G(2,n)$ is independent of the flag $\mathbf{V}$ and is denoted σ_{ab}. It is customary to abbreviate σ_{a0} as σ_a.

2 Intersections of cycles

Given equivalence classes σ_{ab} and $\sigma_{a'b'}$, their intersection product $\sigma_{ab}\cdot\sigma_{a'b'}$ is the equivalence class of the cycle $\sigma_{ab}(\mathbf{V})\cap\sigma_{a'b'}(\mathbf{V}')$ for a generic pair of flags $\mathbf{V}$ and $\mathbf{V}'$ on $\mathbb{C}^n$. Similarly to the previous paragraph, any two pairs of such flags are homotopic, so that the equivalence class of the cycle $\sigma_{ab}(\mathbf{V})\cap\sigma_{a'b'}(\mathbf{V}')$ is independent of the generic pair $(\mathbf{V},\mathbf{V}')$. The codimension of the cycle $\sigma_{ab}\cdot\sigma_{a'b'}$ is given by

$$\operatorname{codim}\sigma_{ab}\cdot\sigma_{a'b'} = \operatorname{codim}\sigma_{ab} + \operatorname{codim}\sigma_{a'b'} = (a+b) + (a'+b').$$

If $\sigma_{a_1b_1},\ldots,\sigma_{a_kb_k}$ are k cycles, the codimension of the cycle $\sigma_{a_1b_1}\cdot\ldots\cdot\sigma_{a_kb_k}$ is thus

$$\operatorname{codim}\big(\sigma_{a_1b_1}\cdot\ldots\cdot\sigma_{a_kb_k}\big) = \sum_{i=1}^{i=k}(a_i+b_i).$$

If this number equals $2(n-2)$, then the dimension of this cycle in $G(2,n)$ is

$$\dim\big(\sigma_{a_1b_1}\cdot\ldots\cdot\sigma_{a_kb_k}\big) = \dim G(2,n) - \operatorname{codim}\big(\sigma_{a_1b_1}\cdot\ldots\cdot\sigma_{a_kb_k}\big) = 0,$$

i.e. $\sigma_{a_1b_1}\cdot\ldots\cdot\sigma_{a_kb_k}$ is simply a finite collection of points. The number of these points is denoted by

$$\big\langle\sigma_{a_1b_1}\cdot\ldots\cdot\sigma_{a_kb_k}, G(2,n)\big\rangle \in \mathbb{Z}.$$

These numbers are called intersection numbers of Schubert cycles.

The above intersection numbers satisfy the following identities:

$$\big\langle\sigma_{a_1b_1}\cdot\ldots\cdot\sigma_{a_kb_k}, G(2,n)\big\rangle = \big\langle\sigma_{a_1-b_1}\cdot\ldots\cdot\sigma_{a_k-b_k}, G(2,n-b_1-\ldots-b_k)\big\rangle; \tag{S1}$$

$$\big\langle\sigma_{n-2}\cdot\sigma_{n-2}, G(2,n)\big\rangle = 1; \tag{S2}$$

$$\big\langle\sigma_{a_1}\sigma_{a_2}\sigma_{a_3}, G(2,n)\big\rangle = 1 \quad \text{if} \quad n-2\ge a_1,a_2,a_3\ge 0,\ a_1+a_2+a_3 = 2n-4; \tag{S3}$$

$$\sigma_{a_1}\cdot\sigma_{a_2} = \sum_{c\ge a_1,a_2}\sigma_{c,a_1+a_2-c}\,. \tag{S4}$$

It is straightforward to verify the first three identities above directly directly from the definitions. We verified (S1)-(S3) directly from the definition during one of the discussion sessions. The second identity is the $a_1,a_2=n-2$, $a_3=0$ case of (S3).

The relation (S4) is known as Pieri's formula. It is actually an immediate consequence of (S1), (S3), and the structure of the cohomology of $G(2,n)$. The latter implies that two cycles A and B in $G(2,n)$ are equivalent if and only if

$$\big\langle A\cdot\sigma_{de}, G(2,n)\big\rangle = \big\langle B\cdot\sigma_{de}, G(2,n)\big\rangle \qquad \forall\, d,e\in\mathbb{Z}.$$

Thus, in order to verify (S4), it is sufficient to show that

$$\big\langle\sigma_{a_1}\cdot\sigma_{a_2}\cdot\sigma_{de}, G(2,n)\big\rangle = \sum_{c\ge a_1,a_2}\big\langle\sigma_{c,a_1+a_2-c}\cdot\sigma_{de}\,, G(2,n)\big\rangle \qquad \forall\, d,e\in\mathbb{Z}. \tag{4}$$

We can assume that $n-2\geq a_1, a_2\geq 0$, $a_1+a_2+d+e=2(n-2)$, and $n-2\geq d\geq e\geq 0$; otherwise, both sides of (4) are zero. By (S1) and then (S3),

$$\begin{aligned}\big\langle\sigma_{a_1}\cdot\sigma_{a_2}\cdot\sigma_{de}, G(2,n)\big\rangle &= \big\langle\sigma_{a_1}\cdot\sigma_{a_2}\cdot\sigma_{d-e}, G(2,n-e)\big\rangle\\ &=\begin{cases}1, & \text{if } \min(n-2, 2(n-2)-(a_1+a_2))\geq d\geq(n-2)-\min(a_1,a_2),\\ 0, & \text{otherwise,}\end{cases}\end{aligned} \tag{5}$$

with the above assumptions on d and e. Similarly, if $\min(n-2, a_1+a_2)\geq c\geq\max(a_1,a_2)$,

$$\begin{aligned}\big\langle\sigma_{c,a_1+a_2-c}\cdot\sigma_{de}, G(2,n)\big\rangle &= \big\langle\sigma_{2c-a_1-a_2}\cdot\sigma_{d-e}, G(2,n+c-a_1-a_2-e)\big\rangle\\ &=\begin{cases}1, & \text{if } d=n-2+c-a_1-a_2;\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

This gives

$$\begin{aligned}&\sum_{c\geq a_1,a_2}\big\langle\sigma_{c,a_1+a_2-c}\cdot\sigma_{de}, G(2,n)\big\rangle\\ &\qquad=\begin{cases}1, & \text{if } \min(n-2, 2(n-2)-(a_1+a_2))\geq d\geq(n-2)-\min(a_1,a_2);\\ 0, & \text{otherwise.}\end{cases}\end{aligned} \tag{6}$$

Comparing (5) with (6), we obtain (4) and thus (S4). Unfortunately, this argument requires deep facts about cycles in $G(2,n)$. We give a direct argument below.

3 The intersection $\sigma_1\cdot\sigma_1$ in $G(2,4)$

A special case of the direct argument is used in the book to show that

$$\sigma_1\cdot\sigma_1=\sigma_2+\sigma_{11} \tag{7}$$

in $G(2,4)$. The argument in the book views the elements of $G(2,4)$ as lines in $\mathbb{P}^3$ by taking their projectivizations. Here is the argument by considering them as 2-planes in $\mathbb{C}^4$. We need to intersect

$$\begin{aligned}\sigma_1(\mathbf{V}^{\mathrm{std}}) &\equiv \big\{P\in G(2,4)\colon P\cap\mathbb{C}^2\neq\{0\}\big\},\\ \sigma_1(\mathbf{V}) &\equiv \big\{P\in G(2,4)\colon P\cap V_2\neq\{0\}\big\},\end{aligned}$$

for a generic flag $\mathbf{V}$. For such a flag $\mathbb{C}^2\cap V_2=\{0\}$. However, we can move $\mathbf{V}$ so that $\mathbb{C}^2\cap V_2=\mathbb{C}$ and thus $\mathbb{C}^2+V_2$ is a three-dimensional linear subspace of $\mathbb{C}^4$, which we can assume to be $\mathbb{C}^3$; it is spanned by the one-dimensional linear subspace $L_0=\mathbb{C}$, a one-dimensional linear subspace L_1 in $\mathbb{C}^2$ different from L_0, and a one-dimensional linear subspace L_2 in V_2 different from L_0. An element of

$$\sigma_1(\mathbf{V}^{\mathrm{std}})\cap\sigma_1(\mathbf{V})\equiv\big\{P\in G(2,4)\colon P\cap\mathbb{C}^2\neq\{0\},\ P\cap V_2\neq\{0\}\big\}$$

must either contain the line $L_0=\mathbb{C}$ common to $\mathbb{C}^2$ and V_2 or intersect $\mathbb{C}^2$ and $\mathbf{V}_2$ along some one-dimensional linear subspace $L_1\subset\mathbb{C}^2$ and $L_2\subset V_2$. In the latter case, P must lie in $\mathbb{C}^2+V_2$.

Conversely, if P is a two-dimensional linear subspace of $\mathbb{C}^2 + V_2 = \mathbb{C}^3$, then P must intersect $\mathbb{C}^2$ and V_2 at least in a one-dimensional linear subspace, since by linear algebra

$$\begin{aligned} \dim P \cap \mathbb{C}^2 &\geq \dim P + \dim \mathbb{C}^2 - \dim \mathbb{C}^3 \geq 1; \\ \dim P \cap V_2 &\geq \dim P + \dim V_2 - \dim \mathbb{C}^3 \geq 1. \end{aligned}$$

From this, we obtain

$$\begin{aligned} \sigma_1(\mathbf{V}^{\text{std}}) \cap \sigma_1(\mathbf{V}) &= \big\{P \in G(2,n)\colon\ P \cap \mathbb{C}^{4-1-2} \neq \{0\}\big\} \\ &\qquad \cup \big\{P \in G(2,n)\colon\ P \subset \mathbb{C}^{4-1},\ P \cap \mathbb{C}^{4-1-1} \neq \{0\}\big\} \\ &= \sigma_2(\mathbf{V}^{\text{std}}) \cup \sigma_{11}(\mathbf{V}^{\text{std}}). \end{aligned}$$

This implies (7) in the case of $G(2,4)$.

4 Proof of (S4) for $G(2,n)$

A similar argument extends to the general case of (S4) with some care. The formula (S4) holds for $G(2,2)$, with the only choices for a_1, a_2 being $a_1, a_2 = 0$; otherwise, both sides of (S4) vanish by definition. Suppose $n \geq 3$ and the formula holds for all $G(2,m)$ with $m < n$. We need to determine the intersection of

$$\begin{aligned} \sigma_a(\mathbf{V}^{\text{std}}) &\equiv \big\{P \in G(2,n)\colon\ P \cap \mathbb{C}^{n-1-a} \neq \{0\}\big\}, \\ \sigma_b(\mathbf{V}) &\equiv \big\{P \in G(2,n)\colon\ P \cap V_{n-1-b} \neq \{0\}\big\}, \end{aligned}$$

for a generic flag $\mathbf{V}$ as in (1).

Case 1: $a+b > n-2$. In this case,

$$\dim(\mathbb{C}^{n-1-a} + V_{n-1-b}) \leq \dim \mathbb{C}^{n-1-a} + \dim V_{n-1-b} = n + (n-2) - (a+b) < n.$$

Thus, $\mathbb{C}^{n-1-a} \cap V_{n-1-b} = \{0\}$ if V_{n-1-b} is generic. We can also assume that $\mathbb{C}^{n-1-a} + \mathbf{V}_{n-1-b} \subset \mathbb{C}^{n-1}$. Then,

$$\begin{aligned} \sigma_a(\mathbf{V}^{\text{std}}) \cap \sigma_b(\mathbf{V}) &\equiv \big\{P \in G(2,n)\colon\ P \cap \mathbb{C}^{n-1-a} \neq \{0\},\ P \cap V_{n-1-b} \neq \{0\}\big\} \\ &= \big\{P \in G(2,n-1)\colon\ P \cap \mathbb{C}^{(n-1)-1-(a-1)} \neq \{0\},\ P \cap V_{(n-1)-1-(b-1)} \neq \{0\}\big\}. \end{aligned}$$

The last set is precisely the intersection of $\sigma_{a-1}(\mathbf{V}^{\text{std}})$ with $\sigma_{b-1}(\mathbf{V})$ in $G(2,n-1)$.

By the inductive assumption,

$$\sigma_{a-1} \cdot \sigma_{b-1} = \sum_{c=\max(a,b)}^{\min(n-2,a+b-1)} \sigma_{c-1,a+b-c-1} \qquad \text{in} \ \ G(2,n-1).$$

Thus, as a cycle in $G(2,n-1)\subset G(2,n)$, this intersection is equivalent to

$$\begin{aligned}
&\sum_{c=\max(a,b)}^{\min(n-2,a+b-1)} \big\{P\in G(2,n-1)\colon P\subset\mathbb{C}^{(n-1)-(a+b-c-1)},\ P\cap C^{(n-1)-1-(c-1)}\neq\{0\}\big\}\\
&\quad=\sum_{c=\max(a,b)}^{\min(n-2,a+b-1)} \big\{P\in G(2,n)\colon P\subset\mathbb{C}^{n-(a+b-c)},\ P\cap\mathbb{C}^{n-1-c}\neq\{0\}\big\} \qquad (8)\\
&\quad=\sum_{c=\max(a,b)}^{\min(n-2,a+b-1)} \sigma_{c,a+b-c}(\mathbf{V}_{\rm std}) = \sum_{c\geq a_1,a_2}\sigma_{c,a_1+a_2-c}.
\end{aligned}$$

The last equality above holds because $a_1+a_2\ c\leq n-24$ and $a_1+a_2>n-2$ This gives (S4).

Case 2: $a+b\leq n-2$. If $\mathbf{V}$ is a generic flag,

$$\dim(\mathbb{C}^{n-1-a}\cap V_{n-1-b}) = \dim\mathbb{C}^{n-1-a}+\dim V_{n-1-b}-\dim\mathbb{C}^n = n-2-(a+b)\geq 0.$$

However, we can move $\mathbf{V}$ so that $\mathbb{C}^{n-1-a}\cap V_{n-1-b}=\mathbb{C}^{n-1-(a+b)}$, and thus by linear algebra

$$\begin{aligned}
\dim(\mathbb{C}^{n-1-a}+V_{n-1-b}) &= \dim\mathbb{C}^{n-1-a}+\dim V_{n-1-b}-\dim(\mathbb{C}^{n-1-a}\cap V_{n-1-b})\\
&=(n-1-a)+(n-1-b)-(n-1-(a+b))\\
&=n-1.
\end{aligned}$$

Thus, we can assume that $\mathbb{C}^{n-1-a}+V_{n-1-b}=\mathbb{C}^{n-1}$.

Under the above assumption, the set

$$\sigma_a(\mathbf{V}^{\rm std})\cap\sigma_b(\mathbf{V})\equiv\big\{P\in G(2,n)\colon\ P\cap\mathbb{C}^{n-1-a}\neq\{0\},\ P\cap V_{n-1-b}\neq\{0\}\big\}$$

consists of the elements $P\in G(2,n)$ such that P either

- has a one-dimensional linear subspace in common with $\mathbb{C}^{n-1-a}\cap V_{n-1-b}=\mathbb{C}^{n-1-(a+b)}$ or
- intersects $\mathbb{C}^{n-1-a}$ and V_{n-1-b} along some one-dimensional linear subspaces $L_1\subset\mathbb{C}^{n-1-a}$ and $L_2\subset V_{n-1-b}$.

Since P is a two-dimensional linear subspace of $\mathbb{C}^n$, it must lie in $\mathbb{C}^{n-1-a}+V_{n-1-b}=\mathbb{C}^{n-1}$ in the latter case. From this, we obtain

$$\begin{aligned}
\sigma_a(\mathbf{V}^{\rm std})\cap\sigma_b(\mathbf{V}) = \big\{P\in G(2,n)\colon\ P\cap\mathbb{C}^{n-1-(a+b)}\neq\{0\}\big\}&\\
\cup\big\{P\in G(2,n)\colon\ P\subset\mathbb{C}^{n-1},\ P\cap\mathbb{C}^{n-1-a}\neq\{0\},&\\
P\cap V_{n-1-b}\neq\{0\}\big\}.& \qquad (9)
\end{aligned}$$

The first set on the right-hand side of (9) is precisely the cycle $\sigma_{a+b}(\mathbf{V}^{\rm std})$ in $G(2,n)$; this set is empty if $a+b>n-2$. The second set is the intersection of the cycles

$$\begin{aligned}
\sigma_{a-1}(\mathbf{V}^{\rm std}) &\equiv \big\{P\in G(2,n-1)\colon\ P\cap\mathbb{C}^{(n-1)-1-(a-1)}\neq\{0\}\big\},\\
\sigma_{b-1}(\mathbf{V}) &\equiv \big\{P\in G(2,n-1)\colon\ P\cap V_{(n-1)-1-(b-1)}\neq\{0\}\big\},
\end{aligned}$$

in $G(2,n-1)$. By the inductive assumption,

$$\sigma_{a-1}\cdot\sigma_{b-1}=\sum_{c=\max(a,b)}^{\min(n-2,a+b-1)}\sigma_{c-1,a+b-c-1}\qquad\text{in}\ \ G(2,n-1).$$

By (8), the second set in (9) as a cycle in $G(2,n-1)\subset G(2,n)$, is equivalent to

$$\begin{split}\sum_{c=\max(a,b)}^{\min(n-2,a+b-1)}&\big\{P\in G(2,n-1)\colon P\subset\mathbb{C}^{(n-1)-(a+b-c-1)},\ P\cap C^{(n-1)-1-(c-1)}\neq\{0\}\big\}\\ &=\sum_{c=\max(a,b)}^{\min(n-2,a+b-1)}\sigma_{c,a+b-c}(\mathbf{V}_{\rm std}).\end{split}\tag{10}$$

Combining (9) and (10), we obtain (S4).

5 Comments on notation

Let $k,n\in\mathbb{Z}^+$ with $k\le n$, $\mathbb{F}=\mathbb{R},\mathbb{C}$, and $G(k,n)$ be the Grassmannian of k-planes in $\mathbb{F}^n$. The maps

$$\begin{aligned}\iota_0\colon G(k,n)&\longrightarrow G(k,n+1), &\qquad \iota_0(P)&=P, \qquad\text{and}\\ \iota_1\colon G(k,n)&\longrightarrow G(k+1,n+1), &\qquad \iota_1(P)&=P\times\mathbb{F},\end{aligned}\tag{11}$$

are embeddings. If $\mathbb{F}=\mathbb{R}$ (resp. $\mathbb{F}=\mathbb{C}$), $G(k,n)$ is a smooth (resp. complex) manifold of dimension $k(n-k)$. We take the coefficient ring to be

$$R=\begin{cases}\mathbb{Z}_2, & \text{if } \mathbb{F}=\mathbb{R};\\ \mathbb{Z}, & \text{if } \mathbb{F}=\mathbb{C}.\end{cases}$$

For a partition $\mathfrak{d}\equiv(d_1\le\ldots\le d_k)$ of $d\in\mathbb{Z}^{\ge0}$ with $0\le d_1$ and $d_k\le n-k$ and a flag $\mathbf{V}$ in $\mathbb{F}^n$ as in (1), Milnor&Stasheff define

$$\sigma'_{\mathfrak{d}}(\mathbf{V})=\big\{P\in G(k,n)\colon \dim_{\mathbb{F}}\big(P\cap V_{i+d_i}\big)\ge i\ \forall\, i=1,\ldots,k\big\}.\tag{12}$$

This is a closed cell in the CW decomposition of $G(k,n)$ determined by $\mathbf{V}$; this cell is a singular $\mathbb{F}$-variety of dimension d. The R-homology class $\sigma'_{\mathfrak{d}}(n)$ of $\sigma'_{\mathfrak{d}}(\mathbf{V})$ is independent of the choice of flag $\mathbf{V}$. Since these homology classes freely generate $H_*(G(k,n);R)$ as an R-module, $H^*(G(k,n);R)$ is the dual module of $H_*(G(k,n);R)$ and is freely generated by the duals $\sigma'^*_{\mathfrak{d}}(n)$ of these homology classes. If $\mathbf{V}'$ and $\mathbf{V}''$ are the flags in $\mathbb{F}^{n+1}$ with $V'_i=V_i$ for $i\le n$ and $V''_i=V_{i-1}\times\mathbb{F}$ for $i\ge1$, then

$$\iota_0\big(\sigma'_{\mathfrak{d}}(\mathbf{V})\big)=\sigma'_{\mathfrak{d}}(\mathbf{V}')\subset G(k,n+1)\quad\text{and}\quad \iota_1\big(\sigma'_{\mathfrak{d}}(\mathbf{V})\big)=\sigma'_{0\mathfrak{d}}(\mathbf{V}'')\subset G(k,n+1).$$

Thus, there exist (unique) elements $\sigma'_{\mathfrak{d}}$ and $\sigma'^*_{\mathfrak{d}}$ in the R-homology and R-cohomology of the Grassmannian $G(k)\equiv G(k,\infty)$ of k-planes in $\mathbb{F}^\infty$ so that

$$\iota_*\big(\sigma'_{\mathfrak{d}}(n)\big)=\sigma'_{\mathfrak{d}},\quad \iota^*\sigma'^*_{\mathfrak{d}}=\sigma'^*_{\mathfrak{d}}(n)\quad\forall\ n\ge k+d_k,\qquad \iota_{1*}\big(\sigma'_{\mathfrak{d}}(n)\big)=\sigma'_{0\mathfrak{d}},\quad \iota_1^*\sigma'^*_{0\mathfrak{d}}=\sigma'^*_{\mathfrak{d}}\,,$$

where $\iota\colon G(k,n)\longrightarrow G(k)$ and $\iota_1\colon G(k)\longrightarrow G(k+1)$ are the embeddings induced by the first map in (11) and the map $P\longrightarrow \mathbb{F}\times P$, respectively.

For a partition $\mathfrak{c}\equiv(c_1\ge c_2\ge\ldots)$ of $c\in\mathbb{Z}^{\ge 0}$ with $\mathfrak{c}_i\ge 0$ for all $i\in\mathbb{Z}^+$ and a flag $\mathbf{V}$ in $\mathbb{F}^n$ as in (1), Griffthis&Harris and most other places define

$$\sigma_{\mathfrak{c}}(k,\mathbf{V})=\big\{P\in G(k,n)\colon \dim\big(P\cap V_{n-k+i-c_i}\big)\ge i\ \forall\, i\in\mathbb{Z}^+\ \text{s.t.}\ c_i\ne 0\big\}. \tag{13}$$

This set is empty unless $n-k\ge c_1$ and $c_i=0$ for all $i>k$. If $n-k\ge c_1$ and $c_i=0$ for all $i>k$, then $\sigma_{\mathfrak{c}}(k,\mathbf{V})$ is a closed cell in the CW decomposition of $G(k,n)$ determined by $\mathbf{V}$; this cell is a singular $\mathbb{F}$-subvariety of $G(k,n)$ of codimension c. The Poincaré dual $\sigma_{\mathfrak{c}}(k,n)$ of the R-homology class of $\sigma_{\mathfrak{c}}(k,\mathbf{V})$ in $G(k,n)$ is independent of the choice of flag $\mathbf{V}$. If $\mathbf{V}'$ and $\mathbf{V}''$ are the flags in $\mathbb{F}^{n+1}$ as above, then

$$\begin{aligned}\iota_0\big(\sigma_{\mathfrak{c}}(k,\mathbf{V})\big)&=\sigma_{\mathfrak{c}}(k,\mathbf{V}'')\cap\iota_0\big(G(k,n)\big)\subset G(k,n+1)\qquad\text{and}\\ \iota_1\big(\sigma_{\mathfrak{c}}(k,\mathbf{V})\big)&=\sigma_{\mathfrak{c}}(k+1,\mathbf{V}')\cap\iota_1\big(G(k,n)\big)\subset G(k+1,n+1).\end{aligned}$$

Since all strata of $\sigma_{\mathfrak{c}}(k,\mathbf{V}'')$ and $\sigma_{\mathfrak{c}}(k+1,\mathbf{V}')$ are transverse to the smooth embeddings ι_0 and ι_1, respectively, it follows that

$$\iota_0^*\sigma_{\mathfrak{c}}(k,n+1),\iota_1^*\sigma_{\mathfrak{c}}(k+1,n+1)=\sigma_{\mathfrak{c}}(k,n)\in H^*\big(G(k,n);R\big).$$

Thus, there exists a (unique) element $\sigma_{\mathfrak{c}}$ in the R-cohomology of $G(k)$ so that

$$\iota^*\sigma_{\mathfrak{c}}=\sigma_{\mathfrak{c}}(k,n)\quad\forall\ n\ge k,\qquad \iota_1^*\sigma_{\mathfrak{c}}=\sigma_{\mathfrak{c}}\,, \tag{14}$$

where $\iota\colon G(k,n)\longrightarrow G(k)$ and $\iota_1\colon G(k)\longrightarrow G(k+1)$ are as before.

By (13) as stated and with $\mathfrak{c}$ and $\mathbf{V}$ replaced by another tuple $\mathfrak{c}'\equiv(c_1'\ge c_2'\ge\ldots)$ of nonnegative integers and a generic flag $\mathbf{V}'$ in $\mathbb{F}^n$, respectively,

$$\big\langle\sigma_{\mathfrak{c}}(k,n)\sigma_{\mathfrak{c}'}(k,n),G(k,n)\big\rangle=\begin{cases}1, & \text{if } c_i+c'_{k+1-i}=n-k\,\forall\, i\le k,\ c_i,c_i'=0\,\forall\, i>k;\\ 0, & \text{otherwise.}\end{cases}$$

By (12) with $\mathfrak{d}\equiv(0\le d_1\le\ldots\le d_k\le n-k)$ as before and (13),

$$\sigma'_{\mathfrak{d}}(\mathbf{V})=\sigma_{(n-k-d_1,\ldots,n-k-d_k,0,0,\ldots)}(k,\mathbf{V}).$$

Along with the previous statement, this implies that

$$\sigma'^*_{(d_1,\ldots,d_k)}(n)=\sigma_{(d_k,\ldots,d_1,0,0,\ldots)}(k,n)\in\begin{cases}H^d(G(k;n);\mathbb{Z}_2), & \text{if } \mathbb{F}=\mathbb{R};\\ H^{2d}(G(k;n);\mathbb{Z}), & \text{if } \mathbb{F}=\mathbb{C}.\end{cases}$$

Thus, a partition of c into nonnegative integers determines the same cohomology class in $G(k,n)$ and $G(k)$ in the notation of M&S (via duals of cells) and in the notation of G&H (via Poincaré duals of cells in finite Grassmannians). One of the advantage of the G&H notation is that the cohomology class $\sigma_{\mathfrak{c}}$ can be used consistently in all Grassmannians, as indicated by (14).