MAT 566: Characteristic Classes

Problem Set 3

Due by Tuesday, 3/11, in class

(if you have not passed the orals yet)

Problem (iv): We have computed the $\mathbb{Z}_2$-cohomology ring of the real Grassmannian and showed that the natural homomorphism

$$f_n^*\colon H^*\big(\mathrm{Gr}_n\mathbb{R}^\infty;\mathbb{Z}_2\big) \longrightarrow H^*\big((\mathbb{R}P^\infty)^n;\mathbb{Z}_2\big)$$

is injective by assuming that Stiefel-Whitney classes exist (classes that satisfy the required Naturality, Normalization, and Product Formula properties, as on pp37,38). The aim of this problem is to reverse the relevant arguments in order to construct Stiefel-Whitney classes for *paracompact* spaces.

The Naturality Property implies that SW-classes arise from the cohomology rings of the Grassmannians. So, for each $n, r \in \mathbb{Z}^+$, we could start by picking some classes

$$w_r(\gamma_n) \in H^r\big(\mathrm{Gr}_n\mathbb{R}^\infty;\mathbb{Z}_2\big).$$

The Naturality property would then be forced by Theorems 5.6 and 5.7. The Normalization property is easy to achieve: simply take

$$w_1(\gamma_1) \in H^1\big(\mathbb{R}P^\infty;\mathbb{Z}_2\big)$$

to be the nonzero element. The tricky part is the Product Formula; by the uniqueness of the SW-classes there is only one way to choose $w_r(\gamma_n)$ for $(n,r) \neq (1,1)$ so that it is satisfied.

(a) Assuming that SW-classes exist, show that $w_r(\gamma_n)$ is represented in $H^*(\mathrm{Gr}_n\mathbb{R}^\infty;\mathbb{Z}_2)$ by the cell corresponding to the partition $(1,\ldots,1)$ of r; see Problem 7-A and the preceeding paragraph.

(b) Without assuming that SW-classes exist, determine the image under f_n^* of the class $w_r(\gamma_n) \in H^*(\mathrm{Gr}_n\mathbb{R}^\infty;\mathbb{Z}_2)$ represented by the partition $(1,\ldots,1)$ of r.
Hint: Proceed by induction on n and use Problem 6-C.

(c) Conclude that

$$H^*\big(\mathrm{Gr}_n\mathbb{R}^\infty;\mathbb{Z}_2\big) \approx \mathbb{Z}_2\big[w_1(\gamma_1),\ldots,w_n(\gamma_n)\big]$$

as graded algebras, with $\deg w_r(\gamma_n) = r$, and that f_n^* is injective.
Note: Here $w_1(\gamma_n),\ldots,w_n(\gamma_n) \in H^*(\mathrm{Gr}_n\mathbb{R}^\infty;\mathbb{Z}_2)$ are the elements chosen in (b).

(d) Conclude that the classes $w_r(\gamma_n)$ chosen in (b) induce characteristic classes of vector bundles over paracompact spaces that satisfy all properties of the Stiefel-Whitney classes.

Note: a comparison of M&S vs. G&H conventions has been added to the Schubert Calculus notes.