MAT 566: Characteristic Classes

Problem Set 2

Due by Tuesday, 2/25, in class

(if you have not passed the orals yet)

Do 7-C plus any *one* of the following problems: 5-B,5-E,6-B,7-A,7-B, (ii), or (iii) below.

Problem (ii): Let X be a paracompact locally contractible topological space. Thus, $\check{H}^1(X;\mathbb{Z}_2)$ is naturally isomorphic to $H^1(X;\mathbb{Z}_2)$; see Chapter 5 in Warner. An equivalence class $[L]$ of real line bundles corresponds to some element $\breve{w}_1(L)\in\check{H}^1(X;\mathbb{Z}_2)$. Show that $\breve{w}_1(L)$ corresponds to $w_1(L)\in H^1(X;\mathbb{Z}_2)$ under the natural isomorphism.
Hint: there is a *very* short solution via naturality.

Problem (iii): Let $\gamma_{\mathbb{C}}\equiv\big\{(\ell,v)\in\mathbb{CP}^1\times\mathbb{C}^2:\, v\in\ell\subset\mathbb{C}^2\big\}$ be the total space of the complex tautological line bundle over $\mathbb{CP}^1\approx S^2$. For each $a\in\mathbb{Z}$, define

$$V_a\equiv\gamma_{\mathbb{C}}^{\otimes_{\mathbb{C}}2a}/\mathbb{Z}_2\longrightarrow\mathbb{CP}^1/\mathbb{Z}_2\approx\mathbb{RP}^2,$$
$$\big([z_0,z_1],(v_0,v_1)^{\otimes_{\mathbb{C}}2a}\big)\sim\big([-\overline{z_1},\overline{z_0}],(-\overline{v_1},\overline{v_0})^{\otimes_{\mathbb{C}}2a}\big),\quad [z_0,z_1]\sim[-\overline{z_1},\overline{z_0}].$$

Show that V_a is a rank 2 real vector bundle, is not orientable for every $a\in\mathbb{Z}$, and does not split as a sum of line bundles if $a\neq0$. Furthermore, V_a and $V_{a'}$ are not isomorphic as real vector bundles if $a\neq a'$. For the last statement, you may need to use that

$$\big\langle e\big(\gamma_{\mathbb{C}}^{\otimes_{\mathbb{C}}2a}\big),[\mathbb{CP}^1]\big\rangle=\big\langle c_1\big(\gamma_{\mathbb{C}}^{\otimes_{\mathbb{C}}2a}\big),[\mathbb{CP}^1]\big\rangle=-2a,$$

as will be shown later in the course (this implies that $\gamma_{\mathbb{C}}^{\otimes_{\mathbb{C}}2a}$ and $\gamma_{\mathbb{C}}^{\otimes_{\mathbb{C}}2a'}$ are not isomorphic as real vector bundles if $a\neq a'$).

Remark 1: Problem 7-C is an example of the *Splitting Principle*: if a natural formula involving characteristic classes holds for split vector bundles (i.e. direct sums of line bundles), then it holds for all vector bundles.

Remark 2: Problem (ii) implies that the real line bundles over a paracompact topological space are classified by their w_1. Problem (iii) implies that the real vector bundles of rank 2 (and higher) are generally not distinguished by their total Stiefel-Whitney classes w, because $H^2(\mathbb{RP}^2;\mathbb{Z}_2)$ contains only 2 elements in total.