# MAT 562: Symplectic Geometry

## Solution to Problem I

(a) *Let $q:\mathbb{C}^n-\{0\}\longrightarrow\mathbb{C}P^{n-1}$ be the quotient projection as in Problem A on PS1. Suppose $U\subset\mathbb{C}P^{n-1}$ is an open subset and $s:U\longrightarrow\mathbb{C}^n-\{0\}$ is a holomorphic section of $q$, i.e. $q\circ s=\mathrm{id}_U$. Show that the 2-form*

$$\omega_{\mathrm{FS};n-1}\big|_U \equiv \frac{\mathfrak{i}}{2\pi}\partial\overline{\partial}\ln|s|^2, \tag{1}$$

*where $|\cdot|$ is the standard (round) norm on $\mathbb{C}^n$, is independent of the choice of $s$.*

(b) *By (a), (1) determines a global 2-form $\omega_{\mathrm{FS};n-1}$ on $\mathbb{C}P^{n-1}$, called the Fubini-Study symplectic form. Show that this form is indeed symplectic and*

$$\omega_{\mathrm{FS};n-1}=\frac{1}{\pi}\omega_{\mathbb{C}P^{n-1}}, \tag{2}$$

*where $\omega_{\mathbb{C}P^{n-1}}$ is the symplectic form on $\mathbb{C}P^{n-1}$ provided by Problem A on PS1.*

(c) *Show that the action of $S^1\equiv\mathbb{R}/\mathbb{Z}$ on $\mathbb{C}^n$ given by*

$$\mathrm{e}^{2\pi\mathrm{i}t}\cdot(z_1,\ldots,z_n)=\big(z_1,\ldots,z_{k-1},\mathrm{e}^{2\pi\mathrm{i}t}z_k,z_{k+1},\ldots,z_n\big) \tag{3}$$

*is Hamiltonian with respect to the standard symplectic form $\omega_{\mathbb{C}^n}$ with a Hamiltonian*

$$\widetilde{H}_k:\mathbb{C}^n\longrightarrow\mathbb{R},\qquad \widetilde{H}_k(z_1,\ldots,z_n)=\pi|z_k|^2.$$

(d) *Show that the actions of $\mathbb{T}^n\equiv(S^1)^n$ and $\mathbb{T}^{n-1}\equiv(S^1)^{n-1}$ on $\mathbb{C}P^{n-1}$ given by*

$$\begin{aligned}\big(\mathrm{e}^{2\pi\mathrm{i}t_1},\ldots,\mathrm{e}^{2\pi\mathrm{i}t_n}\big)\cdot[z_1,\ldots,z_n]&=\big[\mathrm{e}^{2\pi\mathrm{i}t_1}z_1,\ldots,\mathrm{e}^{2\pi\mathrm{i}t_n}z_n\big],\\ \big(\mathrm{e}^{2\pi\mathrm{i}t_1},\ldots,\mathrm{e}^{2\pi\mathrm{i}t_{n-1}}\big)\cdot[z_1,\ldots,z_n]&=\big[\mathrm{e}^{2\pi\mathrm{i}t_1}z_1,\ldots,\mathrm{e}^{2\pi\mathrm{i}t_{n-1}}z_{n-1},z_n\big]\end{aligned} \tag{4}$$

*are Hamiltonian with respect to the symplectic form $\omega_{\mathrm{FS};n-1}$. Determine the moment polytopes for these actions; draw the moment polytopes in the $n=3$ case, labeling everything clearly.*

(a) If $\widetilde{s}:U\longrightarrow\mathbb{C}^n-\{0\}$ is another holomorphic section of $q$, $\widetilde{s}=fs$ for some holomorphic map $f:U\longrightarrow\mathbb{C}^*$. Thus,

$$\partial\overline{\partial}\ln|\widetilde{s}|^2=\partial\overline{\partial}\ln\big(f\overline{f}|s|^2\big)=\partial\overline{\partial}\ln f-\overline{\partial}\partial\ln\overline{f}+\partial\overline{\partial}\ln|s|^2=\partial 0-\overline{\partial}0+\partial\overline{\partial}\ln|s|^2.$$

This shows that the form (1) is independent of the choice of $s$.

(b) Let $p=q|_{S^{2n-1}}: S^{2n-1}\longrightarrow\mathbb{C}P^{n-1}$. Since the 2-form $\omega_{\mathbb{C}P^{n-1}}$ is symplectic, (2) implies that so is $\omega_{\mathrm{FS};n-1}$. By Problem A(f) on HW1, (2) is equivalent to

$$p^*\omega_{\mathrm{FS};n-1}\big|_{TS^{2n-1}}=\frac{1}{\pi}\omega_{\mathbb{C}^n}\big|_{TS^{2n-1}}\,, \tag{5}$$

where $\omega_{\mathbb{C}^n}\equiv\frac{\mathfrak{i}}{2}\sum_{j=1}^{n}\mathrm{d}z_j\wedge\mathrm{d}\bar{z}_j$ is the standard symplectic form on $\mathbb{C}^n$. For each $k=1,\ldots,n$, the map

$$s_k\colon U_k\equiv\big\{[z_1,\ldots,z_n]\in\mathbb{C}P^{n-1}\colon z_k\neq0\big\}\longrightarrow\mathbb{C}^n-\{0\},$$
$$s_k\big([z_1,\ldots,z_n]\big)=\big(z_1/z_k,\ldots,z_{k-1}/z_k,1,z_{k+1}/z_k,\ldots,z_n/z_k\big),$$

is a holomorphic section of $q$ and

$$\begin{split}\big\{q|_{q^{-1}(U_k)}\big\}^*\big(\partial\bar\partial\ln|s_k|^2\big)&=\partial\bar\partial\ln\big|\big\{q|_{q^{-1}(U_k)}\big\}^*s_k\big|^2=\bigg(\partial\bar\partial\ln\sum_{j=1}^{n}|z_j|^2-\partial\bar\partial\ln|z_k|^2\bigg)\bigg|_{q^{-1}(U_k)}\\&=\sum_{j=1}^{n}\partial\frac{z_j\mathrm{d}\bar{z}_j}{|z|^2}\bigg|_{q^{-1}(U_k)}=\sum_{j=1}^{n}\frac{\mathrm{d}z_j\wedge\mathrm{d}\bar{z}_j}{|z|^2}\bigg|_{q^{-1}(U_k)}-\frac{\partial|z|^2\wedge\bar\partial|z|^2}{|z|^4}\bigg|_{q^{-1}(U_k)};\end{split}\tag{6}$$

the first equality above holds because $q$ is holomorphic. Since $\mathrm{d}=\partial+\bar\partial$ and the restriction of $\mathrm{d}|z|^2$ to $TS^{2n-1}$ vanishes (because $|z|^2=1$ on $S^{2n-1}$),

$$\partial|z|^2\wedge\bar\partial|z|^2\big|_{TS^{2n-1}}=-\partial|z|^2\wedge\partial|z|^2\big|_{TS^{2n-1}}=0.$$

Combining this with (6), we obtain

$$p^*\omega_{\mathrm{FS};n-1}\big|_{TS^{2n-1}|_{p^{-1}(U_k)}}=\frac{\mathfrak{i}}{2\pi}\sum_{j=1}^{n}\mathrm{d}z_j\wedge\mathrm{d}\bar{z}_j\big|_{TS^{2n-1}|_{p^{-1}(U_k)}}=\frac{1}{\pi}\omega_{\mathbb{C}^n}\big|_{TS^{2n-1}|_{p^{-1}(U_k)}}\,.$$

Since the open subsets $U_k\subset\mathbb{C}P^{n-1}$ cover $\mathbb{C}P^{n-1}$, this gives (5).

(c) Let $z_j=x_j+\mathfrak{i}y_j$, as usual. Thus,

$$\omega_{\mathbb{C}^n}=\sum_{j=1}^{n}\mathrm{d}x_j\wedge\mathrm{d}y_j,\quad\widetilde{\zeta}_k\equiv\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{2\pi\mathfrak{i}t}\cdot(z_1,\ldots,z_n)\bigg|_{t=0}=2\pi\bigg(-y_k\frac{\partial}{\partial x_k}+x_k\frac{\partial}{\partial y_k}\bigg),\quad-\iota_{\widetilde{\zeta}_k}\omega_{\mathbb{C}^n}=\pi\mathrm{d}|z_k|^2.$$

Thus, $\widetilde{H}_k$ is a Hamiltonian for the $S^1$-action on $(\mathbb{C}^n,\omega_{\mathbb{C}^n})$ given by (3).

(d) The first action in (4) is Hamiltonian if and only if its restriction to each component $S^1 \subset \mathbb{T}^n$ is Hamiltonian. For $k=1,\ldots,n$, define

$$H_k\colon \mathbb{C}P^{n-1} \longrightarrow \mathbb{R}, \qquad H_k\big([z_1,\ldots,z_n]\big) = \frac{\pi|z_k|^2}{|z_1|^2+\ldots+|z_k|^2}\,,$$
$$\zeta_k \equiv \frac{\mathrm{d}}{\mathrm{d}t}\big(\underbrace{1,\ldots,1}_{k-1},\mathrm{e}^{2\pi\mathrm{i}t},1,\ldots,1\big)\cdot[z_1,\ldots,z_n]\bigg|_{t=0} \in \Gamma\big(\mathbb{C}P^{n-1};T(\mathbb{C}P^{n-1})\big).$$

The $S^1$-action (3) restricts to an action on $S^{2n-1}\subset\mathbb{C}^n$ and the projection $p\colon S^{2n-1}\longrightarrow \mathbb{C}P^{n-1}$ is $S^1$-equivariant with respect to the restriction of the first action in (4) to the $k$-th component $S^1\subset\mathbb{T}^n$. Thus,

$$\zeta_k\big(p(z)\big) = \mathrm{d}_z p\big(\widetilde{\zeta}_k(z)\big) \in T_{p(z)}(\mathbb{C}P^{n-1}) \qquad \forall\, z\in S^{2n-1}.$$

Along with (b), (c), and $\widetilde{H}_k|_{S^{2n-1}} = H_k\circ p$, this gives

$$p^*\big(\iota_{\zeta_k}\omega_{\mathrm{FS};n-1}\big) = \iota_{\widetilde{\zeta}_k}\big(p^*\omega_{\mathrm{FS};n-1}\big) = \frac{1}{\pi}\iota_{\widetilde{\zeta}_k}\big(\omega_{\mathbb{C}^n}|_{TS^{2n-1}}\big) = -\frac{1}{\pi}\mathrm{d}\widetilde{H}_k|_{TS^{2n-1}} = -\frac{1}{\pi}p^*\mathrm{d}H_k.$$

Since $q^*$ is injective, it follows that $-\iota_{\zeta_k}\omega_{\mathrm{FS};n-1} = (1/\pi)\mathrm{d}H_k$, i.e. $(1/\pi)H_k$ is the Hamiltonian for the restriction of the first action in (4) to the $k$-th component $S^1\subset\mathbb{T}^n$. Thus,

$$H_{\mathrm{FS};n-1} \equiv \frac{1}{\pi}\big(H_1,\ldots,H_k\big)\colon \mathbb{C}P^{n-1}\longrightarrow\mathbb{R}^k, \quad H_{\mathrm{FS};n-1}\big([z_1,\ldots,z_n]\big) = \frac{(|z_1|^2,\ldots,|z_n|^2)}{|z_1|^2+\ldots+|z_n|^2}\,,$$

is a Hamiltonian for the first action in (4) with respect to $\omega_{\mathrm{FS};n-1}$. The moment polytope $H_{\mathrm{FS};n-1}(\mathbb{C}P^{n-1})$ is then the $(n-1)$-simplex in $\mathbb{R}^k$ with the vertices at $(1,0,\ldots,0)$, $(0,1,\ldots,0)$, etc.; see the first picture below for the $n=3$ case. Since the second action in (4) is the restriction of the first to the subtorus $\mathbb{T}^{n-1}\times\{1\}\subset\mathbb{T}^n$, the composition of $H_{\mathrm{FS};n-1}$ with the projection to the first $(n-1)$ components,

$$\pi_{\mathbb{R}^{n-1}}\circ H_{\mathrm{FS};n-1}\colon \mathbb{C}P^{n-1}\longrightarrow\mathbb{R}^{k-1}, \quad \pi_{\mathbb{R}^{n-1}}\circ H_{\mathrm{FS};n-1}\big([z_1,\ldots,z_n]\big) = \frac{(|z_1|^2,\ldots,|z_{n-1}|^2)}{|z_1|^2+\ldots+|z_n|^2}\,,$$

is a Hamiltonian for the second action in (4) with respect to $\omega_{\mathrm{FS};n-1}$. The moment polytope $\pi_{\mathbb{R}^{n-1}}(H_{\mathrm{FS};n-1}(\mathbb{C}P^{n-1}))$ is then the standard $(n-1)$-simplex in $\mathbb{R}^{k-1}$, i.e. the simplex with the vertices at the origin and $(1,0,\ldots,0)$, $(0,1,\ldots,0)$, etc.; see the second picture below for the $n=3$ case.