MAT 562: Symplectic Geometry

Solution to Problem E

(a) *Suppose a compact Lie group G acts smoothly on a symplectic manifold (M,ω). Show that there exists an ω-compatible almost complex structure J on M preserved by G, i.e.*

$$\mathrm{d}g\circ J = J\circ \mathrm{d}g\colon TM \longrightarrow g^*TM \qquad \forall\, g\in G\,.$$

(b) *Suppose S^1 acts smoothly on a compact almost complex manifold (M,J), i.e. preserving J. Let $\xi\in\Gamma(M;TM)$ be the vector field generating this action, i.e.*

$$\xi(x) = \frac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{e}^{2\pi\mathrm{i}t}\cdot x\big)\bigg|_{t=0} \in T_xM \qquad \forall\, x\in M.$$

Show that the flow of $-J\xi$ extends this action to a well-defined smooth action of $\mathbb{C}^*\supset S^1$ on M,

$$\psi\colon \mathbb{C}^*\times M \longrightarrow M, \quad \psi\big(\mathrm{e}^{2\pi\mathrm{i}t},x\big) = \mathrm{e}^{2\pi\mathrm{i}t}\cdot x, \quad \frac{\mathrm{d}}{\mathrm{d}s}\psi\big(\mathrm{e}^{2\pi(s+\mathrm{i}t)},x\big) = -J\xi\big(\psi(\mathrm{e}^{2\pi(s+\mathrm{i}t)},x)\big),$$

and $u_x\equiv\psi(\cdot,x) : \mathbb{C}^*\longrightarrow M$ is a J-holomorphic map for every $x\in M$. If in addition J is integrable, show that this $\mathbb{C}^*$-action is J-holomorphic.

(c) *Suppose S^1 acts smoothly on a compact symplectic manifold (M,ω) with an associated Hamiltonian $H : M \longrightarrow \mathbb{R}$, i.e. $\iota_\xi\omega = -\mathrm{d}H$, where $\xi\in\Gamma(M;TM)$ is the vector field generating the S^1-action, and $x\in M$. Let J be an S^1-invariant ω-tamed almost complex structure on M and u_x be as in (b). Show that there exist S^1-fixed points $x_-,x_+\in M$ such that*

$$\lim_{s\longrightarrow\pm\infty} u_x\big(\mathrm{e}^{2\pi(s+\mathrm{i}t)}\big) = x_\pm \qquad \text{and} \qquad \mathrm{E}_{g_J^\omega}(u_x) = \int_{\mathbb{C}^*} u_x^*\omega = H(x_+)-H(x_-)\,,$$

where g_J^ω is the metric on M determined by ω and J.

Note. This corrects and sharpens Exercise 5.1.5 in the main book. By (c), u_x extends to a continuous map $\widetilde{u}_x\colon\mathbb{CP}^1\longrightarrow M$ with bounded energy on $\mathbb{C}^*$. By (b), the restriction of this map to $\mathbb{C}^*$ is J-holomorphic. The Removal of Singularity Theorem (Proposition 4.8 in the notes) then implies that the extension $\widetilde{u}_x$ is J-holomorphic as well. Thus, a compact connected symplectic manifold (M,ω) with a non-trivial Hamiltonian S^1-action contains a non-constant J-holomorphic sphere through every point $x\in M$ and for every ω-tamed almost complex structure J. This provides the motivation for MR2484280, one of the rare established connections between Gromov-Witten invariants and geometric properties of the symplectic manifold.

(a) By the solution to Problem C on HW2, there exists a Riemannian metric g on M preserved by G. The almost complex structure $J_{g,\omega}\in\mathcal{J}_{\mathrm{cm}}(\omega)$ constructed in the proof of Proposition 2.3 in the *Notes* is then also preserved by G.

(b) Since M is compact, the flow of the vector field $-J\xi$ on M,

$$\psi_s\colon M\longrightarrow M, \quad \psi_0=\mathrm{id}_M, \quad \frac{\mathrm{d}}{\mathrm{d}s}\psi_s = -J\xi\circ\psi_s\,,$$

is defined for every $s\in\mathbb{R}$. Thus, the map

$$\psi\colon \mathbb{C}^*\times M\longrightarrow M, \quad \psi\big(\mathrm{e}^{2\pi(s+\mathrm{i}t)},x\big) = \psi_s\big(\mathrm{e}^{2\pi\mathrm{i}t}\cdot x\big),$$

is also well-defined and smooth. The resulting $\mathbb{R}$-action on M commutes with the original S^1-action if and only if the Lie bracket of the vector fields generating the two actions vanishes. Since the S^1 preserves J, $\psi_t^*J\!=\!J$, and thus $\mathcal{L}_\xi J\!=\!0$. It follows that

$$[\xi,-J\xi]=-\mathcal{L}_\xi(J\xi)=-(\mathcal{L}_\xi J)\xi-J(\mathcal{L}_\xi\xi)=0-J[\xi,\xi]=0.$$

Thus, the $\mathbb{R}$- and S^1-actions indeed commute and thus correspond to an action of $\mathbb{C}^*\!\approx\!\mathbb{R}\!\times\!S^1$.

Let $x\!\in\!M$. Since the S^1- and $\mathbb{R}$-actions above are the flows of the vector fields ξ and $-J\xi$, respectively,

$$\frac{\partial}{\partial s}u_x\big(\mathrm{e}^{2\pi(s+\mathrm{i}t)}\big)=-J\xi\big(u_x\big(\mathrm{e}^{2\pi(s+\mathrm{i}t)}\big)\big),\;\;\frac{\partial}{\partial t}u_x\big(\mathrm{e}^{2\pi(s+\mathrm{i}t)}\big)=\xi\big(u_x\big(\mathrm{e}^{2\pi(s+\mathrm{i}t)}\big)\big)=J\frac{\partial}{\partial s}u_x\big(\mathrm{e}^{2\pi(s+\mathrm{i}t)}\big). \tag{1}$$

Since $J_{\mathbb{C}}\frac{\partial}{\partial s}\mathrm{e}^{2\pi(s+\mathrm{i}t)}\!=\!\frac{\partial}{\partial t}\mathrm{e}^{2\pi(s+\mathrm{i}t)}$, where $J_{\mathbb{C}}$ is the standard (almost) complex structure on $\mathbb{C}$, it follows that the map u_x is J-holomorphic, i.e.

$$\mathrm{d}u_x\!\circ\!J_{\mathbb{C}}=J\!\circ\!\mathrm{d}u_x\colon T\mathbb{C}^*\longrightarrow u_x^*TM.$$

Suppose J is integrable. Exercise 2.2 in the *Notes* then gives

$$\mathcal{L}_{-\xi J}J=-J(\mathcal{L}_\xi J)=0\,,$$

i.e. the flow of $-\xi J$ preserves J and so does the $\mathbb{R}$-action. Thus, the entire $\mathbb{C}^*$-action defined above preserves J, i.e. this action is J-holomorphic.

(c) Since the $\mathbb{R}$-action is generated by the vector field $-J\xi$,

$$\begin{aligned}\frac{\partial}{\partial s}H\big(u_x(\mathrm{e}^{2\pi(s+\mathrm{i}t)})\big)=\mathrm{d}_{u_x(\mathrm{e}^{2\pi(s+\mathrm{i}t)})}H\big(-J\xi\big(u_x(\mathrm{e}^{2\pi(s+\mathrm{i}t)})\big)\big)&=\omega\big(\xi,J\xi\big)\big|_{u_x(\mathrm{e}^{2\pi(s+\mathrm{i}t)})}\\&=g_J^\omega(\xi,\xi)\big|_{u_x(\mathrm{e}^{2\pi(s+\mathrm{i}t)})}\,.\end{aligned} \tag{2}$$

Since M is compact, H is bounded above and below. Thus, the right-hand side above approaches 0 as $s\!\longrightarrow\!\pm\infty$ and for each $\mathrm{e}^{2\pi\mathrm{i}t}\!\in\!S^1$ there exists $x_\pm(\mathrm{e}^{2\pi\mathrm{i}t})\!\in\!M$ such that

$$\lim_{s\longrightarrow\pm\infty}u_x\big(\mathrm{e}^{2\pi(s+\mathrm{i}t)}\big)=x_\pm\big(\mathrm{e}^{2\pi\mathrm{i}t}\big).$$

Since $\xi\big(u_x(\mathrm{e}^{2\pi(s+\mathrm{i}t)}\big)\longrightarrow0$ as $s\longrightarrow\pm\infty$ and the S^1-action is generated by the vector field ξ, the length of diameter of the set $u_x(\mathrm{e}^{2\pi(s+\mathrm{i}t)})$ with $s\!\in\!\mathbb{R}$ fixed and t varying approaches 0 as $s\!\longrightarrow\!\pm\infty$. Thus, the above limits must be independent of $\mathrm{e}^{2\pi\mathrm{i}t}$. This establishes the first claim.

The first equality in the second equation holds by (2.13) in the *Notes*. By (1) and (2),

$$u_x^*\omega=\omega(-J\xi,\xi)\big|_{u_x(\mathrm{e}^{2\pi(s+\mathrm{i}t)})}\mathrm{d}s\!\wedge\!\mathrm{d}t=\frac{\partial}{\partial s}H\big(u_x(\mathrm{e}^{2\pi(s+\mathrm{i}t)})\big)\,\mathrm{d}s\!\wedge\!\mathrm{d}t\,.$$

Thus,

$$\begin{aligned}\mathrm{E}_{g_J^\omega}(u_x)=\int_{\mathbb{C}^*}u_x^*\omega=\int_0^1\bigg(\int_{-\infty}^{\infty}\frac{\partial}{\partial s}H\big(u_x(\mathrm{e}^{2\pi(s+\mathrm{i}t)})\big)\mathrm{d}s\bigg)\mathrm{d}t&=\int_0^1\big(H(x_+)\!-\!H(x_-)\big)\mathrm{d}t\\&=H(x_+)\!-\!H(x_-)\,.\end{aligned}$$