MAT 562: Symplectic Geometry

Partials Solutions to Problem Set 3

Notes Exercise 2.9

Suppose $u,v\in\mathbb{O}$ are purely imaginary and linearly independent over $\mathbb{R}$. Show that the subalgebra of $\mathbb{O}$ generated by u and v is isomorphic to $\mathbb{H}$.

We can assume that u,v are of unit length and are orthogonal to each other. Thus,

$$\overline{x}=-x,\quad \overline{y}=-y,\quad x^2,y^2=-1,\quad xy=-x\overline{y}=y\overline{x}=-yx,\quad (xy)y=xy^2=-x;$$

the last two equations use (2.17). Thus, the map $u\longrightarrow\mathfrak{i}$ and $v\longrightarrow\mathfrak{j}$ determines an isomorphism between the subalgebra of $\mathbb{O}$ generated by u and v and $\mathbb{H}$.

Notes Exercise 2.10

Show that the Nijenhuis tensor of the almost complex manifold (X,J) of Example 2.8 is given by

$$A_J\big|_u(v_1,v_2)=\frac{1}{4}\big(v_1(v_2u)-(v_1v_2)u-v_2(v_1u)+(v_2v_1)u\big)\qquad\forall\,u\in X,\ v_1,v_2\in T_uX.$$

Conclude that $A_J|_{u=\epsilon}(\mathfrak{i},\mathfrak{j})=(\mathfrak{j}\mathfrak{i})\epsilon$ if $\mathfrak{i},\mathfrak{j}\in\mathbb{H}$ are orthogonal purely imaginary quaternions.

By definition,

$$A_J\big|_u(v_1,v_2)=\frac{1}{4}\Big([\xi_1,\xi_2]+J[\xi_1,J\xi_2]+J[J\xi_1,\xi_2]-[J\xi_1,J\xi_2]\Big)_u,$$

where $\xi_1,\xi_2\in\Gamma(X;TX)$ are vector fields on X so that $\xi_i(u)=v_i$. It is straightforward to compute the Lie bracket of vector fields on $\mathbb{R}^8\supset X$:

$$[\widetilde{\xi}_1,\widetilde{\xi}_2]\equiv\bigg[\sum_{i=1}^{8}f_i\frac{\partial}{\partial x_i},\sum_{i=1}^{8}g_j\frac{\partial}{\partial x_j}\bigg]=\sum_{i,j=1}^{8}f_i\frac{\partial g_j}{\partial x_i}\frac{\partial}{\partial x_j}-\sum_{i,j=1}^{8}g_j\frac{\partial f_i}{\partial x_j}\frac{\partial}{\partial x_i}\equiv\langle\widetilde{\xi}_1\rangle(\widetilde{\xi}_2)-\langle\widetilde{\xi}_2\rangle(\widetilde{\xi}_1)\,.$$

In particular, $\langle\widetilde{\xi}_1\rangle(\widetilde{\xi}_2)=\widetilde{\xi}_1$ if $\widetilde{\xi}_2(x)=x$ for all $x\in\mathbb{R}^8$, i.e. $\widetilde{\xi}_2$ is the canonical vector field, denoted simply by x below. Furthermore, if $\widetilde{\xi}_1,\widetilde{\xi}_2\in\Gamma(\mathbb{R}^8;T\mathbb{R}^8)$ are vector fields on $\mathbb{R}^8$ such that $\widetilde{\xi}_1|_X,\widetilde{\xi}_2|_X\in\Gamma(X;TX)$, then

$$[\widetilde{\xi}_1,\widetilde{\xi}_2]\big|_X=\big[\widetilde{\xi}_1|_X,\widetilde{\xi}_2|_X\big]\in\Gamma(X;TX);$$

see Proposition 1.55 in Warner's book.

For $x\in\mathbb{R}^8=\mathbb{O}$, define

$$\widetilde{J}_x\colon\mathbb{O}\longrightarrow\mathbb{O},\quad \widetilde{J}_x(v)=vx,\qquad \widetilde{\xi}_1(x)=v_1-\langle v_1,x\rangle x,\quad \widetilde{\xi}_2(x)=v_2-\langle v_2,x\rangle x.$$

Thus, $\widetilde{J}$ is an endomorphism of the real vector bundle $T\mathbb{R}^8$ over $\mathbb{R}^8$ such that $\widetilde{J}|_{TX}=J$, while $\widetilde{\xi}_1,\widetilde{\xi}_2\in\Gamma(\mathbb{R}^8;T\mathbb{R}^8)$ are vector fields on $\mathbb{R}^8$ such that $\widetilde{\xi}_i|_X,(\widetilde{J}\widetilde{\xi}_i)|_X\in\Gamma(X;TX)$ and $\widetilde{\xi}_i(u)=v_i$. Viewing v_1,v_2 as constant vector fields on $\mathbb{R}^8$, we obtain

$$\begin{aligned}
&\big[v_1,\langle v_2,x\rangle x\big]=\langle v_2,\langle v_1\rangle(x)\rangle x+\langle v_2,x\rangle\langle v_1\rangle(x)=\langle v_2,v_1\rangle x+\langle v_2,x\rangle v_1,\quad [\widetilde{\xi}_1,\widetilde{\xi}_2]=\langle v_1,x\rangle v_2-\langle v_2,x\rangle v_1,\\
&[\widetilde{\xi}_1,\widetilde{J}\widetilde{\xi}_2]=v_2v_1-\langle v_2,v_1\rangle x^2-\langle v_2,x\rangle\big(v_1x+xv_1\big)+\langle v_1,v_2x\rangle x+2\langle v_1,x\rangle\langle v_2,x\rangle x^2-\langle v_2,x\rangle\langle v_1,x^2\rangle x,\\
&[\widetilde{J}\widetilde{\xi}_1,\widetilde{J}\widetilde{\xi}_2]=v_2(v_1x)-\langle v_2,v_1x\rangle x^2-\langle v_2,x\rangle\big((v_1x)x+x(v_1x)\big)-\langle v_1,x\rangle v_2x^2+\langle v_1,x\rangle\langle v_2,x^2\rangle x^2,\\
&\qquad-v_1(v_2x)+\langle v_1,v_2x\rangle x^2+\langle v_1,x\rangle\big((v_2x)x+x(v_2x)\big)+\langle v_2,x\rangle v_1x^2-\langle v_2,x\rangle\langle v_1,x^2\rangle x^2.
\end{aligned}$$

Evaluating these expressions at $x = u$ and using $\langle v_i, u\rangle, \langle v_i, 1\rangle = 0$ and $u^2 = -1$, we obtain the claimed expression for $A_J|_u(v_1, v_2)$. If $\mathfrak{i}, \mathfrak{j} \in \mathbb{H}$, $\mathfrak{i}(\mathfrak{j}\epsilon) = (\mathfrak{j}\mathfrak{i})\epsilon$ by the definition of octonion multiplication. If in addition $\mathfrak{i}, \mathfrak{j}$ are purely imaginary and orthogonal, $\mathfrak{i}\mathfrak{j} = -\mathfrak{j}\mathfrak{i}$. This yields the last claim.

Problem D

(a) *Let Σ be a connected oriented closed surface (2-dimensional manifold). Show that a continuous map $f\colon \Sigma \longrightarrow S^2$ is null-homotopic if and only if it has degree 0.*

(b) *Let Σ be a connected oriented closed genus g surface embedded in a standard way in $\mathbb{R}^3$ (you can choose what this means). Let $\nu\colon \Sigma \longrightarrow S^2$ be the Gauss map, i.e. $\nu(x)$ is the oriented unit normal vector to $T_x\Sigma \subset T_x\mathbb{R}^3$ for each $x \in \Sigma$. Show that the degree of ν is $1-g$.*

Let $n \in \mathbb{Z}$ with $n \geq 2$. Suppose the 2-torus $\mathbb{T}^2$ is embedded in the open unit ball

$$B_1^3 \subset \mathbb{R}^3 = \mathbb{R}^3 \times \{0\} \times \{0\} \subset \mathbb{R}^3 \times \mathbb{R} \times \mathbb{C}^{n-2} = \mathbb{C}^n$$

in a standard way. Let $z_j \equiv x_j + \mathfrak{i} y_j$ be the standard coordinates on $\mathbb{C}^n$.

(c) *Show that the Gauss map $\nu\colon \mathbb{T}^2 \longrightarrow S^2$ for this $\mathbb{T}^2 \subset \mathbb{R}^3$ extends to a smooth null-homotopic map*

$$\widetilde{\nu}\colon \big(\mathbb{C}^n, \mathbb{C}^n - B_2^{2n}\big) \longrightarrow \big(S^2, (0,0,1)\big).$$

(d) *Let $J_{\mathbb{C}^n}$ be the standard complex structure on $\mathbb{C}^n$ and $\mathfrak{j}$ be an almost complex structure on $\mathbb{T}^2$. Show that there exists a continuous family $(J_t)_{t\in[0,1]}$ of almost complex structures on $\mathbb{C}^n$ so that*

$$J_t\big(\mathbb{C}^n \times (\mathbb{C}^2 \times \{0\})\big) \subset \mathbb{C}^n \times (\mathbb{C}^2 \times \{0\}), \quad J_t\big(\mathbb{C}^n \times (\{0\} \times \mathbb{C}^{n-2})\big) \subset \mathbb{C}^n \times (\{0\} \times \mathbb{C}^{n-2}),$$
$$J_t\big|_{\mathbb{C}^n - B_2^{2n}} = J_{\mathbb{C}^n}\big|_{\mathbb{C}^n - B_2^{2n}}, \quad J_0\big|_{T\mathbb{T}^2} = \mathfrak{j}, \quad J_0\widetilde{\nu} = \frac{\partial}{\partial y_2}, \quad J_1 = J_{\mathbb{C}^n}.$$

Let (M, J) be an almost complex manifold of dimension at least 4 and $U \subset M$ be an nonempty open subset.

(e) *Let $x \in U$. Show that there exists a continuous family $(J_t)_{t\in[0,1]}$ of almost complex structures on M so that $J_0 = J$, $J_t|_{M-U} = J|_{M-U}$ for every $t \in [0,1]$, and J_1 is integrable on some neighborhood of x.*

(f) *Show that there exist an embedded null-homologous 2-torus $\mathbb{T}^2 \subset U$ and a continuous family $(J_t)_{t\in[0,1]}$ of almost complex structures on M so that $J_0 = J$, $J_t|_{M-U} = J|_{M-U}$ for every $t \in [0,1]$, and $\mathbb{T}^2$ is J_1-holomorphic (i.e. $J_1(T\mathbb{T}^2) \subset T\mathbb{T}^2$).*

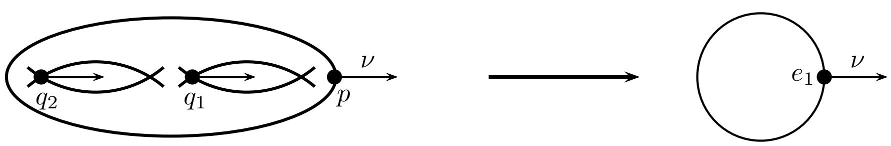
Note. This problem details the proof of Proposition 2.7 in math/2401.17381. Its implication is that every almost complex structure J on a manifold of dimension at least 4 can be homotoped within any nonempty open subset U of M to an almost complex structure J' not tamed by a symplectic form.

(a) Since homotopic maps induce the same maps in homology, a null-homotopic map $f\colon \Sigma \longrightarrow S^2$ induces the trivial map on the second homology and is thus of degree 0. The surface Σ can be written as a CW complex with a single 2-cell attached to the 1-skeleton, which is a union of circles joined at finitely many points. Since S^2 is simply connected, the restriction of any continuous map $f\colon \Sigma \longrightarrow S^2$ to the 1-skeleton of Σ can be homotoped to a constant map. Thus, any continuous map $f\colon \Sigma \longrightarrow S^2$ is a homotopic to a continuous map g which sends the entire 1-skeleton of Σ and thus the boundary of the 2-cell to a point. Such a map g is equivalent to a continuous map $S^2 \longrightarrow S^2$. Since the Hurewicz and degree homomorphisms

$$\pi_2(S^2) \longrightarrow H_2(S^2; \mathbb{Z}) \longrightarrow \mathbb{Z}$$

are isomorphisms, f and g are null-homotopic if and only if they have degree 0.
Note: The statement of (a) is the $m=2$ of *Theorem of Hopf* on p51 in Milnor's *Topology from the Differentiable Viewpoint.* The proof for arbitrary m is the same.

(b) Let x, y, z be the usual coordinates on $\mathbb{R}^3$. Embed Σ into $\mathbb{R}^3$ so that it is "lying horizontally" and centered around the x-axis, as indicated in the picture below. Its intersection with the xy-plane then consists of a large ellipse and g ellipses inside of it, all symmetric about the x-axis. The preimage of $e_1 \equiv (1,0,0)$ under the Gauss map ν consists of $1+g$ points of Σ lying in the xy-plane: the rightmost point q of Σ (and of the large ellipse) and the leftmost points $q_1, \ldots, q_g$ of the g inside ellipses. We can choose local *oriented* coordinates (s,t) centered at $e_1, p, q_1, \ldots, q_g$ (using the exponential map, for example) so that the s-axis stays in the xy-plane with $y'(s) > 0$ and the t-axis stays in the xz-plane with $z'(t) > 0$. The Gauss map ν sends the t-axis at $p, q_1, \ldots, q_g$ to the t-axis at e_1 preserving the direction. It sends the s-axis at $p, q_1, \ldots, q_g$ to the s-axis at e_1, preserving the direction at p and reversing the direction at $q_1, \ldots, q_g$. Thus, ν is orientation-preserving at p and orientation-reversing at $q_1, \ldots, q_g$. Therefore, the degree of ν is $1-g$. The advantage of this approach over those in Notes 2 and 3 below is that it is not based on other nontrivial results (that the signed number of zeros of a vector field on a closed oriented manifold is its Euler characteristic in Note 2 or the Gauss-Bonnet Theorem in Note 3).



Note 1. The degree of the Gauss map ν is in fact independent of the embedding of Σ. Let $Y \subset \mathbb{R}^3$ be the bounded region cut out by $\Sigma \subset \mathbb{R}^3$; it is a compact 3-manifold with boundary Σ. Take a vector field ξ that equals some constant $v \in S^2$ outside of a tubular neighborhood U of Σ, ν on Σ, and connect the two by a straight line homotopy along the fibers of $U \longrightarrow \Sigma$. The zeros of ξ then lie in fibers of U over the points $x \in \Sigma$ with $\nu(x) = -v$. The signed number of these zeros is the signed cardinality of $\nu^{-1}(-v)$, i.e. the degree of ν if v is chosen generically; this is the $m=3$ case of Lemma 6.3 in Milnor's *Topology from the Differentiable Viewpoint.* If Y_1 and Y_2 are the bounded regions for two different embeddings of Σ in $\mathbb{R}^3$, we can glue them together along with Σ to obtain a closed manifold Y with an orientation that agrees with that of Y_1 and with the opposite of that of Y_2. We can also glue the vector field ξ_1 on Y_1 above with the vector field $-\xi_2$ on Y_2 to obtain a vector field ξ. The signed number of zeros of ξ is the signed number of zeros of ξ_1 minus the signed number of zeros on ξ_2 because the multiplication by -1 is orientation-reversing on the 3-dimensional fibers of $TY_2 \longrightarrow Y_2$. This signed number is also the Euler characteristic $\chi(Y)$ of Y, which is 0 because Y is an odd-dimensional manifold. Thus, the degree of the Gauss map ν for the first embedding of Σ (which is the signed number of zeros of ξ_1) is the same as the degree of the Gauss map ν for the

second embedding of Σ (which is the signed number of zeros of ξ_2).

Note 2. Let $\xi\in\Gamma(S^2;TS^2)$ be a vector field with transverse zeros that are regular values of the Gauss map $\nu\colon\Sigma\longrightarrow S^2$. The signed number of zeros of ξ is $\chi(S^2)=2$. Since $T_{\nu(x)}S^2=T_x\Sigma$ for every $x\in\Sigma$, $\zeta\equiv\xi\circ\nu$ is a vector field on Σ. The zeros of ζ are the preimages of the zeros of ξ under ν. The sign of $x\in\zeta^{-1}(0)$ as a zero of ζ is the sign of $\nu(x)\in\xi^{-1}(0)$ as a zero of ξ times the sign of $\mathrm{d}_x\nu$. Thus, the signed number of zeros of ζ is the signed number of zeros of ξ times the degree of ν, i.e. $2(\deg\nu)$. On the other hand, this number is also $\chi(\Sigma)=2(1-g)$. This argument extends to $2n$-dimensional closed manifolds embedded in $\mathbb{R}^{2n+1}$ (the degree of the Gauss map is half the Euler characteristic of the manifold).

Note 3. Another way to obtain (b) for any embedding $\Sigma\subset\mathbb{R}^3$ is via the Gauss-Bonnet Theorem (Corollary 2 on p276 in do Carmo's *Differential Geometry of Curves and Surfaces*). For $x\in\Sigma$, the Gaussian curvature K_Σ of Σ at x is the determinant of the differential

$$\mathrm{d}_x\nu\colon T_x\Sigma\longrightarrow T_{\nu(x)}S^2=T_x\Sigma$$

of ν at x. Let ω_Σ and ω_{S^2} be the volume forms on Σ and S^2, respectively, determined by the Riemannian metric on $\mathbb{R}^3$. Thus,

$$4\pi(\deg\nu)=(\deg\nu)\int_{S^2}\omega_{S^2}\equiv\int_\Sigma\nu^*\omega_{S^2}=\int_\Sigma(\det\mathrm{d}\nu)\omega_\Sigma=\int_\Sigma K_\Sigma\omega_\Sigma=2\pi\chi(\Sigma)=4\pi(1-g);$$

the penultimate equality above is the Gauss-Bonnet Theorem.

(c) Since the normal bundle of Σ in $B_1^3\subset\mathbb{R}^3$ is trivial, there are an open subset $U\subset B_1^3$ and a diffeomorphism

$$\Psi\colon(-1,1)\times\Sigma\longrightarrow U\qquad\text{s.t.}\quad\Psi(0,x)=x\ \ \forall\,x\in\Sigma.$$

We define an open subset $W\subset\mathbb{C}^n=\mathbb{R}^3\times\mathbb{R}^{2n-3}$ and a compact subset $K\subset W$ by

$$\begin{aligned}W&=\big\{(\Psi(s,x),w)\colon(s,x)\in(-1,1)\times\Sigma,\ w\in\mathbb{R}^{2n-3},\ |s|+|w|<1\big\},\\ K&=\big\{(\Psi(s,x),w)\colon(s,x)\in(-1,1)\times\Sigma,\ w\in\mathbb{R}^{2n-3},\ |s|+|w|\le3/4\big\}.\end{aligned}$$

By (b), the degree of ν is 0. By (a), there thus exists a smooth map

$$H\colon[0,1]\times\Sigma\longrightarrow S^2\subset\mathbb{R}^3\quad\text{s.t.}\quad H(t,x)=\nu(x),\ H(1-t,x)=(0,0,1)\ \ \forall\,t\in[0,1/4],\,x\in\Sigma.$$

The map $\widetilde{\nu}\colon\mathbb{C}^n\longrightarrow S^2$ given by

$$\widetilde{\nu}(z)=\begin{cases}H(|s|+|w|,x), & \text{if } z=(\Psi(s,x),w)\in W;\\(0,0,1), & \text{if } z\in\mathbb{C}^n-K;\end{cases}$$

is then a well-defined smooth extension of ν with $\nu(\mathbb{C}^n-B_2^{2n})=\{(0,0,1)\}$. The map

$$\widetilde{H}_\nu\colon[0,1]\times\mathbb{C}^n\longrightarrow S^2,\quad\widetilde{H}_\nu(t,z)=\begin{cases}H(t+(1-t)(|s|+|w|),x), & \text{if } z=(\Psi(s,x),w)\in W;\\(0,0,1), & \text{if } z\in\mathbb{C}^n-K;\end{cases}$$

is a well-defined smooth homotopy from $\widetilde{\nu}$ to a constant map such that $\widetilde{H}_\nu(t,z)=(0,0,1)$ for all $t\in[0,1]$ and $z\in\mathbb{C}^n-B_2^{2n}$.

(d) Since $\mathbb{C}^n$ is a vector space, $T\mathbb{C}^n$ is canonically isomorphic to $\mathbb{C}^n\times\mathbb{C}^n$ as real vector bundles over $\mathbb{C}^n$. The latter bundle is the direct sum of the subbundles $\mathbb{C}^n\times(\mathbb{C}^2\times\{0\}^{n-2})$ and $\mathbb{C}^n\times(\{0\}^2\times\mathbb{C}^{n-2})$ with trivializing frames

$$\frac{\partial}{\partial x_1},\frac{\partial}{\partial y_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial y_2} \qquad \text{and} \qquad \frac{\partial}{\partial x_3},\frac{\partial}{\partial y_3},\ldots,\frac{\partial}{\partial x_n},\frac{\partial}{\partial y_n},$$

respectively. With $\widetilde{H}_\nu$ as in the proof of (c), let $\widetilde{\nu}_t=H(t,\cdot)\colon \mathbb{C}^n\longrightarrow S^2$ for each $t\in[0,1]$. We define the almost complex structure J_t on the span of $\widetilde{\nu}_t$, $\frac{\partial}{\partial y_2}$, and the second summand above by

$$J_t\widetilde{\nu}_t=\frac{\partial}{\partial y_2},\quad J_t\frac{\partial}{\partial y_2}=-\widetilde{\nu}_t,\qquad J_t\frac{\partial}{\partial x_j}=\frac{\partial}{\partial y_j},\quad J_t\frac{\partial}{\partial y_j}=-\frac{\partial}{\partial x_j}\quad \forall\, j=3,\ldots,n.$$

Let $g_{\mathbb{R}^3}$ be the standard metric on the real vector bundle $\mathbb{C}^n\times\mathbb{R}^3$ over $\mathbb{C}^n$ and $g_{\mathbb{R}^3;\mathbb{T}^2}$ be a metric on $T\mathbb{R}^3|_\Sigma=T\Sigma\oplus\mathbb{R}\nu$ so that the metric $g_{\mathbb{R}^3;\mathbb{T}^2}|_{T\Sigma}$ is compatible with the almost complex structure $\mathfrak{j}$ on Σ and $T\Sigma$ is $g_{\mathbb{R}^3;\mathbb{T}^2}$-orthogonal to $\mathbb{R}\nu$. Since the space of metrics is convex and $\Sigma\subset B_1^{2n}$ is a neighborhood deformation retract, there exists a continuous family of metrics $(g_t)_{t\in[0,1]}$ on the real vector bundle $\mathbb{C}^n\times\mathbb{R}^3$ over $\mathbb{C}^n$ such that

$$g_t\big|_{\mathbb{C}^n-B_2^{2n}}=g_{\mathbb{R}^3},\qquad g_0\big|_\Sigma=g_{\mathbb{R}^3;\mathbb{T}^2}\,,\qquad g_1=g_{\mathbb{R}^3}\,.$$

Let $\pi_t\subset\mathbb{C}^n\times\mathbb{R}^3$ be the g_t-orthogonal complement of $\mathbb{R}\widetilde{\nu}_t$. The standard orientation of $\mathbb{R}^3$ and $\widetilde{\nu}_t$ determine an orientation on π_t. Along with the latter, the metric $g_t|_{\pi_t}$ determines a compatible complex structure J_t on π_t.

Since g_0 restricts to $g_{\mathbb{R}^3;\mathbb{T}^2}$ over Σ, the restriction of J_0 to $\pi_0|_{\Sigma_0}=T\Sigma$ agrees with $\mathfrak{j}$ (because $\mathfrak{j}$ is *the* complex structure on $T\Sigma$ compatible with the metric $g_{\mathbb{R}^3;\mathbb{T}^2}|_{T\Sigma}$ and the orientation of $T\Sigma$). Since g_t and ν_t restrict to $g_{\mathbb{R}^3}$ and $\frac{\partial}{\partial x_2}$, respectively, over $\mathbb{C}^n-B_2^{2n}$, the restriction of J_t to

$$\pi_t\big|_{\mathbb{C}^n-B_2^{2n}}=\big(\mathbb{C}^n-B_2^{2n}\big)\times\big(\mathbb{C}\times\{0\}^{n-1}\big)$$

agrees with $J_{\mathbb{C}^n}$. By the same reasoning, $J_1=J_{\mathbb{C}^n}$.

(e) We can assume that $M=B_2^{2n}$, $U=B_1^{2n}$, $x=0$, J_0 agrees with the standard complex structure $J_{\mathbb{C}^n}$ at $0\in\mathbb{C}^n$, and J_0 is tamed by the standard symplectic form $\omega_{\mathbb{C}^n}$ on B_2^{2n}. Let $\eta\colon\mathbb{R}\longrightarrow[0,1]$ be a smooth function such that

$$\eta(r)=\begin{cases}1, & \text{if } r<1/4;\\ 0, & \text{if } r>1.\end{cases}$$

With Φ as in Exercise 2.4 of the *Notes*, the map

$$J_\bullet\colon[0,1]\times B_2^{2n}\longrightarrow \mathrm{GL}_{2n}\mathbb{R},\qquad J_t(x)=J_{\mathbb{C}^n}\Phi\big((1-t\eta(|x|))\Phi(J_{\mathbb{C}^n}^{-1}J(x))\big),$$

is then a smooth family of almost complex structures on B_2^{2n} with the required properties.

(f) By (e), we can take $M=B_3^{2n}$, $U=B_2^{2n}$, and $J=J_{\mathbb{C}^n}$. The claim now follows from (d).