MAT 562: Symplectic Geometry

Solution to Problem C

Let M be a smooth manifold and G be a compact Lie group acting smoothly on M, i.e. there is a group homomorphism $\rho\colon G\longrightarrow \mathrm{Diff}(M)$ such that the map

$$G\times M \longrightarrow M, \qquad (g,x)\longrightarrow \{\rho(g)\}(x),$$

is smooth.

(a) *Suppose $x\in M$ is a fixed point of this action, i.e. $\{\rho(g)\}(x)=x$ for every $g\in G$. Show that the G-action on M induces a linear G-action on T_xM and there are a neighborhood U of $0\in T_xM$, a neighborhood W of $x\in M$, and a G-equivariant diffeomorphism $h\colon U\longrightarrow W$ with $h(0)=x$.*

(b) *Show that every connected component of the G-fixed locus,*

$$M^G \equiv \big\{x\in M\colon \{\rho(g)\}(x)=x \ \forall\, g\in G\big\},$$

is a smooth submanifold of M.

(c) *Suppose in addition ω is a symplectic form on M such that $\{\rho(g)\}^*\omega=\omega$ for every $g\in G$. Show that every connected component of M^G is an ω-symplectic submanifold of M.*

The compactness of the Lie group G is needed only to guarantee the existence of a G-invariant Riemannian metric on M, i.e. a positive-definite symmetric inner-product $\langle\cdot,\cdot\rangle$ on the fibers of TM so that

$$\big\langle\{\mathrm{d}_x\rho(g)\}(v),\{\mathrm{d}_x\rho(g)\}(w)\big\rangle=\langle v,w\rangle \qquad \forall\ x\in M,\ v,w\in T_xM.$$

Let μ be the (left-invariant) Haar measure of volume 1 on G, i.e. $\mu(G)=1$ and $\mu(gS)=\mu(S)$ for every measurable subset $S\subset G$ and $g\in G$. Let $\langle\cdot,\cdot\rangle'$ be any Riemannian metric on M. We then define a G-invariant Riemannian metric on M by

$$\langle v,w\rangle=\int_G\big\langle\{\mathrm{d}_x\rho(g^{-1})\}(v),\{\mathrm{d}_x\rho(g^{-1})\}(w)\big\rangle'\mathrm{d}\mu \qquad \forall\, x\in M,\ v,w\in T_xM.$$

If $v\in TM$, $\gamma_v\colon(a,b)\longrightarrow M$ with $a<0<b$ is the geodesic with respect to the Levi-Civita connection ∇ of a G-invariant metric $\langle\cdot,\cdot\rangle$ on M with $\gamma_v'(0)=v$, and $g\in G$, then

$$g\cdot\gamma_v\colon(a,b)\longrightarrow M, \qquad \{g\cdot\gamma_v\}(t)=\{\rho(g)\}\big(\gamma_v(t)\big) \ \ \forall\, t\in(a,b),$$

is the geodesic with respect to ∇ with $(g\cdot\gamma_v)'(0)=\{\mathrm{d}_{\gamma_v(0)}\rho(g)\}(v)$, i.e. $g\cdot\gamma_v=\gamma_{\{\mathrm{d}_{\gamma_v(0)}\rho(g)\}(v)}$. Thus, the exponential map

$$\exp\colon\mathcal{U}\longrightarrow M, \qquad \exp(v)=\gamma_v(1) \ \ \forall\, v\in\mathcal{U}\subset TM, \tag{1}$$

with respect to ∇ satisfies

$$\{\mathrm{d}\rho(g)\}(\mathcal{U})=\mathcal{U} \quad \text{and} \quad \exp\big(\{\mathrm{d}\rho(g)\}(v)\big)=g\cdot\exp(v) \qquad \forall\ g\in G,\ v\in\mathcal{U}, \tag{2}$$

i.e. it is G-equivariant with respect to the G-action on TM given by

$$\mathrm{d}\rho\colon G\longrightarrow\mathrm{Diff}(TM), \qquad g\longrightarrow \mathrm{d}\rho(g), \tag{3}$$

and the original G-action on M. By the Chain Rule, the above map $\mathrm{d}\rho$ is indeed a (left) group action.

(a) Since $\{\rho(g)\}(x)=x$ for every $g\in G$, $\{\mathrm{d}\rho(g)\}(T_xM)\subset T_xM$ for every $g\in G$. Thus, the $\mathrm{d}\rho$ action of G on M in (3) restricts to a G-action on T_xM by the linear automorphisms $\{\mathrm{d}_x\rho(g)\}$. By (2), the restriction

$$\exp_x \equiv \exp|_{T_xM\cap\mathcal{U}}\colon T_xM\cap\mathcal{U} \longrightarrow M$$

of the exponential map (1) with respect to a G-invariant metric on M is G-equivariant. Since the differential of this map at $0\in T_xM$ is the identity on T_xM, the Inverse Function Theorem implies that there exists a neighborhood U' of 0 in $T_xM\cap\mathcal{U}$ so that $\exp_x|_{U'}$ is a diffeomorphism on an open subset of M. Since G is compact,

$$U \equiv \bigcap_{g\in G}\{\mathrm{d}_x\rho(g)\}(U') \subset T_xM\cap\mathcal{U}$$

is a neighborhood of 0 preserved by the G-action. The restriction

$$h\equiv \exp_x|_U\colon U \longrightarrow W\equiv \exp_x(U)$$

a G-equivariant diffeomorphism onto a neighborhood of x in M such that $h(0)=x$.

(b) Let $x\in M^G$ and $h\colon U\longrightarrow W$ be as in part (a). Since h is G-equivariant,

$$h\colon U^G\equiv\big\{v\in U\colon \{\mathrm{d}_x\rho(g)\}(v)=v\ \forall\, g\in G\big\} \longrightarrow W^G=M^G\cap W$$

is a homeomorphism. Since G acts linearly on T_xM, U^G is an open neighborhood of 0 in the linear subspace

$$(T_xM)^G \equiv \big\{v\in T_xM\colon \{\mathrm{d}_x\rho(g)\}(v)=v\ \forall\, g\in G\big\} \subset T_xM$$

and is thus a submanifold of U in the subspace topology. We conclude that every point $x\in M^G$ has a neighborhood $W_x\subset M$ so that $M^G\cap W_x$ is a submanifold of M in the subspace topology. This implies that every connected component of M^G is a smooth submanifold of M; see 1.33(b) in Warner's textbook.

(c) Let $x\in M^G$. By part (b), $T_x(M^G)=(T_xM)^G$. With $\langle\cdot,\cdot\rangle$ denoting a G-invariant metric on M, define

$$A_x\colon T_xM \longrightarrow T_xM \qquad \text{by} \quad \big\langle A_x(v),w\big\rangle = \omega(v,w)\ \ \forall\, v,w\in T_xM.$$

Since $\langle\cdot,\cdot\rangle$ and ω are G-invariant, so is A_x, i.e.

$$A_x\big(\{\mathrm{d}_x\rho(g)\}(v)\big) = \{\mathrm{d}_x\rho(g)\}\big(A_x(v)\big) \qquad \forall\ g\in G,\ v\in T_xM.$$

Thus, $A_x((T_xM)^G)\subset(T_xM)^G$. Since ω is nondegenerate on T_xM, A_x is invertible,

$$A_x^{-1}\big((T_xM)^G\big)\subset(T_xM)^G, \qquad \text{and} \qquad \omega\big(A_x^{-1}(v),v\big)=\langle v,v\rangle>0\ \ \forall\, v\in T_xM-\{0\}.$$

Thus, the restriction of ω to $T_x(M^G)=(T_xM)^G$ is nondegenerate as well. This establishes the claim.