MAT 562: Symplectic Geometry

Solution to Problem A

The $\mathbb{C}^*$-action on $\mathbb{C}^n$ by the coordinate multiplication restricts to a $\mathbb{C}^*$-action on $\mathbb{C}^n-\{0\}$ and S^1-actions on $\mathbb{C}^n$ and the unit sphere $S^{2n-1}\subset\mathbb{C}^n$. Show that

(a) the quotient topologies on $\mathbb{C}P^{n-1}$ given by $(\mathbb{C}^n-\{0\})/\mathbb{C}^*$ and S^{2n-1}/S^1 are the same (i.e. the map $S^{2n-1}/S^1\longrightarrow(\mathbb{C}^n-\{0\})/\mathbb{C}^*$ induced by inclusions is a homeomorphism);

(b) $\mathbb{C}P^{n-1}$ is a compact topological $2(n-1)$-manifold that admits a complex structure so that the quotient projections

$$q\colon \mathbb{C}^n-\{0\}\longrightarrow \mathbb{C}P^{n-1}=(\mathbb{C}^n-\{0\})/\mathbb{C}^* \quad \textit{and} \quad p\colon S^{2n-1}\longrightarrow \mathbb{C}P^{n-1}=S^{2n-1}/S^1$$

are a holomorphic submersion and a smooth submersion, respectively;

(c) the S^1-action on $\mathbb{C}^n$ preserves the standard symplectic form $\omega_{\mathbb{C}^n}$ on $\mathbb{C}^n$;

(d) the orbits of the restriction of this action to S^{2n-1} are compact connected one-dimensional submanifolds of S^{2n-1};

(e) for each $z\in S^{2n-1}$ the $\omega_{\mathbb{C}^n}$-symplectic complement of T_zS^{2n-1},

$$\big(T_zS^{2n-1}\big)^{\omega_{\mathbb{C}^n}} \equiv \big\{v\in T_z\mathbb{C}^n\colon \omega_{\mathbb{C}^n}(v,w)=0\ \forall\, w\in T_zS^{2n-1}\big\},$$

is the tangent space to the S^1-orbit at z;

(f) there is a unique 2-form $\omega_{\mathbb{C}P^{n-1}}$ on $\mathbb{C}P^{n-1}$ such that $p^*\omega_{\mathbb{C}P^{n-1}}=\omega_{\mathbb{C}^n}|_{TS^{2n-1}}$, and this form $\omega_{\mathbb{C}P^{n-1}}$ is symplectic.

(a) Let $\widetilde{i}\colon S^{2n-1}\longrightarrow\mathbb{C}^{n-1}-0$ and $\widetilde{r}\colon\mathbb{C}^{n-1}-0\longrightarrow S^{2n-1}$ denote the inclusion and the natural retraction, i.e. $\widetilde{r}(v)=v/|v|$. We show that these maps descend to continuous maps between the quotients, i and r below,

$$\begin{array}{ccc} S^{2n-1} & \xrightarrow{\ \widetilde{i}\ } & \mathbb{C}^{n-1}-0 \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ S^{2n-1}/S^1 & \overset{i}{\dashrightarrow} & (\mathbb{C}^{n-1}-0)/\mathbb{C}^* \end{array} \qquad\qquad \begin{array}{ccc} \mathbb{C}^{n-1}-0 & \xrightarrow{\ \widetilde{r}\ } & S^{2n-1} \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ (\mathbb{C}^{n-1}-0)/\mathbb{C}^* & \overset{r}{\dashrightarrow} & S^{2n-1}/S^1 \end{array}$$

that are inverses of each other.

The map $q\circ\widetilde{i}$ is constant on the fibers of p, since if $v,w\in S^{2n-1}$ and $w=g\cdot v$ for some $g\in S^1$, then $\widetilde{i}(w)=g'\cdot\widetilde{i}(v)$ for some $g'\in\mathbb{C}^*$ (in fact, $g'=g$). Thus, $q\circ\widetilde{i}$ induces a map i from the quotient space S^{2n-1}/S^1 (so that the first diagram commutes); since the map $q\circ\widetilde{i}$ is continuous, so is the induced map i. Similarly, the map $p\circ\widetilde{r}$ is constant on the fibers of q, since if $v,w\in\mathbb{C}^{n-1}-0$ and $w=g\cdot v$ for some $g\in\mathbb{C}^*$, then $\widetilde{r}(w)=g'\cdot\widetilde{r}(v)$ for some $g'\in S^1$ (in fact, $g'=g/|g|$). Thus, $p\circ\widetilde{r}$

induces a map r from the quotient space $(\mathbb{C}^{n-1}-0)/\mathbb{C}^*$; since the map $p\circ\widetilde{r}$ is continuous, so is the induced map r. Since $\widetilde{r}\circ\widetilde{i}=\mathrm{id}_{S^{2n-1}}$, $r\circ i=\mathrm{id}_{S^{2n-1}/S^1}$. Similarly, for all $v\in\mathbb{C}^{n-1}-0$,

$$\widetilde{i}\circ\widetilde{r}(v)=(1/|v|)v,\quad 1/|v|\in\mathbb{C}^* \quad\Longrightarrow\quad q\big(\widetilde{i}\circ\widetilde{r}(v)\big)=q(v) \quad\Longrightarrow\quad i\circ r=\mathrm{id}_{(\mathbb{C}^{n-1}-0)/\mathbb{C}^*}.$$

(b-i) Since S^{2n-1} is compact, so is the quotient space $\mathbb{C}P^{n-1}=S^{2n-1}/S^1$ (being the image of S^{2n-1} under the continuous map p). For any $A\subset S^{2n-1}$,

$$p^{-1}\big(p(A)\big)=S^1\cdot A\equiv\big\{g\cdot v\colon v\in A,\ g\in S^1\big\}=\bigcup_{g\in S^1}g^{-1}(A).$$

Thus, $p^{-1}(p(A))$ is the image of the subset $S^1\times A$ in S^{2n-1} under the continuous multiplication map

$$S^1\times S^{2n-1}\longrightarrow S^{2n-1},\qquad (g,v)\longrightarrow g\cdot v,$$

and is the union over $g\in S^1$ of the preimages $g^{-1}(A)$ of A under the continuous map

$$g\colon S^{2n-1}\longrightarrow S^{2n-1},\qquad v\longrightarrow g\cdot v.$$

If A is closed in S^{2n-1}, then $S^1\times A$ is closed in the compact space $S^1\times S^{2n-1}$ and thus compact. It then follows from the first statement above that $p^{-1}(p(A))$ is a compact subset of the Hausdorff space S^{2n-1} and thus closed. We conclude that $p(A)\subset S^{2n-1}/S^1$ is closed for all closed subsets $A\subset S^{2n-1}$, i.e. the quotient map p is a closed map. Since S^{2n-1} is normal, by Lemma 73.3 in Munkres's *Topology* the quotient space $\mathbb{C}P^{n-1}$ is normal as well (and in particular, Hausdorff). If A is open in S^{2n-1}, then $g^{-1}(A)$ is also open in S^{2n-1}. It then follows from the second statement above that $p^{-1}(p(A))$ is open in S^{2n-1} as well. We conclude that $p(A)\subset S^{2n-1}/S^1$ is open for all open subsets $A\subset S^{2n-1}$, i.e. the quotient map p is an open map. Since S^{2n-1} is second countable, the quotient space $\mathbb{C}P^{n-1}$ is therefore also second countable.

(b-ii) We now construct a collection of charts $\{(\mathcal{U}_i,\varphi_i)\}_{i=1,\ldots,n}$ on $\mathbb{C}P^{n-1}$ that covers $\mathbb{C}P^{n-1}$. Given a point $(X_1,\ldots,X_n)\in\mathbb{C}^n-0$, we denote its equivalence class in

$$\mathbb{C}P^{n-1}=(\mathbb{C}^n-0)/\mathbb{C}^*$$

by $[X_1,\ldots,X_n]$. For $i=1,\ldots,n$, let

$$\mathcal{U}_i=\big\{[X_1,\ldots,X_n]\in\mathbb{C}P^{n-1}\colon X_i\neq 0\big\}.$$

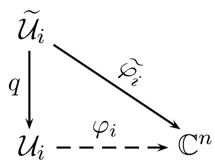
Since

$$q^{-1}(\mathcal{U}_i)=\big\{(X_1,\ldots,X_n)\in\mathbb{C}^n-0\colon X_i\neq 0\big\}\equiv\widetilde{\mathcal{U}}_i$$

is an open subset of $\mathbb{C}^n-0$, $\mathcal{U}_i$ is an open subset of $\mathbb{C}P^{n-1}$. Define

$$\widetilde{\varphi}_i\colon\widetilde{\mathcal{U}}_i\longrightarrow\mathbb{C}^{n-1}=\mathbb{R}^{2(n-1)}\qquad\text{by}$$
$$\widetilde{\varphi}_i(X_1,\ldots,X_n)=\big(X_1/X_i,X_2/X_i,\ldots,X_{i-1}/X_i,X_{i+1}/X_i,\ldots,X_n/X_i\big).$$

Since $\widetilde{\varphi}_i(c\cdot v)=\widetilde{\varphi}_i(v)$, the map $\widetilde{\varphi}_i$ induces a map φ_i from the quotient space $\mathcal{U}_i$ of $\widetilde{\mathcal{U}}_i$:



Since $\widetilde{\varphi}_i$ is continuous, so is φ_i. Define

$$\psi_i\colon \mathbb{C}^{n-1} \longrightarrow \mathcal{U}_i \qquad \text{by} \qquad \psi_i(z_1,\ldots,z_n) = \big[z_1,\ldots,z_{i-1},X_i\!=\!1,z_i,\ldots,z_{n-1}\big].$$

Since ψ_i is a composition of two continuous maps, ψ_i is continuous. Since $\psi_i\circ\varphi_i = \mathrm{id}_{\mathcal{U}_i}$ and $\varphi_i\circ\psi_i = \mathrm{id}_{\mathbb{C}^{n-1}}$, the map

$$\varphi_i\colon \mathcal{U}_i \longrightarrow \mathbb{C}^{n-1}$$

is a homeomorphism. For every $p\equiv[X_1,\ldots,X_n]\in\mathbb{C}P^{n-1}$, there exists $i=1,\ldots,n$ such that $X_i\neq 0$, i.e. $p\in\mathcal{U}_i$. Thus, $\{(\mathcal{U}_i,\varphi_i)\}_{i=1,\ldots,n}$ is a collection of charts on $\mathbb{C}P^n$ that covers $\mathbb{C}P^{n-1}$. In particular, $\mathbb{C}P^{n-1}$ is locally Euclidean of dimension $2n$. Since this collection of charts is countable (actually, finite), it follows that $\mathbb{C}P^{n-1}$ is second countable (since each open subset $\mathcal{U}_i$ is second countable).

(b-iii) We now determine the overlap maps

$$\varphi_i\circ\varphi_j^{-1} = \varphi_i\circ\psi_j\colon \varphi_j(\mathcal{U}_i\cap\mathcal{U}_j) \longrightarrow \varphi_i(\mathcal{U}_i\cap\mathcal{U}_j).$$

Assume that $j<i$. Then,

$$\begin{aligned}
\mathcal{U}_i\cap\mathcal{U}_j &= \big\{[X_1,\ldots,X_n]\in\mathbb{C}P^{n-1}\colon X_i,X_j\neq 0\big\} \qquad \Longrightarrow\\
\varphi_j(\mathcal{U}_i\cap\mathcal{U}_j) &= \big\{(z_1,\ldots,z_{n-1})\in\mathbb{C}^{n-1}\colon z_{i-1}\neq 0\big\} \equiv \mathbb{C}^{n-1}_{i-1},\\
\varphi_i(\mathcal{U}_i\cap\mathcal{U}_j) &= \big\{(z_1,\ldots,z_{n-1})\in\mathbb{C}^{n-1}\colon z_j\neq 0\big\} \equiv \mathbb{C}^{n-1}_{j};
\end{aligned}$$

the assumption $j<i$ is used on the last two lines. By (b-ii), the map

$$\varphi_i\circ\varphi_j^{-1}\colon \mathbb{C}^n_{i-1} \longrightarrow \mathbb{C}^n_j$$

is given by

$$\begin{aligned}
\varphi_i\circ\varphi_j^{-1}(z_1,\ldots,z_{n-1}) &= \varphi_i\circ\psi_j(z_1,\ldots,z_{n-1}) = \varphi_i\big([z_1,\ldots,z_{j-1},X_j\!=\!1,z_j,\ldots,z_{n-1}]\big)\\
&= \big(z_1/z_i,\ldots,z_{j-1}/z_i,1/z_i,z_j/z_i,\ldots,z_{i-1}/z_i,z_{i+1}/z_i,\ldots,z_{n-1}/z_i\big).
\end{aligned}$$

Thus, the overlap map $\varphi_i\circ\varphi_j^{-1}$ is holomorphic on its domain, as is its inverse, $\varphi_j\circ\varphi_i^{-1}$; both maps are given by rational functions on $\mathbb{C}^{n-1}$. We conclude that the collection $\mathcal{F}_0=\{(\mathcal{U}_i,\varphi_i)\}_{i=1,\ldots,n}$ determines a complex structure on $\mathbb{C}P^{n-1}$.

(b-iv) For each $i=1,\ldots,n$, the composition $\varphi_i\circ q|_{\widetilde{\mathcal{U}}_i}=\widetilde{\varphi}_i$ is a holomorphic submersion (even when restricted to the slices with X_i fixed). Thus, $q|_{\widetilde{\mathcal{U}}_i}$ is a holomorphic submersion. Since the open subsets $\mathcal{U}_1,\ldots,\mathcal{U}_n$ cover $\mathbb{C}^n-\{0\}$, it follows that the entire projection q is a submersion. Since $p=q\circ\widetilde{i}$, the projection p is also smooth. Since $q=p\circ\widetilde{r}$, p is a submersion as well.

(c) For $u \in S^1$,

$$u^*\omega_{\mathbb{C}^n} = u^*\bigg(\frac{\mathrm{i}}{2}\sum_{j=1}^{n}\mathrm{d}z_j\wedge\mathrm{d}\overline{z_j}\bigg) = \frac{\mathrm{i}}{2}\sum_{j=1}^{n}\mathrm{d}(u^*z_j)\wedge\mathrm{d}(u^*\overline{z_j}) = \frac{\mathrm{i}}{2}\sum_{j=1}^{n}\mathrm{d}(uz_j)\wedge\mathrm{d}(\overline{uz_j})$$
$$= \frac{\mathrm{i}}{2}\sum_{j=1}^{n}u\overline{u}\,\mathrm{d}z_j\wedge\mathrm{d}\overline{z_j} = \frac{\mathrm{i}}{2}\sum_{j=1}^{n}\mathrm{d}z_j\wedge\mathrm{d}\overline{z_j} = \omega_{\mathbb{C}^n}.$$

(d) The orbit through $z \in \mathbb{C}^n$ is the image of the map

$$S^1 \longrightarrow \mathbb{C}^n, \qquad e^{\mathrm{i}t} \longrightarrow e^{\mathrm{i}t}z.$$

This is an embedding (it is smooth, injective, with everywhere injective differential) if $z \neq 0$. Thus, the orbits of the restriction of this S^1-action to S^{2n-1} are embedded circles.

(e) Define $H\colon \mathbb{C}^n \longrightarrow \mathbb{R}$ by $H(z) = |z|^2/2$. For each $z \in \mathbb{C}^n$, the composition of the map in (d) with the projection

$$\mathbb{R} \longrightarrow S^1, \qquad t \longrightarrow e^{\mathrm{i}t},$$

determines the time t flow $\phi_t : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ for the Hamiltonian vector field ζ_H. For each $z \in S^{2n-1} = H^{-1}(2)$, the tangent space to the S^1-orbit at z is thus $\mathbb{R}\zeta_H(z)$. If in addition $w \in T_zS^{2n-1}$, then

$$\omega\big(\zeta_H(z), w\big) = -\mathrm{d}_zH(w) = 0.$$

Thus, the tangent space to the S^1-orbit at z is contained in $(T_zS^{2n-1})^{\omega_{\mathbb{C}^n}}$. Since

$$\dim T_zS^{2n-1} + \dim\big(T_zS^{2n-1}\big)^{\omega_{\mathbb{C}^n}} = \dim T_z\mathbb{C}^n,$$

it follows that $(T_zS^{2n-1})^{\omega_{\mathbb{C}^n}}$ is the tangent space to the S^1-orbit at z.

(f) Since p is a smooth submersion, the homomorphisms

$$p^*\colon \Lambda^*\big(T^*(\mathbb{C}P^{n-1})\big) \longrightarrow \Lambda^*\big(T^*S^{2n-1}\big) \qquad \text{and} \qquad p^*\colon \Omega^*\big(\mathbb{C}P^{n-1}\big) \longrightarrow \Omega^*\big(S^{2n-1}\big)$$

are injective. Thus, there is at most one 2-form $\omega_{\mathbb{C}P^{n-1}}$ on $\mathbb{C}P^{n-1}$ with $p^*\omega_{\mathbb{C}P^{n-1}} = \omega_{\mathbb{C}^n}|_{TS^{2n-1}}$. Furthermore, if such a form $\omega_{\mathbb{C}P^{n-1}}$ does exist, it must be smooth and satisfy

$$p^*\mathrm{d}\omega_{\mathbb{C}P^{n-1}} = \mathrm{d}\big(p^*\omega_{\mathbb{C}P^{n-1}}\big) = \mathrm{d}\big(\omega_{\mathbb{C}^n}|_{TS^{2n-1}}\big) = \big(\mathrm{d}\omega_{\mathbb{C}^n}\big)\big|_{TS^{2n-1}} = 0.$$

Since p^* is injective, $\omega_{\mathbb{C}P^{n-1}}$ must also be closed.

We define a 2-form $\omega_{\mathbb{C}P^{n-1}}$ on $\mathbb{C}P^{n-1}$ by the condition $p^*\omega_{\mathbb{C}P^{n-1}} = \omega_{\mathbb{C}^n}|_{TS^{2n-1}}$, i.e.

$$\omega_{\mathbb{C}P^{n-1}}\big|_{p(z)}\big(\mathrm{d}_zp(v), \mathrm{d}_zp(w)\big) = \omega_{\mathbb{C}^n}\big|_z(v,w) \qquad \forall\ v,w \in T_zS^{2n-1},\ z \in S^{2n-1}.$$

Since $\ker \mathrm{d}_zp = \mathbb{R}\zeta_H(z)$ is the tangent space to the S^1-orbit at z and is contained in $(T_zS^{2n-1})^{\omega_{\mathbb{C}^n}}$ by (e), the right-hand side above depends only on z, $\mathrm{d}_zp(v)$, and $\mathrm{d}_zp(w)$. Since the S^1-action on $\mathbb{C}^n$ preserves $\omega_{\mathbb{C}^n}$, the right-hand side depends only on $\mathrm{d}_zp(v)$ and $\mathrm{d}_zp(w)$, i.e. the 2-form $\omega_{\mathbb{C}P^{n-1}}$ is well-defined by the above. If $z \in S^{2n-1}$, $v \in T_zS^{2n-1}$, and $\mathrm{d}_zp(v) \neq 0$, then v is not tangent to the S^1-orbit at z, i.e.

$$v \in T_zS^{2n-1} - \mathbb{R}\zeta_H(z) \subset T_zS^{2n-1} - \big(T_zS^{2n-1}\big)^{\omega_{\mathbb{C}^n}}$$

by (e). Thus, there exists $w \in T_zS^{2n-1}$ such that $\omega_{\mathbb{C}^n}(v,w) \neq 0$. We conclude that $\omega_{\mathbb{C}P^{n-1}}$ is nondegenerate.