MAT 562: Symplectic Geometry

Problem Set 7

Due by 12/09, in class

(if you have not passed the orals yet)

Two of the exercises from Chapter 2 of Ana's *Seminar on Symplectic Toric Manifolds*, Exercises IV.12-14 from Audin's book, and/or the following. You do not need to copy the statements of problems (just indicate clearly what problems you are doing).

For $N\in\mathbb{Z}^+$, define $[N]=\{1,\ldots,N\}$ and

$$\widetilde{H}_{\mathbb{C}^N}\colon \mathbb{C}^N \longrightarrow \mathbb{R}^N, \quad \widetilde{H}_{\mathbb{C}^N}(z_1,\ldots,z_N)=\pi\big(|z_1|^2,\ldots,|z_N|^2\big), \qquad \widetilde{\mu}_{\mathbb{C}^N}=\sum_{j=1}^{N}\big(-y_j\mathrm{d}x_j+x_j\mathrm{d}y_j\big).$$

For a $n\times N$ $\mathbb{Z}$-matrix A, let

$$\iota_A\colon \mathbb{T}_A\equiv\ker\big(A\colon \mathbb{T}^N\equiv\mathbb{R}^N/\mathbb{Z}^N\longrightarrow\mathbb{T}^n\equiv\mathbb{R}^n/\mathbb{Z}^n\big)\longrightarrow\mathbb{T}^N$$

be the inclusion, $\iota_A^*\colon\mathbb{R}^N=T_{\mathbb{1}}^*\mathbb{T}^N\longrightarrow T_{\mathbb{1}}^*\mathbb{T}_A$ be the restriction homomorphism induced by ι_A, and

$$\widetilde{H}_A\equiv\iota_A^*\circ\widetilde{H}_{\mathbb{C}^N}\colon\mathbb{C}^N\longrightarrow T_{\mathbb{1}}^*\mathbb{T}_A \tag{1}$$

be a moment map for the restriction of the standard coordinate-wise $\mathbb{T}^N$-action on $\mathbb{C}^N$ to $\mathbb{T}_A\subset\mathbb{T}^N$. Let $(\mathbb{T}_A)_{\mathbb{C}}\subset\mathbb{T}^N_{\mathbb{C}}$ be the complexification of $\mathbb{T}_A\subset\mathbb{T}^N$ and $(\mathbb{T}_A)_{\mathfrak{i}}\subset(\mathbb{T}_A)_{\mathbb{C}}$ be the purely imaginary subgroup (it corresponds to a subgroup of $(\mathbb{R}^*)^N\subset(\mathbb{C}^*)^N$ via $\mathrm{e}^{2\pi\mathfrak{i}\cdot}$). The group $\mathbb{T}^N_{\mathbb{C}}\approx(\mathbb{C}^*)^N$ acts on $\mathbb{C}^N$ by the coordinate-wise multiplication in the usual way. If in addition $\alpha\in T_1^*\mathbb{T}_A$, let

$$P_A^\alpha=\{s\in\big(\mathbb{R}^{\ge0}\big)^N\colon\iota_A^*(s)=\alpha\},\quad \widetilde{Z}_A^\alpha=\widetilde{H}_A^{-1}(\alpha)=\widetilde{H}_{\mathbb{C}^N}^{-1}\big(P_A^\tau\big)\subset\mathbb{C}^N, \tag{2}$$

$$\mathscr{V}_A^\alpha=\big\{J\subset[N]\colon|J|=n,\ P_A^\tau\cap(\mathbb{R}^{\ge0})^{[N]-J}\neq\emptyset\big\},\quad \widetilde{M}_A^\alpha=\mathbb{C}^N-\bigcup_{\substack{J\subset[N]\\ P_A^\tau\cap(\mathbb{R}^{\ge0})^J=\emptyset}}\mathbb{C}^J,\quad M_A^\alpha\equiv\widetilde{M}_A^\alpha/(\mathbb{T}_A)_{\mathbb{C}}\,.$$

For $J\subset[N]$, let A_J be the $n\times|J|$ submatrix of A consisting of the columns indexed by J.

Problem P (counts as 2 exercises)

Let $P\equiv\{r\in\mathbb{R}^n\colon v_k\cdot r\ge c_k\ \forall\, k\in[N]\}$ be a Delzant polytope with the inward normals $v_1,\ldots,v_N$ to the facets (codimension 1 faces) meeting at each vertex of P forming a $\mathbb{Z}$-basis for $\mathbb{Z}^N$. Define

$$A=\big(v_1\ \ldots\ v_N\big)\colon(\mathbb{R}^N,\mathbb{Z}^N)\longrightarrow(\mathbb{R}^n,\mathbb{Z}^n),\quad c=(c_1,\ldots,c_N)\in\mathbb{R}^N,\quad \alpha=-\iota_A^*(c)\in T_{\mathbb{1}}^*\mathbb{T}_A.$$

Thus, $\widetilde{Z}_A^\alpha\subset\mathbb{C}^N$ is the preimage of the regular value $\alpha\in T_{\mathbb{1}}\mathbb{T}^*$ of $\widetilde{H}_A$, $\mathbb{T}_A$ acts freely on $\widetilde{Z}_A^\alpha$, and

$$(M_P,\omega_P)\equiv\big(\widetilde{Z}_A^\alpha,\omega_{\mathbb{C}^N}|_{T\widetilde{Z}_A^\alpha}\big)/\mathbb{T}_A$$

is the compact connected symplectic manifold obtained from P via the Delzant construction in class. Show that

(a) the open subspace $\widetilde{M}_A^\alpha\subset\mathbb{C}^N$ contains $\widetilde{Z}_A^\alpha$ and is path-connected, simply connected, preserved by the $\mathbb{T}_{\mathbb{C}}^N$-action, and acted on freely by $(\mathbb{T}_A)_{\mathbb{C}}\subset\mathbb{T}_{\mathbb{C}}^N$;

(b) the smooth map $(\mathbb{T}_A)_{\mathfrak{i}}\times\widetilde{Z}_A^\alpha\longrightarrow\mathbb{C}^N$, $(g,z)\longrightarrow g\cdot z$, is a diffeomorphism onto $\widetilde{M}_A^\alpha$;

(c) the inclusions $\widetilde{Z}_A^\alpha\longrightarrow\widetilde{M}_A^\alpha$ and $\mathbb{T}_A\longrightarrow(\mathbb{T}_A)_{\mathbb{C}}$ induce a homeomorphism

$$M_A^\alpha\equiv\widetilde{M}_A^\alpha/(\mathbb{T}_A)_{\mathbb{C}}\longrightarrow\widetilde{Z}_A^\alpha/\mathbb{T}_A\equiv M_P$$

with respect to the quotient topologies;

(d) the smooth manifold $M_A^\alpha=M_P$ is simply connected and admits a complex manifold structure, compatible with the smooth and symplectic structures, so that the quotient projection $q\colon\widetilde{M}_A^\alpha\longrightarrow M_A^\alpha$ is a holomorphic submersion and $(\mathbb{T}_A)_{\mathbb{C}}$ acts on M_A^α by biholomorphisms.

Problem Q (counts as 2 exercises)

Let A be a $n\times N$ $\mathbb{Z}$-matrix, $\mathcal{R}_A\subset T_{\mathbb{1}}^*\mathbb{T}_A$ be the subset of regular values of the map $\widetilde{H}_A$ defined in (1), and $\alpha\in T_{\mathbb{1}}^*\mathbb{T}_A$. Show that

(a) the subset $P_A^\alpha\subset\mathbb{C}^N$ defined in (2) is bounded if and only if $P_A^0=\{0\}$;

(b) $\alpha\in\mathcal{R}_A$ if and only if $\alpha\notin\iota_A^*((\mathbb{R}^{\ge0})^J)$ for any $J\subset[N]$ with $|J|<N-n$;

(c) the subset $\mathcal{R}_A\subset T_{\mathbb{1}}^*\mathbb{T}_A$ of regular values of $\widetilde{H}_A$ is open.

Suppose in addition $\alpha\in T_{\mathbb{1}}^*\mathbb{T}_A$ is regular value of $\widetilde{H}_A$. Let $\mathcal{R}_A^\alpha\subset\mathcal{R}_A$ be the connected component containing α. Show that

(d) $\mathcal{R}_A^\alpha\subset T_{\mathbb{1}}^*\mathbb{T}_A$ is preserved by the multiplication by $\mathbb{R}^+$;

(e) $\mathbb{T}_A$ acts freely on $\widetilde{Z}_A^\alpha$ if and only if $\det A_J\in\{\pm1\}$ for every $J\in\mathscr{V}_A^\alpha$;

(f) $(\mathbb{T}_A)_{\mathbb{C}}$ acts freely on $\widetilde{M}_A^\alpha$ if and only if $\det A_J\in\{\pm1\}$ for every $J\in\mathscr{V}_A^\alpha$;

(g) $\mathscr{V}_A^{\alpha'}=\mathscr{V}_A^\alpha$ and $\widetilde{M}_A^{\alpha'}=\widetilde{M}_A^\alpha$ if $\alpha'\in\mathcal{R}_A^\alpha$;

(h) $(\mathbb{T}_A)_{\mathbb{C}}$ acts freely on $\widetilde{M}_A^{\alpha'}$ and $M_A^{\alpha'}=M_A^\alpha$ as complex manifolds if $\alpha'\in\mathcal{R}_A^\alpha$ and $(\mathbb{T}_A)_{\mathbb{C}}$ acts freely on $\widetilde{M}_A^\alpha$.

Problem R (counts as 2 exercises)

Let M be a smooth manifold. A connection in a (smooth) complex line bundle $L\longrightarrow M$ is a $\mathbb{C}$-linear map

$$\nabla\colon \Gamma(M;L)\longrightarrow\Gamma\big(M;T^*M\otimes L\big)\quad\text{s.t.}\quad \nabla(fs)=\mathrm{d}f\otimes s+f\nabla s\ \forall\, s\in\Gamma(M;L),\ f\in C^\infty(M;\mathbb{C}).$$

Such a map extends to L-valued p-forms by

$$\nabla\colon \Gamma\big(M;\Lambda^p(T^*M)\otimes L\big)\longrightarrow\Gamma\big(M;\Lambda^{p+1}(T^*M)\otimes L\big),\quad \nabla(\eta\otimes s)=(\mathrm{d}\eta)\otimes s+(-1)^p\eta\otimes(\nabla s).$$

A connection ∇ in L is compatible with a Hermitian inner-product $\langle\cdot,\cdot\rangle$ on L if

$$w\big(\langle s_1,s_2\rangle\big)=\big\langle\nabla_w s_1,s_2\big\rangle+\big\langle s_1,\nabla_w s_2\big\rangle\quad\forall\ s_1,s_2\in\Gamma(M;L),\ w\in TM.$$

A connection 1-form on a (smooth) principal S^1-bundle $S\longrightarrow M$ is an S^1-invariant 1-form μ on (the total space of) S such that

$$\mu(\zeta_S)=2\pi,\qquad\text{where}\quad \zeta_S\in\Gamma(S;TS),\ \zeta_S(\widetilde{x})=\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{e}^{2\pi\mathrm{i}t}\cdot\widetilde{x}\big)\Big|_{t=0}\quad\forall\,\widetilde{x}\in S.$$

(a) Suppose ∇ is a connection in a complex line bundle $L\longrightarrow M$. Show that there exists a $\mathbb{C}$-valued 2-form κ_∇ on M so that

$$\nabla(\nabla\widetilde{\eta})=\kappa_\nabla\wedge\widetilde{\eta}\quad\forall\ \widetilde{\eta}\in\Gamma\big(M;\Lambda^p(T^*M)\otimes L\big),\ p\in\mathbb{Z}^{\geq 0}.$$

(b) Show that the S^1-invariance condition on a connection 1-form μ in a principal S^1-bundle $\pi_S\colon S\longrightarrow M$ can be equivalently replaced by the condition $\iota_{\zeta_S}\mathrm{d}\mu=0$. Furthermore, for any such μ, there exists an $\mathbb{R}$-valued 2-form κ_μ on M so that $\mathrm{d}\mu=\pi_S^*\kappa_\mu$.

(c) Show that a complex line bundle $L\longrightarrow M$ with a Hermitian inner-product $\langle\cdot,\cdot\rangle$ corresponds to a principal S^1-bundle $S\longrightarrow M$ (and vice versa), while a connection ∇ in L compatible with $\langle\cdot,\cdot\rangle$ corresponds to a connection 1-form μ in the associated principal S^1-bundle S so that $\kappa_\nabla=\mathrm{i}\kappa_\mu$.

Note. The 2-forms κ_∇ and κ_μ above are called the curvature forms of ∇ and α, respectively. By a Čech cohomology computation (p141 in Griffthis&Harris) and (c),

$$c_1(L)=\frac{\mathrm{i}}{2\pi}\big[\kappa_\nabla\big]=-\frac{1}{2\pi}\big[\kappa_\mu\big]\in H^2_{\mathrm{deR}}(M),$$

if ∇ is a connection in a complex line bundle $L\longrightarrow M$ and α is a connection 1-form in an associated principal S^1-bundle.

Problem S (counts as 2 exercises)

Let A be a $n\times N$ $\mathbb{Z}$-matrix, $c\equiv(c_1,\ldots,c_N)\in\mathbb{Z}^N$, $\alpha=-\iota_A^*(c)\in T_{\mathbb{1}}^*\mathbb{T}_A$, and

$$\zeta_{\mathbb{C}}=-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}\in\Gamma(\mathbb{C};T\mathbb{C}).$$

For $r\equiv(r_1,\ldots,r_N)\in\mathbb{R}^N$, let $\zeta_r\in\Gamma(\mathbb{C}^N;T\mathbb{C}^N)$ be the vector field given by

$$\zeta_r(z)=\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{e}^{2\pi\mathrm{i}r_1t}z_1,\ldots,\mathrm{e}^{2\pi\mathrm{i}r_Nt}z_N\big)\Big|_{t=0}\qquad\forall\,z\equiv(z_1,\ldots,z_N)\in\mathbb{C}^n.$$

(a) Show that the 1-form $\widetilde{\mu}\equiv(-\widetilde{\mu}_{\mathbb{C}^N})\oplus\widetilde{\mu}_{\mathbb{C}}$ on $\mathbb{C}^N\times\mathbb{C}$ is invariant under the $\mathbb{T}^N$-action on $\mathbb{C}^N$ given by

$$\big(\mathrm{e}^{2\pi\mathrm{i}t_1},\ldots,\mathrm{e}^{2\pi\mathrm{i}t_N}\big)\cdot(z,z')=\Big(\big(\mathrm{e}^{-2\pi\mathrm{i}t_1}z_1,\ldots,\mathrm{e}^{-2\pi\mathrm{i}t_N}z_N\big),\mathrm{e}^{2\pi\mathrm{i}(c\cdot t)}z'\Big)\tag{3}$$

and $\widetilde{\mu}(\zeta_r,2\pi a\zeta_{\mathbb{C}})=-2r\cdot\widetilde{H}_{\mathbb{C}^N}+2a\widetilde{H}_{\mathbb{C}}$ for every $a\in\mathbb{R}$.

Suppose in addition $\mathbb{T}_A$ acts freely on $\widetilde{Z}_A^\alpha$. Show that

(b) the quotient $\widetilde{Z}_A^\alpha\times_{\mathbb{T}_A}S^1$ of $\widetilde{Z}_A^\alpha\times S^1$ by the restriction of the action of (3) to $\mathbb{T}_A$ is a principal S^1-bundle over $\widetilde{Z}_A^\alpha/\mathbb{T}_A$;

(c) $\widetilde{\mu}$ descends to a connection 1-form μ_α on the above principal S^1-bundle with $p^*\kappa_{\mu_\alpha}=-2\omega_{\mathbb{C}^N}|_{\widetilde{Z}_A^\alpha}$, where $p\colon\widetilde{Z}_A^\alpha\longrightarrow\widetilde{Z}_A^\alpha/\mathbb{T}_A$ is the quotient projection;

(d) the quotient $L_{-\alpha}$ of $\widetilde{M}_A^\alpha\times\mathbb{C}$ by the restriction of the complexification of the action of (3) to $(\mathbb{T}_A)_{\mathbb{C}}$ is a holomorphic line bundle over M_A^α with

$$c_1(L_{-\alpha})=\frac{1}{\pi}[\omega_A^\alpha]\in H^2_{\mathrm{deR}}(\widetilde{M}_A^\alpha),$$

where ω_A^α is the symplectic form on $\widetilde{Z}_A^\alpha/\mathbb{T}_A=M_A^\alpha$ determined by $p^*\omega_A^\alpha=\omega_{\mathbb{C}^N}|_{\widetilde{Z}_A^\alpha}$.

(e) Let $P\subset\mathbb{C}^N$, A, and $\alpha\in T_{\mathbb{1}}^*\mathbb{T}_A$ be as in Problem P. Show that the complex manifold M_A^α is projective.