MAT 562: Symplectic Geometry

Problem Set 6

Due by 11/20, in class

(if you have not passed the orals yet)

Two of the exercises from Sections 5.4 of the main book and/or the following. You do not need to copy the statements of problems (just indicate clearly what problems you are doing).

Problem M (counts as two exercises)

Let $n\in\mathbb{Z}^+$ and $\omega_{\mathbb{C}^n}\equiv\sum_{j=1}^{n}\mathrm{d}x_j\wedge\mathrm{d}y_j$ be the standard symplectic form on $\mathbb{C}^n$. For $a_1,\ldots,a_n\in\mathbb{Z}$, the smooth function

$$H_{a_1\ldots a_n}\colon\mathbb{C}^n\longrightarrow\mathbb{R}^n,\qquad H_{a_1\ldots a_n}(z_1,\ldots,z_n)=\pi\big(a_1|z_1|^2,\ldots,a_k|z_n|^2\big),$$

is a Hamiltonian for the action of $\mathbb{T}^n\equiv(S^1)^n$ on $(\mathbb{C}^n,\omega_{\mathbb{C}^n})$ given by

$$(u_1,\ldots,u_n)\cdot(z_1,\ldots,z_n)=\big(u_1^{a_1}z_1,\ldots,u^{a_n}z_n\big). \tag{1}$$

(a) Suppose $a_1,\ldots,a_n\in\mathbb{Z}-\{0\}$ and $U\subset\mathbb{C}^n$ is a $\mathbb{T}^n$-invariant open subset. Show that $\omega_{\mathbb{C}^n}|_U$ is the unique symplectic form ω on U so that $H_{a_1\ldots a_n}|_U$ is a Hamiltonian for the restriction of the action (1) to U with respect to ω (you do not need to verify that $H_{a_1\ldots a_n}$ is a Hamiltonian for the action (1) with respect to $\omega_{\mathbb{C}^n}$; this was pretty much done in Problem I).

(b) Suppose ψ is an irreducible linear $\mathbb{T}^n$-action on $(\mathbb{C}^n,\omega_{\mathbb{C}^n})$ preserving the standard complex structure on $\mathbb{C}^n$, $H\colon\mathbb{C}^n\longrightarrow\mathbb{R}^n$ is a Hamiltonian for ψ with respect to $\omega_{\mathbb{C}^n}$, and $U\subset\mathbb{C}^n$ is a $\mathbb{T}^n$-invariant open subset. Show that $\omega_{\mathbb{C}^n}|_U$ is the unique symplectic form ω on U so that $H|_U$ is a Hamiltonian for the restriction of the action ψ to U with respect to ω.

(c) Suppose ψ is an irreducible $\mathbb{T}^n$-action on a smooth manifold M of dimension $2n$ preserving symplectic forms ω_1 and ω_2 and H is a Hamiltonian for ψ with respect to ω_1 and ω_2. Show that every $x\in M^\psi$ has a $\mathbb{T}^n$-invariant neighborhood U_x such that $\omega_1|_{U_x}=\omega_2|_{U_x}$.

(d) Suppose $(M_1,\omega_1,\psi_1,H_1)$ and $(M_2,\omega_2,\psi_2,H_2)$ are symplectic manifolds of dimension $2n$ with effective $\mathbb{T}^n$-actions ψ_i and Hamiltonians H_i for ψ_i so that $H_1(M_1)=H_2(M_2)$. Show that for every $x_1\in M_1^{\psi_1}$ there exist $x_2\in M_2^{\psi_2}$, $\mathbb{T}^n$-invariant neighborhoods $U_{x_1}\subset M_1$ of x_1 and $U_{x_2}\subset M_2$ of x_2, and $\mathbb{T}^n$-equivariant symplectomorphism $f\colon U_{x_1}\longrightarrow U_{x_2}$ such that $H_1|_{U_1}=H_2\circ f$.

Note. This problem is intended to provide a local motivation for the uniqueness part of Delzant's theorem.

Problem N (counts as one exercise)

Suppose $\widetilde{\psi}$ is a smooth action of $\mathbb{T}^{k_1}\times\mathbb{T}^{k_2}$ on a symplectic manifold $(\widetilde{M},\widetilde{\omega})$ with Hamiltonian

$$\widetilde{H}\equiv\big(\widetilde{H}_1,\widetilde{H}_2\big)\colon\widetilde{M}\longrightarrow\mathbb{R}^{k_1}\times\mathbb{R}^{k_2},$$

$c_1\in\mathbb{R}^{k_1}$ is a regular value of $\widetilde{H}_1$, and $\mathbb{T}^{k_1}$ acts freely on $\widetilde{H}_1^{-1}(c_1)$. Let $(M,\omega)\equiv(\widetilde{H}_1^{-1}(c_1),\widetilde{\omega})/\mathbb{T}^{k_1}$ be the associated symplectic quotient.

(a) Show that $\widetilde{\psi}|_{\mathbb{T}^{k_2}}$ descends to a smooth $\mathbb{T}^{k_2}$-action ψ_2 on (M,ω), $\widetilde{H}_2$ descends to a smooth map $H_2\colon M\longrightarrow\mathbb{R}^{k_2}$, and H_2 is a moment map for ψ_2.

(b) Suppose $\widetilde{M}$ is compact and connected, and thus so is M. How are the moment polytopes for $\widetilde{H}$ and H_2 related?

Problem O (counts as one exercise)

The fixed locus of the S^1-action on $\mathbb{C}P^2$ given by

$$S^1 \times \mathbb{C}P^2 \longrightarrow \mathbb{C}P^2, \qquad u\cdot\big[z_0, z_1, z_2\big] = \big[z_0, u^2 z_1, u^3 z_2\big],$$

consists of three points. This action also preserves the complex projective lines $\overline{P_iP_j}$ through any pair of these points. The same applies to the complexification of this action, which is given by the above formula with S^1 replaced by $\mathbb{C}^*$. Show that

(a) the above S^1-action on $\mathbb{C}P^2$ is effective and Hamiltonian with respect to ω_{FS} (*hint:* use PS5).

(b) show that the closure $\overline{\mathcal{O}_x}$ of the $\mathbb{C}^*$-orbit $\mathcal{O}_x$ is a rational cubic curve for any $x \in \mathbb{C}P^2 - \bigcup_{i,j} \overline{P_iP_j}$ and thus is not a smooth submanifold of $\mathbb{C}P^2$.