MAT 562: Symplectic Geometry

Problem Set 5

Due by 11/06, in class

(if you have not passed the orals yet)

Two of the exercises from Sections 5.5 of the main book, Exercises IV.1-5 from Audin's book, and/or the following. You do not need to copy the statements of problems (just indicate clearly what problems you are doing).

Problem I (counts as two exercises)

(a) Let $q:\mathbb{C}^n-\{0\}\longrightarrow\mathbb{C}P^{n-1}$ be the quotient projection as in Problem A on PS1. Suppose $U\subset\mathbb{C}P^{n-1}$ is an open subset and $s\colon U\longrightarrow\mathbb{C}^n-\{0\}$ is a holomorphic section of q, i.e. $q\circ s=\mathrm{id}_U$. Show that the 2-form

$$\omega_{\mathrm{FS};n-1}\big|_U\equiv\frac{\mathfrak{i}}{2\pi}\partial\bar\partial\ln|s|^2, \tag{1}$$

where $|\cdot|$ is the standard (round) norm on $\mathbb{C}^n$, is independent of the choice of s.

(b) By (a), (1) determines a global 2-form $\omega_{\mathrm{FS};n-1}$ on $\mathbb{C}P^{n-1}$, called the Fubini-Study symplectic form. Show that this form is indeed symplectic and

$$\omega_{\mathrm{FS};n-1}=\frac{1}{\pi}\omega_{\mathbb{C}P^{n-1}},$$

where $\omega_{\mathbb{C}P^{n-1}}$ is the symplectic form on $\mathbb{C}P^{n-1}$ provided by Problem A on PS1.

(c) Show that the action of $S^1\equiv\mathbb{R}/\mathbb{Z}$ on $\mathbb{C}^n$ given by

$$\mathrm{e}^{2\pi\mathfrak{i}t}\cdot(z_1,\ldots,z_n)=\big(z_1,\ldots,z_{k-1},\mathrm{e}^{2\pi\mathfrak{i}t}z_k,z_{k+1},\ldots,z_n\big)$$

is Hamiltonian with respect to the standard symplectic form $\omega_{\mathbb{C}^n}$ with a Hamiltonian

$$\widetilde{H}_k\colon\mathbb{C}^n\longrightarrow\mathbb{R},\qquad \widetilde{H}_k(z_1,\ldots,z_n)=\pi|z_k|^2.$$

(d) Show that the actions of $\mathbb{T}^n\equiv(S^1)^n$ and $\mathbb{T}^{n-1}\equiv(S^1)^{n-1}$ on $\mathbb{C}P^{n-1}$ given by

$$\begin{aligned}\big(\mathrm{e}^{2\pi\mathfrak{i}t_1},\ldots,\mathrm{e}^{2\pi\mathfrak{i}t_n}\big)\cdot[z_1,\ldots,z_n]&=\big[\mathrm{e}^{2\pi\mathfrak{i}t_1}z_1,\ldots,\mathrm{e}^{2\pi\mathfrak{i}t_n}z_n\big],\\ \big(\mathrm{e}^{2\pi\mathfrak{i}t_1},\ldots,\mathrm{e}^{2\pi\mathfrak{i}t_{n-1}}\big)\cdot[z_1,\ldots,z_n]&=\big[\mathrm{e}^{2\pi\mathfrak{i}t_1}z_1,\ldots,\mathrm{e}^{2\pi\mathfrak{i}t_{n-1}}z_{n-1},z_n\big]\end{aligned}$$

are Hamiltonian with respect to the symplectic form $\omega_{\mathrm{FS};n-1}$. Determine the moment polytopes for these actions; draw the moment polytopes in the $n=3$ case, labeling everything clearly.

Problem J (counts as one exercise)

Suppose ψ is an $\mathbb{R}^k$-action on a symplectic manifold (M,ω) with Hamiltonian $H:M\longrightarrow\mathbb{R}^k$ and A is a real $k\times m$-matrix (determining a linear map from $\mathbb{R}^m$ to $\mathbb{R}^k$). Show that $\psi\circ A$ is an $\mathbb{R}^m$-action on (M,ω) with Hamiltonian $A^{\mathrm{tr}}\circ H\colon M\longrightarrow\mathbb{R}^m$.

Problem K (counts as two exercises)

Let $e_j \in \mathbb{R}^k$ denote the j-th standard coordinate vector. An action ψ of $\mathbb{R}^k$ on a smooth manifold M is called irreducible if the associated vector fields

$$\xi_j = \frac{\mathrm{d}}{\mathrm{d}t}\psi_{te_j}\bigg|_{t=0} \in \Gamma(M;TM)$$

are linearly independent over $\mathbb{R}$; otherwise, ψ is called irreducible. An action of $\mathbb{T}^k \equiv \mathbb{R}^k/\mathbb{Z}^k$ is called irreducible (resp. reducible) if the composition of this action with the projection $\mathbb{R}^k \longrightarrow \mathbb{T}^k$ is irreducible (resp. reducible).

(a) Suppose ψ is a nontrivial $\mathbb{R}^k$-action on a smooth manifold M. Show that there exists an irreducible $\mathbb{R}^m$-action ψ' on M and a full-rank real $m\times k$-matrix A so that $\psi = \psi' \circ A$. If in addition ψ preserves some structure on M (e.g. a metric, symplectic form, almost complex structure), show that so does ψ'.

(b) Suppose ψ is a nontrivial $\mathbb{T}^k$-action on a smooth manifold M. Show that there exists an irreducible $\mathbb{T}^m$-action ψ' on M and a full-rank integer $m\times k$-matrix A so that $\psi = \psi' \circ A$.

Hint. Let G be a compact Lie group. For every $v \in T_1G$, the closure of the one-parameter subgroup $\{\mathrm{e}^{tv} : t \in \mathbb{R}\}$ is a torus.

Problem L (counts as one exercise)

Suppose $\mathbb{T}^k \equiv (S^1)^k \equiv (\mathbb{R}/\mathbb{Z})^k$ acts smoothly on a compact complex manifold (M, J), i.e. preserving integrable J. Show that this action extends to an action of the complexified torus $\mathbb{T}^k_{\mathbb{C}} \equiv (\mathbb{C}^*)^k$ so that the associated map

$$\mathbb{T}^k_{\mathbb{C}} \times M \longrightarrow M, \qquad (g, x) \longrightarrow g \cdot x,$$

is holomorphic (with respect to J and the standard complex structure on $\mathbb{C}^* \subset \mathbb{C}$).
Hint: see Problem E(b) on PS4 and its solution on the course website.