MAT 562: Symplectic Geometry

Problem Set 4

Due by 10/23, in class

(if you have not passed the orals yet)

Two of the exercises from Sections 5.1-5.3,5.5 of the main book and/or the following. You do not need to copy the statements of problems (just indicate clearly what problems you are doing).

Problem E (counts as two exercises)

(a) Suppose a compact Lie group G acts smoothly on a symplectic manifold (M,ω). Show that there exists an ω-compatible almost complex structure J on M preserved by G, i.e.

$$\mathrm{d}g\circ J = J\circ\mathrm{d}g\colon TM \longrightarrow g^*TM \qquad \forall\, g\in G\,.$$

(b) Suppose S^1 acts smoothly on a compact almost complex manifold (M,J), i.e. preserving J. Let $\xi\in\Gamma(M;TM)$ be the vector field generating this action, i.e.

$$\xi(x) = \frac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{e}^{2\pi\mathrm{i}t}\cdot x\big)\bigg|_{t=0} \in T_xM \qquad \forall\, x\in M.$$

Show that the flow of $-J\xi$ extends this action to a well-defined smooth action of $\mathbb{C}^*\supset S^1$ on M,

$$\psi\colon \mathbb{C}^*\times M \longrightarrow M, \quad \psi\big(\mathrm{e}^{2\pi\mathrm{i}t},x\big)=\mathrm{e}^{2\pi\mathrm{i}t}\cdot x, \quad \frac{\mathrm{d}}{\mathrm{d}s}\psi\big(\mathrm{e}^{2\pi(s+\mathrm{i}t)},x\big) = -J\xi\big(\psi(\mathrm{e}^{2\pi(s+\mathrm{i}t)},x)\big),$$

and $u_x\equiv\psi(\cdot,x):\mathbb{C}^*\longrightarrow M$ is a J-holomorphic map for every $x\in M$. If in addition J is integrable, show that this $\mathbb{C}^*$-action is J-holomorphic.

(c) Suppose S^1 acts smoothly on a compact symplectic manifold (M,ω) with an associated Hamiltonian $H:M\longrightarrow\mathbb{R}$, i.e. $\iota_\xi\omega=-\mathrm{d}H$, where $\xi\in\Gamma(M;TM)$ is the vector field generating the S^1-action, and $x\in M$. Let J be an S^1-invariant ω-tamed almost complex structure on M and u_x be as in (b). Show that there exist S^1-fixed points $x_-,x_+\in M$ such that

$$\lim_{s\longrightarrow\pm\infty} u_x\big(\mathrm{e}^{2\pi(s+\mathrm{i}t)}\big) = x_\pm \qquad \text{and} \qquad \mathrm{E}_{g_J^\omega}(u_x) = \int_{\mathbb{C}^*} u_x^*\omega = H(x_+)-H(x_-)\,,$$

where g_J^ω is the metric on M determined by ω and J.

Note. This corrects and sharpens Exercise 5.1.5 in the main book. By (c), u_x extends to a continuous map $\widetilde{u}_x\colon\mathbb{CP}^1\longrightarrow M$ with bounded energy on $\mathbb{C}^*$. By (b), the restriction of this map to $\mathbb{C}^*$ is J-holomorphic. The Removal of Singularity Theorem (Proposition 4.8 in the notes) then implies that the extension $\widetilde{u}_x$ is J-holomorphic as well. Thus, a compact connected symplectic manifold (M,ω) with a non-trivial Hamiltonian S^1-action contains a non-constant J-holomorphic sphere through every point $x\in M$ and for every ω-tamed almost complex structure J. This provides the motivation for MR2484280, one of the rare established connections between Gromov-Witten invariants and geometric properties of the symplectic manifold.

Problem F (counts as one exercise)

Suppose $\pi:\mathcal{N}\longrightarrow Y$ is a smooth oriented rank m real vector bundle over a compact oriented dimension k manifold Y. For $y\in Y$, denote by $\mathcal{N}_y\subset\mathcal{N}$ the fiber over Y.

(a) Let $\alpha\in\Omega^k(\mathcal{N})$ and $\beta\in\Omega^m(\mathcal{N})$ be closed forms with the support of β being compact. Show that

$$\int_{\mathcal{N}}\alpha\wedge\beta=\int_{\mathcal{N}}\big(\pi^*(\alpha|_Y)\big)\wedge\beta=\bigg(\int_Y\alpha\bigg)\bigg(\int_{\mathcal{N}_y}\beta\bigg)\qquad\forall\,y\in Y.$$

(b) Suppose in addition $Y=\mathbb{T}^k\equiv(\mathbb{R}/\mathbb{Z})^k$ and the action of $\mathbb{T}^k$ on itself by translations lifts to an action on $\mathcal{N}$ preserving the fibers with characteristic vector fields $\xi_1,\ldots,\xi_k\in\Gamma(\mathcal{N};T\mathcal{N})$, i.e.

$$\xi_j(x)=\frac{\mathrm{d}}{\mathrm{d}t}\big(\underbrace{1,\ldots,1}_{j-1},\mathrm{e}^{2\pi\mathrm{i}t},1,\ldots,1)\cdot x\bigg|_{t=0}\qquad\forall\;x\in\mathcal{N}.$$

Show that $\mathcal{N}$ is a trivializable vector bundle and that

$$\int_{\mathcal{N}_y}\iota_{\xi_{j_1}}(\ldots(\iota_{\xi_{j_\ell}}\sigma)\ldots)=\int_{\mathcal{N}}\sigma$$

for every $\mathbb{T}^k$-invariant compactly supported $(k+m)$-form σ on $\mathcal{N}$ and every $y\in\mathcal{N}$.

Problem G (counts as one exercise)

Let M be a closed oriented n-manifold. The de Rham Poincaré dual of a class $A\in H_k(M;\mathbb{R})$ is the unique class $\eta_A\in H^{n-k}_{\mathrm{deR}}(M)$ so that

$$\int_M\alpha\wedge\eta_A=\langle\alpha,A\rangle\qquad\forall\,\alpha\in H^{n-k}_{\mathrm{deR}}(M)\,.$$

Suppose $Y\subset M$ is a connected compact oriented submanifold (w/o boundary) of dimension k and

$$\Psi\colon\mathcal{N}_MY\equiv\frac{TM|_Y}{TY}\longrightarrow U$$

is a tubular neighborhood identification for Y in M, i.e. $U\subset M$ is an open subset and Ψ is a diffeomorphism such that $\Psi(y)=y$ for every $y\in Y$ and the composition

$$\mathcal{N}_MY\longrightarrow T(\mathcal{N}_MY)\big|_Y\xrightarrow{\mathrm{d}\Psi}TM|_Y\longrightarrow\mathcal{N}_MY\,,$$

where the first map is the inclusion as the vertical tangent bundle and the last map is the quotient projection, is the identity. The fibers and the total space of the normal bundle $\pi\colon\mathcal{N}_MY\longrightarrow Y$ inherit orientations from the orientations of Y and M. Suppose $\beta\in\Omega^{n-k}(M)$ is a closed form supported in U such that

$$\int_{(\mathcal{N}_MY)_y}\Psi^*\beta=1$$

for some (or every) $y\in Y$. Show that $[\beta]\in H^{n-k}_{\mathrm{deR}}(M)$ is the Poincaré dual $\eta_{[Y]_M}$ of the class $[Y]_M\in H_k(M;\mathbb{R})$ determined by Y.
Hint. Use Problem F(a).

Problem H (counts as two exercises)

Suppose $\mathbb{T}^k \equiv (\mathbb{R}/\mathbb{Z})^k$ acts smoothly on a symplectic $2n$-manifold (M,ω) with characteristic vector fields $\xi_1,\ldots,\xi_k \in \Gamma(M;TM)$, i.e.

$$\xi_j(x) = \frac{\mathrm{d}}{\mathrm{d}t}\big(\underbrace{1,\ldots,1}_{j-1},\mathrm{e}^{2\pi \mathrm{i}t},1,\ldots,1)\cdot x\Big|_{t=0} \qquad \forall\; x\in M.$$

For each $x\in M$, define $u_x\colon \mathbb{T}^k \longrightarrow M$ by $u_x(g)=g\cdot x$.

(a) Show that the form $\iota_{\xi_{j_1}}(\ldots(\iota_{\xi_{j_\ell}}\omega^m)\ldots)$ is closed for all $\ell,m\in\mathbb{Z}^+$ and $j_1,\ldots,j_\ell=1,\ldots,k$.

(b) Let σ be a $\mathbb{T}^k$-invariant $2n$-form. Show that the form $\iota_{\xi_{j_1}}(\ldots(\iota_{\xi_{j_\ell}}\sigma)\ldots)$ is closed for all $\ell\in\mathbb{Z}^+$ and $j_1,\ldots,j_\ell=1,\ldots,k$.

(c) Let $U\subset M$ be an nonempty open subset preserved by the $\mathbb{T}^k$-action, i.e. $g(U)=U$ for every $g\in\mathbb{T}^k$. Show that there exists a $\mathbb{T}^k$-invariant $2n$-form σ on M compactly supported in U such that $\int_M\sigma=1$.

(d) Suppose in addition that M is compact. Show that the Poincaré dual of the class $u_{x*}[\mathbb{T}^k]\in H_k(M;\mathbb{R})$ is given by

$$\eta_{u_{x*}[\mathbb{T}^k]} = \frac{1}{\int_M\omega^n}\big[\iota_{\xi_k}(\ldots(\iota_{\xi_1}\omega^n)\ldots)\big].$$

Note. Problems F-H establish a more general version of the Poincaré dual statement buried in the proof of Theorem 5.1.6 in the main book, with most of the proof of this statement omitted. However, the part done in the book would be helpful with parts of Problem H.