MAT 562: Symplectic Geometry

Problem Set 3

Due by 10/09, in class

(if you have not passed the orals yet)

Two of the exercises from Sections 4.3-4.5 of the main book, Sections 2.2,2.3 of the notes, or the following. You do not need to copy the statements of problems (just indicate clearly what problems you are doing).

Problem D (counts as two exercises)

(a) Let Σ be a connected oriented closed surface (2-dimensional manifold). Show that a continuous map $f\colon \Sigma \longrightarrow S^2$ is null-homotopic if and only if it has degree 0.

(b) Let Σ be a connected oriented closed genus g surface embedded in a standard way in $\mathbb{R}^3$ (you can choose what this means). Let $\nu\colon \Sigma \longrightarrow S^2$ be the Gauss map, i.e. $\nu(x)$ is the oriented unit normal vector to $T_x\Sigma \subset T_x\mathbb{R}^3$ for each $x\in\Sigma$. Show that the degree of ν is $1-g$.

Let $n\in\mathbb{Z}$ with $n\geq 2$. Suppose the 2-torus $\mathbb{T}^2$ is embedded in the open unit ball

$$B_1^3 \subset \mathbb{R}^3 = \mathbb{R}^3\times\{0\}\times\{0\} \subset \mathbb{R}^3\times\mathbb{R}\times\mathbb{C}^{n-2} = \mathbb{C}^n$$

in a standard way. Let $z_j \equiv x_j + \mathfrak{i}y_j$ be the standard coordinates on $\mathbb{C}^n$.

(c) Show that the Gauss map $\nu\colon \mathbb{T}^2 \longrightarrow S^2$ for this $\mathbb{T}^2 \subset \mathbb{R}^3$ extends to a smooth null-homotopic map

$$\widetilde{\nu}\colon \big(\mathbb{C}^n, \mathbb{C}^n - B_2^{2n}\big) \longrightarrow \big(S^2, (0,0,1)\big).$$

(d) Let $J_{\mathbb{C}^n}$ be the standard complex structure on $\mathbb{C}^n$ and $\mathfrak{j}$ be an almost complex structure on $\mathbb{T}^2$. Show that there exists a continuous family $(J_t)_{t\in[0,1]}$ of almost complex structures on $\mathbb{C}^n$ so that

$$J_t\big(\mathbb{C}^n\times(\mathbb{C}^2\times\{0\})\big) \subset \mathbb{C}^n\times(\mathbb{C}^2\times\{0\}), \quad J_t\big(\mathbb{C}^n\times(\{0\}\times\mathbb{C}^{n-2})\big) \subset \mathbb{C}^n\times(\{0\}\times\mathbb{C}^{n-2}),$$

$$J_t\big|_{\mathbb{C}^n - B_2^{2n}} = J_{\mathbb{C}^n}\big|_{\mathbb{C}^n - B_2^{2n}}, \quad J_0\big|_{T\mathbb{T}^2} = \mathfrak{j}, \quad J_0\widetilde{\nu} = \frac{\partial}{\partial y_2}, \quad J_1 = J_{\mathbb{C}^n}.$$

Let (M,J) be an almost complex manifold of dimension at least 4 and $U\subset M$ be an nonempty open subset.

(e) Let $x\in U$. Show that there exists a continuous family $(J_t)_{t\in[0,1]}$ of almost complex structures on M so that $J_0 = J$, $J_t|_{M-U} = J|_{M-U}$ for every $t\in[0,1]$, and J_1 is integrable on some neighborhood of x.

(f) Show that there exist an embedded null-homologous 2-torus $\mathbb{T}^2 \subset U$ and a continuous family $(J_t)_{t\in[0,1]}$ of almost complex structures on M so that $J_0 = J$, $J_t|_{M-U} = J|_{M-U}$ for every $t\in[0,1]$, and $\mathbb{T}^2$ is J_1-holomorphic (i.e. $J_1(T\mathbb{T}^2)\subset T\mathbb{T}^2$).

Note. This problem details the proof of Proposition 2.7 in math/2401.17381. Its implication is that every almost complex structure J on a manifold of dimension at least 4 can be homotoped within any nonempty open subset U of M to an almost complex structure J' not tamed by a symplectic form.