MAT 562: Symplectic Geometry

Problem Set 2

Due by 09/25, in class

(if you have not passed the orals yet)

Two of the exercises from Sections 3.2,3.4,4.1-4.3 of the main book, Sections 2.1-2.3 of the notes, and/or the following. You do not need to copy the statements of problems (just indicate clearly what problems you are doing).

Problem B (counts as one exercise)

An automorphism $\phi\colon M\longrightarrow M$ of a set M is an involution if $\phi^2=\mathrm{id}_M$.

(a) Suppose ϕ is an involution on a neighborhood of $0\in\mathbb{R}^n$ with $\phi(0)=0$. Let $\mathrm{Jac}_0(\phi)\colon\mathbb{R}^n\longrightarrow\mathbb{R}^n$ be its Jacobian at 0 so that

$$\phi(x)=\big\{\mathrm{Jac}_0(\phi)\big\}x+Q(x)$$

for some quadratic term $Q\colon\mathbb{R}^n\longrightarrow\mathbb{R}^n$ ($Q(0)=0$, $\mathrm{Jac}_0(Q)=0$) and all x in a neighborhood of $0\in\mathbb{R}^n$. Show that there exist neighborhoods U and W of $0\in\mathbb{R}^n$ so that

$$h\colon U\longrightarrow W,\qquad h(x)=x+\frac{1}{2}\big\{\mathrm{Jac}_0(\phi)\big\}Q(x),$$

is a well-defined diffeomorphism satisfying $h\circ\phi=\{\mathrm{Jac}_0(\phi)\}h$.

(b) Let M be a smooth manifold and $\phi: M\longrightarrow M$ be a smooth involution. Show that every connected component of the fixed locus of ϕ,

$$M^\phi\equiv\big\{x\in M\colon\phi(x)=x\big\},$$

is a smooth submanifold of M.

(c) Suppose in addition ω is a nondegenerate 2-form on M such that $\phi^*\omega=-\omega$. Show that $M^\phi\subset M$ is an ω-Lagrangian submanifold of M.

Problem C (counts as one exercise)

Let M be a smooth manifold and G be a compact Lie group acting smoothly on M, i.e. there is a group homomorphism $\rho\colon G\longrightarrow\mathrm{Diff}(M)$ such that the map

$$G\times M\longrightarrow M,\qquad (g,x)\longrightarrow\{\rho(g)\}(x),$$

is smooth.

(a) Suppose $x\in M$ is a fixed point of this action, i.e. $\{\rho(g)\}(x)=x$ for every $g\in G$. Show that the G-action on M induces a linear G-action on T_xM and there are a neighborhood U of $0\in T_xM$, a neighborhood W of $x\in M$, and a G-equivariant diffeomorphism $h\colon U\longrightarrow W$ with $h(0)=x$.

(b) Show that every connected component of the G-fixed locus,

$$M^G\equiv\big\{x\in M\colon\{\rho(g)\}(x)=x\ \forall\, g\in G\big\},$$

is a smooth submanifold of M.

(c) Suppose in addition ω is a symplectic form on M such that $\{\rho(g)\}^*\omega=\omega$ for every $g\in G$. Show that every connected component of M^G is an ω-symplectic submanifold of M.