MAT 562: Symplectic Geometry

Problem Set 1
Due by 09/11, in class
(if you have not passed the orals yet)

Two of the exercises from Sections 3.1 and 3.2 of the main book or the following.

Problem A (counts as 2 exercises)

The $\mathbb{C}^*$-action on $\mathbb{C}^n$ by the coordinate multiplication restricts to an $\mathbb{C}^*$-action on $\mathbb{C}^n-\{0\}$ and S^1-actions on $\mathbb{C}^n$ and the unit sphere $S^{2n-1}\subset\mathbb{C}^n$. Show that

(a) the quotient topologies on $\mathbb{C}P^{n-1}$ given by $(\mathbb{C}^n-\{0\})/\mathbb{C}^*$ and S^{2n-1}/S^1 are the same (i.e. the map $S^{2n-1}/S^1\longrightarrow(\mathbb{C}^n-\{0\})/\mathbb{C}^*$ induced by inclusions is a homeomorphism);

(b) $\mathbb{C}P^{n-1}$ is a compact topological $2(n-1)$-manifold that admits a complex structure so that the projection

$$\mathbb{C}^n-\{0\}\longrightarrow\mathbb{C}P^{n-1}=(\mathbb{C}^n-\{0\})/\mathbb{C}^*$$

is a holomorphic submersion;

(c) the S^1-action on $\mathbb{C}^n$ preserves the standard symplectic form $\omega_{\mathbb{C}^n}$ on $\mathbb{C}^n$;

(d) the orbits of the restriction of this action to S^{2n-1} are compact connected one-dimensional submanifolds of S^{2n-1};

(e) for each $z\in S^{2n-1}$ the $\omega_{\mathbb{C}^n}$-symplectic complement of T_zS^{n-1},

$$\big(T_zS^{n-1}\big)^{\omega_{\mathbb{C}^n}}\equiv\big\{v\in T_z\mathbb{C}^n\colon\omega_{\mathbb{C}^n}(v,w)=0\ \forall\,w\in T_zS^{n-1}\big\},$$

is the tangent space to the S^1-orbit at z;

(f) there is a unique 2-form $\omega_{\mathbb{C}P^{n-1}}$ on $\mathbb{C}P^{n-1}$ such that $q^*\omega_{\mathbb{C}P^{n-1}}=om_{\mathbb{C}^n}$, where

$$q\colon S^{2n-1}\longrightarrow\mathbb{C}P^{n-1}=S^{2n-1}/S^1$$

is the quotient projection, and this form $\omega_{\mathbb{C}P^{n-1}}$ is symplectic.