

MAT 562: Symplectic Geometry

Partials Solutions to Problem Set 3

Exercise 1.19

- (a) Suppose $u, v \in \mathbb{O}$ are purely imaginary and linearly independent over $\mathbb{R}$. Show that the subalgebra of $\mathbb{O}$ generated by u and v is isomorphic to $\mathbb{H}$.
- (b) Show that the Nijenhuis tensor of the almost complex manifold (X, J) of Example 1.18 is given by

$$A_J|_u(v_1, v_2) = \frac{1}{4}(v_1(v_2u) - (v_1v_2)u - v_2(v_1u) + (v_2v_1)u) \quad \forall u \in X, v_1, v_2 \in T_uX.$$

Conclude that $A_J|_{u=\epsilon}(\mathbf{i}, \mathbf{j}) = (\mathbf{j}\mathbf{i})\epsilon$ if $\mathbf{i}, \mathbf{j} \in \mathbb{H}$ are orthogonal purely imaginary quaternions.

- (a) We can assume that u, v are of unit length and are orthogonal to each other. Thus,

$$\bar{x} = -x, \quad \bar{y} = -y, \quad x^2, y^2 = -1, \quad xy = -x\bar{y} = y\bar{x} = -yx, \quad (xy)y = xy^2 = -x;$$

the last two equations use (2.17). Thus, the map $u \rightarrow \mathbf{i}$ and $v \rightarrow \mathbf{j}$ determines an isomorphism between the subalgebra of $\mathbb{O}$ generated by u and v and $\mathbb{H}$.

- (b) By definition,

$$A_J|_u(v_1, v_2) = \frac{1}{4} \left([\xi_1, \xi_2] + J[\xi_1, J\xi_2] + J[J\xi_1, \xi_2] - [J\xi_1, J\xi_2] \right)_u,$$

where $\xi_1, \xi_2 \in \Gamma(X; TX)$ are vector fields on X so that $\xi_i(u) = v_i$. It is straightforward to compute the Lie bracket of vector fields on $\mathbb{R}^8 \supset X$:

$$[\tilde{\xi}_1, \tilde{\xi}_2] \equiv \left[\sum_{i=1}^8 f_i \frac{\partial}{\partial x_i}, \sum_{j=1}^8 g_j \frac{\partial}{\partial x_j} \right] = \sum_{i,j=1}^8 f_i \frac{\partial g_j}{\partial x_i} \frac{\partial}{\partial x_j} - \sum_{i,j=1}^8 g_j \frac{\partial f_i}{\partial x_j} \frac{\partial}{\partial x_i} \equiv \langle \tilde{\xi}_1 \rangle(\tilde{\xi}_2) - \langle \tilde{\xi}_2 \rangle(\tilde{\xi}_1).$$

In particular, $\langle \tilde{\xi}_1 \rangle(\tilde{\xi}_2) = \tilde{\xi}_1$ if $\tilde{\xi}_2(x) = x$ for all $x \in \mathbb{R}^8$, i.e. $\tilde{\xi}_2$ is the canonical vector field, denoted simply by x below. Furthermore, if $\tilde{\xi}_1, \tilde{\xi}_2 \in \Gamma(\mathbb{R}^8; T\mathbb{R}^8)$ are vector fields on $\mathbb{R}^8$ such that $\tilde{\xi}_1|_X, \tilde{\xi}_2|_X \in \Gamma(X; TX)$, then

$$[\tilde{\xi}_1, \tilde{\xi}_2]|_X = [\tilde{\xi}_1|_X, \tilde{\xi}_2|_X] \in \Gamma(X; TX);$$

see Proposition 1.55 in Warner's book.

For $x \in \mathbb{R}^8 = \mathbb{O}$, define

$$\tilde{J}_x: \mathbb{O} \rightarrow \mathbb{O}, \quad \tilde{J}_x(v) = vx, \quad \tilde{\xi}_1(x) = v_1 - \langle v_1, x \rangle x, \quad \tilde{\xi}_2(x) = v_2 - \langle v_2, x \rangle x.$$

Thus, $\tilde{J}$ is an endomorphism of the real vector bundle $T\mathbb{R}^8$ over $\mathbb{R}^8$ such that $\tilde{J}|_{TX} = J$, while $\tilde{\xi}_1, \tilde{\xi}_2 \in \Gamma(\mathbb{R}^8; T\mathbb{R}^8)$ are vector fields on $\mathbb{R}^8$ such that $\tilde{\xi}_i|_X, (\tilde{J}\tilde{\xi}_i)|_X \in \Gamma(X; TX)$ and $\tilde{\xi}_i(u) = v_i$. Viewing v_1, v_2 as constant vector fields on $\mathbb{R}^8$, we obtain

$$\begin{aligned} [v_1, \langle v_2, x \rangle x] &= \langle v_2, \langle v_1 \rangle(x) \rangle x + \langle v_2, x \rangle \langle v_1 \rangle(x) = \langle v_2, v_1 \rangle x + \langle v_2, x \rangle v_1, \quad [\tilde{\xi}_1, \tilde{\xi}_2] = \langle v_1, x \rangle v_2 - \langle v_2, x \rangle v_1, \\ [\tilde{\xi}_1, \tilde{J}\tilde{\xi}_2] &= v_2 v_1 - \langle v_2, v_1 \rangle x^2 - \langle v_2, x \rangle (v_1 x + x v_1) + \langle v_1, v_2 x \rangle x + 2 \langle v_1, x \rangle \langle v_2, x \rangle x^2 - \langle v_2, x \rangle \langle v_1, x^2 \rangle x, \\ [\tilde{J}\tilde{\xi}_1, \tilde{J}\tilde{\xi}_2] &= v_2(v_1 x) - \langle v_2, v_1 x \rangle x^2 - \langle v_2, x \rangle ((v_1 x) x + x(v_1 x)) - \langle v_1, x \rangle v_2 x^2 + \langle v_1, x \rangle \langle v_2, x^2 \rangle x^2, \\ &\quad - v_1(v_2 x) + \langle v_1, v_2 x \rangle x^2 + \langle v_1, x \rangle ((v_2 x) x + x(v_2 x)) + \langle v_2, x \rangle v_1 x^2 - \langle v_2, x \rangle \langle v_1, x^2 \rangle x^2. \end{aligned}$$

Evaluating these expressions at $x=u$ and using $\langle v_i, u \rangle, \langle v_i, 1 \rangle = 0$ and $u^2 = -1$, we obtain the claimed expression for $A_J|_u(v_1, v_2)$. If $i, j \in \mathbb{H}$, $i(j\epsilon) = (ji)\epsilon$ by the definition of octonion multiplication. If in addition i, j are purely imaginary and orthogonal, $ij = -ji$. This yields the last claim.

Problem C

- (a) Let Σ be a connected oriented closed surface (2-dimensional manifold). Show that a continuous map $f: \Sigma \rightarrow S^2$ is null-homotopic if and only if it has degree 0.
- (b) Let Σ be a connected oriented closed genus g surface embedded in a standard way in $\mathbb{R}^3$ (you can choose what this means). Let $\nu: \Sigma \rightarrow S^2$ be the Gauss map, i.e. $\nu(x)$ is the oriented unit normal vector to $T_x\Sigma \subset T_x\mathbb{R}^3$ for each $x \in \Sigma$. Show that the degree of ν is $1-g$.

Let $n \in \mathbb{Z}$ with $n \geq 2$. Suppose the 2-torus $\mathbb{T}^2$ is embedded in the open unit ball

$$B_1^3 \subset \mathbb{R}^3 = \mathbb{R}^3 \times \{0\} \times \{0\} \subset \mathbb{R}^3 \times \mathbb{R} \times \mathbb{C}^{n-2} = \mathbb{C}^n$$

in a standard way. Let $z_j \equiv x_j + iy_j$ be the standard coordinates on $\mathbb{C}^n$.

- (c) Show that the Gauss map $\nu: \mathbb{T}^2 \rightarrow S^2$ for this $\mathbb{T}^2 \subset \mathbb{R}^3$ extends to a smooth null-homotopic map

$$\tilde{\nu}: (\mathbb{C}^n, \mathbb{C}^n - B_2^{2n}) \rightarrow (S^2, (0, 0, 1)).$$

- (d) Let $J_{\mathbb{C}^n}$ be the standard complex structure on $\mathbb{C}^n$ and j be an almost complex structure on $\mathbb{T}^2$. Show that there exists a continuous family $(J_t)_{t \in [0,1]}$ of almost complex structures on $\mathbb{C}^n$ so that

$$J_t(\mathbb{C}^n \times (\mathbb{C}^2 \times \{0\})) \subset \mathbb{C}^n \times (\mathbb{C}^2 \times \{0\}), \quad J_t(\mathbb{C}^n \times (\{0\} \times \mathbb{C}^{n-2})) \subset \mathbb{C}^n \times (\{0\} \times \mathbb{C}^{n-2}),$$

$$J_t|_{\mathbb{C}^n - B_2^{2n}} = J_{\mathbb{C}^n}|_{\mathbb{C}^n - B_2^{2n}}, \quad J_0|_{T\mathbb{T}^2} = j, \quad J_0\tilde{\nu} = \frac{\partial}{\partial y_2}, \quad J_1 = J_{\mathbb{C}^n}.$$

Let (X, J) be an almost complex manifold of dimension at least 4 and $U \subset X$ be a nonempty open subset.

- (e) Let $x \in U$. Show that there exists a continuous family $(J_t)_{t \in [0,1]}$ of almost complex structures on X so that $J_0 = J$, $J_t|_{X-U} = J|_{X-U}$ for every $t \in [0,1]$, and J_1 is integrable on some neighborhood of x .
- (f) Show that there exist an embedded null-homologous 2-torus $\mathbb{T}^2 \subset U$ and a continuous family $(J_t)_{t \in [0,1]}$ of almost complex structures on X so that $J_0 = J$, $J_t|_{X-U} = J|_{X-U}$ for every $t \in [0,1]$, and $\mathbb{T}^2$ is J_1 -holomorphic (i.e. $J_1(T\mathbb{T}^2) \subset T\mathbb{T}^2$).

Note. This problem details the proof of Proposition 2.2 in math/2401.17381v2. Its implication is that every almost complex structure J on a manifold of dimension at least 4 can be homotoped within any nonempty open subset U of X to an almost complex structure J' not tamed by a symplectic form.

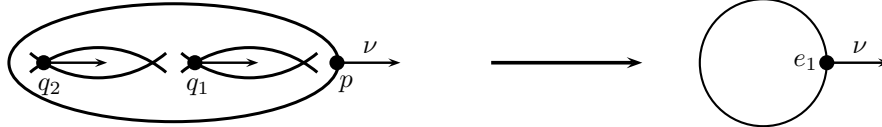
(a) Since homotopic maps induce the same maps in homology, a null-homotopic map $f: \Sigma \rightarrow S^2$ induces the trivial map on the second homology and is thus of degree 0. The surface Σ can be written as a CW complex with a single 2-cell attached to the 1-skeleton, which is a union of circles joined at finitely many points. Since S^2 is simply connected, the restriction of any continuous map $f: \Sigma \rightarrow S^2$ to the 1-skeleton of Σ can be homotoped to a constant map. Thus, any continuous map $f: \Sigma \rightarrow S^2$ is homotopic to a continuous map g which sends the entire 1-skeleton of Σ and thus the boundary of the 2-cell to a point. Such a map g is equivalent to a continuous map $S^2 \rightarrow S^2$. Since the Hurewicz and degree homomorphisms

$$\pi_2(S^2) \rightarrow H_2(S^2; \mathbb{Z}) \rightarrow \mathbb{Z}$$

are isomorphisms, f and g are null-homotopic if and only if they have degree 0.

Note: The statement of (a) is the $m=2$ of *Theorem of Hopf* on p51 in Milnor's *Topology from the Differentiable Viewpoint*. The proof for arbitrary m is the same.

(b) Let x, y, z be the usual coordinates on $\mathbb{R}^3$. Embed Σ into $\mathbb{R}^3$ so that it is “lying horizontally” and centered around the x -axis, as indicated in the picture below. Its intersection with the xy -plane then consists of a large ellipse and g ellipses inside it, all symmetric about the x -axis. The preimage of $e_1 \equiv (1, 0, 0)$ under the Gauss map ν consists of $1+g$ points of Σ lying in the xy -plane: the rightmost point p of Σ (and of the large ellipse) and the leftmost points $q_1, \dots, q_g$ of the g inside ellipses. We can choose local *oriented* coordinates (s, t) centered at $e_1, p, q_1, \dots, q_g$ (using the exponential map, for example) so that the s -axis stays in the xy -plane with $y'(s) > 0$ and the t -axis stays in the xz -plane with $z'(t) > 0$. The Gauss map ν sends the t -axis at $p, q_1, \dots, q_g$ to the t -axis at e_1 preserving the direction. It sends the s -axis at $p, q_1, \dots, q_g$ to the s -axis at e_1 , preserving the direction at p and reversing the direction at $q_1, \dots, q_g$. Thus, ν is orientation-preserving at p and orientation-reversing at $q_1, \dots, q_g$. Therefore, the degree of ν is $1-g$. The advantage of this approach over those in Notes 2 and 3 below is that it is not based on other nontrivial results (that the signed number of zeros of a vector field on a closed oriented manifold is its Euler characteristic in Note 2 or the Gauss-Bonnet Theorem in Note 3).



Note 1. The degree of the Gauss map ν is in fact independent of the embedding of Σ . Let $Y \subset \mathbb{R}^3$ be the bounded region cut out by $\Sigma \subset \mathbb{R}^3$; it is a compact 3-manifold with boundary Σ . Take a vector field ξ that equals some constant $v \in S^2$ outside of a tubular neighborhood U of Σ , ν on Σ , and connect the two by a straight line homotopy along the fibers of $U \rightarrow \Sigma$. The zeros of ξ then lie in fibers of U over the points $x \in \Sigma$ with $\nu(x) = -v$. The signed number of these zeros is the signed cardinality of $\nu^{-1}(-v)$, i.e. the degree of ν if v is chosen generically; this is the $m=3$ case of Lemma 6.3 in Milnor's *Topology from the Differentiable Viewpoint*. If Y_1 and Y_2 are the bounded regions for two different embeddings of Σ in $\mathbb{R}^3$, we can glue them together along with Σ to obtain a closed manifold Y with an orientation that agrees with that of Y_1 and with the opposite of that of Y_2 . We can also glue the vector field ξ_1 on Y_1 above with the vector field $-\xi_2$ on Y_2 to obtain a vector field ξ . The signed number of zeros of ξ is the signed number of zeros of ξ_1 minus the signed number of zeros on ξ_2 because the multiplication by -1 is orientation-reversing on the 3-dimensional fibers of $TY_2 \rightarrow Y_2$. This signed number is also the Euler characteristic $\chi(Y)$ of Y , which is 0 because Y is an odd-dimensional manifold. Thus, the degree of the Gauss map ν for the first embedding of Σ (which is the signed number of zeros of ξ_1) is the same as the degree of the Gauss map ν for the

second embedding of Σ (which is the signed number of zeros of ξ_2).

Note 2. Let $\xi \in \Gamma(S^2; TS^2)$ be a vector field with transverse zeros that are regular values of the Gauss map $\nu: \Sigma \rightarrow S^2$. The signed number of zeros of ξ is $\chi(S^2) = 2$. Since $T_{\nu(x)}S^2 = T_x\Sigma$ for every $x \in \Sigma$, $\zeta \equiv \xi \circ \nu$ is a vector field on Σ . The zeros of ζ are the preimages of the zeros of ξ under ν . The sign of $x \in \zeta^{-1}(0)$ as a zero of ζ is the sign of $\nu(x) \in \xi^{-1}(0)$ as a zero of ξ times the sign of $d_x\nu$. Thus, the signed number of zeros of ζ is the signed number of zeros of ξ times the degree of ν , i.e. $2(\deg \nu)$. On the other hand, this number is also $\chi(\Sigma) = 2(1 - g)$. This argument extends to $2n$ -dimensional closed manifolds embedded in $\mathbb{R}^{2n+1}$ (the degree of the Gauss map is half the Euler characteristic of the manifold).

Note 3. Another way to obtain (b) for any embedding $\Sigma \subset \mathbb{R}^3$ is via the Gauss-Bonnet Theorem (Corollary 2 on p276 in do Carmo's *Differential Geometry of Curves and Surfaces*). For $x \in \Sigma$, the Gaussian curvature K_Σ of Σ at x is the determinant of the differential

$$d_x\nu: T_x\Sigma \rightarrow T_{\nu(x)}S^2 = T_x\Sigma$$

of ν at x . Let ω_Σ and ω_{S^2} be the volume forms on Σ and S^2 , respectively, determined by the Riemannian metric on $\mathbb{R}^3$. Thus,

$$4\pi(\deg \nu) = (\deg \nu) \int_{S^2} \omega_{S^2} \equiv \int_\Sigma \nu^* \omega_{S^2} = \int_\Sigma (\det d\nu) \omega_\Sigma = \int_\Sigma K_\Sigma \omega_\Sigma = 2\pi\chi(\Sigma) = 4\pi(1-g);$$

the penultimate equality above is the Gauss-Bonnet Theorem.

(c) Since the normal bundle of Σ in $B_1^3 \subset \mathbb{R}^3$ is trivial, there are an open subset $U \subset B_1^3$ and a diffeomorphism

$$\Psi: (-1, 1) \times \Sigma \rightarrow U \quad \text{s.t.} \quad \Psi(0, x) = x \quad \forall x \in \Sigma.$$

We define an open subset $W \subset \mathbb{C}^n = \mathbb{R}^3 \times \mathbb{R}^{2n-3}$ and a compact subset $K \subset W$ by

$$\begin{aligned} W &= \{(\Psi(s, x), w) : (s, x) \in (-1, 1) \times \Sigma, w \in \mathbb{R}^{2n-3}, |s| + |w| < 1\}, \\ K &= \{(\Psi(s, x), w) : (s, x) \in (-1, 1) \times \Sigma, w \in \mathbb{R}^{2n-3}, |s| + |w| \leq 3/4\}. \end{aligned}$$

By (b), the degree of ν is 0. By (a), there thus exists a smooth map

$$H: [0, 1] \times \Sigma \rightarrow S^2 \subset \mathbb{R}^3 \quad \text{s.t.} \quad H(t, x) = \nu(x), \quad H(1-t, x) = (0, 0, 1) \quad \forall t \in [0, 1/4], x \in \Sigma.$$

The map $\tilde{\nu}: \mathbb{C}^n \rightarrow S^2$ given by

$$\tilde{\nu}(z) = \begin{cases} H(|s| + |w|, x), & \text{if } z = (\Psi(s, x), w) \in W; \\ (0, 0, 1), & \text{if } z \in \mathbb{C}^n - K; \end{cases}$$

is then a well-defined smooth extension of ν with $\nu(\mathbb{C}^n - B_2^{2n}) = \{(0, 0, 1)\}$. The map

$$\tilde{H}_\nu: [0, 1] \times \mathbb{C}^n \rightarrow S^2, \quad \tilde{H}_\nu(t, z) = \begin{cases} H(t + (1-t)(|s| + |w|), x), & \text{if } z = (\Psi(s, x), w) \in W; \\ (0, 0, 1), & \text{if } z \in \mathbb{C}^n - K; \end{cases}$$

is a well-defined smooth homotopy from $\tilde{\nu}$ to a constant map such that $\tilde{H}_\nu(t, z) = (0, 0, 1)$ for all $t \in [0, 1]$ and $z \in \mathbb{C}^n - B_2^{2n}$.

(d) Since $\mathbb{C}^n$ is a vector space, $T\mathbb{C}^n$ is canonically isomorphic to $\mathbb{C}^n \times \mathbb{C}^n$ as real vector bundles over $\mathbb{C}^n$. The latter bundle is the direct sum of the subbundles $\mathbb{C}^n \times (\mathbb{C}^2 \times \{0\}^{n-2})$ and $\mathbb{C}^n \times (\{0\}^2 \times \mathbb{C}^{n-2})$ with trivializing frames

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2} \quad \text{and} \quad \frac{\partial}{\partial x_3}, \frac{\partial}{\partial y_3}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n},$$

respectively. With $\tilde{H}_\nu$ as in the proof of (c), let $\tilde{\nu}_t = H(t, \cdot): \mathbb{C}^n \rightarrow S^2$ for each $t \in [0, 1]$. We define the almost complex structure J_t on the span of $\tilde{\nu}_t, \frac{\partial}{\partial y_2}$, and the second summand above by

$$J_t \tilde{\nu}_t = \frac{\partial}{\partial y_2}, \quad J_t \frac{\partial}{\partial y_2} = -\tilde{\nu}_t, \quad J_t \frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j}, \quad J_t \frac{\partial}{\partial y_j} = -\frac{\partial}{\partial x_j} \quad \forall j = 3, \dots, n.$$

Let $g_{\mathbb{R}^3}$ be the standard metric on the real vector bundle $\mathbb{C}^n \times \mathbb{R}^3$ over $\mathbb{C}^n$ and $g_{\mathbb{R}^3; \mathbb{T}^2}$ be a metric on $T\mathbb{R}^3|_\Sigma = T\Sigma \oplus \mathbb{R}\nu$ so that the metric $g_{\mathbb{R}^3; \mathbb{T}^2}|_{T\Sigma}$ is compatible with the almost complex structure j on Σ and $T\Sigma$ is $g_{\mathbb{R}^3; \mathbb{T}^2}$ -orthogonal to $\mathbb{R}\nu$. Since the space of metrics is convex and $\Sigma \subset B_1^{2n}$ is a neighborhood deformation retract, there exists a continuous family of metrics $(g_t)_{t \in [0, 1]}$ on the real vector bundle $\mathbb{C}^n \times \mathbb{R}^3$ over $\mathbb{C}^n$ such that

$$g_t|_{\mathbb{C}^n - B_2^{2n}} = g_{\mathbb{R}^3}, \quad g_0|_\Sigma = g_{\mathbb{R}^3; \mathbb{T}^2}, \quad g_1 = g_{\mathbb{R}^3}.$$

Let $\pi_t \subset \mathbb{C}^n \times \mathbb{R}^3$ be the g_t -orthogonal complement of $\mathbb{R}\tilde{\nu}_t$. The standard orientation of $\mathbb{R}^3$ and $\tilde{\nu}_t$ determine an orientation on π_t . Along with the latter, the metric $g_t|_{\pi_t}$ determines a compatible complex structure J_t on π_t .

Since g_0 restricts to $g_{\mathbb{R}^3; \mathbb{T}^2}$ over Σ , the restriction of J_0 to $\pi_0|_{\Sigma_0} = T\Sigma$ agrees with j (because j is the complex structure on $T\Sigma$ compatible with the metric $g_{\mathbb{R}^3; \mathbb{T}^2}|_{T\Sigma}$ and the orientation of $T\Sigma$). Since g_t and ν_t restrict to $g_{\mathbb{R}^3}$ and $\frac{\partial}{\partial x_2}$, respectively, over $\mathbb{C}^n - B_2^{2n}$, the restriction of J_t to

$$\pi_t|_{\mathbb{C}^n - B_2^{2n}} = (\mathbb{C}^n - B_2^{2n}) \times (\mathbb{C} \times \{0\}^{n-1})$$

agrees with $J_{\mathbb{C}^n}$. By the same reasoning, $J_1 = J_{\mathbb{C}^n}$.

(e) We can assume that $X = B_2^{2n}$, $U = B_1^{2n}$, $x = 0$, J_0 agrees with the standard complex structure $J_{\mathbb{C}^n}$ at $0 \in \mathbb{C}^n$, and J_0 is tamed by the standard symplectic form $\omega_{\mathbb{C}^n}$ on B_2^{2n} . Let $\eta: \mathbb{R} \rightarrow [0, 1]$ be a smooth function such that

$$\eta(r) = \begin{cases} 1, & \text{if } r < 1/4; \\ 0, & \text{if } r > 1. \end{cases}$$

With Φ as in Exercise 2.4 of the *Notes*, the map

$$J_\bullet: [0, 1] \times B_2^{2n} \rightarrow \text{GL}_{2n}\mathbb{R}, \quad J_t(x) = J_{\mathbb{C}^n} \Phi((1 - t\eta(|x|))\Phi(J_{\mathbb{C}^n}^{-1}J(x))),$$

is then a smooth family of almost complex structures on B_2^{2n} with the required properties.

(f) By (e), we can take $X = B_3^{2n}$, $U = B_2^{2n}$, and $J = J_{\mathbb{C}^n}$. The claim now follows from (d).