

MAT 562: Symplectic Geometry

Problem Set 3

Due by 10/07, in class

(if you have not passed the orals yet)

Two of the exercises from Sections 1.3, 1.6 of the main text, and/or MS Sections 4.1-4.5, and/or the following. You do not need to copy the statements of problems (just indicate clearly what problems you are doing).

Problem C (counts as two exercises)

- (a) Let Σ be a connected oriented closed surface (2-dimensional manifold). Show that a continuous map $f: \Sigma \rightarrow S^2$ is null-homotopic if and only if it has degree 0.
- (b) Let Σ be a connected oriented closed genus g surface embedded in a standard way in $\mathbb{R}^3$ (you can choose what this means). Let $\nu: \Sigma \rightarrow S^2$ be the Gauss map, i.e. $\nu(x)$ is the oriented unit normal vector to $T_x \Sigma \subset T_x \mathbb{R}^3$ for each $x \in \Sigma$. Show that the degree of ν is $1-g$.

Let $n \in \mathbb{Z}$ with $n \geq 2$. Suppose the 2-torus $\mathbb{T}^2$ is embedded in the open unit ball

$$B_1^3 \subset \mathbb{R}^3 = \mathbb{R}^3 \times \{0\} \times \{0\} \subset \mathbb{R}^3 \times \mathbb{R} \times \mathbb{C}^{n-2} = \mathbb{C}^n$$

in a standard way. Let $z_j \equiv x_j + iy_j$ be the standard coordinates on $\mathbb{C}^n$.

- (c) Show that the Gauss map $\nu: \mathbb{T}^2 \rightarrow S^2$ for this $\mathbb{T}^2 \subset \mathbb{R}^3$ extends to a smooth null-homotopic map

$$\tilde{\nu}: (\mathbb{C}^n, \mathbb{C}^n - B_2^{2n}) \rightarrow (S^2, (0, 0, 1)).$$

- (d) Let $J_{\mathbb{C}^n}$ be the standard complex structure on $\mathbb{C}^n$ and $\mathfrak{j}$ be an almost complex structure on $\mathbb{T}^2$. Show that there exists a continuous family $(J_t)_{t \in [0,1]}$ of almost complex structures on $\mathbb{C}^n$ so that

$$J_t(\mathbb{C}^n \times (\mathbb{C}^2 \times \{0\})) \subset \mathbb{C}^n \times (\mathbb{C}^2 \times \{0\}), \quad J_t(\mathbb{C}^n \times (\{0\} \times \mathbb{C}^{n-2})) \subset \mathbb{C}^n \times (\{0\} \times \mathbb{C}^{n-2}),$$

$$J_t|_{\mathbb{C}^n - B_2^{2n}} = J_{\mathbb{C}^n}|_{\mathbb{C}^n - B_2^{2n}}, \quad J_0|_{T\mathbb{T}^2} = \mathfrak{j}, \quad J_0 \tilde{\nu} = \frac{\partial}{\partial y_2}, \quad J_1 = J_{\mathbb{C}^n}.$$

Let (X, J) be an almost complex manifold of dimension at least 4 and $U \subset X$ be a nonempty open subset.

- (e) Let $x \in U$. Show that there exists a continuous family $(J_t)_{t \in [0,1]}$ of almost complex structures on X so that $J_0 = J$, $J_t|_{X-U} = J|_{X-U}$ for every $t \in [0,1]$, and J_1 is integrable on some neighborhood of x .
- (f) Show that there exist an embedded null-homologous 2-torus $\mathbb{T}^2 \subset U$ and a continuous family $(J_t)_{t \in [0,1]}$ of almost complex structures on X so that $J_0 = J$, $J_t|_{X-U} = J|_{X-U}$ for every $t \in [0,1]$, and $\mathbb{T}^2$ is J_1 -holomorphic (i.e. $J_1(T\mathbb{T}^2) \subset TJ\mathbb{T}^2$).

Note. This problem details the proof of Proposition 2.2 in math/2401.17381v2. Its implication is that every almost complex structure J on a manifold of dimension at least 4 can be homotoped within any nonempty open subset U of X to an almost complex structure J' not tamed by a symplectic form.