Foundations of Symplectic Geometry

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Preface

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Chapter 1

Preliminaries

We begin by defining the notions that are most central to symplectic geometry and Lie group actions in this setting. After recalling the basic facts of the Lie group theory, we introduce the most commonly used notation and terminology and then collect a large number of concrete examples in the form of exercises. This chapter concludes with a section on polyhedra and a formal way to encode them; this section is needed only for the later parts that concern the images of moment maps, i.e. Chapters 3 and 6 and Sections 4.2 and 4.3.

1.1 Key notions

A closed manifold is a compact manifold without boundary. We call a closed subset Z of a smooth manifold X a closed submanifold if every topological component, i.e. a maximal connected subset, of Z is open in Z and is a smooth manifold without boundary smoothly embedded into X. In other words, every topological component of Z has an open neighborhood in X disjoint from the rest of Z and is a submanifold of X in the usual sense, but the dimensions of these submanifolds may not be the same. In such a case, the components of Z are also its path components; as usual, we denote the set of these components by $\pi_0(Z)$.

A 2-form ω on (the fibers of) a smooth vector bundle V over a smooth manifold X is called nondegenerate if the contraction of ω in the first input,

$$V \longrightarrow V^*, \qquad v \longrightarrow \iota_v\omega \equiv \omega(v, \cdot),$$

is an isomorphism. This is the case if and only if for every $x \in X$ and $v \in V_x$ nonzero there exists $v' \in V_x$ such that $\omega(v, v') \neq 0$. If ω is a nondegenerate 2-form on V, then the rank of V is even and $\omega^{\mathrm{rk}\,V/2}$ is an orientation form on (the fibers of) V; in particular, $V \longrightarrow X$ is then an orientable vector bundle. A symplectic form on a smooth manifold X is a (smooth) *closed* nondegenerate 2-form on X, i.e. on the fibers of the vector bundle $TX \longrightarrow X$. A symplectic manifold is a pair consisting of a smooth manifold X and a symplectic form ω. A symplectomorphism of a symplectic manifold (X, ω) is a diffeomorphism ψ of the smooth manifold X preserving the 2-form ω, i.e. $\psi^*\omega = \omega$. For a smooth manifold X, we denote by $\mathrm{Diff}(X)$ the group of diffeomorphisms of X, with the product given by the composition. For a symplectic manifold (X, ω), we denote by $\mathrm{Symp}(X, \omega) \subset \mathrm{Diff}(X)$ the group of symplectomorphisms of (X, ω).

A smooth action of a Lie group G on a smooth manifold X (resp. symplectic manifold (X,ω)) is a group homomorphism

$$\psi\colon G \longrightarrow \mathrm{Diff}(X) \quad \big(\text{resp. } G \longrightarrow \mathrm{Symp}(X,\omega)\big), \qquad u \longrightarrow \psi_u\,, \tag{1.1}$$

such that the map

$$\Psi\colon G\times X \longrightarrow X, \qquad \Psi(u,x)=\psi_u(x),$$

is smooth. For each $v\in T_{\mathbb{1}}G$,

$$\zeta_v \equiv \mathrm{d}_{\mathbb{1}}\psi(v) \in \Gamma(X;TX), \qquad \zeta_v(x)=\mathrm{d}_{(\mathbb{1},x)}\Psi(v,0) \in T_xX \quad \forall\, x\in X, \tag{1.2}$$

is then a well-defined (smooth) vector field on X. For any action ψ as in (1.1), let

$$X^{\psi} \equiv \big\{x\in X\colon \psi_u(x)=x \ \forall\, u\in G\big\}$$

denote its fixed locus. If in addition $x\in X$, let

$$G_x(\psi) \equiv \big\{u\in G\colon \psi_u(x)=x\big\} \tag{1.3}$$

denote the stabilizer of x in G. An action ψ in (1.1) is called effective if the group homomorphism ψ is injective. It is called irreducible if the associated vector space homomorphism

$$\mathrm{d}_{\mathbb{1}}\psi\colon T_{\mathbb{1}}G \longrightarrow \Gamma(X;TX), \qquad v \longrightarrow \zeta_v\,,$$

is injective; otherwise, it is called reducible. An effective action is irreducible, but an irreducible action may have a discrete nontrivial kernel and thus not be effective.

Exercise 1.1. Suppose ψ is a smooth action of a Lie group G on a smooth manifold X as in (1.1). Show that the maps

$$\mathrm{d}\psi\colon G \longrightarrow \mathrm{Diff}(TX), \quad u \longrightarrow \mathrm{d}\psi_u, \quad \text{and} \quad \mathrm{d}\psi^*\colon G \longrightarrow \mathrm{Diff}(T^*X), \quad u \longrightarrow \mathrm{d}\psi_u^* \equiv \big\{\mathrm{d}\psi_u^{-1}\big\}^*,$$

are smooth actions of G on TX and T^*X, respectively, lifting the G-action ψ on X and linear on the fibers of the vector bundles $TX, T^*X \longrightarrow X$.

A smooth action of the standard k-torus $\mathbb{T}^k \equiv \mathbb{R}^k/\mathbb{Z}^k$ on a smooth manifold X is equivalent to a smooth $\mathbb{R}^k$-action ψ on X as in (1.1) such that

$$\psi(v+w)=\psi(v) \qquad \forall\ v\in\mathbb{R}^k,\ w\in\mathbb{Z}^k.$$

Following [3], we call a smooth $\mathbb{R}^k$-action ψ on a smooth manifold X almost periodic if there exists a smooth action ψ' of a torus $\mathbb{T}$ on X and a group homomorphism (1.4)

$$\rho\colon \mathbb{R}^k \longrightarrow \mathbb{T} \qquad \text{s.t.} \quad \psi=\psi'\circ\rho\colon \mathbb{R}^k \longrightarrow \mathrm{Diff}(X)\,. \tag{1.4}$$

If the image of ρ is dense in $\mathbb{T}$ (which can be achieved by replacing $\mathbb{T}$ with the closure of $\rho(\mathbb{R}^k)$), then $X^{\psi}=X^{\psi'}$. If in addition ψ preserves a symplectic form ω on X, then so does ψ'. This implies the existence of an $\mathbb{R}^k$-invariant Riemannian metric on X and thus of an $\mathbb{R}^k$-invariant ω-compatible almost complex structure J on X; see Exercises 2.6 and 2.13.

Let G be a Lie group. For any map $\mu\colon X\longrightarrow T^*_{\mathbb{1}}G$ and $v\in T_{\mathbb{1}}G$, we define

$$\mu_v\equiv\{\mu(\cdot)\}(v)\colon X\longrightarrow\mathbb{R},\qquad \mu_v(x)\equiv\big\{\{\mu(\cdot)\}(v)\big\}(x)=\{\mu(x)\}(v). \tag{1.5}$$

A moment map for a smooth action ψ of G on a symplectic manifold (X,ω) is a smooth map

$$\mu\colon X\longrightarrow T^*_{\mathbb{1}}G\qquad\text{s.t.}\qquad \begin{aligned} -\,\mathrm{d}\mu_v=\iota_{\zeta_v}\omega\equiv\omega(\zeta_v,\cdot)&\qquad\forall\,v\in T_{\mathbb{1}}G,\\ \mu\big(\psi_u(x)\big)=\mathrm{Ad}^*_{u^{-1}}\big(\mu(x)\big)&\qquad\forall\,x\in X,\,u\in G,\end{aligned} \tag{1.6}$$

where $\zeta_v\in\Gamma(X;TX)$ is as in (1.2) and $\mathrm{Ad}^*_{u^{-1}}\colon T^*_{\mathbb{1}}G\longrightarrow T^*_{\mathbb{1}}G$ is the dual of the adjoint action $\mathrm{Ad}_{u^{-1}}$ of G on $T_{\mathbb{1}}G$ as in Section 1.2. On the right-hand side of this equation, $\omega(\zeta_v,\cdot)$ denotes the 1-form on X given by

$$\omega(\zeta_v,\cdot)\colon TX\longrightarrow\mathbb{R},\qquad \big\{\omega(\zeta_v,\cdot)\big\}(w)=\omega\big(\zeta_v(x),w\big)\ \ \forall\,x\in X,\,w\in T_xX.$$

If G is abelian, the second equation in (1.6) is equivalent to μ being G-invariant. If G is connected, Exercise 2.12(a) implies that a smooth action of G on X that admits a smooth map $\mu\colon X\longrightarrow T^*_{\mathbb{1}}G$ satisfying the first condition in (1.6) is in fact an action on (X,ω).

A smooth G-action on (X,ω) that admits a smooth map $\mu\colon X\longrightarrow T^*_{\mathbb{1}}G$ satisfying the first condition in (1.6) (resp. admits a moment map) is called weakly Hamiltonian (resp. Hamiltonian). In such a case, μ is determined up to an additive constant (respectively, an additive constant fixed by the Ad^*-action of G on $T^*_{\mathbb{1}}G$). By Proposition 2.26 and Corollary 2.28, weakly Hamiltonian actions are in fact Hamiltonian in the most common settings. In particular, a map $\mu\colon X\longrightarrow T^*_{\mathbb{1}}G$ satisfying the first condition in (1.6) is G-invariant (and so is a moment map) if G is connected, abelian, and either $G\approx\mathbb{R}$ or either G or X is compact. If G is connected and X is compact, then a smooth map $\mu\colon X\longrightarrow T^*_{\mathbb{1}}G$ satisfying the first condition in (1.6) can be replaced by a moment map $\widehat{\mu}\colon X\longrightarrow T^*_{\mathbb{1}}G$ for the G-action ψ on (X,ω). If G is semisimple, e.g. the special orthogonal group SO_n or the unitary group U_n, then every smooth G-action on (X,ω) is Hamiltonian. We call a tuple (X,ω,ψ,μ) a Hamiltonian G-manifold if (X,ω) is a symplectic manifold, ψ is a smooth G-action on (X,ω), and μ is a moment map for this action. We call a Hamiltonian G-manifold (X,ω,ψ,μ) closed (resp. connected) if the manifold X is closed (resp. connected).

1.2 Lie groups and algebras

We recall that a Lie algebra (over $\mathbb{R}$) is a vector space with an anti-symmetric bilinear product

$$[\cdot,\cdot]\colon\mathfrak{a}\times\mathfrak{a}\longrightarrow\mathfrak{a}\qquad\text{s.t.}\quad \big[v_1,[v_2,v_3]\big]+\big[v_2,[v_3,v_1]\big]+\big[v_3,[v_1,v_2]\big]=0\ \ \forall\,v_1,v_2,v_3\in\mathfrak{a}. \tag{1.7}$$

The above condition is called the Jacobi identity. The space $\Gamma(X;TX)$ of (smooth) vector fields on a smooth manifold X with the usual Lie bracket,

$$\begin{gathered}[\cdot,\cdot]\colon\Gamma(X;TX)\times\Gamma(X;TX)\longrightarrow\Gamma(X;TX),\\ \big\{[\xi,\zeta]\big\}(f)=\xi\big(\zeta(f)\big)-\zeta\big(\xi(f)\big)\ \ \forall\,\xi,\zeta\in\Gamma(X;TX),\,f\in C^\infty(X;\mathbb{R}),\end{gathered}$$

is a Lie algebra; see [56, 1.45(d)]. For a subset $S\subset\mathfrak{a}$, we denote by

$$\mathrm{Ann}(S)\equiv\big\{\alpha\in\mathfrak{a}^*\colon\alpha(s)=0\ \forall\,s\in S\big\},$$

where $\mathfrak{a}^* \equiv \mathrm{Hom}_{\mathbb{R}}(\mathfrak{a}, \mathbb{R})$ is the dual of $\mathfrak{a}$, the annihilator of S in $\mathfrak{a}^*$.

For a group G and $u \in G$, let

$$L_u, R_u, \mathfrak{c}_u \colon G \longrightarrow G, \qquad L_u(u') = uu', \quad R_u(u') = u'u, \quad \mathfrak{c}_u(u') = uu'u^{-1}, \tag{1.8}$$

denote the left multiplication, right multiplication, and conjugation, respectively, by u. If in addition G is a Lie group, $u' \in G$, and $v' \in T_{u'}G$, define

$$uv' = \mathrm{d}_{u'}L_u(v') \in T_{uu'}G \qquad \text{and} \qquad v'u = \mathrm{d}_{u'}R_u(v') \in T_{u'u}G\,.$$

If G is a Lie group, $u \in G$, and $v \in T_{\mathbb{1}}G$, let

$$\mathrm{Ad}_u(v) = \mathrm{d}_{\mathbb{1}}\mathfrak{c}_u(v) \in T_{\mathbb{1}}G\,. \tag{1.9}$$

If G is abelian, then $\mathrm{Ad}_u = \mathrm{Id}_{T_{\mathbb{1}}G}$ for every $u \in G$.

Exercise 1.2. Let G be a Lie group and $G' \subset G$ be a subgroup.

(a) Show that the closure $\overline{G'} \subset G$ of G' is also a subgroup and every coset $G'u$ with $u \in \overline{G'}$ is dense in $\overline{G'}$.

(b) Suppose in addition that $G' \subset G$ is an embedded submanifold. Show that every coset $G'u$ with $u \in \overline{G'}$ is dense in $\overline{G'}$ and conclude that $G' = \overline{G'}$, i.e. G' is closed in G.

A vector field ξ on a Lie group G is called left-invariant if

$$\xi = L_u^*\xi \equiv \mathrm{d}L_u^{-1}\big(\xi \circ L_u\big) \qquad \forall\, u \in G.$$

By [56, 1.55], the linear subspace $\Gamma(G;TG)^G \subset \Gamma(G;TG)$ of left-invariant vector fields is preserved by the Lie bracket and is thus a Lie algebra on its own. By [56, 3.7(a)], the homomorphism

$$\Gamma(G;TG)^G \longrightarrow T_{\mathbb{1}}G, \qquad \xi \longrightarrow \xi(\mathbb{1}),$$

is an isomorphism and thus determines a Lie algebra structure on $T_{\mathbb{1}}G$. The resulting Lie algebra is called the Lie algebra of G and is often denoted by $\mathfrak{g}$.

For a Lie group G and $v \in T_{\mathbb{1}}G$, let $\xi_v \in \Gamma(G;TG)^G$ be the unique left-invariant vector field with $\xi_v(\mathbb{1}) = v$ and $\mathrm{e}^{tv} \in G$ with $t \in \mathbb{R}$ be the flow of ξ_v originating at $\mathbb{1}$, i.e.

$$\mathrm{e}^{0v} = \mathbb{1} \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{tv} = \xi_v\big(\mathrm{e}^{tv}\big). \tag{1.10}$$

Since $\xi_v(u) = \mathrm{d}_{\mathbb{1}}L_u(v)$ for every $u \in G$, the vector field ξ_v depends smoothly on $v \in T_{\mathbb{1}}G$. Thus, the map

$$\exp\colon T_{\mathbb{1}}G \longrightarrow G, \qquad \exp(v) = \mathrm{e}^v, \tag{1.11}$$

is smooth and

$$\mathrm{d}_0\exp = \mathrm{id}_{T_{\mathbb{1}}G} \colon T_0(T_{\mathbb{1}}G) = T_{\mathbb{1}}G \longrightarrow T_{\mathbb{1}}G\,.$$

In particular, exp restricts to a diffeomorphism from a neighborhood of $0 \in T_{\mathbb{1}}G$ to a neighborhood of $\mathbb{1} \in G$. The map (1.11) may not be surjective onto the identity component $G_0 \subset G$; see [30, Example 3.41].

Exercise 1.3. Let G be a Lie group and $v\in T_{\mathbb{1}}G$.

(a) Show that the flow e^{tv} given by (1.10) exists for all $t\in\mathbb{R}$, even if G is not compact,

$$\mathrm{e}^{sv}\mathrm{e}^{tv}=\mathrm{e}^{(s+t)v}\quad s,t\in\mathbb{R},\qquad \mathrm{e}^{v}v=v\mathrm{e}^{v}\in T_{\mathrm{e}^{v}}G,\quad\text{and}\quad \mathrm{e}^{\mathrm{Ad}_u(v)}=c_u\big(\mathrm{e}^{v}\big)\;\;\forall\,u\in G. \tag{1.12}$$

(b) Suppose $G'\subset G$ is a Lie subgroup and $v\in T_{\mathbb{1}}G'$. Show that $\mathrm{e}^{v}\in G'$.

Exercise 1.4. Let G be a Lie group and $|\cdot|$ be a norm on $T_{\mathbb{1}}G$. Show that there exist an open neighborhood $\mathcal{U}$ of $0\in T_{\mathbb{1}}G$, a smooth map $\varepsilon\colon\mathcal{U}\times\mathcal{U}\longrightarrow T_{\mathbb{1}}G$, and $C\in\mathbb{R}^+$ so that

$$\mathrm{e}^{v_1}\mathrm{e}^{v_2}=\mathrm{e}^{v_1+v_2+\varepsilon(v_1,v_2)},\qquad \big|\varepsilon(v_1,v_2)\big|\le C|v_1||v_2|\,,$$
$$e^{v_1+v_2}=\lim_{n\longrightarrow\infty}\exp\big(n\big((v_1+v_2)/n+\varepsilon(v_1/n,v_2/n)\big)\big)=\lim_{n\longrightarrow\infty}\big(\mathrm{e}^{v_1/n}\mathrm{e}^{v_2/n}\big)^n,$$

for all $v_1,v_2\in\mathcal{U}$.

Exercise 1.5 (Closed Subgroup Theorem)**.** Let G be a Lie group and $G'\subset G$ be a closed subgroup.

(a) Show that $\mathfrak{g}'\equiv\{v\in T_{\mathbb{1}}G\colon \mathrm{e}^{tv}\in G'\;\forall\,t\in\mathbb{R}\}$ is a linear subspace of $T_{\mathbb{1}}G$.

(b) Suppose in addition $|\cdot|$ is a norm on $T_{\mathbb{1}}G$, $v\in T_{\mathbb{1}}G$, and $v_1,v_2,\ldots\in T_{\mathbb{1}}G-\{0\}$ is a sequence converging to 0 such that $\mathrm{e}^{v_i}\in G'$ for every $i\in\mathbb{Z}^+$ and the sequence $v_1/|v_1|,v_2/|v_2|,\ldots\in T_{\mathbb{1}}G$ converges to v. Show that $v\in\mathfrak{g}'$. *Hint:* if $t\in\mathbb{R}^+$ and $n_i\in\mathbb{Z}$ is the integer part of $t/|v_i|$, then $\mathrm{e}^{n_iv_i}\in G'$ for every $i\in\mathbb{Z}^+$ and the sequence $n_1v_1,n_2v_2,\ldots\in T_{\mathbb{1}}G$ converges to tv.

(c) Let $\mathfrak{g}''\subset\mathfrak{g}$ be a complement of $\mathfrak{g}'\subset\mathfrak{g}$. Suppose $v_1,v_2,\ldots\in\mathfrak{g}'$ and $w_1,w_2,\ldots\in\mathfrak{g}''-\{0\}$ are sequences converging to 0 such that $\mathrm{e}^{v_i+w_i}\in G'$ for every $i\in\mathbb{Z}^+$. Show that there are sequences

$$v_1',v_2',\ldots\in\mathfrak{g}'\qquad\text{and}\qquad w_1',w_2',\ldots\in\mathfrak{g}''$$

converging to 0 such that $\mathrm{e}^{v_i'+w_i'}\in G'$ and $|v_i'|<|w_i'|$ for every $i\in\mathbb{Z}^+$. *Hint:* with ε as in Exercise 1.4, define $v_i'\in\mathfrak{g}'$ and $w_i'\in\mathfrak{g}''$ by $w_i=v_i'+w_i'+\varepsilon(v_i,v_i'+w_i')$.

(d) Show that there exist an open neighborhood $\mathcal{U}$ of $0\in T_{\mathbb{1}}G$ so that

$$\exp\colon\mathcal{U}\cap\mathfrak{g}'\longrightarrow\exp(\mathcal{U})\cap G'$$

is a homeomorphism. *Hint:* by (b) and (c), sequences $v_1,v_2,\ldots$ and $w_1,w_2,\ldots$ as in (b) do not exist.

(e) Conclude that $G'\subset G$ is a *Lie* subgroup.

Exercise 1.6. Let $\rho\colon G\longrightarrow G'$ be a Lie group homomorphism. Show that

$$\ker\rho\equiv\big\{u\in G\colon\rho(u)=\mathbb{1}\big\}$$

is a closed Lie subgroup of G with Lie algebra

$$\ker\mathrm{d}_{\mathbb{1}}\rho\equiv\big\{v\in T_{\mathbb{1}}G\colon\mathrm{d}_{\mathbb{1}}\rho(v)=0\big\}.$$

Let G be a Lie group. By [56, §3.46/7],

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Ad}_{\mathrm{e}^{tv}}(v')\bigg|_{t=0}=[v,v'],\quad \mathrm{Ad}_u\big([v,v']\big)=\big[\mathrm{Ad}_u(v),\mathrm{Ad}_u(v')\big]\quad \forall\, u\in G,\ v,v'\in T_{\mathbb{1}}G\,. \tag{1.13}$$

If $h\colon [a,b]\longrightarrow G$ is a smooth path, this implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Ad}_{h^{-1}}(v')=-\mathrm{Ad}_{h^{-1}}\big([h'h^{-1},v']\big)\,,\qquad \text{where}\quad h'=\frac{\mathrm{d}h}{\mathrm{d}t}\colon [a,b]\longrightarrow TG. \tag{1.14}$$

Along with (1.12), this gives

$$\mathrm{Ad}_{\mathrm{e}^{v}}(v')=0,\quad \mathrm{e}^{v}\mathrm{e}^{v'}=\mathrm{e}^{v'}\mathrm{e}^{v}=\mathrm{e}^{v+v'}\qquad \forall\ v,v'\in T_{\mathbb{1}}G \text{ s.t. } [v,v']=0.$$

If the identity component G_0 of G is abelian, then the first equation in (1.13) implies that so is the Lie algebra $\mathfrak{g}$, i.e. its Lie bracket $[\cdot,\cdot]$ is trivial.

Exercise 1.7. Suppose G is a Lie group with Lie algebra $\mathfrak{g}$ and $G'\subset G$ is a normal subgroup with Lie subalgebra $\mathfrak{g}'\subset\mathfrak{g}$. Show that

$$\mathrm{Ad}_u(v')\in\mathfrak{g}'\ \ \forall\, u\in G,\ v'\in\mathfrak{g}',\qquad \text{and}\qquad [v,v']\in\mathfrak{g}'\ \ \forall\, v\in\mathfrak{g},\ v'\in\mathfrak{g}'\,.$$

For a Lie group G, we define

$$(T_{\mathbb{1}}G)_{\mathbb{Z}}=\big\{v\in T_{\mathbb{1}}G\colon \mathrm{e}^{v}=\{\mathbb{1}\}\big\}\quad \text{and}\quad (T_{\mathbb{1}}^*G)_{\mathbb{Z}}=\big\{\alpha\in T_{\mathbb{1}}^*G\colon \alpha(v)\in\mathbb{Z}\ \forall\, v\in(T_{\mathbb{1}}G)_{\mathbb{Z}}\big\}.$$

If $G'\subset G$ is a Lie subgroup, then

$$(T_{\mathbb{1}}G')_{\mathbb{Z}}=(T_{\mathbb{1}}G)_{\mathbb{Z}}\cap T_{\mathbb{1}}G'\subset T_{\mathbb{1}}G$$

by Exercise 1.3(b) and thus the image of $(T_{\mathbb{1}}^*G)_{\mathbb{Z}}$ under the restriction homomorphism $T_{\mathbb{1}}^*G\longrightarrow T_{\mathbb{1}}^*G$ is contained in $(T_{\mathbb{1}}^*G')_{\mathbb{Z}}$.

Exercise 1.8. Let G be an abelian Lie group. Show that

(a) $(T_{\mathbb{1}}G)_{\mathbb{Z}}\subset T_{\mathbb{1}}G$ and $(T_{\mathbb{1}}^*G)_{\mathbb{Z}}\subset T_{\mathbb{1}}^*G$ are lattices (not necessarily of full dimension);

(b) if G is connected, then the map

$$T_{\mathbb{1}}G/(T_{\mathbb{1}}G)_{\mathbb{Z}}\longrightarrow G,\qquad [v]\longrightarrow \mathrm{e}^{v},$$

is a well-defined isomorphism of Lie groups.

Hint: use Exercise 1.6 for (a) and Exercise 1.2(b) for surjectivity in (b).

Exercise 1.9. Let G be a Lie group and $\mathfrak{g}'\subset T_{\mathbb{1}}G$ be an abelian Lie subalgebra.

(a) Show that $\mathfrak{g}'_{\mathbb{Z}}\equiv(T_{\mathbb{1}}G)_{\mathbb{Z}}\cap\mathfrak{g}'$ is a lattice (not necessarily of full dimension).

(b) Show that the closure $\overline{G'}$ of $\exp(\mathfrak{g}')\subset G$ is a connected abelian Lie subgroup.

(c) Let $\mathfrak{g}''\subset T_{\mathbb{1}}G$ be another abelian Lie subalgebra. Suppose $\mathfrak{g}'\subset\mathfrak{g}''$, $\mathfrak{g}''$ is the $\mathbb{R}$-span of $\mathfrak{g}'$ and $\mathfrak{g}''_{\mathbb{Z}}$, and

$$\mathrm{Span}_{\mathbb{R}}\big(\mathfrak{g}''_{\mathbb{Z}}\big)\cap\mathfrak{g}'\not\subset\mathrm{Span}_{\mathbb{R}}\big(\mathfrak{g}'''_{\mathbb{Z}}\big)$$

for any abelian Lie subalgebra $\mathfrak{g}'''\subsetneq\mathfrak{g}''$. Show that $\exp(\mathfrak{g}'')\subset\overline{G'}$.

(d) Show that $T_{\mathbb{1}}\overline{G'}\subset T_{\mathbb{1}}G$ is the maximal abelian Lie subalgebra $\mathfrak{g}''$ as in (c).

By Exercise 1.8, a connected abelian Lie group is isomorphic to the product Lie group $(S^1)^k\times\mathbb{R}^m$ for some $k,m\in\mathbb{Z}^{\ge0}$. A torus $\mathbb{T}$ is a *compact* connected abelian Lie group. In such a case, $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\subset T_{\mathbb{1}}\mathbb{T}$ and $(T^*_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\subset T^*_{\mathbb{1}}\mathbb{T}$ are full-dimensional lattices, i.e. the homomorphisms

$$(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathbb{R}\longrightarrow T_{\mathbb{1}}\mathbb{T},\quad v\otimes c\longrightarrow cv,\qquad\text{and}\qquad (T^*_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathbb{R}\longrightarrow T^*_{\mathbb{1}}\mathbb{T},\quad \alpha\otimes c\longrightarrow c\alpha,$$

are isomorphisms of real vector spaces. The Lie algebra of a torus $\mathbb{T}$ is often denoted by $\mathfrak{t}$. We call $\alpha\in T^*_{\mathbb{1}}\mathbb{T}$ integral if $\alpha\in(T^*_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$, $\alpha\in(T^*_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$ primitive if $\alpha\neq k\alpha'$ for any $\alpha'\in(T^*_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$ and $k\in\mathbb{Z}$ with $k\ge2$, and a line segment in $T^*_{\mathbb{1}}\mathbb{T}$ rational if it is parallel to an integral element of $T^*_{\mathbb{1}}\mathbb{T}$.

Exercise 1.10. Let $\mathbb{T}$ be a torus.

(a) Suppose that $V\subset T_{\mathbb{1}}\mathbb{T}$ is a linear subspace. Show that

$$\mathrm{e}^V\equiv\big\{\mathrm{e}^v\colon v\in V\big\}\subset\mathbb{T}$$

is a subtorus if and only if V is the $\mathbb{R}$-span of a finite subset of $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$.

(b) Suppose $\mathbb{T}'\subset\mathbb{T}$ is a subtorus. Show that there exists a subtorus $\mathbb{T}'^c\subset\mathbb{T}$ so that the Lie group homomorphism

$$\mathbb{T}'\times\mathbb{T}'^c\longrightarrow\mathbb{T},\qquad (u',u'^c)\longrightarrow u'u'^c,$$

is an isomorphism. Conclude that the restriction homomorphism $(T^*_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\longrightarrow(T^*_{\mathbb{1}}\mathbb{T}')_{\mathbb{Z}}$ is surjective.

(c) Suppose $S\subset(T^*_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}-\{0\}$ and $\rho\colon\mathbb{R}^k\longrightarrow\mathbb{T}$ is a homomorphism with dense image. Show that

$$\rho^*\alpha\equiv\alpha\circ\mathrm{d}_0\rho\neq0\in T^*_0\mathbb{R}^k\quad\text{and}\quad\rho^*\alpha\neq\rho^*\alpha'\qquad\forall\ \alpha,\alpha'\in S,\ \alpha\neq\alpha'.$$

Let $\mathfrak{a}$ be a Lie algebra and V be a vector space (over $\mathbb{R}$). For $k\in\mathbb{Z}^{\ge0}$, define

$$\begin{gathered}d_{\mathfrak{a};k}\colon\mathrm{Hom}_{\mathbb{R}}\big(\Lambda^k_{\mathbb{R}}\mathfrak{a};V\big)\longrightarrow\mathrm{Hom}_{\mathbb{R}}\big(\Lambda^{k+1}_{\mathbb{R}}\mathfrak{a};V\big),\\ \{d_{\mathfrak{a};k}f\}\big(v_0,v_1,\ldots,v_k\big)=\sum_{0\le i<j\le k}(-1)^{i+j}f\big([v_i,v_j],v_0,v_1,\ldots,\widehat{v_i},\ldots,\widehat{v_j},\ldots,v_k\big),\end{gathered}\tag{1.15}$$

where $\widehat{v_i},\widehat{v_j}$ denote skipped inputs. By the Jacobi identity (1.7), $d_{\mathfrak{a};k}\circ d_{\mathfrak{a};k-1}=0$ for every $k\in\mathbb{Z}^+$. Thus, the k-th cohomology group of $\mathfrak{a}$ with coefficients in V,

$$H^k(\mathfrak{a};V)\equiv\frac{\ker\big(d_{\mathfrak{a};k}\colon\mathrm{Hom}_{\mathbb{R}}(\Lambda^k_{\mathbb{R}}\mathfrak{a};V)\longrightarrow\mathrm{Hom}_{\mathbb{R}}(\Lambda^{k+1}_{\mathbb{R}}\mathfrak{a};V)\big)}{\mathrm{Im}\big(d_{\mathfrak{a};k-1}\colon\mathrm{Hom}_{\mathbb{R}}(\Lambda^{k-1}_{\mathbb{R}}\mathfrak{a};V)\longrightarrow\mathrm{Hom}_{\mathbb{R}}(\Lambda^k_{\mathbb{R}}\mathfrak{a};V)\big)}$$

is well-defined. Let $H^k(\mathfrak{a})=H^k(\mathfrak{a};\mathbb{R})$. We note that

$$H^1\big(\mathfrak{a};V\big)\equiv\ker\big(d_{\mathfrak{a};1}\colon\mathrm{Hom}_{\mathbb{R}}(\mathfrak{a};V)\longrightarrow\mathrm{Hom}_{\mathbb{R}}(\Lambda^2_{\mathbb{R}}\mathfrak{a};V)\big).$$

If the Lie algebra $\mathfrak{a}$ is abelian, then $d_{\mathfrak{a};k}=0$ for all $k\in\mathbb{Z}^{\ge0}$ and so

$$H^k(\mathfrak{a};V)=\mathrm{Hom}_{\mathbb{R}}(\Lambda^k_{\mathbb{R}}\mathfrak{a};V).$$

Exercise 1.11. Suppose $\mathfrak{a}$ is a Lie algebra and $\mathfrak{a}' \subset \mathfrak{a}$ is a Lie subalgebra such that $[v, v'] \in \mathfrak{a}'$ whenever $v \in \mathfrak{a}$ and $v' \in \mathfrak{a}'$. Let

$$\mathcal{K}^1_{\mathfrak{a}';\mathfrak{a}} = \big\{\alpha \in \mathfrak{a}'^* : \alpha([v, v']) = 0 \ \forall\, v \in \mathfrak{a},\ v' \in \mathfrak{a}'\big\} \subset H^1(\mathfrak{a}').$$

Show that

(a) the short exact sequence

$$0 \longrightarrow (\mathfrak{a}/\mathfrak{a}')^* = \big\{\alpha \in \mathfrak{a} : \alpha|_{\mathfrak{a}'} = 0\big\} \longrightarrow \mathfrak{a}^* \longrightarrow \mathfrak{a}'^* \longrightarrow 0$$

of Lie algebras induces an exact sequence

$$0 \longrightarrow H^1(\mathfrak{a}/\mathfrak{a}') \longrightarrow H^1(\mathfrak{a}) \longrightarrow \mathcal{K}^1_{\mathfrak{a}';\mathfrak{a}} \xrightarrow{\delta_1} H^2(\mathfrak{a}/\mathfrak{a}') \longrightarrow H^2(\mathfrak{a})\,;$$

(b) if $\mathfrak{a}$ is abelian, then $\delta_1 = 0$ above.

1.3 Almost complex structures

An almost complex structure J on a smooth manifold X is a complex structure on (the fibers of) the real vector bundle TX over X, i.e.

$$J \in \Gamma\big(X; \mathrm{End}_{\mathbb{R}}(TX)\big) \qquad \text{and} \qquad J^2 = -\mathrm{Id}_{TX}\,.$$

An almost complex manifold is a pair (X, J) consisting of a smooth manifold X and an almost complex structure J on X. Since there is a canonical identification

$$T\mathbb{C}^n \approx \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$$

of real vector bundles over $\mathbb{C}^n$, the scalar multiplication by $\mathfrak{i}$ on the vector space $\mathbb{C}^n$ determines an almost complex structure $J_{\mathbb{C}^n}$ on $\mathbb{C}^n$. A complex structure on a manifold X, i.e. an atlas of coordinate charts that overlap holomorphically, determines an almost complex structure J on X: if

$$\varphi_\alpha : U_\alpha \longrightarrow \varphi(U_\alpha) \subset \mathbb{C}^n$$

is a chart in the chosen atlas, then

$$J|_{TU_\alpha} = \big\{\mathrm{d}\varphi_\alpha\big\}^{-1} \circ J_{\mathbb{C}^n} \circ \mathrm{d}\varphi_\alpha : TU_\alpha \longrightarrow TU_\alpha\,. \tag{1.16}$$

Such an almost complex structure J is called integrable or simply a complex structure on X.

Exercise 1.12. Let X be a complex manifold with a holomorphic atlas $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ of coordinate charts as above. Show that the definitions of the almost complex structure J in (1.16) agree on the overlaps $U_\alpha \cap U_\beta$.

Let V be a real vector bundle over a smooth manifold X with a complex structure J on the fibers. We call a k-tensor g on V, with values in $\mathbb{R}$ or in some vector bundle over X, J-invariant if

$$g\big(Jv_1, \ldots, Jv_k\big) = g(v_1, \ldots, v_k) \quad \forall\ v_1, \ldots, v_k \in V_x,\ x \in X.$$

We call such a tensor J-anti-invariant if

$$g(Jv_1,\ldots,Jv_k) = -g(v_1,\ldots,v_k) \quad \forall\, v_1,\ldots,v_k \in V_x,\ x\in X.$$

The Nijenhuis tensor A_J of an almost complex manifold (X,J) is defined by

$$A_J(\xi_1,\xi_2) = \frac{1}{4}\Big([\xi_1,\xi_2]+J[\xi_1,J\xi_2]+J[J\xi_1,\xi_2]-[J\xi_1,J\xi_2]\Big) \quad \forall\, \xi_1,\xi_2 \in \Gamma(X;TX). \tag{1.17}$$

Exercise 1.13. Let (X,J) be an almost complex manifold. Show that

(a) equation (1.17) determines a J-anti-invariant alternating 2-tensor

$$A_J \in \Gamma\big(X;\mathrm{Hom}_{\mathbb{C}}\big((\Lambda^2_{\mathbb{C}}(TX,J)),(TX,-J)\big)\big) \subset \Gamma\big(X;\mathrm{Hom}_{\mathbb{R}}(TX\otimes_{\mathbb{R}}TX,TX)\big);$$

(b) if $\xi_1,\xi_2\in\Gamma(M;TM)$, then

$$A_J(\xi_1,\xi_2) = \frac{1}{4}\big(J(\mathcal{L}_{\xi_1}J)\xi_2-(\mathcal{L}_{J\xi_1}J)\xi_2\big),$$

where $\mathcal{L}$ is the Lie derivative;

(c) if J is integrable, then $A_J=0$;

(d) if $Y\subset X$ is a submanifold such that $J(TY)\subset TY$, then $A_{J|TY}=A_J|_{TY}$.

The converse of Exercise 1.13(c) is [50, Theorem 1.1]. Since the Nijenhuis tensor $N_{\mathfrak{j}}$ of any two-dimensional almost complex manifold $(\Sigma,\mathfrak{j})$ vanishes, it follows that every almost complex structure $\mathfrak{j}$ on a two-dimensional manifold is integrable. We call such a pair $(\Sigma,\mathfrak{j})$ a Riemann surface.

Let V be again a real vector bundle over a smooth manifold X. A complex structure J on the fibers of V is said to be compatible with a 2-form ω on (the fibers of) V if ω is J-invariant and

$$\omega(v,Jv) > 0 \quad \forall\, v\in V_x-\{0\},\ x\in X;$$

the last condition implies that the restriction of ω to every complex subbundle V' of (V,J), including V itself, is a nondegenerate 2-form on V'. If J is compatible with a 2-form ω on V, then the 2-tensor g_J^ω on V defined by

$$g_J^\omega(v,v') = \omega(v,Jv') \quad \forall\, v,v'\in V_x,\ x\in X\,, \tag{1.18}$$

is a J-invariant metric on (the fibers of) V. Conversely, if g is a J-invariant metric on V, then J is compatible with the 2-form on V defined by

$$\omega_J^g(v,v') = g(Jv,v') \quad \forall\, v,v'\in V_x,\ x\in X\,.$$

We note that $g_J^{\omega_J^g}=g$ for every J-invariant metric g on V and $\omega_J^{g_J^\omega}=\omega$ for every 2-form ω on V compatible with J.

Exercise 1.14. Let $n\in\mathbb{Z}^+$. The standard symplectic form $\omega_{\mathbb{C}^n}$ and (integrable) almost complex structure $J_{\mathbb{C}^n}$ on $\mathbb{C}^n$ are given by

$$\omega_{\mathbb{C}^n}=\sum_{j=1}^n \mathrm{d}x_j\wedge\mathrm{d}y_j=\frac{\mathfrak{i}}{2}\sum_{j=1}^n \mathrm{d}z_j\wedge\mathrm{d}\bar{z}_j \quad\text{and}\quad J_{\mathbb{C}^n}\frac{\partial}{\partial x_j}=\frac{\partial}{\partial y_j}\quad \forall\, j=1,2,\ldots,n, \tag{1.19}$$

respectively, where as usual $x_j,y_j\colon\mathbb{C}^n\longrightarrow\mathbb{R}$ are the real and imaginary parts of the j-th complex coordinate z_j on $\mathbb{C}^n$, i.e. $z_j=x_j+\mathfrak{i}y_j$, and $\bar{z}_j=x_j-\mathfrak{i}y_j$. Show that the almost complex structure $J_{\mathbb{C}^n}$ is compatible with $\omega_{\mathbb{C}^n}$.

Exercise 1.15 ([41, Proof of Lemma 7.1.11(i)])**.** Let $n\in\mathbb{Z}^+$ and $J_{\mathbb{C}^n}$ be the standard complex structure on $\mathbb{C}^n$ as in (1.19).

(a) Show that a $\mathbb{C}$-valued 2-form ω on an open subset $U\subset\mathbb{C}^n$ is $J_{\mathbb{C}^n}|_U$-invariant if and only if

$$\omega=\sum_{j,k=1}^n \omega_{jk}\mathrm{d}z_j\wedge\mathrm{d}\bar{z}_k$$

for some smooth functions $\omega_{jk}\colon U\longrightarrow\mathbb{C}$.

For a smooth function $\beta\colon\mathbb{R}^+\longrightarrow\mathbb{R}^+$, define

$$F_\beta\colon\mathbb{C}^n-\{0\}\longrightarrow\mathbb{C}^n,\qquad F_\beta(z)=\beta\big(|z|\big)\frac{z}{|z|}\,.$$

Let $\omega_{\mathbb{C}^n}$ be the standard symplectic form on $\mathbb{C}^n$ as in (1.19). Show that

(b) $F_\beta^*\omega_{\mathbb{C}^n}=\dfrac{\mathfrak{i}}{2}\bigg(\dfrac{\beta(|z|)^2}{|z|^2}\displaystyle\sum_{j=1}^n\mathrm{d}z_j\wedge\mathrm{d}\bar{z}_j+\dfrac{|z|\beta(|z|)\beta'(|z|)-\beta(|z|)^2}{|z|^4}\sum_{j,k=1}^n\bar{z}_jz_k\mathrm{d}z_j\wedge\mathrm{d}\bar{z}_k\bigg)$;

(c) the closed 2-form $F_\beta^*\omega_{\mathbb{C}^n}$ on $\mathbb{C}^n-\{0\}$ is compatible with $J_{\mathbb{C}^n}|_{\mathbb{C}^n-\{0\}}$ if and only if $\beta'(r)>0$ for all $r\in\mathbb{R}^+$.

A Kähler manifold is a triple (X,ω,J) consisting of a smooth manifold X, a symplectic form ω on X, and an integrable almost complex J which is compatible with ω. The most basic example of a Kähler manifold is the complex Euclidean space $\mathbb{C}^n$, with the standard symplectic form $\omega_{\mathbb{C}^n}$, and the standard complex structure $J_{\mathbb{C}^n}$, as in (1.19). The paradigmatic example of a compact Kähler manifold is the complex projective space $\mathbb{C}P^n$, with the Fubini-Study symplectic form $\omega_{\mathrm{FS};n}$, and the standard complex structure $J_{\mathbb{C}P^n}$, as in Exercise 1.27.

Let $V\longrightarrow X$ be a real vector bundle over a smooth manifold. We denote by

$$\mathcal{J}(V)\subset\Gamma\big(X;\mathrm{End}_{\mathbb{R}}(V)\big)\qquad\text{and}\qquad\Omega^2_\bullet(V)\subset\Gamma\big(X;\mathrm{Hom}_{\mathbb{R}}(\Lambda^2_{\mathbb{R}}V,\mathbb{R})\big)$$

the spaces of (smooth) complex structures on the fibers of V and of nondegenerate 2-forms on V, respectively, with the C^∞-topologies. For a metric g and a nondegenerate 2-form ω on V, we denote by

$$\mathcal{J}(g)\equiv\big\{J\in\mathcal{J}(V)\colon g\big(Jv,Jv'\big)=g(v,v')\ \forall\ v,v'\in V_x,\ x\in X\big\}$$

and $\mathcal{J}_{\mathrm{cm}}(\omega)\subset\mathcal{J}(V)$ the subspaces of $\mathcal{J}(V)$ consisting of complex structures on V preserving g and compatible with ω, respectively. The next proposition is the main result of this section. Its proof is based on [33, Appendix].

Proposition 1.16. *Let V be a real vector bundle over a smooth manifold X. A metric g on (the fibers of) V determines a homotopy inverse*

$$\Omega^2_\bullet(V) \longrightarrow \mathcal{J}(g), \qquad \omega \longrightarrow J_{g,\omega}\,, \tag{1.20}$$

to the map

$$\mathcal{J}(g) \longrightarrow \Omega^2_\bullet(V), \qquad J \longrightarrow \omega^g_J\,,$$

so that $J_{g,\omega}\in\mathcal{J}_{\mathrm{cm}}(\omega)$ for every $\omega\in\Omega^2_\bullet(V)$. If G is a Lie group acting smoothly on X and preserving the metric g, then the map (1.20) is G-equivariant.

Proof. For a 2-form ω on the fibers of V, we define

$$A_{g,\omega} \in \Gamma\big(X;\mathrm{End}_{\mathbb{R}}(V)\big) \qquad \text{by} \quad g\big(A_{g,\omega}v,v'\big)=\omega(v,v') \;\; \forall\, v,v'\in V_x,\, x\in X. \tag{1.21}$$

Since ω is anti-symmetric in the two inputs,

$$g\big(A_{g,\omega}v,v'\big) = -g\big(v,A_{g,\omega}v'\big) \;\; \forall\, v,v'\in V_x,\, x\in X,$$

i.e. the transpose $A^{\mathrm{tr}}_{g,\omega}$ of $A_{g,\omega}$ with respect to g equals $-A_{g,\omega}$. If $\omega\in\Omega^2_\bullet(V)$, $A_{g,\omega}$ is an automorphism of V. The Polar Decomposition Theorem [30, Proposition 2.19(1)] then implies that there exist unique automorphisms $J_{g,\omega}, S_{g,\omega}$ of V such that

$$\begin{gathered} A_{g,\omega} = J_{g,\omega}\circ S_{g,\omega}, \qquad g\big(S_{g,\omega}v,v\big)>0 \;\; \forall\, v\in V_x-\{0\},\, x\in X,\\ g\big(J_{g,\omega}v,J_{g,\omega}v'\big)=g(v,v') \text{ and } g\big(S_{g,\omega}v,v'\big)=g\big(v,S_{g,\omega}v'\big) \;\; \forall\, v,v'\in V_x,\, x\in X. \end{gathered} \tag{1.22}$$

We denote by $J^{\mathrm{tr}}_{g,\omega}$ the transpose of $J_{g,\omega}$ with respect to g.

By the penultimate equation in (1.22),

$$J^{\mathrm{tr}}_{g,\omega}\circ J_{g,\omega},\, J_{g,\omega}\circ J^{\mathrm{tr}}_{g,\omega} = \mathrm{Id}_V.$$

Combined with the first and last equations in (1.22) and $A^{\mathrm{tr}}_{g,\omega}=-A_{g,\omega}$, this gives

$$\begin{aligned} \big(J_{g,\omega}\circ J_{g,\omega}\big)\circ\big(J^{\mathrm{tr}}_{g,\omega}\circ S_{g,\omega}\circ J_{g,\omega}\big) &= J_{g,\omega}\circ S_{g,\omega}\circ J_{g,\omega}\\ &= A_{g,\omega}\circ J_{g,\omega} = -A^{\mathrm{tr}}_{g,\omega}\circ J_{g,\omega} = -S_{g,\omega}\circ J^{\mathrm{tr}}_{g,\omega}\circ J_{g,\omega} = -S_{g,\omega}\,. \end{aligned}$$

From the last equation in (1.22) and the uniqueness of polar decomposition, it then follows that

$$J^2_{g,\omega} = -\mathrm{Id}_V, \quad J^{\mathrm{tr}}_{g,\omega} = -J_{g,\omega}, \quad J^{\mathrm{tr}}_{g,\omega}\circ S_{g,\omega}\circ J_{g,\omega} = S_{g,\omega}\,.$$

Along with (1.21) and the first three equations in (1.22), this gives

$$\begin{gathered} \omega\big(v,J_{g,\omega}v\big)>0 \qquad \forall\, v\in V-\{0\},\, x\in X,\\ \omega\big(J_{g,\omega}v,J_{g,\omega}v'\big)=\omega\big(v,v'\big), \;\; \omega^g_{J_{g,\omega}}(v,v') = g\big(J_{g,\omega}v,v'\big) = \omega\big(S^{-1}_{g,\omega}v,v'\big) \qquad \forall\, v,v'\in V_x,\, x\in X. \end{gathered}$$

In particular, $J_{g,\omega}\in\mathcal{J}(g)\cap\mathcal{J}_{\mathrm{cm}}(\omega)$. Since the space of all automorphisms of V symmetric with respect to g is contractible (via the straight-line homotopy to the identity), the last identity above implies that the composition

$$\Omega^2_\bullet(V)\longrightarrow\mathcal{J}(g)\longrightarrow\Omega^2_\bullet(V), \qquad \omega\longrightarrow J_{g,\omega}\longrightarrow\omega^g_{J_{g,\omega}}\,,$$

is homotopic to the identity. If $J\in\mathcal{J}_{\mathrm{cm}}(g)$ and $\omega=\omega^g_J$, then $A_{g,\omega}=J$ by (1.21) and thus $J_{g,\omega}=J$ by the uniqueness of polar decomposition. This establishes the first claim. The last statement of the proposition also follows from the uniqueness of polar decomposition. $\square$

Exercise 1.17. Verify the claim made in the last sentence of the proof of Proposition 1.16.

1.4 Notation and terminology

For $k\in\mathbb{Z}^{\ge 0}$, let $[k]=\{1,2,\ldots,k\}$. If $H\colon X\longrightarrow\mathbb{R}$ is a smooth function, we denote by

$$\mathrm{Crit}(H)\equiv\big\{x\in X\colon \mathrm{d}_xH=0\big\}$$

its set of critical points. The gradient of H with respect to a Riemannian metric g on X is the vector field ∇^gH on M defined by

$$g\big(\nabla^gH|_x,w\big)=\mathrm{d}_xH(w)\qquad\forall\,x\in X,\,w\in T_xX. \tag{1.23}$$

Let X be a smooth manifold and G be a Lie group. For a map $\mu\colon X\longrightarrow T_{\mathbb{1}}^*G$ and $v\in T_{\mathbb{1}}G$, define

$$\mu_v\colon X\longrightarrow\mathbb{R},\qquad \mu_v(x)=\big\{\mu(x)\big\}(v). \tag{1.24}$$

A basis $v_1,\ldots,v_k$ for $T_{\mathbb{1}}G$ determines identifications

$$\mathbb{R}^k\longrightarrow T_{\mathbb{1}}G,\;\;(r_1,\ldots,r_k)\longrightarrow\sum_{i=1}^k r_iv_i,\quad\text{and}\quad T_{\mathbb{1}}^*G\longrightarrow\mathbb{R},\;\;\alpha\longrightarrow\big(\alpha(v_1),\ldots,\alpha(v_k)\big). \tag{1.25}$$

The latter isomorphism identifies smooth (G-invariant) maps $\mu:X\longrightarrow T_{\mathbb{1}}^*G$ with smooth (G-invariant) maps $H\colon X\longrightarrow\mathbb{R}^k$ by

$$\mu\quad\longleftrightarrow\quad H\equiv\big(\mu_{v_1},\ldots,\mu_{v_k}\big). \tag{1.26}$$

A smooth (G-invariant) map $\mu\colon X\longrightarrow T_{\mathbb{1}}^*G$ satisfying the first condition in (1.6) with respect to a smooth action ψ of G on a symplectic manifold (X,ω) corresponds via (1.26) to a smooth (G-invariant) map

$$H\equiv(H_1,\ldots,H_k)\colon X\longrightarrow\mathbb{R}^k\qquad\text{s.t.}\quad \mathrm{d}H_i=-\iota_{\zeta_{v_i}}\omega\;\;\forall\,i\in[k], \tag{1.27}$$
$$\text{where}\qquad \zeta_{v_i}=\mathrm{d}_{\mathbb{1}}\psi(v_i)\in\Gamma(X;TX).$$

We call such a smooth function a weak Hamiltonian for the action ψ on (X,ω) with respect to basis $v_1,\ldots,v_k$ for $T_{\mathbb{1}}G$. If H corresponds to a moment map via (1.26), then we call it a Hamiltonian for the action ψ on (X,ω) with respect to basis $v_1,\ldots,v_k$ for $T_{\mathbb{1}}G$. If G is a connected abelian Lie group, a weak Hamiltonian is a Hamiltonian if and only if it is G-invariant.

We denote by $e_1,\ldots,e_k\in\mathbb{R}^k$ the standard orthonormal basis and by $\mathbb{R}_i\subset\mathbb{R}^k$ the $\mathbb{R}$-span of e_i. A smooth action ψ of $\mathbb{R}^k$ on a smooth manifold X (resp. symplectic manifold (X,ω)) is equivalent to k commuting smooth $\mathbb{R}$-actions $\psi_i\equiv\psi|_{\mathbb{R}_i}$ on X (resp. (X,ω)). In such a case, we will call

$$\zeta_i\equiv\mathrm{d}_{\mathbb{1}}\psi(e_i)=\frac{\mathrm{d}}{\mathrm{d}t}\psi_{i;t}\bigg|_{t=0}\in\Gamma(X;TX),\qquad i\in[k], \tag{1.28}$$

the generating vector fields of ψ and a Hamiltonian H as in (1.27) with respect to the basis $e_1,\ldots,e_k$ for $T_0\mathbb{R}^k$ simply a Hamiltonian for ψ. If $k=1$ or either X is compact, such a Hamiltonian corresponds to k smooth functions $H_i\colon X\longrightarrow\mathbb{R}$ satisfying the first condition in (1.27) with $\zeta_{v_i}=\zeta_i$; see

Proposition 2.26(3).

Suppose $\mathbb{T}$ is a torus and $v_1,\ldots,v_k\in T_{\mathbb{1}}\mathbb{T}$ is a $\mathbb{Z}$-basis for the lattice $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\subset T_{\mathbb{1}}\mathbb{T}$ and thus an $\mathbb{R}$-basis for $T_{\mathbb{1}}\mathbb{T}$. The isomorphisms in (1.25) then identify $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$ and $(T^*_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\subset T^*_{\mathbb{1}}\mathbb{T}$ with $\mathbb{Z}^k\subset\mathbb{R}^k$. The first isomorphism in (1.25) also induces a Lie group identification of $\mathbb{T}$ with the standard torus $\mathbb{T}^k$,

$$\phi_{v_1\ldots v_k}\colon \mathbb{T}^k\equiv(\mathbb{R}/\mathbb{Z})^k=\mathbb{R}^k/\mathbb{Z}^k\longrightarrow\mathbb{T},\quad \phi_{v_1\ldots v_k}\big([r_1,\ldots,r_k]\big)=\prod_{i=1}^{k}\mathrm{e}^{r_iv_i}\,. \tag{1.29}$$

Exercise 1.18. Suppose $\mathbb{T}$ is a torus and $v_1,\ldots,v_k\in T_{\mathbb{1}}\mathbb{T}$ is $\mathbb{Z}$-basis for the lattice $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\subset T_{\mathbb{1}}\mathbb{T}$. Let $\alpha_1,\ldots,\alpha_k\in(T^*_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$ be the dual basis. For $u\in\mathbb{T}$, let $m_u\colon\mathbb{T}\longrightarrow\mathbb{T}$ be the multiplication by u. Show that the diffeomorphism

$$\Phi_{v_1\ldots v_k}\colon\mathbb{R}^k\times\mathbb{T}^k\longrightarrow T^*_{\mathbb{1}}\mathbb{T}\times\mathbb{T},\quad \Phi_{v_1\ldots v_k}\big((x_1,\ldots,x_k),[r]\big)=\bigg(\sum_{i=1}^{k}x_i\alpha_i,\phi_{v_1\ldots v_k}\big([r]\big)\bigg), \tag{1.30}$$

is equivariant with respect to the identification $\phi_{v_1\ldots v_k}$ in (1.29) and satisfies

$$\begin{gathered}\mathrm{dpr}_2\big(\{\mathrm{d}\Phi_{v_1\ldots v_k}\}(\partial_{x_i})\big)=0\qquad\text{and}\\ \big\{\mathrm{pr}_1\big(\Phi_{v_1\ldots v_k}\big((x_1,\ldots,x_k),[r]\big)\big)\big\}\Big(\mathrm{d}_{\phi_{v_1\ldots v_k}([r])}m_{\phi_{v_1\ldots v_k}([-r])}\big(\mathrm{dpr}_2\big(\{\mathrm{d}\Phi_{v_1\ldots v_k}\}(\partial_{r_i})\big)\big)\Big)=x_i\end{gathered}\quad\forall\,i\in[k], \tag{1.31}$$

where $\mathrm{pr}_1,\mathrm{pr}_2\colon T^*_{\mathbb{1}}\mathbb{T}\times\mathbb{T}\longrightarrow T^*_{\mathbb{1}}\mathbb{T},\mathbb{T}$ are the projections, ∂_{x_i} is the i-th coordinate vector field on $\mathbb{R}^k$, and ∂_{r_i} is the coordinate vector field on $\mathbb{T}^k$ induced by the i-th coordinate vector field on $\mathbb{R}^k$.

We take the standard $\mathbb{Z}$-basis for $(T_{\mathbb{1}}\mathbb{T}^k)_{\mathbb{Z}}$ to be

$$2\pi\mathrm{i}e_1,\ldots,2\pi\mathrm{i}e_k\in T_{\mathbb{1}}\mathbb{T}^k\subset T_{(1,\ldots,1)}\mathbb{C}^k.$$

A smooth action ψ of $\mathbb{T}^k$ on a smooth manifold X (resp. symplectic manifold (X,ω)) is equivalent to k commuting smooth S^1-actions $\psi_i\equiv\psi|_{S^1_i}$ on X (resp. (X,ω)), where $S^1_i\subset\mathbb{T}^k=(S^1)^k$ is the i-th component subgroup S^1. Similarly to the affine case, we then call

$$\zeta_i\equiv\mathrm{d}_{\mathbb{1}}\psi(2\pi\mathrm{i}e_i)=\frac{\mathrm{d}}{\mathrm{d}t}\psi_{i;\mathrm{e}^{2\pi\mathrm{i}t}}\bigg|_{t=0}\in\Gamma(X;TX),\qquad i\in[k],$$

the generating vector fields of ψ and a Hamiltonian H as in (1.27) with respect to the basis $2\pi\mathrm{i}e_1,\ldots,2\pi\mathrm{i}e_k$ for $T_{\mathbb{1}}\mathbb{T}^k$ simply a Hamiltonian for ψ. By Proposition 2.26(3), such a Hamiltonian corresponds to k smooth functions $H_i\colon X\longrightarrow\mathbb{R}$ satisfying the condition in (1.27) with $\zeta_{v_i}=\zeta_i$.

Exercise 1.19. Suppose ψ is a smooth action of a Lie group G on a manifold X as in (1.1), ω is a symplectic form on X, $\mu\colon X\longrightarrow T^*_{\mathbb{1}}G$ is a smooth map, and $\rho\colon G'\longrightarrow G$ is a Lie group homomorphism. Let

$$\rho^*\equiv(\mathrm{d}_{\mathbb{1}}\rho)^*\colon T^*_{\mathbb{1}}G\longrightarrow T^*_{\mathbb{1}}G'$$

the homomorphism induced by ρ. Show that

(a) $\psi\circ\rho$ is a smooth action of G' on X and the map $\rho^*\circ\mu\colon X\longrightarrow T^*_{\mathbb{1}}G'$ is smooth;

(b) if the action ψ preserves ω, so does the action $\psi\circ\rho$;

(c) if the map μ satisfies the first (resp. second) condition in (1.6), then the map $\rho^*\circ\mu$ satisfies the first (resp. second) condition in (1.6) with (ψ,μ) replaced by $(\psi\circ\rho,\rho^*\circ\mu)$.

Exercise 1.20. Suppose ψ is a smooth $\mathbb{R}^k$-action on a manifold X as in (1.1), ω is a symplectic form on X, $H\colon X\longrightarrow\mathbb{R}^k$ is a smooth map, and A is a real $k\times m$-matrix (determining a linear map from $\mathbb{R}^m$ to $\mathbb{R}^k$). Show that

(a) $\psi\circ A$ is a smooth $\mathbb{R}^m$-action on X and the map $A^{\mathrm{tr}}\circ H\colon X\longrightarrow\mathbb{R}^m$ is smooth;

(b) if the action ψ preserves ω, so does the action $\psi\circ A$;

(c) if the map H is a weak Hamiltonian for the action ψ on (X,ω) (resp. $\mathbb{R}^k$-invariant), then the map $A^{\mathrm{tr}}\circ H$ is a weak Hamiltonian for the action $\psi\circ A$ on (X,ω) (resp. $\mathbb{R}^m$-invariant).

Exercise 1.21. Suppose ψ is a smooth action of a Lie group G on a manifold (X,ω) as in (1.1), ω is a symplectic form on X, $\mu\colon X\longrightarrow T^*_{\mathbb{1}}G$ is a smooth map, and $G'\subset G$ is a subgroup acting trivially on X. Let $q\colon G\longrightarrow G/G'$ be the quotient projection. Show that

(a) if $G'\subset G$ is a normal subgroup, then the action ψ of G determines a smooth action ψ' of the Lie group G/G' on X so that $\psi=\psi'\circ q$;

(b) if the action ψ preserves ω and $G'\subset G$ is a normal subgroup, then the induced action ψ' of G/G' on X also preserves ω;

(c) if X is connected and the map μ satisfies the first condition in (1.6), then

$$\alpha\big|_{T_{\mathbb{1}}G'}=\alpha_0\big|_{T_{\mathbb{1}}G'}\qquad\forall\ \alpha,\alpha_0\in\mu(X);$$

(d) if X is connected, the map μ satisfies the first condition in (1.6), $G'\subset G$ is a normal subgroup, and $\alpha_0\in\mu(X)$, then the smooth map

$$\mu-\alpha_0\colon X\longrightarrow\big\{\alpha\in T^*_{\mathbb{1}}G\colon\alpha|_{T_{\mathbb{1}}G'}=0\big\}=T_{\mathbb{1}}(G/G')\tag{1.32}$$

satisfies the first condition in (1.6) with (ψ,μ) replaced by $(\psi',\mu-\alpha_0)$;

(e) if (X,ω,ψ,μ) is a Hamiltonian G-manifold, $G'\subset G$ is a normal subgroup, and

$$\alpha_0\in(T^*_{\mathbb{1}}G)^G\equiv\big\{\alpha\in T^*_{\mathbb{1}}G\colon\mathrm{Ad}^*_u(\alpha)=\alpha\,\forall\,u\in G\big\}$$

satisfies (1.32), then $(X,\omega,\psi',\mu-\alpha_0)$ is a Hamiltonian G/G'-manifold;

(f) if G is abelian, (X,ω,ψ,μ) is a Hamiltonian G-manifold, and $\alpha_0\in T^*_{\mathbb{1}}G$ satisfies (1.32), then $(X,\omega,\psi',\mu-\alpha_0)$ is a Hamiltonian G/G'-manifold.

Exercise 1.22. Suppose $\widetilde\psi$ and ψ are smooth actions of a Lie group G on manifolds $\widetilde X$ and X, respectively, $\widetilde Y\subset\widetilde X$ is a smooth submanifold preserved by $\widetilde\psi$, $p\colon\widetilde Y\longrightarrow X$ is a G-equivariant surjective submersion, $\widetilde\omega$ and ω are symplectic forms on $\widetilde X$ and X, respectively, such that $p^*\omega=\widetilde\omega|_{T\widetilde Y}$, and

$$\widetilde\mu\colon\widetilde X\longrightarrow T^*_{\mathbb{1}}G\qquad\text{and}\qquad\mu\colon X\longrightarrow T^*_{\mathbb{1}}G$$

are maps such that $\mu\circ p=\widetilde\mu|_{\widetilde Y}$. Show that

(a) if the action $\widetilde{\psi}$ preserves $\widetilde{\omega}$, then the action ψ also preserves ω;

(b) if the map $\widetilde{\mu}$ is smooth, then the map μ is also smooth;

(c) if the map $\widetilde{\mu}$ is smooth and satisfies the first (resp. second) condition in (1.6) with (X,ω,ψ,μ) replaced by $(\widetilde{X},\widetilde{\omega},\widetilde{\psi},\widetilde{\mu})$, then the map μ is also smooth and satisfies the first (resp. second) condition in (1.6) as stated.

1.5 Paradigmatic examples

This section consists entirely of exercises. They introduce some of the most important examples of symplectic manifolds with Hamiltonian actions by Lie groups. We begin with four (non-compact) affine examples.

Exercise 1.23. Let $n\in\mathbb{Z}^+$, $k\in[n]$, and $\omega_{\mathbb{C}^n}$ be the standard symplectic form on $\mathbb{C}^n$ as in (1.19). Show that the action of S^1 on $\mathbb{C}^n$ given by

$$u\cdot(z_1,\ldots,z_n)=\big(z_1,\ldots,z_{k-1},uz_k,z_{k+1},\ldots,z_n\big)\qquad\forall\,u\in S^1\equiv\mathbb{R}/\mathbb{Z}\subset\mathbb{C}^*,$$

is Hamiltonian with respect to $\omega_{\mathbb{C}^n}$ with a Hamiltonian

$$H\colon\mathbb{C}^n\longrightarrow\mathbb{R},\qquad H(z_1,\ldots,z_n)=\pi|z_k|^2.$$

Exercise 1.24. With n and $\omega_{\mathbb{C}^n}$ as in Exercise 1.23, let ψ be the action of $\mathbb{C}^n$ on $(\mathbb{C}^n,\omega_{\mathbb{C}^n})$ by translations, i.e.

$$\psi_u\colon\mathbb{C}^n\longrightarrow\mathbb{C}^n,\quad \psi_u(z)=z+u,\qquad\forall\,u\in\mathbb{C}^n.$$

Let $e_1,\ldots,e_n$ be the standard $\mathbb{C}$-basis for $\mathbb{C}^n$ and $z_j=x_j+\mathfrak{i}y_j$ be the split of the j-th complex coordinate on $\mathbb{C}^n$ into real ones. Show that

$$H\equiv\big(-y_1+a_1,x_1+b_1,\ldots,-y_n+a_n,x_n+b_n\big)\colon\mathbb{C}^n\longrightarrow\mathbb{R}^{2n}\tag{1.33}$$

is a weak Hamiltonian for the action ψ of $\mathbb{C}^n$ on $(\mathbb{C}^n,\omega_{\mathbb{C}^n})$ with respect to the $\mathbb{R}$-basis $e_1,\mathfrak{i}e_1,\ldots,e_n,\mathfrak{i}e_n$ for $T_0\mathbb{C}^n=\mathbb{C}^n$ for any $a_1,b_1,\ldots,a_n,b_n\in\mathbb{R}$ and that every weak Hamiltonian for the action ψ of $\mathbb{C}^n$ on $(\mathbb{C}^n,\omega_{\mathbb{C}^n})$ with respect to this basis is given by (1.33) for some $a_1,b_1,\ldots,a_n,b_n\in\mathbb{R}$. Conclude that the action ψ of $\mathbb{C}^n$ on $(\mathbb{C}^n,\omega_{\mathbb{C}^n})$ is not Hamiltonian.

Exercise 1.25. Suppose $(V,\mathfrak{i})$ is a finite-dimensional complex vector space and Ω is a nondegenerate 2-form on V compatible with $\mathfrak{i}$, i.e.

$$\Omega(w,\mathfrak{i}w)>0\ \ \forall\,w\in V-\{0\},\quad \Omega(\mathfrak{i}w,\mathfrak{i}w')=\Omega(w,w')\ \ \forall\,w,w'\in V.$$

Via the canonical identification $T_wV\approx V$ for each $w\in V$, $\mathfrak{i}$ and Ω determine an almost complex structure J on V and a symplectic form ω compatible with J. Let $\psi\colon\mathbb{T}\longrightarrow\mathrm{GL}_{\mathbb{C}}V$ be a complex representation of a torus $\mathbb{T}$ on V. Show that

(a) there exist a subset $S\subset(T_{\mathbb{1}}^*\mathbb{T})_{\mathbb{Z}}$ and a splitting

$$V=\bigoplus_{\alpha\in S}V_\alpha\qquad\text{s.t.}\quad \psi_{\mathrm{e}^v}\big((w_\alpha)_{\alpha\in S}\big)=\big(\mathrm{e}^{2\pi\mathfrak{i}\,\alpha(v)}w_\alpha\big)_{\alpha\in S}\ \ \forall\,v\in T_{\mathbb{1}}\mathbb{T},\ (w_\alpha)_{\alpha\in S}\in\bigoplus_{\alpha\in S}V_\alpha\,;$$

(b) the action ψ is Hamiltonian with respect to ω with a moment map

$$\mu\colon V \longrightarrow T_{\mathbb{1}}^*\mathbb{T}, \qquad \mu\big((w_\alpha)_{\alpha\in S}\big) = \pi\sum_{\alpha\in S}|w_\alpha|^2\alpha \quad \forall\ (w_\alpha)_{\alpha\in S}\in\bigoplus_{\alpha\in S}V_\alpha\,,$$

where $|\cdot|$ is the norm on V with respect to the metric $g_{\mathfrak{i}}^{\Omega}(\cdot,\cdot)\equiv\Omega(\cdot,\mathfrak{i}\cdot)$.

Exercise 1.26. Let $k,n\in\mathbb{Z}^+$, $\mathrm{Mat}_{n\times k}\mathbb{C}\approx\mathbb{C}^{nk}$ be the vector space of $n\times k$ complex matrices, and $\omega_{n\times k}$ be the standard symplectic form on $\mathrm{Mat}_{n\times k}\mathbb{C}$ as (1.19). For $z\in\mathrm{Mat}_{n\times k}\mathbb{C}$, denote by $z^*\in\mathrm{Mat}_{k\times n}\mathbb{C}$ the adjoint (conjugate transpose) of z. In particular,

$$\mathrm{U}_k = \big\{u\in\mathrm{Mat}_{k\times k}\mathbb{C}\colon u^*u=\mathrm{Id}_k\big\} \quad\text{and}\quad \mathfrak{u}_k \equiv T_{\mathbb{1}}\mathrm{U}_k = \big\{v\in\mathrm{Mat}_{k\times k}\mathbb{C}\colon v^*=-v\big\}.$$

Show that

(a) $\omega_{n\times k}(w_1,w_2)=\dfrac{1}{2\mathfrak{i}}\mathrm{tr}\big(w_1^*w_2-w_2^*w_1\big)$ for all $w_1,w_2\in T_z\big(\mathrm{Mat}_{n\times k}\mathbb{C}\big)=\mathrm{Mat}_{n\times k}\mathbb{C}$ and $z\in\mathrm{Mat}_{n\times k}\mathbb{C}$ (the trace $\mathrm{tr}A$ of a square matrix A is the sum of its diagonal entries);

(b) the maps

$$\begin{aligned}
\psi_{n\times k}^{\mathrm{L}}&\colon \mathrm{U}_n \longrightarrow \mathrm{GL}_{\mathbb{C}}\big(\mathrm{Mat}_{n\times k}\mathbb{C}\big), & \psi_{n\times k;u}^{\mathrm{L}}(z) &= uz & &\forall\,u\in\mathrm{U}_n,\ z\in\mathrm{Mat}_{n\times k}\mathbb{C}\,,\\
\psi_{n\times k}^{\mathrm{R}}&\colon \mathrm{U}_k \longrightarrow \mathrm{GL}_{\mathbb{C}}\big(\mathrm{Mat}_{n\times k}\mathbb{C}\big), & \psi_{n\times k;u}^{\mathrm{R}}(z) &= zu^{-1} & &\forall\,u\in\mathrm{U}_k,\ z\in\mathrm{Mat}_{n\times k}\mathbb{C}\,,
\end{aligned}$$

are commuting smooth actions of U_n and U_k, respectively, on $\mathrm{Mat}_{n\times k}\mathbb{C}$ with the corresponding vector fields as in (1.2) given by

$$\zeta_v^{\mathrm{L}}(z)=vz \quad \forall\ v\in\mathfrak{u}_n,\ z\in\mathrm{Mat}_{n\times k}\mathbb{C}, \qquad \zeta_v^{\mathrm{R}}(z)=-zv \qquad \forall\ \ v\in\mathfrak{u}_k,\ z\in\mathrm{Mat}_{n\times k}\mathbb{C}\,;$$

(c) the smooth maps

$$\begin{aligned}
\mu_{n\times k}^{\mathrm{L}}&\colon \mathrm{Mat}_{n\times k}\mathbb{C}\longrightarrow\mathfrak{u}_n^*, & \big\{\mu_{n\times k}^{\mathrm{L}}(z)\big\}(v) &= \frac{1}{2\mathfrak{i}}\mathrm{tr}\big(zz^*v\big) & &\forall\,v\in\mathfrak{u}_n,\ z\in\mathrm{Mat}_{n\times k}\mathbb{C}\,,\\
\mu_{n\times k}^{\mathrm{R}}&\colon \mathrm{Mat}_{n\times k}\mathbb{C}\longrightarrow\mathfrak{u}_k^*, & \big\{\mu_{n\times k}^{\mathrm{R}}(z)\big\}(v) &= \frac{\mathfrak{i}}{2}\mathrm{tr}\big(z^*zv\big) & &\forall\,v\in\mathfrak{u}_k,\ z\in\mathrm{Mat}_{n\times k}\mathbb{C}\,,
\end{aligned}$$

are a $\psi_{n\times k}^{\mathrm{R}}$-invariant moment map for the action $\psi_{n\times k}^{\mathrm{L}}$ and a $\psi_{n\times k}^{\mathrm{L}}$-invariant moment map for the action $\psi_{n\times k}^{\mathrm{R}}$, respectively, with respect to $\omega_{n\times k}$.

The next two examples introduce the paradigmatic compact Hamiltonian $\mathbb{T}$-manifold and a model (necessarily non-compact) symplectic manifold with a free Hamiltonian action of a torus $\mathbb{T}$ (the necessity of the latter to be non-compact is due to Exercise 2.22). Along with Exercise 4.5, the first example below illustrates Delzant's construction of symplectic toric manifolds and the study of their properties detailed in Chapter 6. The second example is in a sense the base for the proof of Delzant's Theorem 4.20 sketched in [42] and detailed in Chapter 4.

Exercise 1.27. Let $n\in\mathbb{Z}^+$, $\omega_{\mathbb{C}^n}$ be the standard symplectic form on $\mathbb{C}^n$ as in (1.19), and

$$q\colon \mathbb{C}^n-\{0\}\longrightarrow\mathbb{C}P^{n-1}$$

be the usual quotient projection.

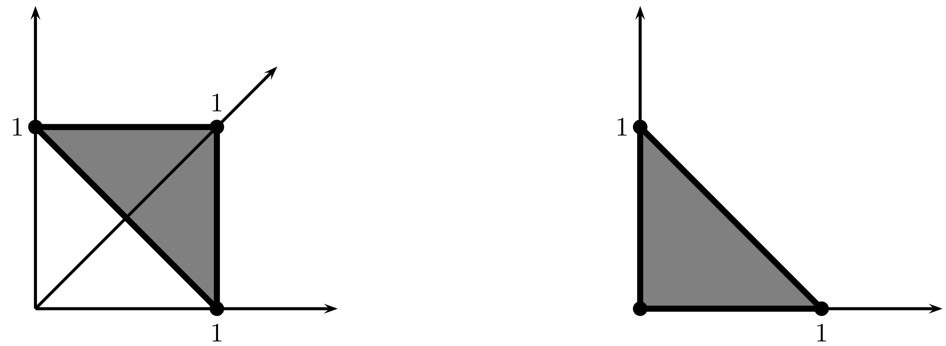


Figure 1.1: The images of $\mathbb{C}P^2$ under Hamiltonians for the torus actions of Exercise 1.27.

(a) Suppose $U\subset\mathbb{C}P^{n-1}$ is an open subset and $s\colon U\longrightarrow\mathbb{C}^n-\{0\}$ is a holomorphic section of q, i.e. $q\circ s=\mathrm{id}_U$. Show that the 2-form

$$\omega_{\mathrm{FS};n-1}\big|_U \equiv \frac{\mathfrak{i}}{2\pi}\partial\overline{\partial}\ln|s|^2, \tag{1.34}$$

where $|\cdot|$ is the standard (round) norm on $\mathbb{C}^n$, is independent of the choice of s.

(b) By (a), (1.34) determines a global 2-form $\omega_{\mathrm{FS};n-1}$ on $\mathbb{C}P^{n-1}$, called the Fubini-Study symplectic form. Show that this form is indeed symplectic,

$$q^*\omega_{\mathrm{FS};n-1}=\frac{\mathfrak{i}}{2\pi}\bigg(\frac{1}{|z|^2}\sum_{j=1}^{n}\mathrm{d}z_j\wedge\mathrm{d}\overline{z}_j-\frac{1}{|z|^4}\sum_{j,k=1}^{n}\overline{z}_jz_k\mathrm{d}z_j\wedge\mathrm{d}\overline{z}_k\bigg)\bigg|_{\mathbb{C}^n-\{0\}};$$
$$q^*\omega_{\mathrm{FS};n-1}\big|_{TS^{2n-1}}=\frac{1}{\pi}\omega_{\mathbb{C}^n}\big|_{TS^{2n-1}},\qquad\text{and}\qquad\int_{\mathbb{C}P^1}\omega_{\mathrm{FS};1}=1.$$

Hint. The restriction of q to the interior of the upper hemisphere $S^2_+\subset S^2=S^3\cap(\mathbb{C}\times\mathbb{R})$ is a diffeomorphism onto the complement of a point in $\mathbb{C}P^1$.

(c) Show that the actions of $\mathbb{T}^n\equiv(S^1)^n$ and $\mathbb{T}^{n-1}\equiv(S^1)^{n-1}$ on $\mathbb{C}P^{n-1}$ given by

$$\begin{gathered}(u_1,\ldots,u_n)\cdot[z_1,\ldots,z_n]=\big[u_1z_1,\ldots,u_nz_n\big],\\ (u_1\ldots,u_{n-1})\cdot[z_1,\ldots,z_n]=\big[u_1z_1,\ldots,u_{n-1}z_{n-1},z_n\big]\end{gathered}\tag{1.35}$$

are Hamiltonian with respect to the symplectic form $\omega_{\mathrm{FS};n-1}$. Determine the images of $\mathbb{C}P^{n-1}$ under moment maps for these action, in particular showing that in the $n=3$ they are as depicted in Figure 1.1, up to translations. *Hint:* use Exercises 1.22 and 1.23.

Exercise 1.28. Let $n\in\mathbb{Z}^+$ and

$$\mathrm{pr}_{\gamma_{n-1}}\colon\gamma_{n-1}\equiv\big\{(\ell,v)\in\mathbb{C}P^{n-1}\times\mathbb{C}^n\colon v\in\ell\big\}\longrightarrow\mathbb{C}P^{n-1},\qquad\mathrm{pr}_{\gamma_{n-1}}(\ell,v)=\ell,$$

be the tautological complex line bundle.

(a) Show that $\gamma_{n-1}\subset\mathbb{C}P^{n-1}\times\mathbb{C}^n$ is a complex submanifold.

(b) For each $i\in[n]$, define

$$\widetilde\varphi_{n;i}\equiv\big(\widetilde\varphi_{n;i;1},\ldots,\widetilde\varphi_{n;i;n}\big)\colon \widetilde U_{n;i}\equiv\big\{\big([z_1,\ldots,z_n],v\big)\in\gamma_{n-1}\colon z_i\neq0\big\}\longrightarrow\mathbb C^n, \tag{1.36}$$
$$\widetilde\varphi_{n;i;j}\big([z_1,\ldots,z_n],(v_1,\ldots,v_n)\big)=\begin{cases}z_j/z_i, & \text{if } j\in[n]-\{i\};\\ v_i, & \text{if } j=i.\end{cases}$$

Show that $\{\widetilde\varphi_{n;i}\colon i\in[n]\}$ is an atlas of holomorphic coordinate charts for γ_{n-1}.

(c) Conclude that $\mathrm{pr}_{\gamma_{n-1}}\colon\gamma_{n-1}\longrightarrow\mathbb CP^{n-1}$ is a holomorphic line bundle.

Exercise 1.29 ([41, Proof of Lemma 7.1.11(ii)])**.** Suppose n, $\omega_{\mathbb C^n}$, and $(\mathbb CP^{n-1},\omega_{\mathrm{FS};n-1})$ are as in Exercise 1.27 and $\mathrm{pr}_{\gamma_{n-1}}\colon\gamma_{n-1}\longrightarrow\mathbb CP^{n-1}$ is as in Exercise 1.28. Let $\mathrm{pr}_2\colon\gamma_{n-1}\longrightarrow\mathbb C^n$ be the restriction of the projection to the second component, $c\in\mathbb R^+$,

$$\widetilde\omega_{n-1;c}=c\,\mathrm{pr}_{\gamma_{n-1}}^*\omega_{\mathrm{FS};n-1}+\mathrm{pr}_2^*\omega_{\mathbb C^n}\,,\quad\text{and}\quad F_c\colon\mathbb C^n-\{0\}\longrightarrow\mathbb C^n,\quad F_c(z)=\sqrt{|z|^2+c}\,\frac{z}{|z|}\,.$$

Show that

(a) $\widetilde\omega_{n-1;c}$ is a symplectic form on (the total space of) γ_{n-1};

(b) $\widetilde\omega_{n-1;c}|_{\gamma_{n-1}-\mathbb CP^{n-1}}=\big\{\mathrm{pr}_2|_{\gamma_{n-1}-\mathbb CP^{n-1}}\big\}^*F_c^*\omega_{\mathbb C^n}$. *Hint:* use Exercises 1.15(b) and 1.27(b).

Note: it follows from part (b) that $(\gamma_{n-1}-\mathbb CP^{n-1},\widetilde\omega_{n-1;c}|_{\gamma_{n-1}-\mathbb CP^{n-1}})$ is symplectomorphic to $(\mathbb C^n-\overline B^n_{\sqrt c}(0),\omega_{\mathbb C^n}|_{\mathbb C^n-\overline B^n_{\sqrt c}(0)})$, where $\overline B^n_{\sqrt c}(0)=\big\{z\in\mathbb C^n\colon|z|\le\sqrt c\big\}$.

Exercise 1.30. Suppose (Y,ω) is a symplectic manifold, $\mathrm{pr}\colon E\longrightarrow Y$ is a complex vector bundle, Ω is a fiberwise nondegenerate 2-form on E compatible with the fiberwise complex structure $\mathfrak i$ on E, so that $g_{\mathfrak i}^{\Omega}(\cdot,\cdot)\equiv\Omega(\cdot,\mathfrak i\cdot)$ is a fiberwise metric on E, and ∇ is a ($\mathbb C$-linear) connection in E with the associated projection $\mathrm{pr}_\nabla\colon TE\longrightarrow\mathrm{pr}^*E$ as in Lemma B.7. Thus, $\Omega_\nabla\equiv\mathrm{pr}_\nabla^*\Omega$ is a 2-form on the total space of E. Let $\zeta_E\in\Gamma(E;TE)$ be the canonical vertical vector field on E as in (B.5) and

$$\widetilde\omega=\mathrm{pr}^*\omega+\frac12\mathrm d\big(\iota_{\zeta_E}\Omega_\nabla\big). \tag{1.37}$$

Show that

(a) $\widetilde\omega|_{TE|_Y}=\omega\oplus\Omega$ under the canonical decomposition $TE|_Y=TY\oplus E$ as in (B.4);

(b) there exists a smooth function $\rho\colon Y\longrightarrow\mathbb R^+$ so that the restriction of the closed 2-form $\widetilde\omega$ to the neighborhood
$$\mathcal U_\rho\equiv\big\{w\in E\colon g_{\mathfrak i}^{\Omega}(w,w)<\rho(\mathrm{pr}(w))\big\}$$
of the zero section $Y\subset E$ is nondegenerate (and thus $\widetilde\omega|_{\mathcal U}$ is a symplectic form on $\mathcal U$);

(c) the smooth function
$$H\colon\mathcal U_\rho\longrightarrow\mathbb R,\qquad H(w)=\pi g_{\mathfrak i}^{\Omega}(w,w),$$
is a Hamiltonian for the restriction of the standard S^1-action on E to $\mathcal U_\rho$ with respect to $\widetilde\omega|_{\mathcal U}$.

Hint: use (B.19) and Exercise B.12. *Note:* by Exercise B.15, the closed 2-form $\widetilde\omega$ is nondegenerate on the entire total space E for a good choice of the connection ∇.

Exercise 1.31. Suppose n, $\mathrm{pr}_{\gamma_{n-1}}\colon \gamma_{n-1}\longrightarrow\mathbb{C}P^{n-1}$, $c\in\mathbb{R}^+$, $\omega_{\mathbb{C}^n}$, $\omega_{\mathrm{FS};n-1}$, and $\widetilde{\omega}_{n-1;c}$ are as in Exercise 1.29,

$$p\colon \mathbb{C}P^{n-1}\times\mathbb{C}^n \longrightarrow \gamma_{n-1}$$

is the orthogonal projection with respect to the standard inner-product on $\mathbb{C}^n$, and ∇ is the trivial connection in the trivial vector bundle $\mathbb{C}P^{n-1}\times\mathbb{C}^n\longrightarrow\mathbb{C}P^{n-1}$ as in Example B.2. Let Ω be the restriction of $\omega_{\mathbb{C}^n}$ to each fiber of $\mathrm{pr}_{\gamma_{n-1}}$ and ∇' be the connection in γ_{n-1} induced by ∇ and p as in Exercise B.4. With the notation as in (1.37), show that

$$\widetilde{\omega}_{n-1;c} = c\,\mathrm{pr}_{\gamma_{n-1}}^*\omega_{\mathrm{FS};n-1}+\frac{1}{2}\mathrm{d}\big(\iota_{\zeta_{\gamma_{n-1}}}\Omega_{\nabla'}\big).$$

Hint: use Exercise B.9.

Exercise 1.32. Let $k\in\mathbb{Z}^+$, $\mathbb{T}^k\equiv\mathbb{R}^k/\mathbb{Z}^k$ is the standard k-torus, and

$$x\equiv(x_1,\ldots,x_k), y\equiv(y_1,\ldots,y_k)\colon \mathbb{R}^k\times\mathbb{R}^k \longrightarrow \mathbb{R}^k$$

be the projections to the two components. Show that

(a) $\lambda_k\equiv\sum\limits_{i=1}^{k}x_i\wedge\mathrm{d}y_i$ is a well-defined 1-form on $\mathbb{R}^k\times\mathbb{T}^k$;

(b) the 2-form $\omega_k\equiv\mathrm{d}\lambda_k$ on $\mathbb{R}^k\times\mathbb{T}^k$ is symplectic;

(c) the action ψ_k of $\mathbb{T}^k$ on $\mathbb{R}^k\times\mathbb{T}^k$ given by

$$\psi_{\mathbb{T};[r]}\big(x,[y]\big) = \big(x,[y+r]\big),$$

is well-defined, free, and smooth, preserves λ_k and ω_k, and has

$$H_k\colon \mathbb{R}^k\times\mathbb{T}^k \longrightarrow \mathbb{R}^k, \qquad H_k\big(x,[y]\big) = x,$$

as a Hamiltonian with respect to ω_k.

The next two exercises construct a canonical 1-form λ_{T^*Y} and a canonical symplectic form ω_{T^*Y} on (the total space of) the cotangent bundle of any smooth manifold Y. Exercise 1.35 then shows that for any torus $\mathbb{T}$ the smooth manifold $T^*_{\mathbb{1}}\mathbb{T}\times\mathbb{T}$ carries a canonical Hamiltonian $\mathbb{T}$-manifold structure $(\omega_{\mathbb{T}},\psi_{\mathbb{T}},\mu_{\mathbb{T}})$ with free action of $\mathbb{T}$ by group multiplication on the second component. The same reasoning shows that $T^*_{\mathbb{1}}\mathbb{T}\times\mathbb{T}$ carries a canonical 1-form $\lambda_{\mathbb{T}}$, but it is not needed for our purposes.

Exercise 1.33. Let Y be a smooth manifold and λ_{T^*Y} be the 1-form on (the total space of) its cotangent bundle $\mathrm{pr}\colon T^*Y\longrightarrow Y$ given by

$$\lambda_{T^*Y}\big|_\theta(w) = \{\mathrm{pr}^*\theta\}(w) \equiv \theta\big(\{\mathrm{d}_\theta\mathrm{pr}\}(w)\big) \qquad \forall\ \theta\in T^*Y,\ w\in T_\theta(T^*Y).$$

For $c\in\mathbb{R}$, let $m_c\colon T^*Y\longrightarrow T^*Y$ be the scalar multiplication by c as in (B.1). Show that

(a) $\alpha^*\lambda_{T^*Y}=\alpha$ for every 1-form α on Y;

(b) $m_c^*\lambda_{T^*Y}=c\lambda_{T^*Y}$ for every $c\in\mathbb{R}$;

(c) $\{\{\mathrm{d}f\}^*\}^*\lambda_{T^*Y} = \lambda_{T^*Y'}$ for every diffeomorphism $f\colon Y \longrightarrow Y'$ between smooth manifolds (thus $\{\mathrm{d}f\}^*\colon T^*Y' \longrightarrow T^*Y$ is well-defined).

Exercise 1.34. Let Y be a smooth manifold and λ_{T^*Y} be the canonical 1-form on (the total space of) T^*Y as in Exercise 1.33. For $c\in\mathbb{R}$, let $m_c\colon T^*Y \longrightarrow T^*Y$ be the scalar multiplication by c as in (B.1). Show that

(a) $\omega_{TY^*} \equiv -\mathrm{d}\lambda_{T^*Y}$ is a symplectic form on (the total space of) T^*Y and $\omega_{T^*\mathbb{R}^n} = \omega_{\mathbb{C}^n}$ under the natural identification of $T^*\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ with $\mathbb{C}^n$;

(b) for every $y\in Y$, there is a canonical decomposition $T_y(T^*Y) = T_yY \oplus T_y^*Y$ and

$$\omega_{T^*Y}\big|_y(v,w) = \begin{cases} 0, & \text{if } v,w\in T_yY \text{ or } v,w\in T_y^*Y; \\ w(v), & v\in T_yY \text{ and } w\in T_y^*Y; \end{cases}$$

(c) $m_c^*\omega_{T^*Y} = c\,\omega_{T^*Y}$ for every $c\in\mathbb{R}$;

(d) $\{\{\mathrm{d}f\}^*\}^*\omega_{T^*Y} = \omega_{T^*Y'}$ for every diffeomorphism $f\colon Y \longrightarrow Y'$ between smooth manifolds (thus $\{\mathrm{d}f\}^*\colon T^*Y' \longrightarrow T^*Y$ is well-defined).

Let ψ be a smooth action of a Lie group G on a smooth manifold Y and $\mathrm{d}\psi^*$ be its lift to T^*Y as in Exercise 1.1. By Exercises 1.33(c) and 1.34(d), $\mathrm{d}\psi^*$ preserves the canonical 1-form λ_{T^*Y} on T^*Y and the canonical symplectic form ω_{T^*Y} on T^*Y, respectively.

Exercise 1.35. Let $\mathbb{T}$, $v_1,\ldots,v_k$, $\Phi_{v_1\ldots v_k}$, and m_u for $u\in\mathbb{T}$ be as in Exercise 1.18, λ_k, ω_k, ψ_k, and H_k be as in Exercise 1.32, and $\lambda_{T^*\mathbb{T}}$ and $\omega_{T^*\mathbb{T}}$ be as in Exercises 1.33 and 1.34, respectively. Define

$$\Phi_{\mathbb{T}}\colon T_{\mathbb{1}}^*\mathbb{T}\times\mathbb{T} \longrightarrow T^*\mathbb{T}, \quad \Phi_{\mathbb{T}}(\alpha,u) = \big\{\mathrm{d}_u m_{u^{-1}}\big\}^*\alpha, \quad \text{and} \quad \omega_{\mathbb{T}} = \big\{\Phi_{v_1\ldots v_k}^{-1}\big\}^*\omega_k\,.$$

Denote by $\psi_{\mathbb{T}}$ the action of $\mathbb{T}$ on $T_{\mathbb{1}}^*\mathbb{T}\times\mathbb{T}$ by the multiplication on the second component and by $\mu_{\mathbb{T}}\colon T_{\mathbb{1}}^*\mathbb{T}\times\mathbb{T} \longrightarrow T_{\mathbb{1}}^*\mathbb{T}$ the projection to the first component. Show that

(a) $\Phi_{v_1\ldots v_k}^*\Phi_{\mathbb{T}}^*\lambda_{T^*\mathbb{T}} = \lambda_k$ and $\omega_{\mathbb{T}} = -\Phi_{\mathbb{T}}^*\omega_{T^*\mathbb{T}}$;

(b) $(T_{\mathbb{1}}^*\mathbb{T}\times\mathbb{T}, \omega_{\mathbb{T}}, \psi_{\mathbb{T}}, \mu_{\mathbb{T}})$ is a Hamiltonian $\mathbb{T}$-manifold which does not depend on the choice of a $\mathbb{Z}$-basis $v_1,\ldots,v_k\in T_{\mathbb{1}}\mathbb{T}$ for the lattice $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \subset T_{\mathbb{1}}\mathbb{T}$ and is identified with $(\mathbb{R}^k\times\mathbb{T}^k, \omega_k, \psi_k, H_k)$ via $\Phi_{v_1\ldots v_k}$.

An automorphism $\phi\colon X \longrightarrow X$ of a set X is an involution if $\phi^2 = \mathrm{id}_X$. If (X,ω) is a symplectic manifold, a smooth involution ϕ on X is called anti-symplectic if $\phi^*\omega = -\omega$. A Lagrangian submanifold of a symplectic manifold (X,ω) is a submanifold $Y\subset X$ such that

$$\dim Y = \frac{1}{2}\dim X \qquad \text{and} \qquad \omega|_{TY} = 0.$$

Exercise 1.36. (a) Suppose ϕ is an involution on a neighborhood of $0\in\mathbb{R}^n$ with $\phi(0)=0$. Let $\mathrm{Jac}_0(\phi)\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be its Jacobian at 0 so that

$$\phi(x) = \big\{\mathrm{Jac}_0(\phi)\big\}x + Q(x)$$

for some quadratic term $Q\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ ($Q(0)=0$, $\mathrm{Jac}_0(Q)=0$) and all x in a neighborhood of $0\in\mathbb{R}^n$. Show that there exist neighborhoods U and W of $0\in\mathbb{R}^n$ so that

$$h\colon U \longrightarrow W, \qquad h(x) = x+\frac{1}{2}\big\{\mathrm{Jac}_0(\phi)\big\}Q(x),$$

is a well-defined diffeomorphism satisfying $h\circ\phi=\{\mathrm{Jac}_0(\phi)\}h$.

(b) Let X be a smooth manifold and $\phi: X \longrightarrow X$ be a smooth involution. Show that every connected component of the fixed locus of ϕ,

$$X^\phi \equiv \big\{x\in X\colon \phi(x)=x\big\},$$

is a smooth submanifold of X.

(c) Suppose in addition ω is a nondegenerate 2-form on X such that $\phi^*\omega=-\omega$. Show that $X^\phi\subset X$ is a Lagrangian submanifold of (X,ω).

Exercise 1.37. Suppose (X,ω) is a symplectic manifold, $Y\subset X$ is a Lagrangian submanifold, J is an ω-compatible almost complex structure on X, and ω_{T^*Y} is the canonical symplectic form on T^*Y as in Exercise 1.34. Show that

(a) $J(TY)\subset TX|_Y$ is a subbundle complementary to TY;

(b) the map $\Phi_{Y;J}\colon J(TY)\longrightarrow T^*Y$, $\Phi_{Y;J}(w)=\omega(\cdot,w)$, is an isomorphism of vector bundles over Y;

(c) $\Phi_{Y;J}^*\omega_{T^*Y}|_{T_y(J(TY))}=\omega|_{T_yX}$ under the canonical identification

$$T_y\big(J(TY)\big) = T_yY\oplus J(T_yY) = T_yX.$$

Exercise 1.38. Suppose (X,ω) and (X',ω') are symplectic manifolds of the same dimension and $f\colon X\longrightarrow X'$ is a smooth map. Show that f is a symplectomorphism with respect to ω and ω' if and only if the graph of f,

$$\mathrm{Gr}(f) \equiv \big\{\big(x,f(x)\big)\colon x\in X\big\} \subset X\times X',$$

is a Lagrangian submanifold of $X\times X'$ with respect to the symplectic form $\mathrm{pr}_1^*\omega-\mathrm{pr}_2^*\omega'$, where

$$\mathrm{pr}_1,\mathrm{pr}_2\colon X\times X' \longrightarrow X, X'$$

are the component projections.

Exercise 1.39. Let Y be a smooth manifold, λ_{T^*Y} and ω_{T^*Y} be the canonical 1-form and canonical symplectic form, respectively, on (the total space of) the cotangent bundle $\mathrm{pr}\colon T^*Y\longrightarrow Y$ of Y as in Exercise 1.34, and α be a 1-form on Y. Show that

(a) the submanifold $\alpha(Y)\subset T^*Y$ is Lagrangian with respect to ω_{T^*Y} if and only if $\mathrm{d}\alpha=0$;

(b) the map

$$\phi_\alpha\colon T^*Y \longrightarrow T^*Y, \quad \phi_\alpha(\theta) = 2\alpha_{\mathrm{pr}(\theta)}-\theta,$$

is a smooth involution satisfying $\phi_\alpha^*\lambda_{T^*Y}=2\mathrm{pr}^*\alpha-\lambda_{T^*Y}$;

(c) the involution ϕ_α above is anti-symplectic with respect to the symplectic form ω_{T^*Y} of Exercise 1.34 if and only if $\mathrm{d}\alpha=0$.

1.6 Polyhedra and defining collections

The image of a moment map $\mu\colon X \longrightarrow T_{\mathbb{1}}^*\mathbb{T}$ for a smooth action ψ of a torus $\mathbb{T}$ on a symplectic manifold (X,ω) is often a polyhedron, i.e. the intersection of a finite set of closed half-spaces in $T_{\mathbb{1}}^*\mathbb{T}$. Each such half-space $\mathcal{H}_{\upsilon}$ is specified by a pair $\upsilon \equiv (v,c)$ with $v \in T_{\mathbb{1}}\mathbb{T}$ nonzero and $c \in \mathbb{R}$; see (1.40). Every polyhedron P can thus be described by a finite subset

$$\mathscr{H} \subset \big(T_{\mathbb{1}}^*\mathbb{T} - \{0\}\big) \times \mathbb{R}\,; \tag{1.38}$$

any property of P is then equivalent to some property of $\mathscr{H}$. While each polyhedron $P \subset T_{\mathbb{1}}^*\mathbb{T}$ can be specified by a minimal subset $\mathscr{H}$ as in (1.38), with no unneeded elements, it is sometimes convenient to consider non-minimal subsets $\mathscr{H}$ as in (1.38). For the purposes of the comparisons of Exercises 6.11 and 6.25, it is necessary to consider finite "subsets" of $T_{\mathbb{1}}^*\mathbb{T}\times\mathbb{R}$ that contain pairs $(v=0,c)$ as well as multiple copies of the same element of $T_{\mathbb{1}}^*\mathbb{T}\times\mathbb{R}$. We call such "subsets" finite subcollections; they are formally defined below. However, aside from Exercises 1.44, 6.11, and 6.25, this more general notion is not needed and does not need to be distinguished from a subset in the usual sense.

The dimension of a polyhedron P in a vector space V is the dimension of the minimal affine subspace of V containing P. A (closed) face of such a polyhedron P is the intersection of P with the hyperplane $L^{-1}(c)$ for some nonzero linear functional $L\colon V \longrightarrow \mathbb{R}$ and $c \in \mathbb{R}$ such that

$$L(\alpha) \ge c \qquad \forall\, \alpha \in P\,. \tag{1.39}$$

A face of P is again a polyhedron (possibly empty). The interior P° of P is the complement of the proper faces of P in P. An open face of P is the interior of a face of P. A vertex (resp. edge, facet) of P is a face of P of dimension 0 (resp. dimension 1, codimension 1). We denote by $\mathrm{Ver}(P)$ and $\mathrm{Edg}(P)$ the sets of vertices and edges, respectively, of P, and by $\mathrm{Edg}_\eta(P) \subset \mathrm{Edg}(P)$ for each $\eta \in \mathrm{Ver}(P)$ the subset of the edges containing η. For $e \in \mathrm{Edg}(P)$, we call $\alpha_e \in V$ an edge vector for e if

$$P \cap \big\{\eta + t\alpha_e : t \in \mathbb{R}\big\} = e \subset V$$

for a vertex $\eta \in \mathrm{Ver}(e)$. A full tuple of edge vectors for P is an element $(\alpha_e)_{e\in\mathrm{Edg}(P)}$ of $V^{\mathrm{Edg}(P)}$ so that α_e is an edge vector for each $e \in \mathrm{Edg}_v(P)$.

The convex hull of a subset S of a vector space V is the subset

$$\mathrm{CH}(S) \equiv \bigg\{ \sum_{i=1}^m r_i s_i : m \in \mathbb{Z}^+,\ s_1,\ldots,s_m \in S,\ r_1,\ldots,r_m \in \mathbb{R}^{\ge 0},\ \sum_{i=1}^m r_i = 1 \bigg\} \subset V.$$

A polytope in V is the convex hull $\mathrm{CH}(S)$ of a finite subset S of V. A polytope in a finite-dimensional vector space is easily seen to be a compact polyhedron. The converse, which is not needed for our purposes, follows from the Minkowski-Weyl Theorem [18, Theorem 3.13], which states that a polyhedron in a finite-dimensional vector space is a finitely generated cone on a polytope. For a finite subset S of a finite-dimensional vector space V, we denote the set $\mathrm{Ver}(\mathrm{CH}(S))$ of the vertices of the polytope $\mathrm{CH}(S)$ simply by $\mathrm{Ver}(S)$; this is the minimal subset of S so that $\mathrm{CH}(\mathrm{Ver}(S)) = \mathrm{CH}(S)$. A face $\mathrm{CH}(S) \cap L^{-1}(c)$ of $\mathrm{CH}(S)$, where L is as in (1.39) with $P = \mathrm{CH}(S)$, is the convex hull of

$\mathrm{Ver}(S)\cap L^{-1}(c)$ and thus is a polytope in itself (possibly empty).

Let $\mathbb{T}$ be a torus. A polytope $P\subset T^*_{\mathbb{1}}\mathbb{T}$ is called Delzant if there exists a full tuple $(\alpha_e)_{e\in\mathrm{Edg}(P)}$ of (integral) edge vectors for P such that for each vertex η of P the components α_e with $e\in\mathrm{Edg}_\eta(P)$ form a $\mathbb{Z}$-basis for $(T^*_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$; this property of P is typically called smoothness in the literature. In particular, all edges of a Delzant polytope are rational. Furthermore, the number of edges containing any given vertex is the same as the dimension of $\mathbb{T}$; this property of P is typically called simplicity.

Let S be any set. By a finite subcollection $\mathscr{H}\subset S$, we mean an element of a finite symmetric product of S, i.e. the quotient of the Cartesian product S^k of k copies of S for some $k\in\mathbb{Z}^{\ge 0}$ by the action of the k-th symmetric group $\mathbb{S}_k$ permuting the components of the elements of S^k. For a finite subcollection $\mathscr{H}\subset S$ with a representative $\widetilde{\mathscr{H}}\in S^k$, we set $|\mathscr{H}|=k$. In such a case, we write $v\in\mathscr{H}$ and $\mathscr{H}'\subset\mathscr{H}$ if $v\in S$ is a component of $\widetilde{\mathscr{H}}$ and $\mathscr{H}'$ is a finite subcollection of S obtained by dropping some of the components of $\widetilde{\mathscr{H}}$, respectively. In the latter case, $\mathscr{H}-\mathscr{H}'$ denotes the finite subcollection of S obtained from the dropped components of $\widetilde{\mathscr{H}}$. We say $\mathscr{H}=\mathscr{H}_1\sqcup\mathscr{H}_2$ is a partition of $\mathscr{H}$ if $\mathscr{H}_1\subset\mathscr{H}$ and $\mathscr{H}_2=\mathscr{H}-\mathscr{H}_1$. We call finite subcollections $\mathscr{H}_1,\mathscr{H}_2\subset S$ disjoint if the sets of the components of their representatives are disjoint subsets of S. In particular, $\mathscr{H}'$ and $\mathscr{H}-\mathscr{H}'$ above may not be disjoint if $\mathscr{H}\subset S$ is not a subset in the usual sense. Informally, a finite subcollection $\mathscr{H}\subset S$ is a "subset" which may contain several copies of an element of S. This generalization of subset is relevant only for Exercises 1.44, 6.11, and 6.25 and can be replaced with subset in the usual sense in other statements, unless they are applied to the three exercises.

Suppose $\mathbb{T}$ is a torus. For $\upsilon\equiv(v,c)\in T_{\mathbb{1}}\mathbb{T}\times\mathbb{R}$, let

$$c_\upsilon=c,\quad \mathcal{H}_\upsilon\equiv\big\{\alpha\in T^*_{\mathbb{1}}\mathbb{T}\colon\alpha(v)\ge c\big\},\quad\text{and}\quad\partial\mathcal{H}_\upsilon\equiv\big\{\alpha\in T^*_{\mathbb{1}}\mathbb{T}\colon\alpha(v)=c\big\};\tag{1.40}$$

the subspaces $\mathcal{H}_\upsilon,\partial\mathcal{H}_\upsilon\subset T^*_{\mathbb{1}}\mathbb{T}$ are a (closed) half-space and an affine hyperplane, respectively, if $v\neq 0$. For a subcollection $\mathscr{H}\subset T_{\mathbb{1}}\mathbb{T}\times\mathbb{R}$ and $\mathscr{H}'\subset\mathscr{H}$, let

$$\langle\mathscr{H}\rangle=\bigcap_{\upsilon\in\mathscr{H}}\mathcal{H}_\upsilon\subset T^*_{\mathbb{1}}\mathbb{T},\quad\langle\mathscr{H}\rangle^\partial=\bigcap_{\upsilon\in\mathscr{H}}\partial\mathcal{H}_\upsilon\subset\langle\mathscr{H}\rangle,\quad\langle\mathscr{H}'\rangle^\partial_{\mathscr{H}}=\langle\mathscr{H}'\rangle^\partial\cap\langle\mathscr{H}\rangle-\bigcap_{\mathscr{H}'\subsetneq\mathscr{H}''\subset\mathscr{H}}\langle\mathscr{H}''\rangle^\partial.$$

In particular, $\langle\emptyset\rangle=\langle\emptyset\rangle^\partial=\langle\emptyset\rangle^\partial_\emptyset=T^*_{\mathbb{1}}\mathbb{T}$, $\langle\mathscr{H}'\rangle^\partial_{\mathscr{H}}\subset\langle\mathscr{H}'\rangle^\partial$ is an open subset,

$$\langle\mathscr{H}'\rangle\cap\langle\mathscr{H}-\mathscr{H}'\rangle=\langle\mathscr{H}\rangle,\quad\langle\mathscr{H}'\rangle^\partial\cap\langle\mathscr{H}-\mathscr{H}'\rangle=\langle\mathscr{H}'\rangle^\partial\cap\langle\mathscr{H}\rangle,\quad\text{and}\quad\langle\mathscr{H}'\rangle^\partial\cap\langle\emptyset\rangle^\partial_{\mathscr{H}-\mathscr{H}'}=\langle\mathscr{H}'\rangle^\partial_{\mathscr{H}}\,.$$

We call a subcollection $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T}-\{0\})\times\mathbb{R}$ minimal if it is a subset of $(T_{\mathbb{1}}\mathbb{T}-\{0\})\times\mathbb{R}$ in the usual sense and $\partial\mathcal{H}_\upsilon\cap\langle\mathscr{H}\rangle\neq\emptyset$ for every $\upsilon\in\mathscr{H}$. Every subcollection $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T}-\{0\})\times\mathbb{R}$ with $\langle\mathscr{H}\rangle\neq\emptyset$ contains a unique minimal subcollection $\mathscr{H}'$ with $\langle\mathscr{H}'\rangle=\langle\mathscr{H}\rangle$.

For a finite subcollection $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$, define

$$\begin{gathered}L_{\mathscr{H}}\colon\mathbb{R}^{\mathscr{H}}\longrightarrow T_{\mathbb{1}}\mathbb{T},\quad L_{\mathscr{H}}\big((r_{v,c})_{(v,c)\in\mathscr{H}}\big)=\sum_{(v,c)\in\mathscr{H}}r_{v,c}v\,,\\ \Phi_{\mathscr{H}}\colon\mathbb{T}^{\mathscr{H}}\equiv\mathbb{R}^{\mathscr{H}}/\mathbb{Z}^{\mathscr{H}}\longrightarrow\mathbb{T},\quad\Phi_{\mathscr{H}}\big([\mathbf{r}]\big)=\mathrm{e}^{L_{\mathscr{H}}(\mathbf{r})},\qquad\mathbb{T}_{\mathscr{H}}=\mathrm{Im}\,\Phi_{\mathscr{H}}.\end{gathered}\tag{1.41}$$

In particular, $\mathbb{T}_{\mathscr{H}}\subset\mathbb{T}$ is a subtorus. We call a finite subcollection $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ Delzant (resp. regular) if $\Phi_{\mathscr{H}'}$ is injective for every subcollection $\mathscr{H}'\subset\mathscr{H}$ such that $\langle\mathscr{H}'\rangle^\partial\cap\langle\mathscr{H}\rangle\neq\emptyset$ (resp. if

$L_{\mathscr{H}}$ is surjective). The former implies in particular that $v\in(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$ is primitive for every element $(v,c)\in\mathscr{H}$ such that $\partial\mathcal{H}_{v,c}\cap\langle\mathscr{H}\rangle\neq\emptyset$ and $\mathbb{T}_{\mathscr{H}'}\subset\mathbb{T}$ is a subtorus of dimension $|\mathscr{H}'|$ for every subset $\mathscr{H}'\subset\mathscr{H}$ such that $\langle\mathscr{H}'\rangle^{\partial}\cap\langle\mathscr{H}\rangle\neq\emptyset$. However, the disjoint union of any Delzant subcollection $\mathscr{H}$ of $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ and a pair $(0,c)$ with $c\in\mathbb{R}^{\pm}$ is still a Delzant subcollection of $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$.

Exercise 1.40. Suppose $\mathbb{T}$ is a torus and $P\subset T^*_{\mathbb{1}}\mathbb{T}$ is a polytope. Show that P is Delzant if and only if $P=\langle\mathscr{H}\rangle$ for some regular minimal Delzant subcollection $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$.

In light of Exercise 1.40, we call any polyhedron $P\subset T^*_{\mathbb{1}}\mathbb{T}$ Delzant if $P=\langle\mathscr{H}\rangle$ for some Delzant subcollection $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$. This means that we can choose a vector $v_F\in T_{\mathbb{1}}\mathbb{T}$ for each for each facet F of P so that $F=P\cap L_{v_F}^{-1}(c_F)$ for some $c_F\in\mathbb{R}$ and the vectors $c_{F_1},\ldots,c_{F_m}$ can be extended to a $\mathbb{Z}$-basis for $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$ whenever the facets $F_1,\ldots,F_m$ of P are distinct and the face $P\cap F_1\cap\ldots\cap F_m$ of P is not empty.

Exercise 1.41. Let $\mathbb{T}$ be a torus and $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ be a finite subcollection.

(a) Suppose $\alpha_0\in\langle\mathscr{H}\rangle^{\partial}$. Show that $\langle\mathscr{H}\rangle^{\partial}\subset T^*_{\mathbb{1}}\mathbb{T}$ is the preimage of $\alpha_0|_{T_{\mathbb{1}}\mathbb{T}_{\mathscr{H}}}$ under the restriction homomorphism $T^*_{\mathbb{1}}\mathbb{T}\longrightarrow T^*_{\mathbb{1}}\mathbb{T}_{\mathscr{H}}$.

(b) Suppose $\mathbb{T}'\subset\mathbb{T}$ is a subtorus of $\mathbb{T}$ so that $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T}')_{\mathbb{Z}}\times\mathbb{R}$ and $\mathscr{H}'\subset\mathscr{H}$. Show that the images of

$$\langle\mathscr{H}\rangle,\langle\mathscr{H}\rangle^{\partial},\langle\mathscr{H}'\rangle^{\partial}\cap\langle\mathscr{H}\rangle,\langle\mathscr{H}'\rangle^{\partial}_{\mathscr{H}}\subset T^*_{\mathbb{1}}\mathbb{T}$$

under the restriction homomorphism $T^*_{\mathbb{1}}\mathbb{T}\longrightarrow T^*_{\mathbb{1}}\mathbb{T}'$ are the corresponding subsets of $T^*_{\mathbb{1}}\mathbb{T}'$. Conclude that $\mathscr{H}$ is Delzant with respect to $\mathbb{T}$ if and only if $\mathscr{H}$ is Delzant with respect to $\mathbb{T}'$.

Exercise 1.42. Let $\mathbb{T}$ be a torus and $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ be a finite subcollection.

(a) Suppose $\mathscr{H}'\subset\mathscr{H}$ and $\langle\mathscr{H}'\rangle^{\partial}\cap\langle\mathscr{H}\rangle\neq\emptyset$. Show that there exists a subcollection $\mathscr{H}_{\bullet}\subset\mathscr{H}$ such that

$$\mathscr{H}'\subset\mathscr{H}_{\bullet},\qquad \langle\mathscr{H}_{\bullet}\rangle^{\partial}\cap\langle\mathscr{H}\rangle\neq\emptyset\subset T^*_{\mathbb{1}}\mathbb{T},\qquad\text{and}\qquad \mathrm{Im}\,L_{\mathscr{H}_{\bullet}}=\mathrm{Im}\,L_{\mathscr{H}}\subset T_{\mathbb{1}}\mathbb{T}.$$

(b) Suppose $\mathscr{H}$ is Delzant with $\langle\mathscr{H}\rangle\neq\emptyset$. Show that $\ker\Phi_{\mathscr{H}}\subset\mathbb{T}^{\mathscr{H}}$ is a subtorus of codimension equal to the dimension of $\mathrm{Im}\,L_{\mathscr{H}}\subset T_{\mathbb{1}}\mathbb{T}$.

Exercise 1.43. Suppose $\mathbb{T}$ is a torus and $\mathscr{H}\subset\mathscr{H}^{\dagger}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ are finite subcollections. Show that

(a) if $\mathscr{H}$ is regular, then the sequence

$$0\longrightarrow\ker\Phi_{\mathscr{H}}\longrightarrow\ker\Phi_{\mathscr{H}^{\dagger}}\longrightarrow\mathbb{T}^{\mathscr{H}^{\dagger}-\mathscr{H}}\longrightarrow 0$$

of abelian Lie groups, where the second homomorphism is the restriction of the projection $\mathbb{T}^{\mathscr{H}^{\dagger}}\longrightarrow\mathbb{T}^{\mathscr{H}^{\dagger}-\mathscr{H}}$, is exact;

(b) if $\mathscr{H}^{\dagger}$ is Delzant and $\mathscr{H}'\subset\mathscr{H}$ is a finite subcollection such that $\langle\mathscr{H}'\rangle^{\partial}\cap\langle\mathscr{H}\rangle=\langle\mathscr{H}'\rangle^{\partial}\cap\langle\mathscr{H}^{\dagger}\rangle$, then

$$\alpha(v)>c\qquad\forall\ (v,c)\in\mathscr{H}^{\dagger}-\mathscr{H},\ \alpha\in\langle\mathscr{H}'\rangle^{\partial}\cap\langle\mathscr{H}\rangle.$$

Exercise 1.44. Suppose $\mathbb{T}$ is a torus, $\mathscr{H}_0 \subset (T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ is a finite subcollection, $\alpha_0 \in \langle\mathscr{H}_0\rangle^{\partial}$, and $q\colon \mathbb{T} \longrightarrow \mathbb{T}/\mathbb{T}_{\mathscr{H}_0}$ is the quotient projection. Define

$$q_{\alpha_0}\colon \big(T_{\mathbb{1}}\mathbb{T}\big)_{\mathbb{Z}} \times \mathbb{R} \longrightarrow \big(T_{\mathbb{1}}(\mathbb{T}/\mathbb{T}_{\mathscr{H}_0})\big)_{\mathbb{Z}} \times \mathbb{R}, \quad q_{\alpha_0}(v,c) = \big(\mathrm{d}_{\mathbb{1}} q(v), c - \alpha_0(v)\big).$$

If $\mathscr{H} \subset (T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ is a finite subcollection such that $\mathscr{H}_0 \subset \mathscr{H}$, let $\mathscr{H}/\mathscr{H}_0 = q_{\alpha_0}(\mathscr{H} - \mathscr{H}_0)$ and

$$\phi_{\mathscr{H};\mathscr{H}_0} \equiv \big(\phi_{\mathscr{H};\mathscr{H}_0;v}\big)_{v \in \mathscr{H}/\mathscr{H}_0}\colon \mathbb{T}^{\mathscr{H}} \longrightarrow \mathbb{T}^{\mathscr{H}/\mathscr{H}_0}, \quad \phi_{\mathscr{H};\mathscr{H}_0;q_{\alpha_0}(v)}\big((u_{v'})_{v' \in \mathscr{H}}\big) = u_v \ \ \forall\, v \in \mathscr{H} - \mathscr{H}_0.$$

Show that

(a) if $\mathscr{H}_0 \subset \mathscr{H}' \subset \mathscr{H} \subset (T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ are finite subcollections, then $\langle\mathscr{H}'/\mathscr{H}_0\rangle^{\partial} \cap \langle\mathscr{H}/\mathscr{H}_0\rangle \neq \emptyset$ if and only if $\langle\mathscr{H}'\rangle^{\partial} \cap \langle\mathscr{H}\rangle \neq \emptyset$;

(b) if $\mathscr{H} \subset (T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ is a finite subcollection such that $\mathscr{H}_0 \subset \mathscr{H}$, then the diagram

$$\begin{tikzcd}
\{\mathbb{1}\} \arrow[r] & \ker \Phi_{\mathscr{H}} \arrow[r] \arrow[d] & \mathbb{T}^{\mathscr{H}} \arrow[r, "\Phi_{\mathscr{H}}"] \arrow[d, "\phi_{\mathscr{H};\mathscr{H}_0}"] & \mathbb{T} \arrow[r] \arrow[d, "q"] & \{\mathbb{1}\} \\
\{\mathbb{1}\} \arrow[r] & \ker \Phi_{\mathscr{H}/\mathscr{H}_0} \arrow[r] & \mathbb{T}^{\mathscr{H}/\mathscr{H}_0} \arrow[r, "\Phi_{\mathscr{H}/\mathscr{H}_0}"] & \mathbb{T}/\mathbb{T}_{\mathscr{H}_0} \arrow[r] & \{\mathbb{1}\}
\end{tikzcd}$$

of exact rows commutes and the left vertical homomorphism is surjective;

(c) if the homomorphism $\Phi_{\mathscr{H}_0}$ is injective and $\mathscr{H} \subset (T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ is a finite subcollection such that $\mathscr{H}_0 \subset \mathscr{H}$, then the homomorphism $\phi_{\mathscr{H};\mathscr{H}_0}$ restricts to an isomorphism from $\ker \Phi_{\mathscr{H}}$ to $\ker \Phi_{\mathscr{H}/\mathscr{H}_0}$;

(d) if $\mathscr{H} \subset (T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ is a Delzant (resp. regular) subcollection such that $\mathscr{H}_0 \subset \mathscr{H}$, then the subcollection $\mathscr{H}/\mathscr{H}_0 \subset (T_{\mathbb{1}}(\mathbb{T}/\mathbb{T}_{\mathscr{H}_0}))_{\mathbb{Z}} \times \mathbb{R}$ is also Delzant (resp. regular).

Exercise 1.45. Suppose $\mathbb{T}$ is a torus and $\mathscr{H} \subset (T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ is a finite subcollection.

(a) Let $\lambda \in \mathbb{R}^+$. Show that the finite subcollection

$$\lambda\mathscr{H} \equiv \big\{(v, \lambda c)\colon (v,c) \in \mathscr{H}\big\} \subset (T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$$

is Delzant (resp. minimal, regular) if and only if $\mathscr{H}$ is. Show also that

$$\langle\lambda\mathscr{H}\rangle = \lambda\langle\mathscr{H}\rangle \equiv \big\{\lambda\alpha\colon \alpha \in \langle\mathscr{H}\rangle\big\} \subset T_{\mathbb{1}}^*\mathbb{T}.$$

(b) Let $\alpha_0 \in T_{\mathbb{1}}^*\mathbb{T}$. Show that the finite subcollection

$$\mathscr{H} + \alpha_0 \equiv \big\{\big(v, c + \alpha_0(v)\big)\colon (v,c) \in \mathscr{H}\big\} \subset (T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$$

is Delzant (resp. minimal, regular) if and only if $\mathscr{H}$ is. Show also that

$$\langle\mathscr{H} + \alpha_0\rangle = \langle\mathscr{H}\rangle + \alpha_0 \equiv \big\{\alpha + \alpha_0\colon \alpha \in \langle\mathscr{H}\rangle\big\} \subset T_{\mathbb{1}}^*\mathbb{T}.$$

(c) Let $\Theta \in \mathrm{GL}_{\mathbb{Z}}\big((T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\big)$ and $\Theta^* \in \mathrm{GL}_{\mathbb{Z}}\big((T_{\mathbb{1}}^*\mathbb{T})_{\mathbb{Z}}\big)$ be the dual of Θ. Show that the finite subcollection

$$\Theta^{-1}\mathscr{H} \equiv \big\{\big(\Theta^{-1}(v), c\big)\colon (v,c) \in \mathscr{H}\big\} \subset (T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$$

is Delzant (resp. minimal, regular) if and only if $\mathscr{H}$ is. Show also that

$$\langle\Theta^{-1}\mathscr{H}\rangle = \Theta^*\big(\langle\mathscr{H}\rangle\big) \equiv \big\{\Theta^*(\alpha)\colon \alpha \in \langle\mathscr{H}\rangle\big\} \subset T_{\mathbb{1}}^*\mathbb{T}.$$

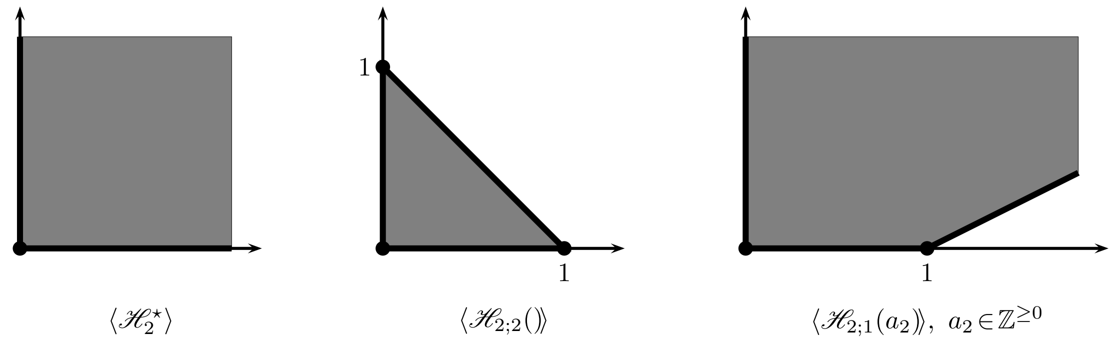


Figure 1.2: Representatives for the equivalence classes of regular minimal Delzant subsets $\mathscr{H}$ of $(T_{\mathbb{1}}\mathbb{T}^2)_{\mathbb{Z}} \times \mathbb{R}$ with $|\mathscr{H}| = 2, 3$ as in Exercise 1.46.

We call two finite subcollections of $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ equivalent if they differ by a composition of rescalings, translations, and automorphisms of $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$ as in Exercise 1.45. A regular finite subcollection of $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ has at least $\dim \mathbb{T}$ elements. The next exercise describes the equivalence classes of regular minimal Delzant subsets $\mathscr{H}$ of $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ of small cardinality relative to the dimension. The subsets $\langle \mathscr{H} \rangle \subset T^*_{\mathbb{1}}\mathbb{T}$ determined by representatives of these classes when $\mathbb{T}$ is two-dimensional are shown in Figures 1.2 and 1.3.

Exercise 1.46. Suppose $n \in \mathbb{Z}^+$ and $\mathscr{H} \subset (T_{\mathbb{1}}\mathbb{T}^n)_{\mathbb{Z}} \times \mathbb{R}$ is a regular minimal Delzant subset. Let $e_1, \ldots, e_n$ be the standard basis for $T_{\mathbb{1}}\mathbb{T}^n = \mathbb{R}^n$. Show that

(a) if $|\mathscr{H}| = n$, then $\mathscr{H}$ is equivalent to $\mathscr{H}_n^\star \equiv \{(e_i, 0) : i \in [n]\}$;

(b) if $|\mathscr{H}| = n+1$, then $\mathscr{H}$ is equivalent to

$$\mathscr{H}_{n;k}\big(a_{k+1}, \ldots, a_n\big) \equiv \mathscr{H}_n^\star \sqcup \big\{(-e_1 - \ldots - e_k + a_{k+1}e_{k+1} + \ldots + a_n e_n, -1)\big\}$$

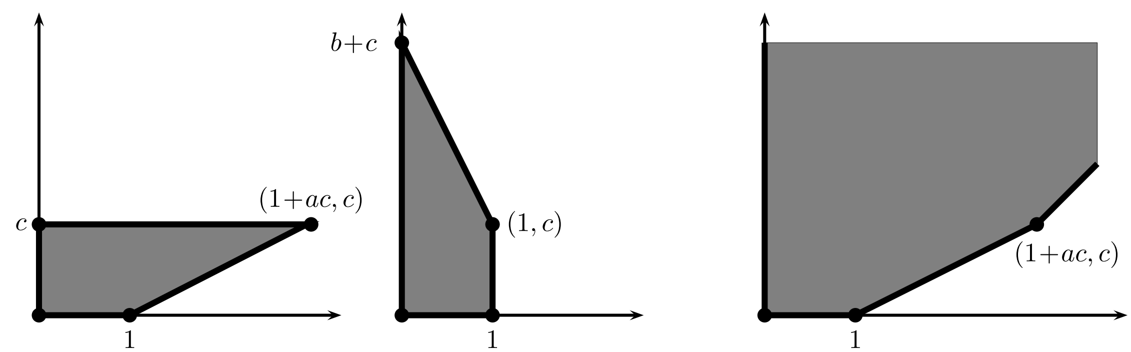
for some $k \in [n]$ and $a_{k+1}, \ldots, a_n \in \mathbb{Z}^{\ge 0}$;

(c) if $n = 2$ and $|\mathscr{H}| = 4$, then $\mathscr{H}$ is equivalent to

$$\mathscr{H}_{2;1}(a; b, c) \equiv \mathscr{H}_{2;1}(a) \sqcup \big\{(-be_1 + (ab-1)e_2, -b-c)\big\}$$

for some $a, b \in \mathbb{Z}^{\ge 0}$ and $c \in \mathbb{R}^+$;

(d) the equivalence class of $\mathscr{H}_{n;k}(a_{k+1}, \ldots, a_n)$ in (b) is invariant under the permutations of $a_{k+1}, \ldots, a_n$, the equivalence classes of $\mathscr{H}_{2;1}(a; b, c)$ and $\mathscr{H}_{2;1}(b; a, 1/c)$ in (c) are the same, and all other pairs of the subsets above represent distinct equivalence classes.



$\langle\mathscr{H}_{2;1}(a;0,c)\rangle\!\rangle,\ a\in\mathbb{Z}^{\geq 0},\ c\in\mathbb{R}^+$ $\langle\mathscr{H}_{2;1}(0;b,c)\rangle\!\rangle,\ b\in\mathbb{Z}^{\geq 0},\ c\in\mathbb{R}^+$ $\langle\mathscr{H}_{2;1}(a;b,c)\rangle\!\rangle,\ a,b\in\mathbb{Z}^{\geq 0},\ c\in\mathbb{R}^+$

Figure 1.3: Representatives for the equivalence classes of regular minimal Delzant subsets $\mathscr{H}$ of $(T_{\mathbb{1}}\mathbb{T}^2)_{\mathbb{Z}}\times\mathbb{R}$ with $|\mathscr{H}|=4$ as in Exercise 1.46; the subsets $\mathscr{H}_{2;1}(a;b,c)$ and $\mathscr{H}_{2;1}(b;a,1/c)$ are equivalent.

Chapter 2

Group Actions on Manifolds

In this chapter, we establish general structure results for Lie group actions on manifolds. We begin with actions on smooth manifolds, then move to arbitrary actions on symplectic manifolds, and then focus on Hamiltonian torus (and almost periodic $\mathbb{R}^k$-) actions on symplectic manifolds. Along the way, we obtain equivariant versions of Darboux Theorem, the most fundamental result in symplectic geometry, which shows that every symplectic manifold is locally isomorphic to the basic symplectic manifold $(\mathbb{C}^n, \omega_{\mathbb{C}^n})$ of Exercise 1.14. Corollaries 2.18 and 2.35 extend this result to actions of compact Lie groups and to Hamiltonian torus actions, respectively, on symplectic manifolds (the conclusions of these corollaries also apply to the almost periodic $\mathbb{R}^k$-actions). Corollary 2.18 is a special case of Proposition 2.15(2), an equivariant version of the Symplectic Tubular Neighborhood Theorem; see also Example 2.17. Another special case of Proposition 2.15(2) is Corollary 2.19, an equivariant version of the Lagrangian Tubular Neighborhood Theorem.

2.1 Group actions and flows of vector fields

We first show the $\mathbb{R}$-actions on smooth manifolds correspond to the flows of vector fields. This basic fact readily yields a number of geometric implications.

Proposition 2.1. *Let X be a smooth manifold.*

(1) The flow of a complete vector field ζ on X determines a smooth $\mathbb{R}$-action ψ on X by

$$\psi_0 = \mathrm{id}_X, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\psi_t(x) = \zeta\big(\psi_t(x)\big) \quad \forall\, t\in\mathbb{R},\, x\in X.$$

Conversely, a smooth $\mathbb{R}$-action ψ on X is the flow of the vector field ζ on X given by

$$\zeta(x) = \frac{\mathrm{d}}{\mathrm{d}t}\psi_t(x)\Big|_{t=0} \quad \forall\, x\in X. \tag{2.1}$$

In particular, $X^\psi = \{x\in X : \zeta(x)=0\}$.

(2) If ψ is a smooth $\mathbb{R}$-action on X with associated vector field ζ and $x\in X^\psi$, the linear $\mathbb{R}$-action $\mathrm{d}_x\psi$ on T_xX is the flow of the vector field

$$\nabla\zeta|_x : T_xX \longrightarrow T_xX, \qquad w \longrightarrow \nabla_w\zeta, \tag{2.2}$$

on T_xX, where ∇ is any connection in the vector bundle $TX \longrightarrow X$. If in addition J is a ψ-invariant endomorphism of this vector bundle, i.e.

$$\big\{\mathrm{d}_{x'}\psi_t\big\}^{-1}\circ J\circ \mathrm{d}_{x'}\psi_t = J\colon T_{x'}X \longrightarrow T_{x'}X \qquad \forall\ t\in\mathbb{R},\, x'\in X, \tag{2.3}$$

then

$$\nabla_{Jw}\zeta = J\nabla_w\zeta \qquad \forall\ w\in T_xX\,. \tag{2.4}$$

Proof. (1) If $\psi_t\colon X\longrightarrow X$ is the time t flow of $\zeta\in\Gamma(X;TX)$ for each $t\in\mathbb{R}$, then

$$\psi_{s+t}=\psi_s\circ\psi_t\colon X\longrightarrow X;$$

see [56, Theorem 1.48]. Thus, the map (1.1) with $G=\mathbb{R}$ is a group homomorphism and ψ is a smooth $\mathbb{R}$-action on X. Conversely, if ψ is a smooth $\mathbb{R}$-action on X and $\zeta\in\Gamma(X;TX)$ is given by (2.1), then

$$\psi_0=\mathrm{id}_X, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\psi_t(x)=\frac{\mathrm{d}}{\mathrm{d}s}\psi_{s+t}(x)\bigg|_{s=0}=\frac{\mathrm{d}}{\mathrm{d}s}\psi_s\big(\psi_t(x)\big)\bigg|_{s=0}\equiv\zeta\big(\psi_t(x)\big);$$

the second equality above holds because (1.1) is a group homomorphism. Thus, ψ_t is the time t flow of ζ. The last claim in (1) follows immediately.

(2) Let $x\in X^\psi$ and $\gamma\colon(-\delta,\delta)\longrightarrow X$ be a smooth curve such that $\gamma(0)=x$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}_x\psi_t\big(\gamma'(0)\big)\bigg|_{t=0}=\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s}\psi_t\big(\gamma(s)\big)\big|_{s,t=0}=\frac{\mathrm{D}}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\psi_t\big(\gamma(s)\big)\bigg|_{s,t=0}=\frac{\mathrm{D}}{\mathrm{d}s}\zeta\big(\gamma(s)\big)\bigg|_{s=0}=\nabla_{\gamma'(0)}\zeta\,,$$

where $\mathrm{D}/\mathrm{d}s$ denotes the covariant derivative with respect to any torsion-free connection in TX; the penultimate equality above holds by the second claim in (1) for the $\mathbb{R}$-action ψ on X. By second claim of (1) for the $\mathbb{R}$-action $\mathrm{d}_x\psi$ on T_xX, $\mathrm{d}_x\psi_t$ is thus the time t flow of the vector field $\nabla\zeta|_x$ on T_xX given by (2.2) (which is independent of the choice of ∇ because $\zeta(x)=0$).

If in addition J is a ψ-invariant endomorphism of the vector bundle $TX\longrightarrow X$, then

$$\mathrm{d}_{x'}\psi_t(Jw)=J\mathrm{d}_{x'}\psi_t(w) \qquad \forall\ w\in T_{x'}X,\, x'\in X\,.$$

Setting $x'=x$ above, differentiating the resulting equation at $t=0$, and using the previous statement, we obtain (2.4). □

Exercise 2.2. Suppose $k\in\mathbb{Z}^{\ge 0}$, ψ is a smooth $\mathbb{R}^k$-action on a smooth manifold X, and $x\in X$. Let $\mathbb{R}^k_x\subset\mathbb{R}^k$ be the largest linear subspace fixing x and $\mathbb{R}^c_x\subset\mathbb{R}^k$ be a complementary linear subspace. Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_{tv}(x)\bigg|_{t=0}\neq 0\in T_xX \qquad \forall\ v\in\mathbb{R}^c_x-\{0\}.$$

Exercise 2.3. Let ψ be a nontrivial smooth action of $\mathbb{R}^k$ (resp. k-torus $\mathbb{T}^k\equiv(\mathbb{R}/\mathbb{Z})^k$) on a smooth manifold X as in (1.1). Show that there exist an irreducible smooth action ψ' of $\mathbb{R}^m$ (resp. $\mathbb{T}^m$) on X for some $m\in\mathbb{Z}^+$ and a full-rank real (resp. integer) $m\times k$-matrix A so that $\psi=\psi'\circ A$, i.e.

$$\psi_v=\psi'_{Av}\in\mathrm{Diff}(X) \qquad \forall\, v\in\mathbb{R}^k \quad \big(\text{resp. } v\in\mathbb{T}^k\big).$$

Furthermore, if the action ψ is almost periodic in the first case, then so is the action ψ'. *Hint:* if G is a compact Lie group and $v\in T_{\mathbb{1}}G$, the closure of the one-parameter subgroup $\{\mathrm{e}^{tv}\colon t\in\mathbb{R}\}$ is a torus; see Exercises 1.8 and 1.9.

Exercise 2.4. Suppose ψ is a smooth action of a Lie group G on a smooth manifold X as in (1.1). For each $v\in T_{\mathbb{1}}G$, let $\zeta_v\in\Gamma(X;TX)$ be the vector field on X generated by v as in (1.2). Show that

(a) if $u\in G$ and $v,v'\in T_{\mathbb{1}}G$, then

$$\zeta_{\mathrm{Ad}_{u^{-1}}(v)}=\psi_u^*\zeta_v\equiv \mathrm{d}\psi_u^{-1}(\zeta_v\circ\psi_u),\quad \zeta_{[v,v']}=\frac{\mathrm{d}}{\mathrm{d}t}\zeta_{\mathrm{Ad}_{e^{tv}}(v')}\bigg|_{t=0}=-\mathcal{L}_{\zeta_v}\zeta_{v'}=-\big[\zeta_v,\zeta_{v'}\big], \tag{2.5}$$

where $\mathcal{L}$ is the Lie derivative;

(b) if $h\colon[a,b]\longrightarrow G$ is a smooth path and $h'=\frac{\mathrm{d}h}{\mathrm{d}t}\colon[a,b]\longrightarrow TG$, then

$$\frac{\mathrm{d}\psi_h}{\mathrm{d}t}=\zeta_{h'h^{-1}}\circ\psi_h. \tag{2.6}$$

Exercise 2.5. Let Y be a smooth manifold, λ_{T^*Y} be the canonical 1-form on T^*Y as in Exercise 1.33, ω_{T^*Y} bbe the canonical symplectic form on T^*Y as in Exercise 1.34, and ψ be a smooth action of a Lie group G on T^*Y.

(a) For each $v\in T_{\mathbb{1}}G$, let $\zeta_v\in\Gamma(T^*Y;T(T^*Y))$ be the vector field on T^*Y generated by v as in (1.2) with $X=T^*Y$. Show that the smooth map

$$\mu\colon T^*Y\longrightarrow T_{\mathbb{1}}^*G \qquad \text{defined by} \quad \big\{\mu(x)\big\}(v)=-\lambda_{T^*Y}\big(\zeta_v(x)\big)\ \forall\, v\in T_{\mathbb{1}}G,$$

satisfies the second condition in (1.6).

(b) Suppose in addition the action ψ preserves λ_{T^*Y}, i.e. $\psi_u^*\lambda_{T^*Y}=\lambda_{T^*Y}$ for every $u\in G$. Show that the map μ above is a moment map for the action ψ on (T^*Y,ω_{T^*Y}).

2.2 Fixed loci of group actions

We next show that the fixed loci of smooth actions of compact Lie groups on smooth manifolds are submanifolds with nice tubular neighborhoods. We also make observations concerning the fixed loci of subgroups of tori acting smoothly on smooth manifolds.

Exercise 2.6. Let ψ be a smooth action of a compact Lie group on a smooth manifold X as in (1.1). Show that there exists a ψ-invariant Riemannian metric on X, i.e. a Riemannian metric g on X such that

$$g\big(\mathrm{d}_x\psi_u(w),\mathrm{d}_x\psi_u(w')\big)=g(w,w') \qquad \forall\ u\in G,\ x\in X,\ w,w'\in T_xX.$$

Suppose $Y\subset X$ is a closed submanifold of a smooth manifold. A tubular neighborhood identification for Y in X is a diffeomorphism $\Phi\colon\mathcal{U}\longrightarrow U$ from an open neighborhood of Y in a subbundle $TY^c\subset TX|_Y$ complementary to TY to an open neighborhood of $Y\subset X$ such that

$$\Phi(y)=y,\ \mathrm{d}_y\Phi=\mathrm{id}\colon T_y\mathcal{U}=T_y(TY^c)=T_yY\oplus TY^c\big|_y\longrightarrow T_yY\oplus TY^c\big|_y=T_yX=T_yU\quad\forall\, y\in Y. \tag{2.7}$$

Proposition 2.7. *Let ψ be a smooth action of a compact Lie group G on a smooth manifold X as in (1.1).*

(1) The fixed locus $X^\psi \subset X$ of ψ is a closed submanifold with

$$T(X^\psi) = (TX)^{\mathrm{d}\psi}. \tag{2.8}$$

(2) If $Y \subset X$ is a closed submanifold preserved by ψ and $TY^c \subset TX|_Y$ is a subbundle complementary to TY and preserved by $\mathrm{d}\psi$, then there exists a tubular neighborhood identification $\Phi\colon \mathcal{U} \longrightarrow U$ for Y in X with $\mathcal{U} \subset TY^c$ which is G-equivariant with respect to the actions ψ on X and $\mathrm{d}\psi$ on TX.

Proof. Let g be a Riemannian metric preserved by the group action ψ, as provided by Exercise 2.6. Its Levi-Civita connection ∇ is also preserved by G. If $w \in TX$, $\gamma_w\colon (a,b) \longrightarrow X$ with $a<0<b$ is the geodesic with respect to ∇ of g with $\gamma_w'(0)=w$, and $u \in G$, then

$$\psi_u \circ \gamma_w\colon (a,b) \longrightarrow X$$

is the geodesic with respect to ∇ with $(\psi_u \circ \gamma_w)'(0) = \{\mathrm{d}_{\gamma_w(0)}\psi_u\}(w)$, i.e. $\psi_u \circ \gamma_w = \gamma_{\{\mathrm{d}_{\gamma_w(0)}\psi_u\}(w)}$. Thus, the exponential map

$$\exp\colon \mathcal{W} \longrightarrow X, \qquad \exp(w) = \gamma_w(1) \quad \forall\, w \in \mathcal{W} \subset TX,$$

with respect to ∇ satisfies

$$\{\mathrm{d}\psi_u\}(\mathcal{W}) = \mathcal{W} \quad \text{and} \quad \exp\big(\{\mathrm{d}\psi_u\}(w)\big) = \psi_u\big(\exp(w)\big) \qquad \forall\, u \in G,\, w \in \mathcal{W}, \tag{2.9}$$

i.e. it is G-equivariant with respect to the actions ψ on X and $\mathrm{d}\psi$ on TX.

(2) Since

$$\{\mathrm{d}_x \exp\}(w) = w \in T_x X \qquad \forall\, x \in X,\, w \in T_x X \cap \mathcal{W}, \tag{2.10}$$

for each $y \in Y$ the restriction of exp to a neighborhood of y in $TY^c \cap \mathcal{W}$ is a diffeomorphism onto an open neighborhood of y in X by the Inverse Function Theorem. Since $Y \subset X$ is closed, it follows that there exists a neighborhood $\mathcal{U}'$ of Y in $TY^c \cap \mathcal{W}$ so that

$$\exp\colon \mathcal{U}' \longrightarrow \Psi(\mathcal{U}')$$

is a diffeomorphism onto an open subset of X. This map satisfies both conditions in (2.7) by the definition of the exponential map. Since G is compact,

$$\mathcal{U} \equiv \bigcap_{u \in G} \mathrm{d}\psi_u(\mathcal{U}') \subset \mathcal{U}' \subset TY^c \cap \mathcal{W}$$

is a neighborhood of $Y \subset TY^c$ preserved by the G-action. By (2.9), the restriction

$$\Phi \equiv \exp|_{\mathcal{U}}\colon \mathcal{U} \longrightarrow U \equiv \exp(\mathcal{U}) \subset X$$

is a G-equivariant diffeomorphism from an open neighborhood of Y in TY^c to an open neighborhood of Y in X with the required properties. This establishes (2).

(1) It is immediate that $X^\psi \subset X$ is a closed subset. For each $y \in X^\psi$, let $\Phi_y : \mathcal{U}_y \longrightarrow U_y$ be a G-equivariant tubular neighborhood identification as in (2) with $Y = \{y\}$ and $\mathcal{U}_y \subset T_y X$. By the G-equivariance of Φ_y,

$$\Phi_y\colon (T_y X)^{\mathrm{d}\psi} \cap \mathcal{U}_y \longrightarrow X^\psi \cap U_y$$

is a homeomorphism for every $y \in X^\psi$. Thus, each topological component of X^ψ is a submanifold of X; see [56, 1.33(b)]. By the G-equivariance of Φ_y, (2.8) holds as well. $\square$

Remark 2.8. The conclusion of Proposition 2.7(1) also holds if $\pi_0(G)$ is finite and X admits a G-invariant metric (but G is not necessarily compact). The first paragraph in the proof of Proposition 2.7 still applies. For (1) in this proof, $\mathcal{U}_y\subset T_yX$ can be taken to be any neighborhood of $0\in T_yX$ on which the map exp is injective.

Corollary 2.9. *Let ψ be an irreducible almost periodic action of $\mathbb{R}^k$ on a smooth manifold X as in (1.1). The subspace* $\mathrm{Crit}(\psi)$ *of points of X with stabilizers containing a one-dimensional linear subspace of $\mathbb{R}^k$ is a countable union of (not necessarily disjoint) closed proper submanifolds of X.*

Proof. Let $\rho\colon\mathbb{R}^k\longrightarrow\mathbb{T}$ and ψ' be as in (1.4). For each one-dimensional linear subspace $L\subset\mathbb{R}^k$, the closure $\mathbb{T}_L\subset\mathbb{T}$ of $\rho(L)$ in $\mathbb{T}$ is a torus; see Exercises 1.8 and 1.9. By (1.4), the fixed locus $X^{\mathbb{T}_L}\subset X$ of the smooth action $\psi'|_{\mathbb{T}_L}$ on X is the same as the fixed locus X^L of the smooth action $\psi|_L$. By Proposition 2.7(1), X^L is thus a closed submanifold of X (possibly empty). Since the action ψ is irreducible, $\psi|_L$ is a nontrivial action and $X^L\neq X$. Thus, $\mathrm{Crit}(\psi)$ is the union of the closed submanifolds $X^{\mathbb{T}'}\neq X$ taken over the subcollection

$$\mathcal{A}\equiv\big\{\mathbb{T}_L\colon L\in\mathbb{RP}^{k-1}\big\}$$

of subtori of $\mathbb{T}$. By Exercise 1.10(a), the subtori of $\mathbb{T}$ are generated by finite sets of vectors in $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$. Thus, the collection $\mathcal{A}$ is (at most) countable. □

For a torus $\mathbb{T}$ and $\alpha\in(T^*_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$, let

$$\mathbb{T}_\alpha=\big\{\mathrm{e}^v\colon v\in T_{\mathbb{1}}\mathbb{T},\ \alpha(v)\in\mathbb{Z}\big\}.$$

If $\alpha\neq0$, $\mathbb{T}_\alpha\subset\mathbb{T}$ is a codimension 1 closed subgroup. If α is primitive, then $\mathbb{T}_\alpha\subset\mathbb{T}$ is a codimension 1 subtorus. For a subset $S\subset(T^*_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$ and a closed subgroup $G\subset\mathbb{T}$, let

$$\mathbb{T}_S=\bigcap_{\alpha\in S}\mathbb{T}_\alpha\qquad\text{and}\qquad S_G=\big\{\alpha\in S\colon G\subset\mathbb{T}_\alpha\big\}.$$

Thus, $\mathbb{T}_S\subset\mathbb{T}$ is a closed subgroup of codimension at most $|S|$, $S_G\subset S$ is the maximal subset so that $G\subset\mathbb{T}_{S_G}$, and $S_{\mathbb{T}_S}=S$.

Proposition 2.10. *Let ψ be a smooth action of a torus $\mathbb{T}$ on a smooth manifold X as in (1.1) and $Y\subset X^\psi$ be a topological component of the ψ-fixed locus.*

(1) There exist a subset $S(Y)\subset(T^*_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}-\{0\}$ and a splitting

$$TX|_Y=TY\oplus\bigoplus_{\alpha\in S(Y)}\mathcal{N}^\alpha_XY\longrightarrow Y \tag{2.11}$$

of $TX|_Y$ into a direct sum of vector bundles preserved by $\mathrm{d}\psi$ so that the bundles $\mathcal{N}^\alpha_XY$ are nonzero and complex with

$$\mathrm{d}\psi_{\mathrm{e}^v}(w)=\mathrm{e}^{2\pi\mathrm{i}\,\alpha(v)}w\qquad\forall\ v\in T_{\mathbb{1}}\mathbb{T},\ w\in\mathcal{N}^\alpha_XY,\ \alpha\in S(Y). \tag{2.12}$$

In particular, $2|S(Y)|\le\mathrm{codim}_XY$. If X is connected and the action ψ is irreducible (resp. effective), then the $\mathbb{R}$-span (resp. $\mathbb{Z}$-span) of $S(Y)$ is $T^*_{\mathbb{1}}\mathbb{T}$ (resp. $(T^*_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$). If $TX|_Y$ is a complex vector bundle and the $\mathbb{T}$-action $\mathrm{d}\psi$ preserves its complex structure J, then the complex structure on each subbundle $\mathcal{N}^\alpha_XY\subset TX|_Y$ can be taken to be the restriction of J.

(2) If $G\subset\mathbb{T}$ is a closed subgroup, $Z\subset X^G$ is a topological component of the G-action $\psi|_G$ on X, and $Y\subset Z$, then

$$TZ|_Y=TY\oplus\bigoplus_{\alpha\in S(Y)_G}\mathcal{N}_X^{\alpha}Y\subset TX|_Y\longrightarrow Y$$

and Z is a topological component of the fixed locus $X^{\mathbb{T}_{S(Y)_G}}$ of the $\mathbb{T}_{S(Y)_G}$-action $\psi|_{\mathbb{T}_{S(Y)_G}}$ on X.

Proof. (1) For each $y\in Y$, $\mathrm{d}_y\psi$ is a real representation of $\mathbb{T}$ on T_yX. Every such representation splits as a direct sum of a trivial real representation and of one-dimensional complex representations with the action on each factor given by (2.12) for some $\alpha\in(T_{\mathbb{1}}^*\mathbb{T})_{\mathbb{Z}}$ nonzero. By (2.8), the trivial representation summand is T_yY. Since $\mathrm{d}_y\psi$ depends smoothly on y, the weights α are independent of $y\in Y$ and the corresponding component representations vary smoothly with $y\in Y$. Thus, the latter form vector subbundles $\mathcal{N}_X^{\alpha}Y\subset TX|_Y$ as in (2.11) with complex structures. If $TX|_Y$ is a complex vector bundle and $\mathrm{d}\psi$ preserves its complex structure, then $\mathrm{d}_y\psi$ is a complex representation of $\mathbb{T}$ on T_yY and the same reasoning applies.

If $S(Y)$ does not span $T_{\mathbb{1}}^*\mathbb{T}$ over $\mathbb{R}$ (resp. $(T_{\mathbb{1}}^*\mathbb{T})_{\mathbb{Z}}$ over $\mathbb{Z}$), there exists $v\in T_{\mathbb{1}}\mathbb{T}-(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$ such that $\alpha(v)=0$ (resp. $\alpha(v)\in\mathbb{Z}$) for all $\alpha\in S(Y)$. Let G be the closure of the subgroup $\{\mathrm{e}^{tv}: t\in\mathbb{R}\}$ (resp. the subgroup generated by v) in $\mathbb{T}$. This subgroup acts trivially on $TX|_Y$. By Proposition 2.7(1), this implies that the connected component of the G-fixed locus X^G containing Y is a connected component of X, i.e. G acts trivially on X (and so the action ψ is not effective) if X is connected. If $\alpha(v)=0$ for all $\alpha\in S(Y)$ and X is connected, then

$$\zeta_v\equiv\mathrm{d}_{\mathbb{1}}\psi(v)=0\in\Gamma(X;TX),$$

i.e. the action ψ is reducible.

(2) By Proposition 2.7(1) applied to $\psi|_G$, ψ, and $\psi|_{\mathbb{T}_{S(Y)_G}}$,

$$TZ|_Y=\big\{w\in TX|_Y:\mathrm{d}\psi_u(w)=w\ \forall\,u\in G\big\}=TY\oplus\bigoplus_{\alpha\in S(Y)_G}\mathcal{N}_X^{\alpha}Y=TX^{\mathbb{T}_{S(Y)_G}}\big|_Y.$$

This establishes both claims. $\square$

Corollary 2.11. *Let ψ be an irreducible almost periodic action of $\mathbb{R}^k$ on a smooth manifold X as in (1.1). For each $L\in\mathbb{RP}^{k-1}$, let $X^L\subset X$ be the fixed locus of the action $\psi|_L$. If X is compact, then the set*

$$\widetilde{\pi}_0^*\big(\mathrm{Crit}(\psi)\big)\equiv\big\{Z\in\pi_0(X^L): L\in\mathbb{RP}^{k-1},\ Z\cap X^{\psi}\neq\emptyset\big\}$$

is finite.

Proof. Let $\rho:\mathbb{R}^k\longrightarrow\mathbb{T}$ and ψ' be as in (1.4). We can assume that the image of ρ is dense in $\mathbb{T}$ and so $X^{\psi}=X^{\psi'}$. For each $L\in\mathbb{RP}^{k-1}$, let $\mathbb{T}_L\subset\mathbb{T}$ be as in the proof of Corollary 2.9. For each subtorus $\mathbb{T}'\subset\mathbb{T}$, let $X^{\mathbb{T}'}\subset X$ be the fixed locus of the action $\psi'|_{\mathbb{T}'}$. In particular, $X^L=X^{\mathbb{T}_L}$. By Proposition 2.10(2), every element Z of $\widetilde{\pi}_0^*(\mathrm{Crit}(\psi))$ intersecting a topological component Y of $X^{\psi}=X^{\psi'}$ is thus the unique topological component of $X^{\mathbb{T}_S}\subset X$ for some $S\subset S(Y)$ intersecting Y. The number of subsets of $S\subset S(Y)$ is finite for each $Y\in\pi_0(X^{\psi})$. Since X is compact, $\pi_0(X^{\psi})$ is finite as well. $\square$

2.3 Group actions on symplectic manifolds

This section provides an analogue of Proposition 2.7 in the symplectic setting, Proposition 2.15, which describes neighborhoods of fixed loci of actions of compact Lie groups on symplectic manifolds. Proposition 2.15 specializes to Example 2.17 and Corollary 2.19, equivariant versions of the Symplectic and Lagrangian Tubular Neighborhood Theorems, respectively. A special case of Example 2.17 is Corollary 2.18, an equivariant version of Darboux Theorem.

Exercise 2.12. Suppose X is a smooth manifold, ω is a closed 2-form on X, and ψ is a smooth action of a Lie group G on X as in (1.1). For each $v\in T_{\mathbb{1}}G$, let $\zeta_v\in\Gamma(X;TX)$ be the vector field on X generated by v as in (1.2)..

(a) Suppose G is connected. Show that ψ preserves ω if and only if $\mathrm{d}(\iota_{\zeta_v}\omega)=0$ for all $v\in T_{\mathbb{1}}G$.

(b) Suppose ψ preserves ω. Show that

$$\mathrm{d}\big(\omega(\zeta_v,\zeta_{v'})\big) = -\iota_{[\zeta_v,\zeta_{v'}]}\omega \quad v,v'\in T_{\mathbb{1}}G, \tag{2.13}$$

$$\omega\big([\zeta_v,\zeta_{v'}],\zeta_{v''}\big)+\omega\big([\zeta_{v'},\zeta_{v''}],\zeta_v\big)+\omega\big([\zeta_{v''},\zeta_v],\zeta_{v'}\big)=0 \quad \forall\, v,v',v''\in T_{\mathbb{1}}G. \tag{2.14}$$

Hint: use Cartan's formula for the Lie derivative.

Exercise 2.13. Let ψ be a smooth action of a compact Lie group on a symplectic manifold (X,ω) as in (1.1). Show that there exists a ψ-invariant ω-compatible almost complex structure on X, i.e. an ω-compatible almost complex structure J on X such that

$$\big\{\mathrm{d}_x\psi_u\big\}^{-1}\circ J\circ \mathrm{d}_x\psi_u = J\colon T_xX\longrightarrow T_xX \qquad \forall\, u\in G,\, x\in X. \tag{2.15}$$

Hint: a Riemannian metric g and a nondegenerate 2-form ω on X determine an ω-compatible almost complex structure $J_{g,\omega}$ on X; see the proof of Proposition 2.3 in [62].

Exercise 2.14. Suppose ψ is a smooth action of a Lie group G on a symplectic manifold (X,ω) as in (1.1), J is a ψ-invariant ω-compatible almost complex structure on X, and $Y\subset X$ is a Lagrangian submanifold. Show that the isomorphism

$$\Phi_{Y;J}\colon J(TY)\longrightarrow T^*Y, \qquad \Phi_{Y;J}(w)=\omega(\cdot,w),$$

of real vector bundles over Y is G-equivariant with respect to the actions $\mathrm{d}\psi$ and $\mathrm{d}\psi^*$ of Exercise 1.1.

Proposition 2.15. *Let ψ be a smooth action of a compact Lie group G on a symplectic manifold (X,ω) as in (1.1).*

(1) The fixed locus $X^\psi\subset X$ of ψ is a closed symplectic submanifold with $T(X^\psi)=(TX)^{\mathrm{d}\psi}$.

(2) Suppose $Y\subset X$ is a closed submanifold preserved by ψ, $TY^c\subset TX|_Y$ is a subbundle complementary to TY and preserved by $\mathrm{d}\psi$, and $\widetilde{\omega}$ is a G-invariant closed 2-form on a neighborhood of Y in TY^c preserved by $\mathrm{d}\psi$. If

$$\widetilde{\omega}\big|_{T_y(TY^c)}=\omega\big|_{T_yX} \qquad \forall\, y\in Y, \tag{2.16}$$

there exists a G-equivariant tubular neighborhood identification $\Phi\colon\mathcal{U}\longrightarrow U$ for Y in X such that $\mathcal{U}\subset TY^c$ and $\Phi^*\omega=\widetilde{\omega}|_{\mathcal{U}}$.

Proof. (1) Let J be a G-invariant ω-compatible almost complex structure on X, as provided by Exercise 2.13. By Proposition 2.7(1), $X^\psi\subset X$ is a closed submanifold with $TX^\psi=(TX)^{\mathrm{d}\psi}$. Since J is G-invariant, $J(TX^\psi)\subset TX^\psi$ by (2.15). Since J is ω-compatible, $\omega(v,Jv)>0$ for all $v\in TX$ nonzero. Thus, $\omega|_{TX^\psi}$ is nondegenerate.

(2) Let $\Phi\colon\mathcal{U}\longrightarrow U$ be a G-equivariant tubular neighborhood identification for Y in X with $\mathcal{U}\subset TY^c$, as provided by Proposition 2.7(2). In particular, $\Phi^*\omega$ is a symplectic form on $\mathcal{U}$. By (2.16) and (2.7),

$$(\Phi^*\omega)\big|_{T_y(TY^c)}=\widetilde{\omega}\big|_{T_y(TY^c)}\qquad\forall\,y\in Y\,. \tag{2.17}$$

Since Φ is G-equivariant, the 2-form $\Phi^*\omega$ is G-invariant. Since the subset of $\mathcal{U}$ on which $\widetilde{\omega}$ is nondegenerate contains Y by (2.16) and is open and preserved by G, we can assume that the 2-form $\widetilde{\omega}$ is nondegenerate on $\mathcal{U}$ (by replacing $\mathcal{U}$ by its subset on which $\widetilde{\omega}$ is nondegenerate) .

Let $m_\tau\colon TY^c\longrightarrow TY^c$ be the scalar multiplication by τ as in (B.1) and $\zeta_{TY^c}\in\Gamma(TY^c;T(TY^c))$ be the canonical vertical vector field as in (B.5). Define a 1-form α on $\mathcal{U}$ by

$$\alpha=\int_0^1 m_\tau^*\big(\iota_{\tau^{-1}\zeta_{TY^c}}(\Phi^*\omega-\widetilde{\omega})\big)\mathrm{d}\tau.$$

By Exercise B.1 and (2.17),

$$\mathrm{d}\alpha=\Phi^*\omega-\widetilde{\omega},\quad \mathrm{d}\alpha\big|_{T(TY^c)|_Y}=0,\quad \alpha\big|_{T(TY^c)|_Y}=0,\quad \nabla\alpha\big|_{T(TY^c)|_Y}=0, \tag{2.18}$$

where ∇ is any connection in $T\mathcal{U}\subset T(TY^c)$.

By the second statement in (2.18) and the compactness of $[0,1]$,

$$\omega_t\equiv\widetilde{\omega}+t\,\mathrm{d}\alpha$$

is a symplectic form on a neighborhood $\mathcal{U}'$ of $Y\subset\mathcal{U}$ for every $t\in[0,1]$. For each $t\in[0,1]$, define

$$\xi_t\in\Gamma\big(\mathcal{U}';T\mathcal{U}'\big)\qquad\text{by}\quad \iota_{\xi_t}\omega_t=-\alpha. \tag{2.19}$$

By the third statement in (2.18), $\xi_t|_Y=0$. Since $[0,1]$ is compact, it follows that there exists a neighborhood $\mathcal{U}''$ of Y in $\mathcal{U}'$ so that the flow of ξ_t,

$$\psi_t\colon\mathcal{U}''\longrightarrow\mathcal{U}',\qquad \psi_0(w)=w\ \ \forall\,w\in\mathcal{U}'',\quad \frac{\mathrm{d}}{\mathrm{d}t}\psi_t=\xi_t\circ\psi_t,$$

is well-defined for every $t\in[0,1]$. By the first, third, and fourth statements in (2.18),

$$\widetilde{\omega}=\psi_t^*\omega_t,\quad \psi_t(y)=y\ \ \forall\,y\in Y\subset\mathcal{U}'',\quad\text{and}\quad \mathrm{d}\psi_t|_{T(TY^c)|_Y}=\mathrm{id}_{T(TY^c)|_Y}\qquad\forall\,t\in[0,1]; \tag{2.20}$$

see Exercise 2.16 below for the first identity.

Since G is compact, the set

$$\mathcal{U}'''\equiv\bigcap_{u\in G}\mathrm{d}\psi_u(\mathcal{U}'')\subset\mathcal{U}'\subset TY^c$$

is a neighborhood of $Y\subset TY^c$ preserved by the G-action so that $\psi_1\!:\mathcal{U}'''\longrightarrow\mathcal{U}$ is a well-defined diffeomorphism onto an open subset of $\mathcal{U}$. Since the 2-forms $\Phi^*\omega$ and $\widetilde{\omega}$ are G-invariant, so are the 1-form α and the vector fields ξ_t. Thus, the smooth map ψ_1 is G-equivariant, as is the diffeomorphism

$$\Phi\circ\psi_1\!:\mathcal{U}'''\longrightarrow\Phi\big(\psi_1(\mathcal{U}''')\big)\subset U\subset X.$$

By the second and third statements in (2.20), this diffeomorphism is a tubular neighborhood identification for Y in X (because Φ is). By the first statement in (2.20), $\{\Phi\circ\psi_1\}^*\omega\!=\!\widetilde{\omega}|_{\mathcal{U}'''}$. $\square$

Exercise 2.16. Suppose X is a smooth manifold, $(\omega_t)_{t\in[0,1]}$ is a smooth family of symplectic forms on X, $(\xi_t)_{t\in[0,1]}$ is the smooth family of vector fields on X satisfying

$$\mathrm{d}\big(\iota_{\xi_t}\omega_t\big)=-\frac{\mathrm{d}}{\mathrm{d}t}\omega_t,$$

and $\psi_t\!:\mathcal{U}\longrightarrow X$ is a flow of $(\xi_t)_{t\in[0,1]}$ on an open subset of X, i.e.

$$\psi_0(x)=x,\quad \frac{\mathrm{d}}{\mathrm{d}t}\big(\psi_t(x)\big)=\xi_t\big(\psi_t(x)\big)\qquad\forall\,x\!\in\!\mathcal{U},\ t\!\in\![0,1].$$

Show that $\psi_t^*\omega_t\!=\!\omega_0|_{\mathcal{U}}$ for all $t\!\in\![0,1]$. *Hint:* differentiate both sides and use Cartan's formula for the Lie derivative.

Example 2.17 (Symplectic Tubular Neighborhood Theorem)**.** Suppose ψ is a smooth action of a compact Lie group G on a symplectic manifold (X,ω) and Y is a symplectic submanifold of (X,ω) preserved by ψ. The restriction $\omega_Y^{\perp}$ of ω to the ω-symplectic complement

$$\mathrm{pr}\!:TY^{\omega}\equiv\big\{w\!\in\!TX|_Y:\omega(w,w')\!=\!0\ \forall\,w'\!\in\!TY\big\}\longrightarrow Y$$

of TY in $TX|_Y$ is then a G-invariant fiberwise nondegenerate 2-form. Let $\zeta_{TY^{\omega}}\!\in\!\Gamma(TY^{\omega};T(TY^{\omega}))$ be the canonical vertical vector field as in (B.5). A G-invariant connection ∇ in the real vector bundle $\mathrm{pr}\!:TY^{\omega}\longrightarrow Y$ extends $\omega_Y^{\perp}$ to a G-invariant 2-form $\omega_{\nabla}^{\perp}$ on (the total space of) TY^{ω}; see Exercise B.8. By Exercise 1.30 with $\Omega\!=\!\omega^{\perp}$, the G-invariant closed 2-form

$$\widetilde{\omega}_{\nabla}\equiv\mathrm{pr}^*\omega+\frac{1}{2}\mathrm{d}\big(\iota_{\zeta_{TY^{\omega}}}\omega_{\nabla}^{\perp}\big)\tag{2.21}$$

satisfies (2.16) with $\widetilde{\omega}\!=\!\widetilde{\omega}_{\nabla}$. If in addition $Y\!\subset\!X$ is a closed subspace, by Proposition 2.15(2) there then exists a G-equivariant tubular neighborhood identification $\Phi\!:\mathcal{U}\longrightarrow U$ for Y in X such that $\mathcal{U}\!\subset\!TY^{\omega}$ and $\Phi^*\omega\!=\!\widetilde{\omega}_{\nabla}|_{\mathcal{U}}$. If $Y\equiv\{x\}$ is a one-point set, $TY^{\omega}\!=\!T_xX$ and $\widetilde{\omega}_{\nabla}\!=\!\mathrm{pr}^*\omega_x$. This yields Corollary 2.18 below.

Corollary 2.18 (Darboux Theorem)**.** *Suppose ψ is a smooth action of a compact Lie group G on a symplectic manifold (X,ω) and $x\in X^{\psi}$. There exist a G-invariant tubular neighborhood $\mathcal{U}$ of 0 in T_xX and a G-equivariant diffeomorphism $\Phi\!:\mathcal{U}\longrightarrow U$ onto a neighborhood U of x in X such that*

$$\Phi(0)=x,\qquad \mathrm{d}_0\Phi\!=\!\mathrm{id}\!:T_0(T_xX)\!=\!T_xX\longrightarrow T_xX,\qquad \textit{and}\qquad \Phi^*\omega=\mathrm{pr}^*\omega_x|_{\mathcal{U}}\,,$$

where $\mathrm{pr}\!:T_xX\longrightarrow\{x\}$ *is the projection.*

Corollary 2.19 (Lagrangian Tubular Neighborhood Theorem)**.** *Suppose ψ is a closed action of a compact Lie group G on a symplectic manifold (X,ω), $Y\subset X$ is a compact Lagrangian submanifold preserved by ψ, and ω_{T^*Y} is the canonical symplectic form on T^*Y as in Exercise 1.34. There exists a G-equivariant diffeomorphism $\Phi\colon \mathcal{U}\longrightarrow U$ from an open neighborhood of Y in T^*Y onto an open neighborhood of Y in X so that*

$$\Phi(y)=y \quad \forall\, y\in Y \qquad \textit{and} \qquad \Phi^*\omega=\omega_{T^*Y}\big|_{\mathcal{U}}\,. \tag{2.22}$$

Proof. By Exercise 2.13, there exists a ψ-invariant ω-compatible almost complex structure J on X. By Exercise 2.14, the isomorphism

$$\Phi_{Y;J}\colon J(TY)\longrightarrow T^*Y, \qquad \Phi_{Y;J}(w)=\omega(\cdot,w),$$

of real vector bundles over Y is G-equivariant with respect to the actions $\mathrm{d}\psi$ and $\mathrm{d}\psi^*$ of Exercise 1.1. Along with the latter exercise, this implies that the closed 2-form $\Phi_{Y;J}^*\omega_{T^*Y}$ on (the total space of) $J(TY)$ is G-invariant. By Exercise 1.37(c), this form satisfies (2.16). By Proposition 2.15(2), there thus exists a G-equivariant tubular neighborhood identification $\Phi\colon\mathcal{U}\longrightarrow U$ for Y in X such that $\mathcal{U}\subset J(TY)$ and $\Phi^*\omega=\Phi_{Y;J}^*\omega_{T^*Y}|_{\mathcal{U}}$. The map

$$\Phi\circ\Phi_{Y;J}^{-1}\colon \Phi_{Y;J}(\mathcal{U})\longrightarrow U$$

is then a G-equivariant diffeomorphism satisfying (2.22) with Φ replaced by $\Phi\circ\Phi_{Y;J}^{-1}$. $\square$

Remark 2.20. The analogue of Proposition 2.15(2) in [41] is Lemma 3.2.1, which unnecessarily requires Y (Q in [41]) to be compact. As a consequence, the Symplectic and Lagrangian Tubular Neighborhood Theorems, i.e. Example 2.17 and Corollary 2.19 above, are restricted to compact submanifolds in [41]; see Theorems 3.4.10 and 3.4.13 in [41]. Even if one is interested only in compact symplectic manifolds, the Symplectic Tubular Neighborhood Theorem without the compactness restriction is needed for the proof of Theorem 4.19. The latter is a key step in the proof of Delzant's Theorem, Theorem 4.20, following the modern efficient approach sketched in [37, 42]; see page 97.

2.4 Weakly Hamiltonian group actions

We next establish some properties of weakly Hamiltonian actions of Lie groups on symplectic manifolds. We then study how close such actions to being Hamiltonian.

Exercise 2.21. Suppose ψ is a smooth action of a Lie group G on a symplectic manifold (X,ω), $\mu\colon X\longrightarrow T_{\mathbb{1}}^*G$ is a smooth map satisfying the first condition in (1.6), and $x\in X$. For each $v\in T_{\mathbb{1}}G$, let $\zeta_v\in\Gamma(X;TX)$ be as in (1.2). Show that

$$\begin{aligned}\ker \mathrm{d}_x\mu &= \big\{\zeta_v(x)\colon v\in T_{\mathbb{1}}G\big\}^{\omega}\equiv\big\{w\in T_xX\colon \omega\big(w,\zeta_v(x)\big)=0\ \forall\, v\in T_{\mathbb{1}}G\big\},\\ \operatorname{Im}\mathrm{d}_x\mu &= \operatorname{Ann}\big(\big\{v\in T_{\mathbb{1}}G\colon \zeta_v(x)=0\big\}\big)\equiv\big\{\alpha\in T_{\mathbb{1}}^*G\colon \alpha(v)=0\ \forall\, v\in T_{\mathbb{1}}G\ \text{s.t.}\ \zeta_v(x)=0\big\}.\end{aligned}\tag{2.23}$$

Conclude that

(a) the G-orbit $Gx\subset X$ of x is open if and only if $\mathrm{d}_x\mu$ is injective;

(b) the stabilizer $G_x(\psi)\subset G$ of x is discrete if and only if $\mathrm{d}_x\mu$ is surjective.

Exercise 2.22. Let ψ be a smooth action of a connected Lie group G on a compact positive-dimensional symplectic manifold (X,ω).

(a) Give an example of such an action with $X^\psi=\emptyset$.

(b) Suppose ψ is a weakly Hamiltonian action. Show that X^ψ contains at least 2 points.

Exercise 2.23. Suppose ψ is a smooth action of a torus $\mathbb{T}$ on a compact symplectic manifold (X,ω), $\mu\colon X\longrightarrow T_{\mathbb{1}}^*G$ is a smooth map satisfying the first condition in (1.6), and $x\in X$. Let $\mathbb{T}_\psi(x)\subset\mathbb{T}$ be the stabilizer of x and $Z\in\pi_0(X^{\mathbb{T}_\psi(x)})$ be the topological component of the $\psi|_{\mathbb{T}_\psi(x)}$-fixed locus containing x. Show that

(a) $Z\cap X^\psi\neq\emptyset$;

(b) if $Y\in\pi_0(X^\psi)$ and $Y\subset Z$, then $\mathbb{T}_x(\psi)=\mathbb{T}_S$ for some $S\subset S(Y)$, with $S(Y)\subset T_{\mathbb{1}}^*\mathbb{T}$ as in Proposition 2.10(1).

Hint: use Exercise 2.22(b) and Proposition 2.10(2).

Exercise 2.24. Let X be a closed oriented manifold.

(1) Suppose $\zeta\in\Gamma(X;TX)$ and Ω is a (top) form on X. Show that

$$\int_X \mathcal{L}_\zeta\Omega=0.$$

(2) Suppose the dimension of X is $2n$, ω is a symplectic form on X, and ψ is a smooth action of a Lie group G on (X,ω). For each $v\in T_{\mathbb{1}}G$, let $\zeta_v\in\Gamma(X;TX)$ be the vector field on X generated by v as in (1.2). If $v_1\in T_{\mathbb{1}}G$ and there exists a smooth map $\mu_{v_1}\colon X\longrightarrow\mathbb{R}$ satisfying the first condition in (1.6) with $v=v_1$, then

$$\int_X \omega\big(\zeta_{v_1},\zeta_{v_2}\big)\omega^n=0 \qquad \forall\ v_2\in T_{\mathbb{1}}G\,.$$

Hint: use Cartan's formula for the Lie derivative.

Exercise 2.25. Suppose (X,ω) is a symplectic manifold, ψ is a smooth action of a connected Lie group G on X as in (1.1), $\mu\colon X\longrightarrow T_{\mathbb{1}}^*G$ is a smooth map satisfying the first condition in (1.6), $v\in T_{\mathbb{1}}G$, and $\zeta_v\in\Gamma(X;TX)$ is the vector field on X generated by v as in (1.2). Let $h\colon[a,b]\longrightarrow G$ be a smooth path. Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_v\circ\psi_h=-\omega\big(\zeta_{\mathrm{Ad}_{h^{-1}}(v)},\zeta_{\mathrm{Ad}_{h^{-1}}(h'h^{-1})}\big)\,,\qquad \text{where}\quad h'=\frac{\mathrm{d}h}{\mathrm{d}t}\colon[a,b]\longrightarrow TG\,.$$

Hint: apply the first condition in (1.6) and (2.6), then the first equation in (2.5) twice, and finally the ψ-invariance of ω.

Proposition 2.26. *Suppose (X,ω) is a symplectic manifold, ψ is a smooth action of a connected Lie group G on X as in (1.1), and $\mu\colon X\longrightarrow T_{\mathbb{1}}^*G$ is a smooth map satisfying the first condition in (1.6). For each $v\in T_{\mathbb{1}}G$, let $\zeta_v\in\Gamma(X;TX)$ be the vector field on X generated by v as in (1.2).*

(1) For all $v,v'\in T_{\mathbb{1}}G$, the function

$$\mu_{[v,v']}+\omega\big(\zeta_v,\zeta_{v'}\big)\colon X\longrightarrow\mathbb{R}$$

is constant on each connected component of X.

(2) The map μ satisfies the second condition in (1.6) if and only if

$$\mu_{[v,v']}=-\omega\big(\zeta_v,\zeta_{v'}\big)\colon X\longrightarrow\mathbb{R}\qquad\forall\ v,v'\in T_{\mathbb{1}}G. \tag{2.24}$$

(3) If G is abelian and either $G\approx\mathbb{R}$, or G is compact, or X is compact, then the map μ satisfies the second condition in (1.6).

(4) If X is closed, connected, and of dimension $2n$, then the smooth map

$$\widehat{\mu}\colon X\longrightarrow T^*_{\mathbb{1}}G,\qquad \widehat{\mu}(x)=\mu(x)-\int_X\mu\,\omega^n\Big/\!\!\int_X\omega^n,$$

is a moment map for the G-action ψ on (X,ω).

Proof. (1) By the first condition in (1.6) and the second identity in (2.5),

$$\mathrm{d}\mu_{[v,v']}=-\omega\big(\zeta_{[v,v']},\cdot\big)=\omega\big([\zeta_v,\zeta_{v'}],\cdot\big).$$

By (2.13),

$$\mathrm{d}\big(\omega(\zeta_v,\zeta_{v'})\big)=-\omega\big([\zeta_v,\zeta_{v'}],\cdot\big).$$

The two statements give the claim.

(2) Let $v\in T_{\mathbb{1}}G$, $h:[0,1]\longrightarrow G$ be a smooth path with $h(0)=\mathbb{1}$, and $h':[0,1]\longrightarrow TG$ be its derivative as in (1.14). Thus,

$$\mu_v\circ\psi_{h(0)}=\mu_{\mathrm{Ad}_{h(0)^{-1}}v}\,,\qquad \begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\mu_v\circ\psi_h=-\omega\big(\zeta_{\mathrm{Ad}_{h^{-1}}(v)},\zeta_{\mathrm{Ad}_{h^{-1}}(h'h^{-1})}\big),\\ &\frac{\mathrm{d}}{\mathrm{d}t}\mu_{\mathrm{Ad}_{h^{-1}}(v)}=-\mu_{\mathrm{Ad}_{h^{-1}}([h'h^{-1},v])}=-\mu_{[\mathrm{Ad}_{h^{-1}}(h'h^{-1}),\mathrm{Ad}_{h^{-1}}(v)]};\end{aligned} \tag{2.25}$$

the equality in the top equation on the right-hand side above holds by Exercise 2.25, while the two equalities in the bottom equation follow from (1.14) and the second identity in (1.13). Since G is connected, the second condition in (1.6) is thus equivalent to the condition that

$$\mu_{[\mathrm{Ad}_{h^{-1}}(v),\mathrm{Ad}_{h^{-1}}(h'h^{-1})]}=-\omega\big(\zeta_{\mathrm{Ad}_{h^{-1}}(v)},\zeta_{\mathrm{Ad}_{h^{-1}}(h'h^{-1})}\big)$$

for every $v\in T_{\mathbb{1}}G$ and every smooth path $h\colon[0,1]\longrightarrow G$ with $h(0)=\mathbb{1}$. Since Ad_u is an automorphism of $T_{\mathbb{1}}G$, the last condition is equivalent to (2.24).

(3) We can assume that X is connected. Let $v,v'\in T_{\mathbb{1}}G$ and $h(t)=\mathrm{e}^{tv'}$ for $t\in\mathbb{R}$. Since G is abelian,

$$\mathrm{Ad}_{h(t)^{-1}}(v)=v\quad\text{and}\quad\mathrm{Ad}_{h^{-1}}(h'h^{-1})=v'\,.$$

Along with (2.25), this implies that

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mu_v\circ\psi_h=-\frac{\mathrm{d}}{\mathrm{d}t}\big(\mu_v\circ\psi_h-\mu_{\mathrm{Ad}_{h^{-1}}(v)}\big)=\omega(\zeta_v,\zeta_{v'})+\mu_{[v,v']}; \tag{2.26}$$

the two sides above are a priori functions on $X\times\mathbb{R}$, but the right-hand side above is explicitly independent of the $\mathbb{R}$-input and is constant in the X-input by (1). If the right-hand side in (2.26) vanishes (as is automatically the case if $G\approx\mathbb{R}$), then μ satisfies the second condition in (1.6) by (2). If the constant on the right-hand side of (2.26) is not zero, then the function $\mu_v\circ\psi_h$ is unbounded on $\mathbb{R}$ (with the X-input fixed), which implies that G and X are not compact.

(4) Since $\widehat{\mu}$ differs from μ by a constant, $\widehat{\mu}$ satisfies the first condition in (1.6) with μ replaced by $\widehat{\mu}$. Let $v,v'\in T_{\mathbb{1}}G$. By (1),

$$\widehat{\mu}_{[v,v']}+\omega\big(\zeta_v,\zeta_{v'}\big)\colon X\longrightarrow\mathbb{R}$$

is a constant function on X. By the definition of $\widehat{\mu}$ and Exercise 2.24(2),

$$\int_X\widehat{\mu}_{[v,v']}\omega^n=0\qquad\text{and}\qquad\int_X\omega\big(\zeta_v,\zeta_{v'}\big)\omega^n=0\,,$$

respectively. Thus,

$$\widehat{\mu}_{[v,v']}=-\omega\big(\zeta_v,\zeta_{v'}\big)\colon X\longrightarrow\mathbb{R}\qquad\forall\ v,v'\in T_{\mathbb{1}}G.$$

Along with (2), this implies that $\widehat{\mu}$ satisfies the second condition in (1.6) with μ replaced by $\widehat{\mu}$. $\square$

Exercise 2.27. Suppose ψ is a smooth action of a Lie group G on a symplectic manifold (X,ω). Let $\mathfrak{g}$ be the Lie algebra of G and $d_{\mathfrak{g};k}$ be as in (1.15) with $\mathfrak{a}=\mathfrak{g}$.

(a) For each $v\in T_{\mathbb{1}}G$, let $\zeta_v\in\Gamma(X;TX)$ be the vector field on X generated by v as in (1.2). By Exercise 2.12(a), the homomorphism

$$\psi_\omega\colon T_{\mathbb{1}}G\longrightarrow H^1_{\mathrm{deR}}(X),\qquad\psi_\omega(v)=\big[\iota_{\zeta_v}\omega\big],$$

is well-defined. Show that ψ_ω lies in the kernel of

$$\mathrm{d}_{\mathfrak{g};1}\colon\mathrm{Hom}_{\mathbb{R}}(T_{\mathbb{1}}G;H^1_{\mathrm{deR}}(X))\longrightarrow\mathrm{Hom}_{\mathbb{R}}(\Lambda^2_{\mathbb{R}}(T_{\mathbb{1}}G);H^1_{\mathrm{deR}}(X)).$$

(b) Suppose in addition X is connected and $\mu\colon X\longrightarrow T^*_{\mathbb{1}}G$ is a smooth map satisfying the first condition in (1.6). For $v,v'\in T_{\mathbb{1}}G$, let $\psi_{\omega;\mu}(v,v')\in\mathbb{R}$ be the constant value of the function in Proposition 2.26(1). Show that

$$\begin{gathered}\psi_{\omega;\mu}\in\ker\big(d_{\mathfrak{g};2}\colon\mathrm{Hom}_{\mathbb{R}}(\Lambda^2_{\mathbb{R}}(T_{\mathbb{1}}G);\mathbb{R})\longrightarrow\mathrm{Hom}_{\mathbb{R}}(\Lambda^3_{\mathbb{R}}(T_{\mathbb{1}}G);\mathbb{R})\big)\\ \text{and}\qquad\psi_{\omega;\mu+\alpha_0}-\psi_{\omega;\mu}=-\mathrm{d}_{\mathfrak{g};1}\alpha_0\ \ \forall\ \alpha_0\in T^*_{\mathbb{1}}G.\end{gathered}$$

Hint: use (2.13) for (a), the second identity in (2.5) and (2.14) for (b).

By Exercise 2.27(a), a smooth action ψ of a Lie group G on a symplectic manifold (X,ω) determines an element

$$\psi^1_\omega\equiv\big[\psi_\omega\big]\in H^1\big(\mathfrak{g};H^1_{\mathrm{deR}}(X)\big).$$

This action is weakly Hamiltonian if and only if $\psi^1_\omega=0$. By Exercise 2.27(b), a weakly Hamiltonian action ψ of a Lie group G on a connected symplectic manifold (X,ω) determines an element

$$\psi^2_\omega\equiv\big[\psi_{\omega;\mu}\big]\in H^2(\mathfrak{g}).$$

By Proposition 2.26(2), this action is Hamiltonian if and only if $\psi_\omega^2 = 0$.

A non-abelian Lie algebra is called simple if it contains no nontrivial ideals and semisimple if it is a direct sum of simple Lie algebras. A connected Lie group G is semisimple if its Lie algebra $\mathfrak{g}$ is semisimple. Examples of semisimple Lie groups include the special orthogonal group SO_n and the unitary group U_n.

Corollary 2.28. *Suppose (X,ω) is a symplectic manifold and G is a semisimple Lie group G. Every smooth G-action on (X,ω) is Hamiltonian.*

Proof. We can assume that X is connected. Let $\mathfrak{g}$ be the Lie algebra of G. By [17, Theorem 21.1], $H^1(\mathfrak{g}), H^2(\mathfrak{g}) = \{0\}$. By Exercise 2.27(a), every smooth G-action on (X,ω) is thus weakly Hamiltonian. By Exercise 2.27(b) and Proposition 2.26(2), every weakly Hamiltonian G-action on (X,ω) is similarly Hamiltonian. $\square$

Example 2.29. With $n \in \mathbb{Z}^+$, let ψ be the action of $\mathbb{C}^n$ on $(\mathbb{C}^n, \omega_{\mathbb{C}^n})$ as in Exercise 1.24. Let $e_1, \mathfrak{i}e_1, \ldots, e_n, \mathfrak{i}e_n$ be the standard $\mathbb{R}$-basis for $T_0\mathbb{C}^n = \mathbb{C}^n$ as before. By Exercise 1.24, this action is weakly Hamiltonian, but not Hamiltonian, with

$$\psi_\omega^2 \in H^2\big(T_{\mathbb{1}}\mathbb{C}^n\big) = \mathrm{Hom}_{\mathbb{R}}\big(\Lambda^2_{\mathbb{R}}(\mathbb{C}^n); \mathbb{R}\big)$$

given by

$$\psi_\omega^2\big(e_j \wedge e_k\big), \psi_\omega^2\big(\mathfrak{i}e_j \wedge \mathfrak{i}e_k\big) = 0, \quad \psi_\omega^2\big(e_j \wedge \mathfrak{i}e_k\big) = \delta_{jk}. \qquad \forall\, j,k \in [n].$$

Exercise 2.30. Suppose ψ is a smooth action of a Lie group G on a nonempty symplectic manifold (X,ω) as in (1.1) and $G' \subset G$ is a normal subgroup acting trivially on X. Let ψ' be the induced action of G/G' as in Exercise 1.21(d) and

$$\delta_1 \colon \mathcal{K}^1_{\mathfrak{g}';\mathfrak{g}} \longrightarrow H^2(\mathfrak{g}/\mathfrak{g}'),$$

where $\mathfrak{g}$ and $\mathfrak{g}'$ are the Lie algebras of G and G', respectively, be as in Exercise 1.11. Show that

(a) $\psi_\omega^1\big|_{\mathfrak{g}'} = 0$ and thus the G/G'-action ψ' on (X,ω) is weakly Hamiltonian if and only if the G-action ψ is;

(b) if (X,ω,ψ,μ) is a Hamiltonian G-manifold and $\alpha_0 \in \mathfrak{g}^*$ satisfies (1.32), then $\alpha_0|_{\mathfrak{g}'} \in \mathcal{K}^1_{\mathfrak{g}';\mathfrak{g}}$;

(c) if (X,ω,ψ,μ) is a connected Hamiltonian G-manifold and $\alpha_0 \in \mathfrak{g}^*$ satisfies (1.32), then

$$\psi'^2_\omega = \delta_1\big(\alpha_0|_{\mathfrak{g}'}\big) \in H^2\big(\mathfrak{g}/\mathfrak{g}'\big).$$

Hint: first show (b) for $\alpha_0 \in \mu(X)$ via Proposition 2.26(2).

Remark 2.31. Let ψ be a smooth action of a Lie group G on a symplectic manifold (X,ω) and $\mu \colon X \longrightarrow T^*_{\mathbb{1}}G$ be a smooth map satisfying the first condition in (1.6). The condition (2.24) is then equivalent to the vector space homomorphism

$$T_{\mathbb{1}}G \longrightarrow C^\infty(X;\mathbb{R}), \qquad v \longrightarrow \mu_v,$$

being a Lie algebra or anti Lie algebra homomorphism, depending on the exact conventions, with respect to the Poisson bracket determined by ω on $C^\infty(X;\mathbb{R})$; see [41, p99]. Some authors use (2.24) instead of the second condition in (1.6) in the definition of moment map. By Proposition 2.26(2), (2.24) is equivalent to the second condition in (1.6) restricted to the identity component of G and completely ignores the remaining components of G. Thus, the second condition in (1.6) is more natural for the notion of moment map than (2.24).

2.5 Hamiltonian group actions

We conclude this chapter with structural results for Hamiltonian almost periodic $\mathbb{R}^k$-actions on symplectic manifolds. Proposition 2.33 establishes a local description of such actions around their fixed loci. It yields Corollary 2.34, which provides topological restrictions on symplectic manifolds admitting Hamiltonian almost periodic $\mathbb{R}^k$-actions with isolated fixed points, and Corollary 2.35, which describes the local structure of moment maps of Hamiltonian almost periodic $\mathbb{R}^k$-actions everywhere.

Exercise 2.32. Suppose G is a Lie group, (X,ω) is a symplectic manifold, ψ is a smooth G-action on (X,ω), J is an ω-compatible almost complex structure on X, and $x\in X$. For each $v\in T_{\mathbb{1}}G$, let $\zeta_v\in\Gamma(X;TX)$ be as in (1.2).

(a) Give an example so that $\dim X>0$ and

$$\big\{\zeta_v(x)\colon v\in T_{\mathbb{1}}G\big\}=\big\{J\zeta_v(x)\colon v\in T_{\mathbb{1}}G\big\}=T_xX.$$

(b) Let $\mu\colon X\longrightarrow T_{\mathbb{1}}^*G$ be a smooth map satisfying the first condition in (1.6) and $g\equiv g_J^\omega$ be the Riemannian metric on X determined by ω and J as in (1.18). Show that

$$\begin{aligned} \nabla^g\mu_v&=-J\zeta_v\in\Gamma(X;TX)\quad\forall\,v\in T_{\mathbb{1}}G\\ \text{and}\qquad \mathrm{d}_x\mu\big(J\zeta_v(x)\big)&\neq0\quad\forall\,v\in T_{\mathbb{1}}G\text{ s.t. }\zeta_v(x)\neq0. \end{aligned}\tag{2.27}$$

(c) Let $\mu:X\longrightarrow T_{\mathbb{1}}^*G$ be a G-invariant smooth map satisfying the first condition in (1.6). Show that

$$\big\{\zeta_v(x)\colon v\in T_{\mathbb{1}}G\big\}\cap\big\{J\zeta_v(x)\colon v\in T_{\mathbb{1}}G\big\}=\{0\}\subset T_xX.\tag{2.28}$$

Suppose in addition that $\alpha\equiv\mu(x)\in T_{\mathbb{1}}^*G$ is a regular value of μ and thus $\mu^{-1}(\alpha)\subset X$ is a smooth submanifold. Show that

$$T_xX=T_x\big(\mu^{-1}(\alpha)\big)\oplus\big\{J\zeta_v(x)\colon v\in T_{\mathbb{1}}G\big\}.\tag{2.29}$$

Proposition 2.33. *Suppose $k\in\mathbb{Z}^+$, (X,ω,ψ,μ) is a Hamiltonian $\mathbb{R}^k$-manifold, ψ' is a smooth action of a torus $\mathbb{T}$ on X, $\rho\colon\mathbb{R}^k\longrightarrow\mathbb{T}$ is a homomorphism with dense image so that $\psi=\psi'\circ\rho$, $Y\subset X$ is a topological component of $X^\psi=X^{\psi'}$, and J is a ψ-invariant (or equivalently ψ'-invariant) ω-compatible almost complex structure on X. Let $S(Y)\subset(T_{\mathbb{1}}^*\mathbb{T})_{\mathbb{Z}}$ and $\mathcal{N}_X^\alpha Y\subset TX|_Y$ for each $\alpha\in S(Y)$ be as in Proposition 2.10(1) with ψ replaced by ψ' so that the complex structure on $\mathcal{N}_X^\alpha Y$ is induced by J. For every $y\in Y$, there exists a $\mathbb{T}$-equivariant tubular neighborhood identification $\Phi_y\colon\mathcal{U}_y\longrightarrow U_y$ for y in X such that*

$$\begin{gathered}\Phi_y^*\omega=\omega_y|_{\mathcal{U}_y},\qquad \mu\big(\Phi_y\big(w_0,(w_\alpha)_{\alpha\in S(Y)}\big)\big)=\mu(Y)+\pi\sum_{\alpha\in S(Y)}|w_\alpha|^2(\rho^*\alpha)\\ \forall\ \big(w_0,(w_\alpha)_{\alpha\in S(Y)}\big)\in\mathcal{U}_y\subset T_yX=T_yY\oplus\bigoplus_{\alpha\in S(Y)}\mathcal{N}_X^\alpha Y\big|_y,\end{gathered}\tag{2.30}$$

where $|\cdot|$ is the norm on TX with respect to the metric g_J^ω.

Proof. By Corollary 2.18 with ψ replaced by ψ', there exists a $\mathbb{T}$-equivariant (or equivalently $\mathbb{R}^k$-equivariant) tubular neighborhood identification $\Phi_y\!:\mathcal{U}_y\longrightarrow U_y$ for y in X satisfying the first condition in (2.30). By Proposition 2.10(1), the complex vector space (T_yX,J_y) splits as

$$T_yX|_Y = T_yY\oplus \bigoplus_{\alpha\in S(Y)}\mathcal{N}_X^{\alpha}Y\big|_y$$

with the $\mathbb{T}$-action $\mathrm{d}_{y'}\psi'$ on $\mathcal{N}_X^{\alpha}Y|_y$ given by (2.12) with ψ replaced by ψ'. By Exercise 1.25, a moment map for this action with respect to ω_y is

$$\mu'\!: T_yX \longrightarrow T_{\mathbb{1}}^*\mathbb{T}, \quad \mu'\big(w_0,(w_\alpha)_{\alpha\in S(Y)}\big)=\pi\sum_{\alpha\in S(Y)}|w_\alpha|^2\alpha \;\;\forall\; \big(w_0,(w_\alpha)_{\alpha\in S(Y)}\big)\in T_yY\oplus\bigoplus_{\alpha\in S(Y)}\mathcal{N}_X^{\alpha}Y\big|_y\,.$$

Since a moment map is unique up to an additive constant on each connected component of the domain, it follows that

$$\Phi_y^*\mu=\mu(y)+\rho^*\circ\mu'\big|_{\mathcal{U}_y}\!:\mathcal{U}_y\longrightarrow T_0^*\mathbb{R}^k\,.$$

This establishes the second condition in (2.30). □

Corollary 2.34. *Suppose $k\in\mathbb{Z}^+$, (X,ω,ψ,μ) is a compact Hamiltonian $\mathbb{R}^k$-manifold, the $\mathbb{R}^k$-action ψ is almost periodic, and the ψ-fixed locus $X^\psi\subset X$ is discrete. Then,*

(1) $H_{\mathrm{odd}}(X;\mathbb{Z})$ is trivial and $H_{\mathrm{even}}(X;\mathbb{Z})$ is a free $\mathbb{Z}$-module of rank $|X^\psi|$;

(2) if $\rho\!:\mathbb{R}^k\longrightarrow\mathbb{T}$, $S(y)\subset(T_{\mathbb{1}}^*\mathbb{T})_{\mathbb{Z}}$ for each $y\in X^\psi$, and $\mathcal{N}_X^{\alpha}\{y\}\subset T_yX$ for each $\alpha\in S(y)$ are as in Proposition 2.33, $v\in T_0\mathbb{R}^k$ is generic, $S_v(y)=\{\alpha\in S(y)\!:\alpha(\rho(v))<0\}$, and $p\in\mathbb{Z}^{\ge0}$, then

$$\mathrm{rk}_{\mathbb{Z}}H_{2p}(X;\mathbb{Z})=\big|X^\psi_{v;2p}\big|, \qquad \textit{where}\quad X^\psi_{v;2p}=\Big\{y\in X^\psi\!: \sum_{\alpha\in S_v(y)}\dim_{\mathbb{R}}\mathcal{N}_X^{\alpha}\{y\}=2p\Big\}; \tag{2.31}$$

(3) if in addition the linear spans of $S_v(y)$ and $S_{-v}(y)$ in $T_0^*\mathbb{R}^k$ are disjoint (except at 0) for every $y\in X^\psi$, $\mathbb{R}^-_{y;v}\subset\mathbb{R}^k=T_0\mathbb{R}^k$ is the annihilator of $S_v(y)$, and $X^-_{y;v}(\psi)\subset X$ is the topological component of the $\psi|_{\mathbb{R}^-_{y;v}}$-fixed locus containing $y\in X^\psi$, then $H_{2p}(X;\mathbb{Z})$ is freely generated by the fundamental homology classes of the submanifolds $X^-_{y;v}(\psi)$ with $y\in X^\psi_{v;2p}$.

Proof. Let ρ, $S(y)$, and $\mathcal{N}_X^{\alpha}\{y\}\subset T_yX$ be as in Proposition 2.33. For each $\alpha\in\mathcal{N}_X^{\alpha}\{y\}$, let $n_{y\alpha}\in\mathbb{Z}^+$ be half the (real) dimension of $\mathcal{N}_X^{\alpha}\{y\}$. For a generic choice of $v\in\mathbb{R}^k$, $\rho(\mathbb{R}v)\subset\mathbb{T}$ is dense and thus $X^{\mathbb{R}v}=X^\psi$. We apply Proposition 2.33 to the restriction of the action ψ to $\mathbb{R}v\subset\mathbb{R}^k$ and its Hamiltonian $\mu_v\!: X\longrightarrow\mathbb{R}$. For every $y\in X^\psi$, there thus exists a coordinate chart

$$(w_\alpha)_{\alpha\in S(y)}\!: U_y\longrightarrow\prod_{\alpha\in S(y)}\mathbb{C}^{n_{y\alpha}} \qquad \text{s.t.}\quad \mu_v=\mu_v(y)+\pi\sum_{\alpha\in S(Y)}\alpha\big(\rho(v)\big)|w_\alpha|^2\!: U_y\longrightarrow\mathbb{R}. \tag{2.32}$$

By Exercise 1.10(c), $\alpha(\rho(v))\neq0$ for all $\alpha\in S(y)$. By the first condition in (1.6) and Proposition 2.1(1),

$$\mathrm{Crit}(\mu_v)\equiv\big\{y\in X\!: \mathrm{d}_y\mu_v=0\big\}=X^\psi\,.$$

Thus, μ_v is a Morse function, i.e. a Morse-Bott function with only isolated critical points. Along with [44, Theorem 3.5] and (2.32), this implies that X has the homotopy type of a CW complex with only even-dimensional cells and the number of cells of dimension $2p$ with $p\in\mathbb{Z}^{\ge0}$ given by (2.31). This establishes (1) and (2).

We now assume that the linear spans of $S_v(y)$ and $S_{-v}(y)$ in $T_0^*\mathbb{R}^k$ are disjoint (except at 0) for every $y\in X^\psi$. Let $\psi_{v;t}\colon X\longrightarrow X$ be the negative gradient flow of μ_v, as in Exercise A.1, with respect to a ψ-invariant Riemannian metric g and $X_y^-(\mu_v)\subset X$ for each $y\in X^\psi$ be the μ_v-unstable manifold of y, as in (A.1). By (A.2) and the second equation in (2.30),

$$T_y\big(X_y^-(\mu_v)\big)=E_y^-(\mu_v)=\bigoplus_{\alpha\in S_v(y)}\mathcal{N}_X^\alpha\{y\}=T_y\big(X_{y;v}^-(\psi)\big)\,.$$

Since the flow $\psi_{v;t}$ commutes with the action ψ, it follows that $X_y^-(\mu_v)\subset X_{y;v}^-(\psi)$ is an open subset and $X_{y;v}^-(\psi)-X_y^-(\mu_v)$ is the union of all other unstable manifolds of $\mu_v|_{X_{y;v}^-(\psi)}$. Since the indices of all critical points are even, $X_y^-(\mu_v)$ is a $2p$-pseudocycle in the sense of [60, Theorem 1.1] for every $y\in X_{v;2p}^\psi$ and the boundary operator in the Witten chain complex of [52, Section 3] vanishes. By [52, Theorem 3.1], $H_{2p}(X;\mathbb{Z})$ is thus freely generated by the homology classes of these pseudocycles. The same applies to every compact connected submanifold $X_{y;v}^-(\psi)\subset X$. In particular, the fundamental homology class of $X_{y;v}^-(\psi)$ and thus its image in X are the homology classes in $X_{y;v}^-(\psi)$ and in X, respectively, determined by the pseudocycle $X_y^-(\mu_v)\subset X_{y;v}^-(\psi),X$ for every $y\in X^\psi$. This gives (3). □

Suppose ψ is a smooth action of $\mathbb{R}^k$ on a smooth manifold X. For $y\in X$, let $\mathbb{R}_y^k\subset\mathbb{R}^k$ be the largest linear subspace preserving y and $\mathbb{R}_y^c\subset\mathbb{R}^k$ be a complementary linear subspace. The decomposition $\mathbb{R}^k=\mathbb{R}_y^k\oplus\mathbb{R}_y^c$ induces decompositions

$$T_0\mathbb{R}^k=T_0\mathbb{R}_y^k\times T_0\mathbb{R}_y^c\qquad\text{and}\qquad \mu\equiv\big(\mu_y,\mu_y^c\big)\colon X\longrightarrow T_0^*\mathbb{R}_y^k\times T_0^*\mathbb{R}_y^c=T_0^*\mathbb{R}^k\,,\tag{2.33}$$

for any map $\mu\colon X\longrightarrow T_0^*\mathbb{R}^k$. The images of the inclusions

$$T_0\mathbb{R}_y^k\hookrightarrow T_0\mathbb{R}^k\qquad\text{and}\qquad T_0^*\mathbb{R}_y^c\hookrightarrow T_0^*\mathbb{R}^k$$

of the factors in (2.33) correspond to the tangent space $T_0(\mathbb{R}_y^k(\psi))$ of the stabilizer $\mathbb{R}_y^k(\psi)\subset\mathbb{R}^k$ of y and to the annihilator of this tangent space, respectively. If μ above is a moment map for the action ψ on X with respect to a symplectic form ω, then μ_y and μ_y^c are moment maps for the $\mathbb{R}_y^k$-action $\psi_y\equiv\psi|_{\mathbb{R}_y^k}$ and $\mathbb{R}_y^c$-action $\psi_y^c\equiv\psi|_{\mathbb{R}_y^c}$, respectively, on X with respect to ω. If the action ψ is almost periodic, then so are the actions ψ_y and ψ_y^c. By Proposition 2.15(1), $(X^{\psi_y},\omega|_{X^{\psi_y}})$ is a symplectic manifold. Since the actions ψ_y and ψ_y^c commute, ψ_y^c preserves X^{ψ_y} and thus $(X^{\psi_y},\omega|_{X^{\psi_y}},\psi_y^c|_{X^{\psi_y}},\mu_y^c|_{X^{\psi_y}})$ is a Hamiltonian $\mathbb{R}_y^c$-manifold. By Exercises 2.21(b) and 2.2, the differential

$$\mathrm{d}_y\mu_y^c\colon T_yX^{\psi_y}\longrightarrow T_0^*\mathbb{R}_y^c\tag{2.34}$$

is surjective in this case.

Corollary 2.35. *Suppose $k\in\mathbb{Z}^+$, (X,ω,ψ,μ) is a Hamiltonian $\mathbb{R}^k$-manifold, the $\mathbb{R}^k$-action ψ is almost periodic, $y\in X$, and $\mathbb{R}_y^k,\mathbb{R}_y^c\subset\mathbb{R}^k$, μ_y, μ_y^c, and ψ_y are as above. There exist a finite subset $S(y)\subset T_0^*\mathbb{R}_y^k$, a positive-dimensional normed complex vector space $(V_\alpha,|\cdot|)$ for each $\alpha\in S(y)$, neighborhoods*

$$\mathcal{U}_{y;1}\subset X^{\psi_y},\qquad \mathcal{U}_{y;2}\subset\bigoplus_{\alpha\in S(y)}V_\alpha,\quad\textit{and}\quad U_y\subset X,$$

of y, 0, and y, respectively, and a diffeomorphism $\Phi_y\colon \mathcal{U}_{y;1}\times\mathcal{U}_{y;2}\longrightarrow U_y$ such that

$$\big|S(y)\big|\le(\dim X)/2-\dim\mathbb{R}^c_y,\quad \Phi_y(x,0)=x,\quad \mu^c_y\big(\Phi_y(x,w)\big)=\mu^c_y(x), \tag{2.35}$$
$$\mu_y\big(\Phi_y\big(x,(w_\alpha)_{\alpha\in S(y)}\big)\big)=\mu_y(y)+\pi\sum_{\alpha\in S(y)}|w_\alpha|^2\alpha$$

for all $x\in\mathcal{U}_{y;1}$ and $w\equiv(w_\alpha)_{\alpha\in S(y)}\in\mathcal{U}_{y;2}$, and the restriction

$$\mu\colon U_y\longrightarrow\mathcal{C}_y(\psi)\equiv\Big\{\mu_y(y)+\sum_{\alpha\in S(y)}t_\alpha\alpha\colon t_\alpha\in\mathbb{R}^{\ge0}\ \forall\,\alpha\in S(y)\Big\}\times\mathrm{Ann}(T_0\mathbb{R}^k_y)\subset T_0^*\mathbb{R}^k \tag{2.36}$$

is an open map with connected fibers.

Proof. Let J be a ψ-invariant ω-compatible almost complex structure on X (as provided by Exercise 2.13), $Y\in\pi_0(X^{\psi_y})$ be the connected component of the ψ_y-fixed locus containing y, and ψ^c_y be as above. With $S(Y)$ and ρ as in Proposition 2.33 with (ψ,μ) replaced by (ψ_y,μ_y), we take

$$S(y)=\rho^*S(Y)\subset T_0^*\mathbb{R}^k_y,\qquad V_{\rho^*\alpha}=\mathcal{N}^\alpha_XY\big|_y\ \ \forall\,\alpha\in S(Y),$$

and the norm $|\cdot|$ on V_α to be induced by the metric g_J^ω as before. By Proposition 2.15(1), $Y\subset X$ is a symplectic submanifold with $\mu_y(Y)=\mu_y(y)$. Since the actions ψ_y and ψ^c_y commute, Y is preserved by ψ^c_y. Since the differential (2.34) is surjective, the second equation in (2.23) and (2.28) applied to $(Y,\omega|_Y,\psi^c_y|_Y,\mu^c_y|_Y)$ imply that

$$\dim Y\ge2\dim\mathbb{R}^c_x\quad\Longrightarrow\quad 2\big|S(y)\big|\le\sum_{\alpha\in S(y)}\dim_{\mathbb{R}}V_\alpha=\dim X-\dim Y\le\dim X-2\dim\mathbb{R}^c_y\,.$$

This establishes the first statement in (2.35). The existence of a diffeomorphism Φ_y with the properties stated in (2.35) follows from Proposition 2.33 with (ψ,μ) replaced by (ψ_y,μ_y) and Exercise 2.36 below with $k=\dim Y$, $\ell=\operatorname{codim}Y$, $m=\dim\mathbb{R}^c_y$, and $f=\mu^c_y\circ\Phi_y$, with Φ_y as in Proposition 2.33. These properties in turn imply the stated properties of the restriction (2.36), after possibly shrinking U_y to achieve fiber connectivity. □

Exercise 2.36. Suppose $k,\ell,m\in\mathbb{Z}^{\ge0}$ and $f:\mathbb{R}^k\times\mathbb{R}^\ell\longrightarrow\mathbb{R}^m$ is a smooth function so that the restriction of the differential $\mathrm{d}_{(0,0)}f$ to $\mathbb{R}^k\times\{0\}$ is surjective. Show that there exist neighborhoods $\mathcal{U}_1$ of $0\in\mathbb{R}^k$ and $\mathcal{U}_2$ of $0\in\mathbb{R}^\ell$ and a smooth map

$$\phi\colon\mathcal{U}_1\times\mathcal{U}_2\longrightarrow\mathbb{R}^k\qquad\text{s.t.}\quad \phi(x,0)=x,\ \ f\big(\phi(x,w),w\big)=f(x,0)\ \ \forall\,x\in\mathcal{U}_1,\,w\in\mathcal{U}_2,$$

and for each $w\in\mathcal{U}_2$ the map $\mathcal{U}_1\longrightarrow\mathbb{R}^k$, $x\longrightarrow\phi(x,w)$, is a diffeomorphism onto an open subset of $\mathbb{R}^k$. *Hint:* assume that the restriction of $\mathrm{d}_{(0,0)}f$ to $\mathbb{R}^m\times\{0\}\times\{0\}\subset\mathbb{R}^k\times\{0\}$ is surjective; show that there exist neighborhoods $\mathcal{U}_1$ of $0\in\mathbb{R}^k$ and $\mathcal{U}_2$ of $0\in\mathbb{R}^\ell$ so that for each $w\in\mathcal{U}_2$ the map

$$\Phi_w\colon\mathcal{U}_1\longrightarrow\mathbb{R}^k,\quad \Phi_w(x_1,x_2)=\big(f\big((x_1,x_2),w\big),x_2\big)\ \ \forall\,(x_1,x_2)\in\mathcal{U}_1\subset\mathbb{R}^m\times\mathbb{R}^{k-m},$$

is a diffeomorphism onto an open subset of $\mathbb{R}^k$.

2.6 Applications of Moser's Trick

The proof of Proposition 2.15(2) applies Moser's Trick, which is summarized in Exercise 2.37 below; this section applies it in other settings. Lemma 2.38 is a technical observation useful for applications of Moser's Trick, such as those in Exercise 2.39 and Proposition 2.42.

For a manifold X and $p\in\mathbb{Z}$, we denote by $\Omega^p_c(X)$ the space of compactly supported p-forms and by $H^p_{\mathrm{deR};c}(X)$ the p-th compactly supported de Rham cohomology of X. We call a (compactly supported) p-form ϖ on X compactly exact if $\varpi=\mathrm{d}\eta$ for some $\eta\in\Omega^{p-1}_c(X)$.

Exercise 2.37 (Moser's Trick)**.** Let X be a manifold and $p\in\mathbb{Z}^{\ge0}$. Suppose $(\omega_t)_{t\in[0,1]}$ is a smooth family of closed p-forms on X, $(\eta_t)_{t\in[0,1]}$ is a smooth family of $(p-1)$-forms on X, $(\xi_t)_{t\in[0,1]}$ is a smooth family of vector fields on X, and $(\psi_t)_{t\in[0,1]}$ is a smooth family of diffeomorphisms of X with $\psi_0=\mathrm{id}_X$ such that

$$\omega_t-\omega_0=\mathrm{d}\eta_t,\quad \iota_{\xi_t}\omega_t=-\frac{\mathrm{d}}{\mathrm{d}t}\eta_t,\quad\text{and}\quad \frac{\mathrm{d}}{\mathrm{d}t}\psi_t=\xi_t\circ\psi_t\qquad\forall\, t\in[0,1]\,.$$

Show that $\psi_t^*\omega_t=\omega_0$ for all $t\in[0,1]$. *Hint:* use Cartan's formula for the Lie derivative.

Lemma 2.38. *Let X be a manifold, B be a manifold (possibly) with boundary, $p\in\mathbb{Z}^{\ge0}$, and $(\varpi_t)_{t\in B}$ be a smooth family of compactly exact p-forms on X. If $U\subset X$ is a precompact open subset such that*

$$\overline{\big\{x\in X:\varpi_t|_x\neq0\text{ for some }t\in B\big\}}\subset U,\tag{2.37}$$

then there exists a smooth family $(\eta_t)_{t\in B}$ of smooth $(p-1)$-forms on X so that

$$\varpi_t=\mathrm{d}\eta_t\ \forall\,t\in B,\quad \eta_t=0\ \forall\,t\in B\ \text{s.t. }\varpi_t=0,\quad \overline{\big\{x\in X:\eta_t|_x\neq0\text{ for some }t\in B\big\}}\subset U.\tag{2.38}$$

Proof. The argument below implements the suggestion in the proof of [41, Theorem 3.2.4]. It is sufficient to prove the claim with the closure dropped in the last condition in (2.38). Suppose $U_1,U_2\subset U$ are open subsets such that $U=U_1\cup U_2$,

$$\dim H^*_{\mathrm{deR};c}(U_1\cap U_2),\dim H^*_{\mathrm{deR};c}(U_1),\dim H^*_{\mathrm{deR};c}(U_2)<\infty,\tag{2.39}$$

and the claim of the lemma holds with U replaced by $U_1\cap U_2$, U_1, and U_2. By the Mayer-Vietoris sequence for the compactly supported de Rham cohomology of $U=U_1\cup U_2$, (2.39) implies that the vector space $H^*_{\mathrm{deR};c}(U_1\cup U_2)$ is also finite-dimensional. By analyzing the short exact sequence

$$0\longrightarrow\Omega^*_c(U_1\cap U_2)\longrightarrow\Omega^*_c(U_1)\oplus\Omega^*_c(U_2)\longrightarrow\Omega^*_c(U_1\cup U_2)\longrightarrow0\tag{2.40}$$

of chain complexes, where the first map sends ϖ to $(\varpi,-\varpi)$ and the second (ϖ_1,ϖ_2) to $\varpi_1+\varpi_2$ (after extensions by 0), we show below that the claim of the lemma with the closure dropped in the last condition in (2.38) holds.

Since $H^*_{\mathrm{deR};c}(U_1\cap U_2)$ is finite-dimensional, we can identify it with a subspace $\widetilde{H}^*_{\mathrm{deR};c}(U_1\cap U_2)$ of $\Omega^*_c(U_1\cap U_2)$ consisting of closed forms. Let

$$\widetilde{H}^\star_{\mathrm{deR};c}(U_1\cap U_2)\subset\widetilde{H}^*_{\mathrm{deR};c}(U_1\cap U_2)$$

be a subspace complementary to the kernel of the homomorphism

$$H^*_{\mathrm{deR};c}(U_1\cap U_2) \longrightarrow H^*_{\mathrm{deR};c}(U_1)\oplus H^p_{\mathrm{deR};c}(U_2)$$

induced by the first homomorphism in (2.40). Let $\beta_1,\beta_2\colon U\longrightarrow\mathbb{R}$ be a partition of unity subordinate to $\{U_1,U_2\}$. For each $t\in B$, define

$$\varpi_{t;1}=\beta_1\varpi_t|_{U_1}\in\Omega^p_c(U_1) \qquad\text{and}\qquad \varpi_{t;2}=\beta_2\varpi_t|_{U_2}\in\Omega^p_c(U_2).$$

Since ϖ_t is exact, $\mathrm{d}\varpi_{t;1}|_{U_1\cap U_2}=\mathrm{d}\varpi_{t;12}$ for some $\varpi_{t;12}\in\Omega^p_c(U_1\cap U_2)$ depending smoothly on $t\in B$. (because the claim of the lemma is assumed to hold for $U_1\cap U_2$). We can view $\varpi_{t;12}$ as a compactly supported p-form on U_1 and U_2 via extension by 0. The p-forms

$$\varpi'_{t;1}\equiv\varpi_{t;1}-\varpi_{t;12}\in\Omega^p_c(U_1) \qquad\text{and}\qquad \varpi'_{t;2}\equiv\varpi_{t;2}+\varpi_{t;12}\in\Omega^p_c(U_2)$$

are closed and $\varpi'_{t;1}+\varpi'_{t;2}\in\Omega^p_c(U)$ is an exact p-form. By the exactness of the Mayer-Vietoris sequence for the compactly supported de Rham cohomology of $U=U_1\cup U_2$, there thus exists a unique p-form $\varpi'_{t;12}\in\widetilde{H}^\star_{\mathrm{deR};c}(U_1\cap U_2)$ depending smoothly on $t\in B$ such that the forms

$$\varpi''_{t;1}\equiv\varpi'_{t;1}-\varpi'_{t;12}\in\Omega^p_c(U_1) \qquad\text{and}\qquad \varpi''_{t;2}\equiv\varpi'_{t;2}+\varpi'_{t;12}\in\Omega^p_c(U_2)$$

are compactly exact. By the assumptions on U_1 and U_2, there thus exist $\eta_{t;1}\in\Omega^{p-1}_c(U_1)$ and $\eta_{t;2}\in\Omega^{p-1}_c(U_2)$ depending smoothly on $t\in B$ such that

$$\varpi''_{t;1}=\mathrm{d}\eta_{t;1} \qquad\text{and}\qquad \varpi''_{t;2}=\mathrm{d}\eta_{t;2}.$$

The $(p-1)$-form $\eta_t\equiv\eta_{t;1}+\eta_{t;2}\in\Omega^{p-1}_c(U)$ thus also depends smoothly on $t\in B$ and satisfies (2.38), with the closure dropped.

Let $U\subset X$ be any precompact open subset. By [11, Theorem 5.1], X admits a good open cover, i.e. a cover by open subsets such that any finite intersection of the elements of the cover is diffeomorphic to $\mathbb{R}^n$, where n is the dimension X. By the proof of Proposition 2.15(2), every such intersection satisfies the claim of the lemma with the closure dropped. Since the closure of U is compact, it is covered by finitely many open subsets of a good open cover of X. By the above, U thus also satisfies the claim of the lemma. □

Exercise 2.39 (Moser's Stability)**.** Let X be a manifold.

(a) Let $(\omega_t)_{t\in[0,1]}$ be a smooth family of symplectic forms on X and $U\subset X$ be a precompact open subset such that

$$\overline{\big\{x\in X\colon \omega_t|_x\neq\omega_0|_x \text{ for some } t\in[0,1]\big\}}\subset U.$$

Suppose the 2-form $\omega_t-\omega_0$ is compactly exact for every $t\in[0,1]$. Show that there exists a smooth family $(\psi_t)_{t\in[0,1]}$ of diffeomorphisms of X such that

$$\psi_t^*\omega_t=\omega_0 \quad\text{and}\quad \psi_t\big|_{X-U}=\mathrm{id}_{X-U} \qquad \forall\, t\in[0,1].$$

(b) Suppose X is closed, connected, and oriented and Ω_0,Ω_1 are volume forms i.e. nowhere vanishing top forms, on X. Show that there exists a diffeomorphism $\psi\colon X\longrightarrow X$ such that $\psi^*\Omega_1=\Omega_0$ if and only if $\displaystyle\int_X\Omega_0=\int_X\Omega_1$.

The assumptions that the 2-forms ω_t agree outside of a compact subset and that their differences are compactly exact are necessary for the conclusion of Exercise 2.39(a). For example, $\mathbb{C}^n$ with $n\ge 2$ admits a symplectic structure ω so that $(\mathbb{C}^n,\omega)$ is not symplectomorphic to $(\mathbb{C}^n,\omega_{\mathbb{C}^n})$; see [28, 0.4.A$_2'$]. A smooth family $(\omega_t)_{t\in\mathbb{R}}$ of symplectic forms on a closed 8-dimensional smooth manifold $\widetilde{Y}$ is constructed in [39] so that all forms ω_k with $k\in\mathbb{Z}$ are cohomologous and the symplectic manifolds $(\widetilde{Y},\omega_k)$ and $(\widetilde{Y},\omega_\ell)$ with $k,\ell\in\mathbb{Z}$ are symplectomorphic if and only if $|k|=|\ell|$; see Theorem 2.1 in [39]. In the case of Exercise 2.39(b), the equality of the integrals implies that

$$\Omega_t\equiv(1-t)\Omega_0+t\Omega_1,\qquad t\in[0,1],$$

is a smooth family of cohomologous volume forms.

For a manifold X, denote by $\mathrm{Diff}_c(X)\subset\mathrm{Diff}(X)$ the subgroup of compactly supported diffeomorphisms, i.e. the diffeomorphisms of X that restrict to the identity outside of a compact subset of X, and by

$$\mathrm{Diff}_c(X)_0\subset\mathrm{Diff}_c(X)$$

the identity component of $\mathrm{Diff}_c(X)$. If in addition ω is a symplectic form on X, we denote by

$$\mathrm{Symp}_c(X,\omega)\equiv\mathrm{Symp}(X,\omega)\cap\mathrm{Diff}_c(X)\subset\mathrm{Diff}(X)$$

the subgroup of compactly supported symplectomorphisms of (X,ω), by

$$\mathrm{Symp}_c(X,\omega)_0'\equiv\mathrm{Symp}_c(X,\omega)\cap\mathrm{Diff}_c(X)_0\subset\mathrm{Diff}_c(X)$$

the subgroup of compactly supported symplectomorphisms of (X,ω) that are isotopic to the identity through compactly supported diffeomorphisms, and by

$$\mathrm{Symp}_c(X,\omega)_0\subset\mathrm{Symp}_c(X,\omega)_0'\tag{2.41}$$

the identity component of the group $\mathrm{Symp}_c(X,\omega)$. Let $\mathcal{S}_\omega(X)$ be the space of symplectic forms ω' on X so that ω' restricts to ω outside of a compact subset of X and the 2-form $\omega'-\omega$ on X is compactly exact. Denote by $\mathcal{S}_\omega^\star(X)\subset\mathcal{S}_\omega(X)$ the connected component containing ω. The subgroup

$$\mathrm{Diff}_c(X)_\omega=\big\{f\in\mathrm{Diff}_c(X)\colon f^*\omega\in\mathcal{S}_\omega\big\}$$

of $\mathrm{Diff}_c(X)$ is a union of topological components of $\mathrm{Diff}_c(X)$ which contains $\mathrm{Symp}_c(X,\omega)$. If $H^2_{\mathrm{deR};c}(X)=0$, then $\mathrm{Diff}_c(X)_\omega$ is the entire group $\mathrm{Diff}_c(X)$. From the definition of $\mathrm{Diff}_c(X)_\omega$, we obtain a continuous map

$$\mathrm{Diff}_c(X)_\omega\longrightarrow\mathcal{S}_\omega(X),\qquad f\longrightarrow f^*\omega.\tag{2.42}$$

Exercise 2.40. Let (X,ω) be a symplectic manifold. Show that $\mathrm{Diff}_c(X)_0$ acts transitively on $\mathcal{S}_\omega^\star(X)$.

Exercise 2.41. Let (X,ω) be a closed two-dimensional symplectic manifold. Show that the inclusion

$$\mathrm{Symp}(X,\omega)\longrightarrow\mathrm{Diff}^+(X),$$

where $\mathrm{Diff}^+(X)\subset\mathrm{Diff}(X)$ is the subgroup of orientation-preserving diffeormophisms, is a homotopy equivalence.

By Exercise 2.41, the inclusion in (2.41) is an equality if the dimension of X is 2. By [28, 0.3.C] and [2, Theorem 1.1], the groups $\mathrm{Symp}(\mathbb{C}P^2,\omega_{\mathrm{FS};2})$ and $\mathrm{Symp}(\mathbb{CP}^1\!\times\!\mathbb{CP}^1,\omega_{\mathrm{FS};1}\!\oplus\! r\omega_{\mathrm{FS};1})$ with $r\!>\!1$ are path-connected. Thus, the inclusion in (2.41) is an equality in these cases as well. By [28, 0.3.C], this is also the case if (X,ω) is $(\mathbb{CP}^1\!\times\!\mathbb{CP}^1,\omega_{\mathrm{FS};1}\!\oplus\!\omega_{\mathrm{FS};1})$, even though

$$\mathrm{Symp}\big(\mathbb{CP}^1\!\times\!\mathbb{CP}^1,\omega_{\mathrm{FS};1}\!\oplus\!\omega_{\mathrm{FS};1}\big)\approx\mathbb{Z}_2\,.$$

However, the inclusion in (2.41) is *not* an equality for most projective complete intersections (X,ω) of complex dimension 2; see [53, Theorem 1.4]. The next proposition collects together the relations between smooth and symplectic topologies pointed out in the proofs of Theorem 1 and Lemma 2.3 in [55].

Proposition 2.42 ([55]). *Let (X,ω) be a symplectic manifold and $G\subset\mathrm{Diff}_c(X)_\omega$ be a subgroup consisting of topological components of $\mathrm{Diff}_c(X)$. For every $k\!\in\!\mathbb{Z}^{\ge0}$, the homomorphism*

$$\pi_k\big(G,\mathrm{id}_X\big)\longrightarrow\pi_k\big(\mathcal{S}_\omega(X),\omega\big) \tag{2.43}$$

induced by (2.42) descends to an injective homomorphism from the cokernel of the homomorphism

$$\pi_k\big(\mathrm{Symp}_c(X,\omega)\!\cap\! G,\mathrm{id}_X\big)\longrightarrow\pi_k\big(G,\mathrm{id}_X\big) \tag{2.44}$$

induced by the inclusion.

Proof. By Exercise 2.40, the map (2.42) induces a fiber bundle

$$\mathrm{Symp}_c(X,\omega)\!\cap\! G\longrightarrow G\longrightarrow\big\{f^*\omega'\colon f\!\in\! G,\,\omega'\!\in\!\mathcal{S}_\omega^\star(X)\big\}=\big\{f^*\omega\colon f\!\in\! G\big\};$$

the base of this bundle is a union of connected components of $\mathcal{S}_\omega(X)$. The claim follows from the homotopy exact sequence for this fiber bundle.

Here is a direct argument, similar to [55]. Let $0\!\in\! S^k$ be a base point. Suppose $(f_s)_{s\in S^k}$ is a smooth family of diffeomorphisms of X and $(\omega_{s,t})_{s\in S^k,t\in[0,1]}$ is a smooth family of symplectic forms on X so that

$$f_0=\mathrm{id}_X,\quad f_s\in G,\;\omega_{s,0}=\omega,\;\omega_{s,1}=f_s^*\omega\;\forall\,s\!\in\! S^k,\quad\text{and}\quad\omega_{s,t}\in\mathcal{S}_\omega(X)\;\;\forall\,s\!\in\! S^k,\,t\!\in\![0,1]\,,$$

i.e. $(f_s)_{s\in S^k}$ represents an element in the kernel of the homomorphism (2.43). By Lemma 2.38 and Exercise 2.37, there exists a smooth family $(\psi_{s,t})_{s\in S^k,t\in[0,1]}$ of diffeomorphisms of X such that

$$\psi_{s,0}=\mathrm{id}_X,\quad\psi_{s,1}^*f_s^*\omega=\omega\quad\forall\,s\!\in\! S^k,\qquad\text{and}\qquad\psi_{s,t}\in\mathrm{Diff}_c(X)_0\;\;\forall\,s\!\in\! S^k,\,t\!\in\![0,1].$$

In particular, $f_s\!\circ\!\psi_{s,1}\!\in\!\mathrm{Symp}_c(X,\omega)$ for every $s\!\in\! S^k$. Thus, $(f_s)_{s\in S^k}$ also represents an element in the image of the homomorphism (2.44). □

For example, let $\omega_{\mathbb{C}\times\mathbb{C}^*}$ be the restriction of the standard symplectic form $\omega_{\mathbb{C}^2}$ on $\mathbb{C}^2$ to $\mathbb{C}\!\times\!\mathbb{C}^*$. By [21, Theorem 1.6], $\mathrm{Symp}_c(\mathbb{C}\!\times\!\mathbb{C}^*,\omega_{\mathbb{C}\times\mathbb{C}^*})$ is weakly contractible. By [12, Theorem 3.15],

$$\mathrm{Diff}_c(\mathbb{C}\!\times\!\mathbb{C}^*)_{\omega_{\mathbb{C}\times\mathbb{C}^*}}=\mathrm{Diff}_c(\mathbb{C}\!\times\!\mathbb{C}^*)\sim\big\{f\!\in\!\mathrm{Diff}(\overline{D}^3\!\times\! S^1)\colon f|_{S^2\times S^1}\!=\!\mathrm{id}_{S^2\times S^1}\big\},$$

where $\overline{D}^3\!\subset\!\mathbb{R}^3$ is the closed unit ball centered at the origin and $\sim$ denotes homotopy equivalence, has infinitely many connected components. By Proposition 2.42 with $G\!=\!\mathrm{Diff}_c(\mathbb{C}\!\times\!\mathbb{C}^*)$ and $k\!=\!0$, there are thus infinitely many symplectic forms on $\mathbb{C}\times\mathbb{C}^*$ that agree with $\omega_{\mathbb{C}\times\mathbb{C}^*}$ outside of a compact subset and are pairwise non-isotopic through such symplectic forms; this statement is [55, Theorem 1].

Let $\overline{D}^2\subset\mathbb{R}^2$ be the closed unit disk centered at the origin and $\omega_{D^2}=\omega_{\mathbb{C}}|_{\overline{D}^2}$. By [1, Theorem 1.9], the group of the symplectomorphisms of $(\overline{D}^2\times\overline{D}^2, r_1\omega_{\overline{D}^2}\oplus r_2\omega_{\overline{D}^2})$ that restrict to the identity on a neighborhood of the boundary of $\overline{D}^2\times\overline{D}^2$ is contractible for all $r_1,r_2\in\mathbb{R}^+$. This implies that the group $\mathrm{Symp}_c(\mathbb{C}^2,\omega_{\mathbb{C}^2})$ of compactly supported symplectomorphisms of $\mathbb{C}^2$ with the standard symplectic form is weakly contractible. By Theorem 1.1 and Remark 2.2 in [57],

$$\mathrm{Diff}_c(\mathbb{C}^2)_{\omega_{\mathbb{C}^2}}=\mathrm{Diff}_c(\mathbb{C}^2)\sim\big\{f\in\mathrm{Diff}(\overline{D}^4)\colon f|_{S^3}=\mathrm{id}_{S^3}\big\},$$

where $\overline{D}^4\subset\mathbb{R}^4$ is the closed unit ball centered at the origin, is not weakly contractible either (in particular, the groups $\pi_1(\mathrm{Diff}_c(\mathbb{C}^2))$ and $\pi_4(\mathrm{Diff}_c(\mathbb{C}^2))$ are infinite). By Proposition 2.42 with $G=\mathrm{Diff}_c(\mathbb{C}^2)$, the space $\mathcal{S}_{\omega_{\mathbb{C}^2}}(\mathbb{C}^2)$ of symplectic forms on $\mathbb{C}^2$ that agree with $\omega_{\mathbb{C}^2}$ outside of a compact subset is not weakly contractible (in particular, the groups $\pi_1(\mathcal{S}_{\omega_{\mathbb{C}^2}}(\mathbb{C}^2))$ and $\pi_4(\mathcal{S}_{\omega_{\mathbb{C}^2}}(\mathbb{C}^2))$ are infinite); this statement is [57, §1.1(3)] and [55, Corollary 2.4]. The same conclusion and reasoning apply with the symplectic form $\omega_{\mathbb{C}^2}$ on $\mathbb{C}^2$ replaced by $r_1\omega_{\mathbb{C}}\oplus r_2\omega_{\mathbb{C}}$ for any $r_1,r_2\in\mathbb{R}^+$.

Examples of closed symplectic 4-manifolds (X,ω) so that $\pi_0(\mathcal{S}_\omega)$ is non-trivial are provided by [38, Theorem 1.5]. Let (Σ,ω_Σ) be a closed connected positive-genus surface with a symplectic form ω_Σ, $(X,\omega)=(\Sigma\times\mathbb{C}P^1,\omega_\Sigma\oplus\omega_{\mathrm{FS};1})$, and

$$G\subset\mathrm{Diff}_c(X)_\omega\subset\mathrm{Diff}(X)=\mathrm{Diff}_c(X)$$

be the subgroup of diffeomorphisms of $\Sigma\times\mathbb{C}P^1$ quasi-isotopic to the identity (diffeomorphisms f of X such that there exists a diffeomorphisms of $X\times[0,1]$ restricting to id_X on $X\times\{0\}$ and to f on $X\times\{1\}$). Fix a point $x_0\in\mathbb{C}P^1$ and let $G_{x_0}\subset G$ be the subgroup of diffeomorphisms f of X so that $f(\Sigma\times\{x_0\})$ is smoothly isotopic to $\Sigma\times\{x_0\}$. Both subgroups $G,G_{x_0}\subset\mathrm{Diff}(X)$ consist of connected components of $\mathrm{Diff}(X)$. By [38, Proposition 7.2],

$$\mathrm{Symp}(X,\omega)\cap G\subset G'.$$

By [38, Lemma 7.3], the quotient G/G' contains an element of infinite order. Thus, there are infinitely many symplectic forms on $\Sigma\times\mathbb{C}P^1$ that lie in the de Rham cohomology class of $\omega_\Sigma\oplus\omega_{\mathrm{FS};1}$ and are pairwise non-isotopic through symplectic forms in this cohomology class. By [38, Theorem 1.5], there are in fact infinitely many symplectic forms on the total space X of any $\mathbb{C}P^1$-fiber bundle over Σ that lie in the de Rham cohomology class of any fixed symplectic form ω on X and are pairwise non-isotopic through symplectic forms in this cohomology class.

Chapter 3

Images of Moment Maps

This chapter establishes important properties of the images of moment maps for Hamiltonian torus actions (and more generally periodic $\mathbb{R}^k$-actions) on symplectic manifolds. The most prominent result in this area is the Atiyah-Guillemin-Sternberg Convexity Theorem, a version of which appears as Theorem 3.1 in this chapter. It is refined in the Kähler setting in Atiyah's original paper [3] and extended to non-compact settings in a number of papers over the three decades after [3, 29]; see Theorems 3.2 and 3.27, respectively.

3.1 Convexity theorems

The statement and proof of the Convexity Theorem for moment maps of Hamiltonian torus actions in the original setting of compact symplectic manifolds first appeared in [3, 29]. The properties of these maps established in [3, 29] are collected in Theorem 3.1 below, with torus actions replaced by the more general almost periodic $\mathbb{R}^k$-actions. As demonstrated in [3], the latter fit more naturally with Theorem 3.1.

Theorem 3.1. *Suppose $k\in\mathbb{Z}^+$, (X,ω,ψ,μ) is a closed connected Hamiltonian $\mathbb{R}^k$-manifold, and the $\mathbb{R}^k$-action ψ is almost periodic.*

(A_k) The subset $\mu^{-1}(\alpha)\subset X$ is connected for every $\alpha\in T_0^*\mathbb{R}^k$.

(B_k) The image $\mu(X)\subset T_0^*\mathbb{R}^k$ of X is a convex subset.

(C_k) The ψ-fixed locus X^ψ is a closed symplectic submanifold of (X,ω), $\mu|_Y$ is constant for each $Y\in\pi_0(X^\psi)$, and $\mu(X)$ is the convex hull of the finite subset $\mu(X^\psi)\subset T_0^*\mathbb{R}^k$ with at most $(\dim X)/2$ edges at each vertex.

(D_k) The map $\mu\colon X\longrightarrow\mu(X)$ is open.

(E_k) The components of a full tuple of edge vectors for the polytope $\mu(X)$ at any given vertex of $\mu(X)$ span $T_0^*\mathbb{R}^k$ if and only if the action ψ is irreducible.

(F_k) If the action ψ is irreducible, then the subset $\mathrm{Crit}(\mu)$ of points $x\in X$ so that $\mathrm{d}_x\mu$ is not surjective is a finite union of (not necessarily disjoint) closed symplectic proper submanifolds of (X,ω), and the image of each such submanifold under μ is contained in a hyperplane of $T_0^*\mathbb{R}^k$.

(G_k) If the action ψ descends to an action of the torus $\mathbb{T}=\mathbb{R}^k/\Lambda$ for some full-dimensional lattice $\Lambda\subset\mathbb{R}^k$, then all edges of the polytope $\mu(X)\subset T^*_{\mathbb{1}}\mathbb{T}=T^*_{\mathbb{1}}\mathbb{R}^k$ are rational.

By the last claim in (C_k), $\mu(X)\subset T^*_0\mathbb{R}^k$ is a polytope, called a moment polytope for the Hamiltonian action ψ on (X,ω). It is determined by (X,ω,ψ) up to translation. Since a torus $\mathbb{T}$ is the quotient of a finite-dimensional vector space by a full-dimensional lattice, Theorem 3.1 immediately implies that (A_k)-(F_k) with $T^*_0\mathbb{R}^k$ replaced by $T^*_{\mathbb{1}}\mathbb{T}$ hold for any closed connected Hamiltonian $\mathbb{T}$-manifold (X,ω,ψ,μ). Figure 1.1 on page 17 shows moment polytopes for the two torus actions on $(\mathbb{C}P^2,\omega_{\mathrm{FS};2})$ of Exercise 1.27.

By Remark 2.8 and the proof of Proposition 2.15(1), the first claim in (C_k) holds for any smooth Lie group action on a symplectic manifold (X,ω) preserving a Riemannian metric on X. The second claim in (C_k) follows from the first and the observation that

$$X^\psi=\big\{x\in X\colon \mathrm{d}_x\mu=0\big\};$$

this identity is an immediate consequence of Proposition 2.1(1) and the first condition in (1.6). As indicated just below, the first claim in (F_k) is a fairly straightforward consequence of the equivariant splitting (2.11) of $TX|_Y$ for each $Y\in\pi_0(X^\psi)$; the second claim in (F_k) then follows readily from Proposition 2.1(1).

Proof of (F_k). For each $L\in\mathbb{R}\mathbb{P}^{k-1}$, let $X^L\subset X$ be the fixed locus of the action $\psi|_L$. By the second equation in (2.23) and Proposition 2.1(1),

$$\mathrm{Crit}(\mu)=\big\{x\in X^L\colon L\in\mathbb{R}P^{k-1}\big\}.$$

By Proposition 2.15(1), $X^L\subset X$ is a compact symplectic submanifold. Every topological component $Z\subset X^L$ is preserved ψ. The restriction of the ψ-action to such a component is Hamiltonian. Thus,

$$Z\cap X^\psi=Z^\psi\neq\emptyset\qquad\forall\ Z\in\pi_0(X^L),\ L\in\mathbb{R}P^{k-1}$$

by Exercise 2.22(b). Along with Corollary 2.11, this implies that $\mathrm{Crit}(\mu)$ is a *finite* union of the topological components Z of the symplectic submanifolds $X^L\subset X$ with $L\in\mathbb{R}P^{k-1}$. By Proposition 2.1(1), the smooth map

$$\mu_v\colon X\longrightarrow T^*_0\mathbb{R}^k,\qquad \mu_v(x)=\{\mu(x)\}(v),\tag{3.1}$$

is constant along each topological component $Z\subset X^L$ for every $v\in L$, i.e. for any $v\in L$ there exists $c_v\in\mathbb{R}$ so that

$$\mu(Z)\subset\big\{\alpha\in T^*_0\mathbb{R}^k\colon \alpha(v)=c_v\big\};$$

the right-hand side above is a *hyperplane* in $T^*_0\mathbb{R}^k$ if $v\neq0$. □

The fundamental reason behind the remaining claims of Theorem 3.1 is the local description of the moment map μ provided by Corollary 2.35, which is in a sense a Hamiltonian version of the Darboux Theorem. It ensures that (A_k), (B_k), and (D_k) hold locally and implies (E_k) and (G_k). Following [3, 41], we extract topological implications from the local properties of the moment map μ provided by Corollary 2.35 via Proposition 3.10 of Morse-Bott theory. After reviewing the setup relevant to this proposition and deducing some implications from it in Section 3.2, we conclude a proof of Theorem 3.1 in the spirit of [3, 41] in Section 3.3.

The complexification of a torus $\mathbb{T}$ is the complex Lie group

$$\mathbb{T}_{\mathbb{C}} \equiv \big(T_{\mathbb{1}}\mathbb{T}\otimes_{\mathbb{R}}\mathbb{C}\big)/(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\otimes_{\mathbb{R}}\{1\}$$

with $T_{\mathbb{1}}\mathbb{T}_{\mathbb{C}} = T_{\mathbb{1}}\mathbb{T}\otimes_{\mathbb{R}}\mathbb{C}$. We denote by $\mathbb{T}_{\mathfrak{i}} \subset \mathbb{T}_{\mathbb{C}}$ the purely imaginary subgroup, i.e. the image of $T_{\mathbb{1}}\mathbb{T}\otimes_{\mathbb{R}}\mathfrak{i}\mathbb{R}$ in the above quotient. An identification of $\mathbb{T}$ with the standard torus $\mathbb{T}^k$ via a $\mathbb{Z}$-basis for $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\subset T_{\mathbb{1}}\mathbb{T}$ as in (1.29) induces an identification of $\mathbb{T}_{\mathbb{C}}$ and $\mathbb{T}_{\mathfrak{i}}$ with $(\mathbb{C}^*)^k$ and $(\mathbb{R}^+)^k$, respectively, so that the rays $\mathfrak{i}\mathbb{R}^+$ associated with the basis vectors correspond the line segments $(0,1)$ in the components of $(\mathbb{R}^+)^k$.

A complexification of a smooth $\mathbb{T}$-action ψ on a symplectic manifold (X,ω) is a smooth $\mathbb{T}_{\mathbb{C}}$-action on X,

$$\begin{aligned} \psi_{\mathbb{C}}\colon \mathbb{T}_{\mathbb{C}} \longrightarrow \mathrm{Diff}(X), \quad u \longrightarrow \psi_{\mathbb{C};u}\,, \qquad \text{s.t.}\\ \mathrm{d}_{\mathbb{1}}\psi_{\mathbb{C}}(v+\mathfrak{i}v') = \mathrm{d}_{\mathbb{1}}\psi(v)+J\mathrm{d}_{\mathbb{1}}\psi(v') \in \Gamma(X;TX) \quad \forall\, v,v'\in T_{\mathbb{1}}\mathbb{T}, \end{aligned} \tag{3.2}$$

for some almost complex structure J on X compatible with ω and preserved by ψ. If X is compact, the conditions (3.2) with $v=0$ determine $\mathbb{R}$-actions in the imaginary directions. However, these actions may not commute with ψ or each other and thus not give rise to a $\mathbb{T}_{\mathbb{C}}$-action.

Theorem 3.2. *Suppose $\mathbb{T}$ is a torus, (X,ω,ψ,μ) is a closed connected Hamiltonian $\mathbb{T}$-manifold, and $\psi_{\mathbb{C}}$ is a complexification of ψ. If $x\in X$ and $\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}\subset X$ is the closure of the $\mathbb{T}_{\mathbb{C}}$-orbit $\mathcal{O}_x(\psi_{\mathbb{C}})\equiv\mathbb{T}_{\mathbb{C}}x$ of x, then*

(1) $\mathrm{Ver}(\mu(\{Y\in\pi_0(X^{\psi})\colon Y\cap\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}\neq\emptyset\}))=\mu(\{Y\in\pi_0(X^{\psi})\colon Y\cap\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}\neq\emptyset\})$*;*

(2) $\mu(\overline{\mathcal{O}_x}(\psi_{\mathbb{C}}))=\mathrm{CH}(\mu(\{Y\in\pi_0(X^{\psi})\colon Y\cap\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}\neq\emptyset\}))$*;*

(3) for every open face σ of the polytope $\mu(\overline{\mathcal{O}_x(\psi_{\mathbb{C}})})$, $\mu^{-1}(\sigma)\cap\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}$ is a single $\mathbb{T}_{\mathbb{C}}$-orbit;

(4) the map $\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}/\mathbb{T}\longrightarrow\mu(\overline{\mathcal{O}_x(\psi_{\mathbb{C}})})$, $[x']\longrightarrow\mu(x')$, is a well-defined homeomorphism.

As illustrated by Exercise 3.4 below, the closure $\overline{\mathcal{O}_x}(\psi_{\mathbb{C}})$ of $\mathcal{O}_x(\psi_{\mathbb{C}})\subset X$ could be singular even in particularly nice cases. If $\mathbb{T}\approx S^1$ or J is an integrable almost complex structure on X compatible with ω and preserved by a smooth $\mathbb{T}$-action ψ on (X,ω), then (3.2) determines a complexification $\psi_{\mathbb{C}}$ of ψ; see Exercise 3.3 below. In the latter case, Theorem 3.2 reduces to [3, Theorem 2], but the argument in [3] applies to the general case of Theorem 3.2. The crucial implication of (3.2) is that the action of the imaginary components (which correspond to the radial direction in $\mathbb{C}^*$) is given by the gradient flow of projections of μ to one-dimensional subspaces of $T^*_{\mathbb{1}}\mathbb{T}$; see (2.27). We establish Theorem 3.1 in Section 3.4 by extracting implications from Morse-Bott theory beyond those in Section 3.2, similarly to the original argument in [3].

Exercise 3.3. Suppose ψ is a smooth action of a torus $\mathbb{T}$ on a compact almost complex manifold (M,J), i.e. ψ preserves J. Show that (3.2) determines a complexification $\psi_{\mathbb{C}}$ of ψ if either $\mathbb{T}$ is one-dimensional or J is integrable. *Hint:* J is preserved by ψ if and only if $\mathcal{L}_{\zeta_v}J=0$ for every $v\in T_{\mathbb{1}}\mathbb{T}$, where $\mathcal{L}$ is the Lie derivative and $\zeta_v\in\Gamma(X;TX)$ is as in (1.2); J is integrable if and only $\mathcal{L}_{J\xi}J=J(\mathcal{L}_{\xi}J)$ for every $\xi\in\Gamma(X;TX)$.

Exercise 3.4. Show that

(a) the S^1-action ψ on $\mathbb{C}P^2$ given by

$$S^1 \times \mathbb{C}P^2 \longrightarrow \mathbb{C}P^2, \qquad u\cdot\big[z_0,z_1,z_2\big]=\big[z_0,u^2z_1,u^3z_2\big],$$

is effective and Hamiltonian with respect to the symplectic form $\omega_{\mathrm{FS};2}$ of Exercise 1.27;

(b) the closure $\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}$ of the $\mathbb{C}^*$-orbit $\mathcal{O}_x(\psi_{\mathbb{C}})$ of x under the complexification of ψ with respect to the standard complex structure on $\mathbb{C}P^2$ is a rational cubic curve (and in particular is not smooth) for any point $x\equiv[z_0,z_1,z_2]$ in $\mathbb{C}P^2$ with $z_0,z_1,z_2\neq 0$.

Exercise 3.5. Let $\psi_{\mathbb{C}}$ be the complexification of either torus action in Exercise 1.27(c) with respect to the standard complex structure $J_{\mathbb{C}P^{n-1}}$ on $\mathbb{C}P^{n-1}$ as in Exercises 3.3 and 3.4.

(a) Determine all possible orbits $\mathcal{O}_x(\psi_{\mathbb{C}})$ and their closures $\overline{\mathcal{O}_x}(\psi_{\mathbb{C}})\subset\mathbb{C}P^{n-1}$.

(b) Verify the claims of Theorem 3.2 directly in this case.

3.2 Morse-Bott theory and moment maps

Let X be a smooth manifold and $H: X\longrightarrow\mathbb{R}$ be a smooth function. If $x\in\mathrm{Crit}(H)$, then the gradient $\nabla^g H$ of H with respect to any Riemannian metric g vanishes at x and the Hessian

$$\nabla^2 H\big|_x\equiv\nabla\big(\nabla^g H\big)\big|_x\colon T_xX\longrightarrow T_xX, \qquad \nabla^2H\big|_x(w)=\nabla_w\big(\nabla^g H\big), \tag{3.3}$$

of H at x does not depend on the choice of a connection ∇ in TX (it does depend on the metric g though). If in addition ξ,ξ' are vector fields on X and ∇ is the Levi-Civita connection of g, then

$$\begin{aligned} g\big(\nabla^2H\big|_x(\xi(x)),\xi'(x)\big) &\equiv g\big(\nabla_{\xi(x)}\nabla^gH,\xi'(x)\big)=\big\{\xi(x)\big\}\big(g(\nabla^gH,\xi')\big)-g\big(\nabla^gH|_x,\nabla_{\xi(x)}\xi'\big)\\ &=\big\{\xi(x)\big\}\big(\mathrm{d}H(\xi')\big)-\mathrm{d}_xH\big(\nabla_{\xi(x)}\xi'\big)=\big\{\xi(x)\big\}\big(\xi'(H)\big)-\big\{\nabla_{\xi(x)}\xi'\big\}(H)\\ &=\big\{\xi'(x)\big\}\big(\xi(H)\big)+\big\{[\xi,\xi'](x)\big\}(H)-\big\{\nabla_{\xi(x)}\xi'\big\}(H)\\ &=\big\{\xi'(x)\big\}\big(\xi(H)\big)-\big\{\nabla_{\xi'(x)}\xi\big\}(H)=g\big(\nabla^2H\big|_x(\xi'(x)),\xi(x)\big)\,. \end{aligned}$$

Thus, the linear automorphism (3.3) is symmetric with respect to the metric g and therefore diagonalizable. We denote by

$$\mathrm{E}_x^0(H),\mathrm{E}_x^-(H),\mathrm{E}_x^+(H)\subset T_xX \qquad\text{and}\qquad n_x^0(H),n_x^-(H),n_x^+(H)\in\mathbb{Z}^{\ge0}$$

the nullspace of $\nabla^2H|_x$, the negative eigenspace of $\nabla^2H|_x$, the positive eigenspace of $\nabla^2H|_x$, and their respective dimensions. In particular,

$$T_xX=\mathrm{E}_x^0(H)\oplus\mathrm{E}_x^-(H)\oplus\mathrm{E}_x^+(H) \qquad\text{and}\qquad \dim X=n_x^0(H)+n_x^-(H)+n_x^+(H)\,.$$

Exercise 3.6. Let X be a smooth manifold, $H\colon X\longrightarrow\mathbb{R}$ be a smooth function, and $x\in\mathrm{Crit}(H)$. Show that

(a) the negative and positive eigenspaces $\mathrm{E}_x^-(H),\mathrm{E}_x^+(H)\subset T_xX$ of $\nabla^2H|_x$ depend on the choice of a Riemannian metric g on X, but

(b) their dimensions $n_x^-(H), n_x^+(H)$ and the nullspace $\mathrm{E}_x^0(H)\subset T_xX$ of $\nabla^2H|_x$ do not.

Definition 3.7. Let X be a smooth manifold. A smooth function $H: X \longrightarrow \mathbb{R}$ is Morse-Bott if $\mathrm{Crit}(H)\subset X$ is a closed submanifold of X with $T_xY = E_x^0(H)$ for all $Y\in\pi_0(\mathrm{Crit}(H))$ and $x\in Y$.

If $H: X\longrightarrow\mathbb{R}$ is a Morse-Bott function and $Y\in\pi_0(\mathrm{Crit}(H))$, $H|_Y$ is constant. Furthermore, the numbers $n_x^-(H), n_x^+(H)$ do not depend on $x\in Y$; we denote them by $n_Y^-(H), n_Y^+(H)$, respectively. The subspaces $\mathrm{E}_x^-(H), \mathrm{E}_x^+(H)$ of T_xX form subbundles $\mathrm{E}_Y^-(H), \mathrm{E}_Y^+(H)$ of $TX|_Y$ so that

$$TX|_Y = TY\oplus\mathrm{E}_Y^-(H)\oplus\mathrm{E}_Y^+(H).$$

Exercise 3.8. Suppose X is a smooth manifold, $H : X \longrightarrow \mathbb{R}$ is a Morse-Bott function, and $Y\in\pi_0(\mathrm{Crit}(H))$. Show that H reaches a local minimum (resp. maximum) on Y if and only if $n_Y^-(H)=0$ (resp. $n_Y^+(H)=0$).

Exercise 3.9. Suppose $H: X\longrightarrow\mathbb{R}$ and $Y\subset\mathrm{Crit}(H)$ are in Exercise 3.8 and $Z\subset X$ is a smooth submanifold transverse to the closed submanifold $Y\subset X$. Show that $Y\cap Z$ is a closed submanifold of Z, is an open subset of $\mathrm{Crit}(H|_Z)$, and

$$T_x(Y\cap Z) = T_xY\cap T_xZ = E_x^0\big(H|_Z\big), \quad n_x^{\pm}\big(H|_Z\big) = n_x^{\pm}(H) \qquad \forall\, x\in Y\cap Z.$$

The next proposition, proved in Appendix A, is the main point-set topology input used in [3, 41] to show that Theorem 3.1(A_k) holds "generically"; see ($\mathrm{A}_k^\star$) on page 57. We apply this proposition to almost periodic Hamiltonian $\mathbb{R}$-actions right after its statement.

Proposition 3.10 ([3, Lemma 2.1], [41, Lemma 5.5.5])**.** *Suppose M is a compact connected smooth manifold and $H: X\longrightarrow\mathbb{R}$ is a Morse-Bott function. If $n_x^{\pm}(H)\neq 1$ for every $x\in\mathrm{Crit}(H)$, then*

(1) H has a unique local minimum and a unique local maximum;

(2) $H^{-1}(c)\subset X$ is connected for every $c\in\mathbb{R}$.

Proposition 3.11. *Let ψ be an almost periodic $\mathbb{R}$-action on a symplectic manifold (X,ω). If $H: X\longrightarrow\mathbb{R}$ is a Hamiltonian for ψ, then $\mathrm{Crit}(H)\subset X$ is a closed symplectic submanifold and H is a Morse-Bott function with $n_x^{\pm}(H)\in 2\mathbb{Z}^{\geq 0}$ for every $x\in\mathrm{Crit}(H)$. If in addition X is compact and connected, then*

(2) H has a unique local minimum and a unique local maximum;

(3) $H^{-1}(c)\subset X$ is connected for every $c\in\mathbb{R}$.

Proof. Let $\rho: \mathbb{R}\longrightarrow\mathbb{T}$ and ψ' be as in (1.4) and $\zeta\in\Gamma(X;TX)$ be the generating vector field for the ψ-action as in (2.1). We can assume that the image of ρ is dense in $\mathbb{T}$ and so $X^{\psi}=X^{\psi'}$. By Exercise 2.13, there exist a ψ-invariant ω-compatible almost complex structure J on X. Let $g(\cdot,\cdot)\equiv\omega(\cdot,J\cdot)$ be the Riemannian metric on X determined by ω and J. By (1.23) and (1.27), the gradient of H with respect to g is then given by

$$\nabla^g H = -J\zeta \in \Gamma(X;TX). \tag{3.4}$$

By (1.27) and Proposition 2.1(1),

$$\mathrm{Crit}(H) \equiv \big\{x\in X: \mathrm{d}_xH=0\big\} = \big\{x\in X: \zeta(x)=0\big\} = X^{\psi}=X^{\psi'}. \tag{3.5}$$

Along with (3.3), (3.4), and (2.4), this implies that the Hessian $\nabla^2 H$ of H satisfies

$$\nabla^2 H\big|_x(w) = -J\nabla_w\zeta, \quad \nabla^2 H\big|_x(Jw) = J\nabla^2 H\big|_x(w) \qquad \forall\, w\in T_xX,\, x\in\mathrm{Crit}(H)\,. \tag{3.6}$$

By (3.5), Propositions 2.15(1) and 2.1(2), and the first equation in (3.6), $\mathrm{Crit}(H)\subset X$ is thus a closed symplectic submanifold of (X,ω) with

$$T_xY = (T_xX)^{\mathrm{d}\psi'} = (T_xX)^{\mathrm{d}\psi} = \big\{w\in T_xX\colon \nabla_w\zeta=0\big\} = E^0_x(H) \quad \forall\, Y\in\pi_0\big(\mathrm{Crit}(H)\big),\, x\in Y.$$

By the second equation in (3.6), the subspaces $\mathrm{E}^{\pm}_x(H)\subset T_xX$ are preserved by J for every $x\in\mathrm{Crit}(H)$ and thus $n^{\pm}_x(H)\in 2\mathbb{Z}^{\ge 0}$ for every $x\in\mathrm{Crit}(H)$. The remaining claims of the proposition now follow immediately from Proposition 3.10. $\square$

Corollary 3.12. *Suppose $k\in\mathbb{Z}^+$, (X,ω,ψ,μ) is a connected closed Hamiltonian $\mathbb{R}^k$-manifold, the $\mathbb{R}^k$-action ψ is almost periodic, and $y\in X$. If $\mathcal{C}_y(\psi)\subset T^*_0\mathbb{R}^k$ is the cone with vertex at $\mu(y)$ as in (2.36), then $\mu(X)\subset\mathcal{C}_y(\psi)$.*

Proof. Let ψ' be a smooth action of a torus $\mathbb{T}$ on X and $\rho\colon\mathbb{R}^k\longrightarrow\mathbb{T}$ be a homomorphism with dense image so that $\psi=\psi'\circ\rho$, J be a ψ-invariant (or equivalently ψ'-invariant) ω-compatible almost complex structure on X (as provided by Exercise 2.13), and $U_y\subset X$ be a neighborhood of y as in Corollary 2.35. Suppose first that $y\in X^{\psi}$ and $Y\subset X$ is the topological component of $X^{\psi}=X^{\psi'}$ containing y. Let $S(Y)\subset(T^*_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$ and $\mathcal{N}^{\alpha}_XY\subset TX|_Y$ for each $\alpha\in S(Y)$ be as in Proposition 2.10(1) with ψ replaced by ψ' so that the complex structure on $\mathcal{N}^{\alpha}_XY$ is induced by J. We show below that

$$\mu(X)\subset\mathcal{C}_y(\psi)\equiv\Big\{\mu(y)+\sum_{\alpha\in S(Y)}t_\alpha(\rho^*\alpha)\colon t_\alpha\in\mathbb{R}^{\ge 0}\ \forall\,\alpha\in S(Y)\Big\}. \tag{3.7}$$

Suppose $\eta_0\in T^*_0\mathbb{R}^k-\mathcal{C}_y(\psi)$. Thus, $\mathcal{C}_y(\psi)$ is contained in a (closed) half-space in $T^*_0\mathbb{R}^k$ and there exists $v\in T_0\mathbb{R}^k$ so that

$$\eta_0(v)<\inf\big\{\eta(v)\colon \eta\in\mathcal{C}_y(\psi)\big\}=\big\{\mu(Y)\big\}(v)\equiv\mu_v(Y). \tag{3.8}$$

By the second equation in (2.30), this implies that $\{\rho^*\alpha\}(v)\ge 0$ for all $\alpha\in S(Y)$. Thus,

$$\mu_v(Y)=\inf\big\{\mu_v(x)\colon x\in U_y\big\},$$

i.e. μ_v has a local minimum along Y. Combining this with Proposition 3.11(2), we conclude that

$$\mu_v(Y)=\inf\big\{\mu_v(x)\colon x\in X\big\},$$

i.e. the above local minimum is in fact a global minimum. Along with (3.8), this implies that $\eta_0\notin\mu(X)$ and establishes (3.7).

For $y\in X$ arbitrary, let $\mathbb{R}^c_y\subset\mathbb{R}^k$, μ_y, and ψ_y be as in Corollary 2.35 and just above. By (3.7) with μ replaced by μ_y and (2.33),

$$\mu(X)\subset\mathcal{C}_y(\psi)\equiv\mathcal{C}_y(\psi_y)\times T^*_0\mathbb{R}^c_y\subset T^*_0\mathbb{R}^k\,,$$

as claimed. $\square$

3.3 Proof of classical convexity theorem

In this section, we complete a proof of Theorem 3.1 along the lines of [3, 41]. Corollaries 2.35 and 3.12 immediately imply (D_k). As in [3, 41], we use the global, Morse-Bott theory statement of Proposition 3.10 and the first, local statement of Proposition 3.11 to obtain

($\mathrm{A}_k^\star$) $\mu^{-1}(\alpha)\subset X$ is connected for every regular value $\alpha\in T_0^*\mathbb{R}^k$ of μ,

via an induction on k. The remaining part of (A_k) follows from ($\mathrm{A}_k^\star$), (D_k), and (F_k) via Exercise 3.15. Claim (B_k) of Theorem 3.1 is obvious for $k=1$, follows readily from (A_k) for $k\ge 2$, and leads to the last claim in (C_k). We conclude this section with proofs of (E_k) and of (G_k) and with remarks on related arguments in [3, 4, 14, 41].

Lemma 3.13. *Suppose $k\in\mathbb{Z}^{\ge 0}$, (X,ω) is a symplectic manifold, $\psi_1,\ldots,\psi_{k+1}$ are $\mathbb{R}$-actions on (M,ω) with Hamiltonians*

$$H_1,\ldots,H_{k+1}\colon X\longrightarrow\mathbb{R},$$

respectively, the $\mathbb{R}$-action ψ_{k+1} is almost periodic, and its Hamiltonian H_{k+1} is ψ_i-invariant for every $i\in[k]$. If $c\in\mathbb{R}^k$ is a regular value of the map

$$H\equiv\big(H_1,\ldots,H_k\big)\colon X\longrightarrow\mathbb{R}^k,$$

then the submanifolds $Z\equiv H^{-1}(c)$ and $\mathrm{Crit}(H_{k+1})$ of X are transverse, $Z\cap\mathrm{Crit}(H_{k+1})$ is an open subset of $\mathrm{Crit}(H_{k+1}|_Z)$, and

$$T_x\big(Z\cap\mathrm{Crit}(H_{k+1})\big)=T_xZ\cap T_x\mathrm{Crit}(H_{k+1})=\mathrm{E}_x^0\big(H_{k+1}|_Z\big),\quad n_x^{\pm}\big(H_{k+1}|_Z\big)=n_x^{\pm}(H_{k+1})\in 2\mathbb{Z}^{\ge 0}$$

for all $x\in Z\cap\mathrm{Crit}(H_{k+1})$.

Proof. By the first statement of Proposition 3.11, $\mathrm{Crit}(H_{k+1})\subset X$ is a closed symplectic submanifold and $H_{k+1}\colon X\longrightarrow\mathbb{R}$ is a Morse-Bott function. Since $c\in\mathbb{R}^k$ is a regular value of H, $Z\equiv H^{-1}(c)$ is a submanifold of X. In light of Exercise 3.9, it thus remains to prove that the submanifolds $\mathrm{Crit}(H_{k+1}),Z\subset X$ are transverse. Let $Y\in\pi_0(\mathrm{Crit}(H_{k+1}))$.

Let $\zeta_1,\ldots,\zeta_k\in\Gamma(X;TX)$ be the vector fields generating the $\mathbb{R}$-actions $\psi_1,\ldots,\psi_k$ and thus satisfying the middle equation in (1.27) with $\zeta_{v_i}=\zeta_i$. Since $H_{k+1}\circ\psi_i=H_{k+1}$ for each $i\in[k]$, ψ_i preserves $\mathrm{Crit}(H_{k+1})$ and thus Y. Therefore,

$$\zeta_i\big|_Y\in\Gamma(Y;TY)\subset\Gamma\big(Y;TX|_Y\big)\qquad\forall\ i\in[k].$$

If $x\in Z$, then $\mathrm{d}_xH_1,\ldots,\mathrm{d}_xH_k$ vanish on T_xZ. Since d_xH is surjective, it follows that

$$\mathrm{d}_xH\equiv\big(\mathrm{d}_xH_1,\ldots,\mathrm{d}_xH_k\big)\colon T_xX/T_xZ\longrightarrow\mathbb{R}^k\tag{3.9}$$

is a well-defined isomorphism. The middle equation in (1.27) with $\zeta_{v_i}=\zeta_i$ then implies that the tangent vectors $\zeta_1(x),\ldots,\zeta_k(x)\in T_xX$ are linearly independent.

Suppose $x\in Y\cap Z$ and $(r_1,\ldots,r_k)\in\mathbb{R}^k-\{0\}$. Since

$$r\cdot\zeta(x)\equiv r_1\zeta_1(x)+\ldots+r_k\zeta_k(x)\in T_xY-\{0\}$$

and $\omega|_{T_xY}$ is a nondegenerate, there exists $w\in T_xY$ so that

$$\sum_{i=1}^{k} r_i \mathrm{d}_x H_i(w) \equiv -\omega\big(r\cdot\zeta(x), w\big) \neq 0.$$

Thus, the restrictions of $\mathrm{d}_xH_1,\ldots,\mathrm{d}_xH_k$ to T_xY are linearly independent. Since (3.9) is a well-defined isomorphism, it follows that $T_xX=T_xY\oplus T_xZ$, i.e. the submanifolds $Y,Z\subset X$ are transverse at x. □

Corollary 3.14. *Suppose $k\in\mathbb{Z}^{\geq 0}$, (X,ω) is a symplectic manifold, $\widetilde{\psi}$ is an almost periodic $\mathbb{R}^{k+1}$-action on (X,ω) with Hamiltonian*

$$\widetilde{H}\equiv(H,H_{k+1})\colon X \longrightarrow \mathbb{R}^k\times\mathbb{R}=\mathbb{R}^{k+1}.$$

If $c\in\mathbb{R}^k$ is a regular value of H, then the restriction of H_{k+1} to the submanifold $Z\equiv H^{-1}(c)$ of X is a Morse-Bott function with $n_x^{\pm}(H)\in 2\mathbb{Z}^{\geq 0}$ for every $x\in\mathrm{Crit}(H_{k+1}|_Z)$.

Proof. Let $H=(H_1,\ldots,H_k)$. Suppose $x\in\mathrm{Crit}(H_{k+1}|_Z)$. Since the map (3.9) is a well-defined isomorphism,

$$\mathrm{d}_xH_{k+1} = r_1\mathrm{d}_xH_1+\ldots+r_k\mathrm{d}_xH_k\colon T_xX\longrightarrow\mathbb{R}$$

for some $r\equiv(r_1,\ldots,r_k)\in\mathbb{R}^k$. The map

$$H_{k+1;r}\equiv H_{k+1}-\big(r_1H_1+\ldots+r_kH_k\big)\colon X\longrightarrow\mathbb{R}$$

is then a Hamiltonian for an almost periodic $\mathbb{R}$-action on (X,ω) so that $x\in Z\cap\mathrm{Crit}(H_{k+1;r})$. This Hamiltonian is preserved by the restriction of the action $\widetilde{\psi}$ to $\mathbb{R}^k\times\{0\}$. Since $H_{k+1}-H_{k+1;r}$ restricts to the constant $r\cdot c$ on Z,

$$\mathrm{Crit}\big(H_{k+1;r}|_Z\big)=\mathrm{Crit}\big(H_{k+1}|_Z\big),\quad \mathrm{E}_x^0\big(H_{k+1;r}|_Z\big)=\mathrm{E}_x^0\big(H_{k+1}|_Z\big),\quad n_x^{\pm}\big(H_{k+1;r}|_Z\big)=n_x^{\pm}\big(H_{k+1}|_Z\big).$$

By Lemma 3.13, the closed submanifold $Z\cap\mathrm{Crit}(H_{k+1;r})$ of Z is thus an open subset of $\mathrm{Crit}(H_{k+1})$,

$$T_x\big(Z\cap\mathrm{Crit}(H_{k+1;r})\big)=\mathrm{E}_x^0\big(H_{k+1}|_Z\big),\qquad\text{and}\qquad n_x^{\pm}\big(H_{k+1}|_Z\big)\in 2\mathbb{Z}^{\geq 0}. \tag{3.10}$$

We conclude that $\mathrm{Crit}(H_{k+1}|_Z)$ is a finite union of submanifolds $Z\cap\mathrm{Crit}(H_{k+1;r})$ of Z with $r\in\mathbb{R}^k$, each of which is a union of the topological components of $\mathrm{Crit}(H_{k+1}|_Z)$ and satisfies (3.10) for all $x\in Z\cap\mathrm{Crit}(H_{k+1;r})$. □

Proof of ($A_k^\star$) on page 57. The claim is trivially true for $k=0$. We assume that it is true for some $k\in\mathbb{Z}^{\geq 0}$ and show that it also holds with k replaced by $k+1$. Let

$$\widetilde{H}\equiv(H,H_{k+1})\colon X\longrightarrow\mathbb{R}^k\times\mathbb{R}=\mathbb{R}^{k+1}$$

be a Hamiltonian for an almost periodic $\mathbb{R}^{k+1}$-action $\widetilde{\psi}$ on (X,ω) and $\widetilde{c}\equiv(c,c_{k+1})\in\mathbb{R}^k\times\mathbb{R}$.

Suppose first that $c\in\mathbb{R}^k$ is a regular value of H and thus $Z\subset H^{-1}(c)$ is a closed submanifold of X. It is connected by the inductive assumption. By Corollary 3.14, $H_{k+1}|_Z$ is a Morse-Bott function with $n_x^{\pm}(H_{k+1}|_Z)\in 2\mathbb{Z}^{\geq 0}$. By Proposition 3.10(2),

$$\widetilde{H}^{-1}(\widetilde{c})\equiv\big\{H_{k+1}|_Z\big\}^{-1}\big(c_{k+1}\big)\subset X$$

is thus connected.

Let $\mathbb{R}^k_H \subset \mathbb{R}^k$ and $\mathbb{R}^{k+1}_{\widetilde{H}} \subset \mathbb{R}^{k+1}$ be the subsets of regular values of H and $\widetilde{H}$, respectively. In particular,

$$\mathbb{R}^{k+1}_{\widetilde{H}} = \big\{\widetilde{c}\in\mathbb{R}^{k+1}\colon \mathrm{d}_x H_1,\ldots,\mathrm{d}_x H_{k+1}\in T_x^*X \text{ are lin. independent } \forall\, x\in\widetilde{H}^{-1}(\widetilde{c})\big\}.$$

Since the subset $\widetilde{H}^{-1}(\widetilde{c})\subset X$ is compact for every $\widetilde{c}\in\mathbb{R}^{k+1}$, the subset $\mathbb{R}^{k+1}_{\widetilde{H}}\subset\mathbb{R}^{k+1}$ is open. The function

$$\mathbb{R}^{k+1}_{\widetilde{H}} \longrightarrow \mathbb{Z}^{\ge 0}, \qquad \widetilde{c}\longrightarrow \big|\pi_0\big(\widetilde{H}^{-1}(\widetilde{c})\big)\big|, \tag{3.11}$$

is constant on the connected components of $\mathbb{R}^{k+1}_{\widetilde{H}}$ and takes value 0 or 1 on $\mathbb{R}^{k+1}_{\widetilde{H}}\cap(\mathbb{R}^k_H\times\mathbb{R})$. Since the subset $\mathbb{R}^k_H\subset\mathbb{R}^k$ is dense by Sard's Theorem and each connected component of $\mathbb{R}^{k+1}_{\widetilde{H}}$ is open in $\mathbb{R}^{k+1}$, the function (3.11) takes value 0 or 1 on each connected component of $\mathbb{R}^{k+1}_{\widetilde{H}}$, i.e. $\widetilde{H}^{-1}(\widetilde{c})\subset X$ is connected for every $\widetilde{c}\in\mathbb{R}^{k+1}_{\widetilde{H}}$. $\square$

Exercise 3.15. Let X be a compact connected manifold (or more generally a topological space which is sequentially compact, connected, and normal). Suppose P is a first countable topological space, $f\colon X\longrightarrow P$ is a continuous open surjective map, and $P^*\subset P$ is a dense subset. Show that if $f^{-1}(c)\subset X$ is connected for every $c\in P^*$, then $f^{-1}(c)\subset X$ is connected for every $c\in P$.

Proof of (A$_k$). By Exercises 2.3 and 1.20, we can assume that the action ψ is irreducible. Let $(T_0^*\mathbb{R}^k)_\mu\subset T_0^*\mathbb{R}^k$ be the subset of regular values of μ. By (F$_k$), $\mu^{-1}((T_0^*\mathbb{R}^k)_\mu)\subset X$ is the complement of a finite union of submanifolds of positive codimensions. Thus, the subset

$$P^*\equiv\mu\big(\mu^{-1}\big((T_0^*\mathbb{R}^k)_\mu\big)\big)\subset\mu(X)\equiv P$$

is dense in P. By (D$_k$), the map $\mu\colon X\longrightarrow P$ is open. By (A$_k^\star$), $\mu^{-1}(\alpha)\subset X$ is connected for every $\alpha\in P^*$. Thus, (A$_k$) now follows from Exercise 3.15. $\square$

Proof of (B$_k$). This claim is trivially true for $k=0$. Suppose $k\in\mathbb{Z}^+$ and $H\colon X\longrightarrow\mathbb{R}^k$ is a Hamiltonian for an almost periodic $\mathbb{R}^k$-action $\widetilde{\psi}$ on (X,ω). For a $k\times(k-1)$ real matrix A, the $\mathbb{R}^{k-1}$-action $\psi_A\equiv\psi\circ A$ is then also almost periodic with Hamiltonian

$$H_A\equiv A^{\mathrm{tr}}\circ H\colon X\longrightarrow\mathbb{R}^{k-1}\,.$$

Suppose $x_0,x_1\in X$. Let A be a $k\times(k-1)$ injective real matrix A so that

$$H(x_1)-H(x_0)\in\ker A^{\mathrm{tr}}.$$

Thus, $x_1\in H_A^{-1}(H_A(x_0))$ and

$$H(x_1)\in H\big(H_A^{-1}\big(H_A(x_0)\big)\big)\subset\big\{H(x_0)+c\colon c\in\ker A^{\mathrm{tr}}\big\}.$$

Since $\ker A^{\mathrm{tr}}$ is a line and $H_A^{-1}\big(H_A(x_0)\big)\subset X$ is connected by (A$_k$),

$$H\big(H_A^{-1}\big(H_A(x_0)\big)\big)\subset H(X)$$

contains the line segment between $H(x_0)$ and $H(x_1)$. Thus, $H(X)\subset\mathbb{R}^k$ is convex. $\square$

Proof of (C$_k$). The first two claims are justified in Section 3.1. By (B$_k$), it remains to show that $\mu(X)\subset\mathrm{CH}(\mu(X^\psi))$. Let $\rho\colon\mathbb{R}^k\longrightarrow\mathbb{T}$ and ψ' be as in (1.4). We can assume that the image of ρ is dense in $\mathbb{T}$ and so $X^\psi=X^{\psi'}$.

Suppose $\eta_0\in T_0^*\mathbb{R}^k-\mathrm{CH}(\mu(X^\psi))$. Thus, there exists $v\in T_0\mathbb{R}^k$ so that

$$\eta_0(v)<\min\big\{\eta(v)\colon\eta\in\mu(X^\psi)\big\}=\min\big\{\eta(v)\colon\eta\in\mathrm{CH}\big(\mu(X^\psi)\big)\big\}. \tag{3.12}$$

Let $y\in X$ be a minimum of the smooth function μ_v as in (3.1) Thus, $\mathrm{d}_y\mu_v=0$, the vector field ζ_v as in (1.2) vanishes at y, and y lies in the fixed locus $X^{\mathbb{R}v}$ of the restriction of the ψ-action to $\mathbb{R}v\subset\mathbb{R}^k$. For a generic choice of $v\in\mathbb{R}^k$ satisfying (3.12), $\rho(\mathbb{R}v)\subset\mathbb{T}$ is dense and thus $X^{\mathbb{R}v}=X^\psi$. It follows that

$$\eta_0(v)<\min\big\{\eta(v)\colon\eta\in\mu(X^\psi)\big\}=\min\big\{\mu_v(x)\colon x\in X\big\}=\min\big\{\eta(v)\colon\eta\in\mu(X)\big\},$$

i.e. $\eta_0\notin\mu(X)$.

Thus, $\mu(X)=\mathrm{CH}(\mu(X^\psi))$. The vertices of this polytope are of the form $\mu(Y)$ with $Y\in\pi_0(X^\psi)$. By Corollary 3.12 and (2.36), the number of edges at any such vertex $\mu(Y)$ is at most $|S(Y)|$. By (2.35), $|S(Y)|\le(\dim X)/2$. □

Proof of (E$_k$). Suppose $Y\in\pi_0(X^\psi)$. Let ρ, J, $S(Y)$, $\mathcal{N}_X^\alpha Y$ for each $\alpha\in S(Y)$, and $\mathcal{C}_{\mu(Y)}(\rho^*S(Y))$ be as in Proposition 2.33. If $\rho^*S(Y)$ does not span $T_0^*\mathbb{R}^k$ over $\mathbb{R}$, there exists

$$v\in T_0\mathbb{R}^k-\{0\}\qquad\text{s.t.}\quad\{\rho^*\alpha\}(v)=0\ \ \forall\,\alpha\in S(Y).$$

The subgroup $\mathbb{R}v\subset\mathbb{R}^k$ then acts trivially on $TX|_Y$. By Proposition 2.7(1), this implies that the connected component of the $\mathbb{R}v$-fixed locus $X^{\mathbb{R}v}$ containing Y is a connected component of X, i.e. $\mathbb{R}v$ acts trivially on X (and so the action ψ is reducible), since X is connected. If the action ψ is reducible, then $\mathbb{R}v\subset\mathbb{R}^k$ acts trivially on X and thus on $TX|_Y$ for some $v\in T_0\mathbb{R}^k$ nonzero and thus $\{\rho^*\alpha\}(v)=0$ for every $\alpha\in S(Y)$, i.e. $\rho^*S(Y)$ does not span $T_0^*\mathbb{R}^k$ over $\mathbb{R}$.

Thus, $\rho^*S(Y)$ spans $T_0^*\mathbb{R}^k$ over $\mathbb{R}$ if and only if the action ψ is irreducible. Suppose $\mu(Y)\in\mathrm{Ver}(\mu(X^\psi))$ is a vertex of the polytope $\mu(X)=\mathrm{CH}(\mu(X^\psi))$. By Proposition 2.33, the edges of $\mu(X)$ at $\mu(Y)$ are the edges of the cone $\mathcal{C}_{\mu(Y)}(\rho^*S(Y))$. A subset $S_\mu(Y)$ of $\rho^*S(Y)$ thus forms a collection of edge vectors of the polytope $\mu(X)$ at the vertex $\mu(Y)$, while the elements of $\rho^*S(Y)-S_\mu(Y)$ lie in the span of $S_\mu(Y)$. We conclude that $S_\mu(Y)$ spans $T_0^*\mathbb{R}^k$ over $\mathbb{R}$ if and only if the action ψ is irreducible. □

Proof of (G$_k$). Suppose $Y\in\pi_0(X^\psi)$. Let ρ, J, $S(Y)$, and $\mathcal{N}_X^\alpha Y$ for each $\alpha\in S(Y)$ be as in Proposition 2.33 with $(\mathbb{R}^k,0)$ replaced by $(\mathbb{T},\mathbb{1})$ and thus $\rho=\mathrm{id}$. As above, a subset $S_\mu(Y)$ of $\rho^*S(Y)=S(Y)$ forms a collection of edge vectors of the polytope $\mu(X)$ at $\mu(Y)$. Since $S(Y)\subset(T_{\mathbb{1}}^*\mathbb{T})_{\mathbb{Z}}$, all edges of the polytope $\mu(X)$ at $\mu(Y)$ are rational. □

Remark 3.16. The three parts of the statement of [3, Theorem 1] are (A$_k$), (B$_k$), and (C$_k$) in Theorem 3.1, while (F$_k$) is stated at the beginning of the proof of (A$_k^\star$) and is justified at the end of Section 2 in [3]. The proof in [3] contains (at least) two gaps, both at the top of page 6.

(G1) It is claimed that ($A_k^\star$) implies (A_k) *by continuity.* This is indeed the case for $k=1$; see Lemma A.3. However, this may not be the case for $k\ge 2$, even if the regular values are dense in the image (as is the case at the top of page 6 in [3]). As an example, pinch the points $+1,-1$ of the unit circle $S^1\subset\mathbb{R}^2$ to the origin (so that the preimage of $0\in\mathbb{R}^2$ consists of two points, while the preimages of all other points contain at most one point). Replacing the circle with a 2-torus, we can ensure that the set of regular values is dense in the image. Statement (D_k) is the key property of μ needed for the *by continuity* claim in [3], but it does not even appear in [3], and neither does the local description of μ of Corollary 2.35 needed for this statement. Both play prominent roles in other approaches to Theorem 3.1; see Section 3.5.

(G2) It is implicitly assumed that if $(c_1,\ldots,c_k)\in\mathbb{R}^k$ is a regular value of an $\mathbb{R}^k$-valued smooth function $(f_1,\ldots,f_k)$ on a smooth manifold X, then $(c_1,\ldots,c_{k-1})\in\mathbb{R}^{k-1}$ is a regular value of $(f_1,\ldots,f_{k-1})$. This need not be the case, including in the setting on page 6 in [3].

Remark 3.17. The statements (B_k) and (C_k) in Theorem 3.1, with $\mathbb{R}^k$ replaced by a torus $\mathbb{T}$, form [4, Theorem IV.4.3] and [41, Theorem 5.5.1]; [14, Theorem 27.1] includes (A_k) as well. The arguments in [4, 14, 41] generally follow [3], with [4] stating (A_k) as part of the proof and replicating the two gaps of Remark 3.16 in the middle of page 115 almost *verbatim.* In [41], only ($A_k^\star$) is stated as part of the argument, making the issue (G1) in Remark 3.16 extraneous, while the gap (G2) is resolved. In order to deduce (B_k) from ($A_k^\star$), it is claimed in the last full sentence on page 239 in [41] that any two points in X (M in [41]) with the same value of a Hamiltonian ($A^{\rm tr}\mu$ in [41]) can be approximated by two points in the preimage of a regular value of the Hamiltonian (the same regular value for both points). However, this is precisely what is needed to deduce (A_k) from ($A_k^\star$), as indicated by the proof of Lemma A.3. This property is implied by the Hamiltonian being an open map onto its image, i.e. (D_k), as suggested by Exercise 3.15, but neither the openness of the Hamiltonian nor its local description as in Corollary 2.35 is ever brought up in [41]. Thus, the attempt in [41] to bypass (G1) while establishing (B_k) contains fundamentally the same gap. In [14], the proof of (A_k) is relegated to Homework 21, which deals only with ($A_k^\star$), while resolving (G2) as in [41]; neither the openness of the Hamiltonian nor its local description is mentioned in [14] either. While (F_k) is also stated at the beginning of the proof of ($A_k^\star$) in [41] with a note that it is established later in the proof, (F_k) is never addressed in [41]. The equivariant splitting (2.11) of $TX|_Y$ for each $Y\in\pi_0(X^\psi)$ needed to establish even the first claim in (F_k) does not appear anywhere in [41].

3.4 Complexified Hamiltonian torus actions

This section establishes a number of properties of complexified Hamiltonian torus actions as in (3.2) and uses them to prove Theorem 3.2. The key steps in the proof are Lemma 3.18, which describes the behavior of the moment map μ on $\mathcal{O}_x(\psi_{\mathbb{C}})$, and Proposition 3.24, which concerns the images of $\mathcal{O}_x(\psi_{\mathbb{C}})$ and $\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}-\mathcal{O}_x(\psi_{\mathbb{C}})$ under μ. Corollary 3.19 below is used in the proof of Corollary 6.16, but not of Theorem 3.2; we state it in this section as it is a ready consequence of Lemma 3.18.

Lemma 3.18. *Suppose $\mathbb{T}$ is a torus, (X,ω,ψ,μ) is a Hamiltonian $\mathbb{T}$-manifold, and $\psi_{\mathbb{C}}$ is a complexification of ψ with respect to a $\mathbb{T}$-invariant ω-compatible almost complex structure J as in (3.2).*

For each $v\in T_{\mathbb{1}}\mathbb{T}$, let $\zeta_v\in\Gamma(X;TX)$ and $\mu_v\in C^\infty(X)$ be as in (1.2) and (1.24), respectively. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_v\big(\psi_{\mathbb{C};[\mathfrak{i}tv']}(x)\big)=-g_J^\omega\big(\zeta_v,\zeta_{v'}\big)\big|_{\psi_{\mathbb{C};[\mathfrak{i}tv']}(x)}\qquad\forall\ v,v'\in T_{\mathbb{1}}\mathbb{T},\ x\in X,\tag{3.13}$$

where g_J^ω is the Riemannian metric determined by ω and J as in (1.18). Furthermore,

$$\mu\big(\psi_{\mathbb{C};[\mathfrak{i}v]}(x)\big)\neq\mu(x)\in T_{\mathbb{1}}\mathbb{T}^*\qquad\forall\ v\in T_{\mathbb{1}}\mathbb{T},\ x\in X\ \text{s.t.}\ \zeta_v(x)\neq 0.\tag{3.14}$$

Proof. By (1.6), (3.2), and Proposition 2.1(1),

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_v\big(\psi_{\mathbb{C};[\mathfrak{i}tv']}(x)\big)\equiv\mathrm{d}_{\psi_{\mathbb{C};[\mathfrak{i}tv']}(x)}\mu_v\bigg(\frac{\mathrm{d}}{\mathrm{d}t}\psi_{\mathbb{C};[\mathfrak{i}tv']}(x)\bigg)=-\omega\Big(\zeta_v\big(\psi_{\mathbb{C};[\mathfrak{i}tv']}(x)\big),\big(J\zeta_{v'}\big(\psi_{\mathbb{C};[\mathfrak{i}tv']}(x)\big)\big)\Big).$$

This gives (3.13). Along with Proposition 2.1(1) again, this implies that

$$\mathbb{R}\longrightarrow\mathbb{R},\qquad t\longrightarrow\mu_v\big(\psi_{\mathbb{C};[\mathfrak{i}tv]}(x)\big),$$

is a strictly decreasing function unless $\zeta_v(x)=0$. This gives the second claim of the lemma. $\square$

Corollary 3.19. *Suppose $\mathbb{T}$, (X,ω,μ,ψ), and $\psi_{\mathbb{C}}$ are as in Lemma 3.18. If $\alpha\in T_{\mathbb{1}}^*\mathbb{T}$ is a regular value of μ, then $\mu^{-1}(\alpha)\subset X$ is a smooth submanifold and the map*

$$\Psi\colon\mathbb{T}_{\mathfrak{i}}\times\mu^{-1}(\alpha)\longrightarrow X,\qquad\Psi(u,x)=\psi_{\mathbb{C};u}(x),$$

is a diffeomorphism onto an open subset of X.

Proof. Since $\mathbb{T}_{\mathfrak{i}}$ is abelian, (2.5) and (3.2) imply that

$$\mathrm{d}_{(u,x)}\Psi\big(\mathrm{d}_{\mathbb{1}}L_u(\mathfrak{i}v),w\big)=\mathrm{d}_x\psi_{\mathbb{C};u}\big(J\zeta_v(x)+w\big)\quad\forall\ u\in\mathbb{T}_{\mathfrak{i}},\ x\in X,\ v\in T_{\mathbb{1}}\mathbb{T},\ w\in T_xX\,,\tag{3.15}$$

with $L_u\colon\mathbb{T}_{\mathfrak{i}}\longrightarrow\mathbb{T}_{\mathfrak{i}}$ as in (1.8). By the regularity assumption, $\mu^{-1}(\alpha)\subset X$ is a smooth submanifold. By (2.29), the homomorphism

$$T_{\mathbb{1}}\mathbb{T}_{\mathfrak{i}}\oplus T_x\big(\mu^{-1}(\alpha)\big)\longrightarrow T_xX,\qquad(\mathfrak{i}v,w)\longrightarrow J\zeta_v(x)+w,\tag{3.16}$$

is surjective for every $x\in\mu^{-1}(\alpha)$. By Exercise 2.21(b), this homomorphism is injective. Along with (3.15), the former implies that Ψ is an open map. By (3.14) and the injectivity of (3.16) on $T_{\mathbb{1}}\mathbb{T}_{\mathfrak{i}}$, Ψ is injective. $\square$

Exercise 3.20. Suppose $\mathbb{T}$, (X,ω,μ,ψ), and $\psi_{\mathbb{C}}$ are as in Lemma 3.18, $v\in T_{\mathbb{1}}\mathbb{T}$, $x\in X$, and ζ_v and μ_v are also as in Lemma 3.18. Show that the function

$$\mathbb{R}\longrightarrow\mathbb{R},\qquad t\longrightarrow\mu_v\big(\psi_{\mathbb{C};\mathfrak{i}tv}(x)\big),\tag{3.17}$$

is strictly decreasing if $\zeta_v(x)\neq0$ and is constant otherwise. If in addition,

$$x_\infty(v)=\lim_{t\longrightarrow\infty}\psi_{\mathbb{C};[\mathfrak{i}tv]}(x)\in X,\tag{3.18}$$

i.e. the limit above also exists, show that

$$\zeta_v\big(x_\infty(v)\big)=0,\quad\mathrm{d}_{x_\infty(v)}\mu_v=0,\quad\big(\psi_{\mathbb{C};u}(x)\big)_\infty(v)=\psi_{\mathbb{C};u}\big(x_\infty(v)\big)\ \ \forall\,u\in\mathbb{T}_{\mathbb{C}}\,.\tag{3.19}$$

Exercise 3.21. With the assumptions as in Exercise 3.20, suppose also that X is compact. Show that the limit in (3.18) exists.

Suppose ψ is a smooth action of a torus $\mathbb{T}$ on a smooth manifold X. For $x\in X$ and $v\in T_{\mathbb{1}}\mathbb{T}$, let $\mathbb{T}_{x;v}(\psi)\subset\mathbb{T}$ be the closed subgroup spanned by $G_x(\psi)$ and the closure of $\{\mathrm{e}^{tv}: t\in\mathbb{R}\}$ in $\mathbb{T}$.

Corollary 3.22. *Suppose $\mathbb{T}$, (X,ω,μ,ψ), and $\psi_{\mathbb{C}}$ are as in Lemma 3.18, $v\in T_{\mathbb{1}}\mathbb{T}$, $x\in X$, μ_v is also as in Lemma 3.18, and the limit in (3.18) exists. There exists a topological component $Z_{x;v}$ of the $\mathbb{T}_{x;v}(\psi)$-fixed locus $X^{\mathbb{T}_{x;v}(\psi)}\subset X$ so that*

$$x'_\infty(v)\in Z_{x;v} \quad \textit{and} \quad \lim_{t\longrightarrow\infty}\mu_v\big(\psi_{\mathbb{C};\mathrm{i}tv}(x')\big)=\mu_v\big(Z_{x;v}\big)=\inf_{\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}}\mu_v \qquad \forall\, x'\in\mathcal{O}_x(\psi_{\mathbb{C}})\,; \tag{3.20}$$

in particular, the limit in (3.18) with x replaced by x' also exists. If in addition X is compact, then

$$\inf_{\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}}\mu_v=\inf_{\substack{Y\in\pi_0(X^\psi)\\ Y\cap\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}\neq\emptyset}}\mu_v(Y)\,. \tag{3.21}$$

Proof. Let $x'\in\mathcal{O}_x(\psi_{\mathbb{C}})$. By the continuity of the action ψ, $x'_\infty(v)$ is fixed by $G_{x'}(\psi)=G_x(\psi)$. By the first equation in (3.19) with x replaced by x', $x'_\infty(v)$ is also fixed by the closure of $\{\mathrm{e}^{tv}: t\in\mathbb{R}\}$ in $\mathbb{T}$. Thus, $x'_\infty(v)$ lies in $X^{\mathbb{T}_{x;v}(\psi)}$, which is a closed symplectic submanifold of (X,ω) by Proposition 2.15(1). Let $Z_{x;v}\subset X^{\mathbb{T}_{x;v}(\psi)}$ be the component containing $x_\infty(v)$. Since $\mathbb{T}_{\mathbb{C}}$ is connected, the last equation in (3.19) implies that $x'_\infty(v)\in Z_{x;v}$ as well.

Since $\mathrm{d}\mu_v$ vanishes on $X^{\mathbb{T}_{x;v}(\psi)}$, μ_v is constant on $Z_{x;v}$. Along with (3.18) with x replaced by x' and the function (3.17) with x replaced by x' being non-increasing, this yields the second equation in (3.20). If $v\in T_{\mathbb{1}}\mathbb{T}$ is generic, $\mathbb{T}_{x;v}(\psi)=\mathbb{T}$ and so $Z_{x;v}\subset X^\psi$. Thus,

$$\inf_{\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}}\mu_v\le\inf_{\substack{Y\in\pi_0(X^\psi)\\ Y\cap\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}\neq\emptyset}}\mu_v(Y)\le\mu_v\big(Z_{x;v}\big)=\inf_{\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}}\mu_v \quad\Longrightarrow\quad \inf_{\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}}\mu_v=\inf_{\substack{Y\in\pi_0(X^\psi)\\ Y\cap\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}\neq\emptyset}}\mu_v(Y)\,.$$

If X is compact, then so is $\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}$. The continuity of μ in both inputs then implies that the last equality holds for all $v\in T_{\mathbb{1}}\mathbb{T}$. $\square$

Corollary 3.23. *Suppose $\mathbb{T}$, (X,ω,ψ,μ), $\psi_{\mathbb{C}}$, and μ_v for each $v\in T_{\mathbb{1}}\mathbb{T}$ are as in Lemma 3.18 and $x\in X$. Then,*

$$\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}-\mathcal{O}_x(\psi_{\mathbb{C}})=\big\{x'\in\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}: \exists\; v\in T_{\mathbb{1}}\mathbb{T}\ \textit{s.t.}\ \mathrm{d}_x\mu_v\neq0,\ \mu_v(x')=\inf_{\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}}\mu_v\big\}. \tag{3.22}$$

Proof. Let $T_{\mathbb{1}}^c\mathbb{T}\subset T_{\mathbb{1}}\mathbb{T}$ be a complement of

$$T_{\mathbb{1}}G_x(\psi)=\big\{v\in T_{\mathbb{1}}\mathbb{T}: \mathrm{d}_x\mu_v=0\big\} \tag{3.23}$$

and $S(T_{\mathbb{1}}^c\mathbb{T})\subset T_{\mathbb{1}}^c\mathbb{T}$ be the unit sphere in $T_{\mathbb{1}}^c\mathbb{T}$ with respect to some metric. In particular,

$$\mathcal{O}_x(\psi_{\mathbb{C}})=\big\{\psi_{\mathbb{C};[v+\mathrm{i}v']}(x): v,v'\in T_{\mathbb{1}}^c\mathbb{T}\big\}.$$

Since the group $\mathbb{T}_{\mathbb{C}}$ is abelian, $\xi_v(x')\neq0$ and $\mathrm{d}_{x'}\mu_v\neq0$ for all $v\in T_{\mathbb{1}}^c\mathbb{T}$ and $x'\in\mathcal{O}_x(\psi_{\mathbb{C}})$. Along with the first condition in (1.6), this implies that the left-hand set in (3.22) contains the right-hand set.

By (3.13), the continuous function

$$S(T_{\mathbb{1}}^c\mathbb{T})\times\mathbb{R}\longrightarrow\mathbb{R},\qquad h(v,t)=\mu_v\big(\psi_{\mathbb{C};\mathfrak{i}tv}(x)\big),$$

is non-increasing in t. By (3.20),

$$\lim_{t\longrightarrow\infty}h(v,t)=\inf_{\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}}\mu_v\qquad\forall\,v\in S(T_{\mathbb{1}}^c\mathbb{T}).\tag{3.24}$$

Suppose $x'\in\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}-\mathcal{O}_x(\psi_{\mathbb{C}})$ is the limit of a sequence $\psi_{\mathbb{C};\mathfrak{i}t_kv_k}(\psi_{u_k}(x))$ with $u_k\in\mathbb{T}$ converging to some u, $v_k\in S(T_{\mathbb{1}}^c\mathbb{T})$ converging to some v, and $t_k\in\mathbb{R}$ converging to ∞. Since μ is a continuous $\mathbb{T}$-invariant function on $T_{\mathbb{1}}\mathbb{T}\times X$ and the functions $h(v_k,t)$ are non-increasing in t,

$$\mu_v(x')=\lim_{k\longrightarrow\infty}\mu_{v_k}\big(\psi_{u_k}\big(\psi_{\mathbb{C};\mathfrak{i}t_kv_k}(x)\big)\big)=\lim_{k\longrightarrow\infty}h(v_k,t_k)\le\lim_{t\longrightarrow\infty}h(v,t).$$

Along with (3.24), this implies that $\mu_v(x')=\inf_{\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}}\mu_v$. $\square$

For a polytope $P\subset T_{\mathbb{1}}^*\mathbb{T}$, we denote by $\partial P\subset P$ the union of the proper faces of P. Let $\operatorname{Int}P=P-\partial P$. With $\mathbb{T}$, (X,ω,ψ,μ), and $\psi_{\mathbb{C}}$ as in Theorem 3.2 and $x\in X$, let

$$P_x\big(\psi_{\mathbb{C}}\big)=\operatorname{CH}\big(\mu\big(\{Y\in\pi_0(X^{\psi})\colon Y\cap\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}\neq\emptyset\}\big)\big).$$

Proposition 3.24. *Suppose $\mathbb{T}$, (X,ω,ψ,μ), and $\psi_{\mathbb{C}}$ are as in Theorem 3.2 and $x\in X$. Then,*

$$\mu\big(\mathcal{O}_x(\psi_{\mathbb{C}})\big)=\operatorname{Int}P_x\big(\psi_{\mathbb{C}}\big),\qquad\mu\big(\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}-\mathcal{O}_x(\psi_{\mathbb{C}})\big)=\partial P_x\big(\psi_{\mathbb{C}}\big),\tag{3.25}$$

and the map $\mathcal{O}_x(\psi_{\mathbb{C}})/\mathbb{T}\longrightarrow T_{\mathbb{1}}^*\mathbb{T}$ induced by μ is injective.

Proof. For each $v\in T_{\mathbb{1}}\mathbb{T}$, the map

$$L_v\colon T_{\mathbb{1}}^*\mathbb{T}\longrightarrow\mathbb{R},\quad L_v(\alpha)=\alpha(v),\tag{3.26}$$

is a linear functional and $\mu_v\equiv L_v\circ\mu\colon X\longrightarrow\mathbb{R}$. Let $\mathbb{T}_x^c\subset\mathbb{T}$ be a subtorus complementary to the identity component $(G_x(\psi))_0$ of $G_x(\psi)$ and $\iota\colon\mathbb{T}_x^c\longrightarrow\mathbb{T}$ be the inclusion so that

$$\mu_x^c\equiv\iota^*\circ\mu\colon X\longrightarrow T_{\mathbb{1}}^*\mathbb{T}_x^c$$

is a moment map for the restriction of the $\mathbb{T}$-action ψ on $\mathbb{T}_x^c$. In particular,

$$\begin{gathered}\mathcal{O}_x(\psi_{\mathbb{C}})\equiv\mathbb{T}_{\mathbb{C}}x=(\mathbb{T}_x^c)_{\mathbb{C}}x,\quad\big\{Y\in\pi_0(X^{\psi})\colon Y\cap\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}\neq\emptyset\big\}=\big\{Y\in\pi_0(X^{\mathbb{T}_x^c})\colon Y\cap\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}\neq\emptyset\big\},\\ \mathcal{O}_x(\psi_{\mathbb{C}})/\mathbb{T}=\mathcal{O}_x(\psi_{\mathbb{C}})/\mathbb{T}_x^c,\\ \mu\big(\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}\big)\subset(T_{\mathbb{1}}^*\mathbb{T})_{\mu;x}\equiv\big\{\alpha\in T_{\mathbb{1}}^*\mathbb{T}\colon L_v(\alpha)=L_v\big(\mu(x)\big)\ \forall\,v\in T_{\mathbb{1}}G_x(\psi)\big\}.\end{gathered}$$

Since $\iota^*\colon(T_{\mathbb{1}}^*\mathbb{T})_{\mu;x}\longrightarrow T_{\mathbb{1}}^*\mathbb{T}_x^c$ is a homeomorphism sending line segments to line segments, it suffices to establish the claims with (ψ,μ) replaced by $(\psi|_{\mathbb{T}_x^c},\mu_x^c)$. We can thus assume that $(G_x(\psi))_0=\{\mathbb{1}\}$, as is done below.

For $v\in T_{\mathbb{1}}\mathbb{T}$, let $\zeta_v\in\Gamma(X;TX)$ be as in (1.2). Since

$$\big(G_{x'}(\psi)\big)_0=\big(G_x(\psi)\big)_0=\{\mathbb{1}\}\qquad\forall\,x'\in\mathcal{O}_x(\psi_{\mathbb{C}}),$$

$\zeta_v(x')\neq 0$ for all $v\in T_{\mathbb{1}}\mathbb{T}-\{0\}$ and $x'\in\mathcal{O}_x(\psi_{\mathbb{C}})$. By (3.13) and Exercise 3.25 below, the map

$$T_{\mathbb{1}}\mathbb{T}\longrightarrow T_{\mathbb{1}}^*\mathbb{T},\qquad v\longrightarrow \mu\big(\psi_{\mathbb{C};[iv]}(x)\big),$$

is thus an immersion. By (3.14), this map is injective and thus a diffeomorphism onto an open subset of $T_{\mathbb{1}}^*\mathbb{T}$. Since μ is $\mathbb{T}$-invariant, this open subset is $\mu(\mathcal{O}_x(\psi_{\mathbb{C}}))$. By (3.21), $\mu(\overline{\mathcal{O}_x(\psi_{\mathbb{C}})})\subset P_x(\psi_{\mathbb{C}})$. Thus, the polytope $P_x(\psi_{\mathbb{C}})\subset T_{\mathbb{1}}^*\mathbb{T}$ is of full dimension, $\mu(\mathcal{O}_x)\subset\mathrm{Int}\,P_x(\psi_{\mathbb{C}})$, and the last claim of the proposition holds.

For $v\in T_{\mathbb{1}}\mathbb{T}-\{0\}$, the level sets of L_v are hyperplanes. Thus, the restriction of $\mu_v\equiv L_v\circ\mu$ to $\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}$ achieves its minimum along the preimage of a proper face of $P_x(\psi_{\mathbb{C}})$ under μ and not at any point of $\mu^{-1}(\mathrm{Int}\,P_x(\psi_{\mathbb{C}}))$; the former preimage contains $Y\cap\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}$ for at least one element $Y\in\pi_0(X^\psi)$ with $Y\cap\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}\neq\emptyset$. From the compactness of $\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}$ and (3.22), we then conclude that

$$\begin{aligned}\overline{\mu(\mathcal{O}_x(\psi_{\mathbb{C}}))}-\mu\big(\mathcal{O}_x(\psi_{\mathbb{C}})\big)&\subset\mu\big(\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}-\mathcal{O}_x(\psi_{\mathbb{C}})\big)\\ &=\big\{\mu(x')\colon x'\in\overline{\mathcal{O}_x(\psi_{\mathbb{C}})},\exists\; v\in T_{\mathbb{1}}\mathbb{T}-\{0\}\text{ s.t. }\mu_v(x')=\inf_{\mathcal{O}_x(\psi_{\mathbb{C}})}\mu_v\big\}\subset\partial P_x(\psi_{\mathbb{C}}).\end{aligned}$$

Thus, $\mu(\mathcal{O}_x(\psi_{\mathbb{C}}))\supset\mathrm{Int}\,P_x(\psi_{\mathbb{C}})$, which establishes the first equality in (3.25). The second equality in (3.25) then follows from the compactness of $\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}$. $\square$

Exercise 3.25. Suppose V is a vector space with an inner-product product $\langle\cdot,\cdot\rangle$ and $v_1,\ldots,v_k\in V$. Show that the matric $(\langle v_i,v_j\rangle)_{i,j\in[k]}$ is nondegenerate if and only if the vectors $v_1,\ldots,v_k$ are linearly independent.

Proof of Theorem 3.2. The two equations in (3.25) give (2), as well as (3) with $\sigma=\mathrm{Int}\,P_x(\psi_{\mathbb{C}})$. Suppose σ is the interior of a proper face F of $P_x(\psi_{\mathbb{C}})$. Choose $v\in T_{\mathbb{1}}\mathbb{T}$ and $c\in\mathbb{R}$ be so that

$$F=P_x(\psi_{\mathbb{C}})\cap L_v^{-1}(c),$$

with L_v as in (3.26). Let $Z_{x;v}\subset X$ be as in (3.20). Thus,

$$\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}\cap\mu^{-1}(F)\subset\mathrm{Crit}(\mu_v),\qquad \mu_v|_{\mu^{-1}(F)}=c,$$

and $Z_{x;v}$ is a topological component of $\mathrm{Crit}(\mu_v)$ with $\mu_v|_{Z_{x;v}}=c$. By (3.2) and (2.27), $\psi_{\mathbb{C};\mathfrak{i}tv}$ is the negative gradient flow $\psi_{\mu_v;t}$ of μ_v with respect to the metric $g(\cdot,\cdot)=\omega(\cdot,J\cdot)$, with J as in (3.2). By the first statement in (3.20),

$$\mathcal{O}_x(\psi_{\mathbb{C}})\subset X^+_{Z_{x;v}}(\mu_v)\equiv\big\{x'\in X\colon \lim_{t\longrightarrow\infty}\psi_{\mathbb{C};[\mathfrak{i}tv]}(x')\in Z_{x;v}\big\}.$$

By the first statement of Proposition 3.11, $\mu_v\colon X\longrightarrow\mathbb{R}$ is a Morse-Bott function. Proposition A.2(6) and the last equation in (3.19) thus imply that

$$\begin{aligned}\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}\cap\mu^{-1}(F)\equiv\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}\cap\mu_v^{-1}(c)&=\overline{\{x'_\infty(v)\colon x'\in\mathcal{O}_x(\psi_{\mathbb{C}})\}}=\overline{\mathcal{O}_{x_\infty(v)}(\psi_{\mathbb{C}})},\\ P_{x_\infty(v)}(\psi_{\mathbb{C}})&\equiv\mathrm{CH}\big(\mu\big(\{Y\in\pi_0(X^\psi)\colon Y\cap\overline{\mathcal{O}_{x_\infty(v)}(\psi_{\mathbb{C}})}\neq\emptyset\}\big)\big)\\ &=\mathrm{CH}\big(\mu\big(\{Y\in\pi_0(X^\psi)\colon Y\cap\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}\neq\emptyset\}\big)\cap F\big)=F.\end{aligned}$$

From (3.25) with x replaced by $x_\infty(v)$, we then conclude that

$$\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}\cap\mu^{-1}(\sigma)=\overline{\mathcal{O}_{x_\infty(v)}(\psi_{\mathbb{C}})}\cap\mu^{-1}(\sigma)=\mathcal{O}_{x_\infty(v)}(\psi_{\mathbb{C}})\,.$$

This establishes (3). Since $\mathcal{O}_y(\psi_{\mathbb{C}})=\{y\}$ for any $y\in X^{\psi}$, (1) follows from (3).

Since the moment map μ is $\mathbb{T}$-invariant, the map

$$\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}/\mathbb{T} \longrightarrow \mu(\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}), \qquad [x'] \longrightarrow \mu(x'), \tag{3.27}$$

is well-defined. It is surjective by definition. Its domain is compact, while the target is Hausdorff. By (3), for every open face σ of the polytope $\mu(\overline{\mathcal{O}_x(\psi_{\mathbb{C}})})$ there exists $x_\sigma\in\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}$ so that

$$\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}\cap\mu^{-1}(\sigma) = \mathcal{O}_{x_\sigma}(\psi_{\mathbb{C}}).$$

By the last statement of Proposition 3.24 with x replaced by x_σ, the restriction of the map (3.27) to $\overline{\mathcal{O}_x(\psi_{\mathbb{C}})}\cap\mu^{-1}(\sigma)/\mathbb{T}$ is thus injective for every open face σ of $\mu(\overline{\mathcal{O}_x(\psi_{\mathbb{C}})})$. It follows that the entire map (3.27) is injective as well and thus a homeomorphism. $\square$

Remark 3.26. As stated, [3, (3.6)] is wrong. For example, it fails if $C\subset N^u-N$ consists of a single point. However, [3, (3.6)] is used in the proof of Theorem 3.2 in [3] only to obtain the second statement in [3, (3.7)]. The latter is correct, as it follows from Proposition A.2(6), which corrects [3, (3.6)].

3.5 Alternative approaches to classical convexity theorem

The most renowned conclusion of Theorem 3.1 on page 51 is that the image $\mu(X)$ of the moment map for a Hamiltonian almost periodic $\mathbb{R}^k$-action ψ on a closed connected symplectic manifold (X,ω) is the convex hull of the finite subset $\mu(X^{\psi})\subset T_0^*\mathbb{R}^k$; this is known as the convexity property of (X,ω,ψ,μ). Other approaches to this statement have appeared in particular in [29, 19, 31, 8, 9]. In contrast to [3, 4, 41, 14], they clearly emphasize the significance of a local description of the moment map μ as in Corollary 2.35. These approaches differ in how they pass from the local versions of (A_k)-(D_k) implied by such a description to the global versions appearing in Theorem 3.1.

The statement of Corollary 3.12 for torus actions is contained in the proof of [29, Theorem 5.2]. Similarly to Corollary 3.12, this statement is obtained in [29] from the local description of the moment map μ at the end of Corollary 2.35 and the global, Morse-Bott theory statement of Proposition 3.11(2), provided by [29, Theorem 4.8] and [29, Lemma 5.1], respectively. As demonstrated in the proof of [29, Theorem 5.2], the last two statements readily imply (B_k) and the first part of the third claim of (C_k) in Theorem 3.1, without going through $(A_k^\star)$ on page 57. They also imply (D_k). However, the remaining conclusions of Theorem 3.1 do not appear in [29], even in the torus case.

A purely topological approach for passing from a local description of the moment map μ as in Corollary 2.35 to global statements was introduced in [19]. It avoids the use of Morse-Bott theory entirely and is instead based on the use of the Reeb space X_μ of the manifold X with the respect to the moment map μ and the metric $d_{\overline{\mu}}$ on X_μ, defined as in (3.28) and (3.29), respectively. This approach has since been extended in [31, 8, 9] to obtain generalizations of the key properties of closed Hamiltonian manifolds provided by Theorem 3.1 to non-compact manifolds with some restrictions on the moment map. The weakest of these restrictions appear in Theorem 3.27 below, i.e. the generalizations in [31, 8, 9] are specializations of this theorem as described below.

Let V be a finite-dimensional vector space. A subset $S\subset V$ is called locally polyhedral in [8] if for every $v\in S$ there exists a polytope $P_v\subset V$ such that $S\cap P_v\subset V$ is also a polytope and $v\in \mathrm{Int}_V(P_v)$, where $\mathrm{Int}_V(P_v)\subset P_v$ is the interior of P_v with respect to V. A continuous map $f\colon X\longrightarrow V$ from a topological space is called convex in [9] if for any $x_0,x_1\in X$ there exists a (continuous) path

$$\gamma\colon \big([0,1],0,1\big)\longrightarrow \big(X,x_0,x_1\big)$$

from x_0 to x_1 in X such that the map $f\circ\gamma$ has connected fibers and $f(\gamma([0,1]))$ is contained in the line segment from $f(x_0)$ to $f(x_1)$ in V. The conditions on γ mean that the path

$$f\circ\gamma\colon \big([0,1],0,1\big)\longrightarrow \big(V,f(x_0),f(x_1)\big)$$

traces the line segment from $f(x_0)$ to $f(x_1)$ in V without reversing the direction at any point in time. If f is convex, its image is a convex subset of V and its fibers are path-connected. Theorem 3.27 follows immediately from the *local-to-global principle* of Proposition 3.33, and the local properties of moment maps provided by Corollary 2.35; see the sentence above Exercise 3.30.

Theorem 3.27. *Suppose $k\in\mathbb{Z}^+$, (X,ω,ψ,μ) is a connected Hamiltonian $\mathbb{R}^k$-manifold, and the $\mathbb{R}^k$-action ψ is almost periodic. If there exists a convex subset $V'\subset T_0^*\mathbb{R}^k$ so that $\mu(X)\subset V'$ and the map $\mu\colon X\longrightarrow V'$ is closed, then the map*

$$\mu\colon X\longrightarrow \mu(X)$$

is open and convex (as a map into $T_0^*\mathbb{R}^k$ in the latter case). If in addition $V'\subset V$ is closed, then $\mu(X)\subset T_0^*\mathbb{R}^k$ is a closed convex locally polyhedral subset.

The versions of Theorem 3.27 appearing in [31, 8, 9] refer only to torus actions, i.e. fully periodic $\mathbb{R}^k$-actions. In [9, Theorem 30], the moment map μ is also required to be proper, i.e. closed with compact fibers; this theorem also does not contain the last statement of Theorem 3.27. The conclusions in [31, Theorem 4.1] and [8, Corollary 3.4] include that the image of μ is a convex subset of $\mathfrak{t}^*$, the dual of the Lie algebra of the torus, and that the fibers of μ are connected, but not that the map μ is convex. In [31, Theorem 4.1], μ is required to be a proper map into the entire vector space $\mathfrak{t}^*$. The conclusions in this theorem in addition include a description of neighborhoods of points in the image $\mu(X)$ of μ, but this description is implicitly contained in Theorem 3.27, given the input of Corollary 2.35, as well as in [9, Theorem 30] and [8, Corollary 3.4]; see the second sentence after Proposition 3.33. In [8, Corollary 3.4], μ is required to be just a closed map into $\mathfrak{t}^*$.

Let V be again a topological vector space. A (closed, convex) subset $S\subset V$ is a (closed, convex) cone with vertex at $v\in V$ if $v+t(v'-v)\in S$ whenever $v'\in S$ and $t\in\mathbb{R}^+$. For any subset $S\subset V$ and $v\in S$, define

$$\mathbb{R}^+(S-v)=\big\{r(v'-v)\colon v'\in S,\, r\in\mathbb{R}^+\big\},\qquad L_v(S)=\overline{\mathbb{R}^+(S-v)}\subset V.$$

If V is finite-dimensional, a convex subset $S\subset V$ is called locally polyhedral in [31] if for every $v\in S$ there exists a neighborhood $U_v\subset V$ of v such that

$$S\cap U_v=\big\{v+v'\colon v'\in L_v(S)\big\}\cap U_v$$

and $L_v(S)\subset V$ is a polyhedron. It is also mentioned in [31] that the requirement on $L_v(S)\subset V$ to be a polyhedron in this definition is shown to be redundant in the unpublished note [49], which does not appear to be readily available.

Exercise 3.28. Let V be a finite-dimensional vector and $S\subset V$ be a convex subset. Show that the definitions for S to be locally polyhedral in [8] and [31] are equivalent.

Exercise 3.29. Let V be a finite-dimensional vector and $S\subsetneq V$ be a convex cone. Show that S is contained in a (closed) half-space, i.e.

$$S\subset\big\{w\in V: L(w)\geq c\big\}$$

for some nonzero linear functional $L\colon V\longrightarrow\mathbb{R}$ and $c\in\mathbb{R}$.

Let V be a topological space. For $v\in V$ and subsets $\mathcal{C},\mathcal{C}'\subset V$ containing v, we define $\mathcal{C}\sim_v\mathcal{C}'$ if there exists a neighborhood $W_v\subset V$ of v such that $\mathcal{C}\cap W_v=\mathcal{C}'\cap W_v$, i.e. if $\mathcal{C}$ and $\mathcal{C}'$ have the same germ at v. We denote by $[\mathcal{C}]_v$ the corresponding equivalence class of $\mathcal{C}$, i.e. the germ of $\mathcal{C}$ at v. We call a continuous map $f\colon X\longrightarrow V$ from a topological space X locally open if for every $x\in X$ there exists a neighborhood $U_x\subset X$ of x so that the restriction

$$f|_{U_x}: U_x\longrightarrow f(U_x)$$

is an open map. The germ $[f(U_x)]_{f(x)}$ of $f(U_x)\subset V$ at $f(x)$ is then independent of the choice of such a neighborhood $U_x\subset X$ of x; we denote it by $[f]_x$. We call a continuous map $f\colon X\longrightarrow V$ between topological spaces locally fiber connected (or LFC) if for every $x\in X$ and every neighborhood $U\subset X$ of x there exists a neighborhood $U'\subset U$ such that all fibers of the restriction $f|_{U'}$ are connected.

Let $f: X\longrightarrow V$ be a continuous map to a topological vector space. We call such a map locally convex if for every $x\in X$ and every neighborhood $U\subset X$ of x there exists a neighborhood $U'\subset U$ so that the restriction

$$f|_{U'}: U'\longrightarrow V$$

is a convex map. We call a locally open map $f\colon X\longrightarrow V$ locally convexly open (resp. locally cone-open) if for every $x\in X$ the germ $[f]_x$ contains a subset of V which is locally convex at $f(x)$ (resp. a cone with vertex at $f(x)$). A locally open and locally convex map is locally convexly open and LFC (in fact, the fibers of local restrictions are then path-connected, not just connected). Our notion of *locally cone-open* is motivated by the intrinsic perspective taken in [9] and corresponds to the notion of *admitting local convexity data* as in [8, Definition 2.6], which is worded in an overly complicated way. In [31, Definition 3.3], the relevant convex cones are required to be closed. By Corollary 2.35, a moment map μ for a Hamiltonian almost periodic $\mathbb{R}^k$-action ψ on a symplectic manifold (X,ω) is locally convex and locally cone-open (and thus LFC and locally convexly open); see also Exercises 3.30-3.32 below.

Exercise 3.30. Suppose $f_i\colon X_i\longrightarrow V_i$ for $i=1,2$ are convex maps. Show that the map

$$f_1\times f_2\colon X_1\times X_2\longrightarrow V_1\times V_2$$

is also convex.

Exercise 3.31. Suppose $k,m\in\mathbb{Z}^{\geq0}$, $f\colon\mathbb{R}^k\longrightarrow\mathbb{R}^m$ is a smooth function so that the differential d_0f is surjective, and $\mathcal{U}$ is a neighborhood of 0 in $\mathbb{R}^k$. Show that f is open and convex on some neighborhood $\mathcal{U}'$ of 0 in $\mathcal{U}$.

Exercise 3.32. Suppose $m\in\mathbb{Z}^{\ge 0}$, $S\subset\mathbb{R}^m$ is a finite subset,

$$f\colon \mathbb{C}^S \longrightarrow \mathbb{R}^m, \qquad f\big((w_\alpha)_{\alpha\in S}\big)=\sum_{\alpha\in S}|w_\alpha|^2\alpha,$$

and $\mathcal{U}$ is a neighborhood of 0 in $\mathbb{C}^S$. Show that the map

$$f\colon \mathbb{C}^S \longrightarrow \mathcal{C}_0(S)\equiv\Big\{\sum_{\alpha\in S}t_\alpha\alpha : t_\alpha\in\mathbb{R}^{\ge 0}\ \forall\,\alpha\in S\Big\}$$

is open and f is convex on some neighborhood $\mathcal{U}'$ of 0 in $\mathcal{U}$.

Proposition 3.33 (Local-to-Global Principle)**.** *Let X be a connected, normal topological space, $V'\subset V$ be a convex subset of a finite-dimensional vector space, and $f\colon X\longrightarrow V'$ be a closed map.*

(1) If f is locally convexly open and LFC, then $f(X)\subset V$ is a convex subset and $f\colon X\longrightarrow f(X)$ is an open map with connected fibers.

(2) If $V'\subset V$ is closed and f is locally cone-open and LFC, then $f(X)\subset V$ is a closed convex locally polyhedral subset.

(3) If f is locally open and locally convex, then $f\colon X\longrightarrow V$ is a convex map.

Since the assumptions in (2) and (3) of this proposition are stronger than in (1), the conclusions in (1) apply in (2) and (3) as well. If f is as in (2) and $\mathcal{C}_x\subset V$ is the cone representing $[f]_x$ for each $x\in X$, it follows from this proposition that

$$L_{f(x)}\big(f(X)\big)=\big\{v-f(x)\colon v\in\mathcal{C}_x\big\}\qquad\forall\ x\in X.$$

Proposition 3.33 captures the local-to-global principles of [31, 8, 9], which in turn build on [19]. In [31, Theorem 3.10], the connected topological space X is required to be only Hausdorff, but the map f is required to be proper into V, locally cone-open, and LFC; the conclusions in this theorem are as in (1) and (2). In [8, Theorem 2.28], the connected normal topological space is required to be first-countable and locally connected, while the map f is required to be closed into V, locally cone-open, and LFC; the conclusions in this theorem are also as in (1) and (2). The first countability condition is used only artificially in the first part of the proof of [8, Lemma 2.21(i)], which obtains the same conclusion as Exercise 3.42 below. The locally connected condition appears as a requirement in [8, Corollary 2.18], but is never used in its proof. The other requirements in [8, Theorem 2.28] imply that X is locally connected, as indicated by Exercise 3.37 below. It is stated in [8, Remark 2.29] that [8, Theorem 2.28] remains valid if f is a closed map into a convex subset of V. However, no indication of the necessary changes in the proof of this theorem are given, and the setup of the proof is not conducive to such changes; see also the paragraphs preceding Lemma 3.46 and Corollary 3.47 below. In [9, Theorem 15], the connected topological space X is again required to be only Hausdorff, while the map f is required to be locally open, locally convex, and proper (but now into a convex subset of V, in contrast to [31]); the conclusions in this theorem are also as in (1) and (3).

As emphasized in [9], the case of Proposition 3.33 with $V'=V$, $X\subset V$ a closed connected locally convex subset, and $f\colon X\longrightarrow V$ the inclusion is the Tietze-Nakajima Theorem from the 1920s recalled below. It is extended to arbitrary topological vector spaces in [34, (5.2)]. This extension is cited in the proof of [8, Theorem 2.13], but in relation to a closed subset of a convex subset of a vector space, i.e. for a more general statement than is actually established in [34]. On the other hand, this is not needed to establish [8, Theorem 2.28], even though its proof cites [8, Corollary 2.18], which in turn cites [8, Theorem 2.13].

Corollary 3.34 (Tietze-Nakajima Theorem, [54, Satz 1], [9, Theorem 1])**.** *A closed connected locally convex subset of* $\mathbb{R}^k$ *is convex.*

Remark 3.35. Nakajima's paper [48] typically credited for Corollary 3.34 contains several statements in the same spirit, most concerning subsets of $\mathbb{R}^2$ and $\mathbb{R}^3$, but not the actual statement of this corollary.

In the remainder of this section, we establish Proposition 3.33. As in [19, 31, 8], its proof revolves around the Reeb quotient X_f of X with respect to f as defined in (3.28) below, the continuous map

$$\overline{f}\colon X_f \longrightarrow V'$$

induced by f, and a metric d_f on X_f induced by $\overline{f}$ as in (3.29) below. We first show that X_f is a normal topological space, the Reeb quotient map

$$q_f\colon X \longrightarrow X_f$$

is open and closed, and $\overline{f}\colon X_f \longrightarrow V'$ is locally a homeomorphism onto a convex subset of V'; see Exercise 3.44(b) and Lemma 3.45. This part of the proof of Proposition 3.33 is a greatly streamlined version of the corresponding parts of [8]. We then show that any proper function with these properties of $\overline{f}$ is a convex homeomorphism onto its image; see Lemma 3.46(2). This part of the proof of Proposition 3.33 follows the first part of the proof of [9, Theorem 15] at the end of [9, Section 6]. The argument in [9] is actually applied with the assumption that $\overline{f}$ is locally a homeomorphism onto a convex subset weakened to $\overline{f}$ (Ψ in [9]) being a locally open and locally convex map, as the use of the Reeb quotient is avoided in [9]. Via the use of the Reeb quotient, the special case of [9, Theorem 6] treated in Lemma 3.46 implies the full statement of [9, Theorem 15] and its analogue with the properness requirement on Ψ and the Hausdorffness requirement on X weakened to closedness and strengthened to normality, respectively; see Exercise 3.48. By Corollary 3.47, the continuous map

$$\overline{f}\colon X_f \longrightarrow f(X)=\overline{f}(X_f)$$

is a convex homeomorphism. This in particular implies that $f(X)\subset V$ is a convex subset. Since the Reeb quotient projection $q_f: X \longrightarrow X_f$ has connected fibers by the definition of X_f (resp. is an open map by Exercise 3.44(b)), it follows that the map

$$f=\overline{f}\circ q_f\colon X \longrightarrow f(X)$$

has connected fibers (resp. is open). This establishes (1) in Proposition 3.33, which implies (2) immediately and (3) via Exercise 3.48.

For a topological space X and $A\subset X$, a connected component of A is a maximal connected subset. Every connected component of A is closed in A. In particular, if A is closed in X, then each of its connected components is also closed in X. The connected components of A partition X. For $x\in A$, we denote by $A_x \subset A$ the connected component of A containing x. The Reeb space of a continuous map $f\colon X \longrightarrow V$ between topological spaces is the quotient topological space

$$X_f \equiv X/\sim, \qquad x \sim x' \ \text{ if } \ f(x) = f(x') \in V, \ \big(f^{-1}(f(x))\big)_x = \big(f^{-1}(f(x))\big)_{x'} \subset X. \tag{3.28}$$

We denote by $q_f : X \longrightarrow X_f$ the quotient projection. The continuous map $\overline{f} : X_f \longrightarrow V$ induced by f is injective if and only if the fibers of f are connected.

Exercises 3.36-3.40 below are (somewhat peculiar) statements concerning basic notions in point-set topology; these statements are used in the proof of Proposition 3.33. Exercises 3.41 and 3.42 note some implications of a map being convex and LFC, respectively. Exercises 3.43 and 3.44 concern the Reeb quotient of a map. The former greatly streamlines [8, Lemma 2.16(ii)]; it is not actually needed for the proof of Proposition 3.33. The latter exercise generalizes Remark 2.12, Lemma 2.16(i), and the implication of Remark 2.22 (in the second sentence after this remark) in [8].

Exercise 3.36. Show that a topological space Y is connected if and only if for every open cover $\mathcal{A}$ of Y and for every pair of points $y, y' \in Y$ there exist $U_0, U_1, \ldots, U_k \in \mathcal{A}$ for some $k \in \mathbb{Z}^+$ such that $y \in U_0$, $y' \in U_k$, and $U_{i-1} \cap U_i \neq \emptyset$ for every $i \in [k]$.

Exercise 3.37. Let $f\colon U \longrightarrow V$ be an open map between topological spaces with connected fibers. Suppose V is locally connected. Show that for every $x \in U$ there exists a connected neighborhood $U' \subset U$ of x such that the fibers of the restriction $f|_{U'}$ are also connected.

Exercise 3.38. Let $f\colon X \longrightarrow V$ be a continuous map between topological spaces. Show that the following are equivalent:

(a) the map f is closed;

(b) for every $B \subset V$ and every neighborhood $U \subset X$ of $f^{-1}(B)$, there exists a neighborhood $W \subset V$ of B such that $f^{-1}(W) \subset U$;

(c) for every $v \in V$ and every neighborhood $U \subset X$ of $f^{-1}(v)$, there exists a neighborhood $W \subset V$ of v such that $f^{-1}(W) \subset U$.

Exercise 3.39. Let $f\colon X \longrightarrow V$ be a proper map between topological spaces, i.e. f is continuous and closed and has compact fibers. Show that the subset $f^{-1}(K) \subset X$ is compact for every compact subset $K \subset V$. *Hint:* use Exercise 3.38.

Exercise 3.40 (Vainstein's Lemma)**.** Let $f\colon X \longrightarrow V$ be a closed continuous map between metrizable topological spaces and $v \in V$. Show that the subset

$$f^{-1}(v) \cap \overline{X - f^{-1}(v)} \subset X$$

is (sequentially) compact. *Note:* it is sufficient to assume that V is just first countable, instead of metrizable, provided $f^{-1}(v)$ in the first appearance above is replaced by its closure in X; the statement of Vainstein's Lemma in [8, Lemma 2.24] is missing the assumption that f is continuous.

Exercise 3.41. Let $f\colon X \longrightarrow V$ be a convex map from a topological space to a vector space. Show that the fibers of f are path-connected.

Exercise 3.42. Let $f: X \longrightarrow V$ be an LFC map between topological spaces. Suppose $v \in V$ is a closed point and $A \subset X$ is a component of $f^{-1}(v)$. Show that A and $f^{-1}(v) - A$ are closed subsets of X. *Note:* the use of first countability of X in the first paragraph of the proof of [8, Lemma 2.21(i)] is unnecessary.

Exercise 3.43. Let $f: X \longrightarrow V$ be a continuous map between topological spaces. Suppose V is Hausdorff and the fibers of f are connected. Show that the Reeb space X_f is Hausdorff.

Exercise 3.44. Let $f: X \longrightarrow V$ be a locally open map between topological spaces.

(a) Show that $[f]_x = [f]_{x'}$ for all $x, x' \in X$ with $q_f(x) = q_f(x')$. *Hint:* use Exercise 3.36.

(b) Suppose in addition that f is LFC. Show that for all $x, x' \in X$ with $q_f(x) = q_f(x')$ there exists an open subset $U_{xx'} \subset X$ containing x, x' so that the fibers of $f|_{U_{xx'}}$ are connected and that the Reeb quotient map q_f is open.

(c) Suppose f is locally convexly open. Show that the Reeb space X_f is locally path-connected.

The conclusion of Lemma 3.45(1) below is the same as of [8, Lemma 2.21], but with much weaker assumptions; its proof is simpler as well. The conclusion of Lemma 3.45(2) is the same as of [8, Remark 2.22], but with the locally cone-open assumption (i.e. admitting local convexity data) weakened to locally convexly open.

Lemma 3.45. *Let $f: X \longrightarrow V$ be an LFC map from a normal topological space. Suppose f is closed onto its image.*

(1) The Reeb quotient map q_f is closed, and the Reeb space X_f is normal.

(2) If in addition f is locally convexly open, then for every $z \in X_f$ there exists a neighborhood $U_z^\dagger \subset X_f$ of z such that $\overline{f}(U_z^\dagger) \subset V$ is a convex subset and the restriction

$$\overline{f}|_{U_z^\dagger}: U_z^\dagger \longrightarrow \overline{f}(U_z^\dagger)$$

is a homeomorphism.

Proof. (1) We use Exercise 3.38. Let $x \in X$,

$$A_x \equiv q_f^{-1}\big(q_f(x)\big) \subset f^{-1}\big(f(x)\big) \subset X$$

be the connected component of $f^{-1}(f(x))$ containing x, and $U \subset X$ be a neighborhood of A_x. By Exercise 3.42, A_x and $A_x' \equiv f^{-1}(f(x)) - A_x$ are closed subsets of X. Let $W, W' \subset X$ be disjoint neighborhoods of A_x, A_x', respectively; it can be assumed that $W \subset U$. Since $f^{-1}(f(x)) \subset W \cup W'$ and f is a closed map, there exists a neighborhood $V' \subset V$ of $f(x)$ so that $f^{-1}(V') \subset W \cup W'$. Since a connected component of any fiber of f intersecting $f^{-1}(V') \cap W$ is contained in $f^{-1}(V') \cap W$,

$$q_f^{-1}\big(q_f\big(f^{-1}(V') \cap W\big)\big) = f^{-1}(V') \cap W \subset U \subset X.$$

In particular, $q_f\big(f^{-1}(V') \cap W\big) \subset X_f$ is a neighborhood of $q_f(x)$. Thus, the Reeb quotient map q_f is closed. This in turn implies that the Reeb space X_f is normal; see [46, Lemma 73.3].

(2) Let $x \in X$ and $z = q_f(x)$. Let $U_x \subset X$ be a neighborhood of x so that $f(U_x) \subset V$ is a convex subset and the restriction

$$f|_{U_x} : U_x \longrightarrow f(U_x)$$

is a surjective open map with connected fibers. By Exercise 3.44(b),

$$U_z^\dagger \equiv q_f(U_x) \subset X_f$$

is a neighborhood of z. Thus, $\overline{f} : U_z^\dagger \longrightarrow \overline{f}(U_z^\dagger) = f(U_x)$ is a bijective open continuous map, i.e. a homeomorphism. □

Let $(V, |\cdot|)$ be a normed vector space. The length of a (continuous) path $\gamma\colon [a,b] \longrightarrow V$ with $a \le b$ is defined to be

$$\ell(\gamma) \equiv \sup\bigg\{ \sum_{i=1}^{k} \big|\gamma(t_i) - \gamma(t_{i-1})\big| : k \in \mathbb{Z}^+,\ a = t_0 \le t_1 \le \ldots \le t_k = b \bigg\}.$$

If in addition $Z\colon Y \longrightarrow V$ is a continuous map from a path-connected topological space, define

$$d_g\colon Z^2 \longrightarrow [0,\infty], \quad d_g(z,z') = \inf\big\{\ell\big(g \circ \gamma\big) \big|\ \gamma\colon [0,1] \longrightarrow Y \text{ is a path, } \gamma(0) = z,\ \gamma(1) = z'\big\}. \tag{3.29}$$

It is immediate that d_f vanishes on the diagonal in Z^2, is symmetric in the two inputs, and satisfies the triangle inequality. Furthermore,

$$d_g(z,z') \ge \big|g(z) - g(z')\big| \qquad \forall\ z, z' \in Z. \tag{3.30}$$

The first part of Lemma 3.46 below generalizes [8, Proposition 2.23], using essentially the same argument. The second part of this lemma systematizes the main conclusion of the proof of [8, Theorem 2.28], but with the properness of $g = \overline{f}$ into the entire vector space V relaxed to properness into a convex subset of V; this is needed to establish the claim of [8, Remark 2.29]. However, the proof of Lemma 3.46(2) is based on the first part of the proof of [9, Theorem 15] at the end of [9, Section 6].

Lemma 3.46. *Let $g\colon Z \longrightarrow V$ be a continuous map from a connected regular topological space to a finite-dimensional vector space. Suppose g is closed onto its image.*

(1) If for every $z \in Z$ there exists a neighborhood $U_z \subset Z$ of z such that $g(U_z) \subset V$ is a convex subset and the restriction

$$g|_{U_z} : U_z \longrightarrow g(U_z)$$

is a homeomorphism, then d_g is a metric on Z inducing the topology of Z.

(2) If in addition there exists a convex subset $V' \subset V$ so that $g(Z) \subset V'$ and the map $g\colon Z \longrightarrow V'$ is proper, then g is a convex homeomorphism onto its image.

Proof. (1) Since Z is connected and locally path-connected, it is path-connected. Fix a norm $|\cdot|$ on V induced by an inner-product; this property is used just after (3.34) below. For $v \in V$ and $\delta \in \mathbb{R}^+$, let

$$B_\delta(v) \equiv \big\{v' \in V : |v - v'| < \delta\big\}$$

denote the open ball in V of radius δ centered at v. Let d_g be as in (3.29). By the assumptions on U_z,

$$d_g(z',z'') = \big|g(z')-g(z'')\big| \qquad \forall\, z',z''\in U_z \tag{3.31}$$

and $d_g(z,z')<\infty$ for all $z,z'\in Z$.

Let $z\in Z$. Since $g^{-1}(g(z))\subset Z$ is a discrete collection of points, $g^{-1}(g(z))-\{z\}\subset Z$ is a closed subset by Exercise 3.42. Let $W,W'\subset Z$ be disjoint neighborhoods of z and $g^{-1}(g(z))-\{z\}$, respectively, and $U\subset W$ be any neighborhood of z. By Exercise 3.38, there exists $\delta\in\mathbb{R}^+$ so that

$$g^{-1}\big(B_\delta\big(g(z)\big)\big) \subset U\cup W' \subset Z\,.$$

Along with (3.30), this gives

$$\begin{aligned} &d_g(z,z') \ge \big|g(z)-g(z')\big| > 0 && \forall\, z'\in g^{-1}\big(B_\delta\big(g(z)\big)\big)\cap U \qquad \text{and}\\ &d_g(z,z') \ge \delta && \forall\, z'\in Z-g^{-1}\big(B_\delta\big(g(z)\big)\big)\cap U\,. \end{aligned}$$

Thus, d_g is a metric and

$$B_\delta(z) \equiv \big\{z'\in Z\colon d_g(z,z')<\delta\big\} \subset g^{-1}\big(B_\delta\big(g(z)\big)\big)\cap U \subset U. \tag{3.32}$$

The latter implies that the induced metric topology on Z is finer than the topology on Z. If $U\subset U_z$, then

$$B_{\delta'}(z) = g^{-1}\big(B_{\delta'}\big(g(z)\big)\big) \quad \forall\ \delta'\in(0,\delta)$$

by (3.32) and (3.31). It follows the open balls with respect to the metric d_g are open in the topology on Z.

(2) Let $z_0,z_1\in Z$ and $d=d(z_0,z_1)\in\mathbb{R}^{\ge0}$. Suppose first that Z is compact. For each $i\in\mathbb{Z}^+$, choose a path

$$\gamma_i\colon \big([0,1],0,1\big) \longrightarrow \big(Z,z_0,z_1\big) \qquad \text{s.t.} \quad \ell\big(g\circ\gamma_i\big) \le d+\frac{1}{i}, \quad \ell\big(g\circ\gamma_i|_{[0,1/2]}\big) = \frac{1}{2}\ell\big(g\circ\gamma_i\big).$$

Since Z is compact, a subsequence of the sequence $\gamma_i(1/2)$ converges to a point

$$z_{1/2}\in Z \qquad \text{s.t.} \quad d_g\big(z_0,z_{1/2}\big), d_g\big(z_{1/2},z_1\big) = d/2.$$

Applying the same procedure to the pairs $(z_0,z_{1/2})$ and $(z_{1/2},z_1)$ of points in Z, we obtain points

$$z_{1/4},z_{3/4}\in X_f \qquad \text{s.t.} \quad d_g\big(z_{(k-1)/4},z_{k/4}\big) = d/4 \ \ \forall\ k=1,2,3,4.$$

Continuing inductively in this way, we obtain points

$$\begin{aligned} z_{k/2^m}\in Z \ \text{ with } m=0,1,\ldots,\ k=0,1,\ldots,2^m \qquad &\text{s.t.}\\ d_g\big(z_{(k-1)/2^m},z_{k/2^m}\big) = d/2^m \ \ \forall\ k\in[2^m].& \end{aligned} \tag{3.33}$$

Since the set of points $k/2^m$ with k,m as above is dense in $[0,1]$, (3.33) implies that there is a unique continuous path

$$\gamma\colon \big([0,1],0,1\big) \longrightarrow \big(Z,z_0,z_1\big) \qquad \text{s.t.} \quad \gamma(k/2^m) = z_{k/2^m} \ \ \forall\ \ m=0,1,\ldots,\ k=0,1,\ldots,2^m.$$

By (3.33), this path satisfies

$$d_g\big(\gamma(t),\gamma(t')\big) = d|t-t'| \quad \forall\, t\in[0,1].$$

Combining this with (3.31), we obtain

$$\big|g\big(\gamma(t)\big)-g\big(\gamma(t')\big)\big| = d|t-t'| \quad \forall\, t\in[0,1], \tag{3.34}$$

i.e. $g\circ\gamma\colon [0,1]\longrightarrow V'$ is a constant-speed parametrization of the line segment from $g(\gamma(0))=g(z_0)$ to $g(\gamma(1))=g(z_1)$.

Suppose now that Z is arbitrary. Let $\gamma\colon ([0,1],0,1)\longrightarrow(Z,z_0,z_1)$ be any path from z_0 to z_1 and

$$K \equiv \mathrm{CH}\big(g\big(\gamma([0,1])\big)\big) \subset V' \subset V$$

be the convex hull of the image of $g\circ\gamma$. Since K is compact and g is proper as a map to V', $g^{-1}(K)\subset Z$ is a compact Hausdorff space by Exercise 3.39 and thus so is its connected component Z' containing z_0; this component also contains z_1. For every $y\in g^{-1}(K)$, the restriction

$$g\colon U_y'\equiv U_y\cap g^{-1}(K) \longrightarrow g(U_y')=g(U_y)\cap K$$

is a homeomorphism onto a convex subset of $V'\subset V$. Thus, $U_y'\equiv U_y\cap Z'$ for every $y\in Z'$. By the previous paragraph, there exists a path

$$\gamma\colon [0,1] \longrightarrow Z' \subset Z$$

such that $g\circ\gamma : [0,1] \longrightarrow V'$ is a constant-speed parametrization of the line segment from $g(z_0)$ to $g(z_1)$. Thus, the map g is convex. □

We now merge Lemmas 3.45 and 3.46 and complete the proof of Proposition 3.33(1). The argument showing that the map $\overline{f}\colon X_f\longrightarrow V'$ is proper in the proof of Corollary 3.47 below is as in the first part of the proof of [8, Proposition 2.27]. This conclusion is the same as the first claim of [8, Proposition 2.27], but with the locally cone-open assumption (i.e. admitting local convexity data) weakened to locally convexly open and the assumption on the map f weakened from the closedness into the entire vector space V to closedness into a convex subset; the latter is needed to establish the claim of [8, Remark 2.29]. The second claim of [8, Proposition 2.27] is not useful if the assumption on the map f in [8, Theorem 2.28] is weakened in this way.

Corollary 3.47. *Let X be a connected, normal topological space, $V'\subset V$ be a convex subset of a finite-dimensional vector space, and $f : X \longrightarrow V'$ be a closed LFC map. If f is locally convexly open, then $\overline{f}\colon X_f\longrightarrow V'$ is a proper convex map which is a homeomorphism onto its image.*

Proof. By Lemma 3.45(1), X_f is a regular topological space and the map $\overline{f}\colon X_f\longrightarrow V'$ is closed. By Lemma 3.45(2), this map also satisfies the local condition of Lemma 3.46(1). Thus, $d_{\overline{f}}$ is a metric on X_f inducing the topology of X_f and $\overline{f}^{-1}(v)\subset X_f$ is a discrete collection of points for any $v\in V$. If $|\overline{f}^{-1}(v)|>1$, then the connectedness of X_f and Exercise 3.40 imply that

$$\overline{f}^{-1}(v) = \overline{f}^{-1}(v) \cap \overline{X_f-\overline{f}^{-1}(v)} \subset X_f$$

is a compact subset. Thus, the map $\overline{f}\colon X_f\longrightarrow V'$ is a proper map and is a convex homeomorphism onto its image by Lemma 3.46(2). □

Exercise 3.48. Let X be a connected, normal topological space, $V' \subset V$ be a convex subset of a finite-dimensional vector space, and $f\colon X \longrightarrow V'$ be a closed map. Suppose f is locally open and locally convex. Show that

(a) the space X and the fibers of f are path-connected;

(b) the map f is convex.

Hint for (b): for each $x \in X$, choose a neighborhood $U_x \subset X$ on which f is open and convex; cover a path in the Reeb quotient X_f by the images of finitely many of these neighborhoods under the Reeb quotient map q_f.

Chapter 4

Symplectic Quotient and Cut Constructions

This chapter describes two of the main operations on symplectic manifolds: the symplectic quotient construction and the sequent symplectic cut construction introduced in [43] and [36], respectively. The latter construction comes in related versions that apply in two different settings: one with a Hamiltonian torus action on an entire symplectic manifold and the other with a Hamiltonian circle action on a neighborhood of a closed real hypersurface in a symplectic manifold; see Sections 4.2 and 4.5, respectively. The first version is in fact a special case of the second version repeated several times. In Section 4.3, we describe a right inverse for the first version of the symplectic cut construction, implementing the sketch in [42]; an inverse for the second version of this construction is described in Chapter 5. As shown in Section 4.4, the first version of the symplectic cut construction and its inverse lead to an efficient proof of Delzant's Theorem identifying certain types of Hamiltonian torus-manifolds with their moment polytopes. In Sections 4.6 and 4.8, we describe two applications of the symplectic cut construction of Section 4.5: a symplectic blowup along a symplectic submanifold and an excision of a Lagrangian submanifold of a certain kind, respectively.

4.1 Symplectic quotient

For a Lie group G, let

$$\big(T^*_{\mathbb{1}}G\big)^G \equiv \big\{\alpha\in T^*_{\mathbb{1}}G\colon \mathrm{Ad}^*_g(\alpha)=\alpha \ \forall\, g\in G\big\}$$

be the fixed locus of the dual of the adjoint action of G on $T_{\mathbb{1}}G$. If ψ is a G-action on a space X, $\mu\colon X\longrightarrow T_{\mathbb{1}}G$ is a map satisfying the second condition in (1.6), and $\alpha\in(T^*_{\mathbb{1}}G)^G$, then ψ restricts to a G-action on $\mu^{-1}(\alpha)\subset X$. If G is abelian, then $(T^*_{\mathbb{1}}G)^G=T^*_{\mathbb{1}}G$.

Theorem 4.1 ([43, Theorems 3,4]). *Suppose G is a compact Lie group, $(\widetilde{X},\widetilde{\omega},\widetilde{\psi},\widetilde{\mu})$ is a Hamiltonian G-manifold, and $\alpha\in(T^*_{\mathbb{1}}G)^G$ is such that G acts freely on $\widetilde{\mu}^{-1}(\alpha)$. Then,*

(SQ0) $\alpha\in T^*_{\mathbb{1}}G$ is a regular value of $\widetilde{\mu}$;

(SQ1) there is a unique smooth structure on $X_\alpha\equiv\widetilde{\mu}^{-1}(\alpha)/G$ so that the quotient projection

$$p_\alpha\colon \widetilde{\mu}^{-1}(\alpha)\longrightarrow X_\alpha$$

is a principal G-bundle;

(SQ2) there exists a unique 2-form ω_α on X_α so that $p_\alpha^*\omega_\alpha = \widetilde{\omega}|_{T\widetilde{\mu}^{-1}(\alpha)}$;

(SQ3) the 2-form ω_α is symplectic.

Suppose in addition that G' is another Lie group and $(\widetilde{X}, \widetilde{\omega}, \widetilde{\psi}', \widetilde{\mu}')$ is a Hamiltonian G'-manifold. If the actions $\widetilde{\psi}$ and $\widetilde{\psi}'$ commute, $\widetilde{\mu}'$ is $\widetilde{\psi}$-invariant, and $\widetilde{\mu}$ is $\widetilde{\psi}'$-invariant, then

(SQ4) $\widetilde{\psi}'$ and $\widetilde{\mu}'$ descend to a G'-action ψ'_α on X_α and a smooth map $\mu'_\alpha\colon X_\alpha \longrightarrow T_{\mathbb{1}}G'$, respectively, so that $(X_\alpha, \omega_\alpha, \psi'_\alpha, \mu'_\alpha)$ is a Hamiltonian G'-manifold;

(SQ5) if $\alpha' \in (T_{\mathbb{1}}^*G')^{G'}$, then G' acts freely on $\mu'^{-1}_\alpha(\alpha')$ if and only if $G \times G'$ acts freely on $(\widetilde{\mu}, \widetilde{\mu}')^{-1}(\alpha, \alpha')$;

(SQ6) if $\alpha' \in (T_{\mathbb{1}}^*G')^{G'}$ and $G \times G'$ acts freely on $(\widetilde{\mu}, \widetilde{\mu}')^{-1}(\alpha, \alpha')$, then

$$p_{(\alpha,\alpha')} = p_{\alpha'} \circ p_\alpha\big|_{\widetilde{\mu}^{-1}(\alpha) \cap \widetilde{\mu}'^{-1}(\alpha')} \colon (\widetilde{\mu}, \widetilde{\mu}')^{-1}(\alpha, \alpha') \longrightarrow X_{(\alpha,\alpha')} = \big(X_\alpha\big)_{\alpha'};$$

(SQ7) if $\alpha' \in (T_{\mathbb{1}}^*G')^{G'}$, $G \times G'$ acts freely on $(\widetilde{\mu}, \widetilde{\mu}')^{-1}(\alpha, \alpha')$, and G' is compact, then

$$\omega_{(\alpha,\alpha')} = (\omega_\alpha)_{\alpha'}.$$

The symplectic manifold $(X_\alpha, \omega_\alpha)$ of Theorem 4.1 is called the symplectic quotient of $(\widetilde{X}, \widetilde{\omega}, \widetilde{\psi}, \widetilde{\mu})$ at α. We will similarly call the Hamiltonian G'-manifold $(X_\alpha, \omega_\alpha, \psi'_\alpha, \mu'_\alpha)$ of this theorem the Hamiltonian quotient of $(\widetilde{X}, \widetilde{\omega}, (\widetilde{\psi}, \widetilde{\psi}'), (\widetilde{\mu}, \widetilde{\mu}'))$ at α. If $G = \mathbb{T}^k$ is the standard torus, then $\widetilde{\mu}$ and α correspond to a Hamiltonian $\widetilde{H}\colon \widetilde{X} \longrightarrow \mathbb{R}^k$ and $c \in \mathbb{R}^k$. In such a case, we may also denote the symplectic manifold $(X_\alpha, \omega_\alpha)$ by (X_c, ω_c) and call it the symplectic quotient of $(\widetilde{X}, \widetilde{\omega}, \widetilde{\psi}, \widetilde{H})$ at c.

Proof of Theorem 4.1. Exercise 2.21(b) establishes (SQ0). Since G is compact, the quotient projection p_α is a closed map. By [46, Lemma 73.3], the quotient space X_α is thus Hausdorff. By (SQ0) and the Implicit Function Theorem, $\widetilde{\mu}^{-1}(\alpha) \subset \widetilde{X}$ is a smooth submanifold on which the compact Lie G acts smoothly and freely. By the Slice Theorem (equivariant version of the Tubular Neighborhood Theorem, as in the proof of Proposition 2.7(2)), for every $x \in \widetilde{\mu}^{-1}(\alpha)$ there thus exists a submanifold $S_x \subset \widetilde{X}$ so that $x \in S_x$ and the map

$$S_x \times G \longrightarrow \widetilde{X}, \qquad (x', u) \longrightarrow \psi_u(x'),$$

is a diffeomorphism onto a neighborhood $\widetilde{U}_x \subset \widetilde{X}$ of x preserved by G. This submanifold is then transverse to the orbits Gx' of G and thus to $\widetilde{\mu}^{-1}(\alpha)$. The restriction

$$p_\alpha\colon \widetilde{\mu}^{-1}(\alpha) \cap S_x \longrightarrow p_\alpha\big(\widetilde{\mu}^{-1}(\alpha) \cap \widetilde{U}_x\big) \subset X_\alpha$$

of the quotient map is a homeomorphism onto an open subset of X_α and induces a smooth structure on $p_\alpha(\widetilde{\mu}^{-1}(\alpha) \cap \widetilde{U}_x)$ so that the restriction

$$p_\alpha\colon \widetilde{\mu}^{-1}(\alpha) \cap \widetilde{U}_x \longrightarrow p_\alpha\big(\widetilde{\mu}^{-1}(\alpha) \cap \widetilde{U}_x\big)$$

is a (trivial) principle G-bundle. Since there is at most one smooth structure on $p_\alpha(\widetilde{\mu}^{-1}(\alpha) \cap \widetilde{U}_x)$ so that the latter restriction is a submersion, the smooth structures on open subset of X_α obtained in this way overlap smoothly. This establishes (SQ1).

By (SQ1), for every $x\in\widetilde{\mu}^{-1}(\alpha)$ the map

$$\mathrm{d}_x p_\alpha\colon T_x\widetilde{\mu}^{-1}(\alpha)/T_x(Gx)\longrightarrow T_{p_\alpha(x)}X_\alpha$$

is a well-defined isomorphism. For each $v\in T_{\mathbb{1}}G$, let $\zeta_v\in\Gamma(\widetilde{X};T\widetilde{X})$ be as in (1.2) with (X,ψ) replaced by $(\widetilde{X},\widetilde{\psi})$. Thus,

$$T_x(Gx)=\big\{\zeta_v(x)\colon v\in T_{\mathbb{1}}G\big\}\quad\forall\,x\in\widetilde{\mu}^{-1}(\alpha)\qquad\text{and}$$
$$-\big(\iota_{\zeta_v(x)}\widetilde{\omega}\big)\big|_{T_x\widetilde{\mu}^{-1}(\alpha)}=\mathrm{d}_x\big(\{\widetilde{\mu}(\cdot)\}(v)\big|_{T_x\widetilde{\mu}^{-1}(\alpha)}\big)=\mathrm{d}_x\big(\alpha(v)\big|_{T_x\widetilde{\mu}^{-1}(\alpha)}\big)=0\quad\forall\,v\in T_{\mathbb{1}}G,\,x\in\widetilde{\mu}^{-1}(\alpha).$$

It follows that there is a unique alternating 2-tensor $\omega_\alpha|_{p_\alpha(x)}$ on $T_{p_\alpha(x)}X_\alpha$ for each $x\in\widetilde{\mu}^{-1}(\alpha)$ so that

$$\omega_\alpha|_{p_\alpha(x)}\big(\mathrm{d}_x p_\alpha(w),\mathrm{d}_x p_\alpha(w')\big)=\widetilde{\omega}|_x(w,w')\quad\forall\,w,w'\in T_x\widetilde{\mu}^{-1}(\alpha),$$

i.e. $p_\alpha^*(\omega_\alpha|_{p_\alpha(x)})=\widetilde{\omega}|_{T_x\widetilde{\mu}^{-1}(\alpha)}$. Since ω is G-invariant, $\omega_\alpha|_{p_\alpha(x)}$ does not depend on the choice of x in $p_\alpha^{-1}(p_\alpha(x))$, i.e. ω_α is a well-defined 2-form on X_α. Since p_α is a submersion and the 2-form $\widetilde{\omega}|_{T\widetilde{\mu}^{-1}(\alpha)}$ is smooth and closed, so is the 2-form ω_α. Since α is a regular value of $\widetilde{\mu}$ and $\widetilde{\omega}$ is nondegenerate on $\widetilde{X}$, the first statement of Exercise 2.21 with (ω,μ) replaced by $(\widetilde{\omega},\widetilde{\mu})$ implies that

$$\big(T_x\widetilde{\mu}^{-1}(\alpha)\big)^{\widetilde{\omega}}=\big(\ker\mathrm{d}_x\widetilde{\mu}\big)^{\widetilde{\omega}}=T_x(Gx).$$

Thus, ω_α is nondegenerate.

Suppose G' is another Lie group and $(\widetilde{X},\widetilde{\omega},\widetilde{\psi}',\widetilde{\mu}')$ is a Hamiltonian G'-manifold such that the actions $\widetilde{\psi}$ and $\widetilde{\psi}'$ commute, $\widetilde{\mu}'$ is ψ-invariant, and $\widetilde{\mu}$ is ψ'-invariant. Since $\widetilde{\mu}$ is $\widetilde{\psi}'$-invariant, $\widetilde{\psi}'$ preserves $\widetilde{\mu}^{-1}(\alpha)\subset X$. Since the actions $\widetilde{\psi}$ and $\widetilde{\psi}'$ commute and $\widetilde{\mu}'$ is $\widetilde{\psi}$-invariant, the restriction of $\widetilde{\psi}'$ and $\widetilde{\mu}'$ to $\widetilde{\mu}^{-1}(\alpha)$ descend to a G'-action ψ'_α on X_α and a smooth map $\mu'_\alpha\colon X_\alpha\longrightarrow T_{\mathbb{1}}G'$. By Exercise 1.22 with $(\widetilde{\psi},\psi,\widetilde{\mu},\mu)$ replaced by $(\widetilde{\psi}',\psi',\widetilde{\mu}',\mu'_\alpha)$ and $\widetilde{Y}=\widetilde{\mu}^{-1}(\alpha)$, $(X_\alpha,\omega_\alpha,\psi'_\alpha,\mu'_\alpha)$ is thus a Hamiltonian G'-manifold. Since G acts freely on $\widetilde{\mu}^{-1}(\alpha)$, (SQ5) follows from

$$\mu_\alpha^{-1}(\alpha')=(\widetilde{\mu},\widetilde{\mu}')^{-1}(\alpha,\alpha')/G.$$

Along with (SQ1) and (SQ2), this identity also implies (SQ6) and (SQ7). $\square$

Exercise 4.2. Suppose G, $(\widetilde{X},\widetilde{\omega},\widetilde{\psi},\widetilde{\mu})$, α, (X_α,ω_α), and p_α are as in Theorem 4.1 and $\widetilde{X}'\subset\widetilde{X}$ is an $\widetilde{\omega}$-symplectic submanifold preserved by the G-action $\widetilde{\psi}$.

(a) Show that the symplectic quotient $(X'_\alpha,\omega'_\alpha)$ of $(\widetilde{X}',\widetilde{\omega}|_{\widetilde{X}'},\widetilde{\psi}|_{\widetilde{X}'},\widetilde{\mu}|_{\widetilde{X}'})$ at α is a symplectic submanifold of (X_α,ω_α) and the bundle homomorphisms

$$\mathcal{N}_{\widetilde{X}}\widetilde{X}'\big|_{\widetilde{\mu}^{-1}(\alpha)\cap\widetilde{X}'}\longleftarrow\mathcal{N}_{\widetilde{\mu}^{-1}(\alpha)}\big(\widetilde{\mu}^{-1}(\alpha)\cap\widetilde{X}'\big)\xrightarrow{\mathrm{d}p_\alpha}\big\{p_\alpha|_{\widetilde{\mu}^{-1}(\alpha)\cap\widetilde{X}'}\big\}^*\mathcal{N}_{X_\alpha}X'_\alpha\tag{4.1}$$

over $\widetilde{\mu}^{-1}(\alpha)\cap\widetilde{X}'$ induced by the inclusions and the quotient projection are isomorphisms.

(b) Suppose in addition that G', $\widetilde{\psi}'$, $\widetilde{\mu}'$, and $(X_\alpha,\omega_\alpha,\psi'_\alpha,\mu'_\alpha)$ are as in Theorem 4.1(SQ6) and the submanifold $\widetilde{X}'\subset\widetilde{X}$ is preserved by the G'-action $\widetilde{\psi}'$. Show that the submanifold $X'_\alpha\subset X_\alpha$ is preserved by the G'-action ψ', $(X'_\alpha,\omega'_\alpha,\psi'_\alpha|_{X'_\alpha},\mu'_\alpha|_{X'_\alpha})$ is the Hamiltonian quotient of

$$\big(\widetilde{X}',\widetilde{\omega}|_{\widetilde{X}'},(\widetilde{\psi},\widetilde{\psi}')|_{\widetilde{X}'},(\widetilde{\mu},\widetilde{\mu}')|_{\widetilde{X}'}\big)$$

at α, and the bundle isomorphisms in (4.1) are G'-equivariant.

Exercise 4.3. Suppose G, $(\widetilde{X},\widetilde{\omega},\widetilde{\psi},\widetilde{\mu})$, α, (X_α,ω_α), and p_α are as in Theorem 4.1 and $\widetilde{J}$ is a $\widetilde{\psi}$-invariant almost complex structure on $\widetilde{X}$ compatible with $\widetilde{\omega}$. Show that

(a) the restriction of the differential

$$\mathrm{d}_x p_\alpha\colon T_x\widetilde{\mu}^{-1}(\alpha)\cap\widetilde{J}\big(T_x\widetilde{\mu}^{-1}(\alpha)\big) \longrightarrow T_{p_\alpha(x)}X_\alpha$$

is an isomorphism for every $x\in\widetilde{\mu}^{-1}(\alpha)$ and thus induces an almost complex structure J_α on X_α compatible with ω_α;

(b) if G', $\widetilde{\psi}'$, $\widetilde{\mu}'$, and $(X_\alpha,\omega_\alpha,\psi'_\alpha,\mu'_\alpha)$ are also as in Theorem 4.1 and the almost complex structure $\widetilde{J}$ on $\widetilde{X}$ is $\widetilde{\psi}'$-invariant as well, then the almost complex structure J_α is ψ'_α-invariant;

(c) if $\widetilde{X}'\subset\widetilde{X}$ is an almost complex submanifold preserved by the G-action $\widetilde{\psi}$ and $(X'_\alpha,\omega'_\alpha)$ is the symplectic quotient of $(\widetilde{X}',\widetilde{\omega}|_{\widetilde{X}'},\widetilde{\psi}|_{\widetilde{X}'},\widetilde{\mu}|_{\widetilde{X}'})$ at α, then X'_α is an almost complex submanifold of X_α and the composite isomorphism from the left-hand side in (4.1) to the right-hand side is $\mathbb{C}$-linear with respect to the complex structures induced by $\widetilde{J}$ and J_α.

Hint for (a): use Exercise 2.32(b).

Example 4.4. Let $n\in\mathbb{Z}^+$. By Exercise 1.23,

$$H\colon\mathbb{C}^n\longrightarrow\mathbb{R},\qquad H(z_1,\ldots,z_n)=\pi\sum_{k=1}^n|z_k|^2,$$

is a Hamiltonian for the standard action ψ of S^1 on $\mathbb{C}^n$,

$$\psi_u\colon\mathbb{C}^n\longrightarrow\mathbb{C}^n,\quad \psi_u(z)=uz,\qquad \forall\, u\in S^1\subset\mathbb{C}.$$

For each $r\in\mathbb{R}^+$, the group S^1 acts freely on $H^{-1}(\pi r^2)$, the sphere of radius r centered at the origin. The associated quotient of Theorem 4.1 is the complex projective space $\mathbb{C}P^{n-1}$ with a symplectic form $\omega_{\mathbb{C}P^{n-1};r}$. By Exercise 1.27(b),

$$\omega_{\mathbb{C}P^{n-1};r}=\pi r^2\omega_{\mathrm{FS};n-1}\,.$$

Exercise 4.5. Let $n\in\mathbb{Z}^+$ and $q\colon\mathbb{C}^n-\{0\}\longrightarrow\mathbb{C}P^{n-1}$ be the usual quotient projection. The $\mathbb{C}^*$-action on $\mathbb{C}^n$ by the coordinate multiplication restricts to a $\mathbb{C}^*$-action on $\mathbb{C}^n-\{0\}$ and S^1-actions on $\mathbb{C}^n$ and the unit sphere $S^{2n-1}\subset\mathbb{C}^n$. Show that

(a) the quotient topologies on $\mathbb{C}P^{n-1}$ given by $(\mathbb{C}^n-\{0\})/\mathbb{C}^*$ and S^{2n-1}/S^1 are the same (i.e. the map $S^{2n-1}/S^1\longrightarrow(\mathbb{C}^n-\{0\})/\mathbb{C}^*$ induced by inclusions is a homeomorphism);

(b) $\mathbb{C}P^{n-1}$ is a compact topological $2(n-1)$-manifold that admits a complex structure so that the quotient projections

$$q\colon\mathbb{C}^n-\{0\}\longrightarrow\mathbb{C}P^{n-1}=(\mathbb{C}^n-\{0\})/\mathbb{C}^*\quad\text{and}\quad p\colon S^{2n-1}\longrightarrow\mathbb{C}P^{n-1}=S^{2n-1}/S^1$$

are a holomorphic submersion and a smooth submersion, respectively.

(c) the above complex structure is compatible with the Fubini-Study symplectic form $\omega_{\mathrm{FS};n-1}$ of Exercise 1.27(b).

Exercise 4.6. Let $k,n\in\mathbb{Z}^+$, $\mathrm{Mat}_{n\times k}\mathbb{C}$, U_k, $\mathfrak{u}_k$, $\omega_{n\times k}$, $\psi^{\mathrm{L}}_{n\times k}$, $\psi^{\mathrm{R}}_{n\times k}$, $\mu^{\mathrm{L}}_{n\times k}$, and $\mu^{\mathrm{R}}_{n\times k}$ be as in Exercises 1.26.

(a) Show that $(\mathfrak{i}c\,\mathrm{tr}\colon\mathfrak{u}_k\longrightarrow\mathbb{R})\in(\mathfrak{u}_k^*)^{\mathrm{U}_k}$ for every $c\in\mathbb{R}$ and that $\mu^{\mathrm{R}\,-1}_{n\times k}(\mathfrak{i}c\,\mathrm{tr})\subset\mathrm{Mat}_{n\times k}\mathbb{C}$ is nonempty if and only if $c\in\mathbb{R}^+$.

(b) Suppose $c\in\mathbb{R}^+$. Show that U_k acts freely on $\mu^{\mathrm{R}\,-1}_{n\times k}(\mathfrak{i}c\,\mathrm{tr})$ if and only if $k\le n$.

(c) Suppose $c\in\mathbb{R}^+$ and $k\le n$. Show that the restrictions of $\omega_{n\times k}$, $\psi^{\mathrm{L}}_{n\times k}$, and $\mu^{\mathrm{L}}_{n\times k}$ to $\mu^{\mathrm{R}\,-1}_{n\times k}(\mathfrak{i}c\,\mathrm{tr})$ determine a Hamiltonian U_n-manifold structure on the Grassmannian of k-planes in $\mathbb{C}^n$,

$$\mathbb{G}_{n,k}\mathbb{C}\approx\mu^{\mathrm{R}\,-1}_{n\times k}(\mathfrak{i}c\,\mathrm{tr})/\mathrm{U}_k\,.$$

Exercise 4.7. Let (Y,ω), $\mathrm{pr}_E:E\longrightarrow Y$, Ω, $\mathfrak{i}$, $g^{\Omega}_{\mathfrak{i}}$, ∇, $\widetilde{\omega}$, $\rho\colon Y\longrightarrow\mathbb{R}^+$, $\mathcal{U}_\rho$, and H be as in Exercise 1.30 and $\widetilde{\psi}$ be the restriction of the standard S^1-action on E to $\mathcal{U}_\rho$. Suppose that the function ρ is constant and $c\in(0,\pi\rho)$. Let (X_c,ω_c) be the symplectic quotient of $(\mathcal{U}_\rho,\widetilde{\omega}|_{\mathcal{U}_\rho},\widetilde{\psi},H)$ at c as in Theorem 4.1. Show that

(a) the projection pr_E descends to a fiber bundle $\mathrm{pr}_{E;c}:X_c\longrightarrow Y$ isomorphic to the complex projectivization of the complex vector bundle $(E,\mathfrak{i})$ over Y;

(b) every fiber of $\mathrm{pr}_{E;c}$ is a symplectic submanifold of (X_c,ω_c) isomorphic to $(\mathbb{C}P^{n-1},c\omega_{\mathrm{FS};n-1})$.

4.2 Hamiltonian symplectic cut

This section concerns the symplectic cut construction for Hamiltonian $\mathbb{T}$-manifolds introduced in [36], where $\mathbb{T}$ is a torus. Since this construction intersects the image of the moment map of the input Hamiltonian $\mathbb{T}$-manifold with a polyhedron in $T^*_{\mathbb{1}}\mathbb{T}$, we summarize its key properties in Theorem 4.11 based on the notation associated with such polyhedra in Section 1.6. This turns out to be convenient for implementing the reverse of this construction sketched in [42] and for an efficient proof of Delzant's Theorem; see Sections 4.3 and 4.4, respectively.

Exercise 4.8. Suppose $\mathbb{T}$ is a torus, $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ is finite subcollection, and (X,ω,ψ,μ) is a Hamiltonian $\mathbb{T}$-manifold. Show that

(a) if $\langle\mathscr{H}\rangle^{\partial}\neq\emptyset$, then $\mu^{-1}(\langle\mathscr{H}\rangle^{\partial})\subset X$ is a fiber of a moment map for the restriction of the action ψ to $\mathbb{T}_{\mathscr{H}}\subset\mathbb{T}$;

(b) if $\mathbb{T}_{\mathscr{H}}$ acts freely on $\mu^{-1}(\langle\mathscr{H}\rangle^{\partial})$, then $\mu^{-1}(\langle\mathscr{H}\rangle^{\partial})\subset X$ is a closed submanifold of codimension equal to the dimension of $\mathbb{T}_{\mathscr{H}}$;

(c) if $\mu(X)\subset\langle\mathscr{H}\rangle^{\partial}$, then $\mathbb{T}_{\mathscr{H}}$ acts trivially on X;

(d) if $\mu(X)\subset\langle\mathscr{H}\rangle^{\partial}$, $x\in X$, and $G_x(\psi)=\mathbb{T}_{\mathscr{H}}$, then the differential $\mathrm{d}_x\mu\colon T_xX\longrightarrow T_{\mu(x)}\langle\mathscr{H}\rangle^{\partial}$ is surjective.

Definition 4.9. Let $\mathbb{T}$ be a torus and $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ be a finite subcollection. A Hamiltonian $\mathbb{T}$-manifold (X,ω,ψ,μ) is

- $\mathscr{H}$-cuttable if for every subcollection $\mathscr{H}'\subset\mathscr{H}$ such that $\mu^{-1}(\langle\mathscr{H}'\rangle^{\partial})\neq\emptyset$ the Lie group homomorphism $\Phi_{\mathscr{H}'}$ as in (1.41) is injective and the subtorus $\mathbb{T}_{\mathscr{H}'}\subset\mathbb{T}$ acts freely on $\mu^{-1}(\langle\mathscr{H}'\rangle^{\partial})$;

- $\mathscr{H}$-cut if $\mu(X)\subset\langle\mathscr{H}\rangle$ and for every $\mathscr{H}'\subset\mathscr{H}$ the subspace $Y_{\mathscr{H}'}\equiv\mu^{-1}(\langle\!\langle\mathscr{H}'\rangle^{\partial})$ is a union of topological components of the fixed locus $X^{\mathbb{T}_{\mathscr{H}'}}\subset X$ of $\psi|_{\mathbb{T}_{\mathscr{H}'}}$ and there is a $\mathbb{T}$-equivariant splitting
$$TX|_{Y_{\mathscr{H}'}}=TY_{\mathscr{H}'}\oplus\bigoplus_{\upsilon\in\mathscr{H}'}\mathcal{N}_X^{\upsilon}Y_{\mathscr{H}'}\longrightarrow Y_{\mathscr{H}'}\tag{4.2}$$
of $TX|_{Y_{\mathscr{H}'}}$ with a ψ-invariant complex structure J compatible with ω so that
$$\begin{gathered}\mathrm{rk}_{\mathbb{C}}\mathcal{N}_X^{\upsilon}Y_{\mathscr{H}'}=1\quad\forall\,\upsilon\in\mathscr{H}'\qquad\text{and}\\ \mathrm{d}\psi_{\Phi_{\mathscr{H}'}([(r_\upsilon)_{\upsilon\in\mathscr{H}'}])}(w)=\mathrm{e}^{2\pi\mathrm{i}r_{\upsilon'}}w\quad\forall\,(r_\upsilon)_{\upsilon\in\mathscr{H}'}\in\mathbb{R}^{\mathscr{H}'},\,w\in\mathcal{N}_X^{\upsilon'}Y_{\mathscr{H}'},\,\upsilon'\in\mathscr{H}'.\end{gathered}\tag{4.3}$$

Suppose $\mathbb{T}$ is a torus, $\mathscr{H}'\subset\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ are finite subcollections, and (X,ω,ψ,μ) is a Hamiltonian $\mathbb{T}$-manifold such that $\mu^{-1}(\langle\!\langle\mathscr{H}'\rangle^{\partial})\neq\emptyset$. If (X,ω,ψ,μ) is an $\mathscr{H}$-cuttable, then $\mu^{-1}(\langle\!\langle\mathscr{H}'\rangle^{\partial})\subset X$ is a closed submanifold of codimension $|\mathscr{H}'|$ by Exercise 4.8(b) and thus
$$\mu^{-1}\big(\langle\!\langle\mathscr{H}'\rangle^{\partial}_{\mathscr{H}}\big)=\mu^{-1}\big(\langle\!\langle\mathscr{H}'\rangle^{\partial}\cap\langle\mathscr{H}\rangle\big)-\bigcap_{\mathscr{H}'\subsetneq\mathscr{H}''\subset\mathscr{H}}\mu^{-1}\big(\langle\!\langle\mathscr{H}''\rangle^{\partial}\big)$$
is an open subset. If (X,ω,ψ,μ) is $\mathscr{H}$-cut, then $\mu^{-1}(\langle\!\langle\mathscr{H}'\rangle^{\partial})\subset X$ is a closed ω-symplectic submanifold of codimension $2|\mathscr{H}'|$ and the Lie group homomorphism $\Phi_{\mathscr{H}'}$ as in (1.41) is injective by Proposition 2.15(1), (4.2), and (4.3). Thus, $\mu^{-1}(\langle\!\langle\mathscr{H}'\rangle^{\partial}_{\mathscr{H}})\subset\mu^{-1}(\langle\!\langle\mathscr{H}'\rangle^{\partial})$ is again an open subset; it is dense in this case, since $\mu(X)\subset\langle\mathscr{H}\rangle$. If in addition $\mathscr{H}_1'\subset\mathscr{H}$ and $\mathscr{H}_2'\subset\mathscr{H}-\mathscr{H}_1'$, then the restriction of μ to the symplectic submanifold $\mu^{-1}(\langle\!\langle\mathscr{H}_2'\rangle^{\partial})\subset X$ is transverse to $\langle\mathscr{H}_1'\rangle^{\partial}\subset T_{\mathbb{1}}^*\mathbb{T}$ by (4.2) and (4.3).

Exercise 4.10. Suppose $\mathbb{T}$ is a torus, $\mathscr{H}=\mathscr{H}_1\sqcup\mathscr{H}_2$ is a partition of a finite subcollection of $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$, and (X,ω,ψ,μ) is an $\mathscr{H}$-cuttable Hamiltonian $\mathbb{T}$-manifold. Show that

(a) (X,ω,ψ,μ) is $\mathscr{H}_1$-cuttable;

(b) for all $\mathscr{H}_1'\subset\mathscr{H}_1$ and $\mathscr{H}_2'\subset\mathscr{H}_2$ such that $\mu^{-1}(\langle\!\langle\mathscr{H}_2'\rangle^{\partial})\neq\emptyset$ the Lie group homomorphism $\Phi_{\mathscr{H}_2'}$ as in (1.41) is injective and the subtorus $\mathbb{T}_{\mathscr{H}_2'}\subset\mathbb{T}$ acts freely on $\mu^{-1}(\langle\!\langle\mathscr{H}_1'\sqcup\mathscr{H}_2'\rangle^{\partial})/\mathbb{T}_{\mathscr{H}_1'}$.

Theorem 4.11 ([37, Proposition 2.4])**.** *Suppose $\mathbb{T}$ is a torus, $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ is a finite subcollection, and (X,ω,ψ,μ) is an $\mathscr{H}$-cuttable Hamiltonian $\mathbb{T}$-manifold. There exists a unique $\mathscr{H}$-cut Hamiltonian $\mathbb{T}$-manifold*
$$(X,\omega,\psi,\mu)_{\mathscr{H}}\equiv\big(X_{\mathscr{H}},\omega_{\mathscr{H}},\psi_{\mathscr{H}},\mu_{\mathscr{H}}\big)\tag{4.4}$$
so that

(HC$_1$) $X_{\mathscr{H}}=\mu^{-1}(\langle\!\langle\mathscr{H}\rangle\!\rangle)/\!\sim$ with $x\sim x'$ if there exist $\mathscr{H}'\subset\mathscr{H}$ and $u\in\mathbb{T}_{\mathscr{H}'}$ so that $\mu(x)\in\langle\mathscr{H}'\rangle^{\partial}$ and $x'=\psi_u(x)$;

(HC$_2$) the quotient projection $p_{\mathscr{H}}\colon\mu^{-1}(\langle\!\langle\mathscr{H}\rangle\!\rangle)\longrightarrow X_{\mathscr{H}}$ is $\mathbb{T}$-equivariant and $\mu|_{\mu^{-1}(\langle\!\langle\mathscr{H}\rangle\!\rangle)}=\mu_{\mathscr{H}}\circ p_{\mathscr{H}}$;

(HC$_3$) for every $\mathscr{H}'\subset\mathscr{H}$, $p_{\mathscr{H}}\colon\mu^{-1}(\langle\!\langle\mathscr{H}'\rangle^{\partial}_{\mathscr{H}})\longrightarrow\mu_{\mathscr{H}}^{-1}(\langle\!\langle\mathscr{H}'\rangle^{\partial})$ is a submersion (onto a dense open subset) and
$$\big\{p_{\mathscr{H}}|_{\mu^{-1}(\langle\!\langle\mathscr{H}'\rangle^{\partial}_{\mathscr{H}})}\big\}^*\omega_{\mathscr{H}}=\omega|_{T(\mu^{-1}(\langle\!\langle\mathscr{H}'\rangle^{\partial}_{\mathscr{H}}))}\,.\tag{4.5}$$

For any partition $\mathscr{H}=\mathscr{H}_1\sqcup\mathscr{H}_2$, $(X,\omega,\psi,\mu)_{\mathscr{H}_1}$ is an $\mathscr{H}_2$-cuttable Hamiltonian $\mathbb{T}$-manifold and

$$(X,\omega,\psi,\mu)_{\mathscr{H}} = \big((X,\omega,\psi,\mu)_{\mathscr{H}_1}\big)_{\mathscr{H}_2}. \tag{4.6}$$

Proof. Let $\mathbf{c}=(c_v)_{v\in\mathscr{H}}\in\mathbb{R}^{\mathscr{H}}$ and $\mathrm{pr}_1,\mathrm{pr}_2: X\times\mathbb{C}^{\mathscr{H}}\longrightarrow X,\mathbb{C}^{\mathscr{H}}$ be the component projections. Denote by $\omega_{\mathscr{H}}$ the standard symplectic form on $\mathbb{C}^{\mathscr{H}}$, analogously to (1.19). The 2-form

$$\widetilde{\omega}\equiv \mathrm{pr}_1^*\omega\oplus\mathrm{pr}_2^*\omega_{\mathscr{H}}$$

on $X\times\mathbb{C}^{\mathscr{H}}$ is then symplectic. Let $\widetilde{\psi}$ be the $\mathbb{T}^{\mathscr{H}}$-action on $X\times\mathbb{C}^{\mathscr{H}}$ given by

$$\widetilde{\psi}_{[(r_v)_{v\in\mathscr{H}}]}\big(x,(z_v)_{v\in\mathscr{H}}\big) = \big(\psi_{\Phi_{\mathscr{H}}([(r_v)_{v\in\mathscr{H}}])}(x),\big(\mathrm{e}^{-2\pi\mathrm{i}r_v}z_v\big)_{v\in\mathscr{H}}\big). \tag{4.7}$$

This action commutes with the $\mathbb{T}$-action ψ on the first component and preserves its moment map

$$\mu\circ\mathrm{pr}_1: X\times\mathbb{C}^{\mathscr{H}}\longrightarrow T_{\mathbb{1}}^*\mathbb{T} \tag{4.8}$$

with respect to $\widetilde{\omega}$. By Exercises 1.19 and 1.23, the smooth function

$$\widetilde{H}_\mu: X\times\mathbb{C}^{\mathscr{H}}\longrightarrow T_{\mathbb{1}}^*\mathbb{T}^{\mathscr{H}}=\mathbb{R}^{\mathscr{H}},\quad \widetilde{H}_\mu\big(x,(z_{v,c})_{(v,c)\in\mathscr{H}}\big) = \big(\mu_v(x)-\pi|z_{v,c}|^2\big)_{(v,c)\in\mathscr{H}}, \tag{4.9}$$

is a Hamiltonian for the action $\widetilde{\psi}$ with respect to $\widetilde{\omega}$. It is preserved by the $\mathbb{T}$-action ψ on the first component.

Suppose $(x,(z_v)_{v\in\mathscr{H}})\in\widetilde{H}_\mu^{-1}(\mathbf{c})$, $u\in\mathbb{T}^{\mathscr{H}}$, and $\widetilde{\psi}_u(x,(z_v)_{v\in\mathscr{H}})=(x,(z_v)_{v\in\mathscr{H}})$. Let

$$\mathscr{H}'=\big\{v\in\mathscr{H}: z_v=0\big\}.$$

From (4.9) and (4.7), we then obtain

$$\mu(x)\in\langle\mathscr{H}'\rangle_{\mathscr{H}}^{\partial}\subset T_{\mathbb{1}}^*\mathbb{T}\qquad\text{and}\qquad u\in\mathbb{T}^{\mathscr{H}'}\subset\mathbb{T}^{\mathscr{H}}. \tag{4.10}$$

Since (X,ω,ψ,μ) is $\mathscr{H}$-cuttable, it follows that $u=\mathbb{1}$. Thus, $\mathbb{T}^{\mathscr{H}}$ acts freely on $\widetilde{H}_\mu^{-1}(\mathbf{c})$ via (4.7). Let

$$\big(X_{\mathscr{H}},\omega_{\mathscr{H}},\psi_{\mathscr{H}},\mu_{\mathscr{H}}\big)\equiv\big(X_\alpha,\omega_\alpha,\psi_\alpha,\mu_\alpha\big)$$

be the associated quotient Hamiltonian $\mathbb{T}$-manifold of Theorem 4.1 and

$$p:\widetilde{H}_\mu^{-1}(\mathbf{c})\longrightarrow X_{\mathscr{H}}\equiv\widetilde{H}_\mu^{-1}(\mathbf{c})/\mathbb{T}^{\mathscr{H}}$$

be the quotient projection.

The map

$$p_{\mathscr{H}}:\mu^{-1}(\langle\!\langle\mathscr{H}\rangle\!\rangle)\longrightarrow X_{\mathscr{H}}\equiv\widetilde{H}_\mu^{-1}(\mathbf{c})/\mathbb{T}^{\mathscr{H}},\quad p_{\mathscr{H}}(x)=\big[x,\big(\sqrt{(\mu_v(x)-c_v)/\pi}\big)_{(v,c)\in\mathscr{H}}\big],$$

is well-defined, continuous, surjective, and $\mathbb{T}$-equivariant and satisfies the last condition in (HC_2). In particular, $\mu_{\mathscr{H}}(X_{\mathscr{H}})\subset\langle\mathscr{H}\rangle$. By the first statement in (4.10), $p_{\mathscr{H}}$ induces an injective map from the quotient $\mu^{-1}(\langle\!\langle\mathscr{H}\rangle\!\rangle)/\!\sim$ in (HC_1) to $X_{\mathscr{H}}$. Since the map

$$\widetilde{p}_{\mathscr{H}}:\mu^{-1}(\langle\!\langle\mathscr{H}\rangle\!\rangle)\longrightarrow\widetilde{H}_\mu^{-1}(\mathbf{c}),\qquad \widetilde{p}_{\mathscr{H}}(x)=\big(x,\big(\sqrt{(\mu_v(x)-c_v)/\pi}\big)_{(v,c)\in\mathscr{H}}\big), \tag{4.11}$$

is closed and the group $\mathbb{T}^{\mathscr{H}}$ is compact, $p_{\mathscr{H}}$ is a closed map and thus so is the induced map from $\mu^{-1}(\langle\!\langle\mathscr{H}\rangle\!\rangle)$. This confirms (HC$_1$) and (HC$_2$).

Let J be a ψ-invariant almost complex structure on X compatible with ω, $J_{\mathscr{H}}$ be the standard complex structure on $\mathbb{C}^{\mathscr{H}}$, and $\widetilde{\psi}'$ be the action of $\mathbb{T}_{\mathscr{H}}$ on $X\times\mathbb{C}^{\mathscr{H}}$ given by

$$\begin{aligned}\widetilde{\psi}'_{\Phi_{\mathscr{H}}([(r_v)_{v\in\mathscr{H}}])}\big(x,(z_v)_{v\in\mathscr{H}}\big) &= \widetilde{\psi}_{[(-r_v)_{v\in\mathscr{H}}]}\big(\psi_{\Phi_{\mathscr{H}}([(r_v)_{v\in\mathscr{H}}])}(x),(z_v)_{v\in\mathscr{H}}\big)\\ &= \big(x,\big(\mathrm{e}^{2\pi\mathrm{i}r_v}z_v\big)_{v\in\mathscr{H}}\big).\end{aligned} \tag{4.12}$$

This action commutes with the $\mathbb{T}^{\mathscr{H}}$-action $\widetilde{\psi}$ and thus induces a $\mathbb{T}_{\mathscr{H}}$-action on $X_{\mathscr{H}}$. By the middle expression in (4.12), the latter is the restriction of the $\mathbb{T}$-action $\psi_{\mathscr{H}}$ to $\mathbb{T}_{\mathscr{H}}$. The almost complex structure $\widetilde{J}\equiv J\oplus J_{\mathscr{H}}$ on $X\times\mathbb{C}^{\mathscr{H}}$ is ψ-, $\widetilde{\psi}$-, and $\widetilde{\psi}'$-invariant and compatible with $\widetilde{\omega}$. By Exercise 4.3, $\widetilde{J}$ thus descends to a $\psi_{\mathscr{H}}$-invariant almost complex structure $J_{\mathscr{H}}$ on $X_{\mathscr{H}}$ which is compatible with $\omega_{\mathscr{H}}$.

Let $\mathscr{H}'\subset\mathscr{H}$. By the first statement in (4.10),

$$\begin{aligned}\widetilde{Y}_{\mathscr{H}'} \equiv \big(\mu^{-1}(\langle\!\langle\mathscr{H}'\rangle\!\rangle^{\partial})\times\mathbb{C}^{\mathscr{H}}\big)\cap\widetilde{H}_{\mu}^{-1}(\mathbf{c}) &=\big\{\widetilde{H}_{\mu}|_{X\times\mathbb{C}^{\mathscr{H}-\mathscr{H}'}}\big\}^{-1}(\mathbf{c})\\ &\subset X\times\mathbb{C}^{\mathscr{H}-\mathscr{H}'}=\big(X\times\mathbb{C}^{\mathscr{H}}\big)^{\widetilde{\psi}'|_{\mathbb{T}_{\mathscr{H}'}}}.\end{aligned} \tag{4.13}$$

Since the moment map $\mu_{\mathscr{H}}\colon X_{\mathscr{H}}\longrightarrow T^*_{\mathbb{1}}\mathbb{T}$ is induced by (4.8),

$$Y_{\mathscr{H}'}\equiv\mu_{\mathscr{H}}^{-1}\big(\langle\!\langle\mathscr{H}'\rangle\!\rangle^{\partial}\big)=p\big(\widetilde{Y}_{\mathscr{H}'}\big)=p_{\mathscr{H}}\big(\mu^{-1}(\langle\!\langle\mathscr{H}'\rangle\!\rangle^{\partial}\big)\subset X_{\mathscr{H}}^{\mathbb{T}_{\mathscr{H}'}}\subset X_{\mathscr{H}}\,;$$

the first inclusion above follows from the last equality in (4.13). The $\widetilde{\omega}$-symplectic submanifold $X\times\mathbb{C}^{\mathscr{H}-\mathscr{H}'}\subset X\times\mathbb{C}^{\mathscr{H}}$ is preserved by the $\mathbb{T}^{\mathscr{H}}$-action $\widetilde{\psi}$, the $\mathbb{T}$-action ψ, and the $\mathbb{T}_{\mathscr{H}}$-action $\widetilde{\psi}'$. By Exercise 4.2, $Y_{\mathscr{H}'}\subset X_{\mathscr{H}}$ is thus an $\omega_{\mathscr{H}}$-symplectic submanifold preserved by the $\mathbb{T}$-action $\psi_{\mathscr{H}}$. By (4.12), the natural splitting

$$\mathcal{N}_{X\times\mathbb{C}^{\mathscr{H}}}\big(X\times\mathbb{C}^{\mathscr{H}-\mathscr{H}'}\big)=\bigoplus_{v\in\mathscr{H}'}\big(X\times\mathbb{C}^{\{v\}}\big)$$

is $\mathbb{T}-$, $\mathbb{T}^{\mathscr{H}}$-, and $\mathbb{T}_{\mathscr{H}}$-equivariant with respect to the actions $\mathrm{d}\psi$, $\mathrm{d}\widetilde{\psi}$, and $\mathrm{d}\widetilde{\psi}'$ on the left-hand side and the actions

$$\begin{gathered}u\cdot\big(x,z_v\big)=\big(\psi_u(x),z_v\big),\quad \big[(r_{v'})_{v'\in\mathscr{H}}\big]\cdot\big(x,z_v\big),=\big(\psi_{\Phi_{\mathscr{H}}([(r_{v'})_{v'\in\mathscr{H}}])}(x),\mathrm{e}^{-2\pi\mathrm{i}r_v}z_v\big),\\ \text{and}\qquad \Phi_{\mathscr{H}}\big([(r_{v'})_{v'\in\mathscr{H}}]\big)\cdot\big(x,z_v\big)=\big(x,\mathrm{e}^{2\pi\mathrm{i}r_v}z_v\big)\end{gathered}$$

on the summand $X\times\mathbb{C}^{\{v\}}$ on the right-hand side. By Exercises 4.2 and 4.3, $TX_{\mathscr{H}}|_{Y_{\mathscr{H}'}}$ thus splits $\mathbb{T}$-equivariantly as in (4.2) and (4.3). It follows that $Y_{\mathscr{H}'}\subset X_{\mathscr{H}}$ is a union of topological components of the fixed locus $X_{\mathscr{H}}^{\mathbb{T}_{\mathscr{H}'}}$ of the restriction of the $\mathbb{T}$-action $\psi_{\mathscr{H}}$ to $\mathbb{T}_{\mathscr{H}'}\subset\mathbb{T}_{\mathscr{H}}$. Thus, (4.4) is an $\mathscr{H}$-cut Hamiltonian $\mathbb{T}$-manifold.

The restriction of the map $\widetilde{p}_{\mathscr{H}}$ in (4.11) to $\mu^{-1}(\langle\!\langle\mathscr{H}'\rangle\!\rangle_{\mathscr{H}}^{\partial})$ is a smooth embedding; its image is $\widetilde{Y}_{\mathscr{H}'}\cap(X\times(\mathbb{R}^+)^{\mathscr{H}-\mathscr{H}'})$. Thus,

$$\big\{\widetilde{p}_{\mathscr{H}}|_{\mu^{-1}(\langle\!\langle\mathscr{H}'\rangle\!\rangle_{\mathscr{H}}^{\partial})}\big\}^*\big(\widetilde{\omega}|_{T\widetilde{Y}_{\mathscr{H}'}}\big)=\omega\big|_{T(\mu^{-1}(\langle\!\langle\mathscr{H}'\rangle\!\rangle_{\mathscr{H}}^{\partial}))}\,. \tag{4.14}$$

Since $p^*(\omega_{\mathscr{H}}|_{TY_{\mathscr{H}'}}) = \widetilde{\omega}|_{T\widetilde{Y}_{\mathscr{H}'}}$ by Theorem 4.1(SQ2) and Exercise 4.2, (4.5) follows from (4.14). The map

$$\widetilde{P}_{\mathscr{H};\mathscr{H}'}\colon \mathbb{T}^{\mathscr{H}-\mathscr{H}'} \times \mu^{-1}\big(\langle\mathscr{H}'\rangle_{\mathscr{H}}^{\partial}\big) \longrightarrow \widetilde{Y}_{\mathscr{H}'} \cap \big(X \times (\mathbb{C}^*)^{\mathscr{H}-\mathscr{H}'}\big),$$
$$\widetilde{P}_{\mathscr{H};\mathscr{H}'}(u,x) = \widetilde{\psi}_u\big(\widetilde{p}_{\mathscr{H}}(x)\big),$$

is a diffeomorphism. Since the map $p\colon \widetilde{Y}_{\mathscr{H}'} \longrightarrow Y_{\mathscr{H}'}$ is a principal $\mathbb{T}^{\mathscr{H}}$-bundle (and thus a submersion) by Theorem 4.1(HC$_2$),

$$\mu_{\mathscr{H}}^{-1}\big(\langle\mathscr{H}'\rangle_{\mathscr{H}}^{\partial}\big) = \widetilde{Y}_{\mathscr{H}'} \cap \big(X \times (\mathbb{C}^*)^{\mathscr{H}-\mathscr{H}'}\big),$$

and $\mathrm{d}p$ vanishes on $\mathrm{d}\widetilde{P}_{\mathscr{H};\mathscr{H}'}(T\mathbb{T}^{\mathscr{H}-\mathscr{H}'})$, the composition

$$p_{\mathscr{H}} = p \circ \widetilde{p}_{\mathscr{H}}\colon \mu^{-1}\big(\langle\mathscr{H}'\rangle_{\mathscr{H}}^{\partial}\big) \longrightarrow \mu_{\mathscr{H}}^{-1}\big(\langle\mathscr{H}'\rangle_{\mathscr{H}}^{\partial}\big) \subset \mu_{\mathscr{H}}^{-1}\big(\langle\mathscr{H}'\rangle^{\partial}\big)$$

is a submersion. This confirms (HC$_3$). The conditions (HC$_1$)-(HC$_3$) ensure the uniqueness of $\mathscr{H}$-cut Hamiltonian $\mathbb{T}$-manifold (4.4) satisfying these properties.

Suppose $\mathscr{H} = \mathscr{H}_1 \sqcup \mathscr{H}_2$. By Exercise 4.10, the Hamiltonian $\mathbb{T}$-manifold (X,ω,ψ,μ) is $\mathscr{H}_1$-cuttable. Let

$$(X,\omega,\psi,\mu)_{\mathscr{H}_1} \equiv \big(X_{\mathscr{H}_1}, \omega_{\mathscr{H}_1}, \psi_{\mathscr{H}_1}, \mu_{\mathscr{H}_1}\big) \tag{4.15}$$

be the corresponding $\mathscr{H}_1$-cut Hamiltonian $\mathbb{T}$-manifold as in (4.4). If $\mathscr{H}_2' \subset \mathscr{H}_2$,

$$\mu_{\mathscr{H}_1}^{-1}\big(\langle\mathscr{H}_2'\rangle^{\partial}\big) = \bigcup_{\mathscr{H}_1'' \subset \mathscr{H}_1} \; \bigcup_{\mathscr{H}_2' \subset \mathscr{H}_2'' \subset \mathscr{H}_2} \mu_{\mathscr{H}_1}^{-1}\big(\langle\mathscr{H}_1'' \sqcup \mathscr{H}_2''\rangle_{\mathscr{H}}^{\partial}\big) = \bigcup_{\mathscr{H}_1'' \subset \mathscr{H}_1} \; \bigcup_{\mathscr{H}_2' \subset \mathscr{H}_2'' \subset \mathscr{H}_2} \mu^{-1}\big(\langle\mathscr{H}_1'' \sqcup \mathscr{H}_2''\rangle_{\mathscr{H}}^{\partial}\big)/\mathbb{T}_{\mathscr{H}_1''};$$

the last equality holds by (HC$_1$) with $\mathscr{H}$ replaced by $\mathscr{H}_1$. If $\mu_{\mathscr{H}_1}^{-1}(\langle\mathscr{H}_2'\rangle^{\partial}) \neq \emptyset$, Exercise 4.10 thus implies that the Lie group homomorphism $\Phi_{\mathscr{H}_2'}$ as in (1.41) is injective and $\mathbb{T}_{\mathscr{H}_2'}$ acts freely on $\mu_{\mathscr{H}_1}^{-1}(\langle\mathscr{H}_2'\rangle^{\partial})$. We conclude that the Hamiltonian $\mathbb{T}$-manifold (4.15) is $\mathscr{H}_2$-cuttable.

By (HC$_1$) and (HC$_2$) with $\mathscr{H}$ replaced by $\mathscr{H}_1$ and $\mathscr{H}_2$,

$$\big(X_{\mathscr{H}_1}\big)_{\mathscr{H}_2} = \mu_{\mathscr{H}_1}^{-1}\big(\langle\langle\mathscr{H}_2\rangle\rangle\big)/\sim_{\mathscr{H}_2} = p_{\mathscr{H}_1}\big(\mu^{-1}(\langle\langle\mathscr{H}_1\rangle\rangle) \cap \mu^{-1}(\langle\langle\mathscr{H}_2\rangle\rangle)\big)/\sim_{\mathscr{H}_2} = \big(\mu^{-1}(\langle\langle\mathscr{H}\rangle\rangle)/\sim_{\mathscr{H}_1}\big)/\sim_{\mathscr{H}_2},$$

with $x,x' \in \mu^{-1}(\langle\langle\mathscr{H}\rangle\rangle)$ being equivalent in the double quotient if there exist $\mathscr{H}_1' \subset \mathscr{H}_1$, $\mathscr{H}_2' \subset \mathscr{H}_2$, and u in the subgroup generated by $\mathbb{T}_{\mathscr{H}_1'}, \mathbb{T}_{\mathscr{H}_2'} \subset \mathbb{T}$ such that

$$\mu(x) \in \langle\mathscr{H}_1'\rangle^{\partial} \cap \langle\mathscr{H}_2'\rangle^{\partial} = \langle\mathscr{H}_1' \sqcup \mathscr{H}_2'\rangle^{\partial} \qquad \text{and} \qquad x' = \psi_u(x).$$

By definition, the subgroup generated by $\mathbb{T}_{\mathscr{H}_1'}, \mathbb{T}_{\mathscr{H}_2'} \subset \mathbb{T}$ is $\mathbb{T}_{\mathscr{H}_1' \sqcup \mathscr{H}_2'}$. Thus,

$$p_{\mathscr{H}_2} \circ p_{\mathscr{H}_1} = p_{\mathscr{H}}\colon X \longrightarrow \big(X_{\mathscr{H}_1}\big)_{\mathscr{H}_2} = X_{\mathscr{H}}, \qquad (\mu_{\mathscr{H}_1})_{\mathscr{H}_2} = \mu_{\mathscr{H}}\colon X_{\mathscr{H}} \longrightarrow T_{\mathbb{1}}^*\mathbb{T},$$

and the $\mathbb{T}$-actions $(\psi_{\mathscr{H}_1})_{\mathscr{H}_2}$ and $\psi_{\mathscr{H}}$ on $X_{\mathscr{H}}$ are the same (as they are induced by the same $\mathbb{T}$-action ψ on X). Since $\langle\emptyset\rangle_{\mathscr{H}}^{\partial} = \langle\emptyset\rangle_{\mathscr{H}_1}^{\partial} \cap \langle\emptyset\rangle_{\mathscr{H}_2}^{\partial}$ is an open subset of $T_{\mathbb{1}}^*\mathbb{T}$, (4.5) gives

$$\begin{aligned}
\big\{p_{\mathscr{H}}|_{\mu^{-1}(\langle\emptyset\rangle_{\mathscr{H}}^{\partial})}\big\}^*(\omega_{\mathscr{H}_1})_{\mathscr{H}_2} &= \big\{p_{\mathscr{H}_1}|_{\mu^{-1}(\langle\emptyset\rangle_{\mathscr{H}}^{\partial})}\big\}^*\big\{p_{\mathscr{H}_2}|_{\mu_{\mathscr{H}_1}^{-1}(\langle\emptyset\rangle_{\mathscr{H}_2}^{\partial})}\big\}^*(\omega_{\mathscr{H}_1})_{\mathscr{H}_2} \\
&= \big\{p_{\mathscr{H}_1}|_{\mu^{-1}(\langle\emptyset\rangle_{\mathscr{H}}^{\partial})}\big\}^*\omega_{\mathscr{H}_1} = \omega\big|_{\mu^{-1}(\langle\emptyset\rangle_{\mathscr{H}}^{\partial})} = \big\{p_{\mathscr{H}}|_{\mu^{-1}(\langle\emptyset\rangle_{\mathscr{H}}^{\partial})}\big\}^*\omega_{\mathscr{H}}.
\end{aligned}$$

Since $p_{\mathscr{H}}$ is a submersion on $\mu^{-1}(\langle\emptyset\rangle_{\mathscr{H}}^{\partial})$, it follows that $(\omega_{\mathscr{H}_1})_{\mathscr{H}_2} = \omega$ on the dense open subset $\mu_{\mathscr{H}}^{-1}(\langle\emptyset\rangle_{\mathscr{H}}^{\partial}) \subset X_{\mathscr{H}}$ and thus everywhere on $X_{\mathscr{H}}$. This establishes (4.6). $\square$

Exercise 4.12. Suppose $\mathbb{T}$ is a torus, $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ is a finite subcollection, and (X,ω,ψ,μ) is an $\mathscr{H}$-cuttable Hamiltonian $\mathbb{T}$-manifold.

(a) Suppose $\mu(X)\subset\langle\mathscr{H}\rangle$. Show that the projection

$$p_{\mathscr{H}}\colon \mu^{-1}(\langle\!\langle\mathscr{H}\rangle\!\rangle)=X\longrightarrow X_{\mathscr{H}}$$

of Theorem 4.11 identifies (X,ω,ψ,μ) with $(X,\omega,\psi,\mu)_{\mathscr{H}}$.

(b) Conclude that $(X,\omega,\psi,\mu)_{\mathscr{H}}$ does not depend on the choice of finite subcollection $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ with $\langle\mathscr{H}\rangle$ fixed so that (X,ω,ψ,μ) is $\mathscr{H}$-cuttable.

Exercise 4.13. Suppose $\mathbb{T}$ is a torus, $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ is a finite subcollection, (X,ω,ψ,μ) is an $\mathscr{H}$-cuttable Hamiltonian $\mathbb{T}$-manifold, and $\widetilde{H}_\mu\colon X\times\mathbb{C}^{\mathscr{H}}\longrightarrow\mathbb{R}^{\mathscr{H}}$ is as in (4.9).

(a) Let $\lambda\in\mathbb{R}^+$ and $\lambda\mathscr{H}$ be as in Exercise 1.45(a). Show that $(X,\lambda\omega,\psi,\lambda\mu)$ is a $\lambda\mathscr{H}$-cuttable Hamiltonian $\mathbb{T}$-manifold and that the map

$$\widetilde{H}_\mu^{-1}(\mathbf{c})\longrightarrow\widetilde{H}_{\lambda\mu}^{-1}(\lambda\mathbf{c})\subset X\times\mathbb{C}^{\lambda\mathscr{H}},\quad\big(x,(z_{v,c})_{(v,c)\in\mathscr{H}}\big)\longrightarrow\big(x,(\sqrt{\lambda}z_{v,c})_{(v,\lambda c)\in\lambda\mathscr{H}}\big),$$

descends to an identification of $(X_{\mathscr{H}},\lambda\omega_{\mathscr{H}},\psi_{\mathscr{H}},\lambda\mu_{\mathscr{H}})$ with $(X,\lambda\omega,\psi,\lambda\mu)_{\lambda\mathscr{H}}$.

(b) Let $\alpha_0\in T_{\mathbb{1}}^*\mathbb{T}$, $\mathbf{c}+\alpha_0=\big(c_v+\alpha_0(v)\big)_{v\in\mathscr{H}}\in\mathbb{R}^{\mathscr{H}}$, and $\mathscr{H}+\alpha_0$ be as in Exercise 1.45(b). Show that $(X,\omega,\psi,\mu+\alpha_0)$ is an $(\mathscr{H}+\alpha_0)$-cuttable Hamiltonian $\mathbb{T}$-manifold and that the map

$$\widetilde{H}_\mu^{-1}(\mathbf{c})\longrightarrow\widetilde{H}_{\mu+\alpha_0}^{-1}(\mathbf{c}+\alpha_0)\subset X\times\mathbb{C}^{\mathscr{H}+\alpha_0},\quad\big(x,(z_{v,c})_{(v,c)\in\mathscr{H}}\big)\longrightarrow\big(x,(z_{v,c})_{(v,c+\alpha_0(v))\in\mathscr{H}+\alpha_0}\big),$$

descends to an identification of $(X_{\mathscr{H}},\omega_{\mathscr{H}},\psi_{\mathscr{H}},\mu_{\mathscr{H}}+\alpha_0)$ with $(X,\omega,\psi,\mu+\alpha_0)_{\mathscr{H}+\alpha_0}$.

(c) Let $\Theta\in\mathrm{GL}_{\mathbb{Z}}((T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}})$, $\Theta^*\in\mathrm{GL}_{\mathbb{Z}}((T_{\mathbb{1}}^*\mathbb{T})_{\mathbb{Z}})$ be the dual of Θ, and ψ_Θ be the $\mathbb{T}$-action on X obtained by composing ψ with the automorphism of $\mathbb{T}$ induced by Θ, and $\Theta^{-1}\mathscr{H}$ be as in Exercise 1.45(c). Show that $(X,\omega,\psi_\Theta,\Theta^*\circ\mu)$ is a $\Theta^{-1}\mathscr{H}$-cuttable Hamiltonian $\mathbb{T}$-manifold and that the map

$$\widetilde{H}_\mu^{-1}(\mathbf{c})\longrightarrow\widetilde{H}_{\Theta^*\circ\mu}^{-1}(\mathbf{c})\subset X\times\mathbb{C}^{\Theta^{-1}\mathscr{H}},\quad\big(x,(z_{v,c})_{(v,c)\in\mathscr{H}}\big)\longrightarrow\big(x,(z_{v,c})_{(\Theta^{-1}(v),c)\in\Theta^{-1}\mathscr{H}}\big),$$

descends to an identification of $(X_{\mathscr{H}},\omega_{\mathscr{H}},(\psi_{\mathscr{H}})_\Theta,\Theta^*\circ\mu_{\mathscr{H}})$ with $(X,\omega,\psi_\Theta,\Theta^*\circ\mu)_{\Theta^{-1}\mathscr{H}}$.

Example 4.14. Let $\omega_{\mathbb{C}^n}$ be the standard symplectic form on $\mathbb{C}^n$ as in (1.19), ψ be the standard S^1-action on $\mathbb{C}^n$, $\mu\colon\mathbb{C}^n\longrightarrow T^*S^1$ be the smooth map corresponding to the Hamiltonian

$$H\colon\mathbb{C}^n\longrightarrow\mathbb{R},\qquad H(x)=\pi|x|^2,$$

for ψ with respect to $\omega_{\mathbb{C}^n}$ under the standard identification of T^*S^1 with $\mathbb{R}$, $c\in\mathbb{R}^+$, and $\mathscr{H}=(e_1,c)$, where $e_1\in T_{\mathbb{1}}\mathbb{R}$ is the standard generator.

(a) Show that $(\mathbb{C}^n,\omega_{\mathbb{C}^n},\psi,\mu)$ is an $\mathscr{H}$-cuttable Hamiltonian S^1-manifold.

Let $(X_{\mathscr{H}},\omega_{\mathscr{H}},\psi_{\mathscr{H}},\mu_{\mathscr{H}})=(\mathbb{C}^n,\omega,\psi,\mu)_{\mathscr{H}}$ be the corresponding $\mathscr{H}$-cut Hamiltonian S^1-manifold as in Theorem 4.11. Show that

(b) the map

$$\mathrm{pr}\colon X_{\mathscr{H}} \equiv \big\{(x,z)\in\mathbb{C}^n\times\mathbb{C}\colon |x|^2-|z|^2=c/\pi\big\}/S^1 \longrightarrow \mathbb{C}P^{n-1}, \quad \mathrm{pr}\big([x,z]\big)=\mathbb{C}x,$$

is a complex line bundle;

(c) the map

$$\Phi\colon X_{\mathscr{H}} \equiv \big\{(x,z)\in\mathbb{C}^n\times\mathbb{C}\colon |x|^2-|z|^2=c/\pi\big\}/S^1 \longrightarrow \gamma_{n-1}, \quad \Phi([x,z])=\big(\mathbb{C}x, zx/|x|\big),$$

is an isomorphism of the above complex line bundle with the tautological complex line bundle;

(d) $\omega_{\mathscr{H}}=\Phi^*\widetilde{\omega}_{n-1;c}$, with $\widetilde{\omega}_{n-1;c}$ as in Exercise 1.29.

Note: along with Exercise 1.31, (d) shows that three natural constructions of a symplectic form on the total space of γ_{n-1} yield the same form. *Hint for (d):* compare the pullbacks of the two sides by $p\colon \widetilde{H}_\mu^{-1}(c)\longrightarrow X_{\mathscr{H}}$ at each point of $\widetilde{H}_\mu^{-1}(c)$.

Exercise 4.15. Let (Y,ω), $\mathrm{pr}_E: E\longrightarrow Y$, Ω, $\mathfrak{i}$, $g_{\mathfrak{i}}^{\Omega}$, ∇, $\widetilde{\omega}$, $\rho\colon Y\longrightarrow\mathbb{R}^+$, $\mathcal{U}_\rho$, and H be as in Exercise 1.30, $\overline{\psi}$ be the restriction of the inverse of the standard S^1-action on E to $\mathcal{U}_\rho$, and $\mu\colon\mathcal{U}_\rho\longrightarrow T^*S^1$ be the smooth map corresponding to H under the standard identification of T^*S^1 with $\mathbb{R}$. Suppose that the function ρ is constant and $c\in(0,\pi\rho)$. Let $\mathscr{H}=(e_1,-c)$, where $e_1\in T_{\mathbb{1}}\mathbb{R}$ is the standard generator.

(a) Show that $(\mathcal{U}_\rho,\widetilde{\omega},\overline{\psi},-\mu)$ is an $\mathscr{H}$-cuttable Hamiltonian S^1-manifold.

Let $(X_{\mathscr{H}},\omega_{\mathscr{H}},\psi_{\mathscr{H}},\mu_{\mathscr{H}})=(\mathcal{U}_\rho,\widetilde{\omega},\overline{\psi},-\mu)_{\mathscr{H}}$ be the corresponding $\mathscr{H}$-cut Hamiltonian S^1-manifold as in Theorem 4.11. Show that

(b) $\mathcal{U}_c\subset X_{\mathscr{H}}$ is a dense open subset, the restrictions of $\psi_{\mathscr{H}}$ and $\mu_{\mathscr{H}}$ to $\mathcal{U}_c$ are the restrictions of $\overline{\psi}$ and $-\mu$, respectively, to $\mathcal{U}_c$, and the projection pr_E extends to a fiber bundle $\widetilde{\mathrm{pr}}_{E;c}\colon X_{\mathscr{H}}\longrightarrow Y$ isomorphic to the complex projectivization of the complex vector bundle $(E,\mathfrak{i})\oplus(Y\times\mathbb{C})$ over Y so that the S^1-action ψ corresponds to the action induced by the standard action of S^1 on $\mathbb{C}$;

(c) every fiber of $\widetilde{\mathrm{pr}}_{E;c}$ is a symplectic submanifold of $(X_{\mathscr{H}},\omega_{\mathscr{H}})$ isomorphic to $(\mathbb{C}P^n, c\omega_{\mathrm{FS};n})$.

Hint: use Exercise 4.7.

4.3 Hamiltonian symplectic uncut

In this section, we show that the Hamiltonian symplectic cut construction is reversible and complete the proof of Theorem 4.20. This formalizes the argument sketched in the proof of [42, Theorem 7.5.10].

Exercise 4.16. Suppose $\mathbb{T}$ is a torus, $\mathscr{H}=\mathscr{H}_1\sqcup\mathscr{H}_2$ is a partition of a finite subcollection of $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$, (X,ω,ψ,μ) is an $\mathscr{H}_2$-cuttable Hamiltonian $\mathbb{T}$-manifold so that $(X,\omega,\psi,\mu)_{\mathscr{H}_2}$ is an $\mathscr{H}$-cut Hamiltonian $\mathbb{T}$-manifold, and $Y\subset X_{\mathscr{H}_2}$ is a topological component of $\mu_{\mathscr{H}_2}^{-1}(\langle\mathscr{H}_1\rangle^{\partial})$. Show that

(a) there exists a (unique) topological component $\widetilde{Y}\subset X$ of $\mu^{-1}(\langle\mathscr{H}_1\rangle^{\partial})$ which contains $p_{\mathscr{H}_2}^{-1}(Y)$;

(b) $\widetilde{Y}$ is a topological component of the fixed locus $X^{\mathbb{T}_{\mathscr{H}_1}} \subset X$ of $\psi|_{\mathbb{T}_{\mathscr{H}_1}}$ with its normal bundle admitting a splitting as in (4.2) and (4.3) with $(Y_{\mathscr{H}'}, \mathscr{H}')$ replaced by $(\widetilde{Y}, \mathscr{H}_1)$.

Corollary 4.17. *Suppose $\mathbb{T}$ is a torus, $\mathscr{H} = \mathscr{H}_1 \sqcup \mathscr{H}_2$ is a partition of a finite subcollection of $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$, and (X, ω, ψ, μ) is an $\mathscr{H}_2$-cuttable Hamiltonian $\mathbb{T}$-manifold. If $(X, \omega, \psi, \mu)_{\mathscr{H}_2}$ is an $\mathscr{H}$-cut Hamiltonian $\mathbb{T}$-manifold, then there exists an open $\mathbb{T}$-invariant subset $X' \subset X$ so that*

$$\big(X', \omega|_{X'}, \psi|_{X'}, \mu|_{X'}\big)_{\mathscr{H}_2} = (X, \omega, \psi, \mu)_{\mathscr{H}_2} \tag{4.16}$$

and the Hamiltonian $\mathbb{T}$-manifold $(X', \omega|_{X'}, \psi|_{X'}, \mu|_{X'})$ is $\mathscr{H}_1$-cut. If $X_{\mathscr{H}_2}$ is connected and/or the restriction

$$\mu\colon \mu^{-1}\big(\langle\!\langle \mathscr{H}_1' \rangle^{\partial}_{\mathscr{H}_1}\big) \longrightarrow \langle \mathscr{H}_1' \rangle^{\partial}_{\mathscr{H}_1} \cap \mu(X) \tag{4.17}$$

is a principal $\mathbb{T}/\mathbb{T}_{\mathscr{H}_1'}$-bundle for every $\mathscr{H}_1' \subset \mathscr{H}_1$, then X' can be chosen so that it is also connected and/or the restriction

$$\mu\colon \mu^{-1}\big(\langle\!\langle \mathscr{H}_1' \rangle^{\partial}_{\mathscr{H}_1}\big) \cap X' \longrightarrow \langle \mathscr{H}_1' \rangle^{\partial}_{\mathscr{H}_1} \cap \mu(X') \tag{4.18}$$

is also a principal $\mathbb{T}/\mathbb{T}_{\mathscr{H}_1'}$-bundle for every $\mathscr{H}_1' \subset \mathscr{H}_1$, respectively.

Proof. Suppose $\mathscr{H}_1' \subset \mathscr{H}_1$ and $Y \subset X_{\mathscr{H}_2}$ is a topological component of $\mu_{\mathscr{H}_2}^{-1}(\langle\!\langle \mathscr{H}_1' \rangle^{\partial})$. It is thus contained in a topological component $Y_{\mathscr{H}_1''} \subset X_{\mathscr{H}_2}$ of $\mu_{\mathscr{H}_2}^{-1}(\langle\!\langle \mathscr{H}_1'' \rangle^{\partial})$ for every $\mathscr{H}_1'' \subset \mathscr{H}_1'$. By Exercise 4.16(a), there exists a (unique) topological component $\widetilde{Y}_{\mathscr{H}_1''} \subset X$ of $\mu^{-1}(\langle\!\langle \mathscr{H}_1'' \rangle^{\partial})$ which contains $p_{\mathscr{H}_2}^{-1}(Y_{\mathscr{H}_1''})$. In particular, $\widetilde{Y}_{\mathscr{H}_1'}$ is disjoint from the closed subsets $\mu^{-1}(\langle\!\langle \mathscr{H}_1'' \rangle^{\partial}) - \widetilde{Y}_{\mathscr{H}_1''}$ of X with $\mathscr{H}_1'' \subset \mathscr{H}_1'$.

By Exercise 4.16(b) and the first part of Proposition 2.33 with $\mathbb{T}$ replaced by $\mathbb{T}_{\mathscr{H}_1'}$, there thus exists a $\mathbb{T}$-invariant neighborhood $\widetilde{U}_Y$ of $\widetilde{Y}_{\mathscr{H}_1'}$ in X so that

$$\mu(\widetilde{U}_Y) \subset \langle \mathscr{H}_1' \rangle \qquad \text{and} \qquad \widetilde{U}_Y \cap \mu^{-1}\big(\langle\!\langle \mathscr{H}_1'' \rangle^{\partial}\big) \subset \widetilde{Y}_{\mathscr{H}_1''} \ \ \forall\, \mathscr{H}_1'' \subset \mathscr{H}_1'\,. \tag{4.19}$$

The $\mathbb{T}$-invariant neighborhood

$$\widetilde{U}_Y' \equiv \widetilde{U}_Y \cap \mu^{-1}\big(\langle\!\langle \emptyset \rangle^{\partial}_{\mathscr{H}_1 - \mathscr{H}_1'}\big) \subset \mu^{-1}\big(\langle\!\langle \mathscr{H}_1' \rangle\!\rangle\big) \cap \mu^{-1}\big(\langle\!\langle \mathscr{H}_1 - \mathscr{H}_1' \rangle\!\rangle\big) = \mu^{-1}\big(\langle\!\langle \mathscr{H}_1 \rangle\!\rangle\big)$$

of $\widetilde{Y}_{\mathscr{H}_1'} \cap \mu^{-1}(\langle\!\langle \mathscr{H}_1' \rangle^{\partial}_{\mathscr{H}_1})$ in X is then disjoint from $\mu^{-1}(\langle\!\langle \mathscr{H}_1'' \rangle^{\partial})$ for every subset $\mathscr{H}_1'' \subset \mathscr{H}_1$ not contained in $\mathscr{H}_1'$. Thus,

$$X' \equiv \bigcup_{\mathscr{H}_1' \subset \mathscr{H}_1} \ \bigcup_{Y \in \pi_0(\mu_{\mathscr{H}_2}^{-1}(\langle\!\langle \mathscr{H}_1' \rangle^{\partial}))} \widetilde{U}_Y' \quad \subset \mu^{-1}\big(\langle\!\langle \mathscr{H}_1 \rangle\!\rangle\big) \tag{4.20}$$

is a $\mathbb{T}$-invariant neighborhood of

$$\bigcup_{\mathscr{H}_1' \subset \mathscr{H}_1} \ \bigcup_{Y \in \pi_0(\mu_{\mathscr{H}_2}^{-1}(\langle\!\langle \mathscr{H}_1' \rangle^{\partial}))} \big(\widetilde{Y}_{\mathscr{H}_1'} \cap \mu^{-1}(\langle\!\langle \mathscr{H}_1' \rangle^{\partial}_{\mathscr{H}_1})\big) \supset \bigcup_{\mathscr{H}_1' \subset \mathscr{H}_1} \ \bigcup_{Y \in \pi_0(\mu_{\mathscr{H}_2}^{-1}(\langle\!\langle \mathscr{H}_1' \rangle^{\partial}))} p_{\mathscr{H}_2}^{-1}(Y) = p_{\mathscr{H}_2}^{-1}(X_{\mathscr{H}_2}) = \mu^{-1}\big(\langle\!\langle \mathscr{H}_2 \rangle\!\rangle\big)$$

in X. By Theorem 4.11(HC$_1$) with $\mathscr{H}$ replaced by $\mathscr{H}_2$, the above inclusion implies (4.16).

By (4.20), $\mu(X')\subset\langle\mathscr{H}_1\rangle$. Let $\mathscr{H}_1''\subset\mathscr{H}_1$ be such that $\mu^{-1}(\langle\!\langle\mathscr{H}_1''\rangle^{\partial})\neq\emptyset$. Since $\widetilde{U}_Y'\subset\widetilde{U}_Y$ with $Y\in\pi_0(\mu_{\mathscr{H}_2}^{-1}(\langle\!\langle\mathscr{H}_1'\rangle^{\partial}))$ is disjoint from $\mu^{-1}(\langle\!\langle\mathscr{H}_1''\rangle^{\partial})$ whenever $\mathscr{H}_1''\not\subset\mathscr{H}_1'$, the second statement in (4.19) implies that

$$X'\cap\mu^{-1}\big(\langle\!\langle\mathscr{H}_1''\rangle^{\partial}\big)\subset\bigcup_{\mathscr{H}_1''\subset\mathscr{H}_1'\subset\mathscr{H}_1}\;\bigcup_{Y\in\pi_0(\mu_{\mathscr{H}_2}^{-1}(\langle\!\langle\mathscr{H}_1'\rangle^{\partial}))}\widetilde{Y}_{\mathscr{H}_1''}=\bigcup_{Y\in\pi_0(\mu_{\mathscr{H}_2}^{-1}(\langle\!\langle\mathscr{H}_1''\rangle^{\partial}))}\widetilde{Y}_{\mathscr{H}_1''}\,. \tag{4.21}$$

Since $(X,\omega,\psi,\mu)_{\mathscr{H}_2}$ is $\mathscr{H}$-cut, it follows that the Lie group homomorphism $\Phi_{\mathscr{H}_1''}$ as in (1.41) is injective. By (4.21), $X'\cap\mu^{-1}(\langle\!\langle\mathscr{H}_1''\rangle^{\partial})\subset X$ is the disjoint union of the open subspaces $X'\cap\widetilde{Y}_{\mathscr{H}_1''}$ of $\widetilde{Y}_{\mathscr{H}_1''}$ with $Y\in\pi_0(\mu_{\mathscr{H}_2}^{-1}(\langle\!\langle\mathscr{H}_1''\rangle^{\partial}))$. By Exercise 4.16(b), $X'\cap\mu^{-1}(\langle\!\langle\mathscr{H}_1''\rangle^{\partial})$ is thus a union of topological components of the fixed locus $X'^{\mathbb{T}_{\mathscr{H}_1''}}$ of the restriction of the $\mathbb{T}$-action ψ to $\mathbb{T}_{\mathscr{H}_1''}\subset\mathbb{T}$ and $X'\subset X$ with its normal bundles admitting a splitting as in (4.2) and (4.3) with $(Y_{\mathscr{H}'},\mathscr{H}')$ replaced by $(X'\cap\mu^{-1}(\langle\!\langle\mathscr{H}_1''\rangle^{\partial}),\mathscr{H}_1'')$. Thus, $(X',\omega|_{X'},\psi|_{X'},\mu|_{X'})$ is $\mathscr{H}_1$-cut.

Since $\mu_{\mathscr{H}_2}(X_{\mathscr{H}_2})\subset\langle\mathscr{H}_2\rangle$, the subspaces $\widetilde{Y}_{\mathscr{H}_1'}\cap\mu^{-1}(\langle\!\langle\mathscr{H}_1'\rangle^{\partial}_{\mathscr{H}_1})\subset X$ above intersect $\mu^{-1}(\langle\!\langle\mathscr{H}_2\rangle\!\rangle)$ and thus so do their neighborhoods $\widetilde{U}_Y'\subset X'\subset X$. If $X_{\mathscr{H}_2}$ is connected, then so is $\mu^{-1}(\langle\!\langle\mathscr{H}_2\rangle\!\rangle)\subset X$ by Theorem 4.11(HC$_1$) with $\mathscr{H}$ replaced by $\mathscr{H}_2$. It follows that $X'\subset X$ is then connected. If the restriction (4.17) is a principal $\mathbb{T}/\mathbb{T}_{\mathscr{H}_1'}$-bundle for some $\mathscr{H}_1'\subset\mathscr{H}_1$, then $\mathbb{T}$ acts transitively on the fibers of μ over $\langle\mathscr{H}_1'\rangle^{\partial}_{\mathscr{H}}\cap\mu(X)$. Since $X'\subset X$ is a $\mathbb{T}$-invariant subset, (4.18) is the restriction of principal $\mathbb{T}/\mathbb{T}_{\mathscr{H}_1'}$-bundle (4.17) to $\langle\mathscr{H}_1'\rangle^{\partial}_{\mathscr{H}_1}\cap\mu(X')\subset\langle\mathscr{H}_1'\rangle^{\partial}_{\mathscr{H}_1}\cap\mu(X)$ and thus is still a principal $\mathbb{T}/\mathbb{T}_{\mathscr{H}_1'}$-bundle. $\square$

Exercise 4.18. Suppose (X,ω) is a symplectic manifold, ψ is a smooth action of $\mathbb{R}^k$ on (X,ω), $X'\subset X$ is an ω-symplectic manifold preserved ψ so that the inclusion $i\colon X'\longrightarrow X$ is a homotopy equivalence, and $\mu'\colon X'\longrightarrow T_0^*\mathbb{R}^k$ is a moment map for the restriction of the action ψ to X'. Show that μ' extends to a moment map $\mu\colon X\longrightarrow T_0^*\mathbb{R}^k$ for ψ. *Hint:* first show this for $k=1$.

Theorem 4.19. *Suppose $\mathbb{T}$ is a torus, $\mathscr{H}=\mathscr{H}_1\sqcup\mathscr{H}_2$ is a partition of a finite subcollection of $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$, and (X,ω,ψ,μ) is an $\mathscr{H}_2$-cut Hamiltonian $\mathbb{T}$-manifold. Then,*

$$(X,\omega,\psi,\mu)=(X',\omega',\psi',\mu')_{\mathscr{H}_2} \tag{4.22}$$

for some $\mathscr{H}_2$-cuttable Hamiltonian $\mathbb{T}$-manifold (X',ω',ψ',μ'). If

(a) X is connected and/or

(b) (X,ω,ψ,μ) is $\mathscr{H}$-cut and the restriction

$$\mu\colon\mu^{-1}\big(\langle\!\langle\mathscr{H}'\rangle^{\partial}_{\mathscr{H}}\big)\longrightarrow\langle\mathscr{H}'\rangle^{\partial}_{\mathscr{H}}\cap\mu(X) \tag{4.23}$$

is a principal $\mathbb{T}/\mathbb{T}_{\mathscr{H}'}$-bundle for every $\mathscr{H}'\subset\mathscr{H}$,

then (X',ω',ψ',μ') can be chosen so that

(a′) X' is connected and/or

(b′) (X',ω',ψ',μ') is $\mathscr{H}_1$-cut and the restriction

$$\mu'\colon\mu'^{-1}\big(\langle\!\langle\mathscr{H}_1'\rangle^{\partial}_{\mathscr{H}_1}\big)\longrightarrow\langle\mathscr{H}_1'\rangle^{\partial}_{\mathscr{H}_1}\cap\mu'(X') \tag{4.24}$$

is a principal $\mathbb{T}/\mathbb{T}_{\mathscr{H}_1'}$-bundle for every $\mathscr{H}_1'\subset\mathscr{H}_1$, *respectively.*

Proof. By (4.6) and Corollary 4.17, it is sufficient to establish this theorem with $\mathscr{H}_2$ consisting of a single element $v\equiv(v,c)$ of $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ with $v\neq 0$. We assume that the closed codimension 2 symplectic submanifold

$$Y\equiv\mu^{-1}\big(\langle\mathscr{H}_2\rangle^{\partial}\big)\equiv\big\{x\in X:\mu_v(x)=c\big\}$$

of (X,ω) is nonempty; otherwise, we can take $(X',\omega',\psi',\mu')=(X,\omega,\psi,\mu)$. Since (X,ω,ψ,μ) is $\mathscr{H}_2$-cut, the Lie group homomorphism $\Phi_{\mathscr{H}_2}$ as in (1.41) is then injective. We establish the proposition by removing Y and continuing the radial directions in the normal bundle of Y in X into the negative values without them coming together at 0. Since (X,ω,ψ,μ) is $\mathscr{H}_2$-cut,

$$X-Y=\big\{x\in X:\mu_v(x)>c\big\}=\mu^{-1}\big(\langle\emptyset\rangle^{\partial}_{\mathscr{H}_2}\big)\,. \tag{4.25}$$

Let J be a $\mathbb{T}^n$-invariant almost complex structure on X compatible with ω and $g(\cdot,\cdot)\equiv\omega(\cdot,J\cdot)$ be the associated $\mathbb{T}^n$-invariant metric compatible with J.

Since (X,ω,ψ,μ) is $\mathscr{H}_2$-cut, Y is a union of topological components of the fixed locus $X^{\mathbb{T}_v}$ of the restriction of the $\mathbb{T}$-action ψ to the circle $\mathbb{T}_v\subset\mathbb{T}$ generated by $v\in(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$ and

$$\mathrm{pr}\colon TY^{\omega}\equiv\big\{w\in TX|_Y:\omega(w,w')=0\ \forall\,w'\in TY\big\}\longrightarrow Y$$

is a complex line bundle complementary to TY. It is preserved by the $\mathbb{T}^n$-action $\mathrm{d}\psi$ and

$$\mathrm{d}\psi_{\mathrm{e}^{tv}}(w)=\mathrm{e}^{2\pi\mathrm{i}t}w\qquad\forall\ t\in\mathbb{R},\ w\in TY^{\omega} \tag{4.26}$$

by (4.3). Let

$$\zeta_v\in\Gamma\big(TY^{\omega};T(TY^{\omega})\big),\qquad \zeta_v(w)=\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}\psi_{\mathrm{e}^{tv}}(w)\Big|_{t=0}=2\pi\mathrm{i}w,$$

be the (vertical) vector field on TY^{ω} generating the S^1-action (4.26). Since this action preserves the unit circle bundle of TY^{ω},

$$\mathrm{pr}\colon S\big(TY^{\omega}\big)\equiv\big\{w\in TY^{\omega}:g(w,w)=1\big\}\longrightarrow Y, \tag{4.27}$$

the vector field $\zeta_v|_{S(TY^{\omega})}$ is tangent to $S(TY^{\omega})$. The maps

$$\begin{gathered}\widetilde{\iota}\colon S\big(TY^{\omega}\big)\times\mathbb{R}^+\longrightarrow S\big(TY^{\omega}\big)\times\mathbb{C},\quad \widetilde{\iota}(w,t)=\big(w,\sqrt{2t}\big),\\ p\colon S\big(TY^{\omega}\big)\times\mathbb{C}\longrightarrow TY^{\omega},\quad p(w,z)=zw,\end{gathered}$$

are smooth. The map p descends to a $\mathbb{T}$-equivariant diffeomorphism from the quotient $S(TY^{\omega})\times_{S^1}\mathbb{C}$ of $S(TY^{\omega})\times\mathbb{C}$ by the S^1-action

$$S^1\times\big(S(TY^{\omega})\times\mathbb{C}\big)\longrightarrow S(TY^{\omega})\times\mathbb{C},\qquad u\cdot(w,z)=\big(uw,u^{-1}z\big), \tag{4.28}$$

to TY^{ω}. The generating vector field for this action is $\widetilde{\zeta}_v\equiv(\zeta_v,-2\pi\partial_\theta)$. The map

$$\iota\equiv p\circ\widetilde{\iota}\colon S\big(TY^{\omega}\big)\times\mathbb{R}^+\longrightarrow TY^{\omega}$$

is a $\mathbb{T}$-equivariant diffeomorphism onto $TY^{\omega}-Y$.

Let λ be a $\mathbb{T}$-invariant connection 1-form on the principal S^1-bundle $S(TY^\omega)\longrightarrow Y$, i.e. λ is a 1-form on $S(TY^\omega)$ so that

$$\lambda(\zeta_v)=2\pi \qquad \text{and} \qquad \iota_{\zeta_v}(\mathrm{d}\lambda)=0\,. \tag{4.29}$$

In light of the first condition above, the second condition is equivalent to λ being S^1-invariant. Let $\widetilde{\omega}$ and $\widetilde{\omega}'$ be the $\mathbb{T}$-invariant closed 2-forms on $S(TY^\omega)\times\mathbb{C}$ and $S(TY^\omega)\times\mathbb{R}$, respectively, given by

$$\widetilde{\omega}=\mathrm{pr}^*\omega+\omega_{\mathbb{C}}+\frac{1}{2}\mathrm{d}\big(|z|^2\lambda\big) \qquad \text{and} \qquad \widetilde{\omega}'=\mathrm{pr}^*\omega+\mathrm{d}(t\lambda),$$

where $\omega_{\mathbb{C}}$ is the standard symplectic form on $\mathbb{C}$ as in (1.19), z is the standard coordinate on $\mathbb{C}$, and t is the standard coordinate on $\mathbb{R}$. Since

$$\widetilde{\omega}'_{(w,0)}=\omega_{\mathrm{pr}(w)}+\mathrm{d}_0t\wedge\lambda_w \qquad \forall\, w\in S(TY^\omega), \tag{4.30}$$

the 2-form $\widetilde{\omega}'$ is nondegenerate (and thus symplectic) on some neighborhood $\mathcal{U}'\subset S(TY^\omega)\times\mathbb{R}$ of $S(TY^\omega)\times\{0\}$.

By (4.29) and the last coordinate of the map $\widetilde{\iota}$ taking only real values,

$$\iota_{\zeta_v}\widetilde{\omega}'=-2\pi\mathrm{d}t \qquad \text{and} \qquad \widetilde{\omega}'\big|_{S(TY^\omega)\times\mathbb{R}^+}=\widetilde{\iota}^*\widetilde{\omega}\,, \tag{4.31}$$

respectively. With (r,θ) denoting the standard radius-angle coordinates on $\mathbb{C}$ so that $\omega_{\mathbb{C}}=r\mathrm{d}r\wedge\mathrm{d}\theta$,

$$\big(\iota_{\widetilde{\zeta}_v}\widetilde{\omega}\big)_{(w,r\mathrm{e}^{\mathrm{i}\theta})}=0-2\pi\iota_{\partial_\theta}\omega_{\mathbb{C}}+\iota_{\zeta_v}\big(r\mathrm{d}r\wedge\lambda+\frac{1}{2}r^2\mathrm{d}\lambda\big)=2\pi r\mathrm{d}r-2\pi r\mathrm{d}r+0=0\,; \tag{4.32}$$

the middle equality above follows again from (4.29). Since the S^1-action (4.28) preserves the 2-form $\widetilde{\omega}$, (4.32) implies that there is a unique 2-form ω_{TY^ω} on (the total space of) TY^ω so that $p^*\omega_{TY^\omega}=\widetilde{\omega}$. This form is $\mathbb{T}$-equivariant and closed and satisfies

$$\widetilde{\omega}'\big|_{S(TY^\omega)\times\mathbb{R}^+}=\iota^*\omega_{TY^\omega}; \tag{4.33}$$

see the second equation in (4.31).

Let λ_S be the 1-form on $L_S=TY^\omega$ determined by λ as in Exercise B.17(a). Thus,

$$\omega_{TY^\omega}=\mathrm{pr}^*\omega+\frac{1}{2}\mathrm{d}\lambda_S.$$

Along with Exercise B.17(b), this implies that the closed 2-form ω_{TY^ω} on TY^ω satisfies (2.16) with $TY^c=TY^\omega$. By Proposition 2.15(2), there thus exists a $\mathbb{T}$-equivariant tubular neighborhood identification $\Phi\colon\mathcal{U}\longrightarrow U$ for Y in X such that $\mathcal{U}\subset TY^\omega$ and $\Phi^*\omega=\omega_{TY^\omega}|_{\mathcal{U}}$. Along with (4.33), the last identity gives

$$\widetilde{\omega}'\big|_{\iota^{-1}(\mathcal{U})}=\iota^*\Phi^*\omega. \tag{4.34}$$

Let $\mathcal{U}''\subset\mathcal{U}'\subset S(TY^\omega)\times\mathbb{R}$ be a $\mathbb{T}$-invariant tubular neighborhood of $S(TY^\omega)\times\{0\}$ so that

$$\mathcal{U}^+\equiv\mathcal{U}''\cap\big(S(TY^\omega)\times\mathbb{R}^+\big)\subset\iota^{-1}(\mathcal{U}).$$

Since the diffeomorphism $\Phi\circ\iota\colon \mathcal{U}^+ \longrightarrow \Phi(\iota(\mathcal{U}^+))$ is $\mathbb{T}$-equivariant and satisfies (4.34) with $\iota^{-1}(\mathcal{U})$ replaced by $\mathcal{U}^+$,

$$\mu\circ\Phi\circ\iota\colon \mathcal{U}^+ \longrightarrow T^*_{\mathbb{1}}\mathbb{T} \tag{4.35}$$

is a moment map for the $\mathbb{T}$-action on $(\mathcal{U}^+,\widetilde{\omega}'|_{\mathcal{U}^+})$. By Exercise 4.18, it extends to a moment map $\widetilde{\mu}'\colon \mathcal{U}''\longrightarrow T^*_{\mathbb{1}}\mathbb{T}$ for the $\mathbb{T}$-action on $(\mathcal{U}'',\widetilde{\omega}'|_{\mathcal{U}''})$. Since

$$\lim_{t\longrightarrow 0^+}\iota(w,t)=\mathrm{pr}(w)\in Y\subset TY^{\omega}, \quad \lim_{t\longrightarrow 0^+}\mu\big(\Phi\big(\iota(w,t)\big)\big)=\mu\big(\mathrm{pr}(w)\big) \qquad \forall\, w\in S(TY^{\omega}),$$

the first equation in (4.31) implies that

$$\widetilde{\mu}'_v(w,t)=2\pi t+c, \qquad \widetilde{\mu}'^{-1}_v\big(\langle\mathscr{H}\rangle^{\partial}\big)=S(TY^{\omega})\times\{0\}, \tag{4.36}$$

and $\widetilde{\mu}'=\mu\circ\mathrm{pr}$ on $S(TY^{\omega})\times\{0\}=S(TY^{\omega})$.

We define

$$X'=\big((X-Y)\sqcup\mathcal{U}''\big)/\sim, \quad \mathcal{U}^+\ni(w,t)\sim\Phi\big(\iota(w,t)\big)\in X-Y, \tag{4.37}$$
$$\omega'_{[x]}=\begin{cases}\omega_x, & \text{if } x\in X-Y;\\ \widetilde{\omega}'_x, & \text{if } x\in\mathcal{U}'';\end{cases} \qquad \mu'([x])=\begin{cases}\mu(x), & \text{if } x\in X-Y;\\ \widetilde{\mu}'(x), & \text{if } x\in\mathcal{U}''.\end{cases}$$

Suppose $x\in X-Y-\Phi(\iota(\mathcal{U}^+))$ and $x'\in\mathcal{U}''-\mathcal{U}^+$. If x' does not lie in the closure $\mathrm{Cl}_{\mathcal{U}''}\mathcal{U}^+$ of $\mathcal{U}^+$ in $\mathcal{U}''$, then the images of $X-Y$ and $\mathcal{U}''-\mathrm{Cl}_{\mathcal{U}''}\mathcal{U}^+$ in X' under the quotient projection

$$q\colon (X-Y)\sqcup\mathcal{U}''\longrightarrow X'$$

are disjoint open subsets around $[x]$ and $[x']$, respectively. If

$$x'\in\mathrm{Cl}_{\mathcal{U}''}\mathcal{U}^+-\mathcal{U}^+=Y\times\{0\}$$

and $U,U'\subset X$ are disjoint open neighborhoods of x and Y, respectively, then

$$q(U),\, q\big(\big(\mathcal{U}''\cap(S(TY^{\omega})\times\mathbb{R}^{\le 0})\big)\cup\iota^{-1}\big(\Phi^{-1}(U')\big)\big)\subset X'$$

are disjoint open subsets around $[x]$ and $[x']$, respectively. Since the restrictions of q to the Hausdorff spaces $X-Y$ and $\mathcal{U}''$ are homeomorphisms onto open subsets of X', it follows that X' is a Hausdorff space and a smooth manifold. By (4.34) and the assumption on $\mathcal{U}'$ below (4.30), ω' is a well-defined symplectic form on X'. Since the smooth map $\widetilde{\mu}'$ is an extension of the map (4.35), $\mu'\colon X'\longrightarrow T^*_{\mathbb{1}}\mathbb{T}$ is a well-defined smooth map.

Since the identification of the spaces $X-Y$ and $\mathcal{U}''$ over $\mathcal{U}^+\subset\mathcal{U}''$ in (4.37) is $\mathbb{T}$-equivariant, the $\mathbb{T}$-action ψ on $X-Y\subset X$ and the $\mathbb{T}$-action $\mathrm{d}\psi$ on $\mathcal{U}''\subset S(TY^{\omega})\times\mathbb{R}$ induce a smooth $\mathbb{T}$-action ψ' on X' which preserves ω'. Since $\mu|_{X-Y}$ and $\widetilde{\mu}'$ are moment maps for the $\mathbb{T}$-actions on $(X-Y,\omega|_{X-Y})$ and $(\mathcal{U}'',\widetilde{\omega}'|_{\mathcal{U}''})$, μ' is a moment map for the $\mathbb{T}$-action ψ' on (X',ω'). By (4.25) and (4.36),

$$\mu'^{-1}\big(\langle\mathscr{H}_2\rangle^{\partial}\big)=q\big(S(TY^{\omega})\times\{0\}\big)\subset X'.$$

Since the restriction of q to $\mathcal{U}''$ is a diffeomorphism onto $q(\mathcal{U}'')$ and $\mathbb{T}_\upsilon$ acts freely on $S(TY^\omega)\times\{0\}$, $\mathbb{T}_\upsilon$ acts freely on $\mu'^{-1}(\langle\!\langle\mathscr{H}_2\rangle^\partial)$ as well. Thus, (X',ω',ψ',μ') is an $\mathscr{H}_2$-cuttable Hamiltonian $\mathbb{T}$-manifold with

$$\begin{gathered}\mu'^{-1}\big(\langle\!\langle\emptyset\rangle^\partial_{\mathscr{H}_2}\big)=q(X-Y)\approx Y,\quad \mu'^{-1}\big(\langle\!\langle\mathscr{H}_2\rangle^\partial\big)=q\big(S(TY^\omega)\times\{0\}\big)\subset X',\\ X=\mu'^{-1}\big(\langle\!\langle\mathscr{H}_2\rangle\!\rangle\big)/\!\sim,\quad x\sim\psi'_u(x)\ \forall\,x\in\mu'^{-1}\big(\langle\!\langle\mathscr{H}_2\rangle^\partial\big),\ u\in\mathbb{T}_\upsilon,\quad Y=\mu'^{-1}\big(\langle\!\langle\mathscr{H}_2\rangle^\partial\big)/\mathbb{T}_\upsilon,\\ \big\{q|_{X-Y}\big\}^*p^*_{\mathscr{H}_2}\omega=\omega|_{X-Y},\quad \big\{q|_{S(TY^\omega)\times\{0\}}\big\}^*\big\{p_{\mathscr{H}_2}|_{\mu'^{-1}(\langle\!\langle\mathscr{H}_2\rangle^\partial)}\big\}^*\big(\omega|_{TY}\big)=\widetilde{\omega}'\big|_{S(TY^\omega)\times\{0\}},\end{gathered}$$

where $p_{\mathscr{H}_2}:\mu'^{-1}(\langle\!\langle\mathscr{H}_2\rangle\!\rangle)\longrightarrow X$ is the quotient projection; the two identities on the first line above follow from (4.25) and the second statement in (4.36). Furthermore, the map $p_{\mathscr{H}_2}$ is $\mathbb{T}$-equivariant and the compositions

$$p_{\mathscr{H}_2}\circ q\colon X-Y\longrightarrow\mu^{-1}\big(\langle\!\langle\emptyset\rangle^\partial_{\mathscr{H}_2}\big)\quad\text{and}\quad p_{\mathscr{H}_2}\circ q\colon S(TY^\omega)\times\{0\}\longrightarrow\mu^{-1}\big(\langle\!\langle\mathscr{H}_2\rangle^\partial\big)$$

are submersions. By the uniqueness statement of Theorem 4.11, (4.22) thus holds.

Every topological component of the tubular neighborhood $q(\mathcal{U}'')\subset X'$ of $q(Y\times\{0\})$ intersects $q(X-Y)$. If X is connected, then so is $X-Y$ (because $Y\subset X$ is a submanifold of codimension 2). It then follows that X' is also connected. If (X,ω,ψ,μ) is $\mathscr{H}$-cut, then (X',ω',ψ',μ') is $\mathscr{H}_1$-cut by Corollary 4.17 if $\mathcal{U}''$ is sufficiently small.

Suppose both conditions in (b) hold and $\mathscr{H}'_1\subset\mathscr{H}_1=\mathscr{H}-\{\upsilon\}$. Let $\mathscr{H}'_1\upsilon=\mathscr{H}'_1\sqcup\{\upsilon\}$. By the above, $\mu'^{-1}(\langle\!\langle\mathscr{H}'_1\rangle^\partial)\subset X'$ is an ω'-symplectic submanifold consisting of components of the fixed locus $X'^{\mathbb{T}_{\mathscr{H}'_1}}$ of the restriction of the action ψ' to the subtorus $\mathbb{T}_{\mathscr{H}'_1}\subset\mathbb{T}$. Since the restriction of the quotient projection q above to $\mathcal{U}''$ is a $\mathbb{T}$-equivariant diffeomorphism onto the open subset $q(\mathcal{U}'')\subset X'$ and $\widetilde{\mu}'=\mu'\circ q$ on $\mathcal{U}''$, it follows that $\widetilde{\mu}'^{-1}(\langle\!\langle\mathscr{H}'_1\rangle^\partial)\subset\mathcal{U}''$ is an $\widetilde{\omega}'$-symplectic submanifold consisting of components of $\mathcal{U}''^{\mathbb{T}_{\mathscr{H}'_1}}$. By (4.23) with $\mathscr{H}'=\mathscr{H}'_1\upsilon$, the restriction

$$\mu\colon\mu^{-1}\big(\langle\!\langle\mathscr{H}'_1\upsilon\rangle^\partial_{\mathscr{H}}\big)=\mu^{-1}\big(\langle\!\langle\mathscr{H}'_1\rangle^\partial_{\mathscr{H}_1}\big)\cap Y\longrightarrow\langle\mathscr{H}'_1\upsilon\rangle^\partial_{\mathscr{H}}\cap\mu(X)=\langle\mathscr{H}'_1\rangle^\partial_{\mathscr{H}_1}\cap\mu(Y)$$

is a principal $\mathbb{T}/\mathbb{T}_{\mathscr{H}'_1\upsilon}$-bundle. Since $\mathbb{T}_\upsilon$ acts freely on the fibers of the circle bundle (4.27), it follows that the restriction

$$\widetilde{\mu}'=\mu\circ\mathrm{pr}\colon S(TY^\omega)\big|_{\mu^{-1}(\langle\!\langle\mathscr{H}'_1\upsilon\rangle^\partial_{\mathscr{H}})}\times\{0\}=\widetilde{\mu}'^{-1}\big(\langle\!\langle\mathscr{H}'_1\rangle^\partial_{\mathscr{H}_1}\big)\cap\big(S(TY^\omega)\times\{0\}\big)\longrightarrow\langle\mathscr{H}'_1\rangle^\partial_{\mathscr{H}_1}\cap\mu(Y)$$

is a principal $\mathbb{T}/\mathbb{T}_{\mathscr{H}'_1}$-bundle. By Exercise 4.8(d) with $\mathscr{H}$ replaced by $\mathscr{H}'_1$ and the submanifold $\langle\mathscr{H}'_1\upsilon\rangle^\partial\subset\langle\mathscr{H}'_1\rangle^\partial$ being closed, the moment map

$$\widetilde{\mu}'\colon\widetilde{\mu}'^{-1}\big(\langle\!\langle\mathscr{H}'_1\rangle^\partial_{\mathscr{H}_1}\big)\longrightarrow\langle\mathscr{H}'_1\rangle^\partial$$

is then a submersion and thus also a principal $\mathbb{T}/\mathbb{T}_{\mathscr{H}'_1}$-bundle over its image if $\mathcal{U}''$ is sufficiently small. Since $q|_{\mathcal{U}''}$ is a $\mathbb{T}$-equivariant diffeomorphism onto $q(\mathcal{U}'')\subset X'$ and $\widetilde{\mu}'=\mu'\circ q$ on $\mathcal{U}''$, it follows that

$$\mu'\colon\mu'^{-1}\big(\langle\!\langle\mathscr{H}'_1\rangle^\partial_{\mathscr{H}_1}\big)\cap q(\mathcal{U}'')\longrightarrow\langle\mathscr{H}'_1\rangle^\partial_{\mathscr{H}_1}\cap\widetilde{\mu}'(\mathcal{U}'')=\langle\mathscr{H}'_1\rangle^\partial_{\mathscr{H}_1}\cap\mu'\big(q(\mathcal{U}'')\big).$$

By (4.23) with $\mathscr{H}'=\mathscr{H}'_1$ and (4.25), the restriction

$$\mu\colon\mu^{-1}\big(\langle\!\langle\mathscr{H}'_1\rangle^\partial_{\mathscr{H}}\big)\cap(X-Y)=\mu^{-1}\big(\langle\!\langle\mathscr{H}'_1\rangle^\partial_{\mathscr{H}_1}\big)\cap(X-Y)\longrightarrow\langle\mathscr{H}'_1\rangle^\partial_{\mathscr{H}_1}\cap\mu(X-Y)$$

is a principal $\mathbb{T}/\mathbb{T}_{\mathscr{H}_1'}$-bundle. Since the restriction of q to $X-Y$ is a $\mathbb{T}$-equivariant diffeomorphism onto the open subset $q(X-Y)\subset X'$ and $\mu=\mu'\circ q$ on $X-Y$, it follows that the restriction

$$\mu'\colon \mu'^{-1}\big(\langle\mathscr{H}_1'\rangle^{\partial}_{\mathscr{H}_1}\big)\cap q(X-Y) \longrightarrow \langle\mathscr{H}_1'\rangle^{\partial}_{\mathscr{H}_1}\cap\mu(X-Y) = \langle\mathscr{H}_1'\rangle^{\partial}_{\mathscr{H}_1}\cap\mu'\big(q(X-Y)\big)$$

is also a principal $\mathbb{T}/\mathbb{T}_{\mathscr{H}_1'}$-bundle. Since $X'=q(\mathcal{U}'')\cup q(X-Y)$, we conclude that the restriction (4.24) is a principal $\mathbb{T}/\mathbb{T}_{\mathscr{H}_1'}$-bundle. $\square$

4.4 Delzant's Theorem

Let $\mathbb{T}$ be a torus. A symplectic toric $\mathbb{T}$-manifold is a connected Hamiltonian $\mathbb{T}$-manifold (X,ω,ψ,μ) so that

$$\dim X = 2\dim\mathbb{T}, \tag{4.38}$$

the action ψ on X is effective, and the moment map $\mu\colon X\longrightarrow T_{\mathbb{1}}^*\mathbb{T}$ is proper. By Delzant's Theorem, Theorem 4.20 below, the map

$$(X,\omega,\psi,\mu)\longrightarrow\mu(X)$$

induces a bijection between the equivalence classes of compact symplectic toric $\mathbb{T}$-manifolds and Delzant polytopes in $T_{\mathbb{1}}^*\mathbb{T}$.

Theorem 4.20 ([20, Théorème 2.1, Section 3.2])**.** *Let $\mathbb{T}$ be a torus.*

(0) If (X,ω,ψ,μ) is a compact symplectic toric $\mathbb{T}$-manifold, then the moment polytope $\mu(X)\subset T_{\mathbb{1}}^*\mathbb{T}$ is Delzant.

(1) For every Delzant polytope $P\subset T_{\mathbb{1}}^*\mathbb{T}$, there exists a (necessarily compact) symplectic toric $\mathbb{T}$-manifold (X,ω,ψ,μ) with $\mu(X)=P$.

(2) If (X,ω,ψ,μ) and (X',ω',ψ',μ') are compact symplectic toric $\mathbb{T}$-manifolds with $\mu(X)=\mu(X')$, then there exists a $\mathbb{T}$-equivariant diffeomorphism

$$\Phi\colon X\longrightarrow X' \qquad s.t.\quad \omega=\Phi^*\omega',\quad \mu=\mu'\circ\Phi.$$

Claim 0 of this theorem is a consequence of the following description of the moment map $\mu\colon X\longrightarrow T_{\mathbb{1}}^*\mathbb{T}$.

(0^+) If (X,ω,ψ,μ) is a compact symplectic toric $\mathbb{T}$-manifold, then there exist a subtorus $\mathbb{T}_F\subset\mathbb{T}$ for every face F of the polytope $\mu(X)$ and a full tuple $(\alpha_e)_{e\in\mathrm{Edg}(P)}$ of integral edge vectors for $\mu(X)$ so that

- $(0^+\mathrm{a})$ $\mathbb{T}_F=\big\{\mathrm{e}^v\colon v\in T_{\mathbb{1}}\mathbb{T},\ \alpha_e(v)=0\,\forall\, e\in\mathrm{Edg}_\eta(F)\big\}$ for every face F of P and $\eta\in\mathrm{Ver}(F)$;
- $(0^+\mathrm{b})$ $\mathbb{T}_x(\psi)=\mathbb{T}_F$ for every face F of P and every $x\in\mu^{-1}(F^\circ)$;
- $(0^+\mathrm{c})$ the restriction $\mu\colon\mu^{-1}(F^\circ)\longrightarrow F^\circ$ is a principal $\mathbb{T}/\mathbb{T}_F$-bundle with ω-isotropic fibers, i.e. $\omega|_{T\mu^{-1}(\eta)}=0$ for every $\eta\in F^\circ$.

Similarly to (E_k) and (G_k) in Theorem 3.1, this statement is a direct consequence of the equivariant splitting (2.11) of $TX|_Y$ for each $Y\in\pi_0(X^\psi)$ and Corollary 2.35; we establish it below. By (4.38) and $(0^+\mathrm{c})$, the dimension of $\mu(X)$ is the same as the dimension of $\mathbb{T}$ and $\mathbb{T}_{\mu(X)}=\{\mathbb{1}\}$. Along with $(0^+\mathrm{a})$, the latter implies that for each vertex η of $\mu(X)$ the components α_e with $\eta\in e$ span $(T_{\mathbb{1}}^*\mathbb{T})_{\mathbb{Z}}$ over $\mathbb{Z}$. Combining this with the last claim of Theorem 3.1(C_k) and (4.38) again, we conclude that these components form a $\mathbb{Z}$-basis for $(T_{\mathbb{1}}^*\mathbb{T})_{\mathbb{Z}}$.

Exercise 4.21. Suppose (X,ω,ψ,μ) is a compact symplectic toric $\mathbb{T}$-manifold, $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ is a Delzant subcollection so that $\mu(X)=\langle\mathscr{H}\rangle$, and $\mathscr{H}'\subset\mathscr{H}$. Show that $\mathbb{T}_{\mathscr{H}'}\subset\mathbb{T}$ is the subtorus $\mathbb{T}_{\langle\mathscr{H}'\rangle^{\partial}\cap\langle\mathscr{H}\rangle}\subset\mathbb{T}$ as in (0^+), $Y_{\mathscr{H}'}\equiv\mu^{-1}(\langle\mathscr{H}'\rangle^{\partial})$ is a connected component of the fixed locus $X^{\mathbb{T}_{\mathscr{H}'}}\subset X$ of $\psi|_{\mathbb{T}_{\mathscr{H}'}}$, and $TX|_{Y_{\mathscr{H}'}}$ splits as in (4.2) and (4.3). Conclude that the Hamiltonian $\mathbb{T}$-manifold (X,ω,ψ,μ) is $\mathscr{H}$-cut.

Proof of (0^+). Let J be a ψ-invariant ω-compatible almost complex structure on X (as provided by Exercise 2.13). For $Y\in\pi_0(X^{\psi})$, let

$$S(Y),\mathcal{C}_{\mu(Y)}\big(\rho^*S(Y)\big)\subset T^*_{\mathbb{1}}\mathbb{T}$$

and $\mathcal{N}^{\alpha}_X Y$ for each $\alpha\in S(Y)$ be as in Proposition 2.33 with $(\mathbb{R}^k,0)$ replaced by $(\mathbb{T},\mathbb{1})$ and thus with $\rho=\mathrm{id}$. By (2.35) and (4.38), $|S(Y)|$ is at most the dimension of $\mathbb{T}$. Since the action ψ is effective, Proposition 2.10(1) then implies that $S(Y)$ is a $\mathbb{Z}$-basis for $T^*_{\mathbb{1}}\mathbb{T}$ for every $Y\in\pi_0(X^{\psi})$; this remains the case if some of the elements of $S(Y)$ are negated. In particular, the polytope $\mu(X)\subset T^*_{\mathbb{1}}\mathbb{T}$ is of full dimension. Furthermore, for every $Y\in\pi_0(X^{\psi})$, the cone $\mathcal{C}_{\mu(Y)}(S(Y))$ contains no lines, $\mu(Y)\in\mathrm{Ver}(\mu(X))$, and for every $S\subset S(Y)$ the μ-image of the topological component X^S_Y of the $\psi|_{\mathbb{T}_S}$-fixed locus X^S containing Y lies in the cone $\mathcal{C}_{\mu(Y)}(S)$ of dimension $|S|$ and contains a neighborhood of the vertex $\mu(Y)$ in this cone.

By Proposition 2.15(1), $(X^S_Y,\omega|_{X^S_Y},\psi|_{X^S_Y},\mu|_{X^S_Y})$ is a closed connected Hamiltonian $\mathbb{T}$-manifold for every $Y\in\pi_0(X^{\psi})$ and $S\subset S(Y)$. Thus,

$$\mu(X^S_Y)=\mathrm{CH}\big(\mu\big((X^S_Y)^{\psi}\big)\big)=\mathrm{CH}\big(\mu(X^{\psi}\cap X^S_Y)\big)$$

by Theorem 3.1(C_k). Since $\mu(X^{\psi}\cap X^S_Y)\subset\mathrm{Ver}(\mu(X))$, it follows that $\mu(X^S_Y)$ is the face $F_{\mu(Y);S}(\mu(X))$ of the polytope $\mu(X)$ containing the edges

$$e_{\mu(Y);\alpha}\equiv\mu(X)\cap\{\mu(Y)+t_{\alpha}\alpha\colon t_{\alpha}\in\mathbb{R}^{\ge0}\}$$

with $\alpha\in S$. Since $S(Y)$ is a $\mathbb{Z}$-basis for $T^*_{\mathbb{1}}\mathbb{T}$ for every $Y\in\pi_0(X^{\psi})$, Exercise 2.23 implies that for each $x\in X$ there exist $Y_x\in\pi_0(X^{\psi})$ and $S_x\subset S(Y_x)$ so that the ψ-stabilizer $\mathbb{T}_x(\psi)\subset\mathbb{T}$ of x is the subtorus $\mathbb{T}_{S_x}\subset\mathbb{T}$ and $x\in X^{S_x}_{Y_x}$. It follows that

$$\begin{aligned}\mu(x)&\in F^{\circ}_{\mu(Y_x);S_x}\big(\mu(X)\big) &&\forall\,x\in X,\\ \mu^{-1}\big(F^{\circ}_{\mu(Y);S}(\mu(X))\big)&=\big\{x\in X^S_Y\colon \mathbb{T}_x(\psi)=\mathbb{T}_S\big\} &&\forall\,Y\in\pi_0(X^{\psi}),\ S\subset S(Y),\end{aligned}\tag{4.39}$$

where $F^{\circ}_{\mu(Y);S}(\mu(X))\subset F_{\mu(Y);S}(\mu(X))$ is the interior as before.

Suppose $e\in\mathrm{Edg}(\mu(X))$ is an edge of the polytope $\mu(X)$ and thus $e=e_{\mu(Y);\alpha}$ for some $Y\in\pi_0(X^{\psi})$ and $\alpha\in S(Y)$. We then set $\alpha_e=\alpha$. If $Y'\in\pi_0(X^{\psi})$ is such that $\mu(Y')$ is the vertex of $e_{\mu(Y);\alpha}$ other than $\mu(Y)$, then $-\alpha\in S(Y')$ and $e=e_{\mu(Y');-\alpha}$. Thus, $(\alpha_e)_{e\in\mathrm{Edg}(\mu(X))}$ is a full tuple of integral edge vectors for the polytope $\mu(X)$ such that for each vertex η of $\mu(X)$ the components α_e with $e\in\mathrm{Edg}_{\eta}(\mu(X))$ form a $\mathbb{Z}$-basis for $(T^*_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$ and

$$\mathbb{T}_S=\bigcap_{\alpha\in S}\mathbb{T}_{\alpha_{e_{\mu(Y);\alpha}}}\qquad\forall\,Y\in\pi_0(X^{\psi}),\ S\subset S(Y).\tag{4.40}$$

Suppose $F \subset \mu(X)$ is a face of the polytope $\mu(X)$ and thus $F = F_{\mu(Y);S}(\mu(X))$ for some $Y \in \pi_0(X^\psi)$ and $S \subset S(Y)$. We then set $\mathbb{T}_F = \mathbb{T}_S$. Thus, $\mathbb{T}_F \subset \mathbb{T}$ is a subtorus. By (4.39), (0^+b) holds. This implies that $\mathbb{T}_F$ is independent of the choice of $\mu(Y) \in F$. By (4.40), (0^+a) thus holds for all $\eta \in F$.

Let $Y \in \pi_0(X^\psi)$, $S \subset S(Y)$, and $F = F_{\mu(Y);S}(\mu(X))$. Since

$$\dim Y + \sum_{\alpha \in S(Y)} \mathrm{rk}\, \mathcal{N}_X^\alpha Y = \dim X, \quad \mathrm{rk}\, \mathcal{N}_X^\alpha Y \geq 2 \ \forall\, \alpha \in S(Y), \quad \text{and} \quad |S(Y)| = \dim \mathbb{T} = (\dim X)/2,$$

we conclude that $\dim Y = 0$ and $\mathrm{rk}\, \mathcal{N}_X^\alpha Y = 2$ for every $\alpha \in S(Y)$. Along with Proposition 2.10(2), this implies that

$$\dim X_Y^S = 2|S| = 2 \dim F = 2 \dim(\mathbb{T}/\mathbb{T}_F)\,. \tag{4.41}$$

Since $\alpha(v) = 0$ for all $\alpha \in S$ and $v \in T_{\mathbb{1}}\mathbb{T}_S$, the affine map

$$\Phi_{Y;S} \colon \big\{\mu(Y) + \sum_{\alpha \in S} t_\alpha \alpha \colon t_\alpha \in \mathbb{R}\big\} \longrightarrow T_{\mathbb{1}}^*(\mathbb{T}/\mathbb{T}_S), \quad \big\{\Phi_{Y;S}(\eta)\big\}(v) = \eta(v) - \big\{\mu(Y)\big\}(v),$$

is a well-defined surjection and thus affine isomorphism by the last equality in (4.41). By (4.39), the torus $\mathbb{T}/\mathbb{T}_F$ acts freely on $\mu^{-1}(F^\circ) \subset X_Y^S$ via ψ; we denote this action by $\psi_{Y;S}$. By Theorem 3.1(A_k) applied to $(X_Y^S, \omega|_{X_Y^S}, \psi|_{X_Y^S}, \mu|_{X_Y^S})$, the fibers of the restriction

$$\mu \colon \mu^{-1}(F^\circ) \longrightarrow F^\circ$$

are connected. Since $F^\circ \subset F$ is open, $\mu^{-1}(F^\circ) \subset X_Y^S$ is a symplectic submanifold and

$$\dim F^\circ = \dim X_Y^S = 2 \dim(\mathbb{T}/\mathbb{T}_F)$$

by (4.41). Thus, (0^+c) follows from Exercise 4.22(a) below with $\mathbb{T}$ and (X, ω, ψ, μ) replaced by $\mathbb{T}/\mathbb{T}_S$ and $(\mu^{-1}(F^\circ), \omega|_{\mu^{-1}(F^\circ)}, \psi_{Y;S}, \Phi_{Y;S} \circ \mu|_{\mu^{-1}(F^\circ)})$, respectively. $\square$

Theorem 4.20(1) is readily obtained by applying the Hamiltonian symplectic cut construction of Theorem 4.11 to the Hamiltonian $\mathbb{T}$-manifold $(T_{\mathbb{1}}^*\mathbb{T} \times \mathbb{T}, \omega_{\mathbb{T}}, \psi_{\mathbb{T}}, \mu_{\mathbb{T}})$ of Exercise 1.35 with respect to the collection $\mathscr{H}$ of half-spaces of $T_{\mathbb{1}}^*\mathbb{T}$ cutting out the polytope P. We establish Theorem 4.20(2) by combining the reverse Hamiltonian symplectic cut construction of Theorem 4.19 with Proposition 4.23 below, which describes certain symplectic toric manifolds; this implements the argument sketched in the proof of [42, Theorem 7.5.10].

Exercise 4.22. Suppose $\mathbb{T}$ is a torus and (X, ω, ψ, μ) is a Hamiltonian $\mathbb{T}$-manifold so that (4.38) holds, the action ψ is free, and the fibers of μ are connected.

(a) Show that $\mu(X) \subset T_{\mathbb{1}}^*\mathbb{T}$ is an open subset, $\mu \colon X \longrightarrow \mu(X)$ is a principal $\mathbb{T}$-bundle, and the fibers of μ are Lagrangian submanifolds of (X, ω), i.e.

$$\dim \mu^{-1}(\alpha) = (\dim X)/2 \quad \text{and} \quad \omega\big|_{T\mu^{-1}(\alpha)} = 0 \qquad \forall\, \alpha \in \mu(X).$$

Hint: use Exercise 2.21.

(b) Let η be a 1-form on $\mu(X)$. Show that the vector field ζ_η on X defined by $\iota_{\zeta_\eta}\omega = \mu^*\eta$ is μ-vertical, i.e.

$$\mathrm{d}\mu(\zeta_\eta) = 0 \in \Gamma\big(\mu(X); T\mu(X)\big).$$

(c) Let $(T_{\mathbb{1}}^*\mathbb{T}\times\mathbb{T},\omega_{\mathbb{T}},\psi_{\mathbb{T}},\mu_{\mathbb{T}})$ be the Hamiltonian $\mathbb{T}$-manifold of Exercise 1.35. Suppose $s\colon \mu(X)\longrightarrow X$ is a (smooth) Lagrangian section of μ, i.e. $\mu\circ s=\mathrm{id}_{\mu(X)}$ and $s^*\omega=0$. Show that the map

$$\Phi\colon \mu(X)\times\mathbb{T}\longrightarrow X, \qquad \Phi(\alpha,u)=\psi_u\big(s(\alpha)\big),$$

is a $\mathbb{T}$-equivariant diffeomorphism such that $\Phi^*\omega=\omega_{\mathbb{T}}|_{\mu(X)\times\mathbb{T}}$ and $\mu\circ\Phi=\mu_{\mathbb{T}}|_{\mu(X)\times\mathbb{T}}$. *Hint:* choose a $\mathbb{Z}$-basis $v_1,\ldots,v_k\in T_{\mathbb{1}}\mathbb{T}$ for the lattice $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\subset T_{\mathbb{1}}\mathbb{T}$ and replace μ by the corresponding Hamiltonian $H\colon X\longrightarrow\mathbb{R}^k$ and $(T_{\mathbb{1}}^*\mathbb{T}\times\mathbb{T},\omega_{\mathbb{T}},\psi_{\mathbb{T}},H_{\mathbb{T}})$ by the Hamiltonian $\mathbb{T}$-manifold $(\mathbb{R}^k\times\mathbb{T}^k,\omega_k,\psi_k,H_k)$ as in Exercise 1.32.

Proposition 4.23. *Suppose $\mathbb{T}$ is a torus and (X,ω,ψ,μ) is a Hamiltonian $\mathbb{T}$-manifold so that (4.38) holds, the action ψ is free, the fibers of μ are connected, and $\mu(X)\subset T_{\mathbb{1}}^*\mathbb{T}$ is contractible. Let $(T_{\mathbb{1}}^*\mathbb{T}\times\mathbb{T},\omega_{\mathbb{T}},\psi_{\mathbb{T}},\mu_{\mathbb{T}})$ be the Hamiltonian $\mathbb{T}$-manifold of Exercise 1.35. Then $\mu(X)\subset T_{\mathbb{1}}^*\mathbb{T}$ is an open subset and there exists a $\mathbb{T}$-equivariant diffeomorphism*

$$\Phi\colon \mu(X)\times\mathbb{T}\longrightarrow X \qquad s.t. \qquad \Phi^*\omega=\omega_{\mathbb{T}}\big|_{\mu(X)\times\mathbb{T}},\quad \mu\circ\Phi=\mu_{\mathbb{T}}\big|_{\mu(X)\times\mathbb{T}}.$$

Proof. The subset $\mu(X)\subset T_{\mathbb{1}}^*\mathbb{T}$ is open by Exercise 2.21(b). By Exercise 4.22(c), it remains to show that μ admits a Lagrangian section $\mu(X)\longrightarrow X$. Since $\mu(X)\subset T_{\mathbb{1}}^*\mathbb{T}$ is contractible, μ admits a section $s\colon\mu(X)\longrightarrow X$ and $s^*\omega=\mathrm{d}\eta$ for some 1-form η on $\mu(X)$. Let $\zeta_\eta\in\Gamma(X;TX)$ be the μ-vertical vector field of Exercise 4.22(b). Since the fibers of μ are compact and the vector field ζ_η is vertical, the flow of $-\zeta_\eta$,

$$\psi_t\colon X\longrightarrow X, \qquad \psi_0=\mathrm{id}_X, \quad \frac{\mathrm{d}}{\mathrm{d}t}\psi_t=-\zeta_\eta\circ\psi_t\,,$$

is defined for all $t\in\mathbb{R}$, $\psi_t\circ s\colon\mu(X)\longrightarrow X$ is a section of μ for every $t\in\mathbb{R}$, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_t^*\omega=\psi_t^*\big(\mathcal{L}_{-\zeta_\eta}\omega\big)=\psi_t^*\big(\mathrm{d}(\iota_{-\zeta_\eta}\omega)+\iota_{-\zeta_\eta}(\mathrm{d}\omega)\big)=\psi_t^*\big(\mathrm{d}(\mu^*(-\eta))+0\big)=-\mathrm{d}(\mu^*\eta)=-\mu^*s^*\omega,$$

where $\mathcal{L}$ is the Lie derivative; the second equality above holds by Cartan's formula for the Lie derivative. Thus,

$$s^*\psi_1^*\omega=s^*\big(\omega-\mu^*s^*\omega\big)=0,$$

i.e. $\psi_1\circ s\colon\mu(X)\longrightarrow X$ is a Lagrangian section of μ. □

Proof of Theorem 4.20. Since (0) in the statement of this theorem follows from (0^+) on page 94, as was shown just after the statement of (0^+), it remains to establish (1) and (2). Suppose $P\subset T_{\mathbb{1}}^*\mathbb{T}$ is a Delzant polytope and $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ is a Delzant subcollection so that $\mu(X)=\langle\mathscr{H}\rangle$, and $\mathscr{H}'\subset\mathscr{H}$. Let $(T_{\mathbb{1}}^*\mathbb{T}\times\mathbb{T},\omega_{\mathbb{T}},\psi_{\mathbb{T}},\mu_{\mathbb{T}})$ be the Hamiltonian $\mathbb{T}$-manifold of Exercise 1.35. By Theorem 4.11, the Hamiltonian $\mathbb{T}$-manifold

$$(X,\omega,\psi,\mu)\equiv\big(T_{\mathbb{1}}^*\mathbb{T}\times\mathbb{T},\omega_{\mathbb{T}},\psi_{\mathbb{T}},\mu_{\mathbb{T}}\big)_{\mathscr{H}}$$

as in (4.4) is then a closed connected Hamiltonian $\mathbb{T}$-manifold so that (4.38) holds, the $\mathbb{T}$-action ψ is effective (it is free on $\mu^{-1}(\langle\emptyset\rangle_{\mathscr{H}}^{\partial})$), and $\mu(X)=P$. This gives (1).

Suppose (X,ω,ψ,μ) is a (necessarily compact) symplectic toric $\mathbb{T}$-manifold with $\mu(X)=P$. In particular, X is connected. By Exercise 4.21 and $(0^+\mathrm{c})$ on page 94, (X,ω,ψ,μ) also satisfies (b) in the statement of Theorem 4.19. By Theorem 4.19 with $\mathscr{H}_1=\emptyset$ and $\mathscr{H}_2=\mathscr{H}$,

$$(X,\omega,\psi,\mu)=(X',\omega',\psi',\mu')_{\mathscr{H}} \tag{4.42}$$

for some $\mathscr{H}$-cuttable Hamiltonian $\mathbb{T}$-manifold (X',ω',ψ',μ') so that X' is connected and

$$\mu'\colon X'=\mu'^{-1}\big(\langle\emptyset\rangle_{\emptyset}^{\partial}\big)\longrightarrow\langle\emptyset\rangle_{\emptyset}^{\partial}\cap\mu'(X')=\mu'(X')$$

is a principal $\mathbb{T}$-bundle. By Exercise 2.21(b) and (4.42), $\mu'(X')\subset T_{\mathbb{1}}^{*}\mathbb{T}$ is an open neighborhood of the polytope P. By replacing X' with the preimage of a contractible neighborhood of P in $\mu'(X')$, we can assume $\mu'(X')$ is contractible. By Proposition 4.23, (X',ω',ψ',μ') is then isomorphic to $(U,\omega_{\mathbb{T}}|_U,\psi_{\mathbb{T}}|_U,\mu_{\mathbb{T}}|_U)$ for an open neighborhood $U\subset T_{\mathbb{1}}^{*}\mathbb{T}\times\mathbb{T}$ of $\mu_{\mathbb{T}}^{-1}(P)$. Along with (4.42) and Theorem 4.11, this implies that

$$(X,\omega,\psi,\mu)\approx\big(U,\omega_{\mathbb{T}}|_U,\psi_{\mathbb{T}}|_U,\mu_{\mathbb{T}}|_U\big)_{\mathscr{H}}=\big(T_{\mathbb{1}}^{*}\mathbb{T}\times\mathbb{T},\omega_{\mathbb{T}},\psi_{\mathbb{T}},\mu_{\mathbb{T}}\big)_{\mathscr{H}}.$$

This gives (2). $\square$

4.5 Twofold symplectic cut

This section presents the symplectic cut construction of [36, Remark 1.8] for a Hamiltonian circle action on neighborhood of a closed real hypersurface in a symplectic manifold. It produces a new symplectic manifold along with an isomorphic pair of symplectic divisors, i.e. closed symplectic submanifolds of codimension 2, with dual complex normal bundles. A generalization of this construction to a Hamiltonian torus action is introduced in [25]; it produces a new symplectic manifold with a simple normal crossings symplectic divisor in the sense of [23, Definition 2.1].

For a closed hypersurface Z in a manifold X, we denote by X_Z the manifold with boundary obtained from X by cutting it along Z and by

$$p_Z\colon X_Z\longrightarrow X$$

the associated surjective submersion that identifies pairs of points on the boundary ∂X_Z of X_Z that differ by a fixed-point-free smooth involution σ_Z. We identify $X_Z-\partial X_Z$ with $X-Z$ via p_Z. If Z is connected and the normal bundle $\mathcal{N}_XZ$ of Z in X is orientable, then ∂X_Z has two connected components. An action ψ of a connected Lie group on a neighborhood $U\subset X$ of Z that preserves Z induces an action $\widetilde{\psi}$ on the neighborhood $p_Z^{-1}(U)\subset X_Z$ of ∂X_Z that preserves ∂X_Z; the restriction of $\widetilde{\psi}$ to ∂X_Z commutes with the involution σ_Z.

If (X,ω) is a symplectic manifold, we call a tuple (U,ψ,H,c) cutting data for (X,ω) if $U\subset X$ is an open subset, ψ is an S^1-action on U, H is a Hamiltonian for ψ with respect to $\omega|_U$, and $c\in\mathbb{R}$ is such that the subset $H^{-1}(c)\subset U$ is closed in X and S^1 acts freely on it. By Exercise 2.21(b), the assumptions on c imply that $H^{-1}(c)\subset X$ is a closed hypersurface.

Theorem 4.24. *Suppose (X,ω) is a symplectic manifold and (U,ψ,H,c) is cutting data for (X,ω). There exists a unique symplectic manifold (X',ω') so that*

(SC_1) $X'=X_{H^{-1}(c)}/\!\sim$ with $x\sim x'$ if $x,x'\in\partial X_{H^{-1}(c)}$ and $x'=\widetilde{\psi}_u(x)$ for some $u\in S^1$;

(SC_2) the image Y' of $\partial X_{H^{-1}(c)}$ under the quotient projection $p\colon X_{H^{-1}(c)}\longrightarrow X'$ is a symplectic divisor in (X',ω');

(SC_3) the restriction $p\colon p_{H^{-1}(c)}^{-1}(U)\longrightarrow p(p_{H^{-1}(c)}^{-1}(U))$ *is equivariant with respect to the* S^1*-action* $\widetilde{\psi}$ *on the domain and an* S^1*-action* ψ' *on the target with a Hamiltonian*

$$H'\colon p\big(p_{H^{-1}(c)}^{-1}(U)\big)\longrightarrow\mathbb{R} \tag{4.43}$$

with respect to ω' *such that* $H\circ p_{H^{-1}(c)}=H'\circ p\colon p_{H^{-1}(c)}^{-1}(U)\longrightarrow\mathbb{R}$*;*

(SC_4) the restrictions $p\colon X-H^{-1}(c)\longrightarrow X'-Y'$ *and* $p\colon \partial X_{H^{-1}(c)}\longrightarrow Y'$ *are submersions,*

$$\big\{p|_{X-H^{-1}(c)}\big\}^*\omega'=\omega\big|_{X-H^{-1}(c)}, \quad \textit{and} \quad \big\{p|_{\partial X_{H^{-1}(c)}}\big\}^*\omega'=p_{X_{H^{-1}(c)}}^*\omega\big|_{T\partial X_{H^{-1}(c)}}. \tag{4.44}$$

Under the assumptions of this theorem, the neighborhood $p_{H^{-1}(c)}^{-1}(U)\subset X_{H^{-1}(c)}$ of $\partial X_{H^{-1}(c)}$ is the disjoint union of open subsets $U_{H;c}^+$ and $U_{H;c}^-$ that are identified by $p_{H^{-1}(c)}$ with

$$\big\{x'\in U\colon H(x')\ge c\big\}\subset X \qquad\text{and}\qquad \big\{x'\in U\colon H(x')\le c\big\}\subset X,$$

respectively. The subsets

$$\partial X_{H^{-1}(c)}\cap U_{H;c}^+,\partial X_{H^{-1}(c)}\cap U_{H;c}^-\subset X_{H^{-1}(c)}$$

are interchanged by the involution $\sigma_{H^{-1}(c)}$ on $\partial X_{H^{-1}(c)}$. Each of these subsets is identified with the real hypersurface $H^{-1}(c)\subset X$ by $p_{H^{-1}(c)}$. By Theorem 4.24,

$$Y'_+\equiv p\big(\partial X_{H^{-1}(c)}\cap U_{H;c}^+\big)\subset X' \qquad\text{and}\qquad Y'_-\equiv p\big(\partial X_{H^{-1}(c)}\cap U_{H;c}^-\big)\subset X' \tag{4.45}$$

are disjoint symplectic divisors in (X',ω') interchanged by the involution on $Y'\equiv p(\partial X_{H^{-1}(c)})$ induced by $\sigma_{H^{-1}(c)}$. The symplectic manifolds $(Y'_+,\omega'|_{Y'_+})$ and $(Y'_-,\omega'|_{Y'_-})$ are the symplectic quotients of the Hamiltonian S^1-manifold $(U,\omega|_U,\psi,H)$ at c as in Theorem 4.1. The sphere (circle) bundles of the normal bundles $\mathcal{N}_{X'}Y'_+$ and $\mathcal{N}_{X'}Y'_-$ of Y'_+ and Y'_-, respectively, in X' are isomorphic to $H^{-1}(c)$, but with opposite orientations and S^1-actions via ψ. The oriented vector bundle $\mathcal{N}_{X'}Y'_\pm$ is isomorphic to the vector bundle

$$\big(H^{-1}(c)\times\mathbb{C}\big)/\!\sim\longrightarrow H^{-1}(c), \quad (x,z)\sim\big(\psi_u(z),u^{\mp1}z\big)\ \ \forall\,(x,z)\in H^{-1}(c)\times\mathbb{C},\,u\in S^1,$$

with its complex orientation.

In the most standard perspective on the symplectic cut of Theorem 4.24, X is a connected manifold, $H^{-1}(c)$ is a separating hypersurface, and ψ is chosen so that all topological components of the divisor $Y'_+\subset X'$ (resp. $Y'_-\subset X'$) are contained in one of the two connected components $X'_+\subset X'$ (resp. $X'_-\subset X'$). However, in general, X' could be a connected manifold containing two copies of the symplectic quotient of the Hamiltonian S^1-manifold $(U,\omega|_U,\psi,H)$ at c; see Example 4.25 below.

Proof of Theorem 4.24. By Exercise 1.23, the smooth maps

$$\widetilde{H}_\pm\colon U\times\mathbb{C}\longrightarrow\mathbb{R}, \qquad \widetilde{H}_\pm(x,z)=H(x)\mp\pi|z|^2, \tag{4.46}$$

are Hamiltonians with respect to the symplectic form $\omega|_U\oplus\omega_{\mathbb C}$ for the S^1-actions

$$\psi_\pm\colon S^1\times(U\times\mathbb C)\longrightarrow U\times\mathbb C,\qquad \psi_{\pm;u}(x,z)=\big(\psi_u(x),u^{\mp1}z\big). \tag{4.47}$$

Since S^1 acts freely on $H^{-1}(c)$ via ψ, it also acts freely on $\widetilde H_\pm^{-1}(c)$ via $\psi_\pm$. Let $(U_\pm,\omega_\pm)$ be the symplectic quotient of the Hamiltonian S^1-manifold $(U\times\mathbb C,\omega|_U\oplus\omega_{\mathbb C},\psi_\pm,\widetilde H_\pm)$ at c as in Theorem 4.1 and

$$p_\pm\colon \widetilde H_\pm^{-1}(c)\longrightarrow U_\pm$$

be the quotient projection. The closed subspace $U\times\{0\}\subset U\times\mathbb C$ is preserved by either S^1-action $\psi_\pm$. By Exercise 4.2, the closed subspace

$$Y'_\pm\equiv p_\pm\big(H^{-1}(c)\times\{0\}\big)\subset U_\pm$$

is thus a symplectic divisor in $(U_\pm,\omega_\pm)$ with

$$\big\{p_\pm|_{H^{-1}(c)\times\{0\}}\big\}^*\omega_\pm=\omega\big|_{TH^{-1}(c)}. \tag{4.48}$$

Since the S^1-action on the first component in $U\times\mathbb C$ via ψ commutes with the action $\psi_\pm$ and preserves the Hamiltonian $\widetilde H_\pm$, it descends to an action $\psi'_\pm$ on $U_\pm$. Since the Hamiltonian H for the former action with respect to $\omega_\pm$ is preserved by $\psi_\pm$, H descends to a Hamiltonian

$$H_\pm\colon U_\pm\longrightarrow\mathbb R$$

for the action $\psi'_\pm$ with respect to $\omega_\pm$.

We define

$$X'=\big(\big(X-H^{-1}(c)\big)\sqcup U_+\sqcup U_-\big)/\sim,\qquad\text{where}$$
$$\big\{x'\in U\colon \pm(H(x')-c)>0\big\}\ni x\sim p_\pm\big(x,\sqrt{\pm(H(x)-c)/\pi}\big)\in U_\pm-Y'_\pm\,.$$

Let $q\colon\big(X-H^{-1}(c)\big)\sqcup U_+\sqcup U_-\longrightarrow X'$ denote the quotient projection. We identify the submanifolds $Y'_\pm\subset U_\pm$ with their images under q. Since $H^{-1}(c)\subset X$ is a closed submanifold and the quotient map is open, X' is a Hausdorff topological space (in particular, any point in $Y'_\pm\subset X'$ can be separated from any point in $q(X-H^{-1}(c))$ by open sets). This space is locally Euclidean with smooth overlap maps since it is obtained by gluing three manifolds by diffeomorphisms on open subsets. The map

$$p\colon X_{H^{-1}(c)}\longrightarrow X',\quad p(x)=\begin{cases}q(x), & \text{if } x\in X-H^{-1}(c);\\ q\big(p_\pm(x,\sqrt{\pm(H(x)-c)/\pi})\big), & \text{if } x\in U^\pm_{H;c};\end{cases}$$

is well-defined, continuous, and surjective. Since the map

$$\widetilde p_\pm\colon U^\pm_{H;c}\longrightarrow \widetilde H_\pm^{-1}(c),\qquad \widetilde p_\pm(x)=\big(x,\sqrt{\pm(H(x)-c)/\pi}\big),$$

is closed and the group S^1 is compact, p is a closed map. This confirms (SC$_1$). The two restrictions in (SC$_4$) are submersions. The S^1-actions $\psi'_\pm$ on $U_\pm$ induce an S^1-action ψ' on $p(p_{H^{-1}(c)}^{-1}(U))$ so that the restriction in (SC$_3$) is equivariant. The smooth functions $H_\pm$ on $U_\pm$ similar induce a smooth function H' as in (4.43) satisfying $H\circ p_{H^{-1}(c)}=H'\circ p$.

Since the smooth map

$$\big\{x'\in U\colon \pm(H(x')-c)>0\big\} \ni x \longrightarrow \big(x, \sqrt{\pm(H(x)-c)/\pi}\big) \in \widetilde{H}_{\pm}^{-1}(c)-H^{-1}(c)\times\{0\} \subset U\times\mathbb{C}$$

pulls back the symplectic form $\omega|_U\oplus\omega_{\mathbb{C}}$ to the restriction of ω to the domain of this map, there is a unique symplectic form ω' on X' so that the first property in (4.44) holds and

$$q^*\omega'\big|_{U_\pm} = \omega_\pm\,.$$

Along with (4.48), this implies that the second property in (4.44) also holds. By the S^1-equivariance of the restriction in (SC$_3$) and the first property in (4.44), H' is the Hamiltonian for the S^1-action ψ' above with respect to ω'. The conditions (SC$_1$)-(SC$_4$) ensure the uniqueness of symplectic manifold (X',ω'). □

Example 4.25. Let $\mathbb{T}^2\equiv\mathbb{R}^2/\mathbb{Z}^2$ be the standard 2-torus with the standard symplectic form ω, i.e. $q^*\omega=\mathrm{d}x\wedge\mathrm{d}y$, where $q\colon\mathbb{R}^2\longrightarrow\mathbb{T}^2$ is the quotient projection and x,y are the standard coordinates on $\mathbb{R}^2$. The free action ψ of $S^1\equiv\mathbb{R}/\mathbb{Z}$ on $\mathbb{T}^2$ given by

$$\psi_{[t]}\big([x,y]\big) = [x,y+t] \quad \forall\ (x,y)\in\mathbb{R}^2,\ t\in\mathbb{R},$$

preserves the open subset $U\equiv q((\mathbb{R}-\mathbb{Z})\times\mathbb{R})$ of $\mathbb{T}^2$. The smooth function

$$H\colon U\longrightarrow\mathbb{R}, \qquad H\big([x,y]\big)=x \ \text{ if } x\in(0,1),$$

is a Hamiltonian for the restriction of ψ to U with respect to ω. Thus, $(U,\psi|_U,H,1/2)$ is cutting data for $(\mathbb{T}^2,\omega)$. The associated cut symplectic manifold (X',ω') of Theorem 4.24 is isomorphic to S^2 with a symplectic form of volume 1 (the space of such symplectic forms is contractible by Moser's trick as in Exercise 2.39(b)). It is obtained from $\mathbb{T}^2$ by collapsing the circle $q(\{1/2\}\times\mathbb{R})$ to a point and then separating the two branches of the resulting node. The associated symplectic divisor Y' consists of the two points in the preimage of the node under this separation.

Exercise 4.26. Suppose ψ is an S^1-action on a symplectic manifold (X,ω) with a Hamiltonian $H\colon X\longrightarrow\mathbb{R}$ with respect to the standard basis vector e_1 for $\mathbb{R}$ and $c\in\mathbb{R}$ is such that S^1 acts freely on $H^{-1}(c)\subset X$. Let $\overline{\psi}$ be the inverse S^1-action on X,

$$\mathscr{H}=(e_1,c), \qquad\text{and}\qquad \overline{\mathscr{H}}=(e_1,-c).$$

Thus, (X,ψ,H,c) is cutting data for (X,ω), (X,ω,ψ,H) is an $\mathscr{H}$-cuttable Hamiltonian S^1-manifold, and $(X,\omega,\overline{\psi},-H)$ is an $\overline{\mathscr{H}}$-cuttable Hamiltonian S^1-manifold. Show that the cut symplectic manifold (X',ω') of Theorem 4.24 is the disjoint union of the $\overline{\mathscr{H}}$-cut symplectic manifold of (X,ω,ψ,H) and the $\mathscr{H}$-cut symplectic manifold of $(X,\omega,\overline{\psi},-H)$ as in Theorem 4.11.

4.6 Symplectic blowup

We now describe an important application of the symplectic cut construction of Section 4.5. It provides a symplectic version of the complex and holomorphic blowup constructions along a closed submanifold.

Suppose Y is a (nonempty) closed symplectic submanifold of a symplectic manifold (X,ω) of codimension $2m$. With $\mathrm{pr}: TY^\omega \longrightarrow Y$ and $\omega_Y^\perp$ as in Example 2.17, let $\mathfrak{i}$ be a fiberwise complex structure in the real vector bundle $TY^\omega \longrightarrow Y$ compatible with $\omega_Y^\perp$, so that

$$g_{\mathfrak{i}}^\omega(\cdot,\cdot) \equiv \omega_Y^\perp(\cdot,\mathfrak{i}\cdot)$$

is a fiberwise metric on TY^ω. Let ∇ be a $\mathbb{C}$-linear connection in TY^ω and $\widetilde{\omega}_\nabla$ be the closed 2-form on the total space of TY^ω defined by (2.21). By the Symplectic Tubular Neighborhood Theorem of Example 2.17, there exist a smooth function $\rho: Y \longrightarrow \mathbb{R}^+$ and a diffeomorphism

$$\Phi: \mathcal{U}_\rho \equiv \big\{w\in TY^\omega: g_{\mathfrak{i}}^\omega(w,w)<\rho(\mathrm{pr}(w))\big\} \longrightarrow U_\rho$$

onto a neighborhood of $Y\subset X$ so that $\Phi(y)=y$ for every $y\in Y$ and $\Phi^*\omega|_{\mathcal{U}_\rho}=\widetilde{\omega}_\nabla|_{\mathcal{U}_\rho}$. By Exercise 1.30, the smooth map

$$H: U_\rho \longrightarrow \mathbb{R}, \qquad H(x) = \pi g_{\mathfrak{i}}^\omega\big(\Phi^{-1}(x),\Phi^{-1}(x)\big), \tag{4.49}$$

is a Hamiltonian for the S^1-action

$$\psi: S^1 \longrightarrow \mathrm{Symp}\big(U_\rho,\omega|_{U_\rho}\big), \qquad \psi_u(x) = \Phi\big(u\Phi^{-1}(x)\big) \quad \forall\, u\in S^1\subset\mathbb{C}^*,\, x\in U_\rho, \tag{4.50}$$

with respect to $\omega|_{U_\rho}$.

Suppose that the function ρ above is constant; this can be achieved in particular if Y is compact. In such a case, (U_ρ,ψ,H,c) is cutting data for (X,ω) for any $c\in\mathbb{R}^*$, with $H^{-1}(c)\neq 0$ whenever $c\in(0,\pi\rho)$. By Exercises 4.26 and 4.15, the cut symplectic manifold $(X'_-,\omega'|_{X'_-})$ of Theorem 4.24 in such a case is the total space of a symplectic fibration over Y with fiber $(\mathbb{C}P^m, c\omega_{\mathrm{FS};m})$. If $c\in(0,\pi\rho)$, the cut symplectic manifold $(X'_+,\omega'|_{X'_+})$ of Theorem 4.24 is called a symplectic blowup of X along Y. The associated symplectic divisor $Y'_+\subset X'_+$ is called the exceptional divisor for this blowup. By Exercise 4.7, it is isomorphic to the complex projectivization of TY^ω as a fiber bundle over Y. Furthermore, every fiber of the projection $Y'_+\longrightarrow Y$ is a symplectic submanifold of $(Y'_+,\omega'|_{Y'_+})$ isomorphic to $(\mathbb{C}P^{m-1}, c\omega_{\mathrm{FS};m-1})$. The oriented normal bundle $\mathcal{N}_{X'_+}Y'_+$ of Y'_+ in X'_+ is isomorphic to the tautological complex line bundle

$$\gamma_{TY^\omega} = \big\{(\ell,w)\in\mathbb{P}(TY^\omega)\times TY^\omega: w\in\ell\big\} \longrightarrow \mathbb{P}(TY^\omega).$$

Suppose Y is a closed submanifold of a smooth manifold X of codimension $2m$ and $\mathfrak{i}$ is a fiberwise complex structure in the normal bundle

$$\mathcal{N}_XY \equiv \frac{TX|_Y}{TY} \longrightarrow Y$$

of Y in X. Let $\Phi:\mathcal{U}\longrightarrow U$ be a tubular neighborhood identification for Y in X, i.e. Φ is a diffeomorphism from an open neighborhood of Y in $\mathcal{N}_XY$ to an open neighborhood of Y such that $\Phi(y)=y$ for every $y\in Y$ and the composition

$$\mathcal{N}_XY \longrightarrow T_y(\mathcal{N}_XY) \xrightarrow{\mathrm{d}_y\Phi} T_yY \xrightarrow{q} \mathcal{N}_XY\big|_y, \tag{4.51}$$

where the first homomorphism is the inclusion of the vertical tangent bundle as in (B.2) and the last homomorphism is the quotient projection, is the identity homomorphism; such a diffeomorphism exists by the Tubular Neighborhood Theorem [32, Theorem 4.5.1].

Let $\mathbb{P}(\mathcal{N}_X Y)$ be the complex projectivization of the complex vector bundle $(\mathcal{N}_X Y, \mathfrak{i})$ over Y and

$$\widetilde{\mathcal{U}} = \{(\ell, w) \in \mathbb{P}(\mathcal{N}_X Y) \times \mathcal{N}_X Y : w \in \ell \cap \mathcal{U}\}.$$

The smooth manifold

$$\mathrm{Bl}_Y X \equiv \big(\widetilde{\mathcal{U}} \sqcup (X - Y)\big)/\sim, \quad \widetilde{\mathcal{U}} - \mathbb{P}(\mathcal{N}_X Y) \ni (\ell, w) \sim \Phi(w) \in X - Y,$$

is a complex blowup of X along Y in the smooth category. The image $\mathrm{E}_Y X \subset \mathrm{Bl}_Y X$ of the smooth submanifold $\mathbb{P}(\mathcal{N}_X Y) \subset \widetilde{\mathcal{U}}$ is the exceptional divisor for this blowup. The oriented normal bundle $\mathcal{N}_{\mathrm{Bl}_Y X}(\mathrm{E}_Y X)$ of $\mathrm{E}_Y X$ in $\mathrm{Bl}_Y X$ is isomorphic to the tautological complex line bundle

$$\gamma_{\mathcal{N}_X Y} = \{(\ell, w) \in \mathbb{P}(\mathcal{N}_X Y) \times \mathcal{N}_X Y : w \in \ell\} \longrightarrow \mathbb{P}(\mathcal{N}_X Y).$$

The blowdown map

$$p_{X|Y} : \mathrm{Bl}_Y X \longrightarrow X, \qquad p_{X|Y}(\widetilde{x}) = \begin{cases} \Phi(w), & \text{if } \widetilde{x} = [\ell, w],\ (\ell, w) \in \widetilde{\mathcal{U}}; \\ x, & \text{if } \widetilde{x} = [x],\ x \in X - Y; \end{cases} \tag{4.52}$$

is smooth. It restricts to a diffeomorphism from $\mathrm{Bl}_Y X - \mathrm{E}_Y X$ to $X - Y$ and a $\mathbb{C}P^{m-1}$-fiber bundle from $\mathrm{E}_Y X$ to Y. By the next exercise, the diffeomorphism class of the pair $(\mathrm{Bl}_Y X, \mathrm{E}_Y X)$ is independent of the choice of a tubular neighborhood identification Φ for Y in X.

Exercise 4.27. Suppose Y is a closed submanifold of a smooth manifold X of even codimension, $\mathfrak{i}$ is a fiberwise complex structure in the normal bundle $\mathcal{N}_X Y$ of Y in X, and $\Phi : \mathcal{U} \longrightarrow U$ is a tubular neighborhood identification for Y in X as above. Let $\mathrm{Bl}_Y X$ be the corresponding complex blowup of X along Y, $\mathrm{E}_Y X \subset \mathrm{Bl}_Y X$ be the associated exceptional divisor, X'_- be the complex projectivization of the complex vector bundle $(\mathcal{N}_X Y, \mathfrak{i}) \oplus (Y \times \mathbb{C})$ over Y,

$$Z = \mathbb{P}\big(Y \oplus (Y \times \mathbb{C})\big) \subset X'_-,$$

and $X \#_{Y=Z} X'_-$ be the connect sum of the smooth manifolds X and X'_- with respect to the canonical identification of the submanifolds $Y \subset X$ and $Z \subset X'_-$. We denote by $\mathrm{E}_Y^- X \subset X \#_{Y=Z} X'_-$ the image of the submanifold

$$\mathbb{P}\big(\mathcal{N}_X Y \oplus Y\big) \subset X'_- - Z.$$

(a) Show that $(\mathrm{Bl}_Y X, \mathrm{E}_Y X)$ is diffeomorphic to $(X \#_{Y=Z} X'_-, \mathrm{E}_Y^- X)$.

(b) Suppose in addition X is an oriented manifold. Its orientation then induces an orientation on the submanifold $Y \subset X$ via the complex orientation of its normal bundle $\mathcal{N}_X Y$ and orientations on the fiber bundles $\mathrm{E}_Y X, \mathrm{E}_Y^- X, X'_- \longrightarrow Y$ via the complex orientations of their fibers. Let $X \#_{Y=Z} \overline{X'_-}$ be the oriented connect sum of X with its orientation and X'_- with the opposite of the induced orientation with respect to the canonical identification of the submanifolds $Y \subset X$ and $Z \subset X'_-$.. Show that $(\mathrm{Bl}_Y X, \mathrm{E}_Y X)$ is diffeomorphic to $(X \#_{Y=Z} \overline{X'_-}, \mathrm{E}_Y^- X)$ as a pair of oriented manifolds.

Proposition 4.28. *Suppose (X,ω) is a symplectic manifold, Y is a connected closed submanifold of X of even codimension, $\mathfrak{i}$ is a fiberwise complex structure in the normal bundle $\mathcal{N}_XY$ of Y in X, and $\Phi\colon\mathcal{U}\longrightarrow U$ is a tubular neighborhood identification for Y in X as above. Let $p_{X|Y}\colon \mathrm{Bl}_YX\longrightarrow X$ be the blowdown map as in (4.52), $\mathrm{E}_YX\subset\mathrm{Bl}_YX$ be the associated exceptional divisor, and $F\approx\mathbb{C}P^1$ be linearly embedded in a fiber of the projection $\mathrm{E}_YX\longrightarrow Y$. If $\widetilde{\omega}$ and $\widetilde{\omega}'$ are symplectic forms on Bl_YX that restrict to $p_{X|Y}^*\omega$ outside of a tubular neighborhood $\widetilde{U}$ of $\mathrm{E}_YX\subset\mathrm{Bl}_YX$ and satisfy $\langle\widetilde{\omega},F\rangle=\langle\widetilde{\omega}',F\rangle$, then there exists a smooth family $(\psi_t)_{t\in[0,1]}$ of diffeomorphisms of Bl_YX that restrict to the identity outside of $\widetilde{U}$ and satisfy $\psi_0=\mathrm{id}_{\mathrm{Bl}_YX}$ and $\psi_1^*\omega'=\omega$.*

Proof. $\square$

Exercise 4.29. Suppose (X,ω), $Y\subset X$, $\mathfrak{i}$, $g_{\mathfrak{i}}^{\omega}$, ∇, $\widetilde{\omega}_\nabla$, $\Phi\colon\mathcal{U}_\rho\longrightarrow U_\rho$ with $\rho\in\mathbb{R}^+$, and $c\in(0,\pi\rho)$ are as in the symplectic blowup setup above. We identify the vector bundle TY^ω with $\mathcal{N}_XY$ via the restriction of the projection q in (4.51). Let $(X'_+,\omega'|_{X'_+})$ and Bl_YX be the corresponding symplectic and complex blowups of X along Y, respectively, $Y'_+\subset X'_+$ and $\mathrm{E}_YX\subset\mathrm{Bl}_YX$ be the associated exceptional divisors,

$$X_+=X-\Phi\big(\{w\in TY^\omega\colon g_{\mathfrak{i}}^{\omega}(w,w)\le c/\pi\}\big),\quad \widetilde{U}_+=\big\{(x,z)\in U_\rho\times\mathbb{C}\colon \big|\Phi^{-1}(x)\big|^2-|z|^2=c/\pi\big\},$$

and $\widetilde{\mathcal{U}_\rho}$ be as above Exercise 4.27 with $\mathcal{U}=\mathcal{U}_\rho$. Let $\beta:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ be a smooth non-decreasing function so that

$$\beta(r)=\begin{cases}r, & \text{if } r^2\le\delta;\\ 1, & \text{if } r^2\ge\rho-c/\pi-\delta;\end{cases}$$

for some $\delta\in\mathbb{R}^+$ (necessarily $\delta\le 1,(\rho-c/\pi)/2$). Show that

(a) $X'_+=((X_+-\overline{U_{\rho-\delta}})\sqcup(\widetilde{U}_+/S^1))/\sim$, where

$$S^1\times\widetilde{U}_+\longrightarrow\widetilde{U}_+,\ u(x,z)=\big(\Phi\big(u\Phi^{-1}(x)\big),u^{-1}z\big),\ \ X_+\cap U_\rho\ni x\sim\big[x,\sqrt{|\Phi^{-1}(x)|^2-c/\pi}\big]\in\widetilde{U}_+/S^1.$$

(b) the map $\widetilde{G}_\beta\colon(X_+-\overline{U_{(\rho-c/\pi)/2}})\sqcup\widetilde{U}_+\longrightarrow(X-Y)\sqcup\widetilde{\mathcal{U}_\rho}$ given by

$$\widetilde{G}_\beta(x)=x\quad\forall\,x\in X_+-\overline{U_{\rho-\delta}}\,,$$
$$\widetilde{G}_\beta(x,z)=\begin{cases}\big(\mathbb{C}\Phi^{-1}(x),z\Phi^{-1}(x)\big), & \text{if } (x,z)\in\widetilde{U}_+,\ |z|^2<\delta;\\ \big(\mathbb{C}\Phi^{-1}(x),\beta(|z|)z\Phi^{-1}(x)/|z|\big), & \text{if } (x,z)\in\widetilde{U}_+,\ z\neq0;\end{cases}$$

descends to a diffeomorphism G_β from (X'_+,Y'_+) to $(\mathrm{Bl}_YX,\mathrm{E}_YX)$.

(c) Let $\beta_0,\beta_1\colon\mathbb{R}^+\longrightarrow\mathbb{R}^+$ be smooth functions as above. Show that there exists a smooth family $(\psi_t)_{t\in[0,1]}$ of diffeomorphisms of Bl_YX that restricts to the identity outside of $U_\delta-U_{\delta'}$ for some $\delta,\delta'\in(0,\rho)$ so that $\{G_{\beta_1}^{-1}\}^*\omega'=\psi_1^*\{G_{\beta_0}^{-1}\}^*\omega'$. *Hint:* use Moser's Trick of Section 2.6.

By Exercise 4.29(c), the de Rham cohomology class $[\widetilde{\omega}]$ of the symplectic form of $\{G_\beta^{-1}\}^*\omega'$ on Bl_YX is independent of the choice of smooth function $\beta\colon\mathbb{R}^+\longrightarrow\mathbb{R}^+$. By the definition of G_β and the first equation in (4.44), $\{G_\beta^{-1}\}^*\omega'=p_{X|Y}^*\omega$ outside of a tubular neighborhood of the exceptional divisor $\mathrm{E}_YX\subset\mathrm{Bl}_YX$; this neighborhood can be made arbitrarily small by choosing the smooth function β appropriately. Thus, the difference $[\widetilde{\omega}]-p_{X|Y}^*[\omega]$ arises entirely from the exceptional divisor E_YX; it is specified by (4.53) below. The closed oriented submanifold $\mathrm{E}_Y X\subset \mathrm{Bl}_Y X$ determines an element $[\mathrm{E}_Y X]_{\mathrm{Bl}_Y X}$ of the Borel–Moore homology $H_*^{\mathrm{lf}}(\mathrm{Bl}_Y X;\mathbb{Z})$ of $\mathrm{Bl}_Y X$, also known as the homology with closed supports and the homology based on locally finite chains, with $\mathbb{Z}$-coefficients; see [15, Section 2.2]. Let

$$\mathrm{PD}_{\mathrm{Bl}_Y X}(\mathrm{E}_Y X)\in H^2\big(\mathrm{Bl}_Y X;\mathbb{Z}\big)$$

be the Poincare dual of $[\mathrm{E}_Y X]_{\mathrm{Bl}_Y X}$ with respect to the symplectic orientation of $\mathrm{Bl}_Y X$; see [15, Proposition 2.13]. We denote the image of $\mathrm{PD}_{\mathrm{Bl}_Y X}(\mathrm{E}_Y X)$ in $H^2(\mathrm{Bl}_Y X;\mathbb{R})$ under the homomorphism induced by the inclusion $\mathbb{Z}\longrightarrow\mathbb{R}$ and the image of this image in $H^2_{\mathrm{deR}}(\mathrm{Bl}_Y X;\mathbb{R})$ under the inverse of the de Rham Isomorphism of [56, 5.35(6)] also by $\mathrm{PD}_{\mathrm{Bl}_Y X}(\mathrm{E}_Y X)$.

Proposition 4.30. *Suppose (X,ω) is a symplectic manifold, Y is a closed symplectic submanifold of (X,ω) of codimension at least 4, $\mathfrak{i}$ is a fiberwise complex structure in the real vector bundle $TY^\omega\approx\mathcal{N}_X Y$ over Y compatible with the fiberwise symplectic form $\omega_Y^\perp\equiv\omega|_{TY^\omega}$, ∇ is a $\mathbb{C}$-linear connection in this vector bundle, $g_{\mathfrak{i}}^\omega$ is the fiberwise metric on TY^ω determined by $\omega^\perp$ and $\mathfrak{i}$, and $\widetilde{\omega}_\nabla$ is the closed 2-form on the total space of TY^ω determined by $\omega_Y^\perp$ and ∇ as in (2.21). Let*

$$\Phi\colon \mathcal{U}_\rho\equiv\big\{w\in TY^\omega\colon g_{\mathfrak{i}}^\omega(w,w)<\rho(\mathrm{pr}(w))\big\}\longrightarrow U_\rho$$

for some $\rho\in\mathbb{R}^+$ be a diffeomorphism onto a neighborhood of $Y\subset X$ so that $\Phi(y)=y$ for every $y\in Y$ and $\Phi^*\omega|_{\mathcal{U}_\rho}=\widetilde{\omega}_\nabla|_{\mathcal{U}_\rho}$, $c\in(0,\pi\rho)$, and (U_ρ,ψ,H,c) be the associated cutting data for (X,ω) with ψ as in (4.50) and H as in (4.49). Let $(X'_+,\omega'|_{X'_+})$ and $p_{X|Y}\colon \mathrm{Bl}_Y X\longrightarrow X$ be the corresponding symplectic blowup of (X,ω) along Y and the complex blowup of X along Y, respectively, and $\mathrm{E}_Y X\subset\mathrm{Bl}_Y X$ be the exceptional divisor for the latter. If $G_\beta\colon X'_+\longrightarrow\mathrm{Bl}_Y X$ is a diffeomorphism as in Exercise 4.29(b), then

$$\big[\{G_\beta^{-1}\}^*\omega'\big]=p_{X|Y}^*[\omega]-c\,\mathrm{PD}_{\mathrm{Bl}_Y X}(\mathrm{E}_Y X)\in H^2_{\mathrm{deR}}(\mathrm{Bl}_Y X;\mathbb{R}).\tag{4.53}$$

Proof. By the de Rham Isomorphism Theorem of [56, 5.36] and the Universal Coefficient Theorem for Cohomology of [46, Corollary 53.6], it is sufficient to check that both sides of (4.53) evaluate the same on a (linearly) generating subset of $H_2(\mathrm{Bl}_Y X;\mathbb{R})$. By the assumption that the codimension of Y is greater than 2 and [16, Proposition 3.3] (or more directly by the proof of this proposition), $H_2(\mathrm{Bl}_Y X;\mathbb{R})$ is generated by cycles disjoint from $\mathrm{E}_Y X$ and by the homology classes of linearly embedded $\mathbb{C}P^1$s in the fibers of the $\mathbb{C}P^{m-1}$-fiber bundle $\mathrm{E}_Y X\longrightarrow Y$, where $2m$ is the codimension of Y in X.

The cycles disjoint from $\mathrm{E}_Y X$ vanish on the last term in (4.53) and evaluate the same on the other two terms, since $\{G_\beta^{-1}\}^*\omega'=p_{X|Y}^*\omega$ outside of a tubular neighborhood of $\mathrm{E}_Y X\subset\mathrm{Bl}_Y X$; this neighborhood can be made arbitrarily small by choosing the smooth function β appropriately. Let $F\approx\mathbb{C}P^1$ be linearly embedded in a fiber of the projection $\mathrm{E}_Y X\longrightarrow Y$. Since this projection is the restriction of $p_{X|Y}$, the middle term in (4.53) vanishes on F. Since $G_\beta^{-1}(F)$ is a symplectic submanifold of $(Y'_+,\omega'|_{Y'_+})$ isomorphic to $(\mathbb{C}P^{m-1},c\,\omega_{\mathrm{FS};m-1})$,

$$\big\langle\{G_\beta^{-1}\}^*\omega',F\big\rangle=\big\langle\omega',G_\beta^{-1}(F)\big\rangle=\big\langle c\,\omega_{\mathrm{FS};1},\mathbb{C}P^1\big\rangle=c\,;$$

the last equality holds by Exercise 1.27(b). Since the restriction of the oriented normal bundle $\mathcal{N}_{\mathrm{Bl}_Y X}\mathrm{E}_Y X$ of $\mathrm{E}_Y X$ in $\mathrm{Bl}_Y X$ to F is isomorphic to the tautological complex line bundle $\gamma_1\longrightarrow\mathbb{C}P^1$,

$$\big\langle\mathrm{PD}_{\mathrm{Bl}_Y X}(\mathrm{E}_Y X),[F]_{\mathrm{Bl}_Y X}\big\rangle=\big\langle e(\gamma_1),\mathbb{C}P^1\big\rangle=-1,$$

where $e(\gamma_1)=c_1(\gamma_1)$ is the Euler class of γ_1. This establishes the claim. $\square$

For $k\in\mathbb{Z}^{\ge 0}$, let $P_k(u_1,\ldots,u_k)\in\mathbb{Z}[u_1,\ldots,u_k]$ be the degree k part of the power series

$$(1+u_1+\ldots+u_k)^{-1}\equiv\sum_{\ell=0}^{\infty}(-1)^{\ell}(u_1+\ldots+u_k)^{\ell}=1-u_1+(u_1^2-u_2)+\ldots$$

if the degree of each u_i is i. For a complex vector bundle $\mathcal{N}$ over a topological space Y, let

$$P_k\big(c(\mathcal{N})\big)=P_k\big(c_1(\mathcal{N}),\ldots,c_k(\mathcal{N})\big)\in H^{2k}(Y;\mathbb{Z}),H^{2k}_{\mathrm{deR}}(Y),$$

where $c_i(\mathcal{N})$ is the i-th Chern class of $\mathcal{N}$.

Corollary 4.31. *Suppose (X,ω), $Y\subset X$, $\mathfrak{i}$, $g^{\omega}_{\mathfrak{i}}$, ∇, $\widetilde{\omega}_{\nabla}$, $\Phi\colon\mathcal{U}_{\rho}\longrightarrow U_{\rho}$ with $\rho\in\mathbb{R}^+$, $c\in(0,\pi\rho)$, $(X'_+,\omega'|_{X'_+})$, and $Y'_+\subset X'_+$ are as in Proposition 4.30, X is compact and of dimension $2n$, and Y is of codimension $2m\ge 4$. Use (4.53) to show that*

$$\big\langle\omega'^n,X'_+\big\rangle=\big\langle\omega^n,X\big\rangle-\sum_{k=0}^{n-m}\binom{n}{k}c^{n-k}\big\langle\omega^kP_{n-m-k}\big(c(\mathcal{N}_XY)\big),Y\big\rangle\,. \tag{4.54}$$

Proof. Here is an indirect approach establishing (4.54) with $\langle\omega'^n,X'_+\rangle$ replaced by $\langle\{G_{\beta}^{-1}\}^*\omega',\mathrm{Bl}_YX\rangle$, with G_{β} as in (4.53). Let s_E be the canonical section of the complex line bundle

$$L_E\equiv\mathcal{O}_{\mathrm{Bl}_YX}(\mathrm{E}_YX)\longrightarrow\mathrm{Bl}_YX$$

determined by the symplectic divisor $\mathrm{E}_YX\subset\mathrm{Bl}_YX$ via the canonical tubular neighborhood identification for E_YX in Bl_YX; see [24, Section 2.1]. We denote by $\overline{s}_E$ the same section of the conjugate complex line bundle $\overline{L}_E$ (same real vector bundle as L_E, but with the opposite multiplication by $\mathfrak{i}$) and by $\overline{\mathrm{E}_YX}$ the submanifold $\mathrm{E}_YX\subset\mathrm{Bl}_YX$ with the opposite orientation; the latter is the oriented zero set of $\overline{s}_E^{-1}(0)$.

First assume that $c\in\mathbb{Z}^+$ and $[\omega]\in H^2_{\mathrm{deR}}(X)$ lies in the image of $H^2(X;\mathbb{Z})$. Thus, $[\omega]=c_1(L_{\omega})$ for some complex line bundle $L_{\omega}\longrightarrow X$. Let $V_{\omega}\equiv nL_{\omega}$ be the direct sum of n copies of L_{ω}, $s_{\omega}\colon X\longrightarrow V_{\omega}$ be a transverse section of this vector bundle with no zeros in Y, and

$$s=\overline{s}_E^{\otimes c}\otimes p_{X|Y}^*s_{\omega}\in\Gamma\big(\mathrm{Bl}_YX;\overline{L}_E^{\otimes c}\otimes p_{X|Y}^*V_{\omega}\big).$$

The zero set $s^{-1}(0)$ consists of E_YX and $p_{X|Y}^{-1}(s_{\omega}^{-1}(0))$. Along with (4.53), this gives

$$\begin{split}\big\langle\big(\{G_{\beta}^{-1}\}^*\omega'\big)^n,\mathrm{Bl}_YX\big\rangle&=\big\langle\{e(\overline{L}_E^{\otimes c}\otimes p_{X|Y}^*V_{\omega}),\mathrm{Bl}_YX\big\rangle={}^{\pm}\big|p_{X|Y}^{-1}(s_{\omega}^{-1}(0))\big|_s+\mathcal{C}_{\overline{\mathrm{E}_YX}}(s)\\&={}^{\pm}\big|s_{\omega}^{-1}(0)\big|_{s_{\omega}}+\mathcal{C}_{\overline{\mathrm{E}_YX}}(s)=\big\langle e(V_{\omega}),X\big\rangle+\mathcal{C}_{\overline{\mathrm{E}_YX}}(s)\,,\end{split} \tag{4.55}$$

where ${}^{\pm}\big|p_{X|Y}^{-1}(s_{\omega}^{-1}(0))\big|_s$ is the signed cardinality of the finite set $p_{X|Y}^{-1}(s_{\omega}^{-1}(0))$ of transverse zeros of s and $\mathcal{C}_{\overline{\mathrm{E}_YX}}(s)$ is the s-contribution of $\mathrm{E}_YX\subset s^{-1}(0)$ to the Euler class of the vector bundle $\overline{L}_{\omega}^{\otimes c}\otimes p_{X|Y}^*V_{\omega}$ over Bl_YX, i.e. the number of zeros of $s+\nu$ that lie near $\overline{\mathrm{E}_YX}$ for a small generic deformation ν of s.

Since $\overline{s}_E$ is transverse to zero along $\overline{\mathrm{E}_Y X}$ and $p_{X|Y}^* s_\omega$ does not vanish along $\overline{\mathrm{E}_Y X}$, $\mathcal{C}_{\overline{\mathrm{E}_Y X}}(s)$ is c times the (signed) number $N(\alpha_{E,\omega})$ of zeros of the affine bundle map $\alpha_{E,\omega}+\nu_{E,\omega}\circ\mathrm{pr}$ over $\overline{\mathrm{E}_Y X}$, where

$$\alpha_{E,\omega}\colon \overline{L}_E^{\otimes c}\longrightarrow \overline{L}_E^{\otimes c}\otimes p_{X|Y}^* V_\omega, \qquad \alpha_{E,\omega}(w^{\otimes c})=\big(\nabla_w \overline{s}_E\big)^{\otimes c}\otimes\big(p_{X|Y}^* s_\omega\circ\mathrm{pr}\big),$$

is an injective bundle homomorphism over $\overline{\mathrm{E}_Y X}$, $\nu_{E,\omega}\in\Gamma(Y;\overline{L}_E^{\otimes c}\otimes p_{X|Y}^* V_\omega)$ is a generic section, and

$$\mathrm{pr}\colon \overline{L}_E^{\otimes c}\otimes p_{X|Y}^* V_\omega, p_{X|Y}^* V_\omega\longrightarrow \overline{\mathrm{E}_Y X}$$

are the bundle projections; see [59, Proposition 2.18B]. Since $\overline{L}_E^{\otimes c}$ is a complex line bundle and $\nu_{E,\omega}$ does not vanish over $\overline{\mathrm{E}_Y X}$, $N(\alpha_{E,\omega})$ is the number of zeros of the section $\alpha_{E,\omega}^\perp$ of the bundle

$$\mathrm{Hom}\big(\overline{L}_E^{\otimes c},(\overline{L}_E^{\otimes c}\otimes p_{X|Y}^* V_\omega)/\mathbb{C}\nu_{E,\omega}\big)\longrightarrow \overline{\mathrm{E}_Y X}$$

given by the composition of $\alpha_{E,\omega}$ with the projection $\overline{L}_E^{\otimes c}\otimes p_{X|Y}^* V_\omega\longrightarrow(\overline{L}_E^{\otimes c}\otimes p_{X|Y}^* V_\omega)/\mathbb{C}\nu_{E,\omega}$. Thus,

$$\begin{aligned}\mathcal{C}_{\overline{\mathrm{E}_Y X}}(s)=cN\big(\alpha_{E,\omega}\big)=c\,{}^{\pm}\big|\alpha_{E,\omega}^{\perp}{}^{-1}(0)\big|&=c\big\langle e\big(\mathrm{Hom}\big(\overline{L}_E^{\otimes c},(\overline{L}_E^{\otimes c}\otimes p_{X|Y}^* V_\omega)/\mathbb{C}\nu_{E,\omega}\big)\big),\overline{\mathrm{E}_Y X}\big\rangle\\ &=-c\big\langle e\big(p_{X|Y}^* V_\omega/\gamma_{\mathcal{N}_X Y}^{\otimes c}\big),\mathrm{E}_Y X\big\rangle.\end{aligned}\tag{4.56}$$

By Lemma 4.33 below,

$$\begin{aligned}\big\langle e\big(p_{X|Y}^* V_\omega/\gamma_{\mathcal{N}_X Y}^{\otimes c}\big),\mathrm{E}_Y X\big\rangle&=\sum_{k=0}^{n-1}c^{n-1-k}\big\langle\big(p_{X|Y}^* c_k(V_\omega)\big)c_1(\gamma_{\mathcal{N}_X Y}^*)^{n-1-k},\mathrm{E}_Y X\big\rangle\\ &=\sum_{k=0}^{n-m}c^{n-1-k}\big\langle\big(p_{X|Y}^* c_k(V_\omega)\big)P_{n-m-k}\big(c(\mathcal{N}_X Y)\big),Y\big\rangle.\end{aligned}$$

Combining this identity with equations (4.55) and (4.56), we obtain (4.54) with $\langle\omega'^n,X'_+\rangle$ replaced by $\langle(\{G_\beta^{-1}\}^*\omega')^n,\mathrm{Bl}_Y X\rangle$. $\square$

Exercise 4.32. Complete the proof of Corollary 4.31.

Lemma 4.33. *If $\mathcal{N}\longrightarrow Y$ is a rank m complex vector bundle over a compact oriented manifold and* $\mathrm{pr}\colon\mathbb{P}\mathcal{N}\longrightarrow Y$ *is the complex projectivization of $\mathcal{N}$, then*

$$\big\langle(\mathrm{pr}^*\mu)c_1(\gamma_{\mathcal{N}}^*)^k,\mathbb{P}\mathcal{N}\big\rangle=\begin{cases}0, & \text{if } k<m-1;\\ \big\langle\mu P_{k-m+1}\big(c(\mathcal{N})\big),Y\big\rangle, & \text{if } k\ge m-1;\end{cases}\qquad\forall\ \mu\in H^*(Y;\mathbb{Z}),\ k\in\mathbb{Z}^{\ge0}.$$

Proof. Since the restriction of $\gamma_{\mathcal{N}}$ to a fiber of pr is isomorphic to the tautological complex line bundle $\gamma_{m-1}\longrightarrow\mathbb{C}P^{m-1}$ and $\langle c_1(\gamma_{m-1}^*)^{m-1},\mathbb{C}P^{m-1}\rangle=1$,

$$\big\langle(\mathrm{pr}^*\mu)c_1(\gamma_{\mathcal{N}}^*)^k,\mathbb{P}\mathcal{N}\big\rangle=\begin{cases}0, & \text{if } k<m-1;\\ \langle\mu,Y\rangle, & \text{if } k=m-1.\end{cases}\tag{4.57}$$

Since $\gamma_{\mathcal{N}}\subset\mathrm{pr}^*\mathcal{N}\longrightarrow\mathbb{P}\mathcal{N}$ is a complex vector subbundle,

$$\big(\mathrm{pr}^*c(\mathcal{N})\big)c(\gamma_{\mathcal{N}})^{-1}=c\big(\mathrm{pr}^*\mathcal{N}/\gamma_{\mathcal{N}}\big)=0\in H^*(\mathbb{P}\mathcal{N};\mathbb{Z})\Big/\bigoplus_{\ell=0}^{2m-2}H^\ell(\mathbb{P}\mathcal{N};\mathbb{Z})\,.$$

Combining this with (4.57), we obtain

$$\begin{aligned}\big\langle(\mathrm{pr}^*\mu)c(\gamma_{\mathcal N})^{-1},\mathbb P\mathcal N\big\rangle &= \big\langle(\mathrm{pr}^*\mu)(\mathrm{pr}^*c(\mathcal N))^{-1}(\mathrm{pr}^*c(\mathcal N))c(\gamma_{\mathcal N})^{-1},\mathbb P\mathcal N\big\rangle\\ &= \big\langle(\mathrm{pr}^*\mu)(\mathrm{pr}^*c(\mathcal N))^{-1}c(\gamma_{\mathcal N}^*)^{m-1},\mathbb P\mathcal N\big\rangle = \big\langle\mu c(\mathcal N)^{-1},Y\big\rangle.\end{aligned}$$

The claim follows by decomposing μ into its components of fixed degree. □

By definition, the first two terms in (4.54) are $n!$ times the symplectic volumes of $(X'_+,\omega'|_{X'_+})$ and (X,ω), respectively. The last term in (4.54) is thus $n!$ times the symplectic volume of $(U_c,\omega|_{U_c})$, since X'_+ is obtained by removing the tubular neighborhood U_c of $Y\subset X$ and collapsing the resulting boundary by a natural S^1-action. If the normal bundle $\mathcal N_XY$ of $Y\subset X$ is trivial in the the setting of Corollary 4.31, then (4.54) gives

$$\mathrm{Vol}\big(U_c,\omega\big)=\frac{c^m}{m!}\mathrm{Vol}\big(Y,\omega\big)=\mathrm{Vol}\big(B^m_{\sqrt{c/\pi}}(0),\omega_{\mathbb C^m}\big)\mathrm{Vol}\big(Y,\omega\big),$$

as expected.

4.7 Holomorphic blowup

There is no tubular neighborhood theorem in the holomorphic category. A blowup along a closed complex submanifold in this category is instead carried out via local charts as described below; see also [27, p603]. The complex blowup in the smooth category can also be carried out through a similar local construction; see [63, Sections 2.2,2.3]. Thus, the blowup in the holomorphic category below is a specialization of the complex blowup in the smooth category.

Suppose X is a complex manifold of complex dimension n and $Y\subset X$ is a closed complex submanifold of complex codimension m. We call a holomorphic coordinate chart

$$\varphi\equiv(\varphi_1,\ldots,\varphi_n)\colon U\longrightarrow\mathbb C^n=\mathbb C^m\times\mathbb C^{n-m}$$

on X a chart for Y in X if

$$U\cap Y=\big\{x\in U\colon \varphi_1(x),\ldots,\varphi_m(x)=0\big\}. \tag{4.58}$$

For such a chart, the subspace

$$\mathrm{Bl}_Y\varphi\equiv\big\{(\ell,x)\in\mathbb CP^{m-1}\times U\colon \big(\varphi_1(x),\ldots,\varphi_m(x)\big)\in\ell\subset\mathbb C^m\big\}$$

of $\mathbb CP^{m-1}\times U$ is a closed submanifold. We denote by

$$\mathrm{pr}_{\mathbb P},\mathrm{pr}_U\colon \mathrm{Bl}_Y\varphi\longrightarrow\mathbb CP^{m-1},U$$

the two projections. The second projection restricts to a diffeomorphism

$$\mathrm{pr}_U\colon \big\{(\ell,x)\in\mathbb CP^{m-1}\times U\colon x\notin Y\big\}\longrightarrow U-Y.$$

For each $i\in[m]$, let $\widetilde\varphi_{m;i;j}$ be as in Exercise 1.28(b). The smooth map

$$\widetilde\varphi_i\equiv(\widetilde\varphi_{i;1},\ldots,\widetilde\varphi_{i;n})\colon \mathrm{Bl}_{Y;i}\varphi\equiv\big\{([v_1,\ldots,v_m],x)\in\mathrm{Bl}_Y\varphi\colon v_i\neq0\big\}\longrightarrow\mathbb C^{n-m}\times\mathbb C^m, \tag{4.59}$$
$$\widetilde\varphi_{i;j}=\begin{cases}\widetilde\varphi_{m;i;j}\circ\mathrm{pr}_{\mathbb P}, & \text{if } j\in[m]-\{i\};\\ \varphi_j\circ\mathrm{pr}_U, & \text{if } j\in\{i\}\cup\big([n]-[m]\big);\end{cases}$$

is then a coordinate chart on $\mathrm{Bl}_Y\varphi$.

Exercise 4.34. With the notation and assumptions as above, suppose

$$\varphi_\alpha\colon U_\alpha \longrightarrow \mathbb{C}^m\times\mathbb{C}^{n-m} \qquad\text{and}\qquad \varphi_{\alpha'}\colon U_{\alpha'} \longrightarrow \mathbb{C}^m\times\mathbb{C}^{n-m}$$

are holomorphic charts for Y in X. Let

$$\varphi_{\alpha\alpha'}\equiv\big(\varphi_{\alpha\alpha';1},\ldots,\varphi_{\alpha\alpha';n}\big)\equiv\varphi_\alpha\circ\varphi_{\alpha'}^{-1}\colon \varphi_{\alpha'}\big(U_\alpha\cap U_{\alpha'}\big)\longrightarrow\varphi_\alpha\big(U_\alpha\cap U_{\alpha'}\big)$$

be the overlap map between the charts φ_α and $\varphi_{\alpha'}$. Show that

(a) there exists a holomorphic map

$$\begin{gathered} h_{\alpha\alpha'}\colon \varphi_{\alpha'}\big(U_\alpha\cap U_{\alpha'}\big)\longrightarrow \mathrm{End}_{\mathbb{C}}(\mathbb{C}^m) \qquad \text{s.t.}\\ \big(\varphi_{\alpha\alpha';1}(v,z),\ldots,\varphi_{\alpha\alpha';\mathfrak{c}}(v,z)\big)=h_{\alpha\alpha'}(v,z)v \quad \forall\,(v,z)\in\varphi_{\alpha'}(U_\alpha\cap U_{\alpha'})\subset\mathbb{C}^m\times\mathbb{C}^{n-m}; \end{gathered} \tag{4.60}$$

(b) $h_{\alpha\alpha'}(v,z)v\neq 0$ for all $(v,z)\in\varphi_{\alpha'}\big(U_\alpha\cap U_{\alpha'}\big)$ with $v\in\mathbb{C}^m-\{0\}$ and $z\in\mathbb{C}^{n-m}$;

(c) $h_{\alpha\alpha'}(0,z)$ is an isomorphism of $\mathbb{C}^m$ for all $(0,z)\in\varphi_{\alpha'}\big(U_\alpha\cap U_{\alpha'}\big)\cap\big(0^m\times\mathbb{C}^{n-m}\big)$;

(d) the map

$$\widetilde\varphi_{\alpha\alpha'}\colon \mathrm{Bl}_Y\big(\varphi_{\alpha'}|_{U_\alpha\cap U_{\alpha'}}\big)\longrightarrow \mathrm{Bl}_Y\big(\varphi_\alpha|_{U_\alpha\cap U_{\alpha'}}\big),\quad \widetilde\varphi_{\alpha\alpha'}(\ell,x)=\big(h_{\alpha\alpha'}(\varphi_{\alpha'}(x))\ell,x\big), \tag{4.61}$$

is well-defined and holomorphic with $\mathrm{pr}_{U_{\alpha'}}|_{U_\alpha\cap U_{\alpha'}}=\mathrm{pr}_{U_\alpha}\circ\widetilde\varphi_{\alpha\alpha'}$.

Suppose $\varphi_{\alpha''}\colon U_{\alpha''}\longrightarrow\mathbb{C}^m\times\mathbb{C}^{n-m}$ is another holomorphic chart for Y in X.

(e) Show that $\widetilde\varphi_{\alpha\alpha''}=\widetilde\varphi_{\alpha\alpha'}\circ\widetilde\varphi_{\alpha'\alpha''}\colon \mathrm{Bl}_Y\big(\varphi_{\alpha''}|_{U_\alpha\cap U_{\alpha'}\cap U_{\alpha''}}\big)\longrightarrow \mathrm{Bl}_Y\big(\varphi_\alpha|_{U_\alpha\cap U_{\alpha'}\cap U_{\alpha''}}\big)$.

Suppose $\{\varphi_\alpha: U_\alpha\longrightarrow\mathbb{C}^n\}_{\alpha\in\mathcal{I}}$ is a collection of holomorphic charts for Y in X covering Y. The holomorphic blowup of X along Y is the complex manifold

$$\begin{gathered} \mathrm{Bl}_Y X\equiv\Big((X-Y)\sqcup\bigsqcup_{\alpha\in\mathcal{I}}\mathrm{Bl}_Y\varphi_\alpha\Big)\Big/\!\!\sim,\quad \mathrm{Bl}_Y\varphi_\alpha\ni(\ell,x)\sim x\in X-Y\ \ \forall\,x\in U_\alpha-Y,\ \alpha\in\mathcal{I},\\ \mathrm{Bl}_Y\big(\varphi_{\alpha'}|_{U_\alpha\cap U_{\alpha'}}\big)\ni(\ell,x)\sim\widetilde\varphi_{\alpha\alpha'}(\ell,x)\in\mathrm{Bl}_Y\big(\varphi_\alpha|_{U_\alpha\cap U_{\alpha'}}\big)\ \ \forall\,\alpha,\alpha'\in\mathcal{I}; \end{gathered}$$

see Exercise 4.35 below. The blowdown map

$$p_{X|Y}\colon \mathrm{Bl}_Y X\longrightarrow X,\qquad p_{X|Y}\big([\widetilde x]\big)=\begin{cases}\widetilde x, & \text{if } \widetilde x\in X-Y;\\ \mathrm{pr}_{U_\alpha}(\widetilde x), & \text{if } \widetilde x\in\mathrm{Bl}_Y\varphi_\alpha,\ \alpha\in\mathcal{I};\end{cases} \tag{4.62}$$

is well-defined, surjective, proper, and holomorphic.

Exercise 4.35. With the notation and assumptions as above, show that

(a) the quotient $\mathrm{Bl}_Y X$ is a Hausdorff topological space;

(b) the quotient $\mathrm{Bl}_Y X$ is a complex manifold;

(c) the blowdown map (4.62) is holomorphic and restricts to a biholomorphism from the complement of the exceptional divisor $\mathrm{E}_Y X\equiv p_{X|Y}^{-1}(Y)$ in $\mathrm{Bl}_Y X$ to the complement of Y in X;

(d) the map $p_{X|Y}\colon \mathrm{Bl}_Y X \longrightarrow X$ is independent of the choice of a collection of holomorphic charts for Y in X covering Y up to biholomorphism.

Exercise 4.36. With the notation and assumptions as above, define

$$\mathcal{O}_{\varphi_\alpha}(\mathrm{E}_Y X) = \mathrm{pr}_{\mathbb{P}}^* \gamma_{m-1} \equiv \big\{(\ell, x, v) \in \mathrm{Bl}_Y \varphi_\alpha \times \mathbb{C}^m \colon v \in \ell \subset \mathbb{C}^m\big\} \longrightarrow \mathrm{Bl}_Y \varphi_\alpha,$$

for each $\alpha \in \mathcal{I}$. Show that

(a) $\mathrm{pr}_\alpha\colon \mathcal{O}_{\varphi_\alpha}(\mathrm{E}_Y X) \longrightarrow \mathrm{Bl}_Y \varphi_\alpha$, $\mathrm{pr}_\alpha(\ell, x, v) = (\ell, x)$, is a holomorphic line bundle for every $\alpha \in \mathcal{I}$;

(b) $s_\alpha\colon \mathrm{Bl}_Y \varphi_\alpha \longrightarrow \mathcal{O}_{\varphi_\alpha}(\mathrm{E}_Y X)$, $s_\alpha(\ell, x) = (\ell, x, \varphi_\alpha(x))$, is a holomorphic section of pr_α for every $\alpha \in \mathcal{I}$;

(c) the map

$$\begin{gathered}\widetilde{\Phi}_{\alpha\alpha'}\colon \mathcal{O}_{\varphi_{\alpha'}|_{U_\alpha \cap U_{\alpha'}}}(\mathrm{E}_Y X) \longrightarrow \mathcal{O}_{\varphi_\alpha|_{U_\alpha \cap U_{\alpha'}}}(\mathrm{E}_Y X),\\ \widetilde{\Phi}_{\alpha\alpha'}(\ell, x, v) = \big(h_{\alpha\alpha'}(\varphi_{\alpha'}(x))\ell, x, h_{\alpha\alpha'}(\varphi_{\alpha'}(x))v\big),\end{gathered}$$

is a well-defined isomorphism of holomorphic line bundles lifting $\widetilde{\varphi}_{\alpha\alpha'}$ for all $\alpha, \alpha' \in \mathcal{I}$;

(d) the quotient

$$\begin{gathered}\mathcal{O}_{\mathrm{Bl}_Y X}(\mathrm{E}_Y X) \equiv \Big(\big((X-Y)\times\mathbb{C}\big) \sqcup \bigsqcup_{\alpha\in\mathcal{I}} \mathcal{O}_{\varphi_\alpha}(\mathrm{E}_Y X)\Big)\Big/\!\sim,\\ \mathcal{O}_{\varphi_\alpha}(\mathrm{E}_Y X) \ni \big(\ell, x, c\big(\phi_{\alpha;1}(x), \ldots, \phi_{\alpha;m}(x)\big)\big) \sim (x, c) \in X - Y \;\; \forall\, x \in U_\alpha - Y,\; \alpha \in \mathcal{I},\\ \mathcal{O}_{\varphi_{\alpha'}|_{U_\alpha \cap U_{\alpha'}}}(\mathrm{E}_Y X) \ni (\ell, x, v) \sim \widetilde{\Phi}_{\alpha\alpha'}(\ell, x, v) \in \mathcal{O}_{\varphi_\alpha|_{U_\alpha \cap U_{\alpha'}}}(\mathrm{E}_Y X) \;\; \forall\, \alpha, \alpha' \in \mathcal{I},\end{gathered}$$

is a complex manifold;

(e) the map $\mathrm{pr}\colon \mathcal{O}_{\mathrm{Bl}_Y X}(\mathrm{E}_Y X) \longrightarrow \mathrm{Bl}_Y X$ given by

$$\mathrm{pr}\big([\widetilde{v}]\big) = \begin{cases} [x], & \text{if } \widetilde{v} = (x, c) \in (X-Y)\times\mathbb{C};\\ [\ell, x], & \text{if } \widetilde{v} = (\ell, x, v) \in \mathcal{O}_{\varphi_\alpha}(\mathrm{E}_Y X);\end{cases}$$

is well-defined and is a holomorphic line bundle;

(f) the map $s_E\colon \mathrm{Bl}_Y X \longrightarrow \mathcal{O}_{\mathrm{Bl}_Y X}(\mathrm{E}_Y X)$ given by

$$s_E\big([\widetilde{x}]\big) = \begin{cases} [x, 1], & \text{if } \widetilde{x} = x \in (X-Y);\\ [\ell, x, (\phi_{\alpha;1}(x), \ldots, \phi_{\alpha;m}(x))], & \text{if } \widetilde{x} = (\ell, x) \in \mathrm{Bl}_Y(\varphi_\alpha);\end{cases}$$

is a well-defined section of pr.

Proposition 4.37. *Suppose (X, ω, J) is a Kähler manifold, $Y \subset X$ is a compact complex submanifold of complex codimension m, and $p_{X|Y}\colon \mathrm{Bl}_Y X \longrightarrow X$ is the complex blowup of X along Y with exceptional divisor $\mathrm{E}_Y X \subset \mathrm{Bl}_Y X$. For every neighborhood $U \subset \mathrm{Bl}_Y X$ of $\mathrm{E}_Y X$, there exist a closed 2-form κ_U on $\mathrm{Bl}_Y X$ and $\epsilon_U \in \mathbb{R}^+$ so that*

(Bl*1*) *κ_U vanishes outside of U and the restriction of κ_U to a fiber of $p_{X|Y}\colon \mathrm{E}_Y X \longrightarrow Y$ is isomorphic to $-\omega_{\mathrm{FS};m-1}$;*

(Bl*2*) *the closed 2-form $\widetilde{\omega}_c \equiv p_{X|Y}^*\omega - c\kappa_U$ is nondegenerate and compatible with the complex structure $\widetilde{J}$ on $\mathrm{Bl}_Y X$ for all $c\in(0,\epsilon_U)$;*

(Bl*3*) *if (X,ω,J) is compact and projective, then so is $(\mathrm{Bl}_Y X, k_c\omega_c, \widetilde{J})$ for every $c\in(0,\epsilon_U)\cap\mathbb{Q}$ and some $k_c\in\mathbb{Z}^+$.*

Proof. We continue with the setup and notation in the holomorphic blowup construction and Exercise 4.36 above. The vector bundle isomorphisms

$$\Phi_\alpha\colon \mathbb{C}^m\times(U_\alpha\cap Y) \longrightarrow \mathcal{N}_Y X|_{U_\alpha\cap Y}, \quad \Phi_\alpha\big((v_1,\ldots,v_m),x\big) \longrightarrow \sum_{j=1}^m v_j \frac{\partial}{\partial\varphi_{\alpha;j}}\bigg|_x + T_x Y, \tag{4.63}$$

with $\alpha\in\mathcal{I}$ determine a holomorphic structure on the normal bundle $\mathcal{N}_X Y$ of Y in X. By the construction in Exercise 4.36, the restriction of the holomorphic line bundle $\mathcal{O}_{\mathrm{Bl}_Y X}(\mathrm{E}_Y X)\longrightarrow \mathrm{Bl}_Y X$ to $\mathrm{E}_Y X = \mathbb{P}(\mathcal{N}_X Y)$ is the tautological holomorphic line bundle,

$$\mathcal{O}_{\mathrm{Bl}_Y X}(\mathrm{E}_Y X)\big|_{\mathrm{E}_Y X} = \gamma_{\mathcal{N}_X Y} \subset \big\{p_{X|Y}|_{\mathrm{E}_Y X}\big\}^*\mathcal{N}_X Y \longrightarrow \mathrm{E}_Y X.$$

The Riemannian metric $g_{\mathfrak{i}}^\omega$ induces a Riemannian metric and a Hermitian metric on $\mathcal{N}_X Y$; the latter induces a Hermitian metric on $\gamma_{\mathcal{N}_X Y}$. Let g_E be a Hermitian metric on $\mathcal{O}_{\mathrm{Bl}_Y X}(\mathrm{E}_Y X)$ that extends the last metric with $g_E(s_E,s_E)=1$ outside of U. Take κ_U to be $\mathfrak{i}/2\pi$ times the curvature of the connection on the holomorphic line bundle $\mathcal{O}_{\mathrm{Bl}_Y X}(\mathrm{E}_Y X)$ compatible with its holomorphic structure and the metric g_E as in Exercise B.22. By Exercise B.13(c), the closed 2-form κ_∇ on $\mathrm{Bl}_Y X$ is real-valued. By Exercise B.23, κ_U is $\widetilde{J}$-invariant and vanishes outside of U. By Exercises B.23 and 1.27, the restriction of κ_U to a fiber of $p_{X|Y}\colon \mathrm{E}_Y X\longrightarrow Y$ is isomorphic to $-\omega_{\mathrm{FS};m-1}$. Since κ_U vanishes outside of U and the blowdown map $p_{X|Y}$ restricts to a biholomorphism from $\mathrm{Bl}_Y X - \mathrm{E}_Y X$ to $X-Y$, the closed 2-form $\widetilde{\omega}_c$ is nondegenerate and compatible with $\widetilde{J}$ outside of U for every $c\in\mathbb{R}$.

Since ω is compatible with J, $p_{X|Y}$ is $(J,\widetilde{J})$-holomorphic, and its restriction to $\mathrm{Bl}_Y X - \mathrm{E}_Y X$ is a biholomorphism onto $X-Y$,

$$p_{X|Y}^*\omega(w,\widetilde{J}w) \ge 0 \ \forall\, w\in T(\mathrm{Bl}_Y X), \quad p_{X|Y}^*\omega(w,\widetilde{J}w) > 0 \ \forall\, w\in T(\mathrm{Bl}_Y X)|_{\mathrm{Bl}_Y X-\mathrm{E}_Y X},\ w\neq 0. \tag{4.64}$$

Let $\widetilde{g}$ be a $\widetilde{J}$-invariant Riemannian metric on $\mathrm{Bl}_Y X$. By the compactness of $\mathrm{E}_Y X$, we can assume that the closure $\overline{U}\subset \mathrm{Bl}_Y X$ of U is compact as well. Thus, there exists $C_U\in\mathbb{R}^+$ so that

$$\big|\kappa_U(w,w')\big| \le C_U\widetilde{g}(w,w)^{1/2}\widetilde{g}(w',w')^{1/2} \qquad \forall\ w,w'\in T_{\widetilde{x}}(\mathrm{Bl}_Y X),\ \widetilde{x}\in U. \tag{4.65}$$

Since $T(\mathrm{E}_Y X)$ is the kernel of $\mathrm{d}p_{X|Y}$ over $\mathrm{E}_Y X$, there exist a neighborhood $U'\subset \mathrm{Bl}_Y X$ of $\mathrm{E}_Y X$ and a complex line subbundle $\gamma\subset T(\mathrm{Bl}_Y X)|_{U'}$ so that the restriction of $p_{X|Y}^*\omega$ to γ is nondegenerate. Thus,

$$\gamma^\omega \equiv \big\{w'\in T_{\widetilde{x}}(\mathrm{Bl}_Y X)\colon \widetilde{x}\in U',\ p_{X|Y}^*\omega(w,w')=0\ \forall\, w\in\gamma_{\widetilde{x}}\big\}$$

is a complex subbundle of $T(\mathrm{Bl}_Y X)|_{U'}$ so that

$$T(\mathrm{Bl}_Y X)|_{U'} = \gamma\oplus\gamma^\omega \qquad \text{and} \qquad \gamma^\omega|_{\mathrm{E}_Y X} = T(\mathrm{E}_Y X).$$

Since $\overline{U}$ is compact and the restriction of κ_U to a fiber of $p_{X|Y}\colon \mathrm{E}_Y X \longrightarrow Y$ is isomorphic to $-\omega_{\mathrm{FS};m-1}$, there exist a neighborhood $U_E \subset U \cap U'$ of $\mathrm{E}_Y X$ and $C_E \in \mathbb{R}^+$ so that

$$p_{X|Y}^*\omega\big(w,\widetilde{J}w\big) \ge C_E^{-1}\widetilde{g}(w,w) \quad \forall\, w \in \gamma, \qquad -\kappa_U\big(w',\widetilde{J}w'\big) \ge C_E^{-1}\widetilde{g}(w',w') \quad \forall\, w' \in \gamma^\omega, \tag{4.66}$$

$$p_{X|Y}^*\omega\big(w,\widetilde{J}w\big) \ge C_E^{-1}\widetilde{g}(w,w) \quad \forall\, w \in T(\mathrm{Bl}_Y X)\big|_{U-U_E}; \tag{4.67}$$

the last bound uses the second statement in (4.64).

Let $c \in \mathbb{R}^+$. Suppose $\widetilde{x} \in U_E$, $w \in \gamma_{\widetilde{x}}$, and $w' \in \gamma^\omega_{\widetilde{x}}$. From (4.66), (4.65), and the first statement in (4.64), we obtain

$$\begin{aligned}
\widetilde{\omega}_c\big(w+w',\widetilde{J}w+\widetilde{J}w'\big) &= \big(p_{X|Y}^*\omega(w,\widetilde{J}w)+p_{X|Y}^*\omega(w',\widetilde{J}w')\big) \\
&\quad -c\big(\kappa_U(w',\widetilde{J}w')+2\kappa_U(w,\widetilde{J}w')+\kappa_U(w,\widetilde{J}w)\big) \\
&\ge C_E^{-1}\big(\widetilde{g}(w,w)+c\widetilde{g}(w',w')\big)-cC_U\big(\widetilde{g}(w,w)+2\widetilde{g}(w,w)^{1/2}\widetilde{g}(w',w')^{1/2}\big) \\
\ge \big(C_E^{-1}-cC_U\big)\widetilde{g}(w,w)&+cC_E^{-1}\widetilde{g}(w',w') - cC_U\Big(2C_EC_U\widetilde{g}(w,w)+\frac{1}{2C_EC_U}\widetilde{g}(w',w')\Big).
\end{aligned}$$

Thus, the 2-form $\widetilde{\omega}_c$ is nondegenerate and compatible with $\widetilde{J}$ over U_E if $cC_EC_U(1+2C_EC_U)<1$. Suppose now $\widetilde{x} \in U-U_E$ and $w \in T_{\widetilde{x}}(\mathrm{Bl}_Y X)$. From (4.67) and (4.65), we then obtain

$$\widetilde{\omega}_c(w,\widetilde{J}w) \ge \big(C_E^{-1}-C_U\big)\widetilde{g}(w,w).$$

Thus, $\widetilde{\omega}_c$ is nondegenerate and compatible with $\widetilde{J}$ over $U-U_E$ if $cC_EC_U<1$. We conclude that the requirement (Bl2) is satisfied with $\epsilon_U = 1/C_EC_U(1+2C_EC_U)$.

Suppose $L_\omega \longrightarrow X$ is a holomorphic line bundle with $[\omega]=c_1(L_\omega) \in H^2_{\mathrm{deR}}(X)$. If $c \in \mathbb{Q}$ and $k_c \in \mathbb{Z}$ are so that $k_c c \in \mathbb{Z}$, then

$$\big[k_c\widetilde{\omega}_c\big] = c_1\big(p_{X|Y}^*L_\omega^{\otimes k_c}\otimes\mathcal{O}_{\mathrm{Bl}_Y X}(\mathrm{E}_Y X)^{k_c c}\big) \in H^2_{\mathrm{deR}}(\mathrm{Bl}_Y X);$$

see Remark B.19. By (Bl2), the holomorphic line bundle $L_\omega^{\otimes k_c}\otimes\mathcal{O}_{\mathrm{Bl}_Y X}(\mathrm{E}_Y X)^{k_c c} \longrightarrow \mathrm{Bl}_Y X$ is positive in the sense of [27, p148] if $c \in (0,\epsilon_U)$ and $k_c \in \mathbb{Z}^+$. Along with the Kodaira Embedding Theorem [27, p181], this establishes (Bl3). $\square$

4.8 Excision of Lagrangian submanifold

As pointed out in [5, Section 4.1], the symplectic cut construction can be used to excise a Lagrangian submanifold admitting a Riemannian metric for which all geodesics (with respect to its Levi-Civita connection) determined by the unit length tangent vectors are periodic with the same period (i.e. all geodesics are closed and all simple geodesics are of the same length) from a symplectic manifold. Examples of such a Lagrangian include the unit sphere $S^n \subset \mathbb{R}^{n+1}$ with the standard metric, the real projective space $\mathbb{R}P^n$ with the induced metric, the complex projective space $\mathbb{C}P^n$ with the Fubini-Study metric, and the quaternionian projective space $\mathbb{H}P^n$ with the metric induced from the standard metric on $\mathbb{H}^{n+1}$; see [7, §3.22,31].

Exercise 4.38. Suppose Z is a smooth manifold, g is a Riemannian metric on Z, and thus

$$\Phi_g\colon TZ \longrightarrow T^*Z, \qquad \Phi_g(v) = g(v,\cdot), \tag{4.68}$$

is an isomorphism of vector bundles over Z. Let ω_{T^*Z} be the canonical symplectic form on T^*Z as in Exercise 1.34, $\omega_g = \Phi_g^*\omega_{T^*Z}$, $\exp_g\colon \mathcal{U}_g \longrightarrow Z$, where $\mathcal{U}_g \subset TZ$ is a neighborhood of $Z \subset TZ$, be the exponential map with respect to the Levi-Civita connection of the metric g, and ζ_g be the geodesic flow vector field on TZ given by

$$\zeta_g(w) = \frac{\mathrm{d}}{\mathrm{d}t}\big\{\mathrm{d}_{tw}\exp\big\}(w)\bigg|_{t=0} \in T_wTZ \qquad \forall\, w\in TZ\,.$$

Show that

$$\iota_{\zeta_g}\omega_g = \frac{1}{2}\mathrm{d}H_g, \qquad \text{where} \quad H_g\colon TZ \longrightarrow \mathbb{R}, \quad H_g(w) = g(w,w).$$

Hint: in the coordinates $(x_1,\ldots,x_n,w_1,\ldots,w_n)$ on TZ induced by coordinates $(x_1,\ldots,x_n)$ on Z, a geodesic flow $(x,w) = (x(t),w(t))$ satisfies

$$\begin{cases} x_k' = w_k, & \forall\, k\in[n];\\ v_k' = -\sum\limits_{i,j=1}^{n}\Gamma_{ij}^k p_i p_j, & \forall\, k\in[n]; \end{cases} \qquad \text{with} \quad \Gamma_{ij}^k = \frac{1}{2}\sum_{\ell=1}^{n} g^{k\ell}\big(\partial_{x_i}g_{\ell j} + \partial_{x_j}g_{i\ell} - \partial_{x_\ell}g_{ij}\big),$$

where $g_{ij} = g(\partial_{x_i},\partial_{x_j})$ and $(g^{ij})_{i,j\in[n]}$ is the inverse of the symmetric matrix $(g_{ij})_{i,j\in[n]}$.

Exercise 4.39. Let $n\in\mathbb{Z}^+$, g be the Riemannian metric on the unit sphere $S^n \subset \mathbb{R}^{n+1}$ induced by the standard metric on $\mathbb{R}^{n+1}$, and ω_g be the symplectic form on TS^n as in Exercise 4.38.

(a) Show that ω_g is the restriction of the standard symplectic form $\omega_{\mathbb{C}^{n+1}}$ to

$$TS^n = \big\{(v,w)\in S^n \times \mathbb{R}^{n+1} \colon v\cdot w = 0\big\} \subset \mathbb{R}^{n+1}\times\mathbb{R}^{n+1} = \mathbb{C}^{n+1}\,.$$

(b) Let ψ be the restriction of the action of $S^1 \subset \mathbb{C}^*$ on $\mathbb{C}^{n+1}$ by the complex multiplication to $TS^n - S^n$. Show that the smooth function

$$H\colon TS^n - S^n \longrightarrow \mathbb{R}, \qquad H(w) = 2\pi g(w,w)^{1/2},$$

is a Hamiltonian for ψ with respect to ω_g. *Hint:* use Exercise 4.38.

Exercise 4.40. With n, g, and ψ as in Exercise 4.39, let $q\colon S^n \longrightarrow \mathbb{R}P^n$ be the usual quotient map, $\overline{g}$ be the Riemannian metric on $\mathbb{R}P^n$ induced by the metric $4g$ on S^n via q, $\omega_{\overline{g}}$ be the symplectic form on $T\mathbb{R}P^n$ induced by $\overline{g}$ as in Exercise 4.38, and $\overline{\psi}$ be the S^1-action on $T\mathbb{R}P^n$ given by

$$\overline{\psi}_u\colon T\mathbb{R}P^n \longrightarrow T\mathbb{R}P^n, \quad \overline{\psi}_u\big(\mathrm{d}q(v,w)\big) = \mathrm{d}q\big(\psi_{\sqrt{u}}(v,w)\big) \;\; \forall\, u\in S^1\subset\mathbb{C}^*,\, (v,w)\in TS^n.$$

Show that

(a) $\{dq\}^*\omega_{\overline{g}} = 4\omega_g$, with ω_g as in Exercise 4.39;

(b) the smooth function

$$\overline{H}\colon T\mathbb{R}P^n - \mathbb{R}P^n \longrightarrow \mathbb{R}, \qquad H(w) = 2\pi\overline{g}(w,w)^{1/2},$$

is a Hamiltonian for $\overline{\psi}$ with respect to $\omega_{\overline{g}}$.

Let (X,ω) be a symplectic manifold (X,ω), $Z\subset X$ be a (nonempty) closed Lagrangian submanifold Z, and g be a Riemannian metric on Z for which all geodesics determined by the unit length tangent vectors are periodic with the same period. After rescaling the metric g, this period can be assumed to be 2π, as in Exercise 4.39. The Riemannian metric g induces a norm $|\cdot|_g$ on the fibers of the vector bundle $\mathrm{pr}\colon T^*Z\longrightarrow Z$ via the isomorphism Φ_g of (4.68). By the Lagrangian Tubular Neighborhood Theorem of Corollary 2.19, there exist a smooth function $\rho\colon Z\longrightarrow\mathbb{R}^+$ and a diffeomorphism

$$\Phi\colon \mathcal{U}_\rho\equiv\big\{\theta\in T^*Z\colon |\theta|_g<\rho(\pi(\theta))\big\}\longrightarrow U_\rho \tag{4.69}$$

onto a neighborhood of $Z\subset X$ so that $\Phi^*\omega$ is the restriction of the canonical symplectic form ω_{T^*Z} on T^*Z to $\mathcal{U}_\rho$.

By Exercise 4.38, the restriction of the continuous function

$$H\colon U_\rho\longrightarrow\mathbb{R},\qquad H(x)=2\pi\big|\Phi^{-1}(x)\big|_g, \tag{4.70}$$

to $U_\rho-Z$ is a Hamiltonian with respect to ω for the free action ψ of $S^1=\mathbb{R}/\mathbb{Z}$ on $U_\rho-Z$ given by

$$\psi_{[t]}(x)=\Phi\big(\Phi_g\big(\big\{\mathrm{d}_{-t\Phi_g^{-1}(\Phi^{-1}(x))/|\Phi^{-1}(x)|_g}\exp\big\}\big(\Phi_g^{-1}(\Phi^{-1}(x))\big)\big)\big)\quad\forall\, t\in\mathbb{R},\ x\in U_\rho-Z. \tag{4.71}$$

Suppose that the function ρ above is constant; this can be achieved in particular if Z is compact. In such a case, $(U_\rho-Z,\psi,H,c)$ is cutting data for (X,ω) for any $c\in\mathbb{R}$, with $H^{-1}(c)\neq 0$ whenever $c\in(0,2\pi\rho)$. If $c\in(0,2\pi\rho)$, the cut symplectic manifold $(X'_-,\omega'|_{X'_-})$ of Theorem 4.24 contains (the image under the quotient projection p of) the Lagrangian submanifold Z. The function $H^2\colon U_\rho\longrightarrow\mathbb{R}$ descends to a (smooth) Morse-Bott function $X'_-\longrightarrow\mathbb{R}$ with the critical locus consisting of the Lagrangian Z and the symplectic divisor Y'_- as in (4.45). If $Z=S^n$ with the standard metric and $c\in(0,\rho/\pi)$, the symplectic manifold $(X'_-,\omega|_{X'_-})$ is isomorphic to the quadric hypersurface

$$Q^n\equiv\Big\{[z_0,\ldots,z_{n+1}]\in\mathbb{C}P^{n+1}\colon z_0^2=\sum_{i=1}^{n+1}z_i^2\Big\}$$

with a properly rescaled Fubini-Study symplectic form $\omega_{\mathrm{FS};n+1}$; see Exercise 4.42 below. If $Z=\mathbb{R}P^n$ with the induced metric (resp. $Z=\mathbb{C}P^n$ with the Fubini-Study metric), then $(X'_-,\omega|_{X'_-})$ is isomorphic to $\mathbb{C}P^n$ (resp. $\mathbb{C}P^n\times\mathbb{C}P^n$); see Exercise 4.43 below and [5, Proposition 4.4].

Exercise 4.41 ([22, Lemma 2.9])**.** Let (X,ω) be a symplectic manifold, $Z\subset X$ be a closed Lagrangian submanifold, and g be a Riemannian metric on Z for which all geodesics determined by unit length tangent vectors are periodic with period 2π.

(a) Suppose $\Phi\colon\mathcal{U}_\rho\longrightarrow U_\rho$ is a tubular neighborhood identification as in (4.69) with ρ being a constant function, ψ is the associated S^1-action on U_ρ–Z given by the negative exponential flow as in (4.71), H is its Hamiltonian with respect to ω, and $c\in(0,2\pi\rho)$. Let $(X'\equiv X'_-\sqcup X'_+,\omega')$ be the cut symplectic manifold determined by the cutting data $(U_\rho$–$Z,\psi,H,c)$ for (X,ω) and

$$p\colon X_{H^{-1}(c)}\longrightarrow X'$$

be the quotient projection as in Theorem 4.24. Show that the anti-symplectic involution

$$\sigma_\Phi\colon U_\rho\longrightarrow U_\rho,\qquad \sigma(x)=\Phi\big(-\Phi^{-1}(x)\big),$$

induces an anti-symplectic involution σ'_- on X'_- such that

$$\sigma'_-\circ p = p\circ\sigma_\Phi\colon \big\{x\in U_\rho\colon |\Phi^{-1}(x)|_g<c/\pi\big\} \longrightarrow X'_-$$

and the fixed locus of σ'_- is (the image under p of) the Lagrangian submanifold Z.

(b) Suppose σ is an anti-symplectic involution on X such that $X^\sigma=Z$. Show that the tubular neighborhood identification Φ above can be chosen so that σ induces an anti-symplectic involution σ' on X' such that

$$\sigma'\circ p = p\circ\sigma\colon X-H^{-1}(c) \longrightarrow X'$$

and the fixed locus of σ' is (the image under p of) the Lagrangian submanifold Z.

Exercise 4.42 ([5, Proposition 4.4],[22, p249])**.** Suppose $n\in\mathbb{Z}^+$, g is the Riemannian metric on the unit sphere $S^n\subset\mathbb{R}^{n+1}$ induced by the standard metric on $\mathbb{R}^{n+1}$, ψ is the associated S^1-action on $T^*S^n-S^n$ given by the negative exponential flow as in (4.71) with $U_\rho=T^*S^n$ and $\Phi=\mathrm{id}_{T^*S^n}$, H is its Hamiltonian with respect to the canonical symplectic form $\omega_{T^*S^n}$ on T^*S^n as in (4.70), and $c\in\mathbb{R}^+$. Let

$$\psi_-\colon S^1 \longrightarrow \mathrm{Symp}\big((T^*S^n-S^n)\times\mathbb{C}, \omega_{T^*S^n}|_{T^*S^n-S^n}\oplus\omega_{\mathbb{C}}\big) \quad\text{and}\quad \widetilde{H}_-\colon (T^*S^n-S^n)\times\mathbb{C} \longrightarrow \mathbb{R}$$

be the associated S^1-action and Hamiltonian as in (4.47) and (4.46). Define

$$\begin{aligned}(TS^n)_c &= \big\{(v,w)\in S^n\times\mathbb{R}^{n+1}\colon v\cdot w=0,\ |w|<c\big\},\\ \widetilde{U}_- &= \big\{(\nu,z)\in\mathbb{C}^{n+1}\times\mathbb{C}\colon |\mathrm{Re}\,\nu|^2, |\mathrm{Im}\,\nu|^2=1,\ (\mathrm{Re}\,\nu)\cdot(\mathrm{Im}\,\nu)=0,\ |z|^2<c\big\}.\end{aligned}$$

(a) Show that $(T^*S^n-S^n,\psi,H,2\pi c)$ is cutting data for $(T^*S^n,\omega_{T^*S^n})$.

Let $(X'\equiv X'_-\sqcup X'_+,\omega')$ be the cut symplectic manifold determined by this cutting data as in Theorem 4.24, $Y'\equiv Y'_-\sqcup Y'_+$ be the associated symplectic divisor, and σ'_- be the involution on X'_- as in Exercise 4.41. Show that

(b) there is an identification of the hypersurface $\widetilde{H}_-^{-1}(\pi c)$ with the submanifold $\widetilde{U}_-\subset\mathbb{C}^{n+1}\times\mathbb{C}$ which is equivariant with respect to the S^1-action ψ_- on the domain and the S^1-action on the target by the complex multiplication in both Cartesian coordinates;

(c) the space

$$X_-\equiv\big((TS^n)_c\sqcup(\widetilde{U}_-/S^1)\big)/\!\sim, \quad (TS^n)_c\ni(v,w\neq0)\sim\big[v+\mathrm{i}w/|w|,\sqrt{c-|w|}\big],$$

is a manifold diffeomorphic to X'_-;

(d) the map $X_-\longrightarrow Q^n$ induced by the maps

$$\begin{aligned}(TS^n)_c\ni(v,w) &\longrightarrow \big[\sqrt{1-|w|^2/c^2}, v+\mathrm{i}w/c\big]\in\mathbb{C}P^{n+1},\\ \widetilde{U}_-/S^1\ni[\nu,z] &\longrightarrow \big[z\sqrt{2/c-|z|^2/c^2}, \nu-\mathrm{i}(z/c)\mathrm{Im}(\bar{z}\nu)\big]\in\mathbb{C}P^{n+1},\end{aligned}$$

is a well-defined diffeomorphism;

(e) there is a diffeomorphism $\Phi: X'_- \longrightarrow Q^n$ intertwining the involution σ'_- on X'_- with the standard conjugation on Q^n so that $\Phi^*\omega_{\mathrm{FS};n+1} = \frac{1}{2\pi c}\,\omega'|_{X'_-}$ and $\Phi(Y'_-) \subset Q^n$ is a hyperplane section.

Hint: use Exercises 1.27(b) and 4.39.

Exercise 4.43 ([5, Proposition 4.4],[22, p250])**.** Suppose $n \in \mathbb{Z}^+$, g is the Riemannian metric on the real projective space $\mathbb{R}P^n \equiv S^n/\mathbb{Z}_2$ induced by four times the standard metric on $\mathbb{R}^{n+1}$, ψ is the associated S^1-action on $T^*\mathbb{R}P^n - \mathbb{R}P^n$ given by the negative exponential flow as in (4.71) with $U_\rho = T^*\mathbb{R}P^n$ and $\Phi = \mathrm{id}_{T^*\mathbb{R}P^n}$, H is its Hamiltonian with respect to the canonical symplectic form $\omega_{T^*\mathbb{R}P^n}$ on $T^*\mathbb{R}P^n$ as in (4.70), and $c \in \mathbb{R}^+$.

(a) Show that $(T^*\mathbb{R}P^n - \mathbb{R}P^n, \psi, H, 2\pi c)$ is cutting data for $(T^*\mathbb{R}P^n, \omega_{T^*\mathbb{R}P^n})$.

(b) Let $(X' \equiv X'_- \sqcup X'_+, \omega')$ be the cut symplectic manifold determined by this cutting data as in Theorem 4.24, $Y' \equiv Y'_- \sqcup Y'_+$ be the associated symplectic divisor, and σ'_- be the involution on X'_- as in Exercise 4.41. Show that there is a diffeomorphism $\Phi: X'_- \longrightarrow \mathbb{C}P^n$ intertwining σ'_- with the standard conjugation on $\mathbb{C}P^n$ so that $\Phi^*\omega_{\mathrm{FS};n+1} = \frac{1}{4\pi c}\omega'|_{X'_-}$ and $\Phi(Y'_-) \subset \mathbb{C}P^n$ is a hyperplane.

Hint: use Exercises 4.39, 4.40, and 4.42.

Remark 4.44. The claimed map $f: U_- \equiv V'_{-;a}/S^1 \longrightarrow Q^n$ in [22, p249] is not well-defined; this error is corrected by Exercise 4.42(d). The relation between $f^*\omega_{\mathrm{FS}}$ and ω_- stated at the bottom of [22, p249] would require ω_{FS} to be $\pi^2\times$ the definition in [27, p31]; the relation stated in Exercise 4.42(e) is consistent with [27]. An action of a group G by isometries of S^n with respect to the standard metric g descends to an action of G on Q^n, as happens in the setting of Exercise 4.43 with $G = \mathbb{Z}_2$. If G is cyclic with $|G| \geq 3$ and acts properly discontinuously on S^n with $n \geq 3$ (necessarily odd) by isometries so that $Z \equiv S^n/G$ is a Lens space, then all geodesics with respect to the Levi-Civita connection determined by the metric on Z induced by g are closed. However, the lengths of simple geodesics are no longer the same: while the length of a generic simple closed geodesics is 2π if $|G|$ is odd and π if $|G|$ is even, some simple closed geodesics are of length $2\pi/|G|$. This implies that the S^1-action ψ on $T^*S^n - S^n$ generated by the geodesic flow as in (4.71) does not induce a free S^1-action on $T^*Z - Z$ and that G does not act freely on the quadric hypersurface $Q^n \subset \mathbb{C}P^{n+1}$, i.e. no Lens space (other than S^n or $\mathbb{R}P^n$) is *archetypal* in the terminology of [22, p245]. Thus, the construction of [5, Section 4.1] does not apply with Z being *any* Lens space at all, contrary to suggestions in the abstract and in Sections 1 and 2 of [22].

Chapter 5

Twofold Symplectic Sum

Lemma 5.1 ([22, Lemma 2.5])**.** *Let (X,ω) be a symplectic manifold with an S^1-action ψ and a Hamiltonian $H: X \longrightarrow \mathbb{R}$ for ψ with respect to ω. If $c\in\mathbb{R}$ is a regular value of H, there exist an S^1-invariant smooth function $\varepsilon: H^{-1}(c)\longrightarrow\mathbb{R}^+$ and an S^1-equivariant diffeomorphism*

$$\Phi: X_\varepsilon(H)\equiv\big\{(x,t)\in H^{-1}(c)\times\mathbb{R}: |t|<\varepsilon(x)\big\} \longrightarrow U$$

onto a neighborhood of $H^{-1}(c)$ in X such that

$$\Phi(x,0)=x \quad \forall\, x\in H^{-1}(c) \qquad \textit{and} \qquad H\big(\Phi(x,t)\big)=c+t \quad \forall\, (x,t)\in X_\varepsilon(H). \tag{5.1}$$

If $H^{-1}(c)$ is compact, then ε can be chosen to be a constant ϵ and $U=H^{-1}\big((c-\epsilon,c+\epsilon)\big)$.

Proof. A direct proof appears in [22]; we instead deduce this lemma from Proposition 2.7(2). Since H is ψ-invariant, the subspace $H^{-1}(c)\subset X$ is preserved by ψ. Since $c\in\mathbb{R}$ is a regular value, $H^{-1}(c)\subset X$ is a smooth hypersurface with a trivial normal bundle (the induced action $\mathrm{d}\psi$ on this line bundle is necessarily trivial). By Proposition 2.7(2), there thus exist an S^1-invariant smooth function $\varepsilon: H^{-1}(c)\longrightarrow\mathbb{R}^+$ and an S^1-equivariant diffeomorphism $\Phi: X_\varepsilon(H)\longrightarrow U$ onto a neighborhood of $H^{-1}(c)$ in X such that the first condition in (5.1) holds. Since c is a regular value of H, there exists S^1-invariant smooth function $\varepsilon': H^{-1}(c)\longrightarrow\mathbb{R}^+$ so that the restriction of the smooth map

$$\Psi: X_\varepsilon(H) \longrightarrow H^{-1}(c)\times\mathbb{R}, \qquad \Psi(x,t)=\big(x, H\big(\Phi(x,t)\big)\big),$$

to a neighborhood of $H^{-1}(c)\times\{0\}$ in $X_\varepsilon(H)$ is a diffeomorphism onto $X_{\varepsilon'}(H)$. The S^1-equivariant diffeomorphism

$$\Phi\circ\Psi^{-1}: X_{\varepsilon'}(H) \longrightarrow \Phi\big(\Psi^{-1}\big(X_{\varepsilon'}(H)\big)\big)\subset X$$

satisfies the requirement of the lemma with (ε,Φ) replaced by $(\varepsilon',\Phi\circ\Psi^{-1})$. The last claim follows from the first by taking the infimum of ε. $\square$

Chapter 6

Symplectic Toric Manifolds

In this chapter, we describe a construction of symplectic toric manifolds along the lines of [20, Section 3.2] and use it to obtain key properties of these manifolds. Example 1.27 is a special case of this construction. The structure of this chapter is motivated by [51, Chapter 2], which efficiently summarizes these properties from a more concrete perspective. We fix a torus $\mathbb{T}$ of dimension n and continue with the notation and terminology introduced at the beginning of Section 4.2.

6.1 Symplectic quotient construction

In this section, we use the symplectic reduction of Theorem 4.1 to construct a symplectic toric $\mathbb{T}$-manifold (X,ω,ψ,μ) with $\mu(X)=\langle\mathscr{H}\rangle$ for any regular Delzant subcollection $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$. By Exercise 1.40, every Delzant polytope P equals $\langle\mathscr{H}\rangle$ for some regular Delzant subcollection $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$. Thus, the construction of this section provides another proof of Theorem 4.20(1).

Suppose $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ is a finite subcollection. Let

$$\mathcal{K}_{\mathscr{H}}\equiv\ker\Phi_{\mathscr{H}}\subset\mathbb{T}^{\mathscr{H}}\equiv\mathbb{R}^{\mathscr{H}}/\mathbb{Z}^{\mathscr{H}}$$

be the kernel of the homomorphism $\Phi_{\mathscr{H}}$ in (1.41) and

$$\iota_{\mathscr{H}}^*:\mathbb{R}^{\mathscr{H}}=T_{\mathbb{1}}^*\mathbb{T}^{\mathscr{H}}\longrightarrow T_{\mathbb{1}}^*\mathcal{K}_{\mathscr{H}}$$

be the composition of the standard identification of $\mathbb{R}^{\mathscr{H}}$ with $T_{\mathbb{1}}^*\mathbb{T}^{\mathscr{H}}$ and of the homomorphism induced by the inclusion $\mathcal{K}_{\mathscr{H}}\longrightarrow\mathbb{T}^{\mathscr{H}}$. Thus, the sequence

$$0\longrightarrow T_{\mathbb{1}}^*\mathbb{T}\xrightarrow{L_{\mathscr{H}}^*}T_{\mathbb{1}}^*\mathbb{T}^{\mathscr{H}}=\mathbb{R}^{\mathscr{H}}\xrightarrow{\iota_{\mathscr{H}}^*}T_{\mathbb{1}}^*\mathcal{K}_{\mathscr{H}}\longrightarrow 0 \tag{6.1}$$

of vector spaces is exact. Since $L_{\mathscr{H}}^*(\alpha)=(\alpha(v))_{(v,c)\in\mathscr{H}}$ for any $\alpha\in T_{\mathbb{1}}^*\mathbb{T}$,

$$\begin{aligned}L_{\mathscr{H}}^*\big(\langle\mathscr{H}'\rangle\big)&=\big\{(s_v)_{v\in\mathscr{H}}\in\ker\iota_{\mathscr{H}}^*:\,s_v\ge c_v\ \forall\,v\in\mathscr{H}'\big\}\qquad\text{and}\\ L_{\mathscr{H}}^*\big(\langle\mathscr{H}'\rangle^{\partial}\big)&=\big\{(s_v)_{v\in\mathscr{H}}\in\ker\iota_{\mathscr{H}}^*:\,s_v=c_v\ \forall\,v\in\mathscr{H}'\big\}\end{aligned} \tag{6.2}$$

for any $\mathscr{H}'\subset\mathscr{H}$. If $\mathscr{H}'\subset\mathscr{H}$ and $\iota_{\mathscr{H};\mathscr{H}'}:\mathbb{T}^{\mathscr{H}'}\longrightarrow\mathbb{T}^{\mathscr{H}}$ is the induced inclusion homomorphism, then

$$\iota_{\mathscr{H};\mathscr{H}'}^*:\mathbb{R}^{\mathscr{H}}=\mathbb{R}^{\mathscr{H}'}\times\mathbb{R}^{\mathscr{H}-\mathscr{H}'}\longrightarrow\mathbb{R}^{\mathscr{H}'}$$

is the projection to the $\mathbb{R}^{\mathscr{H}'}$-factor and

$$L^*_{\mathscr{H}'} = \iota^*_{\mathscr{H};\mathscr{H}'} \circ L^*_{\mathscr{H}} : T^*_{\mathbb{1}}\mathbb{T} \longrightarrow T^*_{\mathbb{1}}\mathbb{T}^{\mathscr{H}'} = \mathbb{R}^{\mathscr{H}'} . \tag{6.3}$$

Denote by $\omega_{\mathscr{H}}$ the standard symplectic form on $\mathbb{C}^{\mathscr{H}}$ as in (1.19). By this example, the map

$$H_{\mathscr{H}} : \mathbb{C}^{\mathscr{H}} \longrightarrow \mathbb{R}^{\mathscr{H}}, \qquad H_{\mathscr{H}}\big((z_v)_{v\in\mathscr{H}}\big) = \big(\pi|z_v|^2 + c_v\big)_{v\in\mathscr{H}}, \tag{6.4}$$

is a Hamiltonian with respect to $\omega_{\mathscr{H}}$ for the standard action $\psi_{\mathscr{H}}$ of $\mathbb{T}^{\mathscr{H}}$ on $\mathbb{C}^{\mathscr{H}}$,

$$\psi_{\mathscr{H};[(r_v)_{v\in\mathscr{H}}]}\big((z_v)_{v\in\mathscr{H}}\big) = \big(\mathrm{e}^{2\pi \mathrm{i} r_v} z_v\big)_{v\in\mathscr{H}} . \tag{6.5}$$

Thus,

$$\mu_{\mathscr{H}} \equiv \iota^*_{\mathscr{H}} \circ H_{\mathscr{H}} : \mathbb{C}^{\mathscr{H}} \longrightarrow T^*_{\mathbb{1}}\mathcal{K}_{\mathscr{H}}$$

is a moment map with respect to $\omega_{\mathscr{H}}$ for the restriction of the action $\psi_{\mathscr{H}}$ to $\mathcal{K}_{\mathscr{H}} \subset \mathbb{T}^{\mathscr{H}}$. By the exactness of (6.1) and by (6.2),

$$\mu^{-1}_{\mathscr{H}}(0) \cap \mathbb{C}^{\mathscr{H}-\mathscr{H}'} = H^{-1}_{\mathscr{H}}\big(L^*_{\mathscr{H}}(\langle\mathscr{H}\rangle \cap \langle\mathscr{H}'\rangle^{\partial})\big) = H^{-1}_{\mathscr{H}}\big(L^*_{\mathscr{H}}(\langle\mathscr{H}'\rangle^{\partial})\big) \quad \forall\ \mathscr{H}' \subset \mathscr{H}. \tag{6.6}$$

Exercise 6.1. Let $\mathscr{H} \subset (T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ be a finite subcollection.

(a) Show that the restriction $H_{\mathscr{H}} : \mu^{-1}_{\mathscr{H}}(0) \longrightarrow \ker \iota^*_{\mathscr{H}}$ is a proper map.

(b) Let $K \subset T^*_{\mathbb{1}}\mathcal{K}_{\mathscr{H}}$ be a bounded subset (with respect to some norm on $\mathcal{K}_{\mathscr{H}}$) such that

$$\iota^{*\,-1}_{\mathscr{H}}(K) \cap \big(\mathbb{R}^{\ge 0}\big)^{\mathscr{H}} \neq \emptyset \subset \mathbb{R}^{\mathscr{H}} .$$

Show that $\iota^{*\,-1}_{\mathscr{H}}(K) \cap (\mathbb{R}^{\ge 0})^{\mathscr{H}}$ is bounded if and only if $(\ker \iota^*_{\mathscr{H}}) \cap (\mathbb{R}^{\ge 0})^{\mathscr{H}} = \{0\}$.

Exercise 6.2. Let $\mathscr{H} \subset (T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ be a finite subcollection and

$$\alpha_{\mathscr{H}} = -\iota^*_{\mathscr{H}}\big((c_v)_{v\in\mathscr{H}}\big) \in T^*_{\mathbb{1}}\mathcal{K}_{\mathscr{H}} . \tag{6.7}$$

Show that the following conditions on $\mathscr{H}$ are equivalent:

(a) $0 \in \mathbb{C}^{\mathscr{H}}$ is a regular value of the map $H_{\mathscr{H}}$ in (6.4),

(b) $0 \in \mathbb{C}^{\mathscr{H}}$ is a regular value of $H_{\mathscr{H}}|_{\mathbb{C}^{\mathscr{H}-\mathscr{H}'}}$ for every $\mathscr{H}' \subset \mathscr{H}$,

(c) $|\mathscr{H}'| \le \dim L_{\mathscr{H}}(\mathbb{R}^{\mathscr{H}})$ for every $\mathscr{H}' \subset \mathscr{H}$ such that $\alpha_{\mathscr{H}} \in \iota^*_{\mathscr{H}}\big((\mathbb{R}^{\ge 0})^{\mathscr{H}-\mathscr{H}'}\big)$.

Exercise 6.3. Let $\mathscr{H}' \subset \mathscr{H} \subset (T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ be finite subcollections.

(a) Show that the subgroup $\mathcal{K}_{\mathscr{H}} \subset \mathbb{T}^{\mathscr{H}}$ acts freely on $\mu^{-1}_{\mathscr{H}}(0) \cap \mathbb{C}^{\mathscr{H}-\mathscr{H}'}$ if and only if the homomorphism $\Phi_{\mathscr{H}''}$ as in (1.41) is injective whenever $\mathscr{H}' \subset \mathscr{H}'' \subset \mathscr{H}$ and $\langle\mathscr{H}''\rangle^{\partial} \cap \langle\mathscr{H}\rangle \neq \emptyset$. *Hint:* proceed similarly to the paragraph containing (4.10).

(b) Suppose the homomorphism $\Phi_{\mathscr{H}''}$ is injective whenever $\mathscr{H}' \subset \mathscr{H}'' \subset \mathscr{H}$ and $\langle\mathscr{H}''\rangle^{\partial} \cap \langle\mathscr{H}\rangle \neq \emptyset$. Show that $\mu^{-1}_{\mathscr{H}}(0) \cap \mathbb{C}^{\mathscr{H}-\mathscr{H}'}$ is a smooth submanifold of $\mathbb{C}^{\mathscr{H}-\mathscr{H}'}$ and $(\mu^{-1}_{\mathscr{H}}(0) \cap \mathbb{C}^{\mathscr{H}-\mathscr{H}'})/\mathcal{K}_{\mathscr{H}}$ is a smooth manifold.

Exercise 6.4. Suppose $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ is a finite subcollection. Let $[\![\mathscr{H}]\!]\approx\mathbb{R}^{\mathscr{H}}$ be the collection of all subcollections $\mathscr{H}^\dagger\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ that differ from $\mathscr{H}$ only in the second input of each element of $\mathscr{H}$ (i.e. there is a bijection $\Psi_{\mathscr{H}^\dagger\mathscr{H}}$ between the elements of $\mathscr{H}$ and of each $\mathscr{H}^\dagger\in[\![\mathscr{H}]\!]$ so that $\Psi_{\mathscr{H}^\dagger\mathscr{H}}(v,c)=(v,c^\dagger)$ for some $c^\dagger\in\mathbb{R}$). Show that

(a) the subset $[\![\mathscr{H}]\!]^*\subset[\![\mathscr{H}]\!]$ of the subcollections $\mathscr{H}^\dagger$ such that $0\in\mathbb{C}^{\mathscr{H}}$ is a regular value of the map $H_{\mathscr{H}^\dagger}$ in (6.4) is open;

(b) every topological component of $[\![\mathscr{H}]\!]^*$ is preserved under the multipication by $\mathbb{R}^+$ (i.e. simultaneous multiplication of the second input of each element of $\mathscr{H}^\dagger\in[\![\mathscr{H}]\!]^*$);

(c) the collections $\big\{\Psi^{-1}_{\mathscr{H}^\dagger\mathscr{H}}(\mathscr{H}')\colon\mathscr{H}'\subset\mathscr{H}^\dagger,\,\langle\mathscr{H}'\rangle^\partial\cap\langle\mathscr{H}^\dagger\rangle\neq\emptyset\big\}$ with $\mathscr{H}^\dagger\in[\![\mathscr{H}]\!]$ are constant functions on the topological components of $[\![\mathscr{H}]\!]^*$;

(d) if the subcollection $\mathscr{H}$ is Delzant, then $\mathscr{H}\in[\![\mathscr{H}]\!]^*$ and every element $\mathscr{H}^\dagger$ of the topological component $[\![\mathscr{H}]\!]^\star\subset[\![\mathscr{H}]\!]^*$ containing $\mathscr{H}$ is also a Delzant subcollection.

Lemma 6.5. *The subspace $\mu^{-1}_{\mathscr{H}}(0)\subset\mathbb{C}^{\mathscr{H}}$ is preserved by the $\mathbb{T}^{\mathscr{H}}$-action (6.5) and is path-connected. If $\langle\emptyset\rangle^\partial_{\mathscr{H}}\neq\emptyset$, i.e. $\langle\mathscr{H}\rangle$ is full-dimensional, then $G_z(\psi_{\mathscr{H}})=\{\mathbb{1}\}$ for some $z\in\mu^{-1}_{\mathscr{H}}(0)$.*

Proof. Since the Hamiltonian $H_{\mathscr{H}}$ is $\mathbb{T}^{\mathscr{H}}$-invariant, the $\mathbb{T}^{\mathscr{H}}$-action (6.5) preserves $\mu^{-1}_{\mathscr{H}}(0)$. By (6.2),

$$\big(\mathrm{Im}\,H_{\mathscr{H}}\big)\cap\big(\ker\iota^*_{\mathscr{H}}\big)=L^*_{\mathscr{H}}\big(\langle\mathscr{H}\rangle\big)\subset\mathbb{R}^{\mathscr{H}}. \tag{6.8}$$

Since the subspace $\langle\mathscr{H}\rangle\subset T^*_{\mathbb{1}}\mathbb{T}$ is path-connected and the fibers of $H_{\mathscr{H}}$ are path-connected, so is the subspace

$$\mu^{-1}_{\mathscr{H}}(0)=H^{-1}_{\mathscr{H}}\big(\iota^{*\,-1}_{\mathscr{H}}(0)\big)\subset\mathbb{C}^{\mathscr{H}}.$$

If $\langle\emptyset\rangle^\partial_{\mathscr{H}}\neq\emptyset$, (6.6) implies that there exists $z\equiv(z_v)_{v\in\mathscr{H}}\in\mu^{-1}_{\mathscr{H}}(0)$ such that $H_{\mathscr{H}}(z)\in L^*_{\mathscr{H}}(\langle\emptyset\rangle^\partial_{\mathscr{H}})$ and thus $z_v\neq0$ for any $v\in\mathscr{H}$. It follows that $G_z(\psi_{\mathscr{H}})=\{\mathbb{1}\}$. $\square$

From now on, we assume that $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ is a regular Delzant subcollection such that $\langle\mathscr{H}\rangle\neq\emptyset$. By Exercise 1.42(b), $\mathcal{K}_{\mathscr{H}}\subset\mathbb{T}^{\mathscr{H}}$ is then a codimension n subtorus. By Exercise 6.3(a), $(\mathbb{C}^{\mathscr{H}},\omega_{\mathscr{H}},\psi_{\mathscr{H}},\mu_{\mathscr{H}})$ is a Hamiltonian $\mathcal{K}_{\mathscr{H}}$-manifold such that $\mathcal{K}_{\mathscr{H}}$ acts freely on $\mu^{-1}_{\mathscr{H}}(0)$. Let $(X,\omega)\equiv(X_0,\omega_0)$ be the quotient symplectic manifold provided by the first part of Theorem 4.1. By (SQ0) and (SQ1) in this theorem,

$$\dim X=\dim_{\mathbb{R}}\mathbb{C}^{\mathscr{H}}-2\dim\mathcal{K}_{\mathscr{H}}=2|\mathscr{H}|-2\big(|\mathscr{H}|-n\big)=2n.$$

By Lemma 6.5, $X\equiv\mu^{-1}_{\mathscr{H}}(0)/\mathcal{K}_{\mathscr{H}}$ is connected.

The torus actions $\psi_{\mathscr{H}}|_{\mathcal{K}_{\mathscr{H}}}$ and $\psi_{\mathscr{H}}$ commute, the Hamiltonian $H_{\mathscr{H}}$ for $\psi_{\mathscr{H}}$ is $\psi_{\mathscr{H}}|_{\mathcal{K}_{\mathscr{H}}}$-invariant and the moment map $\mu_{\mathscr{H}}$ for $\psi_{\mathscr{H}}|_{\mathcal{K}_{\mathscr{H}}}$ is $\psi_{\mathscr{H}}$-invariant. Let

$$\psi_0'\colon\mathbb{T}^{\mathscr{H}}\times X\longrightarrow X\qquad\text{and}\qquad\mu_0'\colon X\longrightarrow T^*_{\mathbb{1}}\mathbb{T}^{\mathscr{H}}$$

be the $\mathbb{T}^{\mathscr{H}}$-action on X induced by $\psi_{\mathscr{H}}$ and its moment map with respect to ω induced by $H_{\mathscr{H}}$, respectively, as provided by the last part of Theorem 4.1.

By Exercise 1.42(a), there exists a subset $\mathscr{H}_\bullet \subset \mathscr{H}$ so that $\langle \mathscr{H}_\bullet \rangle^\partial$ is a vertex of $\langle \mathscr{H} \rangle$, i.e. $\langle \mathscr{H}_\bullet \rangle^\partial \subset \langle \mathscr{H} \rangle$ consists of a single point. Since $\mathscr{H}$ is Delzant, the Lie homomorphism group $\Phi_{\mathscr{H}_\bullet}$ as in (1.41) is an isomorphism and thus so is the Lie group homomorphism

$$\mathcal{K}_{\mathscr{H}} \times \mathbb{T}^{\mathscr{H}_\bullet} \longrightarrow \mathbb{T}^{\mathscr{H}}, \qquad (u, u_\bullet) \longrightarrow u u_\bullet .$$

By the last statement of Lemma 6.5 above and Exercise 6.6 below, the composition ψ of the $\mathbb{T}^{\mathscr{H}}$-action ψ_0' on X with the homomorphism

$$\mathbb{T} \xrightarrow{\Phi_{\mathscr{H}_\bullet}^{-1}} \mathbb{T}^{\mathscr{H}_\bullet} \xrightarrow{\iota_{\mathscr{H};\mathscr{H}_\bullet}} \mathbb{T}^{\mathscr{H}}$$

is therefore an effective $\mathbb{T}$-action on X. The composition

$$\mu\colon X \xrightarrow{\mu_0'} T_{\mathbb{1}}^*\mathbb{T}^{\mathscr{H}} \xrightarrow{\iota_{\mathscr{H};\mathscr{H}_\bullet}^*} T_{\mathbb{1}}^*\mathbb{T}^{\mathscr{H}_\bullet} \xrightarrow{L_{\mathscr{H}_\bullet}^{*\,-1}} T_{\mathbb{1}}^*\mathbb{T}$$

is its moment map with respect to ω. Thus, (X, ω, ψ, μ) is a symplectic toric manifold with moment polytope

$$\begin{aligned} \mu(X) = L_{\mathscr{H}_\bullet}^{*\,-1}\big(\iota_{\mathscr{H};\mathscr{H}_\bullet}^*\big(\mu_0'(X)\big)\big) &= L_{\mathscr{H}_\bullet}^{*\,-1}\big(\iota_{\mathscr{H};\mathscr{H}_\bullet}^*\big(H_{\mathscr{H}}(\mu_{\mathscr{H}}^{-1}(0))\big)\big) \\ &= L_{\mathscr{H}_\bullet}^{*\,-1}\big(\iota_{\mathscr{H};\mathscr{H}_\bullet}^*\big(L_{\mathscr{H}}^*(\langle\!\langle \mathscr{H} \rangle\!\rangle)\big)\big) = L_{\mathscr{H}_\bullet}^{*\,-1}\big(L_{\mathscr{H}_\bullet}^*(\langle\!\langle \mathscr{H} \rangle\!\rangle)\big) = \langle \mathscr{H} \rangle ; \end{aligned}$$

the second equality above holds by (6.6) with $\mathscr{H}' = \emptyset$ and (6.8).

Exercise 6.6. Suppose ψ_1 and ψ_2 are commuting actions of groups G_1 and G_2 on a set Z and $z \in Z$ is a point such that $G_z(\psi_1 \times \psi_2) = \{\mathbb{1}\}$. Let $\overline{\psi}_2$ be the induced G_2-action on the quotient Z/G_1 and $G_1 z \in Z/G_1$ be the G_1-orbit of z. Show that $G_{G_1 z}(\overline{\psi}_2) = \{\mathbb{1}\}$.

Exercise 6.7. Show that the Hamiltonian $\mathbb{T}$-manifold (X, ω, ψ, μ) constructed above does not depend on the choice of subcollection $\mathscr{H}_\bullet \subset \mathscr{H}$ so that $\langle \mathscr{H}_\bullet \rangle^\partial$ is a vertex of $\langle \mathscr{H} \rangle$.

Exercise 6.8. Suppose $\mathscr{H}^\dagger \subset (T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ is a Delzant subcollection such that $\mathscr{H}^\dagger \supset \mathscr{H}$ and $\langle \mathscr{H}^\dagger \rangle = \langle \mathscr{H} \rangle$. Show that

(a) there exist (unique) linear functionals $\ell_{\upsilon'}\colon \operatorname{Im} L_{\mathscr{H}}^* \longrightarrow \mathbb{R}$ with $\upsilon' \in \mathscr{H}^\dagger - \mathscr{H}$ such that

$$L_{\mathscr{H}^\dagger}^*(\alpha) = \big(L_{\mathscr{H}}^*(\alpha), \big(\ell_{\upsilon'}(L_{\mathscr{H}}^*(\alpha))\big)_{\upsilon' \in \mathscr{H}^\dagger - \mathscr{H}}\big) \in \mathbb{R}^{\mathscr{H}} \times \mathbb{R}^{\mathscr{H}^\dagger - \mathscr{H}} = \mathbb{R}^{\mathscr{H}^\dagger} \quad \forall\, \alpha \in T_{\mathbb{1}}^*\mathbb{T};$$

(b) $\ell_{\upsilon'}(s) > c_{\upsilon'}$ for all $\upsilon' \in \mathscr{H}^\dagger - \mathscr{H}$ and $s \in L_{\mathscr{H}}^*(\langle\!\langle \mathscr{H} \rangle\!\rangle)$;

(c) the projection $\mathrm{pr}_{\mathscr{H},\mathscr{H}^\dagger}\colon \mathbb{C}^{\mathscr{H}^\dagger} \longrightarrow \mathbb{C}^{\mathscr{H}}$ restricts to a principal $\mathbb{T}^{\mathscr{H}^\dagger - \mathscr{H}}$-bundle $\mu_{\mathscr{H}^\dagger}^{-1}(0) \longrightarrow \mu_{\mathscr{H}}^{-1}(0)$ with a smooth $\mathbb{T}^{\mathscr{H}}$-equivariant section

$$\begin{gathered} \widetilde{s}\colon \mu_{\mathscr{H}}^{-1}(0) \longrightarrow \mu_{\mathscr{H}^\dagger}^{-1}(0) \subset \mathbb{C}^{\mathscr{H}} \times \mathbb{C}^{\mathscr{H}^\dagger - \mathscr{H}}, \\ \widetilde{s}\big((z_\upsilon)_{\upsilon \in \mathscr{H}}\big) = \Big((z_\upsilon)_{\upsilon \in \mathscr{H}}, \big(\sqrt{\big(\ell_{\upsilon'}(H_{\mathscr{H}}((z_\upsilon)_{\upsilon \in \mathscr{H}})) - c_{\upsilon'}\big)/\pi}\big)_{\upsilon' \in \mathscr{H}^\dagger - \mathscr{H}}\Big); \end{gathered}$$

(d) $\omega_{\mathscr{H}^\dagger}\big|_{T\mu_{\mathscr{H}^\dagger}^{-1}(0)} = \mathrm{pr}_{\mathscr{H},\mathscr{H}^\dagger}^* \omega_{\mathscr{H}}\big|_{T\mu_{\mathscr{H}^\dagger}^{-1}(0)}$ and

$$H_{\mathscr{H}} = \iota_{\mathscr{H}^\dagger;\mathscr{H}}^* \circ H_{\mathscr{H}^\dagger} \circ \widetilde{s}\colon \mu_{\mathscr{H}}^{-1}(0) \longrightarrow \operatorname{Im} L_{\mathscr{H}}^* .$$

Conclude that the Hamiltonian $\mathbb{T}$-manifold (X,ω,ψ,μ) constructed above does not depend on the choice of Delzant subcollection $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ with $\langle\mathscr{H}\rangle$ fixed. *Hint:* using Exercise 1.43(a), show that the section $\widetilde{s}$ descends to an identification of the Hamiltonian $\mathbb{T}$-manifolds determined by $\mathscr{H}$ and $\mathscr{H}^\dagger$ via the above construction.

Remark 6.9. The section $\widetilde{s}$ of Exercise 6.8(c) is equivariant with respect to the inclusion $\mathcal{K}_{\mathscr{H}}\longrightarrow\mathcal{K}_{\mathscr{H}^\dagger}$ in the exact sequence of Exercise 1.43(a). This implies that $\widetilde{s}$ descends to a continuous map

$$X=\mu_{\mathscr{H}}^{-1}(0)/\mathcal{K}_{\mathscr{H}}\longrightarrow X^\dagger\equiv\mu_{\mathscr{H}^\dagger}^{-1}(0)/\mathcal{K}_{\mathscr{H}^\dagger}\,.$$

In contrast, the projection $\mathrm{pr}_{\mathscr{H},\mathscr{H}^\dagger}\colon\mu_{\mathscr{H}^\dagger}^{-1}(0)\longrightarrow\mu_{\mathscr{H}}^{-1}(0)$ does not generally descend to a continuous map $X^\dagger\longrightarrow X$, as $\mathbb{T}^{\mathscr{H}^\dagger-\mathscr{H}}\subset\mathbb{T}^{\mathscr{H}^\dagger}$ is generally not a subgroup of $\mathcal{K}_{\mathscr{H}^\dagger}$ and thus the homomorphisms in the exact sequence of Exercise 1.43(a) are not reversible.

Exercise 6.10. Let $\mathscr{H}_0\subset\mathscr{H}$. Show that

(a) $\mu^{-1}\big(\langle\mathscr{H}_0\rangle^\partial\big)=\big\{\big[(z_v)_{v\in\mathscr{H}}\big]\in X\equiv\mu_{\mathscr{H}}^{-1}(0)/\mathcal{K}_{\mathscr{H}}\colon z_v=0\ \forall\,v\in\mathscr{H}_0\big\}\equiv\big(\mu_{\mathscr{H}}^{-1}(0)\cap\mathbb{C}^{\mathscr{H}-\mathscr{H}_0}\big)/\mathcal{K}_{\mathscr{H}}$;

(b) $\mu^{-1}(\langle\mathscr{H}_0\rangle^\partial)\subset X$ is a connected ω-symplectic submanifold;

(c) the subtorus $\mathbb{T}_{\mathscr{H}_0}\subset\mathbb{T}$ acts trivially on $\mu^{-1}(\langle\mathscr{H}_0\rangle^\partial)$ and thus the $\mathbb{T}$-action ψ induces a $\mathbb{T}/\mathbb{T}_{\mathscr{H}_0}$-action $\psi_{\mathscr{H}_0}$ on $\mu^{-1}(\langle\mathscr{H}_0\rangle^\partial)$;

(d) if $\alpha_0\in\langle\mathscr{H}_0\rangle^\partial$, then

$$\mu_{\alpha_0}\equiv\mu-\alpha_0\colon\mu^{-1}\big(\langle\mathscr{H}_0\rangle^\partial\big)\longrightarrow T_{\mathbb{1}}^*(\mathbb{T}/\mathbb{T}_{\mathscr{H}_0})=\big\{\alpha\in T_{\mathbb{1}}^*\mathbb{T}\colon\alpha|_{T_{\mathbb{1}}\mathbb{T}_{\mathscr{H}_0}}=0\big\}$$

is a moment map for the $\mathbb{T}/\mathbb{T}_{\mathscr{H}_0}$-action $\psi_{\mathscr{H}_0}$ with respect to $\omega|_{\mu^{-1}(\langle\mathscr{H}_0\rangle^\partial)}$.

Exercise 6.11. Suppose $\mathscr{H}_0\subset\mathscr{H}$, $\alpha_0\in\langle\mathscr{H}_0\rangle^\partial$, and q_{α_0} and $\mathscr{H}/\mathscr{H}_0$ are as in Exercise 1.44. Define

$$\begin{gathered}\widetilde{i}_{\mathscr{H};\mathscr{H}_0}\equiv\big(\widetilde{i}_{\mathscr{H};\mathscr{H}_0;v}\big)_{v\in\mathscr{H}-\mathscr{H}_0}\colon\mathbb{C}^{\mathscr{H}/\mathscr{H}_0}\longrightarrow\mathbb{C}^{\mathscr{H}-\mathscr{H}_0}\subset\mathbb{C}^{\mathscr{H}},\\ \widetilde{i}_{\mathscr{H};\mathscr{H}_0;v}\big((z_{v'})_{v'\in\mathscr{H}/\mathscr{H}_0}\big)=z_{q_{\alpha_0}(v)}\quad\forall\,v\in\mathscr{H}-\mathscr{H}_0;\\ \phi^{\mathbb{R}}_{\mathscr{H};\mathscr{H}_0}\equiv\big(\phi^{\mathbb{R}}_{\mathscr{H};\mathscr{H}_0;v}\big)_{v\in\mathscr{H}/\mathscr{H}_0}\colon\mathbb{R}^{\mathscr{H}}\longrightarrow\mathbb{R}^{\mathscr{H}/\mathscr{H}_0},\quad\phi^{\mathbb{R}}_{\mathscr{H};\mathscr{H}_0;q_{\alpha_0}(v)}\big((s_{v'})_{v'\in\mathscr{H}}\big)=s_v\quad\forall\,v\in\mathscr{H}-\mathscr{H}_0\,.\end{gathered}$$

Show that

(a) the smooth map $\widetilde{i}_{\mathscr{H};\mathscr{H}_0}$ restricts to a $\mathbb{T}/\mathbb{T}_{\mathscr{H}_0}$-equivariant diffeomorphism between smooth submanifolds $\mu_{\mathscr{H}/\mathscr{H}_0}^{-1}(0)$ and $\mu_{\mathscr{H}}^{-1}(0)\cap\mathbb{C}^{\mathscr{H}-\mathscr{H}_0}$ of the domain and target, respectively;

(b) $\omega_{\mathscr{H}/\mathscr{H}_0}=\widetilde{i}^{\,*}_{\mathscr{H};\mathscr{H}_0}\omega_{\mathscr{H}}$,

$$\begin{gathered}H_{\mathscr{H}}-L^*_{\mathscr{H}}(\alpha_0)\colon\mu_{\mathscr{H}}^{-1}(0)\cap\mathbb{C}^{\mathscr{H}-\mathscr{H}_0}\longrightarrow\mathbb{R}^{\mathscr{H}-\mathscr{H}_0}\subset\mathbb{R}^{\mathscr{H}},\quad\text{and}\\ H_{\mathscr{H}/\mathscr{H}_0}=\phi^{\mathbb{R}}_{\mathscr{H};\mathscr{H}_0}\circ\big\{H_{\mathscr{H}}\circ\widetilde{i}_{\mathscr{H};\mathscr{H}_0}-L^*_{\mathscr{H}}(\alpha_0)\big\}\colon\mathbb{C}^{\mathscr{H}/\mathscr{H}_0}\longrightarrow T_{\mathbb{1}}^*\mathbb{T}^{\mathscr{H}/\mathscr{H}_0};\end{gathered}$$

(c) if $\mathscr{H}_0\subset\mathscr{H}_\bullet\subset\mathscr{H}$ and $\langle\mathscr{H}_\bullet\rangle^\partial$ is a vertex of $\langle\mathscr{H}\rangle$, then

$$\begin{gathered}L^{*\,-1}_{\mathscr{H}_\bullet}\big(\mathbb{R}^{\mathscr{H}_\bullet-\mathscr{H}_0}\big)=T_{\mathbb{1}}^*\big(\mathbb{T}/\mathbb{T}_{\mathscr{H}_0}\big)=\big\{\alpha\in T_{\mathbb{1}}^*\mathbb{T}\colon\alpha|_{T_{\mathbb{1}}\mathbb{T}_{\mathscr{H}_0}}=0\big\}\quad\text{and}\\ L^{*\,-1}_{\mathscr{H}_\bullet}\circ\iota^*_{\mathscr{H};\mathscr{H}_\bullet}=L^{*\,-1}_{\mathscr{H}_\bullet/\mathscr{H}_0}\circ\iota^*_{\mathscr{H}/\mathscr{H}_0;\mathscr{H}_\bullet/\mathscr{H}_0}\circ\phi^{\mathbb{R}}_{\mathscr{H};\mathscr{H}_0}\colon\mathbb{R}^{\mathscr{H}-\mathscr{H}_0}\longrightarrow T_{\mathbb{1}}^*\big(\mathbb{T}/\mathbb{T}_{\mathscr{H}_0}\big).\end{gathered}$$

Conclude that $\widetilde{i}_{\mathscr{H};\mathscr{H}_0}$ induces an identification of the Hamiltonian $\mathbb{T}/\mathbb{T}_{\mathscr{H}_0}$-manifold determined by the regular Delzant subcollection $\mathscr{H}/\mathscr{H}_0\subset(T_{\mathbb{1}}(\mathbb{T}/\mathbb{T}_{\mathscr{H}_0}))_{\mathbb{Z}}\times\mathbb{R}$ with the Hamiltonian $\mathbb{T}/\mathbb{T}_{\mathscr{H}_0}$-manifold $\big(\mu^{-1}(\langle\!\langle\mathscr{H}_0\rangle^{\partial}),\omega|_{\mu^{-1}(\langle\!\langle\mathscr{H}_0\rangle^{\partial})},\psi_{\mathscr{H}_0},\mu_{\alpha_0}\big)$ of Exercise 6.10. *Hint:* use Exercise 1.44(c).

Exercise 6.12. Suppose (X,ω,ψ,μ) is the Hamiltonian $\mathbb{T}$-manifold determined by a Delzant subcollection $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ and $\mathscr{H}^+\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ is a finite subcollection. Show that

(a) (X,ω,ψ,μ) is $\mathscr{H}^+$-cuttable if and only if $\mathscr{H}\sqcup\mathscr{H}^+$ is a Delzant subcollection;

(b) if $\mathscr{H}\sqcup\mathscr{H}^+$ is a Delzant subcollection, then the Hamiltonian $\mathbb{T}$-manifold determined by $\mathscr{H}\sqcup\mathscr{H}^+$ is the $\mathscr{H}^+$-cut $(X,\omega,\psi,\mu)_{\mathscr{H}^+}$ of (X,ω,ψ,μ) provided by Theorem 4.11.

6.2 Complex quotient construction

As before, let $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ be a regular Delzant subcollection. In this section, we describe the quotient $X\equiv\mu_{\mathscr{H}}^{-1}(0)/\mathcal{K}_{\mathscr{H}}$ constructed in Section 6.1 as the quotient of the complement $\widetilde{X}_{\mathscr{H}}$ of certain coordinate subspaces of $\mathbb{C}^{\mathscr{H}}$ by the complexification $(\mathcal{K}_{\mathscr{H}})_{\mathbb{C}}\subset(\mathbb{C}^*)^{\mathscr{H}}$ of $\mathcal{K}_{\mathscr{H}}\subset\mathbb{T}^{\mathscr{H}}$. This description turns out to be more suitable for constructing (complex) coordinate charts on X, as demonstrated in Section 6.3. We continue with the notation introduced in Section 6.1.

Let $(\mathcal{K}_{\mathscr{H}})_{\mathbb{C}}\subset\mathbb{T}_{\mathbb{C}}^{\mathscr{H}}$ be the complexification of $\mathcal{K}_{\mathscr{H}}\subset\mathbb{T}^{\mathscr{H}}$ and $(\mathcal{K}_{\mathscr{H}})_{\mathfrak{i}}\subset(\mathcal{K}_{\mathscr{H}})_{\mathbb{C}}$ be the purely imaginary subgroup. Under the identification

$$\mathbb{T}_{\mathbb{C}}^{\mathscr{H}}\equiv\mathbb{C}^{\mathscr{H}}/\mathbb{Z}^{\mathscr{H}}\longrightarrow(\mathbb{C}^*)^{\mathscr{H}},\qquad [v]\longrightarrow \mathrm{e}^{2\pi\mathfrak{i}v},$$

the subgroup $(\mathcal{K}_{\mathscr{H}})_{\mathfrak{i}}$ corresponds to a subgroup of $(\mathbb{R}^*)^{\mathscr{H}}\subset(\mathbb{C}^*)^{\mathscr{H}}$. The group $\mathbb{T}_{\mathbb{C}}^{\mathscr{H}}\approx(\mathbb{C}^*)^{\mathscr{H}}$ acts on $\mathbb{C}^{\mathscr{H}}$ by the coordinate-wise multiplication in the usual way, i.e. as in (6.5); we denote this complexified action in the same way. Define

$$\begin{gathered}\mathrm{Ver}(\mathscr{H})=\big\{\mathscr{H}_\bullet\subset\mathscr{H}:\langle\mathscr{H}_\bullet\rangle^{\partial}\cap\langle\mathscr{H}\rangle\neq\emptyset,\ |\mathscr{H}_\bullet|=n\big\},\\ \widetilde{X}_{\mathscr{H}}=\mathbb{C}^{\mathscr{H}}-\bigcup_{\substack{\mathscr{H}'\subset\mathscr{H}\\ \mathbb{C}^{\mathscr{H}'}\cap\mu_{\mathscr{H}}^{-1}(0)=\emptyset}}\mathbb{C}^{\mathscr{H}'},\qquad X_{\mathscr{H}}\equiv\widetilde{X}_{\mathscr{H}}\big/(\mathcal{K}_{\mathscr{H}})_{\mathbb{C}}\,.\end{gathered}\tag{6.9}$$

Thus, $\widetilde{X}_{\mathscr{H}}\subset\mathbb{C}^{\mathscr{H}}$ is a $\mathbb{T}_{\mathbb{C}}^{\mathscr{H}}$-invariant path-connected open subset containing $\mu_{\mathscr{H}}^{-1}(0)$. If $\mu_{\mathscr{H}}^{-1}(0)=\emptyset$, then $\widetilde{X}_{\mathscr{H}}=\emptyset$. By Exercise 1.42(a), $\mathrm{Ver}(\mathscr{H})\neq\emptyset$ if $\mu_{\mathscr{H}}^{-1}(0)\neq\emptyset$. Furthermore, every subset $\mathscr{H}'\subset\mathscr{H}$ such that $\langle\mathscr{H}'\rangle^{\partial}\cap\langle\mathscr{H}\rangle\neq\emptyset$ is contained in some element $\mathscr{H}_\bullet$ of $\mathrm{Ver}(\mathscr{H})$. Along with (6.6), this implies that for every element $z\equiv(z_v)_{v\in\mathscr{H}}$ of $\widetilde{X}_{\mathscr{H}}$ the set

$$\mathscr{H}_z\equiv\big\{v\in\mathscr{H}:z_v=0\big\}$$

is contained in some element $\mathscr{H}_\bullet$ of $\mathrm{Ver}(\mathscr{H})$.

Exercise 6.13. Show that

(a) $(\mathcal{K}_{\mathscr{H}})_{\mathbb{C}}$ acts freely on the subspace $\widetilde{X}_{\mathscr{H}}\subset\mathbb{C}^{\mathscr{H}}$;

(b) the subspace $\widetilde{X}_{\mathscr{H}}\subset\mathbb{C}^{\mathscr{H}}$ is simply connected if $\mathscr{H}$ is minimal.

Hint: proceed similarly to the paragraph containing (4.10).

Lemma 6.14. *Suppose $v_k\equiv(t_{k;v})_{v\in\mathscr{H}}$ with $k\in\mathbb{Z}^+$ is a sequence in $T_{\mathbb{1}}\mathcal{K}_{\mathscr{H}}\subset\mathbb{R}^{\mathscr{H}}$ and $\mathscr{H}_1,\mathscr{H}_2$ are elements of* $\mathrm{Ver}(\mathscr{H})$ *such that*

$$\inf\big\{t_{k;v}\colon k\in\mathbb{Z}^+,\, v\in\mathscr{H}-\mathscr{H}_1\big\}\in\mathbb{R}\quad\textit{and}\quad \sup\big\{t_{k;v}\colon k\in\mathbb{Z}^+,\, v\in\mathscr{H}-\mathscr{H}_1\big\}=+\infty. \tag{6.10}$$

Then, $\sup\{t_{k;v}\colon k\in\mathbb{Z}^+,\, v\in\mathscr{H}-\mathscr{H}_2\}=+\infty.$

Proof. Since $\mathscr{H}$ is Delzant, the projection

$$\mathrm{pr}_{\mathscr{H};\mathscr{H}_\bullet}\colon T_{\mathbb{1}}\mathcal{K}_{\mathscr{H}}\longrightarrow\mathbb{R}^{\mathscr{H}-\mathscr{H}_\bullet},\qquad \mathrm{pr}_{\mathscr{H};\mathscr{H}_\bullet}\big((t_v)_{v\in\mathscr{H}}\big)=(t_v)_{v\in\mathscr{H}-\mathscr{H}_\bullet},$$

is an isomorphism for every $\mathscr{H}_\bullet\in\mathrm{Ver}(\mathscr{H})$. Thus, so is its dual,

$$\mathrm{pr}^*_{\mathscr{H};\mathscr{H}_\bullet}=\iota^*_{\mathscr{H}}\big|_{\mathbb{R}^{\mathscr{H}-\mathscr{H}_\bullet}}\colon\mathbb{R}^{\mathscr{H}-\mathscr{H}_\bullet}\longrightarrow T^*_{\mathbb{1}}\mathcal{K}_{\mathscr{H}}\,.$$

In particular, there is a unique element $s_{\mathscr{H};\mathscr{H}_\bullet}\in\mathbb{R}^{\mathscr{H}-\mathscr{H}_\bullet}$ such that

$$\mathrm{pr}^*_{\mathscr{H};\mathscr{H}_\bullet}\big(s_{\mathscr{H};\mathscr{H}_\bullet}\big)=-\iota^*_{\mathscr{H}}\big((c_v)_{v\in\mathscr{H}}\big)\in T^*_{\mathbb{1}}\mathcal{K}_{\mathscr{H}}.$$

Since $L^*_{\mathscr{H}}(\langle\mathscr{H}_\bullet\rangle^\partial)$ is the single-element set $\{s+(c_v)_{v\in\mathscr{H}}\}$ for some $s\in(\mathbb{R}^+)^{\mathscr{H}-\mathscr{H}_\bullet}$ satisfying the above condition with $s_{\mathscr{H};\mathscr{H}_\bullet}$ replaced by s, $s_{\mathscr{H};\mathscr{H}_\bullet}\in(\mathbb{R}^+)^{\mathscr{H}-\mathscr{H}_\bullet}$ for every $\mathscr{H}_\bullet\in\mathrm{Ver}(\mathscr{H})$.

Let $\mathrm{pr}_{\mathscr{H};\mathscr{H}_2,\mathscr{H}_1}=\mathrm{pr}_{\mathscr{H};\mathscr{H}_2}\circ\mathrm{pr}^{-1}_{\mathscr{H};\mathscr{H}_1}\colon\mathbb{R}^{\mathscr{H}-\mathscr{H}_1}\longrightarrow\mathbb{R}^{\mathscr{H}-\mathscr{H}_2}$. Thus,

$$\begin{aligned}\sup\big\{\big\langle s_{\mathscr{H};\mathscr{H}_2},\mathrm{pr}_{\mathscr{H};\mathscr{H}_2}(v_k)\big\rangle\colon k\in\mathbb{Z}^+\big\}&=\sup\big\{\big\langle s_{\mathscr{H};\mathscr{H}_2},\mathrm{pr}_{\mathscr{H};\mathscr{H}_2,\mathscr{H}_1}\big(\mathrm{pr}_{\mathscr{H};\mathscr{H}_1}(v_k)\big)\big\rangle\colon k\in\mathbb{Z}^+\big\}\\&=\sup\big\{\big\langle\mathrm{pr}^*_{\mathscr{H};\mathscr{H}_2,\mathscr{H}_1}(s_{\mathscr{H};\mathscr{H}_2}),\mathrm{pr}_{\mathscr{H};\mathscr{H}_1}(v_k)\big\rangle\colon k\in\mathbb{Z}^+\big\}\\&=\sup\big\{\big\langle s_{\mathscr{H};\mathscr{H}_1},\mathrm{pr}_{\mathscr{H};\mathscr{H}_1}(v_k)\big\rangle\colon k\in\mathbb{Z}^+\big\}.\end{aligned}$$

Since $s_{\mathscr{H};\mathscr{H}_1}\in(\mathbb{R}^+)^{\mathscr{H}-\mathscr{H}_1}$, the two assumptions in (6.10) imply that the last supremum above is $+\infty$ and so

$$\sup\big\{\big\langle s_{\mathscr{H};\mathscr{H}_2},\mathrm{pr}_{\mathscr{H};\mathscr{H}_2}(v_k)\big\rangle\colon k\in\mathbb{Z}^+\big\}=+\infty\,.$$

Since $s_{\mathscr{H};\mathscr{H}_2}\in(\mathbb{R}^+)^{\mathscr{H}-\mathscr{H}_2}$, this establishes the claim. □

Proposition 6.15. *Suppose $(z_k)_{k\in\mathbb{Z}^+}$ is a sequence in $\mu^{-1}_{\mathscr{H}}(0)\subset\mathbb{C}^{\mathscr{H}}$ and $(v_k)_{k\in\mathbb{Z}^+}$ is a sequence in $T_{\mathbb{1}}\mathcal{K}_{\mathscr{H}}\subset\mathbb{R}^{\mathscr{H}}$.*

(a) If the sequence $\psi_{\mathscr{H};[\mathrm{i}v_k]}(z_k)$ converges to some $y\in\mathbb{C}^{\mathscr{H}}$, then the sequence z_k is bounded.

(b) If the sequence $\psi_{\mathscr{H};[\mathrm{i}v_k]}(z_k)$ converges to some $y\in\widetilde{X}_{\mathscr{H}}$, then subsequences of z_k and v_k converge to some $z\in\mu^{-1}_{\mathscr{H}}(0)$ and $v\in T_{\mathbb{1}}\mathcal{K}_{\mathscr{H}}$, respectively, with $\psi_{\mathscr{H};[\mathrm{i}v]}(z)=y$.

Proof. We denote by $\{e_v\colon v\in\mathscr{H}\}$ the standard basis for $T_{\mathbb{1}}\mathbb{T}^{\mathscr{H}}=\mathbb{R}^{\mathscr{H}}$, by $\{e_v^*\colon v\in\mathscr{H}\}$ the corresponding dual basis for $T^*_{\mathbb{1}}\mathbb{T}^{\mathscr{H}}\approx\mathbb{R}^{\mathscr{H}}$, by $\langle\cdot,\cdot\rangle$ the standard inner-products on $T_{\mathbb{1}}\mathbb{T}^{\mathscr{H}}$ and $T^*_{\mathbb{1}}\mathbb{T}^{\mathscr{H}}$, and by $|\cdot|$ the corresponding norms and the standard norm on $\mathbb{C}^{\mathscr{H}}$. If $\mathscr{H}_\bullet\in\mathrm{Ver}(\mathscr{H})$, the elements $v\in T_{\mathbb{1}}\mathbb{T}$ with $(v,c)\in\mathscr{H}_\bullet$ form an $\mathbb{R}$-basis for $T_{\mathbb{1}}\mathbb{T}$ (in fact, $\mathbb{Z}$-basis for the lattice $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$). Thus, there exists a matrix $A\equiv(a_{vv'})_{v\in\mathscr{H}-\mathscr{H}_\bullet,v'\in\mathscr{H}_\bullet}$ with real (in fact, integer) coefficients so that

$$v=\sum_{(v',c')\in\mathscr{H}_\bullet}a_{(v,c)(v',c')}v'\qquad\forall\,(v,c)\in\mathscr{H}-\mathscr{H}_\bullet\,. \tag{6.11}$$

From (1.41) and the exactness of (6.1), we then obtain

$$\begin{aligned} T_{\mathbb{1}}\mathcal{K}_{\mathscr{H}} \equiv \ker L_{\mathscr{H}} &= \mathrm{Span}_{\mathbb{R}}\Big(\Big\{e_v - \sum_{v'\in\mathscr{H}_\bullet} a_{vv'}e_{v'} \colon v\in\mathscr{H}-\mathscr{H}_\bullet\Big\}\Big), \\ \ker \iota^*_{\mathscr{H}} = \mathrm{Im}\, L^*_{\mathscr{H}} &= \mathrm{Span}_{\mathbb{R}}\Big(\Big\{e^*_{v'} + \sum_{v\in\mathscr{H}-\mathscr{H}_\bullet} a_{vv'}e^*_v \colon v'\in\mathscr{H}_\bullet\Big\}\Big). \end{aligned} \tag{6.12}$$

For each $k\in\mathbb{Z}^+$, let $z_k=(z_{k;v})_{v\in\mathscr{H}}\in\mathbb{C}^{\mathscr{H}}$ and $v_k=(t_{k;v})_{v\in\mathscr{H}}\in\mathbb{R}^{\mathscr{H}}$. By passing to subsequences if necessary, we can assume that

(i) there is a subset $\mathscr{H}_z^\infty\subset\mathscr{H}$ such that $|z_{k;v}|\longrightarrow\infty$ for every $v\in\mathscr{H}_z^\infty$ and $z_{k;v}$ converges to some $z_v\in\mathbb{C}$ for every $v\in\mathscr{H}-\mathscr{H}_z^\infty$;

(ii) there are (disjoint) subsets $\mathscr{H}_v^-,\mathscr{H}_v^+\subset\mathscr{H}$ such that $t_{k;v}\longrightarrow\pm\infty$ for every $v\in\mathscr{H}_v^\pm$ and $t_{k;v}$ converges to some $t_v\in\mathbb{C}$ for every $v\in\mathscr{H}-\mathscr{H}_v^-\cup\mathscr{H}_v^+$.

Let $y=(y_v)_{v\in\mathscr{H}}$. By the assumption in either (a) or (b),

$$\lim_{k\longrightarrow\infty} \mathrm{e}^{-2\pi t_{k;v}} z_{k;v} = y_v \qquad \forall\ v\in\mathscr{H}. \tag{6.13}$$

In particular, $\mathscr{H}_z^\infty\subset\mathscr{H}_v^+$.

(a) Suppose $\mathscr{H}_z^\infty\neq\emptyset$ and thus $|H_{\mathscr{H}}(z_k)|\longrightarrow\infty$. After passing to further subsequences if necessary, we can assume that the sequence $H_{\mathscr{H}}(z_k)/|H_{\mathscr{H}}(z_k)|$ converges to some

$$s\equiv(s_v)_{v\in\mathscr{H}}\in\big(\ker\iota^*_{\mathscr{H}}\big)\cap\big(\mathbb{R}^{\ge0}\big)^{\mathscr{H}}\subset\mathbb{R}^{\mathscr{H}}.$$

Let $\mathscr{H}_s^+=\{v\in\mathscr{H}\colon s_v>0\}$. We note that

$$\emptyset\neq\mathscr{H}_s^+\subset\mathscr{H}_z^\infty\subset\mathscr{H}_v^+. \tag{6.14}$$

Choose $\mathscr{H}_\bullet\in\mathrm{Ver}(\mathscr{H})$. With $A\equiv(a_{vv'})_{v,v'}$ as in (6.11), define

$$A_1\equiv\big(a_{vv'}\big)_{v\in\mathscr{H}_s^+-\mathscr{H}_\bullet,v'\in\mathscr{H}_s^+\cap\mathscr{H}_\bullet} \qquad\text{and}\qquad A_2\equiv\big(a_{vv'}\big)_{v\in\mathscr{H}-\mathscr{H}_\bullet-\mathscr{H}_s^+,v'\in\mathscr{H}_s^+\cap\mathscr{H}_\bullet}.$$

By the second equation in (6.12) and the definition of $\mathscr{H}_s^+$,

$$A_1(s_v)_{v\in\mathscr{H}_s^+\cap\mathscr{H}_\bullet}=(s_v)_{v\in\mathscr{H}_s^+-\mathscr{H}_\bullet}\in(\mathbb{R}^+)^{\mathscr{H}_s^+-\mathscr{H}_\bullet},\quad A_2(s_v)_{v\in\mathscr{H}_s^+\cap\mathscr{H}_\bullet}=0\in\mathbb{R}^{\mathscr{H}-\mathscr{H}_\bullet-\mathscr{H}_s^+}.$$

By the first equation in (6.12),

$$A_1^{\mathrm{tr}}(t_{k;v})_{v\in\mathscr{H}_s^+-\mathscr{H}_\bullet}+A_2^{\mathrm{tr}}(t_{k;v})_{v\in\mathscr{H}-\mathscr{H}_\bullet-\mathscr{H}_s^+}=-(t_{k;v})_{v\in\mathscr{H}_s^+\cap\mathscr{H}_\bullet}$$

for all $k\in\mathbb{Z}^+$. By (6.14), $t_{k;v}\in\mathbb{R}^+$ for all $v\in\mathscr{H}_s^+$ and $k\in\mathbb{Z}^+$ sufficiently large. Thus,

$$\begin{aligned} 0<\big\langle(s_v)_{v\in\mathscr{H}_s^+\cap\mathscr{H}_\bullet},(t_{k;v})_{v\in\mathscr{H}_s^+\cap\mathscr{H}_\bullet}\big\rangle &= -\big\langle(s_v)_{v\in\mathscr{H}_s^+\cap\mathscr{H}_\bullet},A_1^{\mathrm{tr}}(t_{k;v})_{v\in\mathscr{H}_s^+-\mathscr{H}_\bullet}+A_2^{\mathrm{tr}}(t_v)_{v\in\mathscr{H}-\mathscr{H}_\bullet-\mathscr{H}_s^+}\big\rangle \\ &= -\big\langle A_1(s_v)_{v\in\mathscr{H}_s^+\cap\mathscr{H}_\bullet},(t_{k;v})_{v\in\mathscr{H}_s^+-\mathscr{H}_\bullet}\big\rangle-\big\langle A_2(s_v)_{v\in\mathscr{H}_s^+\cap\mathscr{H}_\bullet},(t_{k;v})_{v\in\mathscr{H}-\mathscr{H}_\bullet-\mathscr{H}_s^+}\big\rangle \\ &= -\big\langle(s_v)_{v\in\mathscr{H}_s^+-\mathscr{H}_\bullet},(t_{k;v})_{v\in\mathscr{H}_s^+-\mathscr{H}_\bullet}\big\rangle-0<0 \end{aligned}$$

for all $k\in\mathbb{Z}^+$ large. Since this is a contradiction, $\mathscr{H}_z^\infty=\emptyset$.

(b) After passing to further subsequences if necessary, we can thus assume that the sequence z_k converges to some element $z\equiv(z_v)_{v\in\mathscr{H}}$ of $\mu_{\mathscr{H}}^{-1}(0)\subset\mathbb{C}^{\mathscr{H}}$. By (6.13),

$$\mathscr{H}_v^+\subset\mathscr{H}_y\equiv\big\{v\in\mathscr{H}: y_v=0\big\}\qquad\text{and}\qquad\mathscr{H}_v^-\subset\mathscr{H}_z\equiv\big\{v\in\mathscr{H}: z_v=0\big\}.$$

Let $\mathscr{H}_1,\mathscr{H}_2\in\mathrm{Ver}(\mathscr{H})$ be such that $\mathscr{H}_y\subset\mathscr{H}_1$ and $\mathscr{H}_z\subset\mathscr{H}_2$. By the definition of $\mathscr{H}_v^+$ and $\mathscr{H}_v^-$,

$$\sup\big\{t_{k;v}: k\in\mathbb{Z}^+,\, v\in\mathscr{H}-\mathscr{H}_1\big\}\in\mathbb{R}\quad\text{and}\quad\inf\big\{t_{k;v}: k\in\mathbb{Z}^+,\, v\in\mathscr{H}-\mathscr{H}_2\big\}\in\mathbb{R}.\tag{6.15}$$

By Lemma 6.14 with $\mathscr{H}_1$ and $\mathscr{H}_2$ interchanged, this implies that

$$\sup\big\{t_{k;v}: k\in\mathbb{Z}^+,\, v\in\mathscr{H}-\mathscr{H}_2\big\}\in\mathbb{R}.$$

Along with the first statement in (6.12) and the second statement (6.15), this gives

$$\sup\big\{|t_{k;v}|: k\in\mathbb{Z}^+,\, v\in\mathscr{H}\big\}\in\mathbb{R}.$$

After passing to further subsequences if necessary, we can thus assume that the sequence v_k converges to some element $v\equiv(t_v)_{v\in\mathscr{H}}$ of $T_{\mathbb{1}}\mathcal{K}_{\mathscr{H}}\subset\mathbb{R}^{\mathscr{H}}$. The last claim then follows from the continuity of the complexified action $\psi_{\mathscr{H}}$. $\square$

Corollary 6.16. *The smooth map*

$$\Psi_{\mathscr{H}}:(\mathcal{K}_{\mathscr{H}})_{\mathfrak{i}}\times\mu_{\mathscr{H}}^{-1}(0)\longrightarrow\widetilde{X}_{\mathscr{H}},\qquad\Psi_{\mathscr{H}}(u,z)=\psi_{\mathscr{H};u}(z),$$

is a diffeomorphism. The map

$$\iota: X\equiv\mu_{\mathscr{H}}^{-1}(0)/\mathcal{K}_{\mathscr{H}}\longrightarrow X_{\mathscr{H}}\equiv\widetilde{X}_{\mathscr{H}}/(\mathcal{K}_{\mathscr{H}})_{\mathbb{C}}\tag{6.16}$$

induced by the inclusions $\widetilde{\iota}\colon\mu_{\mathscr{H}}^{-1}(0)\longrightarrow\widetilde{X}_{\mathscr{H}}$ and $\mathcal{K}_{\mathscr{H}}\longrightarrow(\mathcal{K}_{\mathscr{H}})_{\mathbb{C}}$ is a homeomorphism. For every $\mathscr{H}_0\subset\mathscr{H}$,

$$\iota\big(\mu^{-1}(\langle\!\langle\mathscr{H}_0\rangle^{\partial})\big)=\big\{\big[(z_v)_{v\in\mathscr{H}}\big]\in X_{\mathscr{H}}: z_v=0\ \forall\, v\in\mathscr{H}_0\big\}.\tag{6.17}$$

Proof. By the definition of the action $\psi_{\mathscr{H}}$, the image of the map $\Psi_{\mathscr{H}}$ is contained in $\widetilde{X}_{\mathscr{H}}$. By Corollary 3.19, $\Psi_{\mathscr{H}}$ is a diffeomorphism onto an open subset of $\mathbb{C}^{\mathscr{H}}$ and thus of $\widetilde{X}_{\mathscr{H}}$. By Proposition 6.15(b), the image of $\Psi_{\mathscr{H}}$ is closed in the connected space $\widetilde{X}_{\mathscr{H}}$. Since $\widetilde{X}_{\mathscr{H}}\neq\emptyset$ if and only if $\mu_{\mathscr{H}}^{-1}(0)\neq\emptyset$, it follows that the map $\Psi_{\mathscr{H}}$ is a diffeomorphism onto the entire space $\widetilde{X}_{\mathscr{H}}$.

Let $\Psi_{\mathscr{H}}^{-1}=(\phi,\widetilde{\mathcal{R}}):\widetilde{X}_{\mathscr{H}}\longrightarrow(\mathcal{K}_{\mathscr{H}})_{\mathfrak{i}}\times\mu_{\mathscr{H}}^{-1}(0)$. Since the map $\widetilde{\mathcal{R}}$ is $\mathcal{K}_{\mathscr{H}}$-equivariant and $(\mathcal{K}_{\mathscr{H}})_{\mathfrak{i}}$-invariant with respect to the action $\psi_{\mathscr{H}}$, it descends to a continuous map

$$\mathcal{R}:\widetilde{X}_{\mathscr{H}}/(\mathcal{K}_{\mathscr{H}})_{\mathbb{C}}\longrightarrow\mu_{\mathscr{H}}^{-1}(0)/\mathcal{K}_{\mathscr{H}}\,.$$

Since the map $\widetilde{\iota}$ is equivariant with respect to the inclusion $\mathcal{K}_{\mathscr{H}}\longrightarrow(\mathcal{K}_{\mathscr{H}})_{\mathbb{C}}$, it similarly descends to a continuous map ι as in (6.16). Since $\mathcal{R}\circ\iota=\mathrm{id}$ and $\iota\circ\mathcal{R}=\mathrm{id}$, the continuous maps $\mathcal{R}$ and ι are homeomorphisms. Since the action of $(\mathcal{K}_{\mathscr{H}})_{\mathbb{C}}\subset(\mathbb{C}^*)^{\mathscr{H}}$ on $\widetilde{X}_{\mathscr{H}}\subset\mathbb{C}^{\mathscr{H}}$ preserves the sets $\{v\in\mathscr{H}: z_v=0\}$ with $(z_v)_v\in\widetilde{X}_{\mathscr{H}}$, (6.17) follows from Exercise 6.10. $\square$

Exercise 6.17. Show that the manifold X determined by a regular Delzant subcollection $\mathscr{H}$ of $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ as in Section 6.1 is simply connected. *Hint:* use Exercises 6.8 and 6.13, along with the homotopy exact sequence for fibration.

Exercise 6.18. Let

$$\Psi_{\mathscr{H}}^{-1}\equiv\big(\phi,(\widetilde{\mathcal{R}}_{\upsilon})_{\upsilon\in\mathscr{H}}\big)\colon \widetilde{X}_{\mathscr{H}}\longrightarrow(\mathcal{K}_{\mathscr{H}})_{\mathfrak{i}}\times\mu_{\mathscr{H}}^{-1}(0)\subset\mathbb{R}^{+}\times\mathbb{C}^{\mathscr{H}}$$

be the inverse of the diffeomorphism $\Psi_{\mathscr{H}}$ of Corollary 6.16. Show that the maps

$$X_{\mathscr{H}}\equiv\widetilde{X}_{\mathscr{H}}/(\mathcal{K}_{\mathscr{H}})_{\mathbb{C}}\longrightarrow\mathbb{R}^{\ge0},\quad \big[(z_{\upsilon'})_{\upsilon'\in\mathscr{H}}\big]\longrightarrow\big|\widetilde{\mathcal{R}}_{\upsilon}\big((z_{\upsilon'})_{\upsilon'\in\mathscr{H}}\big)\big|^{2},\qquad \upsilon\in\mathscr{H},$$

are well-defined and smooth.

Exercise 6.19. Suppose $n\in\mathbb{Z}^{+}$ and $\mathscr{H}\equiv\mathscr{H}_{n;n}()$ is the regular minimal Delzant subset of $(T_{\mathbb{1}}\mathbb{T}^{n})_{\mathbb{Z}}\times\mathbb{R}$ as in Exercise 1.46(b). Show that the inverse of the diffeomorphism $\Psi_{\mathscr{H}}$ of Corollary 6.16 is given by

$$\begin{gathered}\Psi_{\mathscr{H}}^{-1}\equiv\big(\phi,\widetilde{\mathcal{R}}\big)\colon \widetilde{X}_{\mathscr{H}}=\mathbb{C}^{\mathscr{H}}-\{0\}\longrightarrow(\mathcal{K}_{\mathscr{H}})_{\mathfrak{i}}\times\mu_{\mathscr{H}}^{-1}(0)=\mathbb{R}^{+}\times\big\{z\in\mathbb{C}^{\mathscr{H}}\colon |z|^{2}=1/\pi\big\},\\ \Psi_{\mathscr{H}}^{-1}(z)=\big(\sqrt{\pi}|z|,z/\sqrt{\pi}|z|\big).\end{gathered}$$

6.3 Kähler structure

We now show that the Hamiltonian $\mathbb{T}$-manifold (X,ω,ψ,μ) constructed in Section 6.1 admits a compatible (integrable) complex structure J, i.e. J is compatible with the symplectic form ω and is preserved by the effective $\mathbb{T}$-action ψ on X. From now on, we identify the two quotients in (6.16) via ι.

Proposition 6.20. *The Hamiltonian $\mathbb{T}$-manifold (X,ω,ψ,μ) determined by $\mathscr{H}$ as in Section 6.1 admits a unique compatible complex structure $J_{\mathscr{H}}$ so that the quotient projection*

$$q\colon \widetilde{X}_{\mathscr{H}}\longrightarrow X=X_{\mathscr{H}} \tag{6.18}$$

is a holomorphic submersion. For every $\mathscr{H}_0\subset\mathscr{H}$, $\mu^{-1}(\langle\!\langle\mathscr{H}_0\rangle^{\partial})\subset X$ is a complex submanifold of codimension $|\mathscr{H}_0|$ with respect to this complex structure.

Proof. Let $\mathscr{H}_{\bullet}\in\mathrm{Ver}(\mathscr{H})$ and $(a_{\upsilon\upsilon'})_{\upsilon,\upsilon'}$ be as in (6.11). Define

$$\widetilde{\mathcal{U}}_{\mathscr{H}_{\bullet}}=\big\{(z_{\upsilon})_{\upsilon\in\mathscr{H}}\in\widetilde{X}_{\mathscr{H}}\colon z_{\upsilon}\neq0\ \forall\,\upsilon\in\mathscr{H}-\mathscr{H}_{\bullet}\big\},\qquad \mathcal{U}_{\mathscr{H}_{\bullet}}=q\big(\widetilde{\mathcal{U}}_{\mathscr{H}_{\bullet}}\big).$$

Since the open subset $\widetilde{\mathcal{U}}_{\mathscr{H}_{\bullet}}\subset\widetilde{X}_{\mathscr{H}}$ is $\mathcal{K}_{\mathscr{H}}$-invariant (in fact, $\mathbb{T}^{\mathscr{H}}$-invariant) the subset $\mathcal{U}_{\mathscr{H}_{\bullet}}\subset X_{\mathscr{H}}$ is also open. By the definition of $\widetilde{\mathcal{U}}_{\mathscr{H}_{\bullet}}$ and the first equation in (6.12),

$$\begin{gathered}\widetilde{\phi}_{\mathscr{H}_{\bullet}}\equiv\big(\widetilde{\phi}_{\mathscr{H}_{\bullet};\upsilon'}\big)_{\upsilon'\in\mathscr{H}}\colon\widetilde{\mathcal{U}}_{\mathscr{H}_{\bullet}}\longrightarrow(\mathcal{K}_{\mathscr{H}})_{\mathbb{C}}\subset(\mathbb{C}^{*})^{\mathscr{H}},\\ \widetilde{\phi}_{\mathscr{H}_{\bullet};\upsilon'}\big((z_{\upsilon})_{\upsilon\in\mathscr{H}}\big)=\begin{cases}z_{\upsilon'}, & \text{if } \upsilon'\in\mathscr{H}-\mathscr{H}_{\bullet};\\ \prod\limits_{\upsilon\in\mathscr{H}-\mathscr{H}_{\bullet}} z_{\upsilon}^{-a_{\upsilon\upsilon'}}, & \text{if } \upsilon'\in\mathscr{H}_{\bullet};\end{cases}\end{gathered}$$

is a well-defined map to $(\mathbb{C}^*)^{\mathscr{H}}$ with values in $(\mathcal{K}_{\mathscr{H}})_{\mathbb{C}}$. Since this map is $(\mathcal{K}_{\mathscr{H}})_{\mathbb{C}}$-equivariant, the smooth map

$$\widetilde{\psi}_{\mathscr{H}_\bullet}\colon \widetilde{\mathcal{U}}_{\mathscr{H}_\bullet} \longrightarrow \mathbb{C}^{\mathscr{H}_\bullet}, \qquad \widetilde{\psi}_{\mathscr{H}_\bullet}\big((z_v)_{v\in\mathscr{H}}\big) = \big((z_{v'}/\widetilde{\phi}_{\mathscr{H}_\bullet;v'}((z_v)_{v\in\mathscr{H}})\big)_{v'\in\mathscr{H}_\bullet}\big),$$

is $(\mathcal{K}_{\mathscr{H}})_{\mathbb{C}}$-invariant and thus descends to a continuous map

$$\psi_{\mathscr{H}_\bullet}\colon \mathcal{U}_{\mathscr{H}_\bullet} \longrightarrow \mathbb{C}^{\mathscr{H}_\bullet}.$$

The inverse of this map is $q\circ\widetilde{\vartheta}_{\mathscr{H}_\bullet}$, where

$$\widetilde{\vartheta}_{\mathscr{H}_\bullet} \equiv \big(\widetilde{\vartheta}_v\big)_{v\in\mathscr{H}}\colon \mathbb{C}^{\mathscr{H}_\bullet} \longrightarrow \widetilde{\mathcal{U}}_{\mathscr{H}_\bullet}, \quad \widetilde{\vartheta}_{\mathscr{H}_\bullet;v}\big((z_{v'})_{v'\in\mathscr{H}_\bullet}\big) = \begin{cases} 1, & \text{if } v\in\mathscr{H}-\mathscr{H}_\bullet;\\ z_v, & \text{if } v\in\mathscr{H}_\bullet.\end{cases}$$

Since this map is continuous (in fact, smooth), $q\circ\widetilde{\vartheta}$ is also continuous and thus $\psi_{\mathscr{H}_\bullet}$ is a homeomorphism.

The restriction of the map $\psi_{\mathscr{H}_\bullet}\circ q$ to the slice

$$\widetilde{\mathcal{U}}_{\mathscr{H}_\bullet;1} \equiv \big\{(z_v)_{v\in\mathscr{H}}\in\mathbb{C}^{\mathscr{H}}\colon z_v=1\ \forall\, v\in\mathscr{H}-\mathscr{H}_\bullet\big\} \subset \widetilde{\mathcal{U}}_{\mathscr{H}_\bullet}$$

is the projection onto $\mathbb{C}^{\mathscr{H}_\bullet}$. Since the image of $\widetilde{\mathcal{U}}_{\mathscr{H}_\bullet;1}$ under $(\mathcal{K}_{\mathscr{H}})_{\mathbb{C}}$ is $\widetilde{\mathcal{U}}_{\mathscr{H}_\bullet}$ and the map

$$\psi_{\mathscr{H}_\bullet}\circ q\colon \widetilde{\mathcal{U}}_{\mathscr{H}_\bullet} \longrightarrow \mathbb{C}^{\mathscr{H}_\bullet}$$

is $(\mathcal{K}_{\mathscr{H}})_{\mathbb{C}}$-invariant, it follows that this map is a holomorphic submersion. Thus,

$$q\colon \widetilde{\mathcal{U}}_{\mathscr{H}_\bullet} = q^{-1}\big(\mathcal{U}_{\mathscr{H}_\bullet}\big) \longrightarrow \mathcal{U}_{\mathscr{H}_\bullet}$$

is also a holomorphic submersion with respect to the complex structure $J_{\mathscr{H}_\bullet}$ on $\mathcal{U}_{\mathscr{H}_\bullet}$ induced from the standard complex structure on $\mathbb{C}^{\mathscr{H}_\bullet}$ by $\psi_{\mathscr{H}_\bullet}$. Since the standard complex structure $J_{\mathbb{C}^{\mathscr{H}}}$ on $\mathbb{C}^{\mathscr{H}}$ is $\mathbb{T}^{\mathscr{H}}$-invariant and the $\mathbb{T}$-action on $\mathcal{U}_{\mathscr{H}_\bullet}$ is induced by the $\mathbb{T}^{\mathscr{H}}$-action on $\mathbb{C}^{\mathscr{H}}$, it follows that the complex structure $J_{\mathscr{H}_\bullet}$ is $\mathbb{T}$-invariant.

Let $\mathscr{H}_0\subset\mathscr{H}$. By (6.17),

$$\psi_{\mathscr{H}_\bullet}\big(\mu^{-1}(\langle\!\langle\mathscr{H}_0\rangle^{\partial})\cap\mathcal{U}_{\mathscr{H}_\bullet}\big) = \begin{cases} \emptyset, & \text{if } \mathscr{H}_0\not\subset\mathscr{H}_\bullet;\\ \{(z_v)_{v\in\mathscr{H}_\bullet}\colon z_v=0\ \forall\, v\in\mathscr{H}_0\}, & \text{if } \mathscr{H}_0\subset\mathscr{H}_\bullet.\end{cases}$$

Thus, $\mu^{-1}(\langle\!\langle\mathscr{H}_0\rangle^{\partial})\cap\mathcal{U}_{\mathscr{H}_\bullet}$ is a coordinate slice with respect to $\psi_{\mathscr{H}_\bullet}$.

Suppose $\mathscr{H}_1,\mathscr{H}_2\in\mathrm{Ver}(\mathscr{H})$ and thus

$$\psi_{\mathscr{H}_1}\big(\mathcal{U}_{\mathscr{H}_1}\cap\mathcal{U}_{\mathscr{H}_2}\big) = \big\{(z_v)_{v\in\mathscr{H}_1}\in\mathbb{C}^{\mathscr{H}_1}\colon z_v\neq 0\ \forall\, v\in\mathscr{H}_1-\mathscr{H}_2\big\}.$$

The overlap map between the charts $\psi_{\mathscr{H}_1}$ and $\psi_{\mathscr{H}_2}$ as above,

$$\begin{gathered}\psi_{\mathscr{H}_2}\circ\psi_{\mathscr{H}_1}^{-1} \equiv \big(\psi_{\mathscr{H}_2\mathscr{H}_1;v_2}\big)_{v_2\in\mathscr{H}_2}\colon \psi_{\mathscr{H}_1}\big(\mathcal{U}_{\mathscr{H}_1}\cap\mathcal{U}_{\mathscr{H}_2}\big) \longrightarrow \psi_{\mathscr{H}_2}\big(\mathcal{U}_{\mathscr{H}_1}\cap\mathcal{U}_{\mathscr{H}_2}\big) \subset \mathbb{C}^{\mathscr{H}_2},\\ \psi_{\mathscr{H}_2\mathscr{H}_1;v_2}\big((z_{v_1})_{v_1\in\mathscr{H}_1}\big) = \begin{cases} 1/\widetilde{\phi}_{\mathscr{H}_2;v_2}\big(\widetilde{\vartheta}_{\mathscr{H}_1}((z_{v_1})_{v_1\in\mathscr{H}_1})\big), & \text{if } v_2\in\mathscr{H}_2-\mathscr{H}_1;\\ z_{v_2}/\widetilde{\phi}_{\mathscr{H}_2;v_2}\big(\widetilde{\vartheta}_{\mathscr{H}_1}((z_{v_1})_{v_1\in\mathscr{H}_1})\big), & \text{if } v_2\in\mathscr{H}_2\cap\mathscr{H}_1;\end{cases}\end{gathered}$$

is a holomorphic function. Thus, the charts $\widetilde{\psi}_{\mathscr{H}_\bullet}: \widetilde{\mathcal{U}}_{\mathscr{H}_\bullet} \longrightarrow \mathbb{C}^{\mathscr{H}_\bullet}$ with $\mathscr{H}_\bullet \in \mathrm{Ver}(\mathscr{H})$ determine a ψ-invariant complex structure $J_{\mathscr{H}}$ on $X = X_{\mathscr{H}}$ so that the quotient projection q in (6.18) is a holomorphic submersion. Since the standard complex structure $J_{\mathbb{C}^{\mathscr{H}}}$ on $\mathbb{C}^{\mathscr{H}}$ is compatible with the standard symplectic form $\omega_{\mathscr{H}}$ on $\mathbb{C}^{\mathscr{H}}$ and $q^*\omega = \omega_{\mathscr{H}}|_{T\mu_{\mathscr{H}}^{-1}(0)}$, it follows that $J_{\mathscr{H}}$ is compatible with ω. The map q being a holomorphic submersion implies the uniqueness of $J_{\mathscr{H}}$. By the previous paragraph, $\mu^{-1}(\langle\!\langle \mathscr{H}_0\rangle^{\partial}) \subset X$ is a codimension $|\mathscr{H}_0|$ complex submanifold with respect to $J_{\mathscr{H}}$ for every $\mathscr{H}_0 \subset \mathscr{H}$. $\square$

Exercise 6.21. Let $[\![\mathscr{H}]\!]^\star \subset [\![\mathscr{H}]\!]^* \subset [\![\mathscr{H}]\!] \approx \mathbb{R}^{\mathscr{H}}$ be as in Exercise 6.4 and $\mathscr{H}^\dagger \in [\![\mathscr{H}]\!]^\star$. Show that

(a) $\mathrm{Ver}(\mathscr{H}^\dagger) = \mathrm{Ver}(\mathscr{H})$ and $\widetilde{X}_{\mathscr{H}^\dagger} = \widetilde{X}_{\mathscr{H}}$, with the identification induced by a bijection $\Psi_{\mathscr{H}^\dagger\mathscr{H}}$ as in Exercise 6.4;

(b) $(\mathcal{K}_{\mathscr{H}^\dagger})_{\mathbb{C}}$ acts freely on the subspace $\widetilde{X}_{\mathscr{H}^\dagger} \subset \mathbb{C}^{\mathscr{H}^\dagger}$ and

$$\big(X_{\mathscr{H}^\dagger}, J_{\mathscr{H}^\dagger}, \psi_{\mathscr{H}^\dagger}\big) = \big(X_{\mathscr{H}}, J_{\mathscr{H}}, \psi_{\mathscr{H}}\big),$$

with the identification induced by a bijection $\Psi_{\mathscr{H}^\dagger\mathscr{H}}$ as in Exercise 6.4.

Exercise 6.22. Suppose $\mathscr{H}^\dagger \subset (T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ is a Delzant subcollection such that $\mathscr{H}^\dagger \supset \mathscr{H}$ and $\langle \mathscr{H}^\dagger \rangle = \langle \mathscr{H} \rangle$. Show that

(a) $\mathrm{Ver}(\mathscr{H}^\dagger) = \mathrm{Ver}(\mathscr{H})$ and $\widetilde{X}_{\mathscr{H}^\dagger} = \widetilde{X}_{\mathscr{H}} \times (\mathbb{C}^*)^{\mathscr{H}^\dagger - \mathscr{H}}$;

(b) the $\mathbb{T}^{\mathscr{H}}$-equivariant holomorphic immersion

$$\widetilde{s}\colon \widetilde{X}_{\mathscr{H}} \longrightarrow \widetilde{X}_{\mathscr{H}^\dagger} \subset \mathbb{C}^{\mathscr{H}} \times \mathbb{C}^{\mathscr{H}^\dagger - \mathscr{H}}, \qquad \widetilde{s}(z) = \big(z, 1^{\mathscr{H}^\dagger - \mathscr{H}}\big),$$

descends to a $\mathbb{T}$-equivariant biholomorphism $s\colon (X_{\mathscr{H}}, J_{\mathscr{H}}) \longrightarrow (X_{\mathscr{H}^\dagger}, J_{\mathscr{H}^\dagger})$.

Hint: use Exercise 1.43(a).

Exercise 6.23. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. The regular Delzant subcollections

$$\mathscr{H} \equiv \big\{(1,0)\big\} \qquad \text{and} \qquad \mathscr{H}^\dagger \equiv \big\{(1,0), (1,-\pi)\big\}$$

of $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ satisfy $\mathscr{H} \subset \mathscr{H}^\dagger$ and $\langle \mathscr{H} \rangle = \langle \mathscr{H}^\dagger \rangle$. Let $(X_{\mathscr{H}}, \omega_{\mathscr{H}})$ and $(X_{\mathscr{H}^\dagger}, \omega_{\mathscr{H}^\dagger})$ be the symplectic manifolds determined by $\mathscr{H}$ and $\mathscr{H}^\dagger$, respectively, as in Section 6.1, $s\colon X_{\mathscr{H}} \longrightarrow X_{\mathscr{H}^\dagger}$ be the identification as in Exercise 6.22(b), and

$$f\colon \mathbb{R}^{\ge 0} \longrightarrow \mathbb{R}^+, \qquad f(t) = \sqrt{\frac{2}{1+\sqrt{1+4t}}}\,.$$

Show that

(a) the Hamiltonian Kähler $\mathbb{T}$-manifold $(X_{\mathscr{H}}, \omega_{\mathscr{H}}, J_{\mathscr{H}}, \psi_{\mathscr{H}}, \mu_{\mathscr{H}})$ determined by $\mathscr{H}$ is the symplectic manifold $(\mathbb{C}, \omega_{\mathbb{C}})$ with the standard complex structure $J_{\mathbb{C}}$, the standard S^1-action by multiplication, and the Hamiltonian given by

$$H\colon \mathbb{C} \longrightarrow \mathbb{R}, \qquad H(z) = \pi|z|^2\,;$$

(b) the biholomorphism s is the composition of the map

$$X_{\mathscr{H}}=\mu_{\mathscr{H}}^{-1}(0)\equiv\mathbb{C}\longrightarrow\mu_{\mathscr{H}^\dagger}^{-1}(0)\equiv\big\{(z_1,z_2)\in\mathbb{C}^2\colon |z_2|^2=|z_1|^2+1\big\},\ z\longrightarrow\big(f(|z|^2)z,1/f(|z|^2)\big),$$

with the quotient projection $\mu_{\mathscr{H}^\dagger}^{-1}(0)\longrightarrow X_{\mathscr{H}^\dagger}$;

(c) $s^*\omega_{\mathscr{H}^\dagger}=\big(f(z)^2+2f(|z|^2)f'(|z|^2)|z|^2\big)\omega_{\mathbb{C}}$.

Conclude that the Hamiltonian Kähler $\mathbb{T}$-manifolds determined by $\mathscr{H}$ and $\mathscr{H}^\dagger$ are not isomorphic (there is no $\mathbb{T}$-equivariant identification of $X_{\mathscr{H}}$ with $X_{\mathscr{H}^\dagger}$ which intertwines the symplectic forms and complex structures at the same time).

Remark 6.24. By Exercises 6.8 and 6.22, the Hamiltonian $\mathbb{T}$-manifold (X,ω,ψ,μ) and the complex $\mathbb{T}$-manifold (X,J,ψ) obtained via the constructions of Section 6.1 and Proposition 6.20 do not depend on the choice of Delzant subcollection $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ with $\langle\mathscr{H}\rangle$ fixed. However, the identifications of Exercises 6.8 and 6.22 generally do not provide identifications of the Hamiltonian Kähler $\mathbb{T}$-manifolds determined by regular Delzant subcollections $\mathscr{H},\mathscr{H}^\dagger\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ with $\langle\mathscr{H}^\dagger\rangle=\langle\mathscr{H}\rangle$. By Exercise 6.23, these Hamiltonian Kähler $\mathbb{T}$-manifolds need not be isomorphic at all.

Exercise 6.25. Suppose $\mathscr{H}_0\subset\mathscr{H}$, $\alpha_0\in\langle\mathscr{H}_0\rangle^\partial$, and q_{α_0} and $\mathscr{H}/\mathscr{H}_0$ are as in Exercise 1.44. Show that the identification of the Hamiltonian $\mathbb{T}/\mathbb{T}_{\mathscr{H}_0}$-manifold determined by $\mathscr{H}/\mathscr{H}_0$ with the Hamiltonian $\mathbb{T}/\mathbb{T}_{\mathscr{H}_0}$-manifold $\big(\mu^{-1}(\langle\mathscr{H}_0\rangle^\partial),\omega|_{\mu^{-1}(\langle\mathscr{H}_0\rangle^\partial)},\psi_{\mathscr{H}_0},\mu_{\alpha_0}\big)$ in Exercise 6.11 intertwines the complex structure $J_{\mathscr{H}/\mathscr{H}_0}$ on the former and the restriction of the complex structure $J_{\mathscr{H}}$ to the latter.

Exercise 6.26. Suppose $\mathbb{T}$ is a torus, $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ is a regular Delzant subcollection, and (X,ω,J,ψ,μ) is the Hamiltonian Kähler $\mathbb{T}$-manifold determined by $\mathscr{H}$ as above.

(a) Let $\lambda\in\mathbb{R}^+$ and $\lambda\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ be as in Exercise 1.45(a). Show that the biholomorphism

$$\widetilde{X}_{\mathscr{H}}\longrightarrow\widetilde{X}_{\lambda\mathscr{H}}\subset\mathbb{C}^{\lambda\mathscr{H}},\quad\big(z_{v,c}\big)_{(v,c)\in\mathscr{H}}\longrightarrow\big(\sqrt{\lambda}z_{v,c}\big)_{(v,\lambda c)\in\lambda\mathscr{H}},$$

restricts to a diffeomorphism $\mu_{\mathscr{H}}^{-1}(0)\longrightarrow\mu_{\lambda\mathscr{H}}^{-1}(0)$ and descends to an identification of $(X,\lambda\omega,J,\psi,\lambda\mu)$ with the Hamiltonian Kähler $\mathbb{T}$-manifold determined by $\lambda\mathscr{H}$ as above.

(b) Let $\alpha_0\in T_{\mathbb{1}}^*\mathbb{T}$ and $\mathscr{H}+\alpha_0\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ be as in Exercise 4.13(b). Show that the biholomorphism

$$\widetilde{X}_{\mathscr{H}}\longrightarrow\widetilde{X}_{\mathscr{H}+\alpha_0}\subset\mathbb{C}^{\mathscr{H}+\alpha_0},\quad\big(z_{v,c}\big)_{(v,c)\in\mathscr{H}}\longrightarrow\big(z_{v,c}\big)_{(v,c+\alpha_0(v))\in\mathscr{H}+\alpha_0},$$

restricts to a diffeomorphism $\mu_{\mathscr{H}}^{-1}(0)\longrightarrow\mu_{\mathscr{H}+\alpha_0}^{-1}(0)$ and descends to an identification of $(X,\omega,J,\psi,\mu+\alpha_0)$ with the Hamiltonian Kähler $\mathbb{T}$-manifold determined by $\mathscr{H}+\alpha_0$ as above.

(c) Let $\Theta\in\mathrm{GL}_{\mathbb{Z}}((T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}})$ and $\Theta^*\in\mathrm{GL}_{\mathbb{Z}}((T_{\mathbb{1}}^*\mathbb{T})_{\mathbb{Z}})$, ψ_Θ, and $\Theta^{-1}\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ be as in Exercise 4.13(c). Show that the biholomorphism

$$\widetilde{X}_{\mathscr{H}}\longrightarrow\widetilde{X}_{\Theta^{-1}\mathscr{H}}\subset\mathbb{C}^{\Theta^{-1}\mathscr{H}},\quad\big(z_{v,c}\big)_{(v,c)\in\mathscr{H}}\longrightarrow\big(z_{v,c}\big)_{(\Theta^{-1}(v),c)\in\Theta^{-1}\mathscr{H}},$$

restricts to a diffeomorphism $\mu_{\mathscr{H}}^{-1}(0)\longrightarrow\mu_{\Theta^{-1}\mathscr{H}}^{-1}(0)$ and descends to an identification of $(X,\omega,J,\psi_\Theta,\Theta^*\circ\mu)$ with the Hamiltonian Kähler $\mathbb{T}$-manifold determined by $\Theta^{-1}\mathscr{H}$ as above.

In light of Exercise 6.26, it is sufficient to consider the Hamiltonian Kähler $\mathbb{T}$-manifolds determined by representatives for the equivalence classes of regular Delzant subcollections of $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ as defined above Exercise 1.46. In light of Exercises 6.8 and 6.22, it is usually sufficient to consider the Hamiltonian Kähler $\mathbb{T}$-manifolds determined by minimal regular subsets, up to the nuance pointed out in Remark 6.24. A regular subcollection of $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ has at least $\dim\mathbb{T}$ elements. Exercise 1.46 provides specific representatives $\mathscr{H}$ for the equivalence classes of regular minimal Delzant subsets of $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ of small cardinality relative to the dimension. The associated subsets $\langle\mathscr{H}\rangle\subset T_{\mathbb{1}}^*\mathbb{T}$ in the $\dim\mathbb{T}=2$ case are shown in Figures 1.2 and 1.3.

Exercise 6.27. Suppose $n\in\mathbb{Z}^+$ and $\mathscr{H}_n^\star\subset(T_{\mathbb{1}}\mathbb{T}^n)_{\mathbb{Z}}\times\mathbb{R}$ is the regular minimal Delzant subset of Exercise 1.46(a). Show that the Hamiltonian Kähler $\mathbb{T}$-manifold determined by $\mathscr{H}_n^\star$ is $(\mathbb{C}^n,\omega_{\mathbb{C}^n},J_{\mathbb{C}^n},\psi_{\mathbb{C}^n},H_{\mathbb{C}^n})$, where $\omega_{\mathbb{C}^n}$ is the standard symplectic form on $\mathbb{C}^n$ as in (1.19), $J_{\mathbb{C}^n}$ is the standard complex structure on $\mathbb{C}^n$, $\psi_{\mathbb{C}^n}$ is the standard coordinate-wise $\mathbb{T}^n$-action on $\mathbb{C}^n$, i.e. as in (6.5) with $\mathscr{H}=[n]$, and

$$H_{\mathbb{C}^n}\colon\mathbb{C}^n\longrightarrow\mathbb{R}^n=T_{\mathbb{1}}\mathbb{T}^n,\qquad H_{\mathbb{C}^n}(z_1,\ldots,z_n)=\pi\big(|z_1|^2,\ldots,|z_n|^2\big)\,.$$

Exercise 6.28. Suppose $n\in\mathbb{Z}^+$, $k\in[n]$, $a_{k+1},\ldots,a_n\in\mathbb{Z}^{\ge0}$,

$$\mathscr{H}\equiv\mathscr{H}_{n;k}(a_{k+1},\ldots,a_n)\subset\big(T_{\mathbb{1}}\mathbb{T}^n\big)_{\mathbb{Z}}\times\mathbb{R}$$

is the regular minimal Delzant subset of Exercise 1.46(b), and (X,ω,J,ψ,μ) is the Hamiltonian Kähler $\mathbb{T}$-manifold determined by $\mathscr{H}$. Let

$$\upsilon_k^+=\big(-e_1-\ldots-e_k,-1\big)\in\mathscr{H}_{k;k}(),\quad \upsilon_n^+=\big(-e_1-\ldots-e_k+a_{k+1}e_{k+1}+\ldots+a_ne_n,-1\big)\in\mathscr{H},$$

and

$$\mathrm{pr}_{\gamma_k}\colon\gamma_k\equiv\big\{(\ell,v)\in\mathbb{C}P^k\times\mathbb{C}^{k+1}\colon v\in\ell\big\}\longrightarrow\mathbb{C}P^k,\qquad \mathrm{pr}_{\gamma_k}(\ell,v)=\ell,$$

be the tautological holomorphic line bundle.

(a) Show that the projection

$$\widetilde{\mathrm{pr}}\equiv\big(\widetilde{\mathrm{pr}}_\upsilon\big)_{\upsilon\in\mathscr{H}_{k;k}()}\colon\widetilde{X}_{\mathscr{H}}\longrightarrow\widetilde{X}_{\mathscr{H}_{k;k}()},\quad \widetilde{\mathrm{pr}}_\upsilon\big((z_{\upsilon'})_{\upsilon'\in\mathscr{H}}\big)=\begin{cases}z_\upsilon, & \text{if }\upsilon\in\mathscr{H}_k^\star;\\ z_{\upsilon_n^+}, & \text{if }\upsilon=\upsilon_k^+;\end{cases}$$

is well-defined and descends to a holomorphic vector bundle $\mathrm{pr}\colon X\longrightarrow X_{\mathscr{H}_{k;k}()}$.

(b) Construct an identification of $(X,\omega,J,\psi,\mu,\mathrm{pr})$ with the holomorphic vector bundle

$$\mathrm{pr}_E\colon E\equiv\bigoplus_{j=k+1}^n\gamma^{\otimes a_j}\longrightarrow\mathbb{C}P^k$$

with a symplectic form ω_E such that $\omega_E|_{\mathbb{C}P^k}=\omega_{\mathrm{FS};k}$, with the $\mathbb{T}^n$-action ψ_E given by

$$\begin{aligned}\psi_{E;(u_i)_{i\in[n]}}&\Big(\big[(z_i)_{i\in[k+1]}\big],\big(z_j\big((z_i)_{i\in[k+1]}\big)^{\otimes a_j}\big)_{j\in[n]-[k]}\Big)\\ &=\Big(\big[\big((u_iz_i)_{i\in[k]},z_{k+1}\big),\big(u_jz_j\big((u_iz_i)_{i\in[k]},z_{k+1}\big)^{\otimes a_j}\big)_{j\in[n]-[k]}\Big),\end{aligned}$$

and with a Hamiltonian H_E given by

$$H_E\Big(\big[(z_i)_{i\in[k+1]}\big],\big(z_j\big((z_i)_{i\in[k+1]}\big)^{\otimes a_j}\big)_{j\in[n]-[k]}\Big) = \Big(\pi\big|\widetilde{\mathcal{R}}_{(e_j,0)}\big((z_i)_{(e_i,0)\in\mathscr{H}_n^\star},z_{n+1}\big)\big|^2\Big)_{j\in[n]}\in\mathbb{R}^n=T_{\mathbb{1}}^*\mathbb{T}^n,$$

with $\widetilde{\mathcal{R}}_{(e_j,0)}$ denoting the $\upsilon=(e_j,0)$ component of the inverse of the diffeomorphism $\Psi_{\mathscr{H}}$ of Corollary 6.16 as in Exercise 6.18 and z_{n+1} denoting the υ_n^+-component of the input $\mathscr{H}$-tuple.

(c) Conclude that the Hamiltonian Kähler $\mathbb{T}$-manifold (X,ω,J,ψ,μ) determined by $\mathscr{H}_{n;n}()$ is the complex projective space $\mathbb{C}P^n$ with the Fubini-Study symplectic form $\omega_{\mathrm{FS};n}$, its standard complex structure, the $\mathbb{T}^n$-action given by (1.35) with n replaced by $n+1$, and the Hamiltonian given by

$$H_{\mathbb{C}P^n}\colon\mathbb{C}P^n\longrightarrow\mathbb{R}^n=T_{\mathbb{1}}^*\mathbb{T}^n,\quad H_{\mathbb{C}P^n}\big([z_1,\ldots,z_{n+1}]\big)=\frac{(|z_1|^2,\ldots,|z_n|^2)}{|z_1|^2+\ldots+|z_{n+1}|^2}.$$

Exercise 6.29. Suppose n, k, $a_{k+1},\ldots,a_n\in\mathbb{Z}^{\ge0}$, $\mathscr{H}$, (X,ω,J,ψ,μ), E, and ψ_E are as in Exercise 6.28 and $k<n$. Let $c\in\mathbb{R}^+$ and

$$\mathscr{H}'=\big\{\big(-e_{k+1}-\ldots-e_n,-c\big)\big\}\subset\big(T_{\mathbb{1}}\mathbb{T}^n\big)_{\mathbb{Z}}\times\mathbb{R}.$$

Show that

(a) $\mathscr{H}\sqcup\mathscr{H}'$ is a Delzant subset of $(T_{\mathbb{1}}\mathbb{T}^n)_{\mathbb{Z}}\times\mathbb{R}$ and thus (X,ω,ψ,μ) is $\mathscr{H}'$-cuttable;

(b) if $(X',\omega',J',\psi',\mu')$ is the Hamiltonian Kähler $\mathbb{T}$-manifold determined by $\mathscr{H}\sqcup\mathscr{H}'$, then

$$(X',\omega',\psi',\mu')=(X,\omega,\psi,\mu)_{\mathscr{H}'}$$

and there is an identification of (X',J') with the projectivization $\mathbb{P}(E\oplus\tau^1_{\mathbb{C}P^k})$ of the direct sum of the holomorphic vector bundle E with the trivial complex line bundle $\tau^1_{\mathbb{C}P^k}$ over $\mathbb{C}P^k$ so that ψ' is the $\mathbb{T}^n$-action on $\mathbb{P}(E\oplus\tau^1_{\mathbb{C}P^k})$ induced by the trivial extension of the $\mathbb{T}^n$-action ψ_E to $E\oplus\tau_{\mathbb{C}P^k}$,

$$\int_{\mathbb{C}P^1}\omega'=c$$

for any $\mathbb{C}P^1\subset\mathbb{P}(E\oplus\tau^1_{\mathbb{C}P^k})$ linearly embedded in a fiber of the $\mathbb{C}P^{n-k}$-bundle $\mathbb{P}(E\oplus\tau_{\mathbb{C}P^k})\longrightarrow\mathbb{C}P^k$, and $s_0^*\omega'=\omega_{\mathrm{FS};k}$, where s_0 is the section

$$s_0\colon\mathbb{C}P^k\longrightarrow\mathbb{P}\big(E\oplus\tau^1_{\mathbb{C}P^k}\big),\qquad s_0(x)=[0,1]\in\mathbb{P}(E_x\oplus\mathbb{C}).$$

Exercise 6.30. Suppose $a\in\mathbb{Z}^{\ge0}$, $c\in\mathbb{R}^+$, $\mathscr{H}_{2;1}(a;0,c)\subset(T_{\mathbb{1}}\mathbb{T}^2)_{\mathbb{Z}}\times\mathbb{R}$ is the regular minimal Delzant subset of Exercise 1.46(c), and (X,ω,J,ψ,μ) is the Hamiltonian Kähler $\mathbb{T}$-manifold determined by $\mathscr{H}_{2;1}(a;0,c)$. Let $\gamma_1,\tau^1_{\mathbb{C}P^1}\longrightarrow\mathbb{C}P^1$ be the tautological complex line bundle and the trivial complex line bundle, respectively. Construct an identification of (X,ω,ψ,μ,J) with the a-th Hirzebruch surface,

$$\mathbb{F}_a\equiv\mathbb{P}\big(\gamma_1^{\otimes a}\oplus\tau^1_{\mathbb{C}P^1}\big)\longrightarrow\mathbb{C}P^1,$$

with a symplectic form $\omega_{a;c}$ so that $\omega_{a;c}|_{\mathbb{P}(0\oplus\tau^1_{\mathbb{C}P^1})}=\omega_{\mathrm{FS};1}$ and $\int_F\omega_{a;c}=c$ for every fiber $F\subset\mathbb{F}_a$, with $\mathbb{T}^2$-action $\psi_{a;c}$ given by

$$\psi_{a;c;(u_1,u_2)}\big(\big[\big([z_1,w_1],z_2(z_1,w_1)^{\otimes a},w_2\big)\big]\big)=\big[\big([u_1z_1,w_1],u_2z_2(u_1z_1,w_1)^{\otimes a},w_2\big)\big],$$

and with a Hamiltonian $H_{a;c}\colon\mathbb{F}_a\longrightarrow\mathbb{R}^2$,

$$H_{a;c}\big(\big[\big([z_1,w_1],z_2(z_1,w_1)^{\otimes a},w_2\big)\big]\big)=\bigg(\pi\lambda(z_1,w_1,z_2,w_2)|z_1|^2,\frac{c\lambda(z_1,w_1,z_2,w_2)^{-a}|z_2|^2}{\lambda(z_1,w_1,z_2,w_2)^{-a}|z_2|^2+|w_2|^2}\bigg),$$

where $\lambda(z_1,w_1,z_2,w_2)\in\mathbb{R}^+$ is defined by

$$\pi\lambda(z_1,w_1,z_2,w_2)\big(|z_1|^2+|w_1|^2\big)=\big(1+ac\big)-\frac{ac|w_2|^2}{\lambda(z_1,w_1,z_2,w_2)^{-a}|z_2|^2+|w_2|^2}\,.$$

6.4 Line bundles and projectivity

As before, let $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ be a regular Delzant subcollection, $\mathcal{K}_{\mathscr{H}}\subset\mathbb{T}^{\mathscr{H}}$ be the kernel of the homomorphism $\Phi_{\mathscr{H}}$ in (1.41), and $\widetilde{X}_{\mathscr{H}}\subset\mathbb{C}^{\mathscr{H}}$ be the dense open subset given by (6.9). For $\mathbf{c}'\equiv(c'_\upsilon)_{\upsilon\in\mathscr{H}}\in\mathbb{Z}^{\mathscr{H}}$, define

$$L_{\mathscr{H}\mathbf{c}'}=\big(\widetilde{X}_{\mathscr{H}}\times\mathbb{C}\big)/\!\sim,\qquad\text{where}$$
$$\big((z_\upsilon)_{\upsilon\in\mathscr{H}},z'\big)\sim\bigg(\big(u_\upsilon^{-1}z_\upsilon\big)_{\upsilon\in\mathscr{H}},\bigg(\prod_{\upsilon\in\mathscr{H}}u_\upsilon^{c'_\upsilon}\bigg)z'\bigg)\quad\forall\ (u_\upsilon)_{\upsilon\in\mathscr{H}}\in(\mathcal{K}_{\mathscr{H}})_{\mathbb{C}}\subset(\mathbb{C}^*)^{\mathscr{H}}.$$

By Exercise 6.31 below, the map

$$\mathrm{pr}_{\mathscr{H}\mathbf{c}'}\colon L_{\mathscr{H}\mathbf{c}'}\longrightarrow X_{\mathscr{H}}\equiv\widetilde{X}_{\mathscr{H}}/(\mathcal{K}_{\mathscr{H}})_{\mathbb{C}},\qquad\mathrm{pr}_{\mathscr{H}\mathbf{c}'}\big(\big[(z_\upsilon)_{\upsilon\in\mathscr{H}},z'\big]\big)=\big[(z_\upsilon)_{\upsilon\in\mathscr{H}}\big],\tag{6.19}$$

is a holomorphic line bundle with respect to the complex structure $J_{\mathscr{H}}$ on $X_{\mathscr{H}}$ provided by Proposition 6.20. By definition, this holomorphic line bundle depends only on the restriction $\iota^*_{\mathscr{H}}(\mathbf{c}')\in T^*_{\mathbb{1}}\mathcal{K}_{\mathscr{H}}$ of $\mathbf{c}'\in\mathbb{R}^{\mathscr{H}}\approx T^*_{\mathbb{1}}\mathbb{T}^{\mathscr{H}}$ to $T_{\mathbb{1}}\mathcal{K}_{\mathscr{H}}$.

By Lemma 6.33 below, $L_{\mathscr{H}\mathbf{c}'}$ is a positive line bundle in the sense of [27, p148] if the subcollection $\mathscr{H}'$ obtained from $\mathscr{H}$ by replacing the second input in each element $\upsilon\equiv(v,c)$ of $\mathscr{H}$ with c'_υ lies in $[\![\mathscr{H}]\!]^\star$; see Exercise 6.4. In light of the Kodaira Embedding Theorem [27, p181], this implies that $(X_{\mathscr{H}},J_{\mathscr{H}})$ can be holomorphically embedded into a complex projective space $\mathbb{C}P^N$ if $\langle\mathscr{H}\rangle$ is compact (or equivalently $X_{\mathscr{H}}$ is compact).

Exercise 6.31. Show that the complex charts on $(X_{\mathscr{H}},J_{\mathscr{H}})$ provided by the proof of Proposition 6.20 lift to a holomorphic atlas of trivializations of the complex line bundle (6.19).

Exercise 6.32. Let $\mu^{-1}_{\mathscr{H}}(0)\subset\widetilde{X}_{\mathscr{H}}$ be as before. For $\mathbf{c}'\equiv(c'_\upsilon)_{\upsilon\in\mathscr{H}}\in\mathbb{Z}^{\mathscr{H}}$, define

$$S_{\mathscr{H}\mathbf{c}'}=\big(\mu^{-1}_{\mathscr{H}}(0)\times S^1\big)/\!\sim,\qquad\text{where}$$
$$\big((z_\upsilon)_{\upsilon\in\mathscr{H}},z'\big)\sim\bigg(\big(u_\upsilon^{-1}z_\upsilon\big)_{\upsilon\in\mathscr{H}},\bigg(\prod_{\upsilon\in\mathscr{H}}u_\upsilon^{c'_\upsilon}\bigg)z'\bigg)\quad\forall\ (u_\upsilon)_{\upsilon\in\mathscr{H}}\in\mathcal{K}_{\mathscr{H}}\subset\mathbb{T}^{\mathscr{H}}.\tag{6.20}$$

(a) Show that the map

$$\mathrm{pr}_{\mathscr{H}\mathbf{c}'}\colon S_{\mathscr{H}\mathbf{c}'} \longrightarrow X_{\mathscr{H}} \equiv \mu_{\mathscr{H}}^{-1}(0)/\mathcal{K}_{\mathscr{H}}, \qquad \mathrm{pr}_{\mathscr{H}\mathbf{c}'}\big([(z_v)_{v\in\mathscr{H}}, z']\big) = [(z_v)_{v\in\mathscr{H}}], \tag{6.21}$$

is a principal S^1-bundle.

(b) Let $L'_{\mathscr{H}\mathbf{c}'} \equiv S_{\mathscr{H}\mathbf{c}'} \times_{S^1} \mathbb{C} \longrightarrow X_{\mathscr{H}}$ be the associaited complex line bundle as in Appendix B.3. Show that the map

$$L'_{\mathscr{H}\mathbf{c}'} \longrightarrow L_{\mathscr{H}\mathbf{c}'}, \qquad \big[[(z_v)_{v\in\mathscr{H}}, z'], z''\big] \longrightarrow \big[(z_v)_{v\in\mathscr{H}}, z'z''\big],$$

is a well-defined identification of smooth complex line bundles over $X_{\mathscr{H}}$.

Lemma 6.33. *Suppose $\mathbf{c}=(c_v)_{v\in\mathscr{H}}\in\pi\mathbb{Z}$. There exists a connection ∇ in the complex line bundle $L_{\mathscr{H}(-\mathbf{c}/\pi)}\longrightarrow X_{\mathscr{H}}$ with curvature $\kappa_\nabla=-2\mathrm{i}\omega_{\mathscr{H}}$, where $\omega_{\mathscr{H}}$ is the symplectic form on $X_{\mathscr{H}}$ constructed in Section 6.1.*

Proof. By Exercise B.18, it is sufficient to construct a connection 1-form λ on the principal S^1-bundle (6.21) with $\mathbf{c}'\equiv\mathbf{c}/\pi$ so that its curvature κ_λ is $-2\omega_{\mathscr{H}}$. Let

$$p_{\mathscr{H}}\colon \mu_{\mathscr{H}}^{-1}(0) \longrightarrow X_{\mathscr{H}} \qquad \text{and} \qquad p_{\mathscr{H}\mathbf{c}'}\colon \mu_{\mathscr{H}}^{-1}(0)\times S^1 \longrightarrow S_{\mathscr{H}\mathbf{c}'}$$

be the quotient projections. With $z=x+\mathrm{i}y$ as usual, let

$$\zeta_{\mathbb{C}} = 2\pi\bigg(-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}\bigg) = \frac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{e}^{2\pi\mathrm{i}t}z\big)\bigg|_{t=0} \in \Gamma(\mathbb{C};T\mathbb{C})$$

be the vector field generating the standard action of S^1 on $\mathbb{C}$ and

$$\lambda_{\mathbb{C}} = -y\mathrm{d}x+x\mathrm{d}y \in \Gamma(\mathbb{C};T^*\mathbb{C}).$$

With $z_v=x_v+\mathrm{i}y_v$, define

$$\begin{aligned}\lambda_{\mathscr{H}} &= \sum_{v\in\mathscr{H}}\big(-y_v\mathrm{d}x_v+x_v\mathrm{d}y_v\big) \in \Gamma\big(\mathbb{C}^{\mathscr{H}};T^*\mathbb{C}^{\mathscr{H}}\big) \qquad \text{and}\\ \widetilde{\lambda} &= \big(-\lambda_{\mathscr{H}}\big)\oplus\lambda_{\mathbb{C}} \in \Gamma\big(\mathbb{C}^{\mathscr{H}}\times\mathbb{C};T^*(\mathbb{C}^{\mathscr{H}}\times\mathbb{C})\big).\end{aligned}$$

For $\mathbf{r}\equiv(r_v)_{v\in\mathscr{H}}\in\mathbb{R}^{\mathscr{H}}$, define

$$\mathbf{c}'\cdot\mathbf{r} = \sum_{v\in\mathscr{H}}c'_v r_v \qquad \text{and} \qquad \zeta_{\mathbf{r}} = \big(r_v\zeta_{\mathbb{C}}\big)_{v\in\mathscr{H}} = \frac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{e}^{2\pi\mathrm{i}r_v t}z_v\big)_{v\in\mathscr{H}}\bigg|_{t=0} \in \Gamma\big(\mathbb{C}^{\mathscr{H}};T\mathbb{C}^{\mathscr{H}}\big).$$

In particular,

$$\begin{aligned}\big\{\widetilde{\lambda}(0,\zeta_{\mathbb{C}})\big\}\big((z_v)_{v\in\mathscr{H}},z'\big) &= 2\pi|z'|^2,\\ \big\{\widetilde{\lambda}\big(\zeta_{-\mathbf{r}},(\mathbf{c}'\cdot\mathbf{r})\zeta_{\mathbb{C}}\big)\big\}\big((z_v)_{v\in\mathscr{H}},z'\big) &= 2\sum_{v\in\mathscr{H}}\big(\pi|z_v|^2+c_v|z'|^2\big)r_v.\end{aligned} \tag{6.22}$$

The right-hand side of the last equation above vanishes if $\big((z_v)_{v\in\mathscr{H}},z'\big)\in\mu_{\mathscr{H}}^{-1}(0)\times S^1$ and $\mathbf{r}\in T_{\mathbb{1}}\mathcal{K}_{\mathscr{H}}$.

Since the 1-form $\widetilde{\lambda}$ is invariant under the standard action of $\mathbb{T}^{\mathscr{H}}\times S^1$ on $\mathbb{C}^{\mathscr{H}}\times\mathbb{C}$ by coordinate-wise multiplication, $\widetilde{\lambda}$ is invariant under the action of $\mathcal{K}_{\mathscr{H}}$ on $\mu_{\mathscr{H}}^{-1}(0)\times S^1$ as in (6.20). Along with the second equation in (6.22) and the following sentence, this implies that $\widetilde{\lambda}$ descends to a 1-form λ on $S_{\mathscr{H}\mathbf{c}'}$, i.e. $p_{\mathscr{H}\mathbf{c}'}^*\lambda=\widetilde{\lambda}$. Since $\widetilde{\lambda}$ is invariant under the S^1-action on the second factor in $\mathbb{C}^{\mathscr{H}}\times\mathbb{C}$, λ is invariant under the S^1-action on the principal S^1-bundle (6.21) with $\mathbf{c}'=\mathbf{c}/\pi$. Along with the first equation in (6.22), this implies that λ is a connection 1-form on this principal S^1-bundle. Let κ_λ be its curvature. Since

$$p_{\mathscr{H}\mathbf{c}'}^*\mathrm{pr}_{\mathscr{H}\mathbf{c}'}^*\kappa_\lambda = p_{\mathscr{H}\mathbf{c}'}^*\mathrm{d}\lambda = -2\omega_{\mathbb{C}^{\mathscr{H}}} \in \Gamma\big(\mu_{\mathscr{H}}^{-1}(0)\times S^1;\Lambda^2T^*(\mu_{\mathscr{H}}^{-1}(0)\times S^1)\big),$$

it follows that $p_{\mathscr{H}}^*\kappa_\lambda=-2\omega_{\mathbb{C}^{\mathscr{H}}}|_{T\mu_{\mathscr{H}}^{-1}(0)}$ and so $\kappa_\lambda=-2\omega_{\mathscr{H}}$. This establishes the claim. $\square$

Exercise 6.34. Show that the map

$$\Psi_{\mathscr{H}}\colon \iota_{\mathscr{H}}^*\big(\mathbb{Z}^{\mathscr{H}}\big) \longrightarrow \mathrm{Pic}\big(X_{\mathscr{H}},J_{\mathscr{H}}\big), \quad \Psi_{\mathscr{H}}\big(\iota_{\mathscr{H}}^*(\mathbf{c}')\big) = \big[\mathrm{pr}_{\mathscr{H}\mathbf{c}'}\colon L_{\mathscr{H}\mathbf{c}'} \longrightarrow X_{\mathscr{H}}\big],$$

where $\mathrm{Pic}(X_{\mathscr{H}},J_{\mathscr{H}})$ is the Picard group of isomorphism classes of holomorphic line bundles on $(X_{\mathscr{H}},J_{\mathscr{H}})$, is a well-defined group homomorphism.

Exercise 6.35. Let $v_0\in\mathscr{H}$ and $e_{v_0}\in\mathbb{Z}^{\mathscr{H}}$ be the corresponding coordinate unit vector. Show that the map

$$s_{v_0}\colon X_{\mathscr{H}} \longrightarrow L_{\mathscr{H}(-e_{v_0})}, \qquad s_{v_0}\big(\big[(z_v)_{v\in\mathscr{H}}\big]\big) = \big[(z_v)_{v\in\mathscr{H}},z_{v_0}\big],$$

is a well-defined holomorphic section of the holomorphic line bundle $L_{\mathscr{H}(-e_{v_0})}$, it is transverse to the zero set, and

$$s_{v_0}^{-1}(0) = \mu_{\mathscr{H}}^{-1}(0) \subset X_{\mathscr{H}}\,.$$

Exercise 6.36. Suppose $\mathscr{H}^\dagger \subset (T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ is a Delzant subcollection such that $\mathscr{H}^\dagger\supset\mathscr{H}$ and $\langle\mathscr{H}^\dagger\rangle=\langle\mathscr{H}\rangle$. Let

$$\iota_{\mathscr{H}^\dagger;\mathscr{H}}^*\colon T_{\mathbb{1}}^*\mathbb{T}^{\mathscr{H}^\dagger}=\mathbb{R}^{\mathscr{H}^\dagger} \longrightarrow T_{\mathbb{1}}^*\mathbb{T}^{\mathscr{H}}=\mathbb{R}^{\mathscr{H}}$$

be the restriction/projection homomorphism as before and $\mathbf{c}'\in\mathbb{R}^{\mathscr{H}^\dagger}$. Show that the identification of the complex manifolds $(X_{\mathscr{H}},J_{\mathscr{H}})$ and $(X_{\mathscr{H}^\dagger},J_{\mathscr{H}^\dagger})$ provided by Exercise 6.22 lifts to an identification of the holomorphic line bundles

$$L_{\mathscr{H}\iota_{\mathscr{H}^\dagger;\mathscr{H}}^*(\mathbf{c}')} \longrightarrow X_{\mathscr{H}} \qquad\text{and}\qquad L_{\mathscr{H}^\dagger\mathbf{c}'} \longrightarrow X_{\mathscr{H}^\dagger}\,.$$

Exercise 6.37. Suppose $\mathscr{H}_0\subset\mathscr{H}$, $\alpha_0\in\langle\mathscr{H}_0\rangle^\partial$, and q_{α_0} and $\mathscr{H}/\mathscr{H}_0$ are as in Exercise 1.44. Let

$$\phi_{\mathscr{H};\mathscr{H}_0}^*\colon T_{\mathbb{1}}^*\mathcal{K}_{\mathscr{H}/\mathscr{H}_0} \longrightarrow T_{\mathbb{1}}^*\mathcal{K}_{\mathscr{H}}$$

be the isomorphism induced by the isomorphism of Exercise 1.44(c).

(a) Show that $\phi_{\mathscr{H};\mathscr{H}_0}^*$ restricts to an isomorphism from $\iota_{\mathscr{H}/\mathscr{H}_0}^*(\mathbb{Z}^{\mathscr{H}/\mathscr{H}_0})$ to $\iota_{\mathscr{H}}^*(\mathbb{Z}^{\mathscr{H}})$.

(b) Let $\alpha\in\iota_{\mathscr{H}/\mathscr{H}_0}^*(\mathbb{Z}^{\mathscr{H}/\mathscr{H}_0})$. Show that the identification of the complex manifolds $(X_{\mathscr{H}/\mathscr{H}_0},J_{\mathscr{H}/\mathscr{H}_0})$ and $(\mu^{-1}(\langle\mathscr{H}_0\rangle^\partial),J_{\mathscr{H}}|_{\mu^{-1}(\langle\mathscr{H}_0\rangle^\partial)})$ provided by Exercises 6.11 and 6.25 lifts to an identification of the isomorphic classes of holomorphic line bundles

$$\Psi_{\mathscr{H}/\mathscr{H}_0}(\alpha) \in \mathrm{Pic}\big(X_{\mathscr{H}/\mathscr{H}_0},J_{\mathscr{H}/\mathscr{H}_0}\big) \qquad\text{and}$$
$$\Psi_{\mathscr{H}}\big(\phi_{\mathscr{H};\mathscr{H}_0}^*(\alpha)\big|_{\mu^{-1}(\langle\mathscr{H}_0\rangle^\partial)}\big) \in \mathrm{Pic}\big(\mu^{-1}(\langle\mathscr{H}_0\rangle^\partial),J_{\mathscr{H}}\big|_{\mu^{-1}(\langle\mathscr{H}_0\rangle^\partial)}\big),$$

with Ψ as in Exercise 6.34.

Exercise 6.38. Suppose $n\in\mathbb{Z}^+$, $k\in[n]$, $a_{k+1},\ldots,a_n\in\mathbb{Z}^{\ge0}$, $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T}^n)_{\mathbb{Z}}\times\mathbb{R}$,

$$\mathrm{pr}_{\gamma_k}\colon\gamma_k\longrightarrow\mathbb{C}P^k, \qquad \text{and} \qquad \mathrm{pr}_E\colon E\longrightarrow\mathbb{C}P^k$$

are as in Exercise 6.28. Let $\mathbf{c}=(c_v)_{v\in\mathscr{H}}$ as above.

(a) Show that $\iota_{\mathscr{H}}^*(\mathbb{Z}^{\mathscr{H}})=\iota_{\mathscr{H}}^*(\mathbb{Z}\mathbf{c})\subset T_{\mathbb{1}}^*\mathcal{K}_{\mathscr{H}}$.

(b) Let $a\in\mathbb{Z}$. Show that the identification of $(X_{\mathscr{H}},J_{\mathscr{H}})$ with the total space of the holomorphic vector bundle E provided by Exercise 6.28 lifts to an identification of the holomorphic line bundles

$$L_{\mathscr{H}(a\mathbf{c})}\longrightarrow X_{\mathscr{H}} \qquad \text{and} \qquad \mathrm{pr}_E^*\mathcal{O}_{\mathbb{C}P^k}(a)\equiv\mathrm{pr}_E^*\gamma_k^{*\otimes a}\longrightarrow\mathbb{C}P^k.$$

Exercise 6.39. Suppose $a\in\mathbb{Z}^{\ge0}$, $c\in\mathbb{R}^+$, $\mathscr{H}\equiv\mathscr{H}_{2;1}(a;0,c)\subset(T_{\mathbb{1}}\mathbb{T}^2)_{\mathbb{Z}}\times\mathbb{R}$,

$$\mathrm{pr}_{\gamma_1}\colon\gamma_1\longrightarrow\mathbb{C}P^1, \qquad \text{and} \qquad \mathrm{pr}_{\mathbb{F}_a}\colon\mathbb{F}_a\equiv\mathbb{P}\big(\gamma_1^{\otimes a}\oplus\tau_{\mathbb{C}P^1}^1\big)\longrightarrow\mathbb{C}P^1$$

are as in Exercise 6.30. Let $\mathbf{c}_1=(0,0,-1,0)$, $\mathbf{c}_2=(0,0,0,-1)$, and

$$\mathrm{pr}_{\gamma_{\mathbb{F}_a}}\colon\gamma_{\mathbb{F}_a}\equiv\big\{(\ell,v)\in\mathbb{F}_a\times\big(\gamma_1^{\otimes a}\oplus\tau_{\mathbb{C}P^1}^1\big)\colon v\in\ell\big\}\longrightarrow\mathbb{F}_a, \qquad \mathrm{pr}_{\gamma_{\mathbb{F}_a}}(\ell,v)=\ell,$$

be the tautological holomorphic line bundle.

(a) Show that $\iota_{\mathscr{H}}^*(\mathbb{Z}^{\mathscr{H}})=\iota_{\mathscr{H}}^*(\mathbb{Z}\mathbf{c}_1)\oplus\iota_{\mathscr{H}}^*(\mathbb{Z}\mathbf{c}_2)\subset T_{\mathbb{1}}^*\mathcal{K}_{\mathscr{H}}$.

(b) Let $b\in\mathbb{Z}$. Show that the identification of $(X_{\mathscr{H}},J_{\mathscr{H}})$ with $\mathbb{F}_a$ provided by Exercise 6.30 lifts to identifications of the holomorphic line bundles

$$L_{\mathscr{H}(b\mathbf{c}_1)},L_{\mathscr{H}(b\mathbf{c}_2)}\longrightarrow X_{\mathscr{H}} \qquad \text{and} \qquad \mathrm{pr}_{\mathbb{F}_a}^*\mathcal{O}_{\mathbb{C}P^1}(b),\mathcal{O}_{\mathbb{F}_a}(b)\longrightarrow\mathbb{F}_a,$$

where $\mathcal{O}_{\mathbb{C}P^1}(b)=\gamma_1^{*\otimes b}$ and $\mathcal{O}_{\mathbb{F}_a}(b)=\gamma_{\mathbb{F}_a}^{*\otimes b}$.

Appendix A

Morse-Bott Theory

A.1 Gradient flows

The second proof of Proposition 3.10 is based on standard properties of gradient flows of Morse-Bott functions. As these properties are also used in the proof of Theorem 3.2 in Section 3.4, we collect them in Proposition A.2 below and justify at the end of this section.

Exercise A.1. Suppose (X,g) is a compact Riemannian manifold and $H: X \longrightarrow \mathbb{R}$ is a smooth function. Since X is compact, the negative gradient flow of H,

$$\psi_{H;t}\colon X \longrightarrow X, \qquad \psi_{H;0}=\mathrm{id}_X, \quad \frac{\mathrm{d}}{\mathrm{d}t}\psi_{H;t} = -\nabla^g H\big|_{\psi_{H;t}},$$

is defined for all $t\in\mathbb{R}$. Show that the limits

$$x_H^{\pm} \equiv \lim_{t\longrightarrow\pm\infty}\psi_{H;t}(x)$$

exist for every $x\in X$.

Let (X,g) be a compact Riemannian manifold and $H: X \longrightarrow \mathbb{R}$ be a Morse-Bott function. For $Y\in\pi_0(\mathrm{Crit}(H))$, we denote by

$$\mathrm{pr}^{\pm}_{H;Y}\colon \mathrm{E}^{\pm}_Y(H) \longrightarrow Y$$

the bundle projections and by $S(\mathrm{E}^{\pm}_Y(H))\subset \mathrm{E}^{\pm}_Y(H)$ the sphere bundle of $\mathrm{E}^{\pm}_Y(H)$. Let

$$X^{\pm}_Y(H) \equiv \big\{x\in X\colon x^{\pm}_H\in Y\big\} \supset Y \tag{A.1}$$

be the H-stable and unstable manifolds of Y. For $A\subset X^{\pm}_Y(H)$, let

$$A^{\pm}_H = \big\{x^{\pm}_H\colon x\in A\big\} \subset Y.$$

Proposition A.2. *Suppose (X,g) is a compact Riemannian manifold, $H\colon X\longrightarrow\mathbb{R}$ is a Morse-Bott function, and $Y\in\pi_0(\mathrm{Crit}(H))$.*

(1) The subspaces $X^{\pm}_Y(H)\subset X$ are smooth submanifolds with

$$T\big(X^{\pm}_Y(H)\big)\big|_Y = TY\oplus \mathrm{E}^{\pm}_Y(H) \subset TX|_Y\,. \tag{A.2}$$

(2) *There exist diffeomorphisms* $\Phi^{\pm}_{H;Y}\colon \mathrm{E}^{\pm}_Y(H)\longrightarrow X^{\pm}_Y(H)$ *such that*

$$\begin{gathered}\big(\Phi^{\pm}_{H;Y}(w)\big)^{\pm}_{H}=\mathrm{pr}^{\pm}_{H;Y}(w)\quad\forall\,w\in\mathrm{E}^{\pm}_Y(H),\qquad \Phi^{\pm}_{H;Y}(y)=y\quad\forall\,y\in Y,\\ \mathrm{d}_y\Phi^{\pm}_{H;Y}(w)=w\quad\forall\;y\in Y,\,w\in\mathrm{E}^{\pm}_Y(H)\subset T_y\big(\mathrm{E}^{\pm}_Y(H)\big).\end{gathered}\tag{A.3}$$

(3) *For every* $c\in\mathbb{R}$, *the submanifolds* $X^{\pm}_Y(H)$ *and* $H^{-1}(c)-\mathrm{Crit}(H)$ *of* X *are transverse.*

(4) *For every* $\epsilon\in\mathbb{R}^+$ *such that* $H(Y)$ *is the only critical value of* H *in* $[H(Y)-\epsilon,H(Y)+\epsilon]$, *there exist diffeomorphisms*

$$\Phi^{\pm}_{H;Y;\epsilon}\colon S\big(\mathrm{E}^{\pm}_Y(H)\big)\longrightarrow X^{\pm}_Y(H)\cap H^{-1}\big(H(Y)\pm\epsilon\big)\subset X$$

satisfying the first property in (A.3) with $(\Phi^{\pm}_{H;Y},\mathrm{E}^{\pm}_Y(H))$ *replaced by* $(\Phi^{\pm}_{H;Y;\epsilon},S(\mathrm{E}^{\pm}_Y(H)))$.

(5) *The intersection of the closure* $\overline{X^{\pm}_Y(H)}\subset X$ *of* $X^{\pm}_Y(H)$ *with the level set* $H^{-1}(H(Y))$ *is* Y.

(6) *If* $A\subset X^{\pm}_Y(H)$ *and* $\overline{A}\subset X$ *is the closure of* A, $\overline{A}\cap H^{-1}(H(Y))\subset\overline{A^{\pm}_H}$. *If in addition* A *is preserved by the gradient flow of* H, *i.e.* $\psi_{H;t}(A)=A$ *for all* $t\in\mathbb{R}$, *then* $\overline{A}\cap H^{-1}(H(Y))=\overline{A^{\pm}_H}$.

Proof. By the Tubular Neighborhood Theorem [32, Theorem 4.5.1], there are neighborhoods $\mathcal{U}\subset\mathrm{E}^-_Y(H)\oplus\mathrm{E}^+_Y(H)$ and $U\subset X$ of Y and a diffeomorphism $\Phi\colon\mathcal{U}\longrightarrow U$ such that

$$\Phi(y)=y\quad\forall\,y\in Y,\qquad \mathrm{d}_y\Phi(w)=w\quad\forall\;y\in Y,\,w\in\mathrm{E}^-_Y(H)\oplus\mathrm{E}^+_Y(H)\subset T_y\big(\mathrm{E}^-_Y(H)\oplus\mathrm{E}^+_Y(H)\big).$$

By the statement and proof of [6, Theorem A.9], there are then neighborhoods $\mathcal{U}'\subset\mathcal{U}$ and $U'\subset U$ of Y and smooth embeddings

$$\Phi^{\pm}_{H;Y}\colon\mathcal{U}^{\pm}_{H;Y}\equiv\mathcal{U}'\cap\mathrm{E}^{+}_Y(H)\longrightarrow U$$

so that $\Phi^{\pm}_{H;Y}(\mathcal{U}^{\pm}_{H;Y})=U'\cap X^{\pm}_Y(H)$, the first identity in (A.3) holds whenever $w\in\mathcal{U}^{\pm}_{H;Y}$, and the other two identities in (A.3) hold as stated. For all $\delta\in\mathbb{R}^+$ sufficiently small, $\mathcal{U}'$ contains the closed disk bundle of $\mathrm{E}^-_Y(H)\oplus\mathrm{E}^+_Y(H)$ of radius δ and

$$\pm\frac{\mathrm{d}}{\mathrm{d}t}H\big(\Phi^{\pm}_{H;Y}(tw)\big)\Big|_{t=1}>0\qquad\forall\;w\in\mathrm{E}^{\pm}_Y(H),\,|w|=\delta.\tag{A.4}$$

The smooth maps

$$\Psi^{\pm}_{H;Y}\colon\mathrm{E}^{\pm}_Y(H)-Y\longrightarrow X,\qquad \Psi^{\pm}_{H;Y}(w)=\psi_{H;\mp\ln(|w|/\delta)}\big(\Phi^{\pm}_{H;Y}(\delta w/|w|)\big),$$

are then smooth embeddings onto $X^{\pm}_Y(H)$–Y that agree with $\Phi^{\pm}_{H;Y}$ on the sphere bundle $S_\delta(\mathrm{E}^{\pm}_Y(H))$ in $\mathrm{E}^{\pm}_Y(H)$ of radius δ, satisfy the first identity in (A.3) whenever $w\neq0$, and satisfy (A.4) with $\Phi^{\pm}_{H;Y}$ replaced by $\Psi^{\pm}_{H;Y}$. We can thus paste $\Phi^{\pm}_{H;Y}$ and $\Psi^{\pm}_{H;Y}$ together on a neighborhood of $S_\delta(\mathrm{E}^{\pm}_Y(H))$ in $\mathrm{E}^{\pm}_Y(H)$ to obtain smooth embeddings $\Phi^{\pm}_{H;Y}$ of $\mathrm{E}^{\pm}_Y(H)$ into X with image $X^{\pm}_Y(H)$ which satisfy (A.3). This establishes (1) and (2).

Let $c\in\mathbb{R}$ and $x\in X^{\pm}_Y(H)\cap(H^{-1}(c)-\mathrm{Crit}(H))$. Thus, $\mathrm{d}_xH\neq0$, $H^{-1}(c)-\mathrm{Crit}(H)\subset X$ is a smooth submanifold with

$$T_x\big(H^{-1}(c)-\mathrm{Crit}(H)\big)=\ker\mathrm{d}_xH\,,$$

and $\psi_{H;t}(x)$ is a curve in $X_Y^{\pm}(H)$ with

$$\frac{\mathrm{d}}{\mathrm{d}t}H\big(\psi_{H;t}(x)\big)\Big|_{t=0} = \mathrm{d}_x H(-\nabla^g H) = -g(\nabla^g H, \nabla^g H) \neq 0.$$

This gives (3).

Let $\epsilon, \delta \in \mathbb{R}^+$ be as in (4) and above, respectively, with

$$\Psi^{\pm}_{H;Y}\big(S_\delta(\mathrm{E}^{\pm}_Y(H))\big) \subset H^{-1}\big((H(Y)-\epsilon, H(Y)+\epsilon)\big).$$

Since the norm of $\nabla^g H$ is bounded below on $H^{-1}((H(Y)-\epsilon, H(Y)-\epsilon'))$ and $H^{-1}((H(Y)+\epsilon', H(Y)+\epsilon))$ for every $\epsilon' \in (0,\epsilon)$, (A.4) and the smoothness of the negative gradient flow $\psi_{H;t}$ imply that there is a smooth function

$$\rho\colon S_\delta\big(\mathrm{E}^{\pm}_Y(H)\big) \longrightarrow \mathbb{R} \qquad \text{s.t.} \quad H\big(\psi_{H;\rho(w)}\big(\Phi^{\pm}_{H;Y}(w)\big)\big) = H(Y)\pm\epsilon.$$

Along with (A.4) again, the map

$$\Phi^{\pm}_{H;Y;\epsilon}\colon S\big(\mathrm{E}^{\pm}_Y(H)\big) \longrightarrow X^{\pm}_Y(H) \cap H^{-1}\big(H(Y)\pm\epsilon\big), \quad \Phi^{\pm}_{H;Y;\epsilon} = \psi_{H;\rho(w)}\big(\Phi^{\pm}_{H;Y}(w)\big),$$

is then a diffeomorphism satisfying the first property in (A.3) with $(\Phi^{\pm}_{H;Y}, \mathrm{E}^{\pm}_Y(H))$ replaced by $(\Phi^{\pm}_{H;Y;\epsilon}, S(\mathrm{E}^{\pm}_Y(H)))$.

Suppose $x' \in H^{-1}(H(Y)) - Y$. Choose disjoint open neighborhoods $U, U' \subset X$ of Y and x', respectively. By (A.4), there exists $\epsilon \in \mathbb{R}^+$ so that

$$\big|H(x) - H(Y)\big| \geq \epsilon \qquad \forall\, x \in X^{\pm}_Y(H) - U\,.$$

By shrinking U' if necessary, we can assume that

$$\big|H(x) - H(Y)\big| < \epsilon \qquad \forall\, x \in U'\,.$$

It then follows that U' is disjoint from $X^{\pm}_Y(H)$ and thus from $\overline{X^{\pm}_Y(H)}$. This establishes (5).

Let $A \subset X^{\pm}_Y(H)$. By ((5)), $\overline{A} \cap H^{-1}(H(Y)) = \overline{A} \cap Y$. Suppose $y \in \overline{A} \cap Y$ and $U \subset X$ is a neighborhood of y. By (2), there exists a neighborhood U' of y in U so that $x^{\pm}_H \in U'$ for every $x \in U'$. Since $y \in \overline{A}$, U' contains some $x \in A$ and thus $x^{\pm}_H \in A^{\pm}_H$. We conclude that $y \in \overline{A^{\pm}_H}$.

Suppose A is preserved by the gradient flow of H and $y \in \overline{A^{\pm}_H}$. Let $U \subset X$ be a neighborhood of y and $x \in A$ a point such that $x^{\pm}_H \in U$. Thus, $\psi_{H;\pm}(x) \in A \cap U$ for all $t \in \mathbb{R}$ sufficiently large. Therefore, $y \in \overline{A}$. $\square$

A.2 Fiber connectedness

We give two proofs of Proposition 3.10, which are essentially two different formulations of the same reasoning. The first one is in the style of classical Morse theory, as in [45]. It is based on describing the changes in the homotopy type of $H^{-1}((-\infty, c])$ as c passes through critical values as adding "handles" of various kinds; see (A.5) below. The second proof is in the style of the modern take on Morse theory originating in [58]. It is based on partitioning X into stable or unstable manifolds of the negative gradient flow; see (A.6) below. In both cases, we first show that there are unique connected critical submanifolds $Y_-,Y_+\subset X$ with $n^-_{Y_-}(Y_-)=0$ and $n^+_{Y_+}(Y_+)=0$. The function H reaches its global minimum along Y_- and maximum along Y_+; there are no other local minima or maxima. We then show that $H^{-1}(c)$ is connected whenever $c\in(\min H,\max H)$ is a regular value of H. The claim for arbitrary $c\in\mathbb{R}$ then follows from Lemma A.3 below.

Proof 1 of Proposition 3.10 ([3]). For $Y\in\pi_0(\mathrm{Crit}(H))$, we denote by $D(\mathrm{E}^-_Y(H))\subset\mathrm{E}^-_Y(H)$ the disk bundle of $\mathrm{E}^-_Y(H)$. For $c\in\mathbb{R}$, let

$$X_c(H)=H^{-1}\big((-\infty,c]\big)\subset X.$$

If c is a regular value of H, i.e. $c\notin H(\mathrm{Crit}(H))$, then $X_c(H)$ is a smooth manifold with boundary $\partial X_c(H)=H^{-1}(c)$. If $c_-,c_+\in\mathbb{R}$ are regular values of H with $c_-<c_+$, then

$$X_{c_+}(H)\sim X_{c_-}(H)\cup\bigcup_{\substack{Y\in\pi_0(\mathrm{Crit}(H))\\ H(Y)\in(c_-,c_+)}}D\big(\mathrm{E}^-_Y(H)\big),\tag{A.5}$$

with $\sim$ denoting homotopy equivalence; see [10, Section 1]. The boundaries of the disk bundles on the right-hand side above are attached to $\partial X_{c_-}(H)$; the right-hand side is then a deformation retract of the left-hand side.

If $Y\in\pi_0(\mathrm{Crit}(H))$ and $n^-_Y(H)=0$ (i.e. H has a local minimum along Y), then adding $D(\mathrm{E}^-_Y(H))$ as in (A.5) adds a topological component to $X_{c_-}(H)$. If $Y\in\pi_0(\mathrm{Crit}(H))$ and $n^-_Y(H)\ge 2$, then attaching $D(\mathrm{E}^-_Y(H))$ as in (A.5) has no impact on the topological components of $X_{c_-}(H)$ vs $X_{c_+}(H)$. Since $n^-_Y(H)\neq 1$ for any $Y\in\pi_0(\mathrm{Crit}(H))$ and X is connected, it follows that there is a unique $Y_-\in\pi_0(\mathrm{Crit}(H))$ with $n^-_{Y_-}(H)=0$; thus, $H(Y_-)=\min H$. Since $n^-_Y(-H)=n^+_Y(H)$, the same reasoning shows that there is a unique $Y_+\in\pi_0(\mathrm{Crit}(H))$ with $n^+_{Y_+}(H)=0$; thus, $H(Y_+)=\max H$. Furthermore, $X_c(H)$ is connected for every $c\in\mathbb{R}$.

We can assume that H is not a constant function. Let $c\in(\min H,\max H)$ be a regular value. By (A.5), $X_c(H)$ is a homotopy equivalent to a CW complex with cells of dimension at most the maximum of the numbers

$$\dim D\big(\mathrm{E}^-_Y(H)\big)=n^0_Y(H)+n^-_Y(H)=\dim X-n^+_Y(H)<\dim X-1=\dim\partial X_c(H)$$

taken over $Y\in\pi_0(\mathrm{Crit}(H))$ with $H(Y)<c$. The inequality above holds because $n^+_Y(H)\neq 0$ for $Y\neq Y_+$ and $n^+_Y(H)\neq 1$ for any $Y\in\pi_0(\mathrm{Crit}(H))$. Thus, $H_k(X_c(H);\mathbb{Z}_2)=0$ for $k\ge\dim\partial X_c(H)$ and the boundary homomorphism

$$\partial\colon H_{\dim X}\big(X_c(H);\mathbb{Z}_2\big)\longrightarrow H_{\dim\partial X_c(H)}\big(\partial X_c(H);\mathbb{Z}_2\big)$$

in the homology exact sequence for $(X_c(H),\partial X_c(H))$ is surjective. Since $X_c(H)$ is connected, the domain of this homomorphism is isomorphic to $\mathbb{Z}_2$. It follows that $\partial X_c(H)=H^{-1}(c)$ is connected.

Thus, $H^{-1}(c)\subset X$ is connected for every $c\in\mathbb{R}-H(\mathrm{Crit}(H))$. Since $H(\mathrm{Crit}(H))\subset\mathbb{R}$ is a finite subset, Lemma A.3 below implies that $H^{-1}(c)\subset X$ is connected for every $c\in\mathbb{R}$. $\square$

Proof 2 of Proposition 3.10 (modification of [41, pp233,4]). By definition,

$$X = \bigsqcup_{Y\in\pi_0(\mathrm{Crit}(H))} X_Y^+(H) = \bigsqcup_{Y\in\pi_0(\mathrm{Crit}(H))} X_Y^-(H)\,. \tag{A.6}$$

Since $n_Y^-(H)\neq 1$ for any $Y\in\pi_0(\mathrm{Crit}(H))$,

$$\bigcup_{\substack{Y\in\pi_0(\mathrm{Crit}(H))\\ n_Y^-(H)=0}} X_Y^+(H) = X - \bigcup_{\substack{Y\in\pi_0(\mathrm{Crit}(H))\\ n_Y^-(H)\geq 2}} X_Y^+(H).$$

Since X is connected and each submanifold $X_Y^+(H)\subset X$ on the right-hand side above is of codimension $n_Y^-(H)\geq 2$ by Proposition A.2(1), the union on the left-hand side is connected. Since each submanifold $X_Y^+(H)\subset X$ on the left-hand side is open, it follows that there is a unique $Y_-\in\pi_0(\mathrm{Crit}(H))$ with $n_{Y_-}^-(H)=0$; thus, $H(Y_-)=\min H$. By the same reasoning with the second partition in (A.6), there is a unique $Y_+\in\pi_0(\mathrm{Crit}(H))$ with $n_{Y_+}^+(H)=0$; thus, $H(Y_+)=\max H$.

We can assume that H is not a constant function. Let ϵ be as in Proposition A.2(4) with $Y=Y_-$ and $c\in(\min H,\max H)$ be a regular value. By (A.6),

$$\begin{aligned} H^{-1}(c) - \bigcup_{\substack{Y\in\pi_0(\mathrm{Crit}(H))\\ n_Y^-(H)\geq 2}} X_Y^+(H) &= H^{-1}(c)\cap X_{Y_-}^+(H)\\ &\approx H^{-1}\big(\min H+\epsilon\big) - \bigcup_{\substack{Y\in\pi_0(\mathrm{Crit}(H))\\ H(Y)\in(\min H,c)}} H^{-1}\big(\min H+\epsilon\big)\cap X_Y^-(H)\,; \end{aligned} \tag{A.7}$$

the diffeomorphism $\approx$ above is obtained via the gradient flow. Since $H^{-1}(\min H+\epsilon)$ is transverse to each $X_Y^-(H)$, the codimension of the last intersection above in $H^{-1}(\min H+\epsilon)$ is the codimension of $X_Y^-(H)$ in X, i.e. $n_Y^+(H)\geq 2$. The last inequality holds because $n_Y^+(H)\neq 0$ for $Y\neq Y_+$ and $n_Y^+(H)\neq 1$ for any $Y\in\pi_0(\mathrm{Crit}(H))$. Since $H^{-1}(\min H+\epsilon)$ is diffeomorphic to the connected manifold $S(\mathrm{E}_{Y_-}^+(H))$ by Proposition A.2(2), the right-hand side in (A.7) is connected as well. Since the codimension of $H^{-1}(c)\cap X_Y^+(H)$ in $H^{-1}(c)$ is $n_Y^-(H)$ and $n_Y^-(H)>0$ whenever $Y\neq Y_-$, it then follows from (A.7) that $H^{-1}(c)$ is also connected.

By the above $H^{-1}(c)\subset X$ is connected for every $c\in\mathbb{R}-H(\mathrm{Crit}(H))$. Since $H(\mathrm{Crit}(H))\subset\mathbb{R}$ is a finite subset, Lemma A.3 below implies that $H^{-1}(c)\subset X$ is connected for every $c\in\mathbb{R}$. □

Lemma A.3. *Let X be a compact connected manifold (or more generally a topological space which is sequentially compact, connected, locally connected, and normal). Suppose $f: X\longrightarrow\mathbb{R}$ is a continuous function and $P^*\subset\mathbb{R}$ is a dense subset. If $f^{-1}(c)\subset X$ is connected for every $c\in P^*$, then $f^{-1}(c)\subset X$ is connected for every $c\in\mathbb{R}$.*

Proof. Suppose $c\in\mathbb{R}-P^*$ and $f^{-1}(c)=A\cup B$ for some disjoint nonempty subsets A,B that are closed in $f^{-1}(c)$ and thus in X. Let $U_A,U_B\subset X$ be disjoint open subsets such that $A\subset U_A$ and $B\subset U_B$. Let $W\subset\mathbb{R}$ be a neighborhood of c such that $f^{-1}(W)\subset U_A\cup U_B$ (its existence follows from the first countability of $\mathbb{R}$, sequential compactness of X, and the continuity of f). For each $x \in A \cup B$, choose a connected neighborhood V_x of x in $f^{-1}(W)$. The subsets

$$V_A \equiv \bigcup_{x \in A} V_x \qquad \text{and} \qquad V_B \equiv \bigcup_{x \in B} V_x$$

of X are then open disjoint neighborhoods of A and B, respectively, in X. If $f^{-1}(c_A) \cap V_A \neq \emptyset$ for some $c_A < c$, then

$$f^{-1}(c') \cap V_A \neq \emptyset \qquad \forall\, c' \in (c_A, c).$$

If in addition $f(x) < c$ for some $x \in V_B$, then there exists $c^* \in (c_A, c)$ such that

$$c^* \in P^*, \qquad f^{-1}(c^*) \cap U_A \neq \emptyset, \quad \text{and} \quad f^{-1}(c^*) \cap U_B \neq \emptyset.$$

Since $f^{-1}(c^*) \subset U_A \cap U_B$, this would contradict the assumption that $f^{-1}(c^*) \subset M$ is connected for every $c^* \in P^*$. We can thus assume that $f(x) \leq c$ for all $x \in V_A$ and $f(x) \geq c$ for all $x \in V_B$. Then

$$\widetilde{U}_A \equiv f^{-1}\big((-\infty, c)\big) \cup V_A \qquad \text{and} \qquad \widetilde{U}_B \equiv f^{-1}\big((c, \infty)\big) \cup V_B$$

are disjoint nonempty open subsets of X that cover X. However, this contradicts the assumption that X is connected. $\square$

Appendix B

Bundle Connections

B.1 Connections and splittings

Suppose X is a smooth manifold and $\mathrm{pr}_E\colon E \longrightarrow X$ is a (smooth) real vector bundle. We identify X with the zero section of E. Denote by

$$\mathfrak{a}\colon E\oplus E \longrightarrow E \qquad \text{and} \qquad \mathrm{pr}_{E\oplus E}\colon E\oplus E \longrightarrow X$$

the associated addition map and the induced projection map, respectively. For $f\in C^\infty(X;\mathbb{R})$, define

$$m_f\colon E \longrightarrow E \qquad \text{by} \qquad m_f(v) = f\big(\mathrm{pr}_E(v)\big)\cdot v \quad \forall\, v\in E. \tag{B.1}$$

In particular,

$$\mathrm{pr}_{E\oplus E} = \mathrm{pr}_E\circ\mathfrak{a}, \qquad \mathrm{pr}_E = \mathrm{pr}_E\circ m_f \quad \forall\, f\in C^\infty(X;\mathbb{R}).$$

The total spaces of the vector bundles

$$\mathrm{pr}_{E\oplus E}\colon E\oplus E \longrightarrow X \qquad \text{and} \qquad \mathrm{pr}_E^*E \longrightarrow E$$

consist of the pairs (v,w) in $E\times E$ such that $\mathrm{pr}_E(v)=\mathrm{pr}_E(w)$.

Define a smooth bundle homomorphism

$$\iota_E\colon \mathrm{pr}_E^*E \longrightarrow TE, \qquad \iota_E(v,w) = \frac{\mathrm{d}}{\mathrm{d}t}(v+tw)\Big|_{t=0}. \tag{B.2}$$

Since the restriction of ι_E to the fiber over $v\in E$ is the composition of the isomorphism

$$E_{\mathrm{pr}_E(v)} \longrightarrow T_vE_{\mathrm{pr}_E(v)}, \qquad w \longrightarrow \frac{\mathrm{d}}{\mathrm{d}t}(v+tw)\Big|_{t=0},$$

with the differential of the embedding of the fiber $E_{\mathrm{pr}_E(v)}$ into E, ι_E is an injective bundle homomorphism. Furthermore,

$$\begin{gathered}\mathrm{dpr}_E\circ\iota_E=0\colon \mathrm{pr}_E^*E \longrightarrow \mathrm{pr}_E^*TX, \quad m_f^*\iota_E\circ\mathrm{pr}_E^*m_f = \mathrm{d}m_f\circ\iota_E\colon \mathrm{pr}_E^*E \longrightarrow m_f^*TE,\\ \mathfrak{a}^*\iota_E\circ\mathrm{pr}_{E\oplus E}^*\mathfrak{a} = \mathrm{d}\mathfrak{a}\circ\iota_{E\oplus E}\colon \mathrm{pr}_{E\oplus E}^*(E\oplus E) \longrightarrow \mathfrak{a}^*TE\,,\end{gathered} \tag{B.3}$$

$$TE|_X = TX\oplus\iota_E\big(\mathrm{pr}_E^*E|_X\big) = TX\oplus\iota_E(E)\,. \tag{B.4}$$

Let

$$\zeta_E \in \Gamma(E;TE), \qquad \zeta_E(v) = \iota_E(v,v) \in T_vE, \tag{B.5}$$

be the canonical vertical vector field on E.

Exercise B.1. Suppose $p\in\mathbb{Z}^+$, $\mathrm{pr}_E: E\longrightarrow X$ is a real vector bundle, $\mathcal{U}\subset E$ is a tubular neighborhood of X in E, i.e. $tv\in\mathcal{U}$ whenever $v\in\mathcal{U}$ and $t\in[0,1]$, and ϖ is a closed p-form on $\mathcal{U}$. Show that the family $(m_t^*\varpi)_{t\in[0,1]}$ of p-forms on $\mathcal{U}$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}m_t^*\varpi=m_t^*\big(\mathcal{L}_{t^{-1}\zeta_E}\varpi\big)=\mathrm{d}\big(m_t^*(\iota_{t^{-1}\zeta_E}\varpi)\big)\qquad\forall\,t\in[0,1],$$

where $\mathcal{L}$ is the Lie derivative.

By the first statement in (B.3), the injectivity of ι_E, and surjectivity of dpr_E,

$$0\longrightarrow\mathrm{pr}_E^*E\xrightarrow{\iota_E}TE\xrightarrow{\mathrm{dpr}_E}\mathrm{pr}_E^*TX\longrightarrow0\tag{B.6}$$

is an exact sequence of real vector bundles over E. By the second statement in (B.3), the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{pr}_E^*E \arrow[r, "\iota_E"] \arrow[d, "\mathrm{pr}_E^*m_f"] & TE \arrow[r, "\mathrm{dpr}_E"] \arrow[d, "\mathrm{d}m_f"] & \mathrm{pr}_E^*TX \arrow[r] \arrow[d, "\mathrm{pr}_E^*\mathrm{id}_{TX}"] & 0\\
0 \arrow[r] & \mathrm{pr}_E^*E \arrow[r, "m_f^*\iota_E"] & m_f^*TE \arrow[r, "m_f^*\mathrm{dpr}_E"] & \mathrm{pr}_E^*TX \arrow[r] & 0
\end{tikzcd}\tag{B.7}$$

of real vector bundle homomorphisms over E commutes. By the third statement in (B.3), the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{pr}_{E\oplus E}^*(E\oplus E) \arrow[r, "\iota_{E\oplus E}"] \arrow[d, "\mathrm{pr}_{E\oplus E}^*\mathfrak{a}"] & T(E\oplus E) \arrow[r, "\mathrm{dpr}_{E\oplus E}"] \arrow[d, "\mathrm{d}\mathfrak{a}"] & \mathrm{pr}_{E\oplus E}^*TX \arrow[r] \arrow[d, "\mathrm{pr}_{E\oplus E}^*\mathrm{id}_{TX}"] & 0\\
0 \arrow[r] & \mathrm{pr}_{E\oplus E}^*E \arrow[r, "\mathfrak{a}^*\iota_E"] & \mathfrak{a}^*TE \arrow[r, "\mathfrak{a}^*\mathrm{dpr}_E"] & \mathrm{pr}_{E\oplus E}^*TX \arrow[r] & 0
\end{tikzcd}\tag{B.8}$$

of real vector bundle homomorphisms over $E\oplus E$ commutes.

A connection in E is an $\mathbb{R}$-linear map

$$\begin{split}\nabla:\Gamma(X;E)\longrightarrow\Gamma(X;T^*X\otimes_{\mathbb{R}}E)\qquad\text{s.t.}\\ \nabla(fs)=\mathrm{d}f\otimes s+f\nabla s\quad\forall\,f\in C^\infty(X),\ s\in\Gamma(X;E).\end{split}\tag{B.9}$$

The Leibnitz property (B.9) implies that any two connections in E differ by a 1-form on X. In other words, if ∇ and ∇' are connections in E there exists

$$\begin{split}\theta\in\Gamma\big(X;T^*X\otimes_{\mathbb{R}}\mathrm{Hom}_{\mathbb{R}}(E,E)\big)\qquad\text{s.t.}\\ \nabla'_vs=\nabla_vs+\big\{\theta(v)\big\}\big(s(x)\big)\qquad\forall\,s\in\Gamma(X;E),\ v\in T_xX,\ x\in X.\end{split}\tag{B.10}$$

If U is a neighborhood of $x\in X$ and f is a smooth function on X supported in U such that $f(x)=1$, then

$$\nabla s\big|_x=\nabla\big(fs\big)\big|_x-\mathrm{d}_xf\otimes s(x)\tag{B.11}$$

by (B.9). The right-hand side of (B.11) depends only on $s|_U$. Thus, a connection ∇ in E is a local operator, i.e. the value of $\nabla\xi$ at a point $x\in X$ depends only on the restriction of s to any neighborhood U of x.

Example B.2. Let $\mathrm{pr}_1, \mathrm{pr}_2: E \equiv X \times \mathbb{R}^k \longrightarrow X, \mathbb{R}^k$ be the component projections. The trivial connection in the trivial vector bundle $\mathrm{pr}_1: E \longrightarrow X$ is the map

$$\nabla: \Gamma(X;E) \longrightarrow \Gamma\big(X; T^*X \otimes_{\mathbb{R}} E\big), \quad \nabla s = \big(\mathrm{pr}_1 \circ s, \mathrm{d}(\mathrm{pr}_2 \circ s)\big).$$

Exercise B.3. Suppose ∇, ∇' are connections in real vector bundles $E, E' \longrightarrow X$, respectively. Show that the map

$$\nabla \oplus \nabla': \Gamma\big(X; E \oplus E'\big) \longrightarrow \Gamma\big(X; T^*X \otimes_{\mathbb{R}} (E \oplus E')\big), \quad \nabla \oplus \nabla'(s, s') = \big(\nabla s, \nabla' s'\big),$$

is a connection in the real vector bundle $E \oplus E' \longrightarrow X$.

Exercise B.4. Suppose ∇ is a connection in a real vector bundle $\mathrm{pr}_E: E \longrightarrow X$, $E' \subset E$ is a vector subbundle, and $p: E \longrightarrow E'$ is a linear bundle projection, i.e. p is a bundle homomorphism so that $p(v) = v$ for all $v \in E'$. Show that the map

$$\nabla': \Gamma(X;E') \longrightarrow \Gamma(X; T^*X \otimes_{\mathbb{R}} E'), \qquad \nabla' s = p \circ \nabla s,$$

is a connection in E'.

Exercise B.5. Suppose ∇ is a connection in a real vector bundle $\mathrm{pr}_E: E \longrightarrow X$. Show that

(a) the linear map ∇ extends to a linear map on the E-valued p-forms by

$$\nabla: \Gamma\big(X; \Lambda^p(T^*X) \otimes_{\mathbb{R}} E\big) \longrightarrow \Gamma\big(X; \Lambda^{p+1}(T^*X) \otimes_{\mathbb{R}} E\big), \quad \nabla(\eta \otimes s) = (\mathrm{d}\eta) \otimes s + (-1)^p \eta \otimes (\nabla s);$$

(b) there exists $\kappa_\nabla \in \Gamma(X; \Lambda^2(T^*X) \otimes_{\mathbb{R}} \mathrm{End}_{\mathbb{R}}(E))$ so that

$$\nabla(\nabla \widetilde{\eta}) = \kappa_\nabla \wedge \widetilde{\eta} \quad \forall\, \widetilde{\eta} \in \Gamma\big(X; \Lambda^p(T^*X) \otimes_{\mathbb{R}} E\big),\ p \in \mathbb{Z}^{\geq 0}.$$

Note: the bundle section κ_∇ above is called the curvature of ∇.

Suppose U is an open subset of X and $s_1, \ldots, s_n \in \Gamma(U;E)$ is a frame for E on U, i.e.

$$s_1(x), \ldots, s_n(x) \in E_x$$

is a basis for E_x for all $x \in U$. By definition of ∇, there exist

$$\theta_l^k \in \Gamma(U; T^*U) \qquad \text{s.t.} \qquad \nabla s_\ell = \sum_{k=1}^{k=n} s_k \theta_\ell^k \equiv \sum_{k=1}^{k=n} \theta_\ell^k \otimes s_k \quad \forall\ \ell = 1, \ldots, n.$$

We call

$$\theta \equiv \big(\theta_\ell^k\big)_{k,\ell=1,\ldots,n} \in \Gamma\big(U; T^*U \otimes_{\mathbb{R}} \mathrm{Mat}_n \mathbb{R}\big)$$

the connection 1-form of ∇ with respect to the frame $(s_k)_k$; the entry in the k-th row and ℓ-th column of the matrix θ is θ_ℓ^k.

For an arbitrary section

$$s \equiv \sum_{\ell=1}^{\ell=n} f^\ell s_\ell \in \Gamma(U;E),$$

by (B.9) we have

$$\nabla s = \sum_{k=1}^{k=n} s_k\Big(\mathrm{d}f^k + \sum_{\ell=1}^{\ell=n}\theta_\ell^k f^\ell\Big), \qquad \text{i.e.} \qquad \nabla\big(\underline{s}\cdot\underline{f}^t\big) = \underline{s}\cdot\big\{\mathrm{d}+\theta\big\}\underline{f}^t, \tag{B.12}$$

$$\text{where} \qquad \underline{s} = (s_1,\ldots,s_n), \quad \underline{f} = (f^1,\ldots,f^n). \tag{B.13}$$

This implies that

$$\nabla s\big|_x = \mathrm{pr}_2|_x \circ \mathrm{d}_x s\colon T_xX \longrightarrow E_x \qquad \forall\, x\in X,\ s\in\Gamma(X;E) \text{ s.t. } s(x)=0, \tag{B.14}$$

where $\mathrm{pr}_2|_x\colon T_xE \longrightarrow E_x$ is the projection to the second component in (B.4).

Exercise B.6. With the notation as above, show that

$$\kappa_\nabla\otimes s \equiv \nabla^2 s = \underline{s}\cdot\big(\mathrm{d}\theta+\theta\wedge\theta\big)\cdot\underline{f}^t,$$

where $\theta\wedge\theta \equiv ((\theta\wedge\theta)_\ell^k)_{k,\ell\in[n]}$ is the matrix given by

$$(\theta\wedge\theta)_\ell^k = \sum_{m=1}^{n}\theta_m^k\wedge\theta_\ell^m\,.$$

Lemma B.7. *Suppose X is a smooth manifold and $\mathrm{pr}_E : E \longrightarrow X$ is a real vector bundle. A connection ∇ in E induces a splitting*

$$TE \approx \mathrm{pr}_E^*TX \oplus \mathrm{pr}_E^*E \tag{B.15}$$

of the exact sequence (B.6) extending the splitting (B.4) such that

$$\nabla s\big|_x = \mathrm{pr}_\nabla\circ\mathrm{d}_x s\colon T_xX \longrightarrow E_x \qquad \forall\ s\in\Gamma(X;E),\ x\in X, \tag{B.16}$$

where $\mathrm{pr}_\nabla\colon TE\longrightarrow \mathrm{pr}_E^*E$ is the projection onto the second component in (B.15). Furthermore,

$$\mathrm{d}m_t \approx \mathrm{pr}_E^*\mathrm{id}\oplus\mathrm{pr}_E^*m_t \ \ \forall\ t\in\mathbb{R} \qquad \textit{and} \qquad \mathrm{d}\mathfrak{a} \approx \mathrm{pr}_{E\oplus E}^*\mathrm{id}\oplus\mathrm{pr}_{E\oplus E}^*\mathfrak{a}, \tag{B.17}$$

with respect to the splitting (B.15) and the corresponding splitting for the connection $\nabla\oplus\nabla$ in the real vector bundle $E\oplus E\longrightarrow X$, i.e. these splittings are consistent with the commutative diagrams (B.7) and (B.8).

Proof. Given $x\in X$ and $v\in E_x$, choose $s\in\Gamma(X;E)$ such that $s(x)=v$ and let

$$T_vE^{\mathrm{h}} = \big\{\mathrm{d}_x s(w) - \iota_E\big(\nabla_w s\big)\colon w\in T_xX\big\} \subset T_vE.$$

Since $\mathrm{pr}_E\circ s = \mathrm{id}_X$ and $\mathrm{dpr}_E\circ\iota_E = 0$,

$$\mathrm{d}_v\mathrm{pr}_E\circ\big\{\mathrm{d}s - \iota_E\circ\nabla s\big\}\big|_x = \mathrm{id}_{T_xX} \qquad\Longrightarrow\qquad T_vE \approx T_vE^{\mathrm{h}}\oplus E_x \approx T_xX\oplus E_x.$$

This splitting of T_vE satisfies (B.16) at $v=s(x)$.

With the notation as in (B.12),

$$\big\{\mathrm{d}s - \iota_E\circ\nabla s\big\}\big|_x = \Big(\mathrm{d}_x\mathrm{id}_X, -\sum_{\ell=1}^{\ell=n} f^\ell(x)\theta_\ell^1|_x, \ldots -\sum_{\ell=1}^{\ell=n} f^\ell(x)\theta_\ell^n|_x\Big)\colon T_xX \longrightarrow T_xX\oplus\mathbb{R}^n \tag{B.18}$$

with respect to the identification $E|_U \approx U\times\mathbb{R}^n$ determined by the frame $(s_k)_k$. Thus, T_vE^{h} is independent of the choice of s. Since $T_xE^{\mathrm{h}} = T_xX$ for every $x\in X$, the resulting splitting (B.15) of (B.6) extends (B.4). By (B.18), it also satisfies (B.17). $\square$

Exercise B.8. Suppose $p\in\mathbb{Z}^+$, $\mathrm{pr}_E\colon E\longrightarrow X$ is a real vector bundle, Ω is a fiberwise p-form on E, and ∇ is a connection in E with the associated projection $\mathrm{pr}_\nabla\colon TE\longrightarrow \mathrm{pr}_E^*E$ as in Lemma B.7. Thus, $\Omega_\nabla\equiv\mathrm{pr}_\nabla^*\Omega$ is a p-form on the total space of E. Let $\zeta_E\in\Gamma(E;TE)$ be the canonical vertical vector field on E as in (B.5). Show that

$$\iota_E^*\big(\mathrm{d}(\iota_{\zeta_E}\Omega_\nabla)\big)=p\Omega \qquad\text{and}\qquad \big(\mathrm{d}(\iota_{\zeta_E}\Omega_\nabla)\big)\big|_{T_xX}=0 \;\;\forall\, x\in X. \tag{B.19}$$

Exercise B.9. Suppose $\mathrm{pr}_E\colon E\longrightarrow X$ is a real vector bundle, Ω is a fiberwise linear symplectic (nondegenerate) form on E, ∇ is a connection in E, $E'\subset E$ is a vector subbundle so that $\Omega|_E$ is a nondegenerate 2-form on the fibers of E',

$$E'^{\Omega}\equiv\big\{v\in E\colon \Omega(v,v')=0\;\forall\, v'\in E'\big\}$$

is the Ω-orthogonal complement of E', $p\colon E\longrightarrow E'$ is the linear bundle projection induced by the decomposition $E=E'\oplus E'^{\Omega}$, and $\nabla'\equiv p\circ\nabla$ is the connection in E' induced by ∇ and p as in Exercise B.4. With the notation as in Exercise B.8, show that

$$\iota_{\zeta_{E'}}\big(\Omega|_{E'}\big)_{\nabla'}=\big(\iota_{\zeta_E}\Omega_\nabla\big)\big|_{TE'}\,,\qquad \mathrm{d}\big(\iota_{\zeta_{E'}}\big(\Omega|_{E'}\big)_{\nabla'}\big)=\big(\mathrm{d}\big(\iota_{\zeta_E}\Omega_\nabla\big)\big)\big|_{TE'}.$$

Suppose g is a metric on a real vector bundle $E\longrightarrow X$, i.e.

$$g\in\Gamma\big(X;E^*\otimes_{\mathbb{R}}E^*\big)\quad\text{s.t.}\quad g(v,w)=g(w,v),\;\; g(v,v)>0\quad\forall\, v,w\in E_x,\; v\neq 0,\; x\in X.$$

A connection ∇ in E is g-compatible if

$$\mathrm{d}\big(g(s,s')\big)=g(\nabla s,s')+g(s,\nabla s')\in\Gamma(X;T^*X)\qquad\forall\; s,s'\in\Gamma(X;E).$$

Exercise B.10. Suppose $E\longrightarrow X$ is a vector bundle of rank n, ∇ is a connection in E, g is a metric on E, $U\subset X$ is an open subset, and $s_1,\ldots,s_n\in\Gamma(U;E)$ is a frame for E on U. Let $\theta\equiv(\theta^k_\ell)_{k,\ell\in[n]}$ be the connection 1-form of ∇ with respect to the frame $(s_k)_k$. For $i,j\in[n]$, define

$$g_{ij}=g(s_i,s_j)\in C^\infty(U).$$

Show that ∇ is g-compatible on U if and only if

$$\sum_{k=1}^{k=n}\big(g_{ik}\theta^k_j+g_{jk}\theta^k_i\big)=\mathrm{d}g_{ij}\qquad\forall\; i,j=1,2,\ldots,n. \tag{B.20}$$

B.2 Complex vector bundles

Suppose X is a smooth manifold and $\mathrm{pr}_E : E\longrightarrow X$ is a complex vector bundle. Similarly to Section B.1, there is an exact sequence

$$0\longrightarrow \mathrm{pr}_E^*E\xrightarrow{\iota_E} TE\xrightarrow{\mathrm{dpr}_E}\mathrm{pr}_E^*TX\longrightarrow 0 \tag{B.21}$$

of complex vector bundles over E. The homomorphism ι_E is now $\mathbb{C}$-linear. If $f\in C^\infty(X;\mathbb{C})$ and $m_f\colon E\longrightarrow E$ is defined as in (B.1), there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{pr}_E^*E \arrow[r, "\iota_E"] \arrow[d, "\mathrm{pr}_E^*m_f"] & TE \arrow[r, "\mathrm{dpr}_E"] \arrow[d, "\mathrm{d}m_f"] & \mathrm{pr}_E^*TX \arrow[r] \arrow[d, "\mathrm{pr}_E^*\mathrm{id}_{TX}"] & 0\\
0 \arrow[r] & \mathrm{pr}_E^*E \arrow[r, "m_f^*\iota_E"] & m_f^*TE \arrow[r, "m_f^*\mathrm{dpr}_E"] & \mathrm{pr}_E^*TX \arrow[r] & 0
\end{tikzcd} \tag{B.22}$$

of complex vector bundle maps over E.

Suppose ∇ is a ($\mathbb{C}$-linear) connection in the complex vector bundle $\mathrm{pr}_E\colon E\longrightarrow X$, i.e.

$$\nabla_v(\mathfrak{i}s)=\mathfrak{i}(\nabla_v s)\qquad \forall\, s\in\Gamma(X;E),\ v\in TX.$$

If U is an open subset of X and $s_1,\ldots,s_n\in\Gamma(U;E)$ is a $\mathbb{C}$-frame for E on U, then there exist

$$\theta_\ell^k\in\Gamma\big(U;T^*U\times_{\mathbb{R}}\mathbb{C}\big)\qquad \text{s.t.}\qquad \nabla s_\ell=\sum_{k=1}^{k=n}s_k\theta_\ell^k\equiv\sum_{k=1}^{k=n}\theta_\ell^k\otimes s_k\quad \forall\ \ell=1,\ldots,n.$$

For an arbitrary section

$$s=\sum_{\ell=1}^{\ell=n}f^\ell s_\ell\in\Gamma(U;E),$$

by (B.9) and $\mathbb{C}$-linearity of ∇ we have

$$\nabla\xi=\sum_{k=1}^{k=n}s_k\Big(\mathrm{d}f^k+\sum_{\ell=1}^{\ell=n}\theta_\ell^k f^\ell\Big),\qquad \text{i.e.}\qquad \nabla\big(\underline{s}\cdot\underline{f}^t\big)=\underline{\xi}\cdot\big\{\mathrm{d}+\theta\big\}\underline{f}^t,\tag{B.23}$$

where $\underline{s}$ and $\underline{f}$ are as (B.13).

Exercise B.11. Suppose X is a smooth manifold and $\mathrm{pr}_E\colon E\longrightarrow X$ is a complex vector bundle. Show that a connection ∇ in E induces a splitting of the exact sequence (B.21) of complex bundles as in (B.15) and (B.16) which extends the splitting (B.4) so that

$$\mathrm{d}m_t\approx\mathrm{pr}_E^*\mathrm{id}\oplus\mathrm{pr}_E^*m_t\ \ \forall\ t\in\mathbb{C}\qquad\text{and}\qquad \mathrm{d}\mathfrak{a}\approx\mathrm{pr}_{E\oplus E}^*\mathrm{id}\oplus\mathrm{pr}_{E\oplus E}^*\mathfrak{a},$$

with respect to the splitting (B.15) and the corresponding splitting for the connection $\nabla\oplus\nabla$ in the real vector bundle $E\oplus E\longrightarrow X$.

Exercise B.12. Suppose $p\in\mathbb{Z}^+$, $\mathrm{pr}_E:E\longrightarrow X$ is a complex vector bundle, Ω is a fiberwise p-form on E, and ∇ is a connection in E with the associated projection $\mathrm{pr}_\nabla\colon TE\longrightarrow\mathrm{pr}_E^*E$ as in Lemma B.7. Let $\zeta_E\in\Gamma(E;TE)$ be the canonical vertical vector field on E as in (B.5). Show that

$$\iota_{\mathfrak{i}\zeta_E}\big(\mathrm{d}(\iota_{\zeta_E}\Omega_\nabla)\big)=-\mathrm{d}\big(\iota_{\mathfrak{i}\zeta_E}(\iota_{\zeta_E}\Omega)\big).$$

Hint: use Cartan's formula for the Lie derivative and Exercise B.11.

Let g be a Hermitian metric on E, i.e.

$$g\in\Gamma\big(X;\mathrm{Hom}_{\mathbb{C}}(E\otimes_{\mathbb{C}}\overline{E},\mathbb{C})\big)\quad \text{s.t.}\quad g(v,w)=\overline{g(w,v)},\quad g(v,v)>0\quad \forall\ v,w\in E_x,\ v\neq0,\ x\in X.$$

A ($\mathbb{C}$-linear) connection ∇ in E is g-compatible if

$$\mathrm{d}\big(g(s,s')\big)=g(\nabla s,s')+g(s,\nabla s')\in\Gamma(X;T^*X\otimes_{\mathbb{R}}\mathbb{C})\qquad \forall\ s,s'\in\Gamma(X;E).$$

With the notation as in the previous paragraph, let

$$g_{ij}=g(s_i,s_j)\in C^\infty(U;\mathbb{C})\qquad \forall\ i,j=1,\ldots,n.$$

Then ∇ is g-compatible on U if and only if

$$\sum_{k=1}^{k=n}\big(g_{ik}\theta_j^k+\bar{g}_{jk}\bar{\theta}_i^k\big)=\mathrm{d}g_{ij}\qquad \forall\ i,j=1,2,\ldots,n.\tag{B.24}$$

Exercise B.13. Suppose ∇ is a connection in a complex vector bundle $\mathrm{pr}_E: E \longrightarrow X$. Let $\kappa_\nabla \in \Gamma(X; \Lambda^2(T^*X) \otimes_{\mathbb{R}} \mathrm{End}_{\mathbb{R}}(E))$ be the curvature of ∇ as in Exercise B.5 and $TE^{\mathrm{h}} \subset TE$ be the complement of $\iota_E(\mathrm{pr}_E^* E) \subset TE$ determined by ∇ as in the proof of Lemma B.7.

(a) Show that the splitting (B.15) satisfies the first property in (B.17) for all $t \in \mathbb{C}$ and that $\kappa_\nabla \in \Gamma(X; \Lambda^2(T^*X) \otimes_{\mathbb{R}} \mathrm{End}_{\mathbb{C}}(E))$;

(b) Suppose ∇ is compatible with a Hermitian metric g on E and

$$v \in S_{E,g} \equiv \big\{ w \in E : g(w,w) = 1 \big\}.$$

Show that $T_v E^{\mathrm{h}} \subset T_v S_{E,g}$.

(c) Suppose in addition that $\mathrm{rk}_{\mathbb{C}} E = 1$. Show that κ_∇ is a 2-form on X with values in $\mathfrak{i}\mathbb{R}$.

Exercise B.14. Show that the analogues of Exercises B.3 and B.4 hold for connections in complex vector bundles.

Exercise B.15. Suppose $\mathrm{pr}_E : E \longrightarrow X$ is a complex vector bundle, ω is a symplectic form on X, Ω is a fiberwise linear symplectic (nondegenerate) form on E compatible with the complex structure $\mathfrak{i}$ in E so that

$$g_{\mathfrak{i}}^{\Omega}(\cdot,\cdot) \equiv \Omega(\cdot, \mathfrak{i}\cdot)$$

is an $\mathfrak{i}$-invariant metric on the fibers of E, and ∇ is a ($\mathbb{C}$-linear) connection in E compatible with $g_{\mathfrak{i}}^{\Omega}$. With the notation as in Exercise B.8, show that the closed 2-form

$$\widetilde{\omega} \equiv \mathrm{pr}^*\omega + \frac{1}{2}\mathrm{d}\big(\iota_{\zeta_E}\Omega_\nabla\big)$$

on (the total space of) E is nondegenerate. *Hint.* By choosing a unitary frame for E over an open subset $U \subset E$, it can be assumed that $E = X \times \mathbb{C}^n$, $g_{\mathfrak{i}}^{\Omega}$ is the standard metric on $\mathbb{C}^n$, and Ω is the standard symplectic form on $\mathbb{C}^n$. Show that

$$\Omega\big(v, \theta(w)v\big) = 0 \qquad \forall\, v \in \mathbb{C}^n,\ w \in TX,$$

where θ is the connection 1-form with respect to the canonical frame for E, and then use [56, Proposition 2.25(f)].

B.3 Principal S^1-bundles

Suppose X is a smooth manifold and $\mathrm{pr}_S : S \longrightarrow X$ is a (smooth) principal S^1-bundle. Let

$$\zeta_S \in \Gamma(S; TS), \qquad \zeta_S(v) = \frac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{e}^{2\pi\mathfrak{i}t} \cdot v\big)\Big|_{t=0}, \tag{B.25}$$

be the vector field generating the S^1-action. This vector field generates the vertical tangent bundle of pr_S, i.e.

$$TS^{\mathrm{ver}} \equiv \ker \mathrm{dpr}_S = \big\{ t\zeta_S(v) : v \in S,\ t \in \mathbb{R} \big\} \longrightarrow S.$$

A connection 1-form on S is an S^1-invariant 1-form λ on (the total space of) S such that $\lambda(\zeta_S) = 2\pi$. Such a form determines an S^1-equivariant splitting of the exact sequence

$$0 \longrightarrow TS^{\mathrm{ver}} \longrightarrow TS \xrightarrow{\mathrm{dpr}_S} \mathrm{pr}_S^* TX \longrightarrow 0$$

of real vector bundles over S with

$$TS = TS^{\mathrm{ver}} \oplus (\ker \lambda).$$

Exercise B.16. Suppose $\mathrm{pr}_S\colon S \longrightarrow X$ is a principal S^1-bundle.

(a) Show that the S^1-invariance condition on 1-form λ being a connection 1-form can be equivalently replaced by the condition $\iota_{\zeta_S}\mathrm{d}\lambda = 0$;

(b) Let λ be a connection 1-form on S. Show that there exists a 2-form κ_λ on X so that $\mathrm{d}\lambda = \mathrm{pr}_S^*\kappa_\lambda$.

Note: the 2-form κ_λ above is called the curvature of λ.

A principal S^1-bundle $\mathrm{pr}_S\colon S \longrightarrow X$ determines a complex line bundle

$$\mathrm{pr}_{L_S}\colon L_S \equiv S \times_{S^1} \mathbb{C} \longrightarrow X,$$
$$(v,z) \sim \big(u\cdot v, u^{-1}\cdot z\big)\ \forall\ (v,z) \in S \times \mathbb{C},\ u \in S^1, \qquad \mathrm{pr}_{L_S}\big([v,z]\big) = \mathrm{pr}_S(v),$$

with a Hermitian metric specified by

$$g_S\big([v,z],[v,z']\big) = z\overline{z'}\,.$$

Conversely, a complex line bundle $\mathrm{pr}_L\colon L \longrightarrow X$ with a Hermitian metric g determines a principal S^1-bundle, the unit circle bundle of L,

$$\mathrm{pr}_{S_{L,g}}\colon S_{L,g} \equiv \big\{v \in L\colon g(v,v) = 1\big\} \longrightarrow X, \quad \mathrm{pr}_{S_{L,g}}(v) = \mathrm{pr}_L(v).$$

With S and (L,g) as above, the maps

$$S \longrightarrow S_{L_S,g_S}, \quad v \longrightarrow [v,1], \qquad \text{and} \qquad L_{S_{L_S,g_S}} \longrightarrow L, \quad [v,z] \longrightarrow zv, \tag{B.26}$$

are isomorphism of principal S^1-bundles over X and of complex line bundles with Hermitian metrics over X . Thus, we have constructed a bijective correspondence between the isomorphism classes of principal S^1-bundles over X and the isomorphism classes of complex line bundles with Hermitian metrics over X.

Exercise B.17. Suppose $\mathrm{pr}_S : S \longrightarrow X$ is a principal S^1-bundle and λ is a connection 1-form on S. Let $p\colon S \times \mathbb{C} \longrightarrow L_S$ be the quotient projection and $\iota_{L_S}\colon \mathrm{pr}_{L_S}^* L_S \longrightarrow TL_S$ be as in (B.2) with $E = L_S$. Show that

(a) there is a unique 1-form λ_S on L_S so that

$$p^*\lambda_S\big|_{(v,z)} = |z|^2\lambda_v + \frac{\mathfrak{i}}{2}\big(z\mathrm{d}\overline{z} - \overline{z}\mathrm{d}z\big) \qquad \forall\ (v,z) \in L \times \mathbb{C};$$

(b) $\iota_{L_S}^*(\mathrm{d}\lambda_S) = 2\,\mathrm{Re}\, g_S(\mathfrak{i}\cdot,\cdot)$ and $\iota_v(\mathrm{d}\lambda_S) = 0$ for all $v \in T_xX \subset T_xL_S$, $x \in X$;

(c) there is a unique ($\mathbb{C}$-linear) connection ∇^λ in the complex line bundle $\mathrm{pr}_S\colon L_S \longrightarrow X$ compatible with the Hermitian metric g_S so that the 1-form λ_S vanishes on the complement $TL_S^{\mathrm{h}} \subset TL_S$ of $\iota_{L_S}(\mathrm{pr}_{L_S}^* L_S) \subset TL_S$ determined by ∇^λ as in the proof of Lemma B.7.

Thus, a connection 1-form λ in a principal S^1-bundle $\mathrm{pr}_S\colon S\longrightarrow X$ determines a connection ∇^λ in the associated complex line bundle $\mathrm{pr}_{L_S}\colon L_S\longrightarrow X$ compatible with the Hermitian metric g_S on L_S. Suppose $\mathrm{pr}_L\colon L\longrightarrow X$ is a complex line bundle with a Hermitian metric g and ∇ is a ($\mathbb{C}$-linear) connection in L compatible with g. Let $\zeta_L\in\Gamma(L;TL)$ be the canonical vertical vector field as in (B.5) and $TL^{\mathrm{h}}\subset TL$ be the complement of $\iota_L(\mathrm{pr}_L^*L)\subset TL$ determined by ∇ as in the proof of Lemma B.7. In particular, $2\pi\mathfrak{i}\zeta_L\in\Gamma(L;TL)$ is the vector field generating the S^1-action on L by scalar multiplication. By Exercise B.13, the S^1-action on L preserves the subbundle $TL^{\mathrm{h}}|_{S_{L,g}}$ of $TS_{L,g}$. Thus, the 1-form λ_∇ on $S_{L,g}$ defined by

$$\lambda_\nabla\big(2\pi\mathfrak{i}\zeta_L(v)\big)=2\pi,\quad \lambda_\nabla\big|_{T_vL^{\mathrm{h}}}=0\qquad\forall\,v\in S_{L,g}$$

is a connection 1-form on the principal S^1-bundle $S_{L,g}\longrightarrow X$. By Exercise B.18 below, we have constructed a bijective correspondence between the isomorphism classes of principal S^1-bundles over X with connection 1-forms and the isomorphism classes of complex line bundles with Hermitian metrics over X and compatible connections.

Exercise B.18. Suppose λ is a connection 1-form on a principal S^1-bundle $\mathrm{pr}_S\colon S\longrightarrow X$ and ∇ is a connection in a complex line bundle $\mathrm{pr}_L\colon L\longrightarrow X$ compatible with a Hermitian metric g. Show that

$$\lambda=\lambda_{\nabla^\lambda}\qquad\text{and}\qquad\nabla^{\lambda_\nabla}=\nabla$$

under the isomorphisms (B.26) and that $\kappa_{\nabla^\lambda}=\mathfrak{i}\kappa_\lambda$.

Remark B.19. By a Čech cohomology computation [27, p141] and Exercise B.18,

$$c_1(L)=\frac{\mathfrak{i}}{2\pi}\big[\kappa_\nabla\big]=-\frac{1}{2\pi}\big[\kappa_\lambda\big]\in H^2_{\mathrm{deR}}(X),$$

if ∇ is a connection in a complex line bundle $L\longrightarrow X$ and λ is a connection 1-form in an associated principal S^1-bundle.

B.4 Generalized $\bar\partial$-operators

Let (X,J) be an almost complex manifold. Denote by

$$T^*X^{1,0}\equiv\big\{\eta\in T^*X\otimes_{\mathbb{R}}\mathbb{C}\colon\eta\circ J=\mathfrak{i}\,\eta\big\}\qquad\text{and}\qquad T^*X^{0,1}\equiv\big\{\eta\in T^*X\otimes_{\mathbb{R}}\mathbb{C}\colon\eta\circ J=-\mathfrak{i}\,\eta\big\}$$

the bundles of $\mathbb{C}$-linear and $\mathbb{C}$-antilinear 1-forms on X. If $(E,\widetilde J)$ is another almost complex manifold and $s\colon X\longrightarrow E$ is a smooth function, define

$$\bar\partial_{\widetilde J,J}s\in\Gamma\big(\Sigma;T^*X^{0,1}\otimes_{\mathbb{C}}s^*TE\big)\qquad\text{by}\qquad\bar\partial_{\widetilde J,J}s=\frac12\big(\mathrm{d}s+\widetilde J\circ\mathrm{d}s\circ J\big).\tag{B.27}$$

A smooth map $s\colon(X,J)\longrightarrow(Y,\widetilde J)$ is $(\widetilde J,J)$-holomorphic if $\bar\partial_{\widetilde J,J}s=0$. A $\bar\partial$-operator operator on a complex vector bundle $\mathrm{pr}_E\colon E\longrightarrow X$ is a $\mathbb{C}$-linear map

$$\bar\partial\colon\Gamma(X;E)\longrightarrow\Gamma\big(X;T^*X^{0,1}\otimes_{\mathbb{C}}E\big)$$

such that

$$\bar\partial\big(f\xi\big)=(\bar\partial f)\otimes\xi+f(\bar\partial\xi)\qquad\forall\,f\in C^\infty(X),\ \xi\in\Gamma(X;E),\tag{B.28}$$

where $\bar\partial f = \bar\partial_{\mathfrak{i},J} f$ is the usual $\bar\partial$-operator on complex-valued functions. A ($\mathbb{C}$-linear) connection ∇ in E is $\bar\partial$-compatible if

$$\bar\partial s = \bar\partial_\nabla \xi \equiv \frac{1}{2}\big(\nabla s + \mathfrak{i}\nabla s \circ \mathfrak{j}\big) \qquad \forall\ s \in \Gamma(X;\Sigma). \tag{B.29}$$

Exercise B.20 ([27, p70]). Suppose (X,J) is an almost complex manifold and $\mathrm{pr}_E : E \longrightarrow X$ is a J-holomorphic vector bundle, i.e. a complex vector bundle with a collection of trivializations covering X so that the associated transition maps $g_{\alpha\alpha'} \in C^\infty(U_\alpha \cap U_{\alpha'}; \mathrm{GL}_n\mathbb{C})$ are J-holomorphic. Show that such a collection determines a $\bar\partial$-operator on E.

Exercise B.21 ([61, Lemmas 2.2,2.3]). Suppose (X,J) is an almost complex manifold, $\mathrm{pr}_E : E \longrightarrow X$ is a complex vector bundle, and

$$\bar\partial \colon \Gamma(X;E) \longrightarrow \Gamma\big(X; T^*X^{0,1} \otimes_{\mathbb{C}} E\big)$$

is a $\bar\partial$-operator on E. Show that

(a) there exists an almost complex structure $\widetilde{J} \equiv \widetilde{J}_{\bar\partial}$ on (the total space of) E natural with respect to pullbacks by J-holomorphic maps such that pr is a $(J,\widetilde{J})$-holomorphic map, the restriction of $\widetilde{J}$ to the vertical tangent bundle $TE^{\mathrm{v}} \approx \mathrm{pr}_E^* E \subset TE$ agrees with $\mathfrak{i}$, and

$$\bar\partial_{\widetilde{J},J} s = 0 \in \Gamma\big(X; T^*X^{0,1} \otimes_{\mathbb{C}} s^*TE\big) \qquad \Longleftrightarrow \qquad \bar\partial s = 0 \in \Gamma\big(X; T^*X^{0,1} \otimes_{\mathbb{C}} E\big)$$

for every $s \in \Gamma(U;E)$;

(b) a $\mathbb{C}$-linear connection ∇ in E is $\bar\partial$-compatible if and only if the splitting (B.15) determined by ∇ respects the complex structures.

Hint for (a): assume that E is a trivial complex vector bundle and thus $\bar\partial$ corresponds to a $(0,1)$-form on X with values in $\mathrm{Mat}_n\mathbb{C}$.

Exercise B.22 ([27, p73]). Suppose (X,J) is an almost complex manifold, $\mathrm{pr}_E : E \longrightarrow X$ is a complex vector bundle,

$$\bar\partial \colon \Gamma(X;E) \longrightarrow \Gamma\big(X; T^*X^{0,1} \otimes_{\mathbb{C}} E\big)$$

is a $\bar\partial$-operator on E, and g is a Hermitian metric on E. Show that there exists a unique connection ∇ on E compatible with $\bar\partial$ and g.

Exercise B.23 ([27, p142]). Suppose (X,J) is a complex manifold, $\mathrm{pr}_L : L \longrightarrow X$ is a holomorphic line bundle, g is a Hermitian metric on L, and ∇ is the connection on L compatible with $\bar\partial$ and g as in Exercise B.22. Show that the curvature κ_∇ of ∇ satisfies

$$\kappa_\nabla\big|_U = \bar\partial\partial \log g(s,s)$$

for every open subset $U \subset X$ and a nowhere zero holomorphic section s of L over U. *Hint:* show that

$$\nabla s = \big(\partial \log g(s,s)\big) \otimes s \in \Gamma\big(U; T^*U \otimes_{\mathbb{C}} L\big).$$

Note: along with Exercise 1.27, this implies that $\omega_{\mathrm{FS};n-1} = -\frac{\mathfrak{i}}{2\pi}\kappa_\nabla$, where ∇ is the connection in the tautological holomorphic line bundle $\gamma_{n-1} \longrightarrow \mathbb{C}P^{n-1}$ compatible with its holomorphic structure and the Hermitian metric induced from $\mathbb{C}^n$, and thus this line bundle is *negative*, as defined in [27, p148].

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