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Convexity Properties of the Moment Mapping

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§ 1. Introduction

Let G be a compact Lie group, $\mathfrak{g}$ its Lie algebra and X a compact, connected symplectic manifold. Let $G \times X \to X$ be a Hamiltonian action of G on X and let

$$\Phi\colon X \to \mathfrak{g}^*$$

be its associated moment mapping. (See § 2 for definitions.) The set, $\Phi(X)$, is the union of co-adjoint orbits. The main result of this paper is a description of the orbit structure of this set. To describe this result, we first note that the co-adjoint orbits in $\mathfrak{g}^*$ can be parametrized as follows: Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$ and let B be a positive-definite G-invariant bilinear form on $\mathfrak{g}$. By means of B we get a G-equivariant identification, $\mathfrak{g} \cong \mathfrak{g}^*$. Let $\mathfrak{t}^*$ be the subspace of $\mathfrak{g}^*$ corresponding to $\mathfrak{t}$. Then every co-adjoint orbit intersects $\mathfrak{t}^*$ in a Weyl-group orbit; so there is a one-one correspondence between co-adjoint orbits in $\mathfrak{g}^*$ and Weyl-group orbits in $\mathfrak{t}^*$. To get an even sharper parametrization, let $\mathfrak{t}^*_+$ be a Weyl chamber in $\mathfrak{t}^*$. Then every Weyl-group orbit intersects $\mathfrak{t}^*_+$ in a single point; so $\mathfrak{t}^*_+$ is a "moduli space" for the co-adjoint orbits.

Two orbits in X are said to be of the same type if they are isomorphic as homogeneous G-spaces. Since X is compact, there are at most a finite number of different types of orbits. (See [11].) Let H be a closed subgroup of T of codimension r and let $\mathcal{O}_H$ be the set of all orbits in X of type G/H.

Theorem 1. *$\Phi(\mathcal{O}_H) \cap \mathfrak{t}^*_+$ is the union of a finite number of open r-dimensional convex polytopes.*

We conjecture that $\Phi(X) \cap \mathfrak{t}^*_+$ is itself a convex polytope. We have been able to prove this in some important special cases. (See below.)

We also prove a number of related results. In § 3 we show that if $G = T = $ an r-dimensional torus, then X has a "symplectic stratification" such that Φ maps each stratum submersively onto a polyhedral subset of $\mathfrak{t}^*$. The proof makes use of the following elementary symplectic fact. (See also [2], Theorem 1.)

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Lemma 2. *The range of $d\Phi_x$ is the annihilator in $\mathfrak{g}^*$ of the stabilizer algebra of x.*

In § 4 we show that the moment map has a "local canonical form" in the neighborhood of a fixed point.[1] We use this result to prove the following local convexity theory for $G=T=$an r-torus:

Theorem 3. *Let x be a fixed point of T and let $\alpha_i \in \mathfrak{t}^*$, $i=1, \ldots, n$, be the weights of the linear isotropy representation of T on the tangent space at x. Let*

$$S(\alpha_1, \ldots, \alpha_n)=\{s_1\alpha_1+\ldots+s_n\alpha_n,\ s_1, \ldots, s_n \geqq 0\}.$$

Then there exists a neighborhood, U, of x in X and a neighborhood, V, of $p=\Phi(x)$ in $\mathfrak{t}^*$ such that $\Phi(U)=V\cap(p+S(\alpha_1, \ldots, \alpha_n))$.

In § 5 we prove a global convexity theorem for the case $G=T=$an r-torus.

Theorem 4. *The image of Φ is a convex polytope. Its vertices are the image of the fixed points of T for which $S(\alpha_1, \ldots, \alpha_n) \neq \mathfrak{t}^*$.*

The proof of this theorem is based on the following Morse-theoretic result which is itself of some interest:

Theorem 5. *Let $\xi \in \mathfrak{g}$ and let Φ^ξ be the ξ-th component of Φ. Then Φ^ξ has a unique local maximum.*

In § 6 we will show that the action of G on X admits a "global symplectic cross-section" and use this fact to prove Theorem 1. In § 7 we study the phenomenon of vertex formation at the boundary of $\Phi(X)\cap\mathfrak{t}_+^*$. The interior vertices of $\Phi(X)\cap\mathfrak{t}_+^*$ are associated with fixed points of T just as in the abelian case; however, the boundary vertices are more complicated. To each open face, F, of $\partial\mathfrak{t}_+^*$ corresponds a subgroup H of G such that

$$f \in F \Leftrightarrow H \text{ is the stabilizer group of } f.$$

Let $H=K\times M$ where K is semi-simple and M abelian. If we reduce X with respect to K (in the sense of [9]), we get a symplectic manifold on which M acts. We will show that the vertices on F are associated with the fixed points of M on this reduced manifold.

In Sect. 8 we show that if the action of G on X is "pre-quantizable", the vertices $\Phi(X)\cap\mathfrak{t}_+^*$ are lattice points. Finally, in an appendix we will prove

Theorem 6. *Let X be a connected, non-singular complex projective variety, Ω its Kaehler form and G a closed, connected subgroup of the group of complex symplectic transformations of X. Suppose that for some point, $p\in X$, the stabilizer group of p in G is finite. Then $\Phi(X)\cap\mathfrak{t}_+^*$ is a convex polytope of dimension equal to the rank of G.*

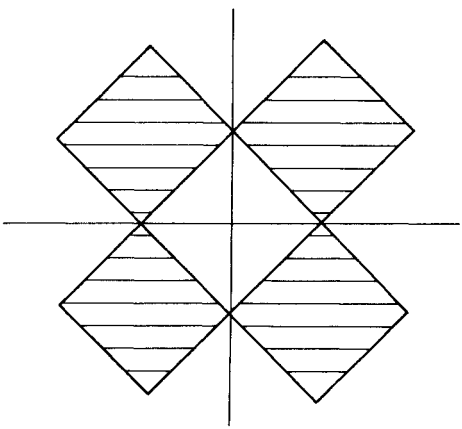
The proof depends upon the fact that one can characterize the *rational* points in $\Phi(X)\cap\mathfrak{t}_+^*$ in terms of the existence of sections of certain line bundles.

The results we have just described were largely inspired by some recent work of Gerrit Heckman: Let K be a compact Lie group and X a co-adjoint

[1] This way also noticed by Arms, Marsden, and Moncrief. See § 2, Lemma 17

orbit in $\mathfrak{k}^*$. If G is a Lie subgroup of K, the projection map, $X \subset \mathfrak{k}^* \to \mathfrak{g}^*$ is the moment map of the action of G on X. Kostant proved several years ago that if $G = T =$ the Cartan subgroup of K, the image of X in $\mathfrak{t}^*$ is a convex polytope. In his thesis Heckman gave a very simple Morse-theoretic proof of this result; and also, for such X's, either proved or conjectured most of the other theorems described above. Our own involvement with these matters started when we noticed that Heckman's proof of Kostant's theorem could be extended with rather little effort to the case of an arbitrary Hamilton T-space[2].

Example:



Shaded region $= \Phi(X) \cap \mathfrak{t}^*$
$G = SO(4)$,
X is a co-adjoint orbit in $so(5)^*$.
The positive quadrant is $\mathfrak{t}^*_+$.

Notice that $\Phi(X) \cap \mathfrak{t}^*$ is *not* convex.

§ 2. Symplectic Preliminaries

Let X be a symplectic manifold with symplectic form, Ω. Given a C^∞ function, f, on X, let ξ_f be the symplectic vector field, $\xi_f \lrcorner \Omega = df$. The map

$$\tau\colon\ C^\infty(X) \to \text{Symplectic vector fields}, \tag{2.1}$$

sending f into ξ_f, is a morphism of the Poisson algebra ($C^\infty(X)$ with its Poisson bracket) into the Lie algebra of symplectic vector fields. Suppose now that a connected Lie group, G, acts symplecticly on X. Let $\mathfrak{g}$ be its Lie algebra. Associated with the action of G is a Lie algebra morphism

$$\beta\colon\ \mathfrak{g} \to \text{Symplectic vector fields}. \tag{2.2}$$

The action of G is said to be *Hamiltonian* if there exists a Lie algebra morphism

$$\alpha\colon\ \mathfrak{g} \to C^\infty(X) \tag{2.3}$$

[2] Atiyah has also proved this result independently. (See [3].) Atiyah's proof has some features in common with ours, but is a good deal shorter due to his more perspicacious use of Morse theory

such that $\alpha=\tau\circ\beta$. To each point, $x\in X$, we can associate an element, l_x, of the dual space, $\mathfrak{g}^*$, by the identity

$$l_x(\xi)=\phi^\xi(x), \qquad \forall\,\xi\in\mathfrak{g} \tag{2.4}$$

where $\phi^\xi=\alpha(\xi)\in C^\infty(X)$. If we let x vary, we get a mapping

$$\Phi\colon X\to\mathfrak{g}^*, \qquad x\to l_x. \tag{2.5}$$

This mapping is called the moment mapping. It is easily seen to be G-equivariant. A somewhat more pedestrian way of describing it is as follows: Let $\xi\in\mathfrak{g}$ and let $e_\xi\colon \mathfrak{g}^*\to\mathbf{R}$ be the linear form on $\mathfrak{g}^*$: $f\in\mathfrak{g}^*\to f(\xi)$. Let $\phi^\xi=e_\xi\circ\Phi$ and let $\xi^\#$ be the vector field on X associated with ξ by (2.2). Then

$$\xi^\# \lrcorner\,\Omega=d\phi^\xi. \tag{2.6}$$

This equation determines ϕ^ξ up to an additive constant.

We will use (2.6) to prove an elementary lemma which is in some sense the key to why the moment map is as rigid as it is.

Lemma 2.1. *Let $x\in X$. Let G_x be the stabilizer group of x in G and let $\mathfrak{g}_x$ be its Lie algebra. Then the image of $d\phi_x\colon T_x\to g^*$ is the annihilator space in $\mathfrak{g}^*$ of $\mathfrak{g}_x$.*

Proof. Consider the map

$$\mathfrak{g}\to T_x, \tag{2.7}$$

which maps ξ into $\xi^\#(x)$, and the linear isomorphism

$$T_x \xrightarrow{\cong} T_x^* \tag{2.7$'$}$$

which sends $v\in T_x$ into $v\lrcorner\,\Omega\in T_x^*$. The composition of (i) and (ii) is a map

$$g\to T_x^*. \tag{2.7$''$}$$

By (2.6) this map is the transpose of $d\Phi_x\colon T_x\to\mathfrak{g}^*$; so the image of $d\Phi_x$ is the annihilator space of the kernel of (2.7)$''$. By (2.7) this kernel is the space of vectors, $\xi\in\mathfrak{g}$ for which $\xi^\#(x)=0$; i.e. it is just the stabilizer algebra of x. Q.E.D.

Corollary. *The moment map is a submersion at x if and only if the stabilizer group of x is discrete* [3].

§3. Global Properties of the Moment Mapping

By combining Lemma 2.1 with some standard facts about actions of compact groups we can, with little effort, obtain quite a lot of information about the moment mapping. Let X be a connected manifold and G a connected compact Lie group acting differentiably on X. For each $x\in X$ let G_x be the stabilizer group of x. We recall the following well-known facts:

[3] Arms, Marsden and Moncrief, [2], also have this result

Proposition 3.1. *Let* $r=\min \dim G_x$. *Let* X_i *be the set of* x*'s for which* $\dim G_x \geqq i+r$. *Then* X_i *is a subset of* X *of codimension* $\geqq i+1$ *for* $i=1,2,\ldots$.

Proof. See Montgomery et al. [10].

Proposition 3.2. *If* X *is compact, then up to conjugacy, only a finite number of subgroups of* G *occur as stabilizer groups of points of* X.

Proof. See Mostow, [11], or Yang, [13].

Proposition 3.3. *Let* H *be a closed subgroup of* G *and let* $X_H=\{x\in X, G_x=H\}$. *Then* X_H *is a submanifold of* X. *Moreover, if* $x\in X_H$, *the tangent space to* X_H *at* x *is the space of* H*-fixed vectors in the tangent space to* X. *If* X *is compact,* X_H *has finite connectivity.*

Proof. See Sternberg [12], p. 212.

Suppose now that X is a symplectic manifold and the action of G on X is Hamiltonian. Let Φ: $X\to\mathfrak{g}^*$ be the moment mapping. Combining proposition with Lemma 2.1 we get

Theorem 3.4. *Suppose* $\min \operatorname{corank} d\Phi_x=r$. *Let* S_i *be the set of* $x\in X$ *for which* $\operatorname{corank} d\Phi_x \geqq r+i$. *Then* S_i *is of codimension* $\geqq i+1$ *in* X *for* $i=1,2,\ldots$.

Corollary. *Suppose* $d\Phi$ *is surjective at some point,* x_0. *Then the set of* $x\in X$ *at which* $d\Phi$ *is surjective is open and connected, and its complement is of codimension* $\geqq 2$.

Theorem 3.5. *Let* H *be a closed subgroup of* G *and let* $X_H=\{x\in X, G_x=H\}$. *Then* X_H *is a symplectic submanifold of* X.

Proof. This is an immediate consequence of Proposition 3.3 and the following elementary lemma.

Lemma 3.6. *Let* V *be a symplectic vector space and* H *a compact subgroup of* $Sp(V)$. *Let* W *be the subspace of* H*-fixed vectors in* V. *Then* W *is symplectic.*

Proof. Let Ω be the symplectic form on V and let B be a positive-definite bilinear form on V which is H-invariant. There exists a unique linear map A: $V\to V$ such that for all $v, w\in V$

$$B(v,w)=\Omega(v,Aw).$$

Clearly this map is H-invariant. Suppose $v\in W$ and $\Omega(v,w)=0$ for all $w\in W$. If $w\in W$, $Aw\in W$; so $B(v,w)=\Omega(v,Aw)=0$ for all $w\in W$. Thus $v=0$. Q.E.D.

Combining Theorem 3.5 with Lemma 2.1 we get

Theorem 3.6. *Let* $\mathfrak{h}$ *be the Lie algebra of* H *and let* $h^\perp$ *be its annihilator in* $\mathfrak{g}^*$. *Then* Φ *maps each connected component of* X_H *into an affine subspace of* $\mathfrak{g}^*$ *of the form* $p+\mathfrak{h}^\perp$. *Moreover, if* H *is a normal subgroup of* G, *this map is a submersion.*

Proof. By Lemma 2.1, $d\Phi_x$ maps the tangent space to X at x onto $\mathfrak{h}^\perp$ for all $x\in X_H$; so the first part of the theorem is obvious.

If H is a normal subgroup of G, X_H is G-invariant, and the restriction, Φ_1, of Φ to X_H is the moment map of the action of G on X_H. By Lemma 2.1, $(d\Phi_1)_x$ maps the tangent space to X_H at x *onto* the space, $\mathfrak{h}^\perp$. Q.E.D.

Remark. Of special interest is the case, $H=G$. The theorem then says that $\Phi(X_G)$ is discrete, and in fact, finite if X is compact. The points in $\Phi(X_G)$ are called the *vertices* of Φ.

If X is compact and $G=T=$an r-dimensional torus, the manifold, X, can be stratified so that on each stratum, Φ has an extremely simple form. Explicitly, let $T_1, \ldots, T_N$ be the subgroups of T which occur as stabilizer groups of points of X. (By Proposition 3.2 these groups are finite in number.) Let X_i be the set of points, x, for which the stabilizer group of X is T_i. By relabeling we can assume that the X_i's are connected. (The T_i's then get relabeled somewhat redundantly; i.e. T_i can equal T_j for $i \neq j$.) With this notation:

$$X = \bigcup_{i=1}^{N} X_i \quad \text{(disjoint union)}. \tag{3.1}$$

Theorem 3.7. *Each X_i is a T-invariant symplectic submanifold of X. Moreover there exists a vector, $a_i \in \mathfrak{t}^*$, such that Φ maps X_i onto an open subset of the affine subspace, $a_i + \mathfrak{t}_i^\perp$ of $\mathfrak{t}^*$.*

Proof. Since T_i is a normal subgroup of T, Theorem 3.6 applies. Q.E.D.

Let $v_1, \ldots, v_k$ be the vertices of Φ: $X \to \mathfrak{t}^*$. As a corollary of Theorem 3.7 we get:

Theorem 3.8. *$\Phi(X_i)$ is the union of a finite number of convex sets each of which is the convex hull of a collection of v_k's.*

Proof. We will prove the assertion by induction on the codimension of T_i in T. For $T_i = T$, $\Phi(X_i)$ is v_k for some k. Suppose the theorem is true when the codimension is less than m. If the codimension of T_i is m, then by Theorem 3.7, $\Phi(X_i)$ is an open subset of the m-dimensional affine plane $a_i + \mathfrak{t}_i^\perp$, and its boundary components are connected components of the set

$$\bigcup \Phi(X_j), \qquad T_j \supset T_i.$$

By induction the theorem is true for each $\Phi(X_j)$; so it is true for $\Phi(X_i)$. Q.E.D.

In Sect. 5 we will prove that each $\Phi(X_i)$ (including $\Phi(X)$ itself) is the convex hull of certain of the v_k's. Not all the v_k's are necessarily genuine vertices of $\Phi(X)$. Some can lie in the interior of $\Phi(X)$. A criterion for a vertex of $\Phi(X)$ to be a genuine vertex will be given in Sect. 4.

§4. Local Convexity Results

Let X be a manifold, G a compact Lie group and $G \times X \to X$ a differentiable G-action. Let $x \in X$ be a fixed point of G. We will need below the following G-equivariant Darboux theorem.

Theorem 4.1. *Let Ω_0 and Ω_1 be G-invariant symplectic forms on X with $\Omega_0 = \Omega_1$ at x. Then there exists a G-invariant neighborhood, U_0, of x and a G-equivariant map $\Psi: (U_0, x) \to (X, x)$ such that $\Psi^* \Omega_1 = \Omega_0$.*

Proof. The proof is by an argument due to Moser and Weinstein. This argument has by now become fairly standard; so we will just give it in its bare outlines: Join Ω_0 to Ω_1 by the linear homotopy, $\Omega_t = (1-t)\Omega_0 + t\Omega_1$. Let us attempt to find a G-invariant neighborhood, U, of x and a G-equivariant mapping $\Psi_t: (U, x) \to (X, x)$ depending smoothly on t such that $\Psi_t^* \Omega_t = \Omega_0$. Let

$$\xi_t = \Psi_t^{-1}(d\Psi_t/dt). \tag{4.1}$$

By differentiating the equation $\Psi_t^* \Omega_t = \Omega_0$ we get

$$\mathscr{L}_{\xi_t} \Omega_t = d(\xi_t \lrcorner \Omega_t) = -\Omega_t = \Omega_1 - \Omega_0. \tag{4.2}$$

The two form, $\Omega_1 - \Omega_0$, is closed, G-invariant and zero at x. Restrict it to a contractible, G-invariant neighborhood, U, of x. By the Poincare lemma we can find a smooth one-form, α_t, on U depending smoothly on t such that $\Omega_1 - \Omega_0 = d\alpha_t$ and $\alpha_t = 0$ at x. By "averaging over the group G" we can arrange that α_t is G-invariant. Since $\Omega_t = \Omega_0 = \Omega_1$ at x, Ω_t can be assumed to be non-degenerate on U. Let ξ_t be the unique vector field on U satisfying

$$\xi_t \lrcorner \Omega_t = \alpha_t. \tag{4.3}$$

Then (4.2) is clearly satisfied. Restricting, if necessary, to a smaller G-invariant neighborhood, U_0, we can integrate the ODE (4.1) with intitial conditions, $\Psi_0 =$ identity, to obtain a map $\Psi_t: (U_0, x) \to (X, x)$ satisfying $(d/dt)\Psi_t^* \Omega_t = 0$. Since $\Psi_t^* \Omega_t = \Omega_0$ when $t = 0$, $\Psi_1^* \Omega_1 = \Omega_0$. Q.E.D.

Now let X be a symplectic manifold with symplectic form, Ω, G a compact Lie group and $G \times X \to X$ a Hamiltonian G-action. Let $\Phi: X \to \mathfrak{g}^*$ be the associated moment mapping and let x be a fixed point of G. We will use the result above to show that Φ is right-equivalent to the moment mapping associated with the linear action of G on the tangent space $V = T_x X$. Let X be equipped with a G-invariant Riemannian metric and let

$$\exp: V \to X \tag{4.4}$$

be the exponential map defined by this metric. This map intertwines the action of G on X and the linear action of G on V, and it maps a G-invariant neighborhood, U_0, of zero in V diffeomorphically onto a neighborhood, U, of x in X. Let Ω_1 be the linear symplectic form on V and let $\Omega_0 = (\exp)^* \Omega$. For U_0 sufficiently small there exists, by Theorem 4.1, a G-equivariant mapping Ψ: $(U_0, 0) \to (V, 0)$ such that $\Psi^* \Omega_1 = \Omega_0$. Let $\Phi_0 = \Phi \exp$ and let Φ_1 be the moment mapping associated with the linear action of G on V and the linear symplectic form, Ω_1. From (2.6) we obtain:

Theorem 4.2. *The mappings $\Phi_0: U_0 \to \mathfrak{g}^*$ and $\Phi_1: U_1 \to \mathfrak{g}^*$ differ by a translation.*

Corollary. *Let $U_1 = \Psi(U_0)$. Then the image of Φ: $U_0 \to \mathfrak{g}^*$ is, up to a translation identical with the image of Φ_1: $U_1 \to \mathfrak{g}^*$.*[4]

The moment mapping, Φ_1: $V \to \mathfrak{g}^*$, is rather easy to describe. Let $Sp(V)$ be the group of symplectic linear transformations of V and let $sp(V)$ be its Lie algebra. There is a standard identification of $sp(V)$ with the space of homogeneous quadratic polynomials of V. (See [1].) For each point $p \in V$, let $\Phi_2(p) \in sp(V)^*$ be the linear functional on $sp(V)$ which assigns to each quadratic polynomial its value at p. As p varies, we get a mapping

$$\Phi_2\colon V \to sp(V)^*. \tag{4.5}$$

Lemma 4.3. *The mapping, Φ_2, is the moment mapping associated with the linear symplectic action of $Sp(V)$ on V.*

Proof. Left to the reader.

Since G injects into $Sp(V)$ via its linear isotropy action, $\mathfrak{g}$ injects into $sp(V)$ and by duality we get a linear map:

$$sp(V)^* \to \mathfrak{g}^*. \tag{4.6}$$

Proposition 4.4. *The moment mapping Φ_1: $V \to \mathfrak{g}^*$ is the composition of* (4.5) *and* (4.6).

Proof. Immediate consequence of Lemma 4.3.

The maximal compact subgroup of $Sp(V)$ is $U(V)$. Let H be the Cartan subgroup of $U(V)$ and $\mathfrak{h}$ its Lie algebra. As we have already pointed out $sp(V)$ can be identified with the space of quadratic polynomials on V. It is possible to choose a system of linear symplectic coordinates, $x_1, \ldots, x_n$, $y_1, \ldots, y_n$, on V such that

$$\begin{aligned} \mathfrak{h} &\cong \mathbf{R}^n \\ &= \{a_1((x_1^2 + y_1^2)/2) + \ldots + a_n((x_n^2 + y_n^2)/2), (a_1, \ldots, a_n) \in \mathbf{R}^n\}. \end{aligned} \tag{4.7}$$

Suppose now that $G = T =$ an r-dimensional torus. One can choose the group, H, so that the image of T in $Sp(V)$ lies in H. Since T injects in H, $\mathfrak{t}$ injects in $\mathfrak{h}$; and, by duality, there is a linear map the other way

$$\mathfrak{h}^* \cong (\mathbf{R}^n)^* \to \mathfrak{t}^* \to 0. \tag{4.8}$$

Proposition 4.5. *The moment mapping Φ_1: $V \to \mathfrak{t}^*$ is the composition of the quadratic mapping*

$$(x, y) \to ((x_1^2 + y_1^2)/2, \ldots, (x_n^2 + y_n^2)/2) \tag{4.9}$$

and the linear mapping (4.8).

Proof. This is just a restatement of Proposition 4.4.

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[4] The recent paper, [2], of Arms, Marsden and Moncrief contains a result similar to Theorem 4.2

We can describe the mapping in still another way: Let ι: $\mathfrak{t} \to \mathfrak{h}$ be the inclusion of $\mathfrak{t}$ in $\mathfrak{h}$. Given $\xi \in \mathfrak{t}$ let

$$\iota(\xi) = \sum \alpha_i(\xi)(x_i^2 + y_i^2)/2. \tag{4.10}$$

The α_i's are linear functionals on $\mathfrak{t}$ or, in other words, elements of $\mathfrak{t}^*$. By definition they are the *weights* of the representation of T on V. Then Proposition 4.5 amounts to the statement:

Proposition 4.6. *Let $\alpha_1, \ldots, \alpha_n$ be the weights of the representation of T on V. Then, in terms of the (x, y) coordinates, the moment mapping Φ_1: $V \to \mathfrak{t}^*$ is the mapping*

$$(x, y) \to \sum ((x_i^2 + y_i^2)/2)\alpha_i. \tag{4.11}$$

Corollary. *The image of Φ_1 is the convex region*

$$S(\alpha_1, \ldots, \alpha_n) = \left\{ \sum_{i=1}^{n} s_i \alpha_i, s_1, \ldots, s_n \geqq 0 \right\} \tag{4.12}$$

in $\mathfrak{t}^*$, the α_i's being the weights of the representation of T on V.

Combining this result with Theorem 4.2 we obtain the main theorem of this section:

Theorem 4.7 *(Local Convexity Theorem). Let X be a symplectic manifold, T an r-dimensional torus, $T \times X \to X$ a Hamiltonian action of T on X and Φ: $X \to \mathfrak{t}^*$ the associated moment mapping. Let x be a fixed point of T and let $p = \Phi(x)$. Then there exists a neighborhood, U, of x in X and a neighborhood U' of p in $\mathfrak{t}^*$ such that*

$$\Phi(U) = U' \cap (p + S(\alpha_1, \ldots, \alpha_n))$$

where $\alpha_1, \ldots, \alpha_n$ are the weights associated with the linear isotropy representation of T on the tangent space of X at x.

We will need later on a relative form of this theorem. As above let X be a symplectic manifold, T an r-dimensional torus, $T \times X \to X$ a Hamiltonian action of T on X and x a point of X. Let T_1 be the stabilizer group of x. The inclusion of $\mathfrak{t}_1$ in $\mathfrak{t}$ dualizes to give a linear mapping the other way

$$\pi\colon \mathfrak{t}^* \to \mathfrak{t}_1^*. \tag{4.12}$$

Let $\alpha_1, \ldots, \alpha_n$ be the weights of the representation of T_1 on the tangent space of X at x. Let $S(\alpha_1, \ldots, \alpha_n)$ be the convex region (4.12) in $\mathfrak{t}_1^*$ and let

$$S'(\alpha_1, \ldots, \alpha_n) = \pi^{-1} S(\alpha_1, \ldots, \alpha_n)$$

in $\mathfrak{t}^*$.

Theorem 4.8. *There exists a neighborhood, U, of x in X and a neighborhood, U', of $p = \Phi(x)$ in $\mathfrak{t}^*$ such that*

$$\Phi(U) = U' \cap (p + S'(\alpha_1, \ldots, \alpha_n)).$$

Proof. Let V be the tangent space to X at x. We will make a few simplifying assumptions. We will assume: i) X is an open subset of V; ii) x is the origin in V; iii) the action of T_1 is the linear isotropy action, and iv) the symplectic form on X is the restriction to X of the linear symplectic form on V. These assumptions are warranted by Theorem 4.1. Let X_1 be the subset of X consisting of all points in X which have T_1 as stabilizer group. It is clear that X_1 is the intersection of X with a linear subspace W of V. By Theorem 3.6, Φ: $X \to \mathfrak{t}^*$ maps X_1 submersively onto an open subset of $p + \mathfrak{t}_1^\perp$. In particular

$$d\Phi_0 \colon W \to t_1^\perp \to 0. \tag{4.13}$$

Let $\Phi_1 = \pi \circ \Phi =$ the moment map associated with the linear action of T_1 on V. The action of T_1 on W is trivial; so Φ_1 is constant on the affine subspaces

$$W + a, \quad a \in V \tag{4.14}$$

of V. Therefore Φ maps the affine subspace, (4.14), into the affine subspace

$$\pi^{-1}(\Phi_1(a)) = \mathfrak{t}_1^\perp + q, \quad q = \Phi(a). \tag{4.15}$$

By (4.13) this mapping is a submersion near $a = 0$; so the image of (4.14) contains an open neighborhood of q in (4.15). This being true for all points, a, sufficiently close to zero, the image of Φ: $X \to \mathfrak{t}^*$ contains an open neighborhood of $\Phi(x)$ in $\pi^{-1}(\Phi_1(X))$. Q.E.D.

§5. Global Convexity

Let X be a connected compact symplectic manifold, G a compact Lie group, $G \times X \to X$ a Hamiltonian action of G on X and Φ: $X \to \mathfrak{g}^*$ the associated moment mapping. For each $\xi \in \mathfrak{g}$ let ϕ^ξ be the ξ-th component of Φ. (See (2.6).) In this section we will prove the first of our main convexity results. The proof is based on a lemma concerning the maximum values of the function, ϕ^ξ. Variants of this lemma occur in all the papers cited in the introduction, i.e. Kostant [7], Heckman [5], and Atiyah [3].

Definition. Let X be a manifold and f: $X \to \mathbf{R}$ a smooth function. A real number, a, is a *local maximum* of f if there exists a point $x_0 \in X$ and a neighborhood, U, of x such that $f(x_0) = a$ and $f(x) \leqq a$ for $x \in U$.

Lemma 5.1. *The function, ϕ^ξ, has a unique local maximum.*

Before proving this lemma we will make an application of it. Let $G = T =$ an r-torus.

Theorem 5.2. *The image of the moment mapping, Φ: $X \to \mathfrak{t}^*$, is a convex polytope.*

Proof. Let p be a point of the boundary of $\Phi(X)$ and let $x \in X$ be a pre-image point of p. Let T_1 be the stabilizer group of x, and let

$$\alpha_i \in \mathfrak{t}_i^*, \quad i = 1, \ldots, n$$

be the weights of the isotropy representation of T_1 on the tangent space to X at x. By Theorem 4.8 there exists a neighborhood, U, of x and a neighborhood, U', of p such that $\Phi(U)=U'\cap(p+S'(\alpha_1,\ldots,\alpha_n))$. For each $\xi\in\mathfrak{t}$, let e_ξ: $\mathfrak{t}^*\to\mathbf{R}$ be the linear functional, $f\in\mathfrak{t}^*\to f(\xi)$. Let S_i be a boundary component of $S'(\alpha_1,\ldots,\alpha_n)$. We can choose $\xi\in\mathfrak{t}$ such that $e_\xi=0$ on S_i and e_ξ is negative on the interior of $S'(\alpha_1,\ldots,\alpha_n)$. Then if $e_\xi(p)=a$,

$$\phi^\xi(x)=(e_\xi\circ\Phi)(x)\leqq a$$

for $x\in U$; so a is a local maximum of ϕ^ξ. By Lemma 5.1, $\phi^\xi(X)\leqq a$. Applying this argument to all the faces, S_i, of $S'(\alpha_1,\ldots,\alpha_n)$ we conclude that $\Phi(X)\subseteq p+S'(\alpha_1,\ldots,\alpha_n)$. This proves that $\Phi(x)$ behaves as a convex set relative to its boundary and hence is convex. Q.E.D.

To prove Lemma 5.1 we will need to review some elementary Morse theory. Let X be a connected compact n-dimensional manifold and f: $X\to\mathbf{R}$ a smooth function. Let C_f be the critical set of f.

Definition. f is *clean* if each connected component of C_f is a submanifold of X and if in addition at each critical point, $x\in C_f$, the Hessian, d^2f_x, of f is non-degenerate in directions normal to C_f at x.

Let C be a connected component of C_f. If f is clean the index of d^2f_x is constant along C; and its value is called the *index of the critical set, C*.

Suppose X is equipped with a Riemannian metric. Then f defines a gradient vector field on X. Let ψ_t: $X\to X$, $-\infty<t<\infty$ be the flow generated by this gradient vector field; and, for each component, C_i, of C_f let

$$W_i=\{x\in X,\psi_t(x)\to C_i \text{ as } t\to+\infty\}.$$

One of the fundamental theorems of Morse theory is the following. (See, for instance, [4].)

Theorem. *If f is clean then each W_i is an immersed cell of dimension equal to the index plus the dimension of C_i, and*

$$X=\bigcup W_i \quad \text{(disjoint union)} \tag{5.1}$$

is a cell decomposition of X.

Corollary. *Let f: $X\to\mathbf{R}$ be clean. Suppose that the indices of all the critical manifolds of f are even. Then f has a unique local maximum.*

Proof. Let $C_1,\ldots,C_k$ be the critical manifolds of local maxima n and $C_{k+1},\ldots,C_N$ the remaining critical manifolds. Then $f=\text{constant}=a_i$ on C_i, and $a_1,\ldots,a_k$ are the local maxima of f. By the theorem quoted above

$$X=W_1\cup\ldots\cup W_k\cup W_{k+1}\cup\ldots\cup W_N$$

with W_i open for $i\leqq k$ and W_i of codimension $\geqq 2$ for $i>k$. But if W_i is of codimension $\geqq 2$ it cannot disconnect X, so

$$X-\bigcup_{i>k} W_i$$

is connected. Hence $k=1$. Q.E.D.

Therefore, to prove the Lemma 5.1 it is enough to prove

Theorem 5.3. *The function, ϕ^ξ, is clean and the indices of its critical manifolds are all even.*

Proof. Let $\xi^\#$ be the vector field on X associated with ξ by (2.2) and let ρ_t: $X \to X$, $-\infty < t < \infty$, be the flow generated by $\xi^\#$. If x is a critical point of ϕ^ξ, then $\xi^\#(x) = 0$ by (2.6) and hence $\rho_t(x) = x$ for all t. Let V be the tangent space to X at x and let L: $V \to V$ be the infinitesimal generator of the linearized flow

$$(d\rho_t)_x\colon\ V \to V. \tag{5.2}$$

Lemma 5.4. *The Hessian of ϕ^ξ at x is the quadratic form*

$$v \in V \to \Omega_x(Lv, v). \tag{5.3}$$

Proof. Left to the reader (See [1], p. 572.)

Let X be equipped with a G-invariant Riemannian metric and let

$$\exp\colon\ V \to X \tag{5.4}$$

be the exponential map. This map intertwines the flow, ρ_t, on X and the linearized flow, (5.2), on V. Therefore, if W is the subspace of V on which $L = 0$, (5.4) maps a neighborhood of the origin in W diffeomorphically onto a neighborhood of the zero set of $\xi^\#$ on X. By (2.6) the zero set of $\xi^\#$ is identical with the critical set of ϕ^ξ; so we have proved that the connected components of the critical set are manifolds. Moreover, by Lemma 5.4, $d^2\phi^\xi_x$ is non-degenerate on V/W; so the critical manifold is "clean" at x. Finally it is clear from (4.7) that the index of $d^2\phi^\xi_x$ is even. Q.E.D.

We will illustrate Theorem 5.2 with a simple example. Let $X = \mathbf{C}P^N$ and let T be the $(N+1)$-dimensional torus group

$$\{(e^{i\theta_1}, \ldots, e^{i\theta_{N+1}}), \theta_1, \ldots, \theta_{N+1} \in \mathbf{R}\}.$$

T acts on $\mathbf{C}P^N$ as a group of collineations, and it is easy to see that it preserves the Kaehler form associated with the Hermitian inner product $\langle z, w\rangle = z_1\bar{w}_1 + \ldots + z_{N+1}\bar{w}_{N+1}$ on $\mathbf{C}^{N+1}$. The one-parameter subgroup

$$T_r = \{(0, \ldots, e^{i\theta_r}, \ldots, 0)\}$$

is globally Hamiltonian, and in terms of homogeneous coordinates its generating function is $|z_r|^2/|z|^2$; so the action of T itself is globally Hamiltonian and its associated moment mapping, Φ: $X \to \mathbf{R}^{N+1}$ is

$$(z_1, \ldots, z_{N+1}) \to (|z_1|^2, \ldots, |z_{N+1}|^2)/|z|^2.$$

Its image is the simplex

$$\{(s_1, \ldots, s_{N+1}),\ s_1 + \ldots + s_{N+1} = 1,\ s_1, \ldots, s_{N+1} \geqq 0\}$$

and the vertices of this simplex are the images of the fixed points, $x_i = (0, \ldots, z_i, \ldots, 0)$, of T.

§ 6. The Moment Mapping and the Cartan Subgroup of G

Let G be a connected, compact Lie group and T a Cartan subgroup of G. Let G act on $\mathfrak{g}^*$ by its co-adjoint action and let $\mathfrak{t}^*$ be the subspace of $\mathfrak{g}^*$ consisting of vectors in $\mathfrak{g}^*$ which are stabilized by T. Let $\mathfrak{t}^*_{\text{reg}}$ be the set of vectors in $\mathfrak{g}^*$ whose stabilizer group is precisely T. It is easy to see that

$$\mathfrak{t}^*_{\text{reg}} = \mathfrak{t}^* - \text{a finite number of hyperplanes.}$$

We will denote by $\mathfrak{t}^*_0$ one of the connected components of $\mathfrak{t}^*_{\text{reg}}$ (for instance the interior of the positive Weyl chamber) and by $\mathfrak{t}^*_+$ its closure in $\mathfrak{t}^*$.

Now let X be a compact connected manifold on which G acts in a Hamiltonian fashion, and let $\Phi: X \to \mathfrak{g}^*$ be the associated moment mapping. Let H be a closed connected subgroup of T and let

$$X_H = \{x \in X, \text{ the stabilizer group of } x \text{ in } G \text{ is } H\}.$$

We will prove the following theorem in this section.

Theorem 6.1. *$\Phi(X_H) \cap \mathfrak{t}^*_+$ is a finite union of r-dimensional convex polytopes where $r = \dim T/H$.*

The first step in the proof will be to describe $\Phi(X_H)$ more explicitly. Let $\mathfrak{h}$ be the Lie algebra of H and let $\mathfrak{h}^\perp$ be the annihilator of $\mathfrak{h}$ in $\mathfrak{g}^*$. Let $\mathfrak{k}^*$ be the subspace of $\mathfrak{g}^*$ consisting of all vectors, $v \in \mathfrak{g}^*$, which are stabilized by H.

Theorem 6.2. *Let $X_1, \ldots, X_m$ be the connected components of X_H. Then there exist points, $p_1, \ldots, p_m$ in $\mathfrak{k}^*$ such that Φ maps X_i submersively onto an open subset of $\mathfrak{k}^* \cap (p_i + \mathfrak{h}^\perp)$.*

Proof. Let $\alpha_1, \ldots, \alpha_n$ be the positive roots of the Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}^*$ can be decomposed into a direct sum of T-invariant subspaces

$$\mathfrak{g}^* = \mathfrak{t}^* + \sum_{i=1}^{n} \mathfrak{g}_i^* \tag{6.1}$$

each $\mathfrak{g}_i^*$ being two-dimensional and the representation of $\mathfrak{t}$ on $\mathfrak{g}_i^*$ being the representation

$$\xi \in \mathfrak{t} \to \alpha_i(\xi) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let $\phi_i: X \to \mathfrak{g}_i^*$ be the composition of the moment mapping and the projection of $\mathfrak{g}^*$ onto $\mathfrak{g}_i^*$. We can think of ϕ_i as a complex valued function which transforms under T according to the rule

$$\mathscr{L}_{\xi^\#} \phi_i = \sqrt{-1}\, \alpha_i(\xi)\, \phi_i. \tag{6.2}$$

If $\xi \in \mathfrak{h}$, then $\xi^\# = 0$ on X_H; so by (6.2) $\phi_i \equiv 0$ on X_H if $\alpha_i(\xi) \neq 0$. This shows that the image of X_H lies in $\mathfrak{k}^*$. To prove the second half of the theorem, let K be the centralizer of H in G. We can canonically identify $\mathfrak{k}^*$ with the dual of the

Lie algebra of K. Since K preserves X_H we can view X_H as a Hamiltonian K-space. By Theorem 3.6, for each connected component, X_i, of X_H there exists a point, p_i, in $\mathfrak{k}^*$ such that Φ_K maps X_H submersively onto an open subset of $\mathfrak{k}^* \cap (p_i + \mathfrak{h}^\perp)$. Q.E.D.

Applying this argument to the boundary components of $\bar{X}_H$ we get:

Corollary. *If the centralizer of H is T, there exist $p_1, \ldots, p_m \in \mathfrak{t}^*$ such that Φ maps X_i onto an open, convex polyhedral subregion of $\mathfrak{t}^* \cap (p_i + \mathfrak{h}^\perp)$.*

To prove Theorem 6.1, we observe first of all that if the centralizer of H in G is T, Theorem 6.1 is a special case of the corollary to Theorem 6.2. If the centralizer of H is bigger than T but is not equal to G, we can assume the theorem is true by induction. (Replace G by K in Theorem 6.2.) Therefore we are reduced to proving the theorem when G centralizes H. Furthermore, by Theorem 6.2 we can replace G by G/H in Theorem 6.1 or, in other words, can assume H is trivial. For the theorem to be non-vacuous, there has to exist some point $p \in X$ such that the stabilizer group of p is trivial. If such a point exists, then by Theorem 3.1 the set of points for which this is not true is of codimension $\geqq 2$ in X. Therefore to prove Theorem 6.1 it is enough to prove

Theorem 6.3. *Let X be a compact connected Hamiltonian G-space with the property that there exists some point, $p \in X$, such that the stabilizer of p in G is trivial. Then $\Phi(X) \cap \mathfrak{t}_+^*$ is a finite union of convex polytopes.*

An element, $\xi \in \mathfrak{g}^*$ is *regular* if its stabilizer is a Cartan subgroup of G. Every regular element of $\mathfrak{g}^*$ is conjugate to a unique element of $\mathfrak{t}_0^*$. Let $\mathfrak{g}_{\text{reg}}^*$ be the set of all regular elements of $\mathfrak{g}^*$ and let $X_{\text{reg}} = \Phi^{-1}(\mathfrak{g}_{\text{reg}}^*)$.

Lemma. *The complement of X_{reg} in X is of codimension $\geqq 2$. In particular X_{reg} is dense.*

Proof. By Theorem 3.1 the complement of $\mathfrak{g}_{\text{reg}}^*$ in $\mathfrak{g}^*$ is of codimension $\geqq 2$. Let X_0 be the set of points, $p \in X$, such that the stabilizer group of p in G is discrete. By Theorem 3.1 the complement of X_0 is of codimension $\geqq 2$ and by Lemma 2.1, $\Phi\colon X_0 \to \mathfrak{g}^*$ is a submersion; so the complement of $X_0 \cap X_{\text{reg}}$ is of codimension $\geqq 2$. Q.E.D.

Let $Y = \Phi^{-1}(\mathfrak{t}_0^*)$. By definition Y is contained in X_{reg}. We will show it is a kind of "symplectic cross-section" to the action of G on X_{reg}.

Theorem 6.4. *Y has the following properties:*

i) *It is a T-invariant symplectic submanifold of X.*

ii) *Every G-orbit in X_{reg} intersects Y.*

iii) *$\Phi(Y)$ is a dense subset of $\Phi(X) \cap \mathfrak{t}_+^*$.*

Proof. It is obvious that Y is T-invariant and that every G-orbit in X_{reg} intersects it. To see that it is a submanifold of X, observe that if $p \in \mathfrak{t}_0^*$, the coadjoint orbit of G through p intersects $\mathfrak{t}_0^*$ transversally; so if $\Phi(x) = p$, Φ is transversal to $\mathfrak{t}_0^*$ at x. Since Y is the pre-image of $\mathfrak{t}_0^*$ it has to be a submanifold of X. To see that $\Phi(Y)$ is dense in $\Phi(X) \cap \mathfrak{t}_+^*$, let p be a point in $\Phi(X) \cap \mathfrak{t}_+^*$ and let $p = \Phi(x)$. Since X_{reg} is dense in X there exists a sequence of points, x_i, in

X_{reg} such that x_i converges to X. The orbit through x_i intersects Y in some point, y_i, and since X is compact we can assume the y_i's converge to a point $z \in X$. Since the y_i's are conjugate to the x_i's, z is conjugate to x, and $q = \Phi(z)$ is conjugate to p. But p and q are both in $\mathfrak{t}_+^*$ so they are Weyl-group conjugate. Replacing the y_i's by their images with respect to an appropriate Weyl-group element we can assume $\Phi(y_i)$ converges to p.

We must still show that Y is symplectic. For this we will need:

Lemma 6.5. *Let x be a point of X and let $\mathfrak{g}_x^\#$ be the image of $\mathfrak{g}$ in T_x under the map* (2.7). *The kernel of the map $d\Phi_x$: $T_x \to \mathfrak{g}^*$ is the annihilator of $\mathfrak{g}_x^\#$ in $T_x X$ with respect to the symplectic form Ω_x.*

Proof. The map (2.7)$''$ is the transpose of $d\Phi_x$: $T_x \mathfrak{g}^*$; so the kernel of $d\Phi_x$ is the perpendicular space to $\mathfrak{g}_x^\#$ in T_x with respect to Ω_x. Q.E.D.

Lemma 6.6. *Let x be a point of X and let $\mathfrak{h}$ be the Lie algebra of the stabilizer group of $\Phi(x)$. Let $\mathfrak{h}_x^\#$ be the image of $\mathfrak{h}$ in T_x under* (2.7). *Then kernel $d\Phi_x \cap \mathfrak{g}_x^\# = \mathfrak{h}_x^\#$.*

Proof. Let $p = \Phi(x)$. If $\xi \in \mathfrak{g}$ then

$$d\Phi_x(\xi_x^\#) = \mathrm{ad}(\xi)^* p \tag{6.3}$$

because Φ is G-equivariant. Thus $\xi_x^\#$ is in the kernel of $d\Phi_x$ if and only if $\mathrm{ad}(\xi)^* p = 0$. Q.E.D.

Lemma 6.7. *Let x be a point of Y. Then $T_x Y \cap \mathfrak{g}_x^\# = \mathfrak{t}_x^\#$.*

Proof. If $\xi_x^\# \in T_x Y$, $d\Phi_x(\xi_x^\#) \in \mathfrak{t}^*$. On the other hand by (6.3), $d\Phi_x(\xi_x^\#)$ is tangent to the co-adjoint orbit of G through p. However, the co-adjoint orbit intersects $\mathfrak{t}_0^*$ transversally in the single point, p, so $d\Phi_x(\xi_x^\#) = \mathrm{ad}(\xi)^* p = 0$ and $\xi \in t$. Q.E.D.

We will now prove that Y is symplectic at x. Clearly kernel $d\Phi_x$ $T_x Y$; so by Lemma 6.3, $(T_x Y)^\perp \subseteq \mathfrak{g}_x^\#$. (The perpendicular sign refers to Ω_x.) By Lemma 6.5

$$(T_x Y) \cap (T_x Y)^\perp \subseteq \mathfrak{t}_x^\# \tag{6.4}$$

so the left hand side of (6.4) is contained in $(\mathfrak{g}_x^\#)^\perp \cap (T_x Y)^\perp$ by Lemmas 6.3 and 6.4. On the other hand the orbit of G through x is transversal to Y; so

$$\mathfrak{g}_x^\# + T_x Y = T_x X;$$

hence $(\mathfrak{g}_x^*)^\perp \cap (T_x Y)^\perp = 0$, and the left hand side of (6.4) is zero. Q.E.D.

Let Φ_T: $X \to \mathfrak{t}^*$ be the moment mapping associated with the action of T on X. It is clear that $\Phi = \Phi_T$ on Y. By Theorem 6.4, Y is a Hamiltonian T-space and Φ_T: $Y \to \mathfrak{t}^*$ is its moment map. Moreover

$$\Phi(X) \cap \mathfrak{t}_+^* = \text{the closure of } \Phi_T(Y); \tag{6.5}$$

so if Y were compact, Theorem 6.3 would be an immediate consequence of Theorem 3.8. Since Y is not compact, we will get at the conclusions of Theo-

rem 3.8 in a slightly roundabout way. If H is a closed subgroup of T, let

$$X'_H = \{x \in X,\ H \text{ is the stabilizer group of } x \text{ in } T\}.$$

By Theorem 3.5, X'_H is a symplectic submanifold of X. Since

$$X'_H \cap Y = \{x \in Y,\ H \text{ is the stabilizer group of } x \text{ in } T\},$$

we can apply Theorem 3.5 to Y to conclude

Theorem 6.8. *$X'_H \cap Y$ is a symplectic submanifold of Y.*

Let

$$T_1, \ldots, T_N \tag{6.6}$$

be a list of all subgroups of T which occur as stabilizer groups of points. (Since X is compact, there are finitely many such groups by Proposition 3.2.) Let

$$X_i = \{x \in X,\ T_i \text{ is the stabilizer group of } x \text{ in } T\}.$$

The X_i's are finitely connected by Proposition 3.3; so, allowing for duplications on the list (6.6), we can assume X_i is connected. The disjoint union

$$X = \bigcup_{i=1}^{N} X_i$$

is a stratification of X by connected symplectic submanifolds. Moreover, by Theorem 3.7, there exists an open set, $_i$, in $\mathfrak{t}^*$ and a point $a_i \in \mathfrak{t}^*$ such that $\Phi_T\colon X \to \mathfrak{t}^*$ maps X_i submersively onto $\mathcal{O}_i \cap (a_i + \mathfrak{t}_i^\perp)$. Let $Y_i = X_i \cap Y$. By Theorem 6.8, the disjoint union

$$Y = \bigcup_{i=1}^{N} Y_i \tag{6.7}$$

is a stratification of Y by symplectic submanifolds. Moreover, by applying Theorem 3.6 directly to Y_i, we get

Theorem 6.9. *There exists an open set $\mathcal{O}'_i$ in $\mathfrak{t}_0^*$ such that Φ_T maps Y_i submersively onto $\mathcal{O}'_i \cap (a_i + \mathfrak{t}_i^\perp)$.*

Consider the set of all affine subspaces of $\mathfrak{t}^*$ which are intersections of $a_i + \mathfrak{t}_i^\perp$'s and walls of $\mathfrak{t}_+^*$. Let $v_1, \ldots, v_M$ be the "vertices" or zero dimensional subspaces in this set.

Theorem 6.10. *The closure of $\Phi_T(Y_i)$ in $a_i + \mathfrak{t}_i^\perp$ is the union of a finite number of convex sets, each of which is the convex hull of a collection of v_i's.*

The proof, an induction on $\dim \mathfrak{t}_i^\perp$, will be left to the reader. In view of (6.5) and the fact that $\bigcup Y_i = Y$ we obtain

Theorem 6.11. *$\Phi(X) \cap \mathfrak{t}_+^*$ is the union of a finite number of convex sets each of which is the convex hull of a collection of v_i's.*

§ 7. The Boundary Vertices

The interior vertices of $\Phi(X)\cap \mathfrak{t}_+^*$ correspond to fixed points of T. Theorem 4.7 provides a simple explanation of how these vertices are formed. In this section we will examine the more complicated phenomenon of "vertex-formation" at the boundary. The main features of this phenomenon are exhibited by the following model. Let $G=G_1\times T_2$ where G_1 is semi-simple and T_2 is abelian. Let T_1 be a Cartan subgroup of G_1. Then $T=T_1\times T_2$ is a Cartan subgroup of G, and if $(\mathfrak{t}_1^*)_+$ is a positive Weyl chamber in $\mathfrak{t}_1^*$, $\mathfrak{t}_+^*=(\mathfrak{t}_1^*)_+\times \mathfrak{t}_2$ is a positive Weyl chamber in $\mathfrak{t}^*$. Let $F=\{0\}\times \mathfrak{t}_2^*=$ the "most degenerate" face of $\partial \mathfrak{t}_+^*$. We will examine the phenomenon of "vertex-formation" on F. Let X be a Hamiltonian G-space and $\Phi\colon X\to \mathfrak{g}^*$ its moment mapping. Also let $\Phi_1\colon X\to \mathfrak{g}_1^*$ and $\Phi_2\colon X\to \mathfrak{t}_2^*$ be the moment mappings associated with the actions of G_1 and T_2. We will assume that *the origin in* $\mathfrak{g}_1^*$ *is a regular value of* Φ_1. This assumption implies that $X_0=\Phi_1^{-1}(0)$ is a submanifold of X. Moreover, by Lemma 2.1 it implies that for every $x\in X_0$ the stabilizer group of x in G_1 is finite. We will strengthen this assumption still further by assuming that *for all points,* $x\in X_0$, *the stabilizer group of* x *in* G_1 is trivial. It is clear from the equivariance of Φ_1 that X_0 is G_1 invariant. Because of the above assumptions, the set

$$M=X_0/G_1 \tag{7.1}$$

is a C^∞ Hausdorff manifold, and the projection mapping $\pi\colon X_0\to M$ makes X_0 into a principal G_1-bundle over M. M is called the "Marsden-Weinstein reduction" of X with respect to the zero in $\mathfrak{g}_1^*$. Marsden and Weinstein proved in [9] that it carries a symplectic form Ω_M such that if Ω_X is the symplectic form on X and $\iota\colon X_0\to X$ the inclusion map

$$\iota^*\Omega_X=\pi^*\Omega_M. \tag{7.2}$$

In fact (6.2) determines Ω_M uniquely. Since the action of T_2 commutes with the action of G_1, T_2 preserves X_0, and there is an induced action of T_2 on M. In fact, this action is Hamiltonian; and its moment mapping

$$\Psi_2\colon M\to \mathfrak{t}_2^*$$

satisfies $\Phi_2\circ\iota=\Psi_2\circ\pi$. This implies that $\Phi(X)\cap F=\Phi_2(X_0)=\Psi_2(M)$. Applying Theorem 5.2 to Ψ_2, we conclude:

Proposition 7.1. $\Phi(X)\cap F$ *is the union of a finite number of convex polytopes, each polytope corresponding to a connected component of* M.

Now let p be a vertex of $\Phi(X)\cap F$. By Theorem 5.2 there exists a fixed point, $m\in M$, of T_2 such that $\Psi_2(m)=p$. Let $\xi_1,\ldots,\xi_n$ be a basis of $\mathfrak{t}_2$ and let x_0 be a point of X_0 lying above m. Since $\pi(x_0)=m$ and m is a fixed point, the vector fields $\xi_1^\#,\ldots,\xi_n^\#$ must be tangent to the fiber of π at x_0; so there exist elements, $\eta_1,\ldots,\eta_n$, of $\mathfrak{g}_1$ such that $\xi_i^\#+\eta_i^\#=0$ at x_0; i.e. $\xi_i+\eta_i$, $i=1,\ldots,n$, are in the stabilizer algebra of x_0. The Lie brackets

$$[\xi_i+\eta_i,\xi_j+\eta_j]$$

are also in the stabilizer algebra; however $[\xi_i, \xi_j]=[\xi_i, \eta_j]=[\xi_j, \eta_i]=0$ and $[\eta_i, \eta_j]\in\mathfrak{g}_1$. By assumption the action of G_1 on X_0 is free; so the stabilizer algebra cannot contain elements of $\mathfrak{g}_1$. Therefore, we have proved that the vectors

$$\xi_i+\eta_i, \quad i=1, \ldots, n \tag{7.3}$$

span an n-dimensional *abelian* subalgebra of $\mathfrak{g}$. Every other point on the fiber above m is the translate of x_0 by some element, g, of G_1, and the algebra corresponding to (7.3) at gx_0 is spanned by

$$\xi_i+\mathrm{Ad}(g)\eta_i, \quad i=1, \ldots, n. \tag{7.4}$$

In particular, there exists a point, x, on the fiber above m such that the algebra, $\mathfrak{h}$, spanned by (7.4) lies in $\mathfrak{t}$. If $\{wx, w\in W_x\}$ are the translates of x by the elements of the Weyl group of G_1, the algebras associated with the wx's are the Weyl-group conjugates of $\mathfrak{h}$, and also lie in $\mathfrak{t}$. Notice also that if H is the stabilizer group of x in T, then its Lie algebra is $\mathfrak{h}$: i.e. its Lie algebra clearly contains $\mathfrak{h}$ and on the other hand if it were bigger than $\mathfrak{h}$ then by (7.3) it would have to contain a non-zero element of $\mathfrak{g}_1$ which, as we saw above, is impossible. Summarizing we have proved:

Proposition 7.2. *There exists a point, x, on the fiber above m in X_0 such that the stabilizer group, H, of x in T is n-dimensional. Moreover, for every element, w, of the Weyl group, the conjugate, $H^w\subset T$, of H stabilizes wx.*

Remark. If K^w is the centralizer of H^w in G_1 then the whole orbit of K^w through wx is stabilized by H^w.

Using Proposition 7.2 we can get a fairly exact picture of what $\Phi(X)\cap\mathfrak{t}^*_+$ looks like in the neighborhood of p. To simplify our exposition a bit we will assume that the centralizer of H in G is T. (However, the following remarks are true without this assumption.) Let

$$X_H=\{x\in X, \text{ the stabilizer of } x \text{ in } T \text{ is } H\}.$$

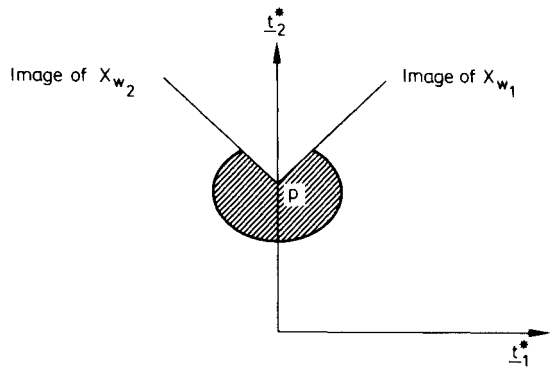
By Proposition 7.2, X_H intersects the fiber above m in a point, x. Let X'_H be the connected component of X_H containing x. By Theorem 6.2, Φ maps X'_H onto an open convex polyhedral subregion of the affine plane, $p+\mathfrak{h}^-$, in $\mathfrak{t}^*$. By (7.3) this plane intersects the face F, of $\partial\mathfrak{t}^*_+$ transversally in the point, p. Also it is of complementary dimension to F; so p is its unique point of intersection with F. The same remarks apply to the Weyl group conjugates of $\mathfrak{h}$: For every element, w, of the Weyl group let X_w be the translate of X'_H with respect to w, H^w the translate of H with respect to w and $\mathfrak{h}^w$ the translate of $\mathfrak{h}$ with respect to w. We have proved

Proposition 7.3. *Φ maps a connected component, X_w of the fixed point set of H_w onto an open convex polyhedral subregion of $p+(\mathfrak{h}^w)^\perp$ in $\mathfrak{t}^*$ and this set intersects F transversally in the single point p.*

By Theorem 6.2 $\Phi(X_w)$ is a codimension n face of the polyhedral region $\Phi(X)\cap \mathfrak{t}^*$; so we have shown the following:

Corollary. *The vertex, $p\in F$, of $\Phi(X)\cap \mathfrak{t}_+^*$ is the point of intersection of the n-dimensional face, F, of $\partial \mathfrak{t}_+^*$ with the codimension n faces, $\Phi(X_w)$ of $\Phi(X)\cap \mathfrak{t}^*$.*

Example. Let T_2 be the circle group and let $G_1 = SU(2)$. The picture below is a typical picture of a boundary vertex for $G_1 \times T_2$



Notice that our description of vertex-formation involves a local aspect (the description of $\Phi(X)\cap F$ near p) and a global aspect (determination of the groups H^w).

We will conclude this section with a few remarks about boundary vertices in general. Let G be an arbitrary connected compact Lie group, T a Cartan subalgebra of G, X a compact, connected Hamiltonian G-space and $\Phi\colon X\to \mathfrak{g}^*$ the associated moment mapping. Let F be an n-dimensional face of $\partial \mathfrak{t}_+^*$. For the following, see [6].

Lemma. *There exists a closed subgroup, H, of G with the following properties:*

i) *H is the stabilizer group of every point on F.*

ii) *H is the direct product of a semi-simple group G_1 and an n-dimensional torus, T_2.*

iii) *H is the centralizer of T_2 in G.*

Let $\mathfrak{h}^*$ be the fixed point set of T_2 in $\mathfrak{g}^*$. Clearly $F\subseteq \mathfrak{t}^*\subseteq \mathfrak{h}^*$.

Lemma. *Let $Y_H = \Phi^{-1}(\mathfrak{h}^*)$. If $x\in Y_H$ and $\Phi(x)\in F$, Y_H is a symplectic submanifold of X in a neighborhood of x.*

(The proof of this lemma, which we will omit, is very much like the proof of the symplecticity of Y in § 6.)

Thus H is a group of the form, $G_1\times T_2$, which we considered above, and Y_H is a Hamiltonian H-space (at least in the vicinity of points whose images are on F). One gets a complete local description of the vertices on F by applying the results of the first part of this section to Y_H and H.

§ 8. Pre-Quantization

Let X be a symplectic manifold with symplectic form, Ω. X is *pre-quantizable* if the cohomology class

$$[\Omega] \in H^2_{\text{DeRham}}(X)$$

is integral, i.e. is in the image of the mapping

$$H^2_{\text{Cech}}(X, \text{integers}) \xrightarrow{\text{Weil}} H^2_{\text{DeRham}}(X).$$

If this is the case, there exists a line bundle, L, on X whose Chern class is $[\Omega]$, and there exists a connection, ∇, on L such that the curvature form of ∇ is Ω. (L, ∇) is called a *pre-quantum structure* on X. (See [8].)

Let $G \times X \to X$ be a Hamiltonian action of G on X, and $\Phi: X \to \mathfrak{g}^*$ its moment mapping. For each $\xi \in \mathfrak{g}$ let $\xi^\#$ be the ξ-th component of Φ. Let

$$D_\xi = \nabla_{\xi^\#} + 2\pi i \phi^\xi. \tag{8.1}$$

This operator operates on C^∞ sections of L and, in fact, defines an infinitesimal action of $\mathfrak{g}$ on L; i.e. $\xi \to D_\xi$ is a Lie algebra morphism. (Loc. cit. §3.) The action of G is *pre-quantizable* if there is a global action of G on L such that the induced action of $\mathfrak{g}$ on sections is given by (8.1). (For example, if G is compact and simply connected, this is always the case.) We will assume below that the action of G has this property.

Let f be an element of $\mathfrak{g}^*$, let G_f be its stabilizer group in G and $\mathfrak{g}_f$ the Lie algebra of G_f. The linear mapping

$$\xi \in \mathfrak{g}_f \to 2\pi i \langle f, \xi \rangle \tag{8.2}$$

is an infinitesimal character of $\mathfrak{g}_f$; that is, it vanishes on $[\mathfrak{g}_f, \mathfrak{g}_f]$. (Loc. cit. §3.) One says that f is *integral* if (8.2) comes from a global character of G_f.

Suppose now that $\Phi(x) = f$. Let H be the stabilizer group of x. Since Φ is equivariant, H is contained in G_f.

Theorem 8.1. *If $H = G_f$, f is integral.*

Proof. Consider the isotropy action of H on L_x. By (8.1) the corresponding action of $\mathfrak{h}$ is

$$\xi \in \mathfrak{h} \to \text{multiplication by } 2\pi i \phi^\xi(x). \tag{8.3}$$

However, $\phi^\xi(x) = \langle \Phi(x), \xi \rangle = 2\pi i \langle f, \xi \rangle$. If $\mathfrak{h} = \mathfrak{g}_f$ then by comparing (8.2) and (8.3) we see that (8.2) is associated with a global character of G_f. Q.E.D.

If $G = T =$ an n-torus, the integral points of $\mathfrak{t}^*$ are lattice points. By Theorem 5.2 the vertices $\Phi(X)$ in $\mathfrak{t}^*$ are the images of fixed points of T; so by Theorem 8.1, we conclude:

Corollary 1. *Let X be a pre-quantizable Hamiltonian T-space. Then the vertices of $\Phi(X)$ are lattice points in $\mathfrak{t}^*$.*

Notice that from the convex polyhedron, $\Phi(X)$, we can read off quite a bit about the isotropy action of T at fixed points. By Theorem 4.7 the shape of the vertices determines the isotropy action on the tangent spaces to X at the fixed points; and, as we have just seen, the location of the vertices determines the isotropy action on L. The statement that $\Phi(X)$ is a convex polyhedron is perhaps the most elegant way of describing the inter-relationships among the isotropy actions at the different fixed points.

For G an arbitrary compact, connected Lie group we have:

Corollary 2. *Let X be a pre-quantizable Hamiltonian G-space. Then the vertices of $\Phi(X) \cap \mathfrak{t}_+^*$ are lattice points.*

Proof. The interior vertices are images of fixed points of T; so they are lattice points for the same reasons as above. As we saw in § 7, the boundary vertices are associated with fixed points of subgroups of T acting on sub-quotients of X. It is easy to see that if X is pre-quantizable, these subquotients are as well; so the boundary vertices are also integral.

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Appendix

In this appendix we will show that $\Phi(X) \cap \mathfrak{t}_+^*$ is a convex polyhedron providing the action of G on X is "quantizable." The ingenious idea behind our proof is due to Gerrit Heckman. (See [A2].)

We will begin by describing what we mean by a quantizable G-action. Let V be a real vector space equipped with a linear symplectic form Ω, and let $\Omega_{\mathbf{C}}$ be the $\mathbf{C}$-linear extension of Ω to $V \otimes \mathbf{C}$. A Lagrangian subspace, F, of $V \otimes \mathbf{C}$ is said to be *positive-definite* if the Hermitian form

$$\sqrt{-1}\,\Omega(v, \bar{w}), \qquad v \cdot w \in F,$$

is positive-definite.

Definition A.1. Let X be a compact symplectic manifold. A subbundle, F, of the complexified tangent bundle of X is said to be a *positive-definite polarization* if

a) F is integrable,

and

b) for all $x \in X$, F_x is a positive-definite Lagrangian subspace of $T_x \otimes \mathbf{C}$.

Definition A.2. X is said to be quantizable if it is prequantizable in the sense described in Sect. 8, and possesses a positive-definite polarization.

Remark. Using the Kodaira imbedding theorem one can show that a quantizable symplectic manifold is essentially just a non-singular complex projective variety. However, in keeping with the symplectic outlook of this article we prefer thinking of it as a special kind of symplectic manifold.

Let G be a compact, connected Lie group and $G \times X \to X$ a Hamiltonian G-action.

Definition A.3. The action of G on X is said to be quantizable if it is pre-quantizable in the sense of Sect. 8 and if, in addition, X possesses a G-invariant positive-definite polarization.

Suppose the action of G on X is quantizable. Let L be the pre-quantum line bundle on X, ∇ its connection and F a G-invariant positive-definite polarization. A section, s, of L is, by definition, *holomorphic* if for all sections, Ξ, of $\bar{F}$

$$\nabla_{\Xi} s = 0.$$

It is easy to see that every point, $p \in X$, possesses a neighborhood on which a non-zero holomorphic section exists; so L has the structure of a holomorphic line bundle. The same is true of the m-th power of the line bundle, L; i.e. the line bundle $\overset{m}{\otimes} L$. By assumption G acts on L and on $\overset{m}{\otimes} L$. In particular there is a representation of G on the space of holomorphic sections of $\overset{m}{\otimes} L$. Let us denote this space by Γ_m.

Theorem A.4. *Let β be a rational point in the interior of $\mathfrak{t}_+^*$. Suppose that for some integer k, the representation of G with maximal weight, $k\beta$, occurs as a subrepresentation of Γ_k. Then β is in the image of the moment map, Φ.*

For the proof of this theorem, see [A1]. This theorem has the following weak converse which is also proved in [A1].

Theorem A.5. *Let β be a rational point in the interior of $\mathfrak{t}_+^*$. Suppose β is in the image of Φ and suppose in addition that for all $p \in X$ with $\Phi(p) = \beta$, the stabilizer group of p in G is finite. Let k be an integer such that $k\beta$ is a lattice point of $\mathfrak{t}_+^*$. Then providing k is sufficiently large the representation of G on Γ_k contains the irreducible representation of G with maximal weight $k\beta$.*

(See [A1], Theorem 5.6.)

Let S be the set of all points, $\beta \in \mathfrak{t}_+^*$, for which the hypotheses of Theorem A.5 hold. We will give Heckman's proof of the following

Theorem A.6. *If S is dense in $\Phi(X) \cap \mathfrak{t}_+^*$ then $\Phi(X) \cap \mathfrak{t}_+^*$ is convex.*

Proof. It is enough to show that if β_1 and β_2 are in S then $(\beta_1 + \beta_2)/2$ is in $\Phi(X) \cap \mathfrak{t}_+^*$. By Theorem A.5 we can find an integer, k, such that $k\beta_1$ and $k\beta_2$ are lattice points, and such that Γ_k contains the representations of G with maximal weights $k\beta_1$ and $k\beta_2$ as subrepresentations. Let s_1 and s_2 be

the non-zero sections of $\overset{k}{\otimes} L$ representing the maximal weight vectors of these two representations. Then $s_1 s_2$ is a non-zero holomorphic section of $\overset{2k}{\otimes} L$. Let W be the smallest G-invariant subspace of Γ_{2k} containing it. It is clear that the representation of G on W is irreducible and has maximal weight $k(\beta_1+\beta_2)$. Hence by Theorem A.4, $(\beta_1+\beta_2)/2$ occurs in the image of Φ. Q.E.D.

Remark. By Sard's theorem, S is dense in $\Phi(X)\cap \mathfrak{t}_+^*$ if and only if for *some* $p\in X$ the stabilizer group of p in G is finite. (Compare the corollary to Lemma 2.1 with Proposition 3.1.)

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