CONFORMAL WELDING OF FLEXIBLE CURVES

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MAThS - Thessaloniki

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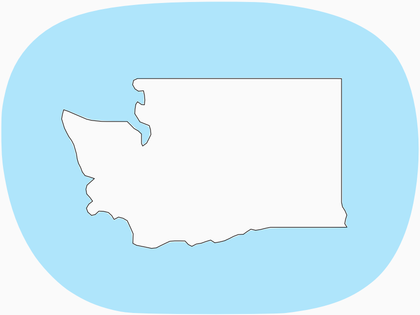
Stony Brook University

CONFORMAL WELDING HOMEOMORPHISMS



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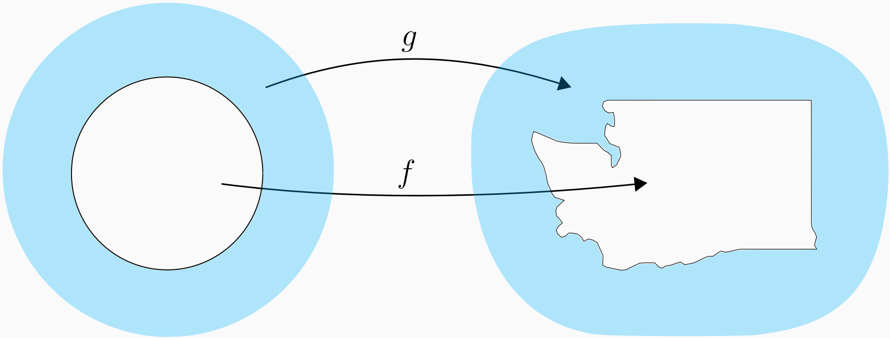

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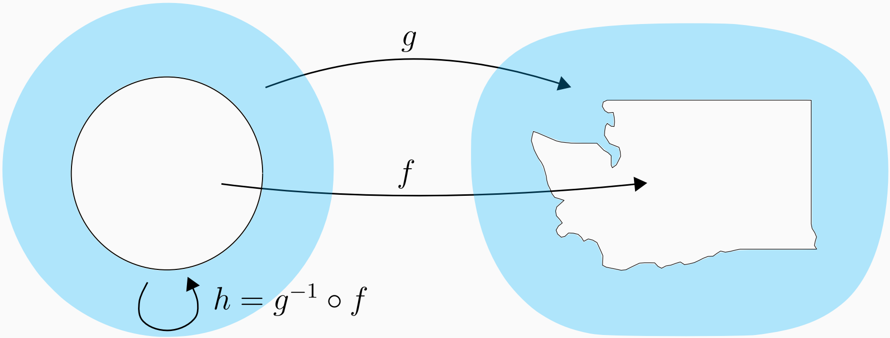

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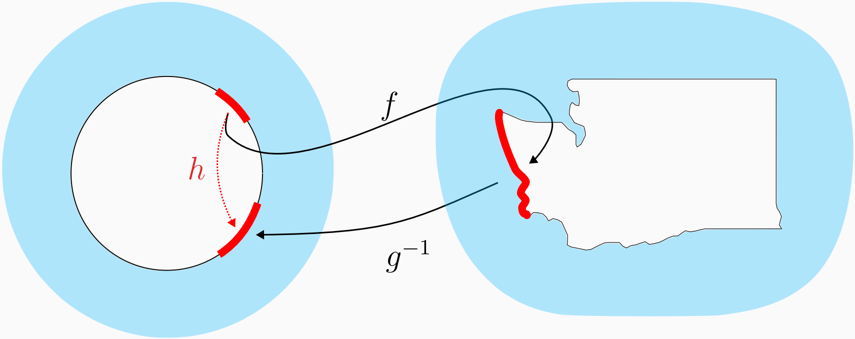
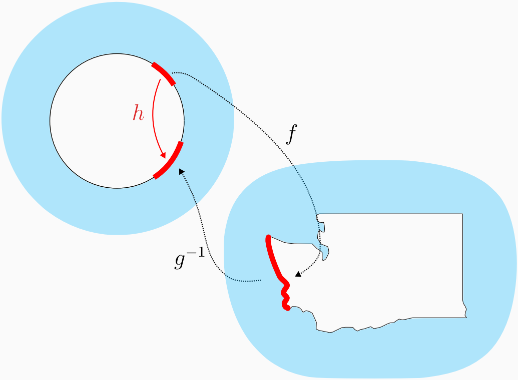




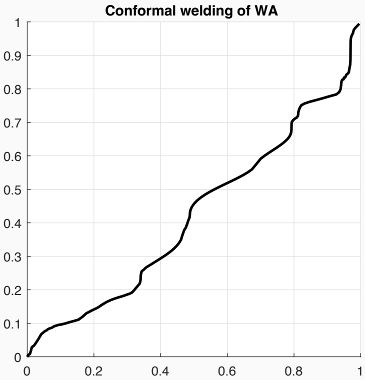
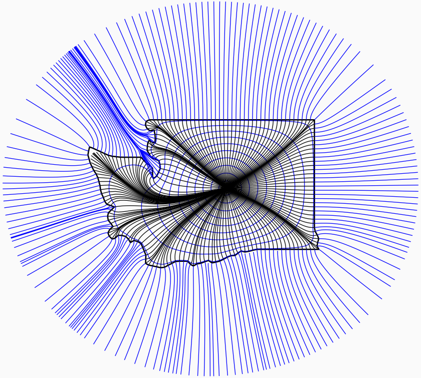
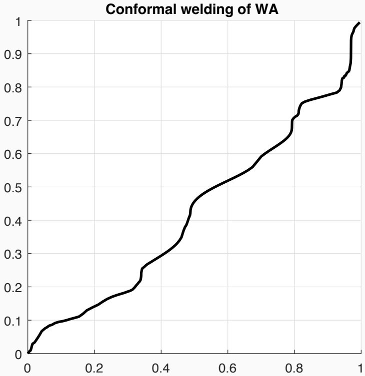


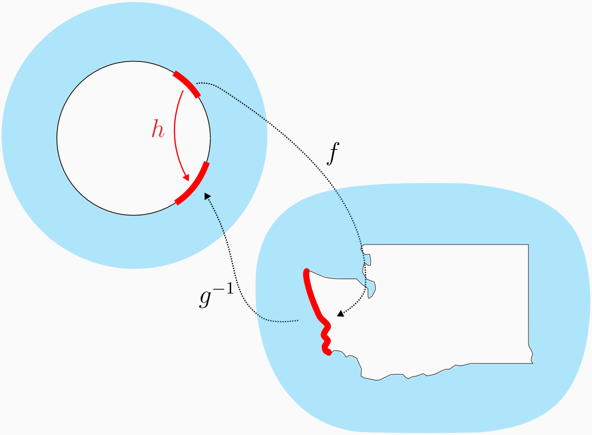
Image by Chris Bishop





Images by Chris Bishop, using Toby Driscoll's code

GOAL



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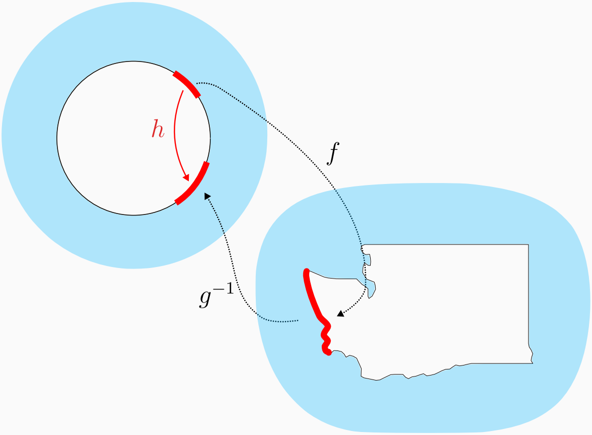

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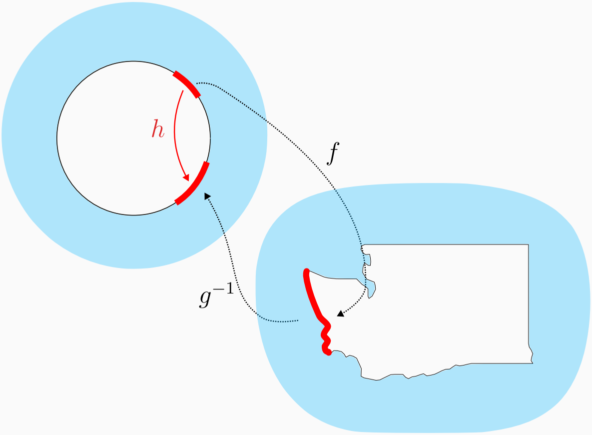

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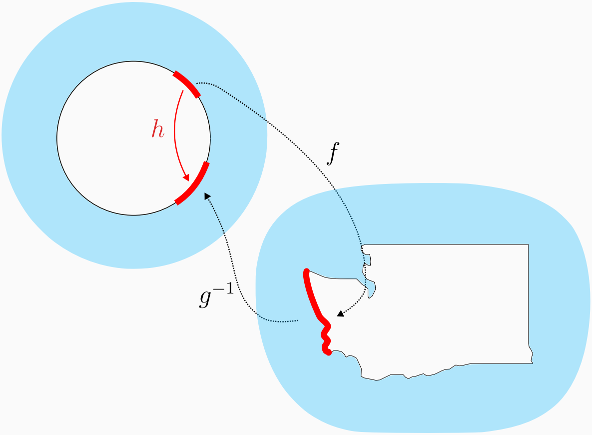

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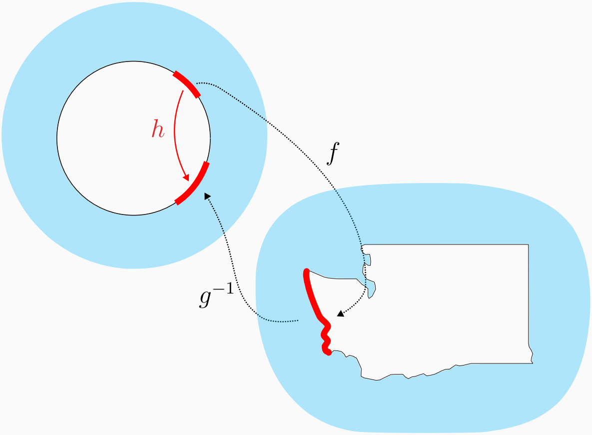

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Theorem (R. 2026)

*For **some** curves $\mathcal{W}$ is not injective in a **quantitative** way.*

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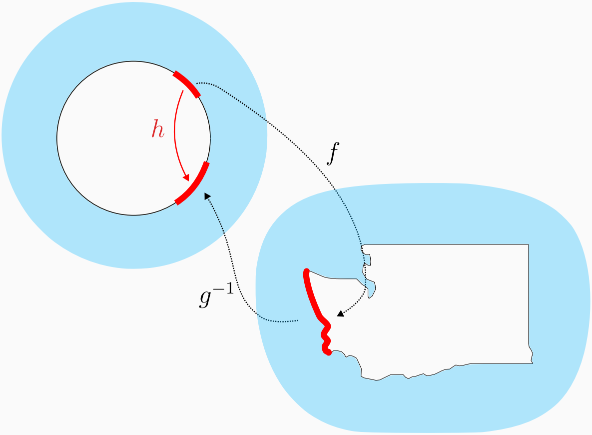

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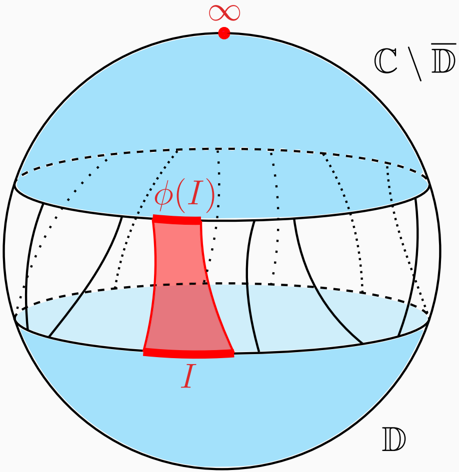
Theorem (R. 2026)

*For **flexible** curves $\mathcal{W}$ is not injective in a **quantitative** way.*

WHEN WILL $\phi\colon \mathbb{S}^1 \to \mathbb{S}^1$ BE A WELDING?



$\phi\colon \mathbb{S}^1 \circlearrowleft$ is quasisymmetric if there is $M < \infty$ s.t.

$$M^{-1} \leq |\phi(I)|/|\phi(J)| \leq M,$$

where $I, J \subset \mathbb{S}^1$ are two adjacent arcs of equal length.

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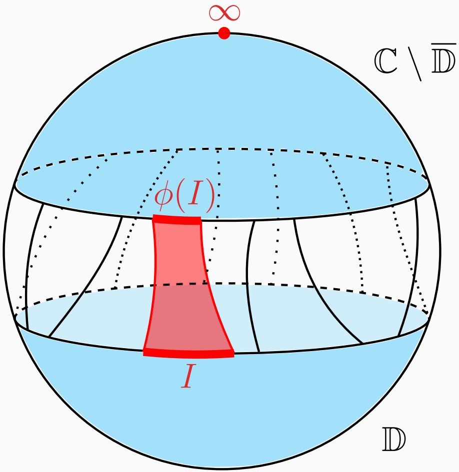

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Non-example:

$$\phi(x) = \begin{cases} x & \text{for } x \le 0 \\ x^3 & \text{for } x \ge 0. \end{cases}$$

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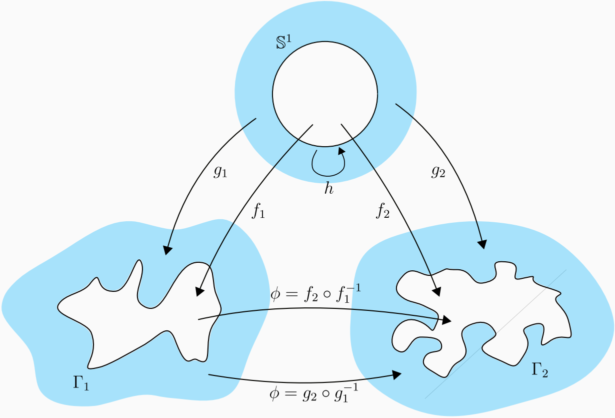
Theorem (Pfluger, 1960):
Every quasisymmetric $\phi\colon \mathbb{S}^1 \circlearrowleft$ is a welding. The Jordan curve is a quasicircle.

Conformal welding is important in:

- Teichmüller theory.
- Kleinian groups.
- Complex dynamics.
- Random geometry (Gluing of Liouville Quantum Gravity disks and SLE).
- Computer vision (work of Mumford).

WHY STUDY $\mathcal{W}: [\Gamma] \to [h]$?



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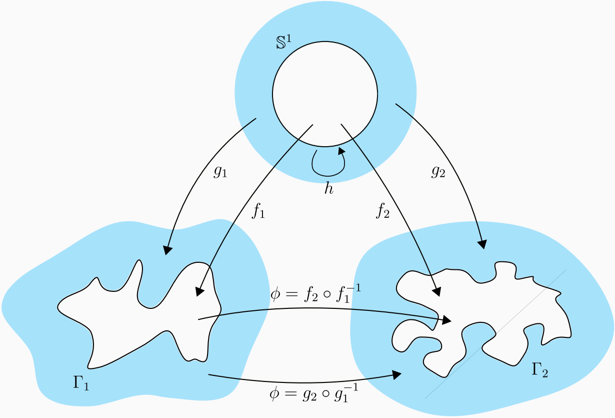

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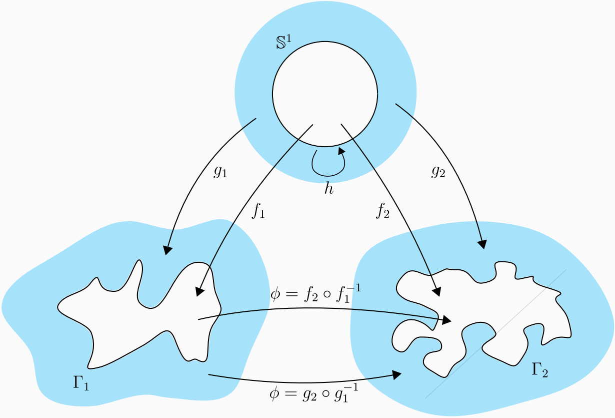

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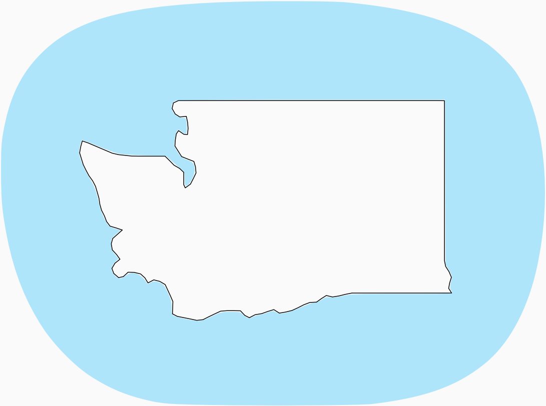
(like $\mathbb{S}^1$)

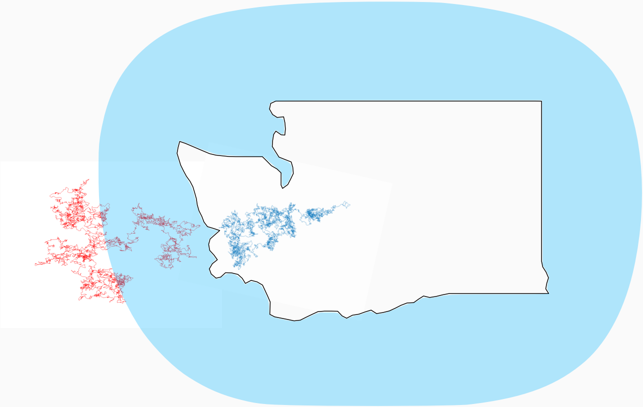
Question: Does $\Leftarrow$ hold?

REGULARITY OF THE WELDING



Harmonic measure: hitting distribution of Brownian motion.

REGULARITY OF THE WELDING



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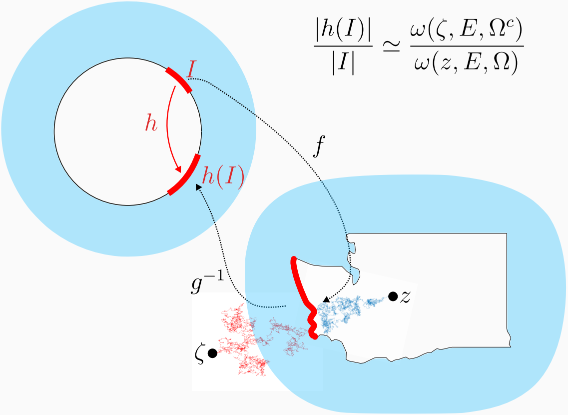

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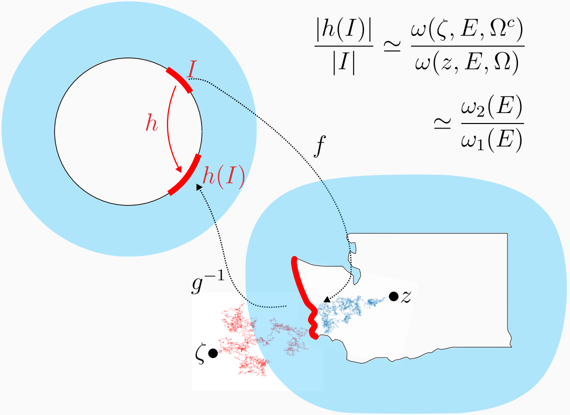

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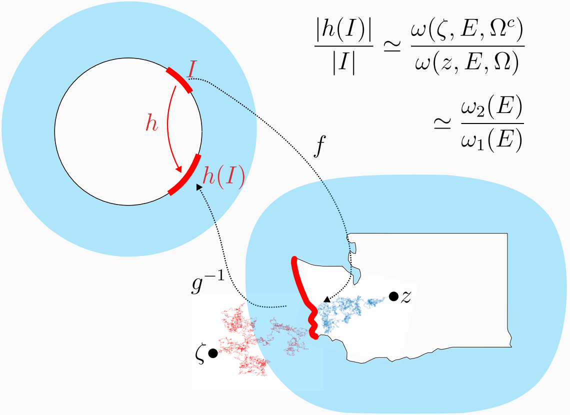

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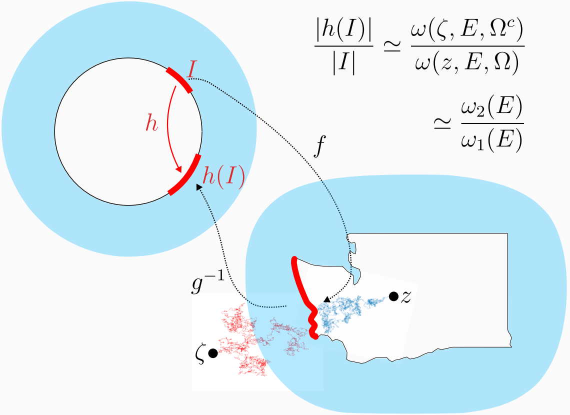

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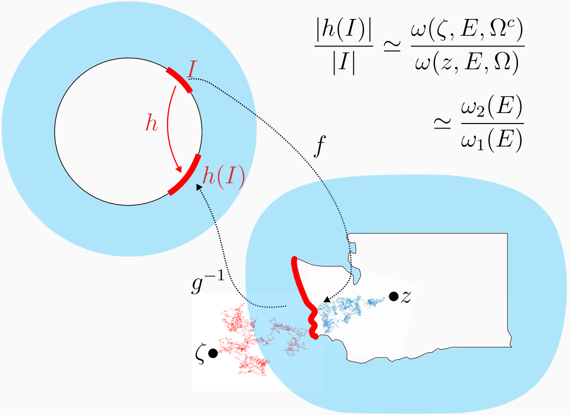

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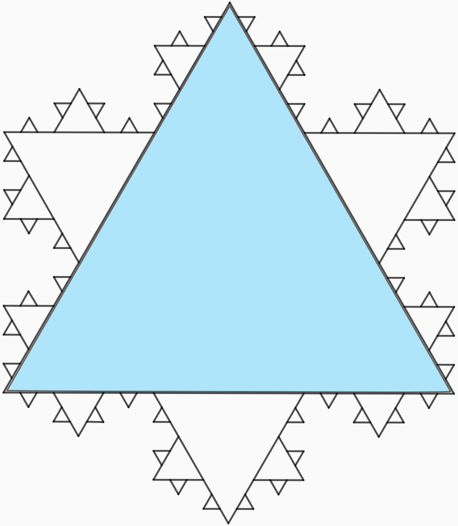

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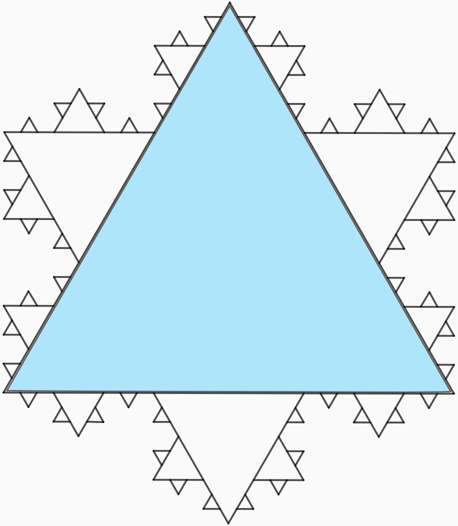

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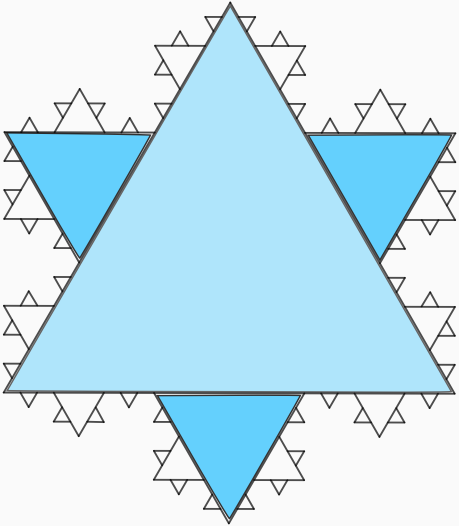

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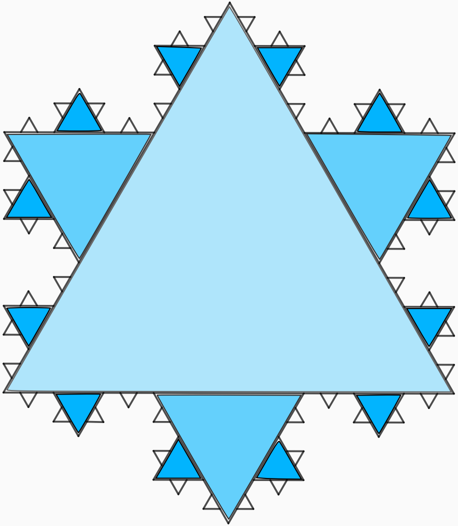

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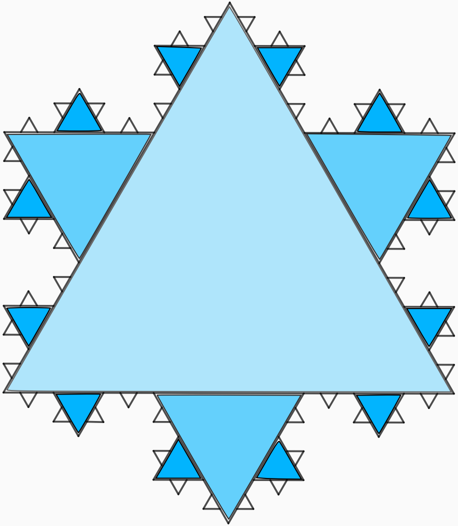

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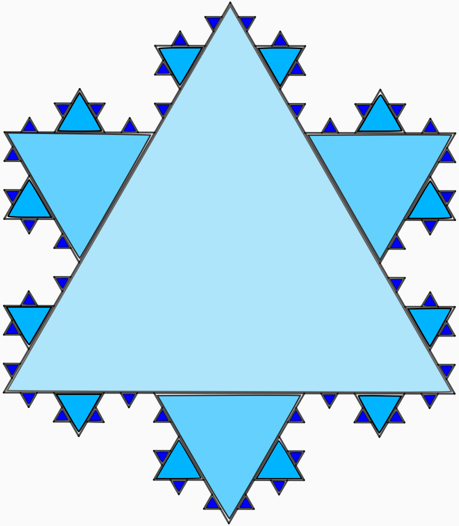

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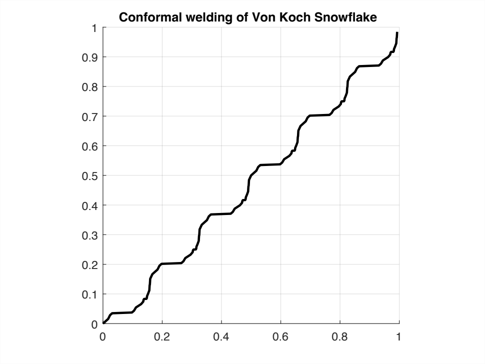
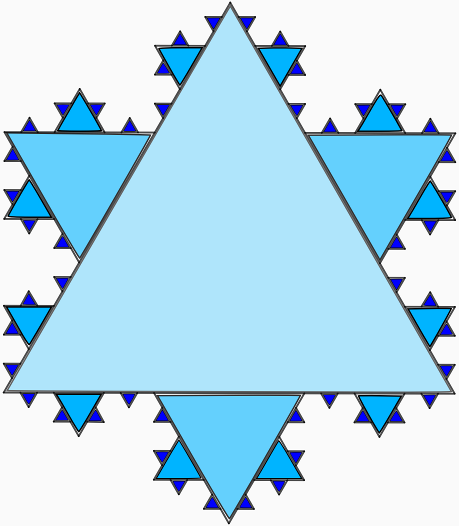


Image by Chris Bishop



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If E is Borel,

$$\mathrm{Cap}(E) = \sup\{\mathrm{Cap}(K) \colon K \subset E \text{ compact}\}.$$

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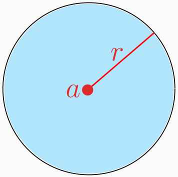

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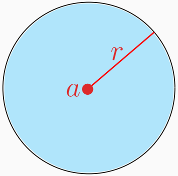

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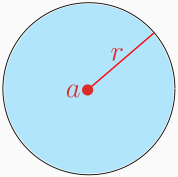

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$\mathrm{Cap}(E) = 0 \implies E$ has Hausdorff dim 0.

LOG-SINGULAR CIRCLE HOMEOMORPHISMS

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Lemma If for $C < \infty$ and $\alpha > 0$

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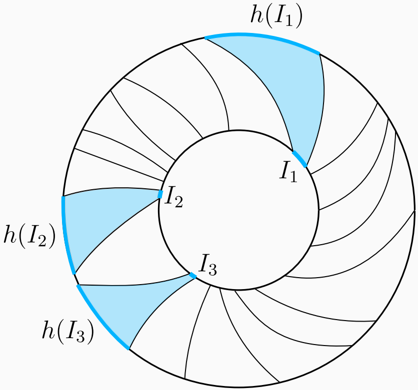

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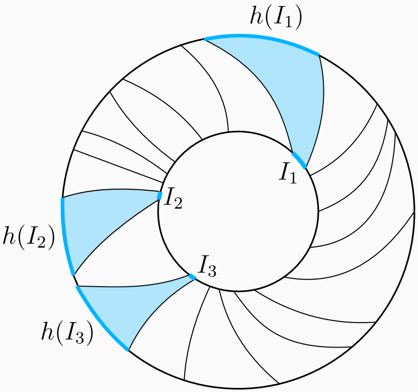

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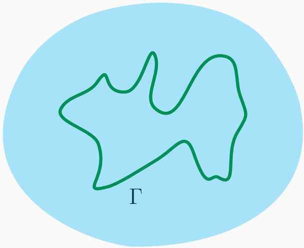

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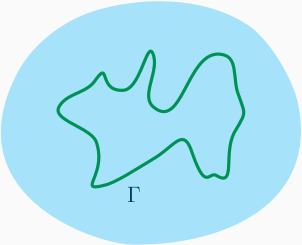

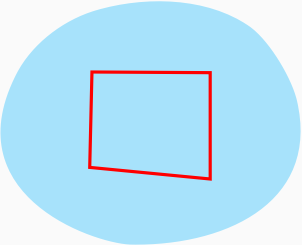

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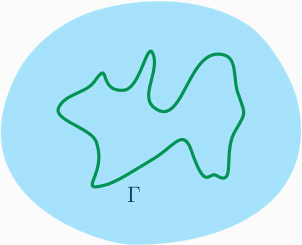

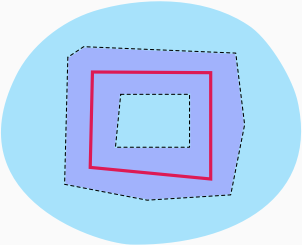

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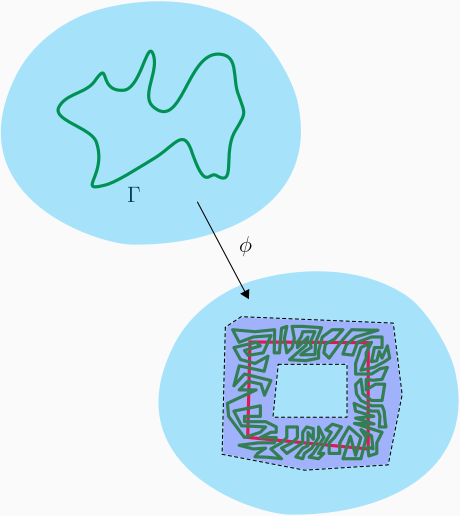

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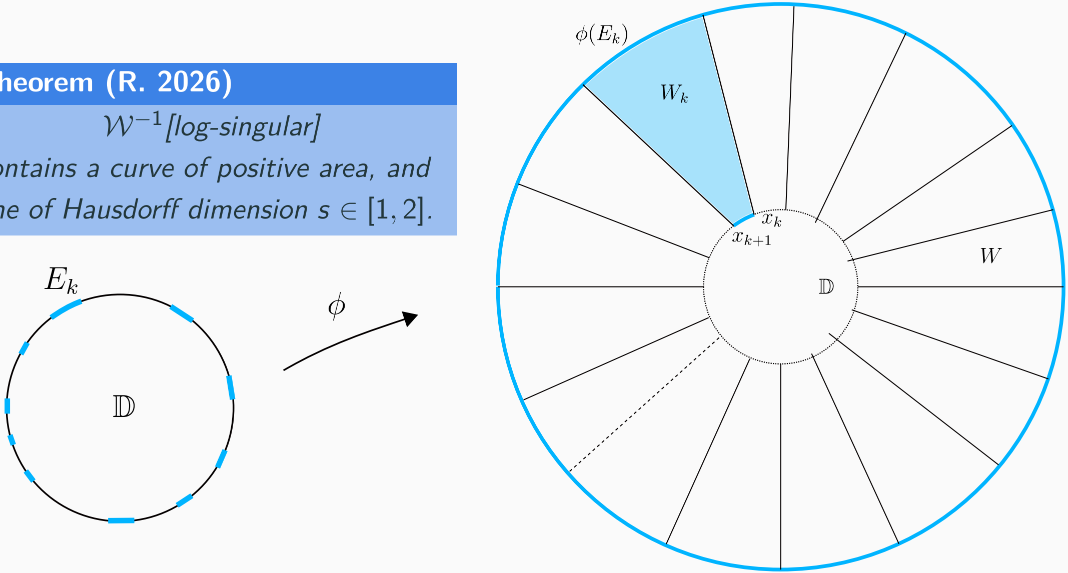

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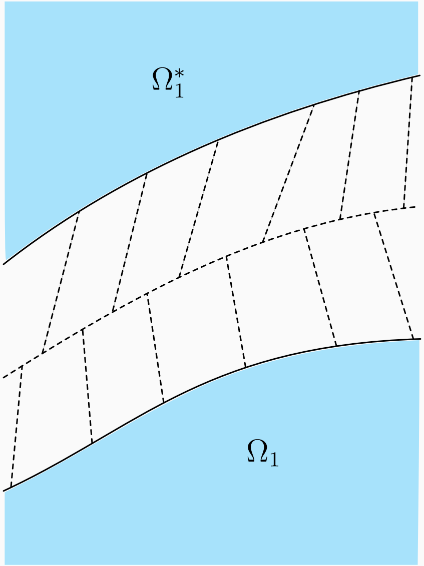

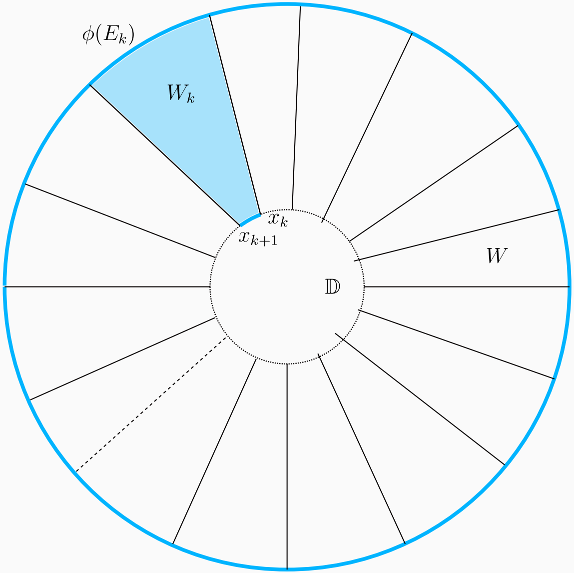

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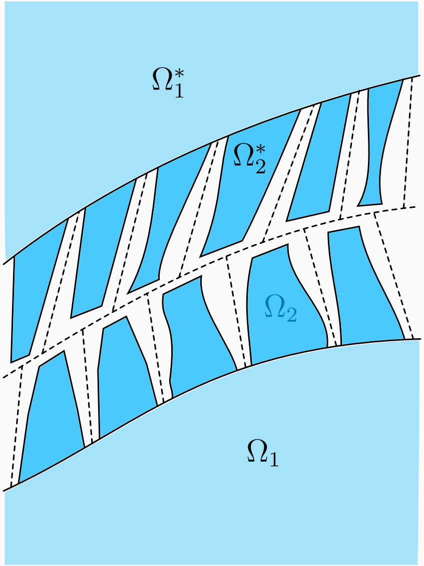

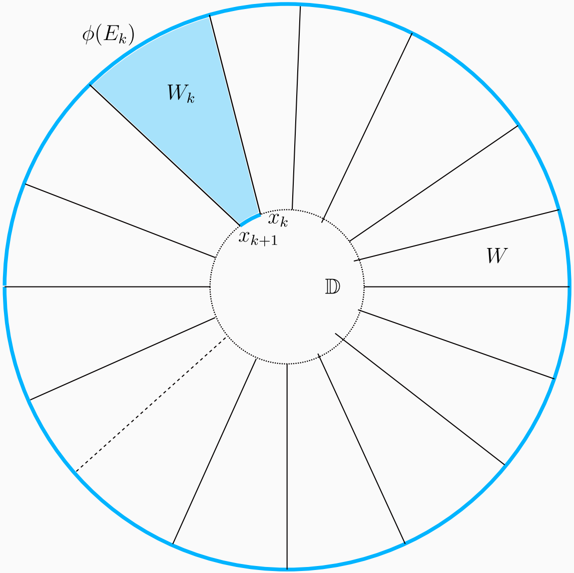

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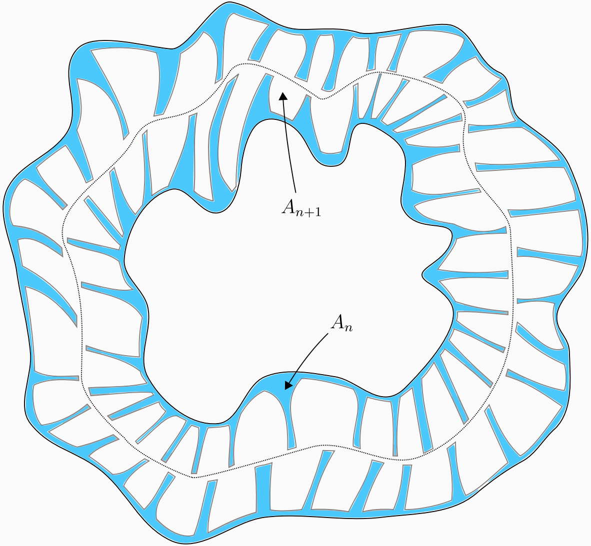

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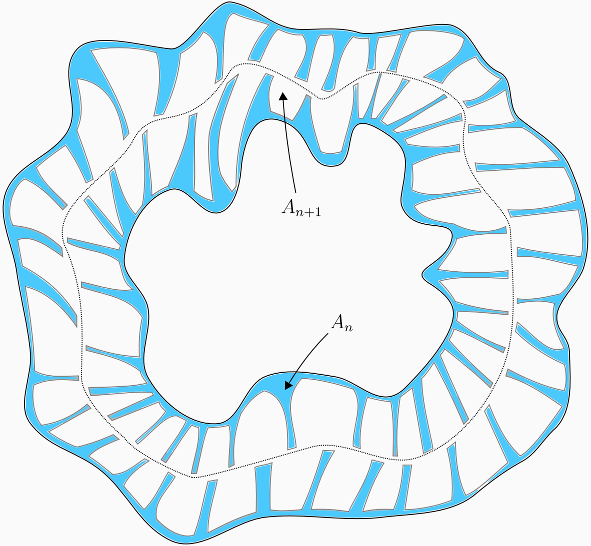

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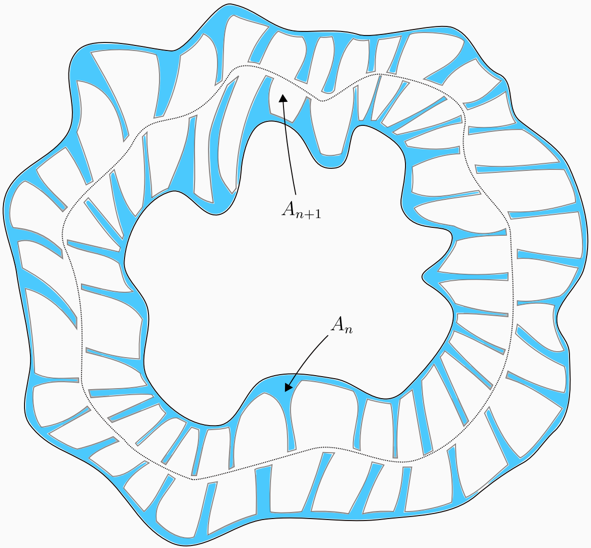

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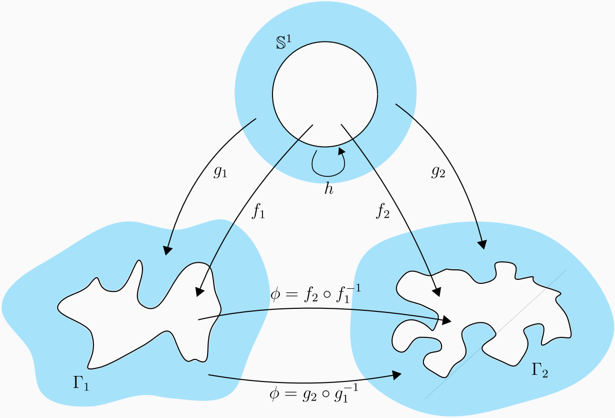
APPLICATIONS AND OPEN PROBLEMS



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$$CH(\Gamma) = \{\phi\colon \text{conformal off } \Gamma\}$$

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Given Γ flexible there is $\phi \in CH(\Gamma)$ so that $\phi(\Gamma)$ has positive area.

Open for Γ non-removable.

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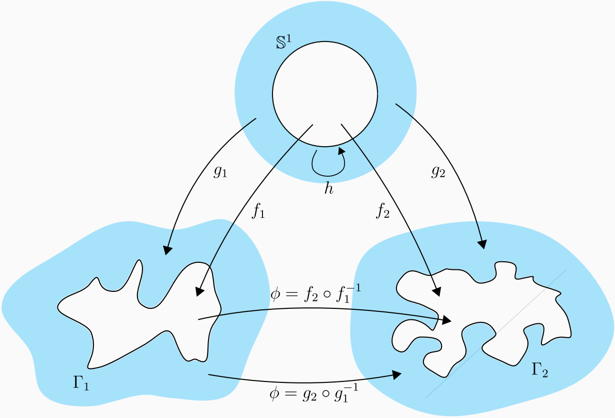

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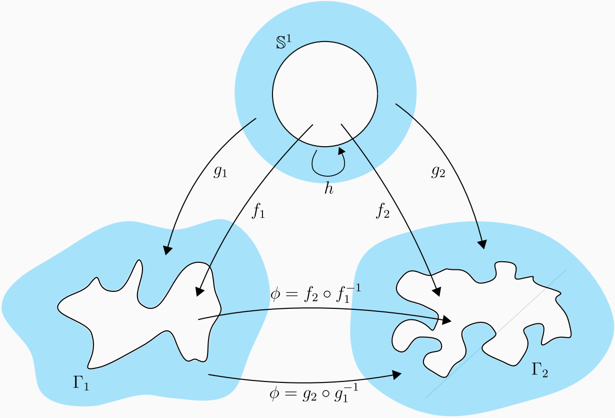
APPLICATIONS AND OPEN PROBLEMS



Question:

Is $\mathcal{W}\colon [\Gamma] \to [h]$ 1-to-1 exactly for removable curves.

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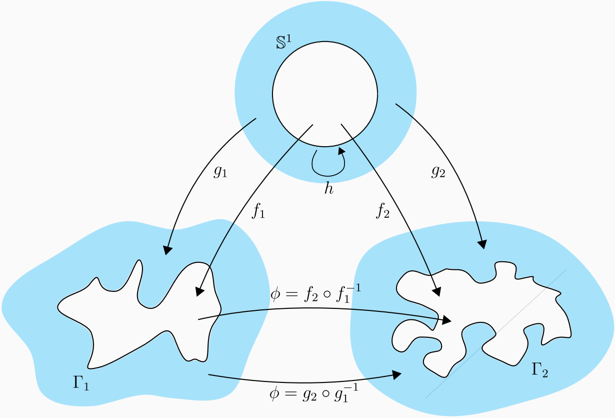

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There exists Γ non-removable and $\phi \in CH(\Gamma)$ non-Möbius so that $\phi(\Gamma) = \Gamma$.

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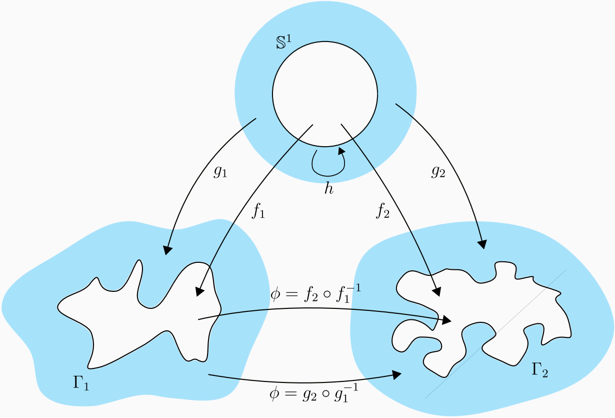

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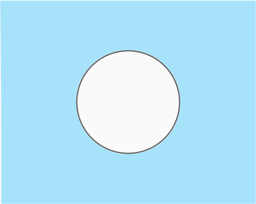
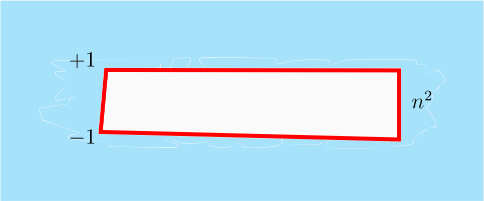
FLEXIBLE $\Rightarrow$ LOG-SINGULAR





Lemma (Balogh, Bonk 1999)

$f\colon \mathbb{D} \to \Omega$ conformal and for $R \geq 1$,

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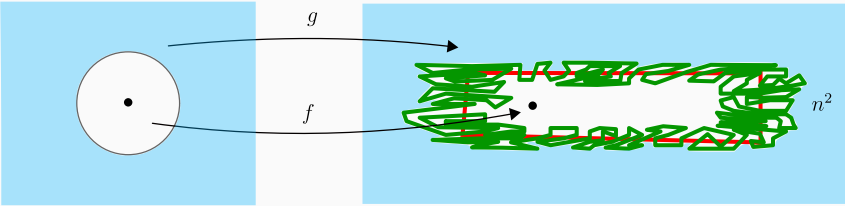
Then $\mathrm{Cap}(E) \leq CR^{-1/2}$.

FLEXIBLE ⇒ LOG-SINGULAR



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Let $E_n = f^{-1}(\{x + iy \in \Gamma_n \colon x \geq n\})$,
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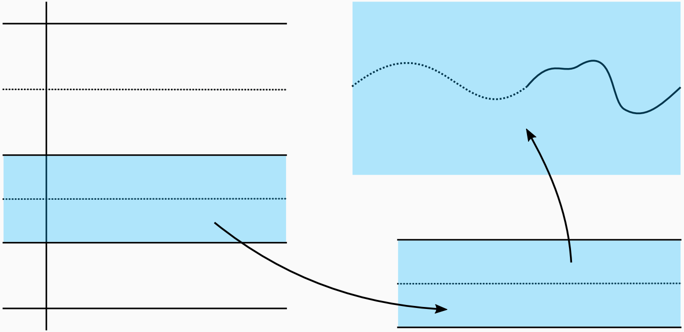
E_n and $h(\mathbb{S}^1 \setminus E_n)$ have small capacity ⇒
h is log-singular.

Thank you!

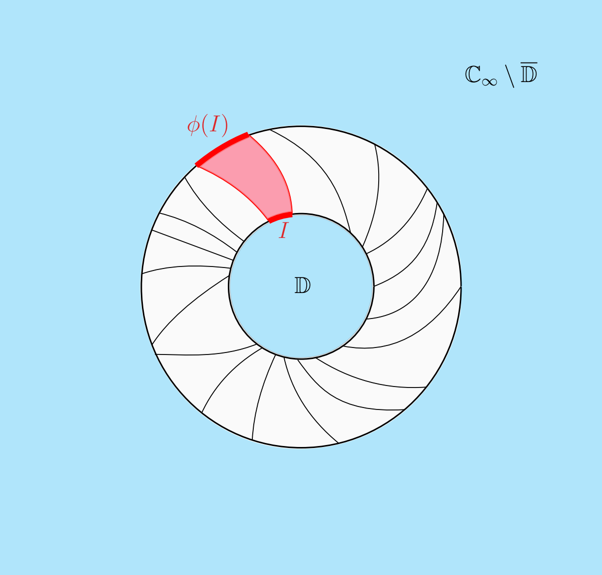
Non-example:

$$\phi(x) = \begin{cases} x & \text{for } x \leq 0 \\ x^3 & \text{for } x \geq 0. \end{cases}$$



Argument from Peter Lin's thesis. Originally proved by Oikawa.

WHEN WILL $\phi\colon \mathbb{S}^1 \to \mathbb{S}^1$ BE A WELDING?



Ahlfors-Beurling map $\psi_\phi\colon \mathbb{D} \circlearrowleft$.

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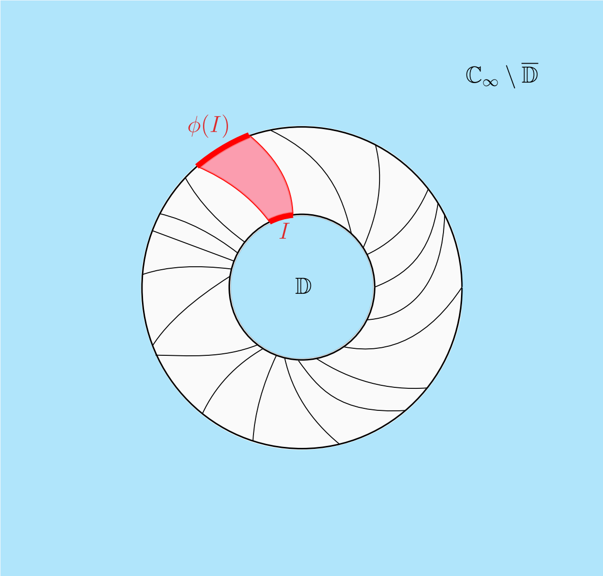

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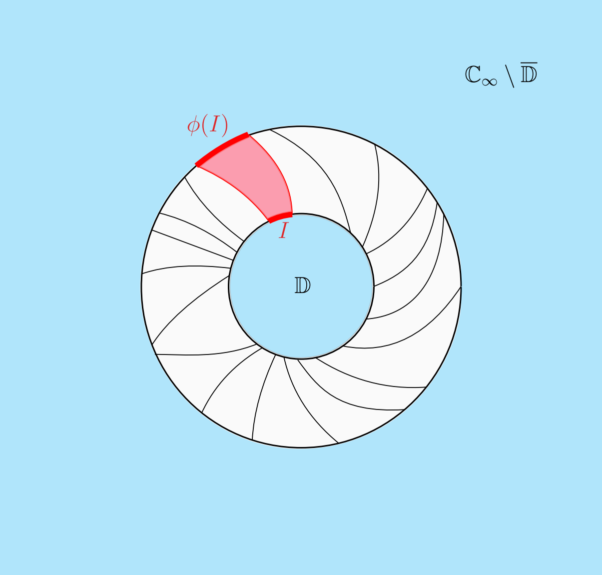

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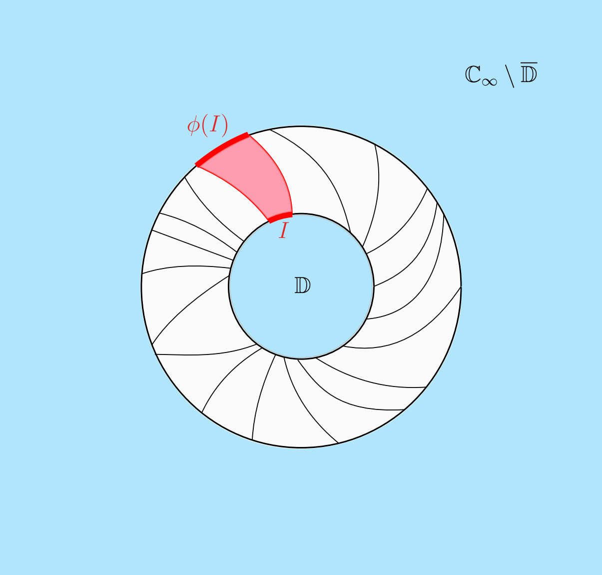

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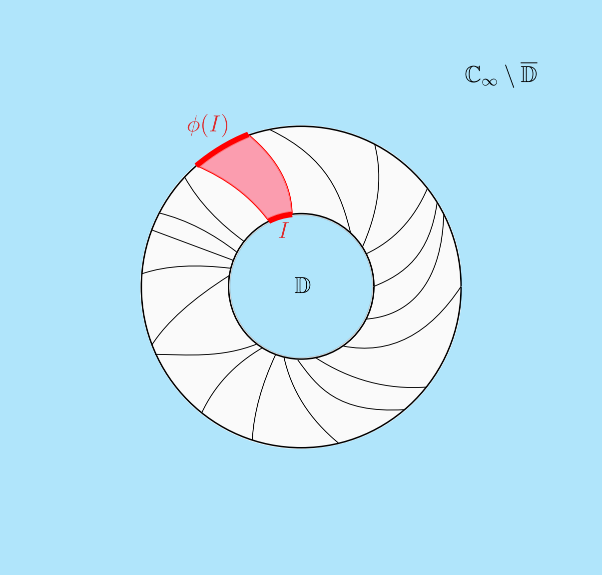

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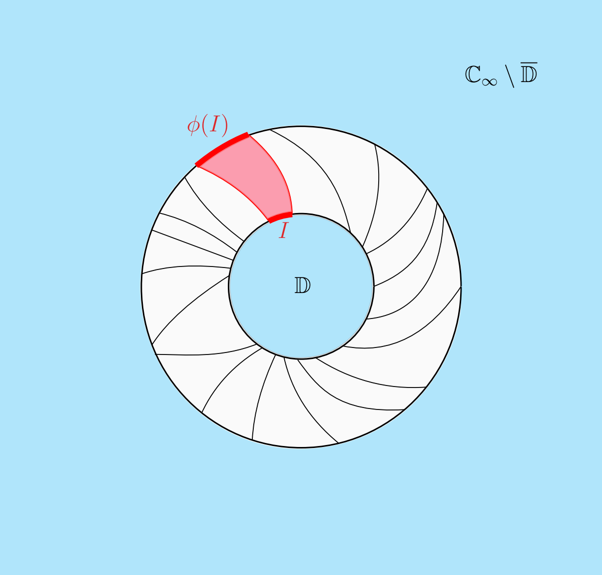

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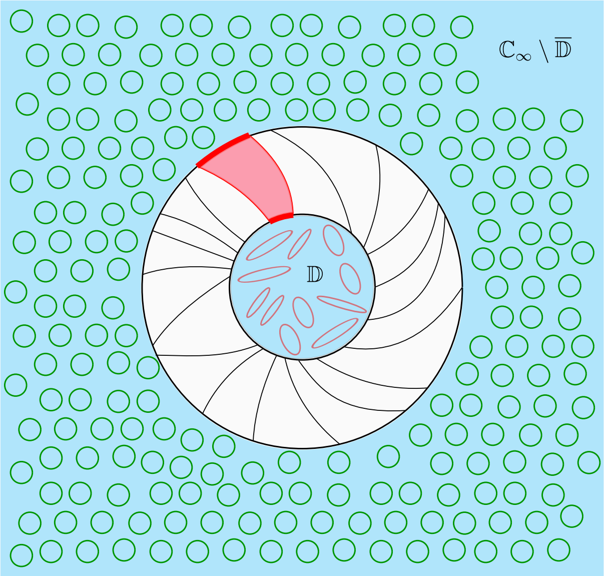

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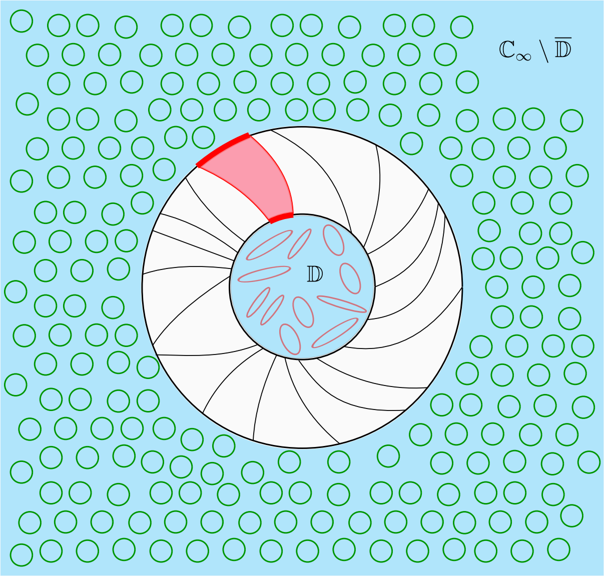

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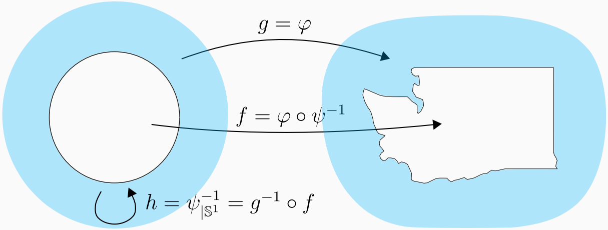

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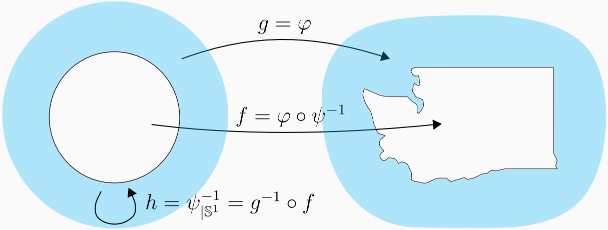

Example (Pfluger, 1960):
Every quasisymmetric $\phi\colon \mathbb{S}^1 \to \mathbb{S}^1$ is a welding.