Heat Flows for Extremal Kähler Metrics

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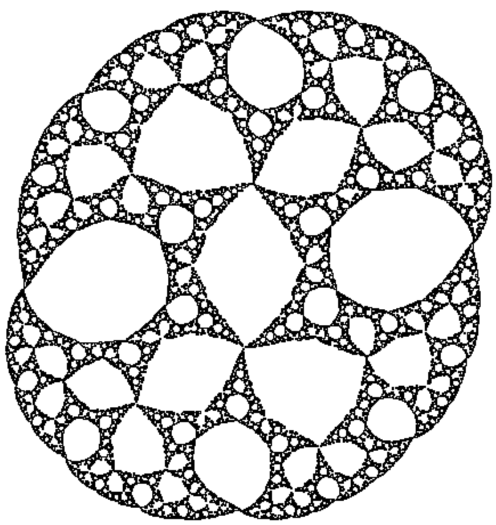

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HEAT FLOWS FOR EXTREMAL KÄHLER METRICS

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Abstract. Let (M, J, Ω) be a polarized complex manifold of Kähler type. Let G be the maximal compact subgroup of the automorphism group of (M, J). On the space of Kähler metrics that are invariant under G and represent the cohomology class Ω, we define a flow equation whose critical points are extremal metrics, those that minimize the square of the L^2-norm of the scalar curvature. We prove that the dynamical system in this space of metrics defined by the said flow does not have periodic orbits, and that its only fixed points, or extremal solitons, are extremal metrics. We prove local time existence of the flow, and conclude that if the lifespan of the solution is finite, then the supremum of the norm of its curvature tensor must blow-up as time approaches it. We end up with some conjectures concerning the plausible existence and convergence of global solutions under suitable geometric conditions.

1. Introduction

We define and study a geometrically motivated dynamical system in the space of Kähler metrics that represent a fixed cohomology class of a given closed complex manifold of Kähler type. The critical points of this flow are extremal metrics, that is to say, minimizers of the functional defined by the L^2-norm of the scalar curvature. We derive the equation, describe some of its general properties, and prove that given an initial data, the equation has a unique classical solution on some time interval. It would be of great interest to know if the solution exist for all time, or whether it develops some singularities in finite time. We have no general answer to this yet. However, we show some evidence indicating that in some specific cases, the solution should exist for all time and converge to an extremal metric as time goes to infinity.

In order to put our equation in the proper perspective, we begin by recalling a different but related one, the Ricci flow. Let M be a compact manifold M of dimension n. Given a metric g, we denote its Ricci tensor by $Ricci_g$ and its average scalar curvature by r_g. The Ricci flow

$$\frac{d}{dt} g = 2\left(\frac{r_g}{n} g - Ricci_g\right) ,$$

was introduced by R. Hamilton [9] as a mechanism to improve the properties of its initial data. It is a non-linear heat equation in the metric, which hopefully becomes better as time passes by in the same way as the heat equation improves an initial distribution of heat in a given region, and makes it uniform all throughout as time goes to infinity. Hamilton used it to show that on a three dimensional manifold, an initial metric of positive Ricci curvature flows according to this equation towards a limit that has constant positive sectional curvature.

In the case of a Kähler manifold, Hamilton's flow equation may be used when seeking a Kähler-Einstein metric on the said manifold. Of course, this would a priori require that the first Chern class c_1 has a sign so that it may be represented by Kähler-Einstein metrics, or their opposites. Regardless of that consideration, the idea inspired Cao [5] to study the equation

$$\frac{d\omega}{dt} = -\rho_t + \eta \, ,$$

for η a fixed real closed $(1, 1)$-form representing the class $c_1(M)$. Using Yau's work on the Calabi conjecture, he proved that solutions exists for all $t \geq 0$ and that the path of metrics so defined

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converges to a Kähler metric with prescribed Ricci form η as $t \to \infty$. He went on and, under the assumption that $c_1(M) < 0$, replaced η in the equation above by $-\omega_t$ and proved that the corresponding solution to the initial value problem exists for all time and converges to a Kähler-Einstein metric as $t \to \infty$.

Given a polarized Kähler manifold (M, J, Ω), we now propose to study the equation

$$\partial_t \omega = -\rho_t + \Pi_t \rho \,,$$

with initial condition a given Kähler metric representing Ω. Here Π_t is a metric dependent projection operator that intertwines the metric trace with the L^2-orthogonal projection π_t onto the space of real holomorphy potentials, these being those real valued functions whose gradients are holomorphic vector fields. The projection Π_t is such that $\Pi_t\rho_t - \rho_t$ is cohomologous to zero, and so all metrics satisfying the equation represent Ω if the same is true of the initial data. For a variety of technical reasons, we define this flow only on the space of metrics representing Ω that are invariant under a fixed maximal compact subgroup of the automorphism group of (M, J). As such, its critical points will be precisely the metrics whose scalar curvatures have holomorphic gradients, or said differently, the extremal metrics of Calabi [3]. This fact constitutes the guiding principle behind our consideration of this new flow equation.

In general, our flow equation is different from the Kähler version of the Ricci flow, even when the class Ω represents $\pm c_1$. This last assertion is illustrated, for instance, by the blow-up of $\mathbb{CP}^2$ at one or two points, and the reason is basically a simple one: extremal metrics, which is what we seek when we consider the new flow, is a more general concept than that of Kähler-Einstein metrics, and when $\Omega = c_1$, these two concepts agree only if we impose an additional restriction on c_1 [7, 16]. It is worth mentioning, however, that on compact connected Riemann surfaces, both the Ricci and extremal flow coincide with one another, and they also coincide with the two-dimensional version of the Yamabe flow. This is so because regardless of the metric g you consider on the given Riemann surface, the holomorphy potential $\pi_g s_g$, where s_g the scalar curvature of g, turns out to be a topological constant. Even in this last case, it is still of some interest to point out that of these three equivalent flows, only the extremal one can be interpreted as the gradient flow of a variational problem.

The main point of the present article will be to show that solutions to the extremal flow equation exists locally in time. However, even if these were going to exist globally, we should not expect that they would converge as time approaches infinity in all possible cases. We already know of examples of Kählerian manifolds that do not admit extremal metrics [2, 13, 18].

We do not have a satisfactory general picture that explains why these examples exist. Those in [13] fail to satisfy a necessary condition on the space of holomorphic vector fields, while those in [2] and [18] are related to stability of the manifold under deformations of the complex structure, and this property does not appear reflected by the Lie algebra of holomorphic vector fields.

At the positive end, we had proven [12] that the set of Kähler classes that can be represented by extremal metrics is open in the Kähler cone. The study of the extremal flow equation above, and its potential convergence to a limit extremal metric, can be seen as a general method that could decide if the the extremal cone is —or is not— closed also.

In proving local time existence of the extremal flow, we also show that if the lifespan is finite, then the pointwise norm of the curvature tensor must blow-up as times approaches it. We leave for later the analysis of global solvability and convergence under suitable geometric conditions.

We organize the paper as follows: in §2 we recall the notion and basic facts about extremal metrics; in §3 we explain in detail the derivation of the extremal flow equation, and prove general results about it; in §4 we linearized this flow equation, showing that it results into a pseudo-differential perturbation of the standard time dependent heat equation. This form of the linearization is an essential fact in our proof of local time existence, done in §5 via a fixed point type of argument.

We end with some remarks justifying our hope that the extremal flow will converge to an extremal metric under suitable general geometric conditions.

2. Extremal Kähler Metrics

Let (M,J,g) be a Kähler manifold of complex dimension n. This means that (M,J) is a complex manifold and that $\omega(X,Y) := g(JX,Y)$, which is skew-symmetric because g is a *Hermitian* Riemannian metric, is a closed 2-form. The differential form ω is called the *Kähler form*, and its cohomology class $[\omega] \in H^2(M,\mathbb{R})$ is called the *Kähler class*.

By complex multi-linearity, we may extend the metric g, the Levi-Civita connection ∇ and the curvature tensor $\mathcal{R}$ to the complexified tangent bundle $\mathbb{C}\otimes TM$. Since $\mathbb{C}\otimes TM$ decomposes into the $\pm i$-eigenspaces of J, $\mathbb{C}\otimes TM = T^{1,0}M \oplus T^{0,1}M$, we can express any tensor field or differential operator in terms of the corresponding decomposition. For example, if $(z^1,\dots,z^n)$ is a holomorphic coordinate system on M, we get induced bases $\{\partial_{z^j}\}$ and $\{\partial_{z^{\bar\jmath}} := \partial_{\bar z^j}\}$ for $T^{1,0}M$ and $T^{0,1}M$, respectively, and if we express the metric g in terms of this basis by setting $g_{\mu\nu} := g\left(\partial_{z^\mu}, \partial_{z^\nu}\right)$, where the indices μ, ν range over $\{1,\dots,n,\bar 1,\dots,\bar n\}$, it follows from the Hermiticity condition that $g_{jk} = g_{\bar\jmath\bar k} = 0$, and that $\omega = \omega_{j\bar k} dz^j \wedge d\bar z^k = i g_{j\bar k} dz^j \wedge d\bar z^k$.

The complexification of the exterior algebra can be decomposed into a direct sum eof of forms of type (p,q). Indeed, we have $\wedge^r M = \bigoplus_{p+q=r} \wedge^{p,q} M$. The integrability of J implies that the exterior derivative d splits as $d = \partial + \bar\partial$, where $\partial : \wedge^{p,q}M \to \wedge^{p+1,q}M$, $\bar\partial : \wedge^{p,q}M \to \wedge^{p,q+1}M$, $\partial^2 = \bar\partial^2 = 0$ and $\partial\bar\partial = -\bar\partial\partial$. Complex conjugation also extends, and we define a form to be real if it is invariant under this operation. An important result in Kähler geometry is that, given a d-exact real form β of type (p,p), there exists a real form α of type $(p-1,p-1)$ such that $\beta = i\partial\overline{\partial}\alpha$.

The *Ricci form* ρ is defined in terms of the Ricci tensor r of g by $\rho(X,Y) = r(JX,Y)$. It is a closed form whose components are given by

$$r_{j\bar k} = -i\rho_{j\bar k} = -\frac{\partial^2}{\partial z^j \partial \bar z^k} \log\det(g_{p\bar q})\,.$$

We have that $\rho/2\pi$ is the curvature of the canonical line bundle $\kappa = \Lambda^n(T^*M)^{1,0}$, and it represents the first Chern class $c_1(M)$.

The *scalar curvature* s is, by definition, the trace $s = r_\mu^\mu = 2g^{j\bar k} r_{j\bar k}$ of the Ricci tensor, and can be conveniently calculated by using the formula

$$s\,\omega^{\wedge n} = 2n\,\rho \wedge \omega^{\wedge(n-1)}\,. \tag{1}$$

Since the volume form is given by $d\mu = \dfrac{\omega^{\wedge n}}{n!}$, this formula implies that, in the compact case,

$$\int_M s\,d\mu = \frac{4\pi}{(n-1)!} c_1 \cup [\omega]^{\cup(n-1)}\,,$$

a quantity that only depends upon the complex structure J and the cohomology class $[\omega]$. Notice that $\displaystyle\int_M d\mu = \frac{1}{n!}[\omega]^{\cup n}$, and so *the average scalar curvature* s_0 is also a quantity that depends only on the Kähler class $[\omega]$ and the homotopy class of the complex-structure tensor J.

Suppose that (M,J) is a closed complex manifold *polarized* by a positive class $\Omega \in H^{1,1}(M,\mathbb{C}) \cap H^2(M,\mathbb{R})$. Let $\mathfrak{M}_\Omega$ be the set of all Kähler forms representing Ω. Since any two elements $\tilde\omega$ and ω of $\mathfrak{M}_\Omega$ are such that $\tilde\omega = \omega + i\partial\overline{\partial}\varphi$ for some real valued potential function φ, we can define a topology on $\mathfrak{M}_\Omega$ by defining a suitable topology on the space of potentials. In what follows, we shall not distinguish between the Kähler metric and its Kähler form, passing from one to the other at will.

Consider the functional

$$
\begin{array}{rcl}
\mathfrak{M}_\Omega & \xrightarrow{E_\Omega} & \mathbb{R} \\
\omega & \mapsto & \displaystyle\int_M s_\omega^2 d\mu_\omega
\end{array}\,, \tag{2}
$$

where the metric associated with the form ω has scalar curvature s_ω and volume form $d\mu_\omega$. A critical point of this functional is by definition an *extremal* Kähler metric [3], a notion introduced with the idea of seeking canonical representatives of Ω.

Given any Kähler metric g, a smooth complex-valued function f gives rise to the (1,0) vector field $f \mapsto \partial^{\#} f = \partial^{\#}_g f$ defined by the expression

$$
g(\partial^{\#} f,\, \cdot\,) = \overline{\partial} f\,.
$$

This vector field is *holomorphic* iff we require that $\bar\partial \partial^{\#} f = 0$, condition equivalent to f being in the kernel of the operator

$$
L_g f := (\bar\partial\partial^{\#})^* \bar\partial\partial^{\#} f = \frac{1}{4}\Delta^2 f + \frac{1}{2} r^{\mu\nu}\nabla_\mu\nabla_\nu f + \frac{1}{2}(\nabla^{\bar\ell}\sigma)\nabla_{\bar\ell} f\,. \tag{3}
$$

We then have that

$$
\frac{d}{dt} E_\Omega(\omega + t i\partial\overline{\partial}\varphi)\mid_{t=0} = -4\int s_\omega L_\omega \varphi d\mu_\omega\,.
$$

Hence, the Euler-Lagrange's equation for a critical point g of (2) is just that the scalar curvature s_g be in the kernel of L_g. In other words, the vector field $\partial^{\#}_g s_g$ must be holomorphic.

3. Derivation of the evolution equation

Calabi [4] showed that the identity component of the isometry group of an extremal Kähler metric g is a maximal compact subgroup of the identity component of the biholomorphism group of (M, J). This implies that, up to conjugation, the identity components of the isometry groups of extremal Kähler metrics coincide [12]. Therefore, modulo biholomorphisms, the search for extremal Kähler metrics is completely equivalent to the search for extremal metrics among those that are invariant under the action of a fixed maximal compact subgroup of the connected biholomorphism group. This last problem, however, turns out to be technically easier to analyze.

3.1. **Holomorphy potentials.** For any given Kähler metric g on (M, J), every complex-valued function f in the kernel of (3) is associated with the holomorphic vector field $\Xi = \partial^{\#} f$, and since the operator is elliptic, the space of such functions is finite dimensional. However, since $(\bar\partial\partial^{\#})^*\bar\partial\partial^{\#}$ is not a real operator, in general, the real and imaginary part of a solution will not be solutions. It has been proven elsewhere [12] that if f is a real valued solution of this equation, then the imaginary part of $\partial^{\#} f$ is a Killing field of g, and that a Killing field arises in this way if, and only if, its zero set is not empty.

Let G be a maximal compact subgroup of the biholomorphism group of (M, J), and g be a Kähler metric on M with Kähler class Ω. Without loss of generality, we assume that g is G-invariant. We denote by $L^2_{k,G}$ the real Hilbert space of G-invariant real-valued functions of class L^2_k, and consider G-invariant deformations of this metric preserving the Kähler class:

$$
\tilde\omega = \omega + i\partial\overline{\partial}\varphi\,, \quad \varphi \in L^2_{k+4,G}\,,\ k > n. \tag{4}
$$

In this expression, the condition $k > n$ ensures that $L^2_{k,G}$ is a Banach algebra, making the scalar curvature of $\tilde\omega$ a well-defined function in the space.

Let $\mathfrak{h}$ be the complex Lie algebra of holomorphic vector fields of the complex manifold (M, J); by compactness of M, this is precisely the Lie algebra of the group of biholomorphism of (M, J). We denote by $\mathfrak{g}$ the Lie algebra of G. If $\mathfrak{z} \subset \mathfrak{g} \subset \mathfrak{h}$ is the center of $\mathfrak{g}$, we let $\mathfrak{z}_0 = \mathfrak{z} \cap \mathfrak{g}_0$, where

$\mathfrak{g}_0 \subset \mathfrak{g}$ is the ideal of Killing fields which have zeroes. If $\tilde{g}$ is any G-invariant Kähler metric on (M, J), then each element of $\mathfrak{z}_0$ is of the form $J\nabla_{\tilde{g}} f$ for a real-valued solution of (3). In fact, $\mathfrak{z}_0$ corresponds to the set of real solutions f which are *invariant under* G, since

$$\partial^{\#} : \ker[(\bar{\partial}\partial^{\#}_{\tilde{g}})^* \bar{\partial}\partial_{\tilde{g}}] \rightarrow \mathfrak{h}_0 \,,$$

$\mathfrak{h}_0 \subset \mathfrak{h}$ the subset of holomorphic vector fields with zeroes, is a homomorphism of G-modules.

The restriction of $\ker(\bar{\partial}\partial^{\#}_{\tilde{g}})^* \bar{\partial}\partial_{\tilde{g}}$ to $L^2_{k+4,G}$ depends smoothly on the G-invariant metric $\tilde{g}$. Indeed, choose a basis $\{X_1, \ldots, X_m\}$ for $\mathfrak{z}_0$, and, for each $(1,1)$-form χ on (M, J), consider the set of functions

$$\begin{array}{rcl} p_0(\chi) & = & 1 \\ p_j(\chi) & = & 2iG_g \overline{\partial}^*_g ((JX_j + iX_j) \lrcorner \chi) \,, \quad j = 1, \ldots, m \end{array}$$

where G_g is the Green's operator of the metric g. If $\tilde{\omega}$ is the Kähler form of the G-invariant metric $\tilde{g}$, then $\partial^{\#}_{\tilde{g}} p_j(\tilde{\omega}) = JX_j + iX_j$, and the set $\{p_j(\tilde{\omega})\}_{j=0}^m$ consists of real-valued functions and forms a basis for $\ker(\bar{\partial}\partial^{\#}_{\tilde{g}})^* \bar{\partial}\partial_{\tilde{g}}$. Furthermore, for metrics $\tilde{\omega}$ as in (4), the map $\varphi \mapsto p_j(\omega + i\partial\overline{\partial}\varphi)$ is, for each j, bounded as a linear map from $L^2_{k+4,G}$ to $L^2_{k+3,G}$.

With respect to the fixed L^2 inner product, let $\{f^0_{\tilde{\omega}}, \ldots, f^m_{\tilde{\omega}}\}$ be the orthonormal set extracted from $\{p_j(\tilde{\omega})\}$ by the Gram-Schmidt procedure. We then let

$$\begin{array}{rcl} \pi_{\tilde{\omega}} : L^2_{k,G} & \rightarrow & L^2_{k,G} \\ u & \mapsto & \displaystyle\sum_{j=0}^{m} \langle f^j_{\tilde{\omega}}, u \rangle_{L^2} f^j_{\tilde{\omega}} \end{array} \tag{5}$$

denote the associated projector. In fact, by the regularity of the functions $\{p_1, \ldots, p_m\}$, this projection can be defined on $L^2_{k+j,G}$ for $j = 0, 1, 2, 3$, and for metrics as in (4), the map $\varphi \mapsto \pi_{\tilde{\omega}}$ is smooth from a suitable neighborhood of the origin in $L^2_{k+4,G}$ to the real Hilbert space $\mathrm{End}(L^2_{k+j,G}) \cong \bigotimes^2 L^2_{k+j,G}$.

Given a Kähler metric ω, its normalized Ricci potential ψ_ω is defined to be the only function orthogonal to the constants such that $\rho = \rho_H + i\partial\overline{\partial}\psi_\omega$, where ρ_H is the ω-harmonic component of ρ. In terms of the scalar curvature and its projection onto the constants, we have that $\psi_\omega = -G_\omega(s_\omega - s_0)$. Given any cohomology class $\Omega = [\omega]$ in the Kähler cone of (M, J), the *Futaki character* of the class is defined to be the map

$$\begin{array}{rcl} \mathfrak{F} : \mathfrak{h} & \longrightarrow & \mathbb{C} \\ \Xi & \mapsto & \mathfrak{F}(\Xi, [\omega]) = \displaystyle\int_M \Xi(\psi_\omega) d\mu = -\int_M \Xi(G_\omega(s_\omega - s_0)) d\mu_\omega \,. \end{array} \tag{6}$$

It is independent of the particular metric ω in $\mathfrak{M}_{[\omega]}$ chosen to calculate it [7, 4], and when applied to a holomorphic vector field of the form $\Xi = \partial^{\#} f$, it yields

$$\mathfrak{F}(\Xi, [\omega]) = -\int_M f(s_\omega - s_0)\, d\mu_\omega \,. \tag{7}$$

A metric $g \in \mathfrak{M}_\Omega$ is extremal iff $\partial^{\#}_g s_g$ is a holomorphic vector field. In other words, g is extremal iff $s_g = \pi_g s_g$. These metrics achieve the infimum of E_Ω over $\mathfrak{M}_\Omega$. Indeed, there is exists a number $E = E(\Omega)$ and a holomorphic vector field X_Ω such that

$$E_\Omega(\omega) \geq E(\Omega) := s_0^2 \frac{\Omega^n}{n!} - \mathfrak{F}(X_\Omega, \Omega) \tag{8}$$

for all $\omega \in \mathfrak{M}_\Omega$. The field X_Ω may depend on the choice of a maximal compact subgroup G of the automorphism group of (M, J), but the value of $\mathfrak{F}(X_\Omega, \Omega)$ does not. Given G and a metric $g \in \mathfrak{M}_\Omega$

that is G-invariant, we may take $X_\Omega = \partial_g^{\#}(\pi_g s_g - s_0)$, and we easily see that

$$E(\Omega) = \int (\pi_g s_g)^2 d\mu_g \, .$$

This way of computing the energy of the class through G-invariant metrics is very convenient and has been used several times elsewhere [15, 16, 17].

Remark 1. Let g be any Kähler metric on any Riemann surface Σ. By the Gauss-Bonnet theorem, the average scalar curvature of g is given by $4\pi\chi(M)/\mu_g(\Sigma)$. The invariance of (6) and (7) imply that if g is a G-invariant metric on Σ, the projection $\pi_g s_g$ of s_g onto the space of real holomorphy potentials is always equal to this constant. Consequently, the vector field X_Ω of a polarized Riemann surface is always trivial. □

From now on, we shall denote by $\mathfrak{M}_{\Omega,G}$ the set of G-invariant Kähler metrics representing the class Ω. When considering a path of metrics $\omega_t \in \mathfrak{M}_{\Omega,G}$, the fact that the kernel of $(\bar\partial\partial_{\tilde g}^{\#})^*\bar\partial\partial_{\tilde g}$ depends smoothly on $\tilde g$ allows us to rightfully compute the differential of $\pi_t s_t$. Here, π_t and s_t are the projection and scalar curvature associated to ω_t, respectively. Since $\pi_t s_t$ is of order four in the potential of the metric, naively we would expect its differential to be an operator of order four on the tangent space to $\mathfrak{M}_{\Omega,G}$ at ω_t. However, we get something significantly better, and gain quite a bit of regularity. This fact that will be very convenient later on.

Lemma 2. *Let $\omega_t = \omega + i\partial\overline{\partial}\varphi_t$ be a path of metrics in $\mathfrak{M}_{\Omega,G}$ with $\omega_0 = \omega$. Consider the projection $\pi_t s_t$ of the scalar curvature s_t onto the space of real holomorphy potentials, and let $\dot\varphi_t = \frac{d}{dt}\varphi_t$. Then*

$$\frac{d}{dt}(\pi_t s_t) = \partial\dot\varphi_t \lrcorner X_\Omega = (\partial^{\#}\dot\varphi_t, X_\Omega)_t = (\partial\dot\varphi_t, \partial(\pi_t s_t))_t \, ,$$

where $X_\Omega = \partial_t^{\#}\pi_t s_t$ is the holomorphic vector field of the class Ω. In particular, this derivative is a differential operator of order one in $\dot\varphi_t$ whose coefficients depend non-linearly on the coefficients of ω_t.

Proof. By the invariance of the Futaki character, if $\pi_\omega s_\omega$ is constant then so will be $\pi_{\tilde\omega} s_{\tilde\omega}$ for any other metric $\tilde\omega$ in $\mathfrak{M}_{\Omega,G}$ (see §4 of [16]). In that case, X_Ω is trivial and both sides of the expression in the statement are zero. The result follows.

So let us assume that $\pi_t s_t$ is not constant. For convenience, we use the subscript t to denote geometric quantities associated with ω_t. Thus, the imaginary part of $X_\Omega = \partial_{\omega_t}^{\#}\pi_t s_t$ is a non-trivial Killing vector field, and in the construction of the projection map above, we can choose a basis $\{X_j\}$ for $\mathfrak{z}_0$ such that $X_\Omega = \partial_{\omega_t}^{\#}\pi_t s_t = JX_1 + iX_1 = X_\Omega$. Hence,

$$\pi_t s_t = 2iG_t\overline{\partial}_t^*(\omega_t \lrcorner X_\Omega) + s_0 \, ,$$

expression that we now know depends differentiably on ω_t. Here, s_0 is the projection of s onto the constants, constant that itself only depends on Ω, J and the volume of metrics in $\mathfrak{M}_{\Omega,G}$. By the Kähler identity $\overline{\partial}_t^* = -i[\Lambda_t, \partial]$, we conclude that

$$\pi_t s_t = 2G_t\Lambda_t\partial(\omega_t \lrcorner X_\Omega) + s_0 \, ,$$

and therefore,

$$\frac{d}{dt}\pi_t s_t = 2G_t\Lambda_t\partial(\dot\omega_t \lrcorner X_\Omega) + 2G_t\dot\Lambda_t\partial(\omega_t \lrcorner X_\Omega) + 2\dot G_t\Lambda_t\partial(\omega_t \lrcorner X_\Omega) \, .$$

The last two terms in the expression above cancel each other out. Indeed, $\omega_t \lrcorner X_\Omega = -i\overline{\partial}(\pi_t s_t)$ and computing the derivative of $\dot\Lambda_t$ in terms of $\dot\varphi_t$, we see that $2G_t\dot\Lambda_t\partial(\omega_t \lrcorner X_\Omega) = 2G_t(i\partial\overline{\partial}\dot\varphi_t, i\partial\overline{\partial}(\pi_t s_t))_t$. On the other hand, the differential of the Green's operator is given by $-G_t\dot\Delta_t G_t$, and we obtain that $2\dot G_t\Lambda_t\partial(\omega_t \lrcorner X_\Omega) = 2G_t\dot\Delta_t G_t\Lambda_t i\partial\overline{\partial}(\pi_t s_t) = -G_t\dot\Delta_t(\pi_t s_t) = -2G_t(i\partial\overline{\partial}\dot\varphi_t, i\partial\overline{\partial}(\pi_t s_t))_t$.

Since the real and imaginary parts of X_Ω are Killing vector fields and the metric potential φ_t is G-invariant, we have that $X_\Omega(\dot\varphi_t) = 0$, and so $\partial\dot\varphi_t \lrcorner X_\Omega = (\partial\dot\varphi_t, i\partial(\pi_t s_t))_t$ is orthogonal to the constants. But notice also that since X_Ω is holomorphic, we have that $\dot\omega_t = i\partial\overline{\partial}\dot\varphi_t \lrcorner X_\Omega = -i\overline{\partial}(\partial\dot\varphi_t \lrcorner X_\Omega)$. Thus, we obtain $2G_t\Lambda_t\partial(\omega_t \lrcorner X_\Omega) = -2G_t\Lambda_t i\partial\overline{\partial}(\partial\dot\varphi_t \lrcorner X_\Omega)$. The desired result follows now because G_t is the inverse of the Laplacian in the complement of the constants. □

Given any Kähler metric g in $\mathfrak{M}_{\Omega,G}$, the extremal vector field X_Ω of the class can be written as $X_\Omega = \partial^{\#}_g(\pi_g s_g)$. Thus, the critical points of $\pi_g s_g$ corresponds to zeroes of X_Ω, and are therefore, independent of g. By the Lemma above, this can be strengthen a bit, and we have the following remarkable consequence. This result is reminiscent of the convexity theorem on the image of moment mappings [1, 8].

Theorem 3. *Let ω be any metric in $\mathfrak{M}_{\Omega,G}$ and consider the function $\pi_\omega s_\omega$ obtained by projection of the scalar curvature onto the space of real holomorphy potentials. Then the range of $\pi_\omega s_\omega$ is a closed interval on the real line that only depends on the class Ω and not on the particular metric $\omega \in \mathfrak{M}_{\Omega,G}$ chosen to represent it.*

Proof. Let $\omega_t = \omega + i\partial\overline{\partial}\varphi_t$ be a path in $\mathfrak{M}_{\Omega,G}$. If we use the subscript t to denote geometric quantities associated with ω_t, by Lemma 2 we have that

$$\frac{d}{dt}\pi_t s_t = (\partial\dot\varphi_t, \partial(\pi_t s_t))_t \,.$$

Since the maximum and minimum of $\pi_t s_t$ occur at critical points, this expression shows that these extrema values do not change with t. The result follows because $\mathfrak{M}_{\Omega,G}$ is path connected. □

We now proceed to lift the projection π_g onto holomorphy potentials to the level of G-invariant $(1,1)$ forms. This lift will be essential in our definition of the extremal flow.

So let us denote by $\wedge^{1,1}_{k,G}$ be the space of real forms of type $(1,1)$, invariant under G and of class L^2_k.

Lemma 4. *Given any G-invariant metric $\tilde g$, there exists a uniquely defined continuous projection map*

$$\Pi_{\tilde g} : \wedge^{1,1}_{k+2,G} \mapsto \wedge^{1,1}_{k+2,G} \,, \tag{9}$$

which intertwines the trace and the projection map $\pi_{\tilde\omega}$ in (5), and such that $\eta - \Pi_{\tilde g}\eta$ is cohomologous to zero for all $\eta \in \wedge^{1,1}_{k+2,G}$. For metrics $\tilde\omega$ as in (4), the map $\varphi \mapsto \Pi_{\tilde\omega}$ from $L^2_{k+4,G}$ to $\mathrm{End}(\wedge^{1,1}_{k+2,G})$ is smooth.

Proof. Let $\eta \in \wedge^{1,1}_{k+2,G}$. Since $\Pi_{\tilde\omega}\,\eta$ must be of the form $\eta + i\partial\overline{\partial}f$ for some real valued function f, the intertwining property of the projection and trace gives that

$$\mathrm{trace}_{\tilde\omega}\,\eta - \frac{1}{2}\Delta_{\tilde\omega} f = \pi_{\tilde\omega}\,\mathrm{trace}_{\tilde\omega}\,\eta \,,$$

and so

$$\Delta_{\tilde\omega} f = -2(\pi_{\tilde\omega} - 1)\mathrm{trace}_{\tilde\omega}\eta \,.$$

The right side of this expression is a G-invariant real valued function in the complement of the constants. We can then solve the equation for f and obtain a real valued function which is invariant under G. By the continuity properties of the map $\pi_{\tilde\omega}$, for metrics as in (4) the map $\varphi \mapsto \Pi_{\tilde\omega}$ is a smooth map from a suitable neighborhood of the origin in $L^2_{k+4,G}$ to the real Hilbert space $\mathrm{End}(\wedge^{1,1}_{k+2,G})$. □

3.2. **Extremal flow equation.** Lemma 4 provides us with the tool needed to set-up the heat flow equation adapted to the extremal metric problem. The idea of using *good* flows to better geometric quantities was originally used by Eells and Sampson [6] in another context, and reconsidered by Hamilton [9] in his definition of the Ricci flow. In our case, we are given a metric in $\mathfrak{M}_{\Omega,G}$ and try to improve it by means of a non-linear *pseudo-differential* heat equation, requiring the velocity of the curve to equal the component of the Ricci curvature that is perpendicular to the image of Π.

More precisely, we fix a maximal compact subgroup G of the automorphism group of (M,J), and work on $\mathfrak{M}_{\Omega,G}$, the space of all G-invariant Kähler forms that represent Ω. Given $\omega \in \mathfrak{M}_{\Omega,G}$, we consider a path ω_t of Kähler metrics that starts at ω at $t=0$ and obeys the flow equation $\partial_t \omega_t = -\rho_t + \Pi_t \rho_t$. Since $-\rho_t + \Pi_t\rho_t$ is cohomologous to zero and G-invariant, for as long as the solution exists, we will have that $\omega_t \in \mathfrak{M}_{\Omega,G}$. Thus, our evolution equation is given by the initial value problem

$$
\begin{array}{rcl} \partial_t \omega_t & = & -\rho_t + \Pi_t \rho_t \, . \\ \omega_0 & = & \omega \, . \end{array} \tag{10}
$$

Critical points of this equation correspond to extremal metrics, that is to say, metrics such that $\rho = \Pi\rho$.

It is easy to reformulate (10) as a scalar equation. For if $\omega_t = \omega + i\partial\overline{\partial}\varphi_t$, we have that $\Pi_t\rho_t - \rho_t = i\partial\overline{\partial}G_t(s_t - \pi_t s_t)$, where G_t is the Green's operator of the metric ω_t, and by compactness of M, we see that the potential φ_t evolves according to

$$
\begin{array}{rcl} \partial_t \varphi_t & = & G_t(s_t - \pi_t s_t) \, , \\ \varphi_0 & = & 0 \, . \end{array} \tag{11}
$$

A critical point of this scalar version of the equation is given by a metric for which $G_\omega(s_\omega - \pi_\omega s_\omega) = 0$, and since $s_\omega - \pi_\omega s_\omega$ is orthogonal to the constant, this condition is equivalent to saying that $s_\omega = \pi_\omega s_\omega$. Thus, a critical point ω is an extremal metric.

3.3. **General properties of the extremal flow.** We begin by making a rather expected observation.

Proposition 5. *Let ω_t be a solution of the initial value problem* (10)*. If $d\mu_t$ is the volume form, we have that*

$$
\frac{d}{dt} d\mu_t = \frac{1}{2}(\pi_t s_t - s_t) d\mu_t \, .
$$

In particular, the volume of ω_t is constant.

Proof. The volume form is given by

$$
d\mu_t = \frac{\omega_t^n}{n!} \, .
$$

Differentiating with respect to t, we obtain:

$$
\frac{d}{dt} d\mu_t = \frac{1}{(n-1)!}\omega_t^{n-1} \wedge \dot{\omega}_t = \frac{1}{(n-1)!}\omega_t^{n-1} \wedge (\Pi_t\rho_t - \rho_t) = \frac{1}{2}(\pi_t s_t - s_t) d\mu_t \, ,
$$

as desired. Notice that this form of maximal rank is exact. □

Our next results address the plausible existence of fixed points or periodic solutions of the flow equation.

Observe that (10) is invariant under the group of diffeomorphims that preserve the complex structure J. An *extremal soliton* is a solution that changes only by such a diffeomorphism. Then, there must be a holomorphic vector field $V = (V^i)$ such that $V_{i,\bar{j}} + V_{\bar{j},i} = \Pi\rho_{i\bar{j}} - \rho_{i\bar{j}}$. If the vector field V has a holomorphy potential f, we refer to the pair (g, V) as a gradient extremal soliton.

Proposition 6. *There are no extremal solitons other than extremal metrics.*

Proof. Suppose we have an extremal gradient soliton (g, V) defined by a holomorphic potential f. Then

$$i\partial\overline{\partial}f = \Pi\rho - \rho\,,$$

and therefore,

$$f = G_g(s - \pi s)\,.$$

This implies that $\Delta f = s - \pi s$ and since Δ is a real operator, the holomorphy potential f must be real. But f is a holomorphy potential, so it is L^2-orthogonal to $s - \pi s$. Hence,

$$\|\nabla f\|^2 = \int f \Delta f d\mu_g = \int f(1-\pi)s\, d\mu = 0\,.$$

Thus, f is constant, and therefore, necessarily zero.

Thus, a non-trivial soliton, if any, must be given by a holomorphic vector field V that is not a gradient. The set of all such vector fields forms an Abelian subalgebra of the algebra of holomorphic vector fields. The group of diffeomorphism they generate must be in the maximal compact subgroup G of isometries of the metric. This vector field does not change the metric and so $\omega_t = (\exp(tV))^*\omega = \omega$. Hence, $\dot{\omega}_t = 0 = \Pi\rho - \rho$, and the metric is extremal. □

We now show that the evolution equation (10) is *almost* the gradient flow of the K-energy that characterizes extremal Kähler metrics [14]. Given two elements ω_0 and ω_1 of $\mathfrak{M}_{\Omega,G}$, there exists a G-invariant function φ, unique modulo constants, such that

$$\omega_1 = \omega_0 + i\partial\overline{\partial}\varphi\,.$$

Let φ_t be a curve of G-invariant functions such that $\omega_t = \omega_0 + i\partial\overline{\partial}\varphi_t \in \mathfrak{M}_{\Omega,G}$ and $\omega(0) = \omega_0$, $\omega(1) = \omega_1$. We set

$$M(\omega_0, \omega_1) = -\int_0^1 dt \int_M \dot{\varphi}_t(s_t - \pi_t s_t) d\mu_t\,,$$

where s_t and $d\mu_t$ are the scalar curvature and volume form of the metric ω_t, π_t is the projection (5) onto the space of G-invariant holomorphic potentials associated with this metric, and $\dot{\varphi}_t = \dfrac{d\varphi_t}{dt}$. This definition is independent of the curve $t \to \varphi_t$ chosen.

Fix $\omega_0 \in \mathfrak{M}_{\Omega,G}$. The K-energy is defined to be

$$\begin{array}{ccc} \mathfrak{M}_{\Omega,G} & \xrightarrow{\kappa} & \mathbb{R} \\ \omega & \to & M(\omega_0, \omega)\,. \end{array} \tag{12}$$

We have (see Proposition 2 in [14]) that

$$\frac{d}{dt}\kappa(\omega_t) = -\int_M \dot{\varphi}_t(s_t - \pi_t s_t) d\mu_t\,.$$

Thus, up to the action of the non-negative Green's operator, the gradient of κ is given by the right-side of (11), and we have

Proposition 7. *Let ω_t be a solution of the initial value problem* (10). *Then*

$$\frac{d}{dt}\kappa(\omega_t) = -\int_M (s_t - \pi_t s_t) G_t(s_t - \pi_t s_t) d\mu_t\,.$$

The flow equation (10) is also invariant under the one-parameter group of homotheties, where time scales like the square of the distance. In principle, such an invariance could give rise to periodic orbits of the flow. However,

Proposition 8. *The only periodic orbits of the flow equation* (10) *are its fixed points, that is to say, the extremal metrics* (*if any*) *in $\mathfrak{M}_{\Omega,G}$.*

Proof. Consider the K-energy suitably normalized by a volume factor to make it scale invariant. If there is a loop solution ω_t of (10) for $t \in [t_1, t_2]$, since the volume remains constant, we will have that $\kappa(\omega_{t_1}) = \kappa(\omega_{t_2})$. By the previous proposition, since G_t is a non-negative operator, we conclude that $G_t(s_t - \pi_t s_t) = 0$ on this time interval. This says that ω_t is extremal for each t on the interval, and so the right side of the evolution equation is zero. Thus, the loop is trivial, a fixed point of the flow. □

We end this section by showing that the functional (2) decreases along the flow (10). This should be clear from the way the equation was set-up, or at the very least, expected.

Proposition 9. *Let ω_t be a path in $\mathfrak{M}_{\Omega,G}$ that solves the flow equation* (10)*. Then*

$$\frac{d}{dt}E_\Omega(\omega_t) = -4\int (s_t - \pi_t s_t) L_t G_t (s - \pi_t s_t) d\mu_t \leq 0\,,$$

and the equality is achieved if and only if ω_t is extremal. In this expression, $L_t = (\overline{\partial}\partial^{\#})^*(\overline{\partial}\partial^{\#})$ and G_t is the Green's operator.

Proof. Given any variation of the metric with potential function φ, we know that

$$\frac{d}{dt}E(\omega_t) = -4\int s L_t \dot{\varphi} d\mu_t\,.$$

But $\dot{\varphi} = G_t(s_t - \pi_t s_t)$, and since $\pi_t s_t$ is a holomorphy potential and L_t is self-adjoint, we see that

$$\frac{d}{dt}E(\omega_t) = -4\int (s_t - \pi_t s_t) L_t G_t (s - \pi_t s_t) d\mu_t\,.$$

Both L_t and G_t are non-negative operators. Then so is $L_t G_t$, and the expression above is non-negative. If it reaches the value zero at some t, then we must have that $f_t = G_t(s_t - \pi_t s_t)$ is a holomorphy potential and $\Delta_t f = (1 - \pi_t)s_t$ is an element of the image of $(1 - \pi_t)$. Thus, f_t is L^2-orthogonal to $(1 - \pi_t)s_t$, and integration by parts yields that the gradient field $\nabla_t f_t$ is zero. Thus, f_t is a constant, which is necessarily zero. We then obtain that $s_t = \pi_t s_t$ and the metric ω_t is extremal. □

It is clear that we could have used also the energy E in the rôle that κ played when proving that the flow does not have periodic orbits other than its fixed points. In fact, it is better to work with E_Ω itself. For we do not know if κ is in general bounded below on $\mathfrak{M}_{\Omega,G}$, but the energy functional E_Ω has this property indeed. If the solution to the flow equation were to exist for all $t \in [0, \infty)$, the monotonicity result above would lead us to expect that, as $t \to \infty$, the sequence ω_t should be getting closer and closer to an extremal metric. In any case, we discussed both κ and E_Ω to show their similar behaviour under the extremal flow.

4. The linearized flow equation

Consider a family of metrics in $\mathfrak{M}_{\Omega,G}$ of the form $\omega_t(v) = \omega_\varphi + i\partial\overline{\partial}\alpha(t, v)$, with $\alpha(t, 0) = 0$. We set $\beta = \beta_t = \dfrac{d\alpha(t,v)}{dv}\mid_{v=0}$. The linearization of (11) at ω_φ in the direction of β is given by

$$\partial_t \beta_t = \frac{d}{dv}(G_{(t,v)}((1 - \pi_{(t,v)})s_{(t,v)})) \mid_{v=0}\ .$$

Of course, before taking the restriction to $t = 0$, the argument of the v-differentiation in the right side involves quantities associated with the metric $\omega_t(v)$.

In the remaining part of this section we use the subscript φ, or no subscript at all, to denote geometric quantities associated with the metric ω_φ. We have that

$$\frac{ds_{(t,v)}}{dv}\mid_{v=0} = -\frac{1}{2}\Delta^2_\varphi \beta - 2(\rho_\varphi, i\partial\overline{\partial}\beta)_\varphi\,.$$

Since the variation of the Green's operator is $-G_\varphi(\frac{d}{dv}\Delta_{(t,v)})G_\varphi$ (keep in mind that this operator needs to be applied only to $s-\pi s$, a function that is orthogonal to the constants), using the relation between ρ_φ and $\Pi_\varphi\rho_\varphi$, we obtain that

$$\partial_t\beta = -\frac{1}{2}\Delta_\varphi\beta - 2G_\varphi(\Pi_\varphi\rho_\varphi, i\partial\overline{\partial}\beta)_\varphi - G_\varphi\left(\frac{d}{dv}\pi_{(t,v)}s_{(t,v)}\mid_{v=0}\right).$$

By Lemma 2, we may write this as

$$\partial_t\beta = -\frac{1}{2}\Delta_\varphi\beta - 2G_\varphi(\Pi_\varphi\rho_\varphi, i\partial\overline{\partial}\beta)_\varphi - G_\varphi(\partial^{\#}_\varphi\beta, X_\Omega)_\varphi\,, \tag{13}$$

where X_Ω is the holomorphic vector field of the class Ω, vector field that can be expressed as $X_\Omega = \partial^{\#}_\varphi(\pi_\varphi s_\varphi)$. Notice that

$$P_\varphi(\beta) = G_\varphi(\partial^{\#}_\varphi\beta, X_\Omega)_\varphi$$

is a pseudo-differential operators of order -1 in β whose coefficients depend non-linearly on the coefficients of the metric ω_φ.

We summarize our discussion into the following

Theorem 10. *Let (M, J, Ω) be a polarized Kähler manifold and let G be a maximal compact subgroup of $Aut(M, J)$. The extremal flow equation* (11) (*or equivalently,* (10)) *in $\mathfrak{M}_{\Omega,G}$ is a non-linear pseudo-differential parabolic equation.*

Remark 11. Generically, the manifold (M, J) carries no non-trivial holomorphic vector fields, and the space of holomorphic potentials reduces to the constants. (For example, this is the situation when the first Chern class $c_1(M, J)$ is negative.) This is still so in the slightly larger case of a manifold (M, J) where all of its non-trivial holomorphic vector fields have no zeroes. Under this hypothesis, the pseudo-differential term of order -1 in the right side of the linearized flow equation (13) vanishes, and the equation reduces to

$$\partial_t\beta_t = -\frac{1}{2}\Delta_\varphi\beta - 2G_\varphi(\Pi_\varphi\rho_\varphi, i\partial\overline{\partial}\beta)_\varphi\,,$$

still a pseudo-differential equation, in this case, a zeroth-order perturbation of pseudo-differential type of the standard time dependent heat equation. Thus, even for generic complex manifolds of Kähler type, the pseudo-differential nature of our flow equation remains. □

Remark 12. Even if Ω is the canonical Kähler class c_1 (which a fortiori must then have a sign), the extremal flow equation (10) (or equivalently, (11)) does not necessarily coincide with the Kähler Ricci flow. This will only be the case if we know a priori that πs is a constant, which is a rather non-trivial condition to impose and only happen if the Futaki character of the canonical class vanishes. This fails to be so in general, as is the case, for instance, of the blow-up of $\mathbb{CP}^2$ at one or two points. □

We now introduced an *approximate* linearized equation whose solution is needed in our study of local solvability of (11). In order to do so, we make some preliminary observations.

Let T be a positive real number to be determined later and set $I = [0, T]$. A scale $\mathcal{Y} = \{\mathcal{Y}_j\}_{j\geq 0}$ of Banach spaces is a countable family of complete normed spaces such that $\mathcal{Y}_j \supset \mathcal{Y}_{j+1}$ and each $\mathcal{Y}_j$ is dense in $\mathcal{Y}_0$. Given one such, we define

$$C_{(j,k)}(I;\mathcal{Y}) = C^0(I;\mathcal{Y}_j)\cap\cdots\cap C^{j-k}(I;\mathcal{Y}_k)\,,$$

and provide it with the norm

$$\|v\|_{j,k} = \sup_{t\in I}\{\sup_{0\leq r\leq j-k}\{\|\partial_t^r v(t)\|_{j-r}\}\}\,.$$

In what follows, where we shall consider metrics of the form $\omega_t = \omega + i\partial\overline{\partial}\varphi_t$ for path of functions φ_t that begin at 0 when $t=0$, we shall always use the scale of Sobolev spaces

$$\mathcal{Y}_j = H^{2j}(M)$$

as defined by the background metric ω. When t varies on the interval $[0,T]$, if we choose T sufficiently small, all the metrics ω_t will be equivalent, and the Sobolev spaces defined by them will be equivalent to each other, with equivalent norms. We let the Sobolev order jump by 2 because the operator $F(\varphi)$ in the right side of (11),

$$F(\varphi) := G_t(s_t - \pi_t s_t)\,,$$

is of second order. That is the reason for the peculiar definition of the scale $\mathcal{Y}_j$. By the Sobolev embedding theorem we know that $H^k(M)$ is a Banach algebra if $k>n$. Thus, for as long as the metric ω_t is equivalent to ω and provided that $k>n$, we have a continuous mapping

$$F : H^{s+4}(M) \mapsto H^{s+2}(M)\,.$$

Proposition 13. *Assume that a solution $\varphi(t)$ of (11) is in $C_{(s+1,0)}(I;\mathcal{Y})$ on the interval I for some integer s such that $2s > n+2$. Then all the values of $\partial_t^r\varphi(t)$ $(1\leq r\leq s+1)$ restricted to $t=0$ are completely determined and $\partial_t^r\varphi(t)\mid_{t=0} := \varphi_r \in \mathcal{Y}_{s+1-r} = H^{2s+2-2r}(M)$.*

Proof. The initial condition $\varphi\mid_{t=0}$ is zero, and the equation itself sets the value of $\partial_t\varphi\mid_{t=0} = F(0) = G_\omega(s_\omega - \pi_\omega s_\omega)$ that is evidently in $H^{2s}(M)$.

The relation (13) for $\beta = \partial_t\varphi_t$ says that

$$\frac{d}{dt}\beta = -\frac{1}{2}\Delta_\varphi\beta + P_0(\varphi)\beta\,,$$

where P_0 is a pseudo-differential operator of order zero whose coefficients depend on the coefficients of the metric ω_φ and its curvature tensor. Since $\varphi_t\in C(I;H^{2s}(M))$ and $2s>n+2$, by the Sobolev embedding theorem, these coefficients are continuous functions. By regularity of pseudo-differential operators on Sobolev spaces, we obtain that $\partial_t\beta = \partial_t^2\varphi_t \in H^{2(s-1)}(M)$, which is still a continuous function because $2(s-1)>n$.

If we differentiate the expression above for $\beta = \partial_t\varphi_t$ with respect to t, we obtain

$$\frac{d^2}{dt^2}\beta = -\frac{1}{2}\Delta_\varphi\frac{d}{dt}\beta - \frac{1}{2}L_\varphi(\beta)\beta + P_0(\varphi)\frac{d}{dt}\beta + P_{0,\varphi}(\beta)\beta\,,$$

where $L_\varphi(\beta)$ and $P_{0,\varphi}(\beta)$ are the linearizations of Δ_φ and $P_0(\varphi)$ at φ in the direction of $\partial_t\varphi$, respectively. The first is an operator of order two whose coefficients are continuous. By the metric dependence of $P_0(\varphi)$, the latter is a pseudo-differential operator of order zero whose coefficients are also continuous functions. Hence, $\partial_t^2\beta = \partial_t^3\varphi_t \in H^{2(s-2)}(M)$.

Iteration of the argument above yields that

$$\partial_t^r\varphi = F_r(\varphi, \partial_t\varphi, \ldots, \partial_t^{r-1}\varphi)\,,$$

where F_r is some operator whose coefficients depend upon the coefficients of the metric $\omega_\varphi = \omega + i\partial\overline{\partial}\varphi$. The desired result for the regularity of $\partial_t^r\varphi$ follows again using the Sobolev embedding theorem and the known regularity of the lower order time derivatives $\partial_t^j\varphi$, $0\leq j\leq r-1$. $\square$

Assume given Cauchy data $\varphi_0 = 0$ for (11) and let $\varphi_r = \partial_t^r\varphi(t)\mid_{t=0}$ be the sequence of coefficients of the Taylor series of $\varphi(t)$ given by the proposition above. The Cauchy data φ_0 determines the sequence φ_r, $1\leq r\leq s+1$. We consider the metric space:

$$W(I) = W^s(I) = \{\psi(t)\in C_{(s+1,0)}(I;\mathcal{Y}) :\ \partial_t^r\psi(t)\mid_{t=0} = \varphi_r,\ 0\leq r\leq s+1\}\,. \tag{14}$$

It is not empty, as can be seen by solving the Cauchy problem for a suitable parabolic equation.

By a continuity argument, for any $\psi(t)\in W(I)$ the form $\omega_\psi = \omega + i\partial\overline{\partial}\psi(t)$ is positive provided that t is sufficiently small. Hence, ω_ψ defines a Kähler metric. This metric is not smooth in general.

However, if $2s > n+2$, by the Sobolev embedding theorem, ω_ψ is at least C^2, and the operator in the right side of (13) will make sense when ψ plays the rôle of φ. Thus, we set

$$P_0(\psi)b = -2G_\psi(\Pi_\psi \rho_\psi, i\partial\overline{\partial} b)_\psi - G_\psi(\partial^{\#}_\psi b, X_\Omega)_\psi\,. \tag{15}$$

Then $P_0(\psi)$ is a pseudo-differential operator of order zero in b, whose coefficients depend upon the coefficients of the metric ω_ψ and its curvature tensor, all of which are continuous functions. For each t on a time interval where all the metrics ω_ψ are uniformly equivalent, we have that

$$-\frac{1}{2}\Delta_\psi + P_0(\psi) : H^2(M) \to H^0(M) = L^2(M) \tag{16}$$

continuously. We shall consider the equation

$$\frac{d}{dt}b = -\frac{1}{2}\Delta_\psi b + P_0(\psi)b\,, \tag{17}$$

whose Cauchy problem will be studied in the next section. We shall refer to it as the *approximate* linearized equation, the reasons being —we hope— clear at this point.

We end this section with the following

Proposition 14. *Let φ_1 be the Cauchy data for* (17). *If $b(t) \in C_{(s,0)}(I, \mathcal{Y})$ is a solution, then $\partial_t^r b\mid_{t=0} = \varphi_{r+1}$, $0 \leq r \leq s$.*

Proof. We have seen above that if $\varphi(t)$ satisfies (11), then

$$\partial_t^r \varphi = F_r(\varphi, \partial_t\varphi, \ldots, \partial_t^{r-1}\varphi)\,, \quad r \geq 2\,,$$

where F_r is some operator whose coefficients depend upon the coefficients of the metric $\omega_\varphi = \omega + i\partial\overline{\partial}\varphi$, and whose restriction at $t = 0$ depends only on the sequence $\varphi_0, \varphi_1, \ldots, \varphi_{r-1}$. The approximate linearized equation (17) is obtained from the linearization of (11) given in (13), when we replace the rôle played by $\varphi(t)$ by that of $\psi(t)$. But $\psi(t)$ and $\varphi(t)$ have the same coefficients in their Taylor expansions up to order $s+1$. Therefore, the solution $b(t)$ to the Cauchy problem of (17) with data $b(0) = \varphi_1$ will have necessarily a Taylor series of order s that agrees with the Taylor series of the solution to the Cauchy problem of linearized equation (13). The conclusion follows by Proposition 13. □

5. Local solvability of the extremal flow equation

In this section, we prove local time existence of solution to the extremal flow (11). We do so by adapting to our situation a method of T. Kato for the solvability of abstract differential equations and non-linear problems [10]. The pseudo-differential nature of our linearized equation (13) makes the task harder. But fortunately enough, the strictly pseudo-differential part of the equation is lower order, and most of the analysis is based on that of the standard time-dependent heat equation.

5.1. **The Cauchy problem for the approximate linearized equation.** From now on, we take s to be an integer such that $2s > n+2$ and $\mathcal{Y}_j = H^{2j}(M)$ as in the previous section. Given Cauchy data $\varphi_0 = 0$ for (11), Proposition 13 determines the sequence $\{\varphi_j\}_{j=0}^{s+1}$ that in turn allows us to define the space $W(I)$ of (14). The interval $I = [0, T]$ will be determined later. For $\psi \in W(I)$, we consider the metrics $\omega_\psi = \omega + i\partial\overline{\partial}\psi$ and the Cauchy problem of the approximate linearized equation (17),

$$\frac{d}{dt}b = -\frac{1}{2}\Delta_\psi b + P_0(\psi)b\,,$$

where $P_0(\psi)$ is given by (15), a pseudo-differential operator of order zero whose coefficients depend non-linearly on the coefficients of the metric ω_ψ and its curvature tensor.

Let $p(t,s)$ be the evolution operator of

$$\frac{d}{dt}b = -\frac{1}{2}\Delta_{\psi} b\,.$$

Thus, $p(t,s)$ is a two-parameter family of strongly continuous operators on $\mathcal{Y}_0$ and $\mathcal{Y}_1$, respectively, such that $p(t,s)p(s,r) = p(t,r)$, $p(t,t) = 1$, and for $b \in \mathcal{Y}_1$ we have

$$\begin{array}{rcl} \partial_t p(t,s)b & = & -\frac{1}{2}\Delta_{\psi(t)} p(t,s)b\,, \\ \partial_s p(t,s)b & = & -\frac{1}{2} p(t,s)\Delta_{\psi(s)} b\,. \end{array} \tag{18}$$

This family of operators exists for $0 \leq s \leq t \leq T$, and their operator norm is bounded uniformly by a constant that only depends upon a bound on $I = [0,T]$ of the coefficients of $\omega_{\psi(t)}$. The function solving (17) with Cauchy data β must satisfy the integral equation

$$b(t) = p(t,0)\beta + \int_0^t p(t,s)P_0(\psi(s))b(s)ds\,. \tag{19}$$

Consider the set of functions $b(t)$ in $C_{(1,0)}(I;\mathcal{Y}) = C(I;\mathcal{Y}_0) \cap C^1(I;\mathcal{Y}_1)$ such that $b(0) = \beta$. The right hand side of the expression above defines an operator in this space,

$$P : b \mapsto p(t,0)\beta + \int_0^t p(t,s)P_0(\psi(s))b(s)ds\,,$$

and by the explicit form of the coefficients of $P_0(\psi)$ discussed above, combined with continuity of pseudo-differential operators on Sobolev spaces, we have that

$$\|Pb - P\tilde{b}\| \leq CT\|b - \tilde{b}\|\,,$$

where C is a constant that depends upon the L^∞-norm of the coefficients of $\omega_{\psi(t)}$ and its curvature tensor on the time interval I. A fixed point argument now yields the following result:

Theorem 15. *Consider the Cauchy problem for* (17) *with Cauchy data $b(t)\mid_{t=0} \in \mathcal{Y}_1$. Then there exists T such that this problem has a unique solution in $C_{(1,0)}(I;\mathcal{Y}) = C(I;\mathcal{Y}_0) \cap C^1(I;\mathcal{Y}_1)$. The value of T only depends on supremum norms of the coefficients of $\omega_{\psi(t)}$ and its curvature tensor.*

Of course, the regularity of the solution in the theorem above can be improved if we start with a better initial condition. For that observe that the coefficients of the operator Δ_ψ are curves in $C_{(s,0)}(I;\mathcal{Y})$, and consequently,

$$\Delta_{\psi(t)} : H^{2j}(M) \mapsto H^{2j-2}(M)\,,\ 1 \leq j \leq s\,,$$

continuously. While the metrics remain equivalent, we can choose a uniform constant for the operator norm of these maps, and (18) holds for $b \in H^{2j}(M)$ with j's in this range. Then we have

Corollary 16. *If the initial data $b(t)\mid_{t=0} = \varphi_1 \in \mathcal{Y}_s$, the solution to the Cauchy problem for* (17) *belongs to $C_{(s,0)}(I;\mathcal{Y}) = C(I;\mathcal{Y}_s) \cap \cdots C^s(I,\mathcal{Y}_0)$.*

Proof. The arguments in the proof of the theorem and the remarks made above show that we now have a solution $b(t)$ to the Cauchy problem for (17) that is in $C(I;\mathcal{Y}_s) \cap C^1(I;\mathcal{Y}_{s-1})$. This solution satisfies (19) with $\beta = \varphi_1$.

We can differentiate repeatedly the identity (17) in order to show that the regularity of $b(t)$ with this initial condition can be improved. Notice that the coefficients of the second order operators $d_t^r \Delta_{\psi(t)}$, $1 \leq r \leq s-1$, are curves in $C(I;H^{2s-2r})$, and so we have $d_t^r \Delta_{\psi(t)} \in C(I;\mathcal{L}(\mathcal{Y}_{j+r+1},\mathcal{Y}_j)$ for $0 \leq j \leq s-1-r$. Here, $\mathcal{L}(X,Y)$ is the space of linear bounded operators from X to Y, and the assertion follows because in the stated range, $H^{2s-2r} \cdot H^{2j+2r} \subset H^{2j}$. This suffices to conclude that the contributions to $\partial_t^{k+1} b$ arising from $\partial_t^k \Delta_\psi b$ are in $H^{2s-2k-2}$ if we already know that $b \in C_{(s,k)}(I;\mathcal{Y})$.

The analysis of the contributions to $\partial_t^{k+1} b$ arising from $d_t^k(P_0(\psi)b)$ is similar. This time, the coefficients of the operators $d_t^r(P_{\psi(t)})$ are curves in $C(I; H^{2s-2r-2})$, one degree worse than those of $d_t^r \Delta_{\psi(t)}$, but the operators are of pseudo-differentials of order zero. The desired improved regularity follows by the same arguments as the ones in the previous paragraph. □

5.2. **An elliptic equation for $\gamma - F$.** Let us recall that $F(\varphi) = G_\varphi(s_\varphi - \pi_\varphi s_\varphi)$ is the second order non-linear operator defined by the right side (11). The derivative L_ψ of this map at a general point ψ in $\mathcal{Y}_{s+1}$ was computed in §4 and equals the operator in the right side of (13):

$$L_\psi b = -\frac{1}{2}\Delta_\psi b - 2G_\psi(\Pi_\psi \rho_\psi, i\partial\overline{\partial} b)_\psi - G_\psi(\partial_\psi^{\#} b, X_\Omega)_\psi \, . \tag{20}$$

Since the top part of this linearization is the negative operator $-\frac{1}{2}\Delta_\psi$, while the lower order term is a pseudo-differential operator of order zero, coercive estimates for this linearization imply that $\lambda - L_\psi$ is an invertible operator as a map, say, from $\mathcal{Y}_1$ to $\mathcal{Y}_0$, for a sufficiently large constant λ.

Let us then take a constant λ, and consider the non-linear elliptic map

$$\begin{array}{ccc} \mathcal{Y}_{s+1} & \longrightarrow & \mathcal{Y}_s \\ \varphi & \mapsto & \lambda\varphi - F(\varphi) \, . \end{array} \tag{21}$$

We remind the reader here of the sequence $\{\varphi_r\}$ given by Proposition 13, whose first element is $\varphi_0 = 0$.

Proposition 17. *For λ sufficiently large, there are neighborhoods $\mathcal{O}$ and $\mathcal{V}$ of φ_0 and $-\varphi_1$ in $\mathcal{Y}_{s+1}$ and $\mathcal{Y}_s$, respectively, such that the restriction of* (21) *to $\mathcal{O}$ is an isomorphism onto $\mathcal{V}$.*

Proof. This is a consequence of the Inverse Function Theorem. Indeed, the linearization $\lambda - L_0$ is an invertible operator from $\mathcal{Y}_1$ to $\mathcal{Y}_0$. Hence, if $f \in \mathcal{Y}_s$, there exists an element $b \in \mathcal{Y}_1$ that satisfies the equation

$$(\lambda - L_0)b = f \, .$$

Thus, the image of b under L_0 is in $\mathcal{Y}_1$, and by the regularity properties of $\lambda - L_0$, we must have $b \in \mathcal{Y}_2$. Iterating this argument, we conclude that $b \in \mathcal{Y}_{s+1}$, and so, b is an element of the tangent space of $\mathcal{Y}_{s+1}$ at 0. The desired result follows. □

Corollary 18. *Let $\psi \in \mathcal{Y}_s$ be sufficiently closed to $-\varphi_1$. Then, for large λ, the equation*

$$\lambda\varphi - F(\varphi) = \psi$$

has a solution $\varphi \in \mathcal{Y}_{s+1}$. The solution is unique if it is required to be closed enough to $\varphi_0 = 0$.

In the sequel, we let $D = D^s$ be the open neighborhood of φ_0 in $\mathcal{Y}_{s+1}$ where the operator $F(\varphi)$ is defined and smooth.

5.3. **The fixed point argument and the non-linear equation.** Proceeding in analogy with [10], we define $E_{\varphi_0}(I)$ to be the set of curves $\psi(t) \in W^s(I) \subset C_{(s+1,0)}(I; \mathcal{Y})$ such that

$$\|\partial_t^k \psi(t) - \varphi_k\|_{s+1-k} \leq R, \; k = 0, \ldots, s, \; t \in I \, ,$$

for some positive constant R. The value of R is chosen so the ball in $\mathcal{Y}_{s+1}$ with center φ_0 and radius R is contained in the domain D of $F(\varphi)$. This space is not empty for some $R > 0$ and some $I = [0, T]$.

By the form (20) of the linearization of $F(\varphi)$ at ψ, we may conclude that if ψ_1 and ψ_2 are elements of $\mathcal{Y}_{s+1}$, then the operator norm, as a map from $\mathcal{Y}_s$ to $\mathcal{Y}_0$, satisfies the estimate

$$\|L_{\psi_1} - L_{\psi_2}\|_{s,0} \leq C\|\psi_1 - \psi_2\|_1 \, ,$$

for some constant C. Indeed, the top part of L_ψ in (20) is half of the Laplacian, and its lower order part is a zeroth order pseudo-differential operator with nicely behaved coefficients. Regularity of pseudo-differential operators on Sobolev spaces yields to the assertion made.

We now define a key mapping in our proof of the local time existence to the extremal flow. Let $\psi(t)$ be an element of $E_{\varphi_0}(I)$, and consider the solution $b(t)$ of (17) given in Theorem 15, with initial data φ_1. We then solve the equation

$$\lambda\varphi - F(\varphi) = -b(t) + \lambda\left(\varphi_0 + \int_0^t b(s)ds\right) . \tag{22}$$

This is a stationary equation in φ that is solved for each $t \in I$. Here λ is a real number such that $\lambda - L_0$ is an isomorphism, where L_0 is the linearization (20) of $F(\varphi)$ at $\varphi = \varphi_0$. Notice that if t is sufficiently small, the right side of the equation lies in a neighborhood of $-\varphi_1$, and Corollary 18 applies to produce a solution $\varphi(t)$ in a neighborhood of φ_0.

The following two results are the versions of Proposition 7.4 and Proposition 7.6 in [10] adapted to our problem. We give proofs here for the sake of completeness.

Proposition 19. *For sufficiently small t, (22) has a unique solution $\varphi(t)$ in a neighborhood of φ_0 in $D \subset \mathcal{Y}_{s+1}$, with $\varphi(0) = \varphi_0 = 0$. Furthermore, $\varphi(t) \in C_{(s+1,0)}(I; \mathcal{Y})$ and $\partial_t^r \varphi(t) \mid_{t=0} = \varphi_r$, $0 \leq r \leq s$ provided T is chosen sufficiently small, uniformly in $\psi \in E_{\varphi_0}(I)$. In that case, $\varphi(t) \in E_{\varphi_0}(I)$.*

Proof. The operator $\varphi \mapsto \lambda\varphi - F(\varphi)$ is a local diffeomorphism of a neighborhood of $\psi(t)$ in $\mathcal{Y}_{s+1}$ into a neighborhood of $\lambda\psi - F(\psi)$ in $\mathcal{Y}_s$. By Theorem 15, the right side of (22) is a curve in $C(I, \mathcal{Y}_s)$ that has value $-\varphi_1$ at $t = 0$. By Corollary 18, we may solve the equation uniquely for $\varphi(t)$ in a $\mathcal{Y}_{s+1}$ neighborhood of φ_0 and obtain that $\varphi(t) \in C(I, \mathcal{Y}_{s+1})$. This requires to choose T sufficiently small but uniformly in $\psi \in E_{\varphi_0}(I)$.

Formal differentiation of the equation solved by $\varphi(t)$ yields that

$$(\lambda - L_{\varphi(t)})\partial_t\varphi = \lambda b - \partial_t b = (\lambda - L_{\psi(t)})b(t) .$$

By the invertibility of the operator $\lambda - L_{\varphi(t)}$ and the known regularity of the right side, it follows that $\partial_t\varphi \in C(I, \mathcal{Y}_s)$ and has value φ_1 at $t = 0$. Iterated differentiation yields that $\varphi(t) \in C_{(s+1,0)}(I; \mathcal{Y})$ and has the desired coefficients in its Taylor series expansion up to order s. Moreover, the way the equation is solved, we have that

$$\|\partial_t^k \varphi(t) - \varphi_r\|_{s+1-k} \leq R$$

for $t \in I$. This completes the proof. ◻

Proposition 20. *For $\psi \in E_{\varphi_0}(I)$, let $\varphi(t) \in E_{\varphi_0}(I)$ be the solution curve given by the previous proposition. If T is sufficiently small, the mapping*

$$\begin{array}{c} E_{\varphi_0}(I) \longrightarrow E_{\varphi_0}(I) \\ \psi(t) \longrightarrow \varphi(t) \end{array} \tag{23}$$

is a contraction in the metric induced by the norm $|||w|||_1 = \sup_{t\in I} \|w(t)\|_1$, relative to which, $E_{\varphi_0}(I)$ is complete.

Proof. Given a curve $b(t)$ in $\mathcal{Y}_k$, we define a norm by $|||b|||_k = \sup_{t\in I} \|b(t)\|_k$. We shall only make use of the 1 and 0 norm, respectively.

Let ψ_1 and ψ_2 be two elements of $E_{\varphi_0}(I)$ and let b_1 and b_2 be the solutions to the corresponding approximate linearized equations with the same initial condition φ_1. We the have that

$$b_1(t) = p_{\psi_1}(t,0)\varphi_1 , \quad b_2(t) = p_{\psi_2}(t,0)\varphi_1 ,$$

where $p_{\psi_1}(t,s)$ and $p_{\psi_2}(t,0)$ are the evolution operators of the linear equations $\partial_t v = L_{\psi_1(t)}v$ and $\partial_t v = L_{\psi_2(t)}v$, respectively. Consequently,

$$b_2(t) - b_1(t) = (p_{\psi_2}(t,0) - p_{\psi_1}(t,0))\varphi_1 ,$$

and using the identity

$$p_{\psi_2}(t,0)\varphi - p_{\psi_1}(t,0)\varphi = -\int_0^t p_{\psi_2}(t,\tau)(L_{\psi_2(\tau)} - L_{\psi_1(\tau)})p_{\psi_1}(\tau,0)\varphi d\tau ,$$

we obtain the estimate

$$\|b_2(t) - b_1(t)\|_0 \leq C\|\varphi_1\|_s \int_0^t \|L_{\psi_2(\tau)} - L_{\psi_1(\tau)}\|_{s,0} d\tau$$

for some constant C. But we have observed that $\|L_{\psi_2(\tau)} - L_{\psi_1(\tau)}\|_{s,0}$ is bounded by a constant times $\|\psi_2(\tau) - \psi_1(\tau)\|_1$. For small enough R, this last constant can be chosen uniformly. We then obtain that

$$|||b_2 - b_1|||_0 \leq CT\|\varphi_1\|_s|||\psi_2 - \psi_1|||_1 \,,$$

showing that the map

$$\psi(t) \mapsto b(t)$$

is a contraction from the 1-norm to the 0-norm, with contraction factor arbitrarily small with T.

That the map $\psi(t) \mapsto \varphi(t)$ is a contraction now follows because the map $b(t) \mapsto \varphi(t)$ is uniformly C^1 from the 0-norm to the 1-norm. This last map is simply the inverse of $\varphi \mapsto \lambda\varphi - F(\varphi)$ from $\mathcal{Y}_1$ to $\mathcal{Y}_0$, and we have that $\lambda - L_{\varphi(t)}$ is an isomorphism from $\mathcal{Y}_1$ to $\mathcal{Y}_0$, uniformly in $\psi(t)$ when $\psi(t)$ is close to φ_0. □

In view of the previous results, there exists a unique fixed point $\varphi(t)$ of the map (23). Since $b(t)$ solves (17) with initial data φ_1, differentiating with respect to t in (22) we obtain:

$$(\lambda - L_{\varphi(t)})\partial_t\varphi(t) = -\dot{b}(t) + \lambda b(t) = (\lambda - L_{\varphi(t)})b(t)\,,$$

and since $\lambda - L_{\varphi(t)}$ is injective, we must have that

$$b(t) = \partial_t\varphi(t)\,.$$

We may now use this fact in carrying the time integral in (22), and conclude that

$$\frac{d}{dt}\varphi(t) = F(\varphi(t))\,.$$

Thus, the fixed point $\varphi(t) \in E_{\varphi_0}(I)$ is a solution to the initial value problem (11).

We thus arrive at the following

Theorem 21. *Let (M, J, Ω) be a polarized Kähler manifold and let G be a maximal compact subgroup of $Aut(M, J)$. The extremal flow equation*

$$\partial_t\omega_t = -\rho_t + \Pi_t\rho_t$$

in $\mathfrak{M}_{\Omega,G}$ with a given initial data has a unique solution for a short time.

In fact, our proof carefully analyses how the time of existence depends upon the coefficients of the metric and its curvature tensor. Indeed, it shows that the local time of existence depends on the L^∞-norm of the coefficients of the initial metric and its curvature operator. We can improve a bit the statement above in relation to the lifespan of the extremal flow.

Corollary 22. *Given an initial condition $\omega \in \mathfrak{M}_{\Omega,G}$, the extremal evolution equation has a unique solution on a maximal time interval $0 \leq t < T \leq \infty$. If $T < \infty$, then the maximum of the pointwise norm of the curvature tensor blows-up as $t \to T$.*

6. Further remarks

It is of course important to know if the extremal flow has solutions for all time. Indeed, once the local time existence is known, the next problem to consider is the use of the flow to show the existence of extremal metrics representing a given cohomology class Ω, task that could be accomplished if we manage to prove global time existence and convergence of the metrics as $t \to \infty$.

This scheme could not possible work in all cases, as we already know of examples of polarized Kähler manifolds without extremal metrics. But as a testing ground of its usefulness, we have started its analysis when in pursue of extremal metrics on polarized manifolds (M, J, Ω) with

$c_1 < 0$, or on polarized complex surfaces with $c_1 > 0$. The partial results obtained so far are quite encouraging.

We have two types of fairly strong reasons supporting our belief that this approach will produce extremal metrics in the said cases. The first of these reasons is directly related to the flow itself, while the other one involves some relation between this flow and the study of *families* of extremal problems as we vary the cohomology class Ω. We discuss them briefly in this section.

The evolution equation (10) implies evolution equations for various metric tensors associated to the varying metrics. For instance, the Ricci form evolves according to the equation

$$\frac{d}{dt}\rho = -\frac{1}{2}\Delta\rho + \frac{i\partial\overline{\partial}(\pi s)}{2}\,,$$

the scalar curvature evolves according to the equation

$$\frac{d}{dt}s = -\frac{1}{2}\Delta(s-\pi s) - 2(\rho, i\partial\overline{\partial}G(s-\pi s))\,,$$

and the Ricci potential evolves according to the equation

$$\frac{d}{dt}\psi = -\frac{1}{2}\Delta\psi - 2G(\rho_H, i\partial\overline{\partial}(\psi + G(\pi s))) - (\pi s - s_0) + \frac{1}{2v}\int \psi(s-\pi s)d\mu\,.$$

Here ρ_H is the harmonic component of ρ, and v is the volume of M relative to ω. In particular, using the first of these equations, we easily obtain the following result:

Proposition 23. *Let (M,J) be a complex manifold of Kähler type polarized by a Kähler class Ω, all of whose non-trivial holomorphic vector fields have no zeroes. If $\pm\rho \geq 0$ at $t=0$, then it remains so along the flow* (10) *for $0 \leq t \leq T$, T the lifespan of the solution.*

Proof. Under the stated hypothesis, the Ricci form evolves according to the heat equation. By Hamilton's maximum principle for tensors (Theorem 9.1 in [9]), the result follows. □

This result applies directly to manifolds with no non-trivial holomorphic vector fields, such as any complex manifold (M,J) with negative first Chern class, or most complex surfaces with positive first Chern classes. We have also verified the analogous result for the blow-up of $\mathbb{CP}^2$ at one point, a complex surface that carries non-trivial holomorphic vector fields. It is likely that this result holds for any complex surface with positive c_1.

We may also refine our earlier Theorem 3 when dealing with a complex surface of positive first Chern class. Indeed, we have the following result, whose proof will be given elsewhere.

Theorem 24. *Let (M,J,Ω) be a polarized complex surface of positive first Chern class. Given any Kähler metric g in $\mathfrak{M}_{\Omega,G}$, the image of the holomorphy potential $\pi_g s_g$ is an interval contained in the set of positive real numbers, interval that only depends on Ω and not on g.*

Thus, if for a given initial condition with positive Ricci curvature the flow (10) would exist for all time and converge to an extremal metric on this type of surfaces, the extremal metric so obtained would have positive scalar curvature, which is what one expects.

The preservation of the sign of the Ricci tensor should have very strong implications on the global analysis of (10). This property has been of utmost importance already in the work of Hamilton [9], and should remain so in the general analysis of our flow equation as well. We venture the following two conjectures.

Conjecture 25. Let (M,J) be a complex manifold of Kähler type polarized by a Kähler class Ω. If $c_1(M,J) < 0$, there exists an initial condition to the extremal flow (11) equation so that the solution exists on $[0,\infty)$ and, as $t \to \infty$, converges to a metric of constant negative scalar curvature representing Ω.

Conjecture 26. Let (M, J) be a complex surface of positive first Chern class polarized by a Kähler class Ω. Then there exists an initial condition to the extremal flow (11) equation so that the solution exists on $[0, \infty)$ and, as $t \to \infty$, converges to an extremal metric of positive scalar curvature representing Ω.

The initial condition we have in mind in these two cases is given by a metric whose Ricci form is negative or positive, respectively.

These conjectures are further supported by the results in [17], that we proceed to describe in brief detail. For a complex manifold (M, J) of complex dimension n, we denote by $\mathfrak{M}$ the space of Kähler metrics on (M, J). As before, given a positive class $\Omega \in H^{1,1}(M, \mathbb{C}) \cap H^2(M, \mathbb{R})$, we let $\mathfrak{M}_\Omega$ be the space of of Kähler metrics whose Kähler forms represent Ω. We shall also consider the space $\mathfrak{M}_1$ of Kähler metrics of volume one, and $\mathcal{K}_1$, the space of cohomology classes that can be represented by Kähler forms of metrics in $\mathfrak{M}_1$:

$$\mathcal{K}_1 = \{\Omega \in H^{1,1}(M, \mathbb{C}) : \ \Omega = [\omega] \text{ for some } \omega \in \mathfrak{M}_1\} \, . \tag{24}$$

Extremal metrics in $\mathfrak{M}_\Omega$ achieve the infimum of the functional E_Ω in (2), and we have the lower bound (8):

$$E(\Omega) = \int (\pi_g s_g)^2 d\mu_g \, .$$

One approach to providing (M, J) with a canonical shape would be to find critical points of the functional

$$\begin{array}{rcl} \mathfrak{M}_1 & \to & \mathbb{R} \\ \omega & \mapsto & \displaystyle\int_M s_\omega^2 d\mu_\omega \, . \end{array} \tag{25}$$

A special metric ω of this type must have the following properties:

a) ω achieves the lower bound $E([\omega])$, that is to say, ω is extremal relative to the polarization defined by the Kähler class $\Omega = [\omega]$ that it represents;
b) the Kähler class $\Omega = [\omega]$ is a critical point of $E(\Omega)$ as a functional defined over $\mathcal{K}_1$.

Thus, the search for critical points of (25) —or *strongly extremal metrics* [15]— achieving an optimal lower bound involves the solution of back-to-back minimization problems: the first solving for critical points of (2) within a fixed cohomology class Ω, and the second solving for those classes that minimize the critical value $E(\Omega)$ as the class Ω varies within $\mathcal{K}_1$. Naturally, we separate the two problems by, in addition to (2), introducing the functional

$$\begin{array}{rcl} \mathcal{K}_1 & \to & \mathbb{R} \\ \Omega & \mapsto & E(\Omega) = \displaystyle\int_M (\pi s)^2 d\mu \, , \end{array} \tag{26}$$

where the geometric quantities in the right are those associated with any G-invariant metric that represents Ω, for G a fixed maximal compact subgroup of the automorphism group of (M, J). Its extremal points will be called either critical or canonical classes. We then have [16] the following

Theorem 27. *Let Ω be a cohomology class that is represented by a Kähler metric g, assumed to be invariant under the maximal compact subgroup G of the biholomorphism group of (M, J). Then Ω is critical class if and only if*

$$\int_M (\pi_g s_g)(\Pi_g \rho, \alpha) d\mu_g = 0$$

for any trace-free harmonic $(1, 1)$-form α. In this expression, ρ is the Ricci form of the metric g, π is the L^2 projection (5) onto the space of holomorphy potentials, and Π is its lift (9) at the level of (1,1)-forms.

This theorem states that Ω is a critical class of (26) if and only if

$$\int_M (\pi_g s_g)(\Pi_g \rho, \alpha) d\mu_g = 0$$

for any trace-free harmonic $(1,1)$-form α. In other words, the form $\pi s \Pi\rho$ is L^2-perpendicular to the space of trace-free harmonic (1,1)-forms, and therefore, by Hodge decomposition, the class must be such that

$$\pi s \Pi \rho = \lambda\omega + \partial G_{\partial}(\partial^*(\pi s\, \Pi\rho)) + \partial^* G_{\partial}(\partial(\pi s\, \Pi\rho))\,, \tag{27}$$

for λ equal to the L^2-projection of $(\pi s)^2$ onto the constants, divided by $2n$:

$$\lambda = \frac{1}{2n}\int (\pi s)^2 d\mu_g\,. \tag{28}$$

In order to study the existence of critical classes, we may consider [17] the evolution equation

$$\frac{d\Omega}{dt} = \pi s\, \Pi\rho - \lambda\omega + \partial G_{\partial}(\partial^*(\pi s\, \Pi\rho)) + \partial^* G_{\partial}(\partial(\pi s\, \Pi\rho))\,. \tag{29}$$

The flow equation (29) defines a dynamical system on $\mathcal{K}_1$ provided the solutions remain in $\mathcal{K}_1$ throughout time. Unfortunately, this is not true in general [17].

In the generic case where all non-trivial holomorphic vector fields of (M, J) have no zeroes, equation (29) can be extended to a dynamical system on

$$\overline{\mathcal{K}}_1 = \{\Omega \in H^{1,1}(M,\mathbb{C}) \cap H^2(M,\mathbb{R}) :\ \frac{\Omega^n}{n!} = 1\}\,.$$

Indeed, given $\Omega \in \overline{\mathcal{K}}_1$, let us define the function

$$s_\Omega := 4\pi n \frac{c_1 \cdot \Omega^{n-1}}{\Omega^n}\,.$$

If Ω were a Kähler class represented by a metric g, this function would be precisely the holomorphy potential $\pi_g s_g$. The equation

$$\frac{d}{dt}\Omega = 2\pi s_\Omega c_1 - \frac{s_\Omega^2}{2n}\Omega\,, \tag{30}$$

extends (29), which as such is defined only on $\mathcal{K}_1$, all the way to a dynamical system on $\overline{\mathcal{K}}_1$.

Solutions to (30) with initial data in $\overline{\mathcal{K}}_1$ remain in $\overline{\mathcal{K}}_1$. In fact, we have that [17]

Theorem 28. *Suppose that all non-trivial holomorphic vector fields of (M, J) have no zeroes. Then solutions to (30) with initial data in $\overline{\mathcal{K}}_1$ converge, as $t \to \infty$, to a stationary point of the equation in the space $\overline{\mathcal{K}}_1$.*

It is then of natural interest to see if solutions to the equation with Cauchy data given by a positive class, that is to say, an element of $\mathcal{K}_1$, remain positive thereafter. We already know [17] of examples where this is not so, with solutions to the flow equation that are initially in the Kähler cone but that, in converging to a critical point of the flow in $\overline{\mathcal{K}}_1$, must eventually leave the cone through its walls.

In fact, this situation occurs already on complex surfaces, where the stability of $\mathcal{K}_1$ under the flow (30) can be analyzed using a criterion giving necessary and sufficient for a cohomology class to be Kähler, criterion that extends that of Nakai for integral classes. Applied to our problem, if the Chern number $c_1^2 \neq 0$, we have that a path Ω_t solving (30) with initial condition in $\mathcal{K}_1$ stays there forever after if, and only if,

$$\Omega_0 \cdot [D] + 8\pi^2 (c_1 \cdot \Omega_0)(c_1 \cdot [D]) \left(\frac{e^{c_1^2 t} - 1}{c_1^2} \right) > 0$$

for all $t \geq 0$ and for all effective divisors D in (M, J). When $c_1^2 = 0$ we still obtain a similar criterion, replacing the expression in parentheses above by its limit t as $c_1^2 \to 0$. This forward stability of the Kähler cone holds in very general situations, as can be seen by a run-down of the various cases in the Enriques-Kodaira classification of complex surfaces [17]. In particular, it holds if the complex surface has a signed first Chern class c_1, condition under which all solutions to the flow (30) that start in $\mathcal{K}_1$ stay there forever after, and as $t \to \infty$, they either converge to the only critical class $\sqrt{2}(\operatorname{sgn} c_1)c_1/c_1^2$ of (26) if $c_1^2 > 0$, or all classes are critical and the flow is constant if $c_1 = 0$.

Notice that the positivity condition above involves the evaluation of c_1 over the divisor D, and only in the case when there are effective divisors D for which $c_1 \cdot [D]$ changes sign from one to another could the condition fail to hold. Merely fixing the sign of c_1 prevents this from happening, but the counterpart to that is of great interest. It shows that the existence of divisors on which c_1 achieves values of opposite signs is in effect part of the reason why the the Kähler cone might be poorly behaved in relation to the flow (29).

When the surface in question has positive first Chern class and carries non-trivial holomorphic fields, the forward stability of the Kähler cone under the flow (29) seems to hold also, though we have only verified that for the case of $\mathbb{CP}^2$ blown-up at one point.

In higher dimension and for manifolds (M, J) where c_1 is either positive or negative, the space of Kähler classes is also forward stable under the flow (30). As a matter of fact, there is a positivity criterion that generalizes the one outlined above for surfaces, which guarantees forward stability of the Kähler cone under the flow. Manifolds with signed first Chern classes meet this criterion, though one can give a direct argument to prove the flow stability of the cone in such a case.

All of these facts combined give further support to the conjectures made earlier. We end up venturing a final one.

Conjecture 29. Suppose the flow equation (29) with initial data in the Kähler cone converges to a stationary point that is outside it. Then the extremal Kähler cone is not a closed subset of the Kähler cone.

In other words, under the given hypothesis, there should exist cohomology classes in the Kähler cone that cannot be represented by extremal metrics.

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