Geometry and Sets

Sets in Geometry and Topology

Math Camp 2024

Department of Mathematics Stony Brook University

July 19, 2024

Overview

1. Sets in Topology
2. Sets in Geometry

Parametrization

- A parametric equation defines a group of quantities as functions of one or more independent variables called parameters.
- Parametric equations are commonly used to express the coordinates of the points that make up a geometric object such as a curve or surface.
- Ex. the equations

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases}$$

form a parametric representation of the unit circle, where t is a parameter: A point (x, y) is on the unit circle if and only if there is a value of t such that these two equations generate that point.

Parametrization

- Ex. the equations

$$\begin{cases} x = \rho \sin u \cos v \\ y = \rho \sin u \sin v \\ z = \rho \cos u \end{cases}$$

$\rho > 0$, $0 \leq u \leq \pi$, $0 \leq v \leq 2\pi$ parametrize the sphere of radius ρ in $\mathbb{R}^3$ centered at origin.

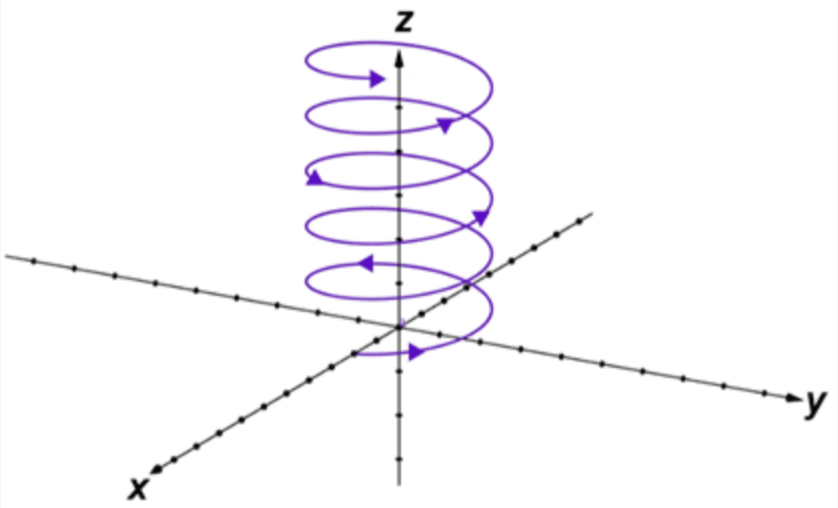
Parametrization

- Ex. the equations

$$\begin{cases} x = r\cos t \\ y = r\sin t \\ z = ct \end{cases}$$

$r > 0$, $t \in [0, 2\pi)$ parametrize a helical "space curve" in $\mathbb{R}^3$. r is the radius of the helix and $2\pi c$ is a constant giving the vertical separation of the helix's loops.

Helical Space Curve



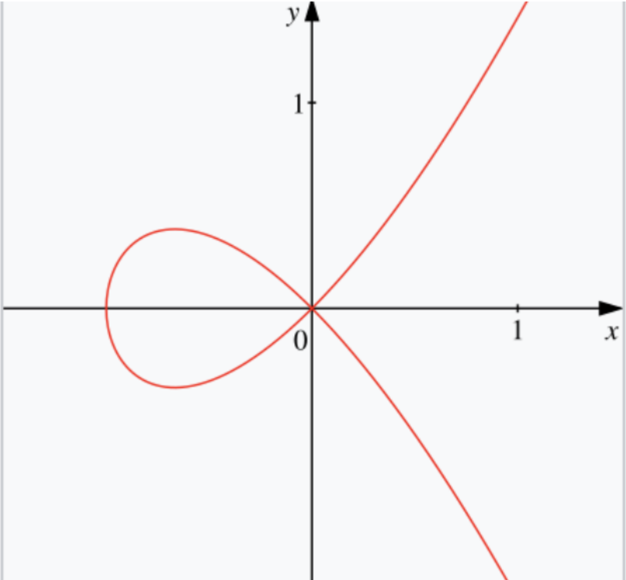
Locus

- In geometry, a locus is a set of all points whose location satisfies or is determined by one or more specified conditions.

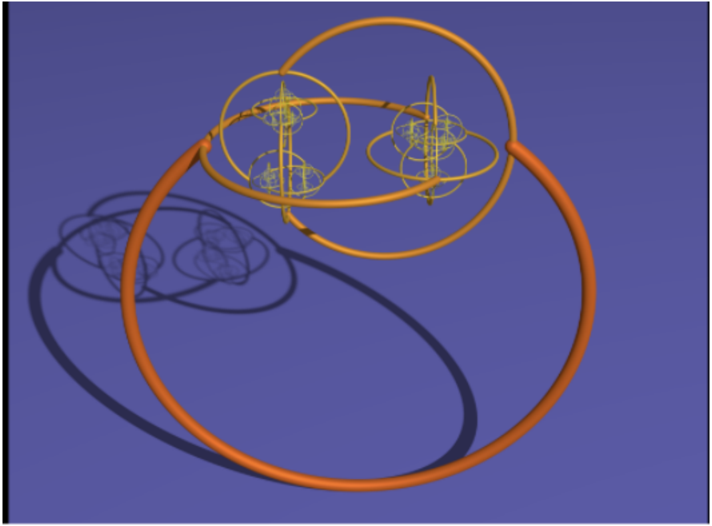
Locus

- A cubic plane curve given by $y^2 = x^2(x+1)$.
- $V(y^2 - x^2(x+1))$.

Locus



Alexander Horned Sphere



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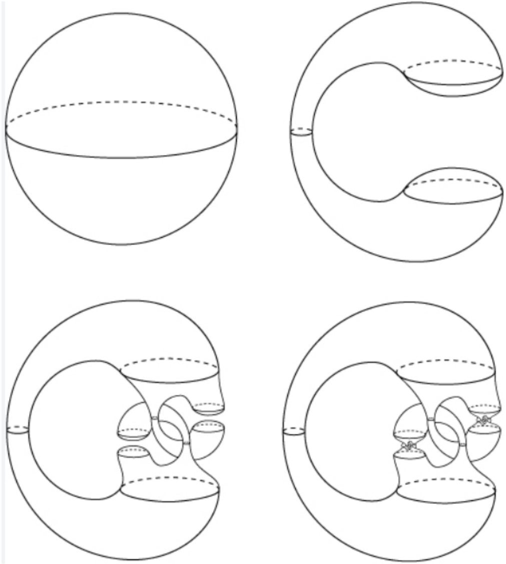

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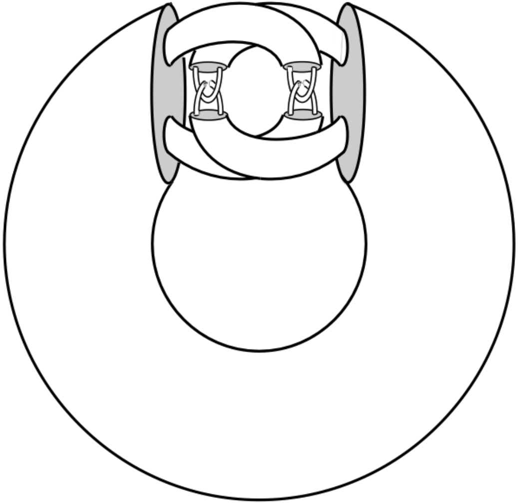

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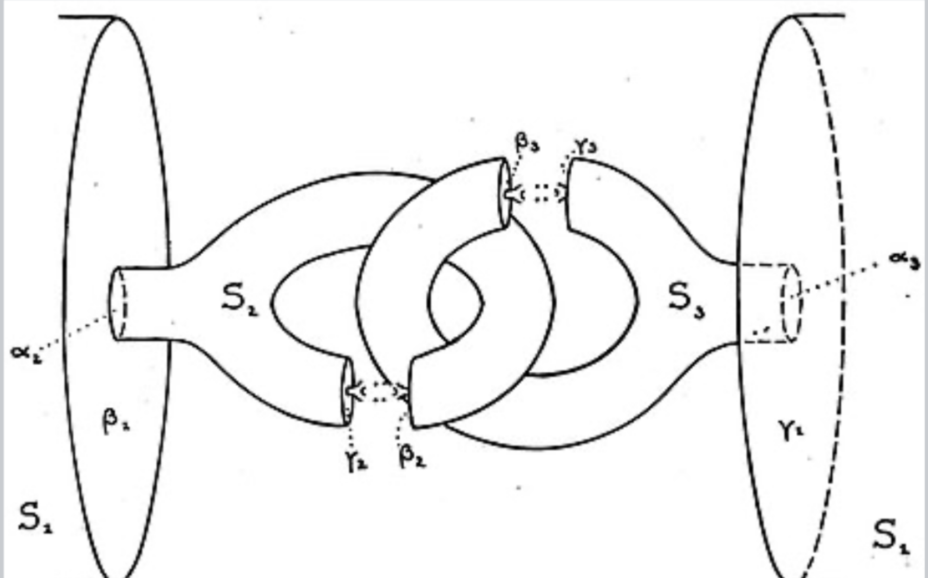
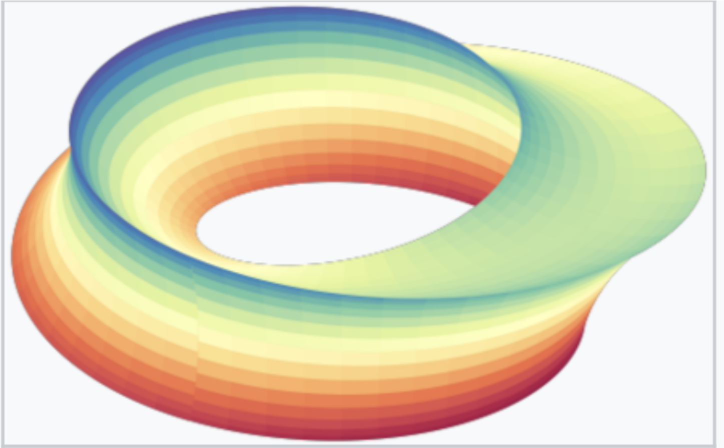


Alexander Horned Sphere

- Discovered by J.W. Alexander (1924).
- Particular embedding of 2− dimensional sphere in 3− dimensional space.
- Together with inside, is homeomorphic to unit ball B^3 (topologically indistinguishable).
- Therefore is simply connected: $\pi_1(X) = 0$ (Every loop can be shrunk to a point while staying inside).
- To construct:
  1. Remove radial slice of torus
  2. Connect a standard punctured torus to each side of the cut, interlinked with the torus on the other side.
  3. Repeat steps 1–2 on the two tori just added ad infinitum.

Umbilical Torus



Umbilical Torus

Umbilical Torus

- Christopher Zeeman named this set "Umbilical Bracelet" (1976).
- Important in singularity theory. In particular in the classification of umbilical points which are determined by real cubic forms:

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3$$

- Has parametric equations

$$\begin{cases} x = \sin u \left(7 + \cos\left(\frac{u}{3} - 2v\right) + 2\cos\left(\frac{u}{3} + v\right)\right) \\ y = \cos u \left(7 + \cos\left(\frac{u}{3} - 2v\right) + 2\cos\left(\frac{u}{3} + v\right)\right) \\ z = \sin\left(\frac{u}{3} - 2v\right) + 2\sin\left(\frac{u}{3} + v\right) \end{cases}$$

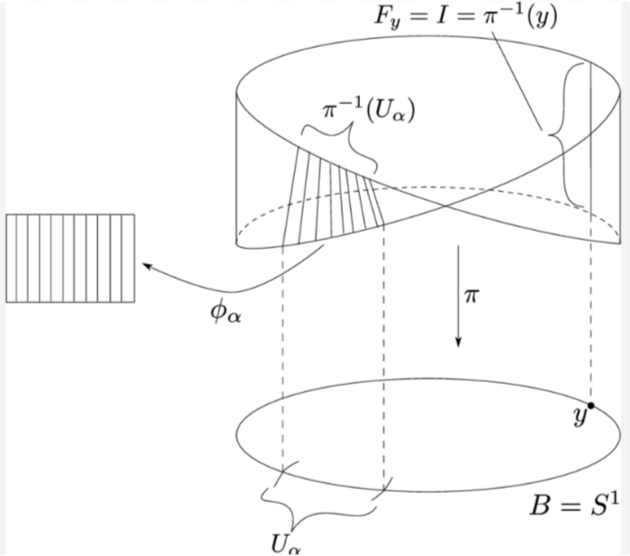
for $-\pi \leq u \leq \pi$, $-\pi \leq v \leq \pi$.

Möbius Band

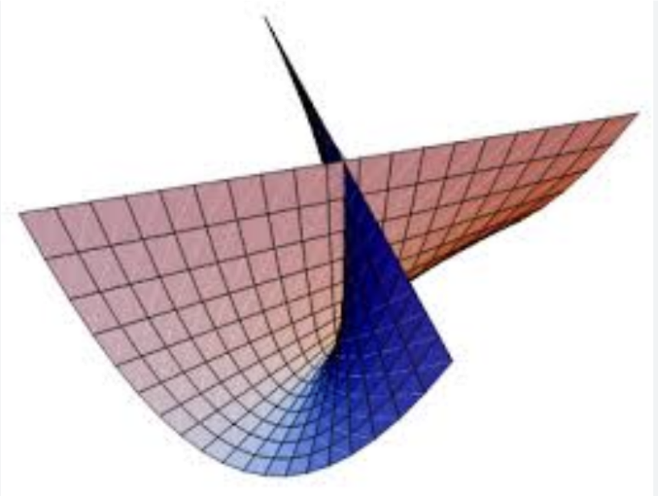
Möbius Band

- Discovered by Johann Benedict Listing and August Ferdinand Möbius (1858).
- Nonorientable surface (cannot distinguish front from back or clockwise from counterclockwise turns).
- $\pi_1(X) = \mathbb{Z}$ ($\exists$ deformation retraction onto center circle).
- Is a fiber bundle over S^1 with fiber $[0, 1]$.

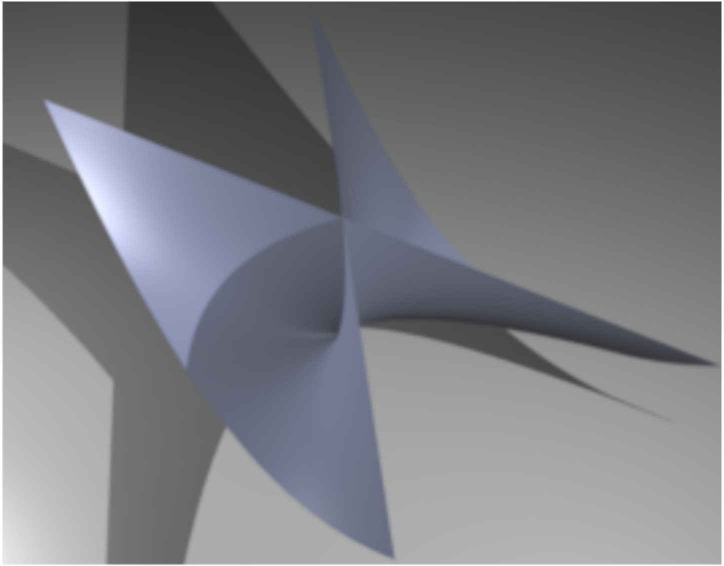
Möbius Band Fiber Bundle Structure



Whitney Umbrella



Whitney Umbrella



Whitney Umbrella

- Discovered by the great Hassler Whitney.
- It is an example of a "ruled surface". A surface in 3− dimensional space, such that through every point is a straight line lying on the surface.
- It's also a right conoid. A ruled surface generated by a family of straight lines that all intersect perpendiculary to a fixed straight line, called the axis.
- Is the affine variety $V(x^2 - y^2 z) \subset \mathbb{R}^3$.
- Has parametric equations

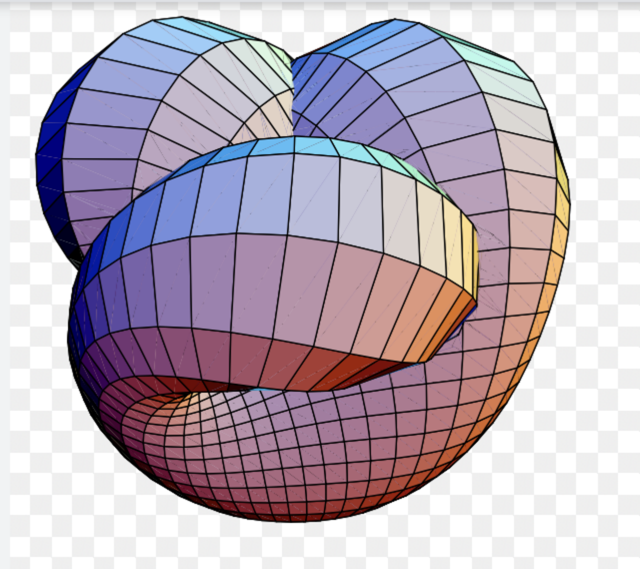
$$\begin{cases} x = uv \\ y = u \\ z = v^2 \end{cases}$$

for $u, v \in \mathbb{R}$.

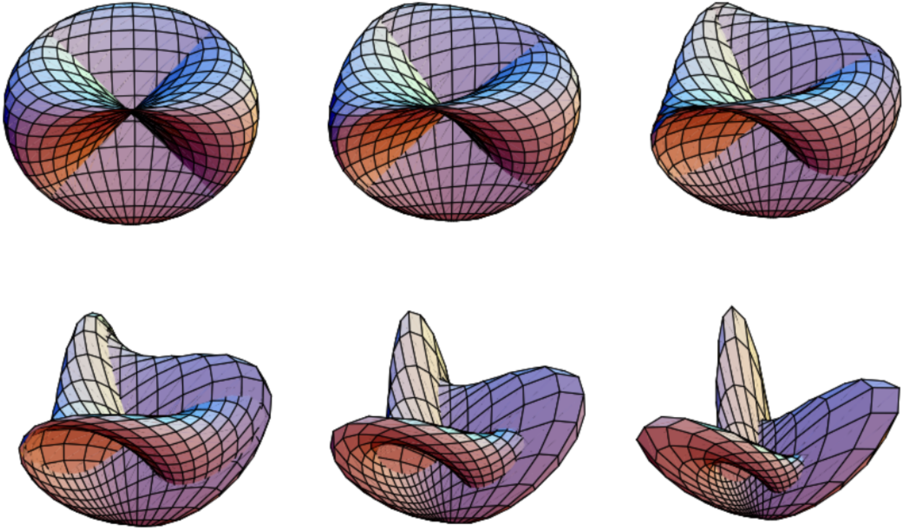
Whitney Umbrella

- This hypersurface is important in singularity theory as an example of a "pinch point singularity".
- The pinch point (in this case the origin) is a limit of normal crossings ($\mathbb{Z}/2\mathbb{Z}$ crossings) singular points (the $z-$axis in this case).
- These singular points are intimately related in the sense that in order to resolve the pinch point singularity one must blow-up the whole $z-$axis and not only the pinch point

Boy's Surface



Boy's Surface



Boy's Surface

- Nonorientable, just like Möbius band.
- Immersion of $\mathbb{R}P^2$ in $3-$ dimensional Euclidean space. So like the Roman surface and cross-cap, is topologically equivalent to $\mathbb{R}P^2$.
- Unlike Roman surface and cross-cap, has no singularities other than self-intersections (no pinch-points).

Boy's Surface Kusner/Bryant Parametrization

- There are several ways to parametrize Boy's surface. Here is the one by Rob Kusner and Robert Bryant:
- Let $w \in \mathbb{C}$ with $|w| \leq 1$.
- Let

$$\begin{cases} g_1 = -\frac{3}{2} \operatorname{Im}\left[\frac{w(1-w^4)}{w^6+\sqrt{5}w^3-1}\right] \\ g_2 = -\frac{3}{2} \operatorname{Re}\left[\frac{w(1+w^4)}{w^6+\sqrt{5}w^3-1}\right] \\ g_3 = \operatorname{Im}\left[\frac{1+w^6}{w^6+\sqrt{5}w^3-1}\right] - \frac{1}{2} \end{cases}$$

- Then

$$(x, y, z) = \frac{(g_1, g_2, g_3)}{g_1^2 + g_2^2 + g_3^2}$$

is a point in Cartesian coordinates (x, y, z) on Boy's surface.