0.1 Pigeon Hole Principle. *If n items are placed into m containers with $n > m$, at least one container must contain more than one item.*

Problem 1
Two people in London have the same number of hairs on their heads.

Solution The max number of hairs on any head is $\approx 1,000,000 \quad (<)$. London has more than $1,000,000$ people. By **Pigeon Hole Principle** at least two people in London have the same number of hairs on their heads.

Problem 2
If n people in a room shake hands, at least two people in the room shake the same number of hands.

Solution Consider a function from the number of people (pigeons) to the number of hands he/she shakes (holes). After fixing a labeling, this is a function

$$\{1, ..., n\} \longrightarrow \{0, ..., n-1\}$$

However, one of the holes 0 or $n-1$ must be empty (there cannot be someone shaking no hands and another person $n-1$ hands). So it's really (without loss of generality) a function

$$\{1, ..., n\} \longrightarrow \{0, ..., n-2\}$$

Since it is an assignment of n people (pigeons) to $n-1$ possible number of hand shakes (holes), we can conclude at least two people (pigeons) shake the same number of hands (are in the same hole).

0.2 Introduction to Graphs. *A graph is (naively) a set with relations between elements of the set. We denote it pictorially using the elements as vertices and relations as edges.*

1 Definition. The degree of a vertex in a graph is the number of edges incident to that vertex.

Exercise 1

Show that given any graph with more than one vertex, at least two vertices have the same degree.

Problem 3
Given a 15×20 rectangle containing 26 points, show that at least two points are no more than 5 units apart.

Solution Note there was a small mistake in the solution provided in class. You can't chop a 15×20 rectangle evenly into 25 squares of the same dimension! Instead chop into rectangles of dimensions $x \times y$. We want 25 of them in order to apply PHP, so we need $x = 3$ and $y = 4$. Then follow the same protocol: By PHP, there are at least two points in a 3×4 rectangle. The max distance between any two points in the rectangle is

$$\sqrt{3^2 + 4^2} = 5$$

(the length of the diagonal) and we're done.

Now we seek a stronger form of PHP. In class I renamed the number of pigeons ℓ and holes m. Still consider $\ell > m$.

- If there are $\ell = nm$ pigeons put into m holes, worst case scenario is when n pigeons are evenly dispersed in the m holes. The best we can say is at least one hole contains

  $$n = \ell/m = \lceil \ell/m \rceil$$

  pigeons.

- If $\ell > m$ but ℓ is not divisible by m, by division we can write

  $$\ell = nm + c \quad (0 < c < m)$$

  In the worst case scenario, we have nm pigeons filled up evenly in m holes. We can put the remaining c pigeons into distinct holes individually. So the best we can say is at least one hole contains $n + 1$ pigeons. But ℓ/k isn't an integer and

  $$n = \frac{\ell - c}{m} < \ell/m < \frac{\ell - c}{m} + 1 = n + 1$$

  So

  $$n + 1 = \lceil \ell/m \rceil$$

  Thus, we have the:

  0.3 Generalized Pigeon Hole Principle. *If ℓ items are placed into m containers with $\ell > m$, at least one container must contain $\lceil \ell/m \rceil$ items.*

Problem 4
Given that London has 9.002 million people, show at least 10 people in London have the same number of hairs on their heads.

Solution By the Generalized Pigeon Hole Principle, at least

$$\begin{aligned} \left\lceil \frac{9.002 \times 10^6}{1,000,000} \right\rceil &= \left\lceil \frac{9,002,000}{1,000,000} \right\rceil \\ &= \left\lceil 9 + \frac{2}{1,000} \right\rceil \\ &= 10 \end{aligned}$$

people in London have the same number of hairs on their heads.

Exercise 2

(Done in class) Given 41 rooks on a 10×10 "chessboard", say two rooks are mutually attacking eachother if they're in the same row or column. Find a "Pacifist Quintet" i.e. 5 rooks not mutually attacking eachother.

Problem 5
Let $\alpha \in \mathbb{R}$ and N a positive integer. Show that there are integers n, m with $1 \leq n \leq N$ and $|\alpha n - m| < \frac{1}{N}$

Solution Consider the function $f : \mathbb{R} \longrightarrow [0,1) \coloneqq f(x) = x + \lfloor x \rfloor$ (reduction mod 1). Consider the $N+1$ element set

$$\{f(i\alpha) \mid i = 0, ..., N\}$$

and the N disjoint intervals

$$[0, 1/N), [1/N, 2/N), ..., [(N-1)/N, 1)$$

partitioning $[0,1)$. By Pigeon Hole Principle, at least two elements $f(i\alpha), f(j\alpha)(i > j)$ of the first set must lie in one of the intervals $[K/N, (K+1)/N)$ having length $1/N$. So

$$\begin{aligned} |f(i\alpha) - f(j\alpha)| &= |\alpha(i-j) - (\lfloor ja \rfloor - \lfloor ia \rfloor)| \\ &< \frac{1}{N} \end{aligned}$$

So pick $n = i - j$ and $m = \lfloor ja \rfloor - \lfloor ia \rfloor$.