Equivariant Lagrangian Floer Theory on Compact Toric Manifolds

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Yao Xiao

We, the dissertation committee for the above candidate for the

Doctor of Philosophy degree, hereby recommend

acceptance of this dissertation

Kenji Fukaya, Dissertation Advisor
Professor, Department of Mathematics

Mark McLean, Chairperson of Defense
Professor, Department of Mathematics

Olga Plamenevskaya
Professor, Department of Mathematics

Christopher T. Woodward
Professor, Department of Mathematics, Rutgers University

This dissertation is accepted by the Graduate School

Celia Marshik
Dean of the Graduate School

Abstract of the Dissertation

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We introduce an equivariant Lagrangian Floer theory on compact symplectic toric manifolds for the subtorus actions. We prove that the set of Lagrangian torus fibers (with weak bounding cochain data) with non-vanishing equivariant Lagrangian Floer cohomology forms a rigid analytic space. We can apply tropical geometry to locate such Lagrangian torus fibers in the moment polytope. In addition, we apply equivariant theory to show that moment Lagrangian correspondences induced by symplectic reduction are unobstructed after bulk deformation, assuming the existence of certain equivariant Kuranishi structures and compatible equivariant CF-perturbations.

To my parents

Table of Contents

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Chapter 1

Introduction

The study of Hamiltonian group actions on symplectic manifolds traces back to classical mechanics, predating the invention of the term "symplectic manifold" and remains a main theme in symplectic geometry. Notably, Noether's principle states that every symmetry on a physical system corresponds to a conservation law. For instance, time translation symmetry corresponds to the conservation of energy, and rotational symmetry corresponds to the conservation of angular momentum. In fact, on general symplectic manifolds that are equipped with Hamiltonian group actions, we can interpret this beautiful principle as a correspondence between smooth functions that are invariant under the Hamiltonian action and smooth functions whose flows preserve the moment maps.

Symplectic structures that arise through symplectic reduction with respect to Hamiltonian group actions, a procedure that generalizes the reduction of dimension of a physical system by exploiting symmetry, are prevalent. Such a construction, however, often results in rather singular spaces. Besides leveraging the symmetry to study the symplectic manifolds with group actions themselves, we are also interested in applying equivariant theories on such manifolds to study their possibly singular symplectic quotients.

Lagrangian Floer cohomology was developed by Floer [16] and generalized by Fukaya, Oh, Ohta, and Ono ([24], [25], [20], and other papers by the authors) to study the topology of intersection of Lagrangian submanifolds in symplectic manifolds. It is the building block of the derived Fukaya category of a symplectic manifold, which is predicted by the celebrated Homological Mirror Symmetry conjecture to be equivalent to the derived category of coherent sheaves on some "mirror" complex algebraic variety dual to the symplectic manifold.

An equivariant version of Lagrangian Floer theory is expected to be useful for the study of the intersection of Lagrangian submanifolds invariant under Hamiltonian group actions. Various constructions of equivariant Lagrangian Floer theory have been made to suit different scenarios. See [49], [33], [34], [10], [26], [36], [35], and [38] for example. By exploiting the symmetry on a symplectic manifold, we expect to apply equivariant Lagrangian Floer theory to investigate the homological mirror symmetry of symplectic manifolds that admit non-trivial Hamiltonian group actions. In this thesis, we explore an equivariant Lagrangian Floer theory on compact symplectic toric manifolds, which are examples of symplectic manifolds with "maximal" symmetry.

The thesis is organized as follows. We prepare the background on equivariant de Rham theory and compact symplectic toric manifolds in Chapter 2 and Chapter 3. In Chapter 4, assuming the existence of certain equivariant Kuranishi data, we prove that moment Lagrangian correspondences are unobstructed after bulk deformation. We define an equivariant Lagrangian Floer theory in Chapter 5 on compact symplectic toric manifolds. In Chapter 6, we prove that the set of Lagrangian torus fibers (with weak bounding cochain data) with non-vanishing equivariant Lagrangian Floer cohomology forms a rigid analytic space, where we also apply tropical geometry to locate such Lagrangian torus fibers in the moment polytope. We also show in Chapter 6) that, in certain cases, that the dimension of such a rigid analytic space is equal to that of the acting group. In Chapter 7, we show that the equivariant Lagrangian Floer cohomology in this setup is Hamiltonian isotopy invariant. Lastly, we discuss equivariant Kuranishi structures in Chapter 8. Section 4.3 is based on [52], and Chapter 5–8 are based on [51].

Chapter 2

Equivariant de Rham theory

In this chapter, we review some properties of equivariant de Rham theory. The equivariant de Rham theory is particularly useful for simplifying geometric problems involving smooth manifolds with symmetry. For instance, when the fixed points are isolated, we can utilize the localization formulas to reduce the integration of certain equivariant differential forms to some local computations at the connected components of the fixed-point set.

An essential concept that underlies some of the most important applications of equivariant cohomology is the notion of equivariant integration along the fiber. The construction of the equivariant integration along the fiber map relies on the existence of equivariant Thom forms on equivariant vector bundles.

After introducing some preliminary concepts in Section 2.1, we review the definition of the equivariant de Rham cohomology in Section 2.2 and the Mathai-Quillen construction of equivariant Thom forms in Section 2.4. Then we discuss the construction and properties of equivariant integration along the fibers in Section 2.3. Much of the content of this chapter is borrowed from [31], [50], [3], and [11]. The interested reader is referred to the aforementioned works for more details.

2.1 Preliminary definitions

We recall the definition of fundamental vector fields associated to smooth Lie group actions.

Definition 2.1 (Fundamental vector fields)**.** Consider a smooth Lie group action on a smooth manifold M. Let $\mathfrak{g}$ be the Lie algebra of G. There is a Lie algebra homomorphism $\sigma : \mathfrak{g} \to \Gamma(TM)$, which assigns to every $X \in \mathfrak{g}$ its **fundamental vector field** $\underline{X}$ on M, as follows.

$$\sigma(X)_p := \underline{X}(p) := \frac{d}{dt}\bigg|_{t=0} (e^{-tX} \cdot p) \qquad p \in M. \tag{2.1.1}$$

Equivalently, if we define $j_p : G \to M$ by

$$j_p(g) = g \cdot p \qquad \forall g \in G, \tag{2.1.2}$$

then

$$\underline{X}(p) = (dj_p)_e(-X).$$

Here the negative signs are used to make σ a Lie algebra homomorphism, rather than an anti-homomorphism.

The smooth G-action on a smooth manifold M induces two $\mathfrak{g}$-actions on the de Rham complex $\Omega(M)$ such that, for all $X \in \mathfrak{g}$ and $\alpha \in \Omega(M)$,

$$\mathcal{L}_X \, \alpha = \mathcal{L}_{\underline{X}} \, \alpha \quad \text{ and } \quad \iota_X \alpha = \iota_{\underline{X}} \alpha, \tag{2.1.3}$$

where $\mathcal{L}_{\underline{X}}, \iota_{\underline{X}}$ are the usual Lie derivative and the usual interior product defined on differential forms.

It is straightforward to see that if $f : M \to N$ is a G-equivariant map and $X \in \mathfrak{g}$, then the fundamental vector fields $\underline{X}^M, \underline{X}^N$ associated to M, N are f-related in the following sense.

Definition 2.2 (f-related vector fields)**.** Let $f : M \to N$ be a smooth map between manifolds. Then two vector fields $Y \in \Gamma(TM), Z \in \Gamma(TN)$ are ***f*-related** if $df \circ Y = Z \circ f$.

We will encounter group actions which are free.

Definition 2.3 (Locally free and free group actions)**.** A continuous group action by a topological group G on a topological space M is **locally free** if the isotropy group G_p is finite $\forall p \in M$, and it is **free** if G_p is trivial $\forall p \in M$.

The quotient of a G-manifold by a free G-action is an example of a principal G-bundle.

Definition 2.4 (Principal G-bundles)**.** Let G be a topological group. A topological **principal G-bundle** is a fiber bundle $\pi : P \to B$ with fiber G and an open cover $\{(U, \phi_U)\}$ of B such that the following holds.

1) G acts continuously and freely on the right of P.

2) For each U, the fiber-preserving homeomorphism $\phi_U : \pi^{-1}(U) \to U \times G$ is G-equivariant, where G acts on $U \times G$ by $(x, a) \cdot g = (x, ag)$.

When we consider smooth G-actions, we can define a principal G-bundle in the smooth category by requiring that G is a smooth Lie group acting smoothly on P, that π is smooth, and that the ϕ_U are diffeomorphisms.

Definition 2.5 (Connection 1-form on a principal G-bundle)**.** A **connection 1-form** on a smooth principal G-bundle $\pi : P \to B$ is an element $A \in \Omega^1(P) \otimes \mathfrak{g}$ such that

$$g^* A = \mathrm{Ad}_{g^{-1}} A, \qquad \iota_{\underline{X}} A = X \qquad \forall X \in \mathfrak{g}\,.$$

Definition 2.6 (Universal G-bundles)**.** Let G be a topological group. A **universal G-bundle** is a principal G-bundle $\pi : EG \to BG$ satisfying the following.

1) Every topological principal G-bundle $P \to B$ is isomorphic to the pullback bundle f^*EG via a continuous map $f : X \to BG$.

2) If the pullback bundles f^*EG, g^*EG are isomorphic for some continuous maps $f, g : X \to BG$, then f, g are homotopic.

The base space BG of the universal G-bundle is called a **classifying space** for G.

The following fact is well-known (see for example [31] Section 1.2).

Proposition 2.1 (Existence of universal bundle)**.** For any compact Lie group G, there exists a universal G-bundle $EG \to BG$ such that EG is contractible.

A compact Lie group G can be embedded in the orthogonal group $O(k)$ for some $k \in \mathbb{N}$. Let

$$EG(m) = V_k(\mathbb{R}^{m+k+1}) \quad \forall m \in \mathbb{N} \tag{2.1.4}$$

be the Stiefel manifold of all orthonormal k-frames on $\mathbb{R}^{m+k+1}$. Then the quotient of the free G-action on

$$EG = \varinjlim_{m} EG(m)$$

is a universal G-bundle

$$EG \to BG = EG/G. \tag{2.1.5}$$

For the rest of the chapter, we will consider the smooth action by a compact connected Lie group G on a smooth manifold M unless otherwise stated.

Definition 2.7 (Derivations)**.** A function $D : A \to A$ on a graded k-algebra $A = \bigoplus_{j\in\mathbb{N}} A_j$ is a **derivation** of degree m if

- $D : A_j \to A_{j+m}$ is k-linear for all $j \in \mathbb{N}$; and
- $D(uv) = (Du)v + (-1)^{m \deg u} uDv$ for all $u, v \in A$.

The derivations $\iota_X, \mathcal{L}_X$, and d on the de Rham complex of a G-manifold M make $\Omega(M)$ into a $\mathfrak{g}$-differential graded algebra, whose definition is given below.

Definition 2.8 ($\mathfrak{g}$-differential graded algebras)**.** A graded commutative algebra $A = \bigoplus_{j\in\mathbb{N}} A_j$ is a **$\mathfrak{g}$-differential graded algebra** if there are derivations $d, \mathcal{L}_X, \iota_X$ of degrees $1, 0, -1$, respectively, such that the following relations hold:

$$[d, d] = 0, \quad [\mathcal{L}_X, d] = 0, \quad [\iota_X, d] = \mathcal{L}_X, \tag{2.1.6}$$

$$[\iota_X, \iota_Y] = 0, \quad [\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]_{\mathfrak{g}}}, \quad [\mathcal{L}_X, \iota_Y] = \iota_{[X,Y]_{\mathfrak{g}}}, \tag{2.1.7}$$

where

- the brackets $[\cdot,\cdot]$ on the left sides of the relations denote the graded commutators on derivations defined by

$$[D_1, D_2] = D_1 D_2 - (-1)^{ab} D_2 D_1$$

  if D_1, D_2 are derivations of degree a, b.

- the brackets $[\cdot,\cdot]_{\mathfrak{g}}$ on the right denote the Lie brackets on the Lie algebra $\mathfrak{g}$.

We will write "$\mathfrak{g}$-differential graded algebra" as "**$\mathfrak{g}$-dga**" for short.

Definition 2.9 (Morphisms of $\mathfrak{g}$-dgas)**.** A map $\phi : \mathcal{A} \to \mathcal{A}'$ between two $\mathfrak{g}$-dgas is a **morphism of $\mathfrak{g}$-dgas** if it commutes with $d, \mathcal{L}_X$, and ι_X for all $X \in \mathfrak{g}$.

We can generalize the definition of a connection on a principal G-bundle to one on a $\mathfrak{g}$-dga.

Definition 2.10 (Locally free $\mathfrak{g}$-dga)**.** A **G-connection** on a $\mathfrak{g}$-dga $\mathcal{A} = \bigoplus_{j\in\mathbb{N}} \mathcal{A}_j$ is a linear map $A : \mathfrak{g}^* \to \mathcal{A}_1$ such that, for all $X \in \mathfrak{g}$ and all $\alpha \in \mathfrak{g}^*$, the following holds.

1) $\mathcal{L}_X(A(\alpha)) = -A(\mathrm{ad}^*_X(\alpha))$.

2) $\iota_X(A(\alpha)) = \alpha(X)$.

The **curvature** associated to a connection A is a linear map $F^A : \mathfrak{g}^* \to \mathcal{A}_2$ given by

$$F^A = dA + \frac{1}{2}[A, A]. \tag{2.1.8}$$

A $\mathfrak{g}$-dga which admits a connection form is said to be **locally free**.

In [31], a locally free $\mathfrak{g}$-dga is called a W^*-module and the existence of a connection is called condition (C).

Lemma 2.1 (*G-action is locally free if and only if $\mathfrak{g}$-action is free*)**.** A smooth action on a smooth manifold M by a compact connected Lie group G is locally free if and only if

$$\mathfrak{g}_p = \{X \in \mathfrak{g} \mid \underline{X}(p) = 0\} = \{0\} \qquad \forall p \in M.$$

Proof. Consider a smooth G-action on a smooth manifold M. It suffices to show that

$$\mathrm{Lie}(G_p) = \mathfrak{g}_p \qquad \forall p \in M.$$

Let $\exp : \mathfrak{g} \to G$ be the exponential map on $\mathfrak{g}$. Then $\exp(\mathfrak{h}) \subset H$ if $H \subset G$ and $\mathfrak{h} = \mathrm{Lie}(H)$. Let $X \in \mathrm{Lie}(G_p)$ be an element of the Lie algebra of the isotropy group of p for some $p \in M$. Then $\exp(-tX) \in G_p$ implies that the integral curve fixes p:

$$\exp(-tX) \cdot p = p \qquad \forall t \in \mathbb{R}\,. \tag{2.1.9}$$

Thus,

$$\underline{X}(p) = \frac{d}{dt}\bigg|_{t=0} \exp(-tX) \cdot p = 0,$$

showing $\mathrm{Lie}(G_p) \subset \mathfrak{g}_p$ for all $p \in M$.

Conversely, if $\underline{X}(p) = 0$, then by uniqueness of the integral curve, (2.1.9) holds. Thus, $X \in \mathrm{Lie}(G_p)$. □

Proposition 2.2. A smooth action on a smooth manifold M by a compact connected Lie group G is locally free if and only if $\Omega(M)$ is a locally free $\mathfrak{g}$-dga.

Proof. We reproduce the proof in [31] §2.3.4. If a G-action is locally free, by Lemma 2.1, for each $p \in M$, there is an injective homomorphism

$$\mathfrak{g} \to T_pM, \qquad X \mapsto \underline{X}(p). \tag{2.1.10}$$

Fix a basis $X_1, \ldots, X_r$ for $\mathfrak{g}$, a dual basis $\theta_1, \ldots, \theta_r$ for $\mathfrak{g}^*$, and a G-invariant metric g on M such that, for all $p \in M$, the vectors $\underline{X_1}(p), \ldots, \underline{X_r}(p)$ are orthonormal. Then we can define $A : \mathfrak{g}^* \to \Omega^1(M)$ by

$$A(\theta_i) = g(\underline{X_i}, -) \quad \forall 1 \leq i \leq r. \tag{2.1.11}$$

Then the one-forms $\Theta_i = A(\theta_i)$, $1 \leq i \leq r$, span the vertical subbundle V of T^*M. Let $H = V^\perp$ be the horizontal subbundle.

Then

$$\iota_{\underline{X_j}}(A(\theta_i)) = g(\underline{X_i}, \underline{X_j}) = \delta_{ij} = \langle \theta_i, X_j \rangle \quad \forall i, j.$$

We now show the G-equivariance

$$\mathcal{L}_X A(\xi) = A(\mathrm{ad}^*_X(\xi)). \tag{2.1.12}$$

Let $k, i \in \{1 \ldots, r\}$. For any $j \in \{1 \ldots, r\}$, $\iota_{\underline{X_j}}(A(\theta_i))$ is constant. Thus,

$$\begin{aligned} 0 &= \mathcal{L}_{\underline{X_k}} \iota_{\underline{X_j}}(A(\theta_i)) \\ &= \left[\mathcal{L}_{\underline{X_k}}, \iota_{\underline{X_j}}\right](A(\theta_i)) + \iota_{\underline{X_j}} \mathcal{L}_{\underline{X_k}}(A(\theta_i)) \\ &= \iota_{[\underline{X_k}, \underline{X_j}]}(A(\theta_i)) + \iota_{\underline{X_j}} \mathcal{L}_{\underline{X_k}}(A(\theta_i)). \end{aligned} \tag{2.1.13}$$

Here $\left[\mathcal{L}_{\underline{X_k}}, \iota_{\underline{X_j}}\right]$ is the commutator $\mathcal{L}_{\underline{X_k}} \iota_{\underline{X_j}} - \iota_{\underline{X_j}} \mathcal{L}_{\underline{X_k}}$. Let the c^i_{kj} be the structure constants defined by

$$[\underline{X_k}, \underline{X_j}] = \sum_{i=1}^r c^i_{kj} \underline{X_i}.$$

By (2.1.13), we have

$$\mathcal{L}_{\underline{X_k}}(A(\theta_i)) = -\sum_{j=1}^r c^i_{kj} \Theta_j + \alpha_{ki},$$

for some horizontal α_{ki}. Since the metric is invariant, and both $\mathcal{L}_{\underline{X_k}}(A(\theta_i))$ and $-\sum_{j=1}^r c^i_{kj} \Theta_j$ are vertical, we have $\alpha_{ki} = 0$. On the other hand, for any j, we have

$$\left\langle \mathrm{ad}^*_{X_k}(\theta_i), X_j \right\rangle = \langle \theta_i, -[X_k, X_j] \rangle = -c^i_{kj}.$$

This implies that $\mathrm{ad}^*_{X_k}(\theta_i) = -\sum_{j=1}^r c^i_{kj} X_j$. Thus,

$$A(\mathrm{ad}^*_{X_k}(\theta_i)) = -\sum_{j=1}^r c^i_{kj} g(\underline{X_j}, -) = -\sum_{j=1}^r c^i_{kj} \Theta_j = \mathcal{L}_{\underline{X_k}}(A(\theta_i)).$$

Therefore, A is a G-connection on $\Omega(M)$.

Conversely, if there exists a G-connection A on $\Omega(M)$, then $\mathfrak{g}_p = \{0\}$ for all $p \in M$ and thus the G-action is locally free by Lemma 2.1. □

2.2 Equivariant cohomology

A good definition of equivariant cohomology is expected to recover the cohomology of the quotient manifold when the group action is free. Moreover, the equivariant cohomology should be a contravariant functor from the category of G-manifolds to the category of rings.

2.2.1 Homotopy quotients and equivariant cohomology

In fact, there is a nice homotopy-theoretic quotient whose cohomology would satisfy these properties. We review the construction and some basic properties of homotopy quotients in §2.2.1. For more details about homotopy quotients, we refer the reader to [31] Chapter 1 and [50] Part I.

Definition 2.11 (Homotopy quotient and equivariant cohomology)**.** Let G be a topological group which acts continuously on a topological space M. Let $EG \to BG$ be the universal G-bundle of the group G. The **homotopy quotient** M_G of the G-space M is obtained by Cartan's mixing construction (also called the Borel construction):

$$M_G := EG \times_G M := EG \times M/\sim, \tag{2.2.1}$$

where

$$(p, x) \sim (pg, g^{-1}x) \qquad \forall p \in EG, \quad \forall x \in M, \quad \forall g \in G.$$

The G-**equivariant cohomology** for the topological G-space M over a ring R is defined by

$$H^*_G(M, R) := H^*(M_G, R), \tag{2.2.2}$$

where the right hand side is the singular cohomology of M_G.

Given the principal G-bundle $EG \to BG$ and a G-manifold M, we obtain the mixing diagram

$$\begin{array}{ccccc} EG & \longleftarrow & EG \times M & \longrightarrow & M \\ \downarrow & & \downarrow & & \downarrow \\ BG & \underset{\sigma_1}{\longleftarrow} & EG \times_G M & \underset{\sigma_2}{\longrightarrow} & M/G \end{array} \quad . \tag{2.2.3}$$

The map $\sigma_1 : M_G \to BG$ is a fiber bundle with fiber M. And the fiber of the map $\sigma_2 : M_G \to M/G$ at Gx is given by

$$\{[p, gx] \mid p \in EG, g \in G\} = \{[pg, x] \mid p \in EG, g \in G\} \cong EG/G_x,$$

where G_x denotes the isotropy group of $x \in M$. If G acts on M freely, then $M \to M/G$ is a principal G-bundle and σ_2 becomes a fiber bundle with fiber EG.

Theorem 2.1 (Properties of equivariant cohomology, [50] Proposition 9.2 and [50] Theorem 9.5)**.** The G-equivariant cohomology of a G-manifold M satisfy the following properties.

1) (Functoriality of homotopy quotients) Every continuous G-equivariant map $f : M \to N$ between G-spaces induces a continuous map

$$f_G : M_G \to N_G, \quad [p, x] \mapsto [p, f(x)]$$

between homotopy quotients. This defines a covariant functor from the category of G-spaces to the category of topological spaces.

2) (Functoriality of equivariant cohomology) Every continuous G-equivariant map $f : M \to N$ between G-spaces induces a pullback map

$$f_G^* : H_G^*(N) \to H_G^*(M),$$

which is a contravariant functor from the category of G-spaces to the category of rings.

3) (Free action) If G acts on M freely, then M_G and M/G are weakly homotopy equivalent. In particular, $H_G^*(M) \cong H^*(M/G)$.

Since $M \to \{pt\}$ is a G-equivariant map for every G-manifold M, the induced pullback map on cohomology endows the equivariant cohomology $H_G^*(M)$ a module structure over the ring

$$H_G^*(pt, R) = H^*(BG, R). \tag{2.2.4}$$

Corollary 2.1. If M is a G-manifold, then $H_G^*(M)$ is a module over the ring $H^*(BG)$.

2.2.2 The Weil model

Since we are working mainly with smooth G-manifolds, we are interested in a model reminiscent of de Rham theory. Our goal now is to extract an algebraic model by looking at the de Rham complex on the homotopy quotient, whose cohomology agrees with equivariant cohomology by the de Rham theorem, more carefully. We refer the reader to [31] Chapter 3-4, [50] Chapter 19-20, and [3] for more details regarding the Weil model.

For the rest of the paper, we consider smooth actions by compact connected Lie groups. Since G acts on $EG \times M$ freely, the quotient $\pi : EG \times M \to EG \times_G M$ is a principal G-bundle. We observe that, for any principal G-bundle, the pullback map identifies the de Rham forms on the base with the "basic" forms on the total space.

Theorem 2.2 (Basic forms are invariant and horizontal, [50] Theorem 12.5)**.** Let $\pi : P \to B$ be a principal G-bundle. Then $\alpha \in \pi^*(\Omega(B))$ if and only if α satisfies the following. For all $X \in \mathfrak{g}$,

1) (invariant) $\mathcal{L}_X\, \alpha = 0$, and

2) (horizontal) $\iota_X \alpha = 0$.

The following lemma is well-known in equivariant de Rham theory. See [29] Chapter IV §2 Proposition III or [50] Theorem 12.5 for instance. We will use it for proving Theorem 5.4.

Lemma 2.2 (Basic forms on principal bundles are pullbacks)**.** Let G be a compact connected Lie group. Let $\pi : P \to B$ be a smooth principal G-bundle. Then $\pi^* : \Omega(B) \to \Omega_{bas}(P)$ is an isomorphism.

Proof. Since π is a surjective submersion, π^* is injective. We now show $\pi^*\Omega(B) = \Omega_{bas}(P)$.

Suppose $\eta = \pi^*\beta$ for some $\beta \in \Omega(B)$. Then, for any $g \in G$ and its induced diffeomorphism $\varphi_g : P \to P$, we have $\varphi_g^*\eta = \varphi_g^*\pi^*\beta = (\pi \circ \varphi_g)^*\beta = \eta$. Since G is connected, this is equivalent to $\mathcal{L}_{\underline{\zeta}}\eta = 0$ for all $\zeta \in \mathfrak{g}$, showing that η is G-invariant. Moreover, $d\pi \circ \underline{\zeta} = 0$ for all $\zeta \in \mathfrak{g}$. Hence, $\iota_{\underline{\zeta}}(\pi^*\beta) = 0$ for all $\zeta \in \mathfrak{g}$. This shows that $\pi^*\Omega(B) \subset \Omega_{bas}(P)$.

Suppose $\eta \in \Omega(P)$ is G-horizontal and G-invariant. Let $\left\{U_\alpha \times G \xrightarrow{\psi_\alpha} \pi^{-1}(U_\alpha)\right\}$ be a trivialization of π. Since ψ_α^* commutes with $\mathcal{L}_{\underline{\zeta}}$ and $\iota_{\underline{\zeta}}$ for all $\zeta \in \mathfrak{g}$, the form $\psi_\alpha^*(\eta\big|_{\pi^{-1}(U_i)})$ is also horizontal and invariant. Thus, there exists a unique $\beta_\alpha \in \Omega(U_\alpha)$ such that $\beta_\alpha \otimes 1 = \psi_\alpha^*(\eta\big|_{\pi^{-1}(U_\alpha)})$. Then $(\beta_\alpha)_x = (\beta_{\alpha'})_x$ if $x \in U_\alpha \cap U_{\alpha'}$, and we can define $\beta \in \Omega(B)$ by $\beta_x = (\beta_\alpha)_x$ for $x \in U_\alpha$. Hence, we have $\pi^*\beta = \eta$. □

Definition 2.12 (Basic subcomplexes)**.** For any $\mathfrak{g}$-dga $\mathcal{A}$, we define its **basic subcomplex** $\mathcal{A}_{bas}$ by

$$\mathcal{A}_{bas} := \{\alpha \in \mathcal{A} \mid \mathcal{L}_X\, \alpha = 0 \text{ and } \iota_X \alpha = 0 \quad \forall X \in \mathfrak{g}\}.$$

Corollary 2.2. If G acts on M freely, then there is an isomorphism

$$\pi^* : (\Omega(M/G), d) \to (\Omega_{bas}(M), d),$$

from the de Rham complex of M/G to the basic subcomplex of the de Rham complex of M. Therefore, we have isomorphisms

$$H_G^*(M) \cong H^*(M/G) \cong H^*(\Omega_{bas}(M), d).$$

Applying the above to the principal G-bundle $EG \times M \to M_G$, we have

$$H^*(M_G, \mathbb{R}) \cong H^*(\Omega_{bas}(EG \times M), d).$$

This inspires an algebraic model, called the Weil model, for equivariant de Rham cohomology.

We first define a $\mathfrak{g}$-dga $(W(\mathfrak{g}), D)$ which resembles $(\Omega(EG), d)$ in the sense that

1) $W(\mathfrak{g})$ is acyclic. This means that $H^0(W(\mathfrak{g}), D) \cong \mathbb{R}$ and $H^i(W(\mathfrak{g}), D) = 0$ for all $i \neq 0$.

2) $W(\mathfrak{g})$ is a locally free $\mathfrak{g}$-dga in the sense of Definition 2.10.

Definition 2.13 (Weil algebra)**.** Let $\mathfrak{g}^*$ be the dual of the Lie algebra $\mathfrak{g}$ of G. The **Weil algebra** $W(\mathfrak{g})$ of $\mathfrak{g}$ is defined by the following:

$$W(\mathfrak{g}) = \Lambda(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*), \tag{2.2.5}$$

where $\Lambda(\mathfrak{g}^*)$, $S(\mathfrak{g}^*)$ denote the exterior algebra and symmetric algebra on $\mathfrak{g}^*$, respectively.

Let $r = \dim G$. Let $X_1, \ldots, X_r$ be a basis for $\mathfrak{g}$ and $\alpha^1, \ldots, \alpha^r$ be a basis for $\mathfrak{g}^*$. Denote

$$\theta_i = \alpha^i \otimes 1 \quad \text{and} \quad u_i = 1 \otimes \alpha^i \qquad \forall 1 \le i \le r.$$

Then we identify $W(\mathfrak{g}) = \Lambda(\theta_1, \ldots, \theta_r) \otimes \mathbb{R}[u_1, \ldots, u_r]$.

The Weil algebra is graded by requiring $\deg \theta_i = 1$ and $\deg u_i = 2$ for all i.

Let the c_{ij}^k be the structure constants in the sense that

$$[X_i, X_j] = \sum_{k=1}^{r} c_{ij}^k X_k \qquad \forall 1 \le i, j \le r. \tag{2.2.6}$$

Thus, if G is abelian, the c_{ij}^k are 0. Then we define a differential $D : W(\mathfrak{g}) \to W(\mathfrak{g})$ by, for all $1 \le k \le r$,

$$D\theta_k = u_k - \frac{1}{2} \sum_{i,j} c_{ij}^k \theta_i \theta_j, \tag{2.2.7}$$

$$Du_k = \sum_{i,j} c_{ij}^k u_i \theta_j. \tag{2.2.8}$$

Moreover, we define

$$\iota_X \theta_k = \alpha^k(X), \quad \iota_X u_k = 0 \qquad \forall X \in \mathfrak{g}, \tag{2.2.9}$$

and

$$\mathcal{L}_X = D\iota_X + \iota_X D \qquad \forall X \in \mathfrak{g}\,. \tag{2.2.10}$$

The derivations $D, \iota_X, \mathcal{L}_X$ make the Weil algebra $W(\mathfrak{g})$ into a $\mathfrak{g}$-dga. These definitions are independent of the chosen basis. (See [50] §19.3.)

Let $\mathcal{A} = \bigoplus_{j \in \mathbb{N}} \mathcal{A}_j$ be a $\mathfrak{g}$-dga with a G-connection $A : \mathfrak{g}^* \to \mathcal{A}_1$. Note that we can identify A with a map $A^* : \mathcal{A}_1^* \to \mathfrak{g}$ and its curvature F^A with $(F^A)^* : \mathcal{A}_2^* \to \mathfrak{g}$. We can define a map $\kappa_A : W(\mathfrak{g}) \to \mathcal{A}$, sometimes called the **Weil map**, as follows. Let $\kappa_{\Lambda(\mathfrak{g}^*)} : \Lambda(\mathfrak{g}^*) \to \mathcal{A}$ be given by

$$\kappa_{\Lambda(\mathfrak{g}^*)}(\beta_1 \wedge \cdots \wedge \beta_k) = (\beta_1 \circ A^*) \wedge \cdots (\beta_k \circ A^*) \qquad \forall \beta_1, \ldots, \beta_k \in \mathfrak{g}^*, \tag{2.2.11}$$

where $\beta_i \circ A^* = (\mathcal{A}_1^* \xrightarrow{A^*} \mathfrak{g} \xrightarrow{\beta_i} \mathbb{R})$. And let $\kappa_{S(\mathfrak{g}^*)} : S(\mathfrak{g}^*) \to \mathcal{A}$ be given by

$$\kappa_{S(\mathfrak{g}^*)}(\gamma_1 \cdots \gamma_k) = (\gamma_1 \circ (F^A)^*) \wedge \cdots \wedge (\gamma_k \circ (F^A)^*) \quad \forall \gamma_1, \ldots, \gamma_k \in \mathfrak{g}^*, \tag{2.2.12}$$

where $\gamma_i \circ F^A = (\mathcal{A}_2^* \xrightarrow{(F^A)^*} \mathfrak{g} \xrightarrow{\gamma_i} \mathbb{R})$. Then we define $\kappa_\theta : W(\mathfrak{g}) \to \mathcal{A}$ by

$$\kappa_A(\gamma \otimes \beta) = \kappa_{\Lambda(\mathfrak{g}^*)}(\gamma) \wedge \kappa_{S(\mathfrak{g}^*)}(\beta) \qquad \forall \gamma \in \Lambda(\mathfrak{g}^*), \quad \forall \beta \in S(\mathfrak{g}^*). \tag{2.2.13}$$

The map κ_A and the map it induces on $S(\mathfrak{g}^*)^G$ are sometimes also called **Chern-Weil homomorphisms**.

One may find (2.2.7) and (2.2.8) resonant of Cartan's second structural equation and the Bianchi identity. In fact, the $\mathfrak{g}$-dga structure on $W(\mathfrak{g})$ is designed such that the Chern-Weil homomorphism is a morphism of locally free $\mathfrak{g}$-dgas, and $\Omega(P)$ for a principal G-bundle P is a primitive example of a locally free $\mathfrak{g}$-dga. It turns out that the Weil algebra is an universal object among all locally free $\mathfrak{g}$-dgas. By [31] Theorem 3.3.1, $W(\mathfrak{g})$ is characterized by the following. For any $\mathfrak{g}$-dga $\mathcal{A}$ with a connection θ, there exists a unique map $\kappa_A : W(\mathfrak{g}) \to \mathcal{A}$, up to chain homotopy, such that the diagram

$$\begin{array}{ccc} W(\mathfrak{g}) & \xrightarrow{\kappa_A} & \mathcal{A} \\ {\scriptstyle \theta_{W(\mathfrak{g})}} \uparrow & \nearrow {\scriptstyle A} & \\ \mathfrak{g}^* & & \end{array} \tag{2.2.14}$$

commutes, where $\theta_{W(\mathfrak{g})} : \mathfrak{g}^* \to W(\mathfrak{g})$ is defined by $\alpha \mapsto \alpha \otimes 1$.

We are now ready to introduce the Weil model, an algebraic model for equivariant de Rham theory.

Definition 2.14 (Weil Model and equivariant de Rham cohomology, v1)**.** The **Weil model** for a G-manifold M is given by the following differential complex

$$(W(\mathfrak{g}) \otimes \Omega(M))_{bas} \tag{2.2.15}$$

with differential $d_W = D \otimes 1 + 1 \otimes d$. Here D is the differential on the Weil algebra as in (2.2.7)–(2.2.8) and d is the de Rham differential on $\Omega(M)$. We define the **equivariant de Rham cohomology** by

$$H_G^{dR}(M) := H^*((W(\mathfrak{g}) \otimes \Omega(M))_{bas}, d_W). \tag{2.2.16}$$

If we take M to be a point, for example, then

$$H_G^{dR}(pt) = H^*(S(\mathfrak{g}^*)^G, d_W),$$

where $S(\mathfrak{g}^*)^G$ consists of elements of $S(\mathfrak{g}^*)$ that are invariant under the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$. If G is compact and connected, then $H_G^*(pt) \cong S(\mathfrak{g}^*)^G$.

As expected, the two definitions of equivariant cohomology agree.

Theorem 2.3 (Equivariant de Rham theorem, [31] Theorem 2.5.1)**.** For a compact connected Lie group G and a smooth G-manifold M, there is an isomorphism

$$H_G^*(M, \mathbb{R}) \cong H_{G,dR}^*(M)$$

between the equivariant cohomology (2.2.2) defined via homotopy quotients and the equivariant de Rham cohomology (2.2.16) defined via the Weil model.

2.2.3 The Cartan model

The Weil model provides a nice model for equivariant de Rham cohomology, but is in general hard to compute. It turns out that there is a simplified model, the Cartan model, which is more computable.

Consider an endomorphism $\gamma : W(\mathfrak{g}) \otimes \Omega(M) \to W(\mathfrak{g}) \otimes \Omega(M)$ by

$$\gamma = \sum_{j=1}^{\dim \mathfrak{g}} \theta_j \otimes \iota_{X_j}.$$

Since $\theta_j^{r+1} = 0$, $\gamma^{r+1} = 0$ as well. Thus, the map $\phi : W(\mathfrak{g}) \otimes \Omega(M) \to W(\mathfrak{g}) \otimes \Omega(M)$ given by

$$\phi = \exp \gamma = \operatorname{Id} + \gamma + \frac{1}{2}\gamma \circ \gamma + \frac{1}{3!}(\gamma \circ \gamma \circ \gamma) + \cdots \tag{2.2.17}$$

is a finite sum. This map ϕ, called the **Mathai-Quillen isomorphism**, transits the Weil model to a more computable model, called the Cartan model.

Theorem 2.4 (Mathai-Quillen isomorphism, [31] Theorem 4.2.1, see also [50] Theorem 21.1)**.** The automorphism

$$\phi : W(\mathfrak{g}) \otimes \Omega(M) \to W(\mathfrak{g}) \otimes \Omega(M)$$

given by (2.2.17) restricts an isomorphism of algebras

$$(W(\mathfrak{g}) \otimes \Omega(M))_{hor} \to W(\mathfrak{g})_{hor} \otimes \Omega(M) = S(\mathfrak{g}^*) \otimes \Omega(M), \tag{2.2.18}$$

$$\alpha + \sum_{|I|>1} \theta_I \alpha_I \mapsto \alpha, \tag{2.2.19}$$

$$\text{with inverse} \quad \prod_{i=1}^{r} (1 - \theta_i \iota_{X_i})\, \alpha \leftarrow\!\shortmid \alpha. \tag{2.2.20}$$

It further restricts an isomorphism of algebras

$$(W(\mathfrak{g}) \otimes \Omega(M))_{bas} \to (S(\mathfrak{g}^*) \otimes \Omega(M))^G,$$

where the latter consists of G-invariant elements of $S(\mathfrak{g}^*) \otimes \Omega(M)$.

The Mathai-Quillen isomorphism carries the differential d_W to

$$d_G = 1 \otimes d - \sum_{i=1}^{r} u_i \otimes \iota_{X_i}. \tag{2.2.21}$$

This motivates the definition of the Cartan model.

Definition 2.15 (Cartan model and equivariant de Rham cohomology, v2)**.** The **Cartan complex** for a smooth G-manifold M is given by

$$\Omega_G(M) := (\Omega(M) \otimes S(\mathfrak{g}^*))^G .$$

We may identify every element in $\Omega_G(M)$ as a polynomial map $\alpha : \mathfrak{g} \to \Omega(M)$ that is G-equivariant:

$$\alpha(\mathrm{Ad}_{g^{-1}} X) = g^* \alpha(X) \qquad \forall X \in \mathfrak{g}, \quad \forall g \in G.$$

Define the **equivariant de Rham differential** $d_G : \Omega_G^*(M) \to \Omega_G^{*+1}(M)$ by

$$(d_G \alpha)(X) = d(\alpha(X)) - \iota_{\underline{X}}(\alpha(X)) \qquad \forall X \in \mathfrak{g}, \quad \forall \alpha \in \Omega_G^*(M).$$

An element of $\Omega_G(M)$ is called a G-equivariant differential form. The grading is given by

$$\Omega_G(M) := \bigoplus_{j\in\mathbb{N}} \Omega_G^j(M), \qquad \text{where} \quad \Omega_G^j(M) = \bigoplus_{0\leq 2l\leq \dim M} \left(\Omega^{j-2l}(M)\otimes S^l(\mathfrak{g}^*)\right)^G.$$

By Theorem 2.4, we can equivalently define the equivariant de Rham cohomology by

$$H^*_{G,dR}(M) := H^*(\Omega_G(M), d_G). \tag{2.2.22}$$

Proposition 2.3 (Properties of the Cartan complex)**.**

1) d_G is zero on $S(\mathfrak{g}^*)^G$.

2) (Functoriality of Cartan models) Every G-equivariant map $f : M \to N$ between G-manifolds induces a pullback map

$$f_G^* : \Omega_G(N) \to \Omega_G(M)$$

on G-equivariant differential forms

$$(f_G^*\alpha)(X) = f^*(\alpha(X)) \qquad \forall X \in \mathfrak{g}, \tag{2.2.23}$$

which is a contravariant functor from the category of G-manifolds to the category of rings.

For a free smooth G-action on M, it is straightforward to see that $H^*_{G,dR}(M) \cong H^*_{dR}(M/G)$ via the equivariant and non-equivariant de Rham theorems. There is, however, an alternative proof which is evocative of the Chern-Weil theory.

Theorem 2.5 (Cartan operator is homotopic to identity, [31] §5)**.** Consider a locally free action of a compact connected Lie group G on a smooth manifold M. Then we can equip $\Omega(M)$ with a G-connection

$$A = \sum_{i=1}^{r} A_i \otimes X_i \in \Omega^1(M)\otimes \mathfrak{g}\,. \tag{2.2.24}$$

Let Car^A be the composition

$$\mathrm{Car}^A : (\Omega(M)\otimes S(\mathfrak{g}^*))^G \xrightarrow{\mathrm{Hor}^A} \left(\Omega^A_{hor}(M)\otimes S(\mathfrak{g}^*)\right)^G \xrightarrow{1\otimes\kappa_{S(\mathfrak{g}^*)}} \Omega_{bas}(M), \tag{2.2.25}$$

where Hor^A is the projection to $\left(\Omega^A_{hor}(M) \otimes S(\mathfrak{g}^*)\right)^G$, and $\kappa_{S(\mathfrak{g}^*)}$ is defined in (2.2.12). Then

$$\mathrm{Car}^A : (\Omega_G(M), d_G) \to (\Omega_{bas}(M), d)$$

is a chain map which is chain homotopic to the identity.

From now on, we will omit the subscript "dR" in and reserve the notation $H^*_G(M)$ for equivariant de Rham cohomology.

2.3 Equivariant integration along the fiber

The equivariant integration along the fiber on manifolds is closely related to Section 8.3. We refer the reader to [28] §VII and [31] §10 for details on integration along the fiber.

We first recall the definition of integration along the fiber via a submersion.

Theorem 2.6 (Integration along the fiber)**.** Let $f : M \to N$ be a submersion between smooth manifolds such that $\dim M - \dim N = d$. Let $\Omega_c(M), \Omega_c(N)$ be the sets of G-equivariant differential forms on M, N with compact support, respectively. Then there exists a map $f_! : \Omega^*_c(M) \to \Omega^{*-d}_c(N)$, called the integration along the fiber, where $\forall \alpha \in \Omega^*_c(M)$, $f_!\alpha \in \Omega^{*-d}_c(N)$ is uniquely determined by, $\forall X \in \mathfrak{g}$,

$$\int_M \alpha(X) \wedge f^*\beta = \int_N f_!\alpha(X) \wedge \beta \qquad \forall \beta \in \Omega(N). \tag{2.3.1}$$

Moreover, it satisfies the following properties:

1) (Adjoint property) $f_!(\alpha \wedge f^*\beta) = (f_!\alpha) \wedge \beta$ for all $\alpha \in \Omega_c(M)$ and all $\beta \in \Omega(N)$.

2) $\iota_Z f_!\alpha = f_!\iota_Y\alpha$ for all $\alpha \in \Omega^*_c(M)$, whenever $Y \in \Gamma(TM), Z \in \Gamma(TN)$, and Y, Z are f-related.

3) $df_!\alpha = f_!d\alpha$ for all $\alpha \in \Omega^*_c(M)$.

4) $\mathcal{L}_Y\, f_!\alpha = f_!\, \mathcal{L}_Z\, \alpha$ for all $\alpha \in \Omega^*_c(M)$, whenever $Y \in \Gamma(TM), Z \in \Gamma(TN)$, and Y, Z are f-related.

5) (Thom isomorphism) If f is a vector bundle over a compact manifold, then $f_!$ induces an isomorphism $f_! : H_c^*(M) \to H^{*-d}(N)$ on cohomology.

Theorem 2.7 (Equivariant integration along the fiber)**.** Let $f : M \to N$ be a G-equivariant submersion between smooth manifolds such that $\dim M - \dim N = d$. Let $\Omega_{G,c}(M), \Omega_{G,c}(N)$ be the sets of G-equivariant differential forms with compact support. Then there exists a map $f_{G!} : \Omega^*_{G,c}(M) \to \Omega^{*-d}_{G,c}(N)$, called equivariant integration along the fiber, where $\forall \alpha \in \Omega_{G,c}(M)$, we define $f_{G!}\alpha : \mathfrak{g} \to \Omega_c(N)$ by

$$(f_{G!}\alpha)(X) = f_{G!}(\alpha(X)) \qquad \forall X \in \mathfrak{g}\,. \tag{2.3.2}$$

It satisfies the following properties.

1) $f_{G!}(\alpha \wedge f_G^*\beta) = (f_{G!}\alpha) \wedge \beta$ for all $\alpha \in \Omega^*_{G,c}(M)$ and all $\beta \in \Omega^*_G(N)$.

2) (Equivariant Thom isomorphism) If f is a G-equivariant vector bundle over a compact manifold, then $f_!$ induces an isomorphism $f_! : H^*_{G,c}(M) \to H^{*-d}_G(N)$ on equivariant cohomology, whose inverse is the map on cohomology induced by wedging with a G-equivariant Thom form:

$$\Omega_G^{*-d}(N) \to \Omega^*_{G,c}(M), \qquad \beta \mapsto \tau \wedge f_G^*\beta.$$

2.4 Equivariant Thom forms

The equivariant integration along the fiber construction, which is essential in the proof of the localization theorem, relies on the existence of equivariant Thom forms.

Definition 2.16 (Equivariant vector bundles)**.** A vector bundle $\pi : E \to M$ is a **G-equivariant vector bundle** if G acts on it by vector bundle automorphisms. In other words, the following holds.

1) The action by each $g \in G$ defines diffeomorphisms of E and M such that the following diagram commutes.

$$\begin{array}{ccc} E & \xrightarrow{g} & E \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ M & \xrightarrow[g]{} & M \end{array}$$

2) The action of g on E restricts to a linear isomorphism $E_x \to E_{gx}$ on the fiber over x for all $x \in M$.

We follow [11] for the definition of equivariant Thom forms.

Definition 2.17 (Equivariant Thom forms)**.** Let $f : E \to B$ be a G-equivariant oriented real vector bundle of rank d. A G-equivariant differential form $\tau \in \Omega^d_G(E)$ is an **equivariant Thom form** if the following holds.

1) τ is equivariantly closed: $d_G\tau = 0$.

2) $\int_{E_x} \tau = 1$ for all $x \in B$, where $E_x = f^{-1}(x)$.

3) There exists a G-invariant open neighborhood $\mathcal{O}$ of the zero section such that

   a) $\operatorname{supp} \tau \subset \mathcal{O}$,

   b) $\mathcal{O} \cap E_x$ is convex for all $x \in B$, and

   c) $\mathcal{O} \cap E|_K$ is precompact for any compact subset $K \subset B$.

Theorem 2.8 (Mathai-Quillen's universal equivariant Thom forms)**.** There exists an SO(d)-equivariant Thom form $\mathrm{Th}_{\mathrm{SO}(d)}(\mathbb{R}^d) \in \Omega_{\mathrm{SO}(d)}(\mathbb{R}^d)$, called the universal Thom form, on the SO(d)-equivariant vector bundle $\mathbb{R}^d \to pt$.

We refer the reader to [31] §7.2 for the detailed construction. On any oriented G-equivariant vector bundle of rank k, one can construct a G-equivariant Thom form from the universal equivariant Thom form $\mathrm{Th}_{\mathrm{SO}(d)}(\mathbb{R}^d)$.

Theorem 2.9 (Universal equivariant Thom form on an equivariant vector bundle, [31] §10)**.** Let $E \to M$ be an oriented G-equivariant real vector bundle of rank d with a G-invariant metric. Let $P \to M$ be the associated orthonormal frame bundle. Then the image $\tau \in \Omega^d_G(E)$ of the universal equivariant Thom form $\mathrm{Th}_{\mathrm{SO}(d)}(\mathbb{R}^d)$ under the map

$$\Omega^d_{\mathrm{SO}(d)\times G}(\mathbb{R}^d) \xrightarrow{pr_2^*} \Omega^d_{\mathrm{SO}(d)\times G}(P \times \mathbb{R}^d) \xrightarrow{\kappa} \Omega^d_G(E) \tag{2.4.1}$$

G-equivariant Thom form of the bundle $E \to M$.

Here pr_2^* is the pullback map induced by the projection $P \times \mathbb{R}^d \to \mathbb{R}^d$. We briefly explain the map κ. Since the actions of $\mathrm{SO}(d)$ and G on $P \times \mathbb{R}^d$ commute, we can identify

$$\Omega_{\mathrm{SO}(d)\times G}(P \times \mathbb{R}^d) = \Big(\big(\Omega(P \times \mathbb{R}^d) \otimes S(\mathfrak{g}^*)\big)^G \otimes S(\mathfrak{so}(d)^*)\Big)^{\mathrm{SO}(d)}.$$

Then, since $\mathrm{SO}(d)$ acts freely on $P \times \mathbb{R}^d$, there exists an $\mathrm{SO}(d)$-connection on $\Omega(P \times \mathbb{R}^d)$, which allows us to define the map in the same way as (2.2.12).

We will refer the interested reader to [11] for the proof of the following theorem.

Theorem 2.10 (Existence of basic Thom forms [11] Theorem 3.8 and Remark 5.2)**.** Let $\tau \in \Omega^d_G(E)$ be an equivariant Thom form on the G-equivariant oriented real vector bundle $E \to B$ of rank d. Suppose the G-actions on E and B are locally free. Then there exists a G-connection $A : \mathfrak{g}^* \to \Omega^1(E)$ such the Cartan operator Car^A as in (2.2.25) carries τ to a G-basic Thom form τ_A, which also satisfies Definition 2.17.

We can generalize Theorem 2.10 to show the existence of equivariant Thom forms on equivariant orbifold vector bundles.

Chapter 3

Compact symplectic toric manifolds

In this chapter, we review some basic definitions and constructions related to compact symplectic toric manifolds.

3.1 Hamiltonian group actions

Every symplectic toric manifold is an example of a Hamiltonian G-manifold.

Definition 3.1 (Moment maps and Hamiltonian group actions)**.** Let G be a Lie group that acts smoothly on a symplectic manifold (M, ω). Denote the diffeomorphism induced by the action of $g \in G$ by φ_g. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^*$ be its dual.

1) We say the G-action is **symplectic** if the action preserves the symplectic structure: $\varphi_g^* \omega = \omega$ for all $g \in G$.

2) A symplectic G-action is said to be **Hamiltonian** if there exists a map

$$\mu : M \to \mathfrak{g}^*$$

such that the following holds.

   a) μ is G-equivariant.

b) For each $X \in \mathfrak{g}$, its fundamental vector field $\underline{X}$ is Hamiltonian:

$$d\left(\mu^*(X)\right) = \iota_{\underline{X}}\omega,$$

with Hamiltonian function $\mu^*(X) = \langle \mu(-), X \rangle : M \to \mathbb{R}$.

We call μ a **moment map** of the Hamiltonian G-action and (M, ω, G, μ) a **Hamiltonian G-manifold**. For a connected symplectic manifold, the moment map is unique up to a constant in $(\mathfrak{g}^*)^G$.

In fact, the concept of moment maps appears naturally in equivariant de Rham theory. By degree reasons, every G-equivariant 2-form on a G-manifold takes the form $\omega + \mu^*$ such that $\omega \in \Omega^2(M)^G$ and $\mu^* : \mathfrak{g} \to \Omega^0(M) = C^\infty(M)$. For any equivariant 2-form $\omega + \mu^*$ to be equivariantly closed, we need

$$0 = d_G(\omega + \mu^*)(X) = d\omega - \iota_X\omega(X) + d(\mu^*(X)).$$

This is equivalent to

$$\begin{cases} d\omega = 0 \\ d(\mu^*(X)) = \iota_{\underline{X}}\omega \qquad \forall X \in \mathfrak{g}\,. \end{cases}$$

Hence, every Hamiltonian G-manifold carries a natural equivariantly closed 2-form, which is an equivariantly closed extension of the symplectic form.

Recall that, if the smooth G-action on a manifold is free and proper, the quotient space will be a nice smooth manifold. A symplectic analog of this is the symplectic reduction construction.

Theorem 3.1 (Marsden-Weinstein, Meyer [31] Theorem 9.6.1, [9] Theorem 23.1)**.** Let G be a compact Lie group, and let (M, ω, G, μ) be a Hamiltonian G-manifold. Suppose G acts freely on $\mu^{-1}(0)$. Then the quotient space of $\pi : \mu^{-1}(0) \to \mu^{-1}(0)/G$ is a smooth manifold. There is a natural symplectic structure ω_{red} on the quotient space

$$M /\!\!/ G := \mu^{-1}(0)/G,$$

which is compatible with the symplectic structure on M:

$$\pi^* \omega_{red} = \omega \Big|_{\mu^{-1}(0)}.$$

The construction is called **symplectic reduction** and the space $M /\!\!/ G$ is called a **symplectic quotient**.

A more general version of Theorem 3.1 is the following.

Theorem 3.2 (Symplectic reduction of orbifolds at a regular level, [40] Lemma 3.9)**.** Let G be a compact Lie group, and let (M, ω, G, μ) be a Hamiltonian G-orbifold. Suppose $c \in \mathfrak{g}^*$ is a regular value of μ and G preserves $\mu^{-1}(c)$ (i.e. c is a fixed point of the coadjoint action on $\mathfrak{g}^*$). Then $\mu^{-1}(c)/G$ is a symplectic orbifold.

Definition 3.2 (Compact symplectic toric manifold)**.** Let T^n be an n-dimensional torus. We say a Hamiltonian T^n-manifold (M, ω, T^n, μ) is **toric** if the T^n-action on the compact connected $2n$-dimensional symplectic manifold (M, ω) is effective and Hamiltonian.

The image of a moment map of a Hamiltonian torus action has particularly nice properties. We will see in Theorem 3.1 that they can be used to classify compact symplectic toric manifolds.

Theorem 3.3 (Atiyah, Guillemin-Sternberg, Convexity theorem, [9] Theorem 27.1)**.** Suppose that (M, ω, T^r, μ) is a Hamiltonian T^r-manifold for an r-torus. Then the following holds.

1) The level sets of μ are connected.

2) The image of μ is a convex polytope.

3) $\mu(M)$ is the convex hull of the fixed-point set.

We call $\mu(M)$ the **moment polytope** of the Hamiltonian T^r-manifold.

In the case of a Hamiltonian action by a non-abelian compact Lie group, Theorem 3.3 has been generalized to the non-abelian convexity theorem (see [37] and [39]), where, instead of the full moment map image, a similar result holds for the intersection of a closed Weyl chamber with the moment map image.

Theorem 3.4 (Arnold-Liouville, Action-angle coordinates, [9] Theorem 18.12)**.** Suppose the smooth functions $f_i : M \to \mathbb{R}$, $1 \le i \le n$, on a $2n$-dimensional symplectic manifold (M, ω) satisfy

$$0 = \{f_i, f_j\} := \omega(X_{f_i}, X_{f_j}) \qquad \forall 1 \le i, j \le n. \tag{3.1.1}$$

Here X_f denotes the Hamiltonian vector field for f. Let $F = (f_1, \ldots, f_n)$. Suppose $c \in \mathbb{R}^n$ is a regular value of F and L is a compact connected component of $F^{-1}(c)$. Then the following holds.

1) L is a Lagrangian torus.

2) There exists an open neighborhood U of L in M and a neighborhood $W \cong V \times T^n$ of the zero section of T^*T^n such that $\Psi : U \to V \times T^n$ is a symplectomorphism, and $\Psi(p) = (\varphi_1(p), \ldots, \varphi_n(p), t_1(p), \ldots, t_n(p))$. The φ_i are called the **action** coordinates, and the t_i are called the **angle** coordinates.

On a symplectic toric manifold (M, ω, T^n, μ), the moment map μ induces such "commuting functions" as follows. Pick a basis $X_1, \ldots, X_n \in \mathrm{Lie}(T^n)$. Then the maps of the form

$$f_i : M \to \mathbb{R}, \quad f_i(p) = \langle \mu(p), X_i \rangle \qquad \forall p \in M$$

satisfy

$$\{f_i, f_j\} = \langle \mu(-), [X_i, X_j] \rangle ,$$

which vanishes because the Lie bracket is trivial on the abelian Lie group T^n.

The regular values of the moment map of a toric manifold corresponds to the interior points of the moment polytope. In fact, more is true.

Proposition 3.1 (Non-regular toric moment map fibers, [4] Proposition IV.4.16)**.** Let $(M^{2n}, \omega, T^n, \mu)$ be a compact symplectic toric manifold. Let $\Delta = \mu(\Delta)$ be the moment polytope and F be a k-dimensional face of P and $\mathring{F}$ be its relative interior. Then $\mu^{-1}(F)$ is a symplectic manifold of dimension $2k$ and $\mu\big|_{\mu^{-1}(\mathring{F})} : \mu^{-1}(\mathring{F}) \to \mathring{F}$ is a Lagrangian torus fibration with fibers diffeomorphic to T^k.

3.2 Constructions of toric manifolds from Delzant polytopes

Definition 3.3 (Delzant polytopes)**.** Let $N_{\mathbb{R}} \cong \mathbb{R}^n$ be an n-dimensional real vector space. Let $\Gamma \cong \mathbb{Z}^n$ be a lattice in $\mathbb{R}^n$. Let $N_{\mathbb{R}}^*, \Gamma^*$ be the duals of $N_{\mathbb{R}}, \Gamma$, respectively. Let $T = \mathbb{R}^n / \Gamma$. A convex polytope $\Delta \subset N_{\mathbb{R}}^*$ is a **Delzant polytope** if the following holds.

1) (Simplicity) Exactly n edges meet at every vertex.

2) (Rationality) For any vertex p of Δ, the edges that meet at p are rays of the form $p + tw_i(p)$, where $t \geq 0$ and $w_i(p) \in \Gamma^*$, $1 \leq i \leq n$.

3) (Smoothness) For any vertex p, $\{w_1(p), \ldots, w_n(p)\}$ form a $\mathbb{Z}$-basis of Γ^*.

By the description, any Delzant polytope Δ with m facets can be described as the intersection

$$\Delta = \bigcap_{i=1}^{m} \{u \in N_{\mathbb{R}}^* \mid \langle u, v_i \rangle - \lambda_i \geq 0\} \tag{3.2.1}$$

of half-spaces for some vectors $v_i \in \Gamma \subset N_{\mathbb{R}}$ and constants $\lambda_i \in \mathbb{R}$.

In fact, in [13], Delzant has classified all compact symplectic toric manifolds by their moment polytopes.

Theorem 3.5 (Delzant's Theorem)**.** There is a one-to-one correspondence between the set of compact symplectic toric manifolds and the set of Delzant polytopes in $\mathbb{R}^n$.

Delzant's construction

We briefly recall Delzant's construction of a compact symplectic toric manifold from a Delzant polytope of the form (3.2.1).

Let $\{e_1, \ldots, e_m\}$ be the standard basis of $\mathbb{R}^m$. Define a linear transformation

$$\pi : \mathbb{R}^m \to N_{\mathbb{R}}^*, \qquad e_i \mapsto v_i.$$

This induces a surjective map $\overline{\pi} : \mathbb{R}^m / \mathbb{Z}^m \to N_{\mathbb{R}}/\Gamma \cong T^n$. The standard effective Hamiltonian T^m-action on $\left(\mathbb{C}^m, -i \sum_{i=1}^{m} dz_i \wedge d\overline{z}_i\right)$ is given by, for all $(\theta_1, \ldots, \theta_m) \in \mathbb{R}^m / \mathbb{Z}^m$ and all $(z_1, \ldots, z_m) \in \mathbb{C}^m$,

$$(\theta_1, \ldots, \theta_m) \cdot (z_1, \ldots, z_m) = \left(e^{2\pi i \theta_1} z_1, \ldots, e^{2\pi i \theta_m} z_m\right).$$

A moment map for this action is

$$\mu(z_1, \ldots, z_m) = \frac{1}{2} \sum_{i=1}^{m} |z_i|^2 e_i^*.$$

Note that $K = \ker \overline{\pi} \cong T^{m-n}$ is also a torus. Then the moment map of the restriction of the T^m-action to K is given by

$$\mu_K = \iota^* \circ \mu(z_1, \ldots, z_m)$$

where $\iota : \ker \overline{\pi} \to \mathbb{R}^m / \mathbb{Z}^m$ is the inclusion map. Since the action of K on $\mu_K^{-1}(0)$ is free, we can apply symplectic reduction (Theorem 3.1) to get a symplectic manifold $(\mu_K^{-1}(0)/K, \omega_{red})$. Since the Hamiltonian T^m-action on $\mathbb{C}^m$ can be regarded as the action by $K \times N_{\mathbb{R}}/\Gamma$, by the general theory of reduction in stages (see for example [9] §24.3), there is an effective Hamiltonian $N_{\mathbb{R}}/\Gamma$-action on $(\mu_K^{-1}(0)/K, \omega_{red})$, making it toric manifold.

Chapter 4

Ordinary Lagrangian Floer theory on compact toric manifolds

A renowned problem in symplectic topology is the Arnold's conjecture, a version of which predicts a lower bound of number of intersection points of two Hamiltonian isotopic Lagrangian submanifolds in a compact symplectic manifold, provided that they intersect transversely. To tackle the problem, Floer [16] considered a Morse-type cohomology, nowadays called Lagrangian Floer cohomology, by considering an action functional, an analog of a Morse function, on certain infinite-dimensional path spaces and solved the Arnold's conjecture under the assumption that the second homotopy group of the symplectic manifold relative to the Lagrangian submanifolds are trivial. It turns out that, when the assumption is removed, Floer's boundary operator may not define a differential on Floer's complex, due to the bubbling phenomenon that appears in the compactification of the moduli spaces of pseudoholomorphic strips. Indeed, in many cases, the appearance of disc bubbles cannot be ruled out and is the source of the obstruction of defining Lagrangian Floer "cohomology". Therefore, A_∞ structures were introduced to encode the information carried by the moduli space of pseudoholomorphic curves. This is explained in detail in the books [24], [25], and [20] by Fukaya, Oh, Ohta, and Ono.

4.1 The A_∞ algebra associated to a Lagrangian submanifold

The study of moduli spaces of pseudoholomorphic curves is crucial in Lagrangian Floer theory. Let (X, ω) be a compact symplectic manifold and J be a compatible almost complex structure. Let L be a compact Lagrangian submanifold of X with a relatively spin structure. For each $\beta \in \pi_2(X, L)$, we denote its symplectic area by $\omega(\beta)$ and its Maslov index by $I_\mu(\beta)$.

Definition 4.1 (Moduli space of pseudoholomorphic discs)**.** Let $k, l \in \mathbb{N}$ and $\beta \in \pi_2(X, L)$. The moduli space $\mathcal{M}_{k+1,l}(L, J, \beta)$ is defined by

$$\left\{ (\Sigma, j_\Sigma, \vec{z}, \vec{w}, u) \;\middle|\; \begin{array}{l} \Sigma \text{ is a genus 0 nodal Riemann surface with} \\ \text{connected boundary and complex structure } j_\Sigma; \\ u : (\Sigma, \partial\Sigma) \to (X, L) \text{ is smooth;} \\ du \circ j_\Sigma = J \circ du; \quad [u] = \beta \in \pi_2(X, L); \\ (\Sigma, j_\Sigma, \vec{z}, \vec{w}, u) \text{ is stable;} \quad E(u) < \infty \\ \vec{z} = (z_0, z_1, \ldots, z_k) \in (\partial\Sigma)^{k+1}, \text{where the } z_i \text{ are} \\ \text{distinct non-nodal boundary marked points and the} \\ \text{enumeration is in counterclockwise order along } \partial\Sigma; \\ \vec{w} = (w_1, \ldots, w_l) \in (\mathring{\Sigma})^l \text{ are distinct non-nodal} \\ \text{interior marked points} \end{array} \right\} \Big/ \sim, \qquad (4.1.1)$$

where $(\Sigma, j_\Sigma, \vec{z}, \vec{w}, u) \sim (\Sigma', j_{\Sigma'}, \vec{z}', \vec{w}', u')$ if and only if there exists a biholomorphism $\varphi : (\Sigma, j_\Sigma) \to (\Sigma', j_{\Sigma'})$ such that $u' \circ \varphi = u$, $\varphi(z_i) = z_i'$ for all $0 \le i \le k$, and $\varphi(w_j) = w_j'$ for all $1 \le j \le l$.

We denote the evaluation map at the i-th boundary marked point by

$$\mathrm{ev}_{i,(k+1,l,\beta)} : \mathcal{M}_{k+1,l}(L, J, \beta) \to L, \quad [\Sigma, j_\Sigma, \vec{z}, \vec{w}, u] \mapsto u(z_i) \qquad \forall 0 \le i \le k,$$

and we denote the evaluation map at the j-th interior marked point by

$$\mathrm{ev}^{j}_{(k+1,l,\beta)} : \mathcal{M}_{k+1,l}(L,J,\beta) \to L, \quad [\Sigma, j_\Sigma, \vec{z}, \vec{w}, u] \mapsto u(w_j) \qquad \forall 1 \le j \le l.$$

To define the A_∞ algebra associated to the Lagrangian submanifold L, we consider moduli spaces of the form $\mathcal{M}_{k+1,0}(L,J,\beta)$ and the evaluation maps of the form $\mathrm{ev}_{i,(k+1,0,\beta)}$.

Definition 4.2 (The A_∞ algebra associated to a Lagrangian submanifold)**.** The A_∞ algebra

$$(\Omega(L,\Lambda_{0,nov}), \{\mathfrak{m}_k\}_{k\in\mathbb{N}})$$

associated to L is given by the following data.

- $\Omega(L,\Lambda_{0,nov}) = \Omega(L)\widehat{\otimes}_{\mathbb{R}}\Lambda_{0,nov}$, where $\Omega(L)$ denotes the de Rham complex of L, $\Lambda_{0,nov}$ denotes the universal Novikov ring defined by (5.1.3), and $\widehat{\otimes}$ denotes completion of the tensor product with respect to the T-adic topology. We specify that $\Omega(L,\Lambda_{0,nov})^{\otimes 0} = \Lambda_{0,nov}$, and the elements of $\Omega(L,\Lambda_{0,nov})[1]$ are the elements of $\Omega(L,\Lambda_{0,nov})$ with degree shifted down by 1.

- For each $k \in \mathbb{N}$, the A_∞ operator

$$\mathfrak{m}_k = \sum_{\beta\in\pi_2(Y,L)} \mathfrak{m}_{k,\beta} T^{\omega(\beta)} e^{\frac{I_\mu(\beta)}{2}} : (\Omega(L,\Lambda_{0,nov})[1])^{\otimes k} \to \Omega(L,\Lambda_{0,nov})[1]$$

is defined by the following. Let $x_1,\ldots,x_k \in \Omega(L,\Lambda_{0,nov})$. For $\beta = 0$,

$$\begin{cases} \mathfrak{m}_{0,\beta=0}(1) = 0 \\ \mathfrak{m}_{1,\beta=0}(x_1) = dx_1, \text{ where } d \text{ is the de Rham differential} \\ \mathfrak{m}_{2,\beta=0}(x_1\otimes x_2) = (-1)^{\deg x_1} x_1 \wedge x_2 \\ \mathfrak{m}_{k,\beta=0} = 0 \qquad \forall k \ge 3. \end{cases} \tag{4.1.2}$$

For $\beta \neq 0$, define

$$\mathfrak{m}_{0,\beta}(1) = (\mathrm{ev}_{0,\beta})_!(1) \tag{4.1.3}$$

and, for $k \ge 1$,

$$\mathfrak{m}_{k,\beta}(x_1\otimes\cdots\otimes x_k) = (-1)^{1+\sum_{j=1}^{k} j(\deg x_j+1)} (\mathrm{ev}_{0,(k+1,0,\beta)})_! \left(\mathrm{ev}^*_{1,(k+1,0,\beta)}\, x_1 \wedge \cdots \wedge \mathrm{ev}^*_{k,(k+1,0,\beta)}\, x_k\right). \tag{4.1.4}$$

We note that the signs in the definition of the $\mathfrak{m}_k$ follow that in [20] Chapter 22. The key observation is that the data in Definition 4.2 satisfies nice algebraic properties, as noted in Theorem 4.1.

Theorem 4.1 ([24] Theorem 3.5.11)**.** Let $(\Omega(L, \Lambda_{0,\mathrm{nov}}), \{\mathfrak{m}_k\}_{k\in\mathbb{N}}, S)$ be the data defined in Definition 4.2 and (5.3.7). It is an S-gapped curved filtered A_∞ algebra.

We recall the notions showing up in the theorem below.

Definition 4.3 (Discrete submonoid)**.** Consider the monoid $(\mathbb{R}_{\geq 0} \times 2\,\mathbb{Z}, +, (0,0))$ and the projection maps $E : \mathbb{R}_{\geq 0} \times 2\,\mathbb{Z} \to \mathbb{R}_{\geq 0}$, $I_\mu : \mathbb{R}_{\geq 0} \to 2\,\mathbb{Z}$. A subset $S \subset \mathbb{R}_{\geq 0} \times 2\,\mathbb{Z}$ is a **discrete submonoid** if the following holds.

i) $(S, +, (0,0))$ is a monoid.

ii) $E(S)$ is discrete.

iii) For each $E_0 \in \mathbb{R}_{\geq 0}$, $S \cap E^{-1}([0, E_0])$ is a finite set.

In the setup of this section, the set

$$S = \{(\omega(\beta), I_\mu(\beta)) \mid \beta \in \pi_2(X, L)\}. \tag{4.1.5}$$

is a discrete submonoid.

Definition 4.4 (S-gapped curved filtered A_∞ algebra)**.** An **S-gapped curved**[1] **filtered A_∞ algebra** is a tuple $(C, \{\mathfrak{m}_k\}_{k\in\mathbb{N}}, S)$ consisting of

- a $\Lambda_{0,\mathrm{nov}}$-module C,
- a family of operators $\mathfrak{m}_k : (C[1])^{\otimes k} \to C[1]$, and
- a discrete submonoid $S \subset \mathbb{R}_{\geq 0} \times 2\,\mathbb{Z}$

such that the following holds.

[1]The word "curved" means an $\mathfrak{m}_0 : \Lambda_{0,nov} \to C[1]$ is included, in contrast to the classical A_∞ algebra.

i) (S-gapped) $\forall k \in \mathbb{N}$, there is a decomposition

$$\mathfrak{m}_k^G = \sum_{(\lambda,n)\in S} \mathfrak{m}_{k,(\lambda,n)}^G T^\lambda e^{\frac{n}{2}}.$$

ii) (Energy filtered) There is an energy filtration on C such that, $\forall p \in \mathbb{N}$, the following holds.

- The filtration on C^p is decreasing: $F^\lambda C^p \subset F^{\lambda'} C^p$ if $\lambda > \lambda'$.
- $\forall \lambda' > 0$, we have $T^{\lambda'} \cdot F^\lambda C^p \subset F^{\lambda+\lambda'} C^p$.
- C^p is complete with respect to the T-adic topology induced by the filtration.
- C^p has a basis whose elements are in $F^0 C^p \setminus \bigcup_{\lambda>0} F^\lambda C^p$.
- $\mathfrak{m}_0(1) \in F^\lambda C[1]$ for some $\lambda > 0$.

Moreover, for each $k \in \mathbb{N}$, $\mathfrak{m}_k$ is filtration-preserving:

$$\mathfrak{m}_k(F^{\lambda_1} C^{p_1} \otimes \cdots \otimes F^{\lambda_k} C^{p_k}) \subset F^{\lambda_1+\cdots+\lambda_k} C^{p_1+\cdots+p_k-k+2}$$

for all $(\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k_{\geq 0}$ and all $(p_1, \ldots, p_k) \in \mathbb{N}^k$.

iii) (A_∞ relations) The family $\{\mathfrak{m}_k\}_{k\in\mathbb{N}}$ satisfy the following A_∞ relations: For any $k \in \mathbb{N}\setminus\{0\}$, $s \in S$,

$$\sum_{\substack{s_1,s_2\in S\\ s_1+s_2=s}} \sum_{\substack{k_1,k_2\in\mathbb{N}\\ k_1+k_2=k+1}} \sum_{i=0}^{k_1} (-1)^* \mathfrak{m}_{k_1,s_1}(x_1 \otimes \cdots \otimes x_i \otimes$$
$$\mathfrak{m}_{k_2,s_2}(h_{i+1} \otimes \cdots \otimes x_{i+k_2}) \otimes x_{i+k_2+1} \otimes \cdots \otimes x_k) = 0, \tag{4.1.6}$$

where $* = \sum_{j=1}^{i} (\deg x_j + 1)$.

4.2 Bulk deformations

Consider the $\Lambda_{0,nov}$-module homomorphisms of the form

$$\mathfrak{q}_{l,k} : (\Omega(X, \Lambda_{0,nov})[2])^{\otimes l} \otimes (\Omega(L, \Lambda_{0,nov})[1])^{\otimes k} \to \Omega(L, \Lambda_{0,nov})[1],$$

$$\mathfrak{q}_{l,k} = \sum_{\beta \in \pi_2(X,L)} \mathfrak{q}_{l,k,\beta} T^{\omega(\beta)} e^{\frac{I_\mu(\beta)}{2}},$$

where $\mathfrak{q}_{l,k,\beta} : \Omega(X, \Lambda_{0,nov})^{\otimes l} \otimes \Omega(L, \Lambda_{0,nov})^{\otimes k} \to \Omega(L, \Lambda_{0,nov})$ is defined by, for all $x_1, \ldots, x_k \in \Omega(L, \Lambda_{0,nov})$ and all $y_1, \ldots, y_l \in \Omega(X, \Lambda_{0,nov})$,

$$\begin{cases} \mathfrak{q}_{0,k,\beta}(x_1 \otimes \cdots \otimes x_k) = \mathfrak{m}_{k,\beta}(x_1 \otimes \cdots \otimes x_k) \\ \mathfrak{q}_{1,0,\beta=0}(y_1) = (-1)^{\dagger} \iota^* y_1, \quad \text{where } \iota : L \hookrightarrow X \text{ is the inclusion map} \\ \mathfrak{q}_{1,0,\beta}(y_1) = 0 \quad \text{if } \beta \neq 0 \\ \mathfrak{q}_{l,k,\beta}(y_1 \otimes \cdots \otimes y_l \otimes x_1 \otimes \cdots \otimes x_k) = \\ (-1)^{\ddagger} \dfrac{1}{l!} (\mathrm{ev}_{0,(k+1,l,\beta)})_! \Big((\mathrm{ev}^1_{(k+1,l,\beta)})^* y_1 \wedge \cdots \wedge (\mathrm{ev}^l_{(k+1,l,\beta)})^* y_l \wedge \\ \quad \mathrm{ev}^*_{1,(k+1,l,\beta)}\, x_1 \wedge \cdots \wedge \mathrm{ev}^*_{k,(k+1,l,\beta)}\, x_k \Big) \\ \quad \text{if } l \neq 0, k \neq 0, \text{ and } (l,k) \neq (1,0). \end{cases} \tag{4.2.1}$$

Here $\dagger$ is an integer depending on the degree of the differential form y, and $\ddagger$ is an integer depending on the degrees of the differential forms $y_1, \ldots, y_l, x_1, \ldots, x_k$.

Let $\mathfrak{b} \in \Omega^{even}(X, \Lambda_{0,nov})$ and $b \in \Omega^{odd}(L, \Lambda_{0,nov})$. The bulk-deformed A_∞ operators are defined by

$$\begin{aligned} &\mathfrak{m}_k^{\mathfrak{b},b}(x_1 \otimes \cdots \otimes x_k) \\ = &\sum_{\beta \in \pi_2(Y,L)} \sum_{l \geq 0} \sum_{r_0, \ldots, r_k \geq 0} \mathfrak{q}_{l, r_0 + \cdots + r_k + k, \beta}(\mathfrak{b}^{\otimes l} \otimes b^{r_0} \otimes x_1 \otimes b^{r_1} \otimes \cdots \otimes x_k \otimes b^{r_k}) T^{\omega(\beta)} e^{\frac{I_\mu(\beta)}{2}}. \end{aligned}$$

In particular, for any $\mathfrak{b} \in \Omega^{even}(X, \Lambda_{0,nov})$ and $b \in \Omega^{odd}(L, \Lambda_{0,nov})$,

$$\mathfrak{m}_0^{\mathfrak{b},b}(1) = \sum_{\substack{l,k \in \mathbb{N} \\ \beta \in \pi_2(X,L)}} \mathfrak{q}_{l,k,\beta}(\mathfrak{b}^{\otimes l} \otimes b^{\otimes k}) T^{\omega(\beta)} e^{\frac{I_\mu(\beta)}{2}}. \tag{4.2.2}$$

We say L is **unobstructed after bulk deformation** by $\mathfrak{b}, b$ if

$$\mathfrak{m}_0^{\mathfrak{b},b}(1) = 0.$$

Since the bulk-deformed A_∞ operators $\mathfrak{m}_k^{\mathfrak{b},b}$ still satisfy the A_∞ relations, there will be no obstruction in defining the Floer cohomology of L by

$$HF(L, L, \Lambda_0) := H^* \left(\Omega(L, \Lambda_{0,nov}), \mathfrak{m}_1^{\mathfrak{b},b} \right).$$

We refer the readers to [24], [25], [22], and [21] for more details on bulk deformation theory.

4.3 Moment Lagrangian correspondences are unobstructed after bulk deformation

In this section, we prove that the moment Lagrangian correspondences induced by the symplectic reduction of level sets with respect to free actions are unobstructed after bulk deformations, under Assumption 1. We refer the reader to Chapter 8 for the notions related to Kuranishi structures.

Let (Y, ω_Y, G, μ) be a Hamiltonian G-manifold consisting of the following data.

- (Y, ω_Y) is a compact symplectic manifold.
- G is a compact connected Lie group acting on (Y, ω_Y) in a Hamiltonian fashion. Let $\mathfrak{g}$ be the Lie algebra of G.
- $\mu : Y \to \mathfrak{g}^*$ is a moment map of the G-action.

Suppose G acts on $\mu^{-1}(0)$ freely. Then by Theorem 3.1, there exist a symplectic reduction map $\pi : \mu^{-1}(0) \to \mu^{-1}(0)/G =: Y /\!\!/ G$ and an induced symplectic form ω_{red} on $Y /\!\!/ G$. Following Abouzaid-Bottman [1], we will call the Lagrangian submanifold

$$L = \left\{(p, [p]) \in Y^- \times Y /\!\!/ G \,\middle|\, p \in \mu^{-1}(0), \pi(p) = [p]\right\} \tag{4.3.1}$$

of $(Y^- \times Y /\!\!/ G, -\omega_Y \oplus \omega_{red})$ the **moment Lagrangian correspondence** induced by this symplectic reduction. Denote the inclusion map by

$$\iota : L \to Y^- \times Y /\!\!/ G. \tag{4.3.2}$$

Let G act on $Y^- \times Y /\!\!/ G$ such that it acts trivially on the second factor $Y /\!\!/ G$ and acts on the first factor Y^- by the original Hamiltonian action.

We denote $Y^- \times Y /\!\!/ G$ by X and $-\omega_Y \oplus \omega_{red}$ by ω. Let J be a G-invariant ω-compatible almost complex structure on Y. And we equip L with a relatively spin structure.

The main theorem of this section is the following.

Theorem 4.2. The Lagrangian submanifold $L \subset (Y^- \times Y /\!\!/ G, -\omega_Y \oplus \omega_{red})$ defined by (4.3.1) is unobstructed after bulk deformation under Assumption 1.

Assumption 1. For each $k, l \in \mathbb{N}$ and $\beta \in \pi_2(X, L)$, we assume that the following holds.

i) $\mathcal{M}_{k+1,l}(L, J, \beta)$ has a G-equivariant Kuranishi structure, and G acts freely on each Kuranishi chart.

ii) The evaluation map at every (interior or boundary) marked point is strongly smooth and is G-equivariant on each chart.

iii) There is a compatible G-equivariant system of CF-perturbations $\widehat{\mathcal{S}}$ such that the Thom forms in the CF-perturbation data (8.2.1) are G-basic.

iv) Moreover, the equivariant Kuranishi structures and equivariant CF-purturbations are compatible with

$$\partial \mathcal{M}_{k+1,l}(L, J, \beta) =$$

$$\bigcup_{\substack{k_1,k_2,l_1,l_2\geq 0\\ k_1+k_2=k+1\\ l_1+l_2=l}} \bigcup_{\substack{\beta_1,\beta_2\in\pi_2(X,L)\\ \beta_1+\beta_2=\beta}} \bigcup_{j=1}^{k_2} \mathcal{M}_{k_1+1,l_1}(L, J, \beta_1)_{\mathrm{ev}_{0,(k_1+1,l_1,\beta_1)}} \times_{\mathrm{ev}_{j,(k_2+1,l_2,\beta_2)}} \mathcal{M}_{k_2+1,l_2}(L, J, \beta_2).$$

v) The evaluation map $\mathrm{ev}_{0,(k+1,l,\beta)} : \mathcal{M}_{k+1,l}(L, J, \beta) \to L$ at the zero-th boundary marked point is strongly submersive with respect to $\widehat{\mathcal{S}}$.

The notions that appear in Assumption 1 involving equivariant Kuranishi structures will be defined in Chapter 8. The key lemma in proving Theorem 5.4 is the following.

Lemma 4.1 (Key lemma)**.** Suppose Assumption 1 holds. For any $l, k \in \mathbb{N}$, the map

$$\mathfrak{q}_{l,k,\beta} : \Omega(X, \Lambda_{0,nov})^{\otimes l} \otimes \Omega(L, \Lambda_{0,nov})^{\otimes k} \to \Omega(L, \Lambda_{0,nov})$$

defined in (4.2.1) maps an element of $\Omega_{bas}(X, \Lambda_{0,nov})^{\otimes l} \otimes \Omega_{bas}(L, \Lambda_{0,nov})^{\otimes k}$ to an element of $\Omega_{bas}(L, \Lambda_{0,nov})$.

Proof of Lemma 4.1. Consider a G-equivariant Kuranishi structure on $\mathcal{M}_{k+1,l}(L, J, \beta)$ and a compatible G-equivariant CF-perturbations which satisfy Assumption 1.

Recall that pullback maps defined via strongly smooth maps of the form $\mathcal{M} \to L$, from a Kuranishi space to a smooth manifold L, are defined to be chart-wise pullback, and the integration along the fiber maps defined via strongly smooth maps of the form $\mathcal{M} \to L$, which are strongly submersive with respect to a CF-perturbation, are defined by taking integration along the fiber maps on suborbifolds that cover the Kuranishi charts and gluing by partitions of unity. Therefore, it suffices to show that the pullback maps of the form $\mathrm{ev}^*_{i,(k+1,l,\beta)}$, $(\mathrm{ev}^j_{(k+1,l,\beta)})^*$, ι^*, preserve G-basicness on the G-equivariant Kuranishi charts, and that the integration along the fiber maps of the form $\mathrm{ev}_{0,(k+1,l,\beta)!}$ preserve G-basicness on G-invariant open subsets of the G-equivariant Kuranishi charts.

Since ι is a G-equivariant map of smooth manifolds, ι^* commutes with $\mathcal{L}_{\underline{\zeta}}$, $\iota_{\underline{\zeta}}$ for all $\zeta \in \mathfrak{g}$, showing that ι^* preserves G-basicness. Similarly, if $\mathcal{M}_{k+1,l}(L, J, \beta)$ satisfies Assumption 1, then pulling back by the equivariant maps of the form $\mathrm{ev}_{i,(k+1,l,\beta)}$, $\mathrm{ev}^j_{(k+1,l,\beta)}$ also preserve G-basicness.

Let $\mathcal{U} = (U, \mathcal{E}, \psi, s)$ be a G-equivariant Kuranishi chart of the moduli space $\mathcal{M}_{k+1,l}(L, J, \beta)$. Let

$$\mathcal{S}_{\mathfrak{r}} = \{\mathcal{S}^{\epsilon}_{\mathfrak{r}} = (W_{\mathfrak{r}} \xrightarrow{\nu_{\mathfrak{r}}} U_{\mathfrak{r}}, \tau_{\mathfrak{r}}, \mathfrak{s}^{\epsilon}_{\mathfrak{r}}) \mid \epsilon \in (0, 1]\}$$

be a CF-perturbation representative on a nonempty G-invariant open subset $U_{\mathfrak{r}} \subset U$. Let $f_{U_{\mathfrak{r}}}$ denote the restriction of $\mathrm{ev}_{0,(k+1,l,\beta)}$ to $U_{\mathfrak{r}}$. By assumption, it is a G-equivariant strongly smooth map which is strongly submersive. By Asssumption 1 and Theorem 2.10, we may assume that the equivariant Thom form $\tau_{\mathfrak{r}}$ is G-basic.

We want to show that $(f_{U_{\mathfrak{r}}})_!$ commutes with $\iota_{\underline{\zeta}}$ and $\mathcal{L}_{\underline{\zeta}}$ for all $\zeta \in \mathfrak{g}$.

Then for all $\rho \in \Omega(L)$ and all $\zeta \in \mathfrak{g}$ we have

$$\begin{aligned}
&\int_L (f_{U_\mathfrak{r}})_!(\iota_{\underline{\zeta}} h; \mathcal{S}_\mathfrak{r}^\epsilon) \wedge \rho \\
=&\int_{(\mathfrak{s}_\mathfrak{r}^\epsilon)^{-1}(0)} \nu_\epsilon^* \iota_{\underline{\zeta}} h \wedge (f_{U_\mathfrak{r}} \circ \nu_\epsilon)^* \rho \wedge \tau_\mathfrak{r} \\
=&\int_{(\mathfrak{s}_\mathfrak{r}^\epsilon)^{-1}(0)} \iota_{\underline{\zeta}} \nu_\epsilon^* h \wedge (f_{U_\mathfrak{r}} \circ \nu_\epsilon)^* \rho \wedge \tau_\mathfrak{r} \\
=&(-1)^{\deg h+1} \int_{(\mathfrak{s}_\mathfrak{r}^\epsilon)^{-1}(0)} \nu_\epsilon^* h \wedge \iota_{\underline{\zeta}} (f_{U_\mathfrak{r}} \circ \nu_\epsilon)^* \rho \wedge \tau_\mathfrak{r} \\
&+ (-1)^{\deg h+\deg\rho+1} \int_{(\mathfrak{s}_\mathfrak{r}^\epsilon)^{-1}(0)} \nu_\epsilon^* h \wedge (f_{U_\mathfrak{r}} \circ \nu_\epsilon)^* \rho \wedge \iota_{\underline{\zeta}} \tau_\mathfrak{r} \\
=&(-1)^{\deg h+1} \int_{(\mathfrak{s}_\mathfrak{r}^\epsilon)^{-1}(0)} \nu_\epsilon^* h \wedge (f_{U_\mathfrak{r}} \circ \nu_\epsilon)^* \iota_{\underline{\zeta}} \rho \wedge \tau_\mathfrak{r} \qquad \text{since } \tau_\mathfrak{r} \text{ is basic} \\
=&(-1)^{\deg h+1} \int_L (f_{U_\mathfrak{r}})_! h \wedge \iota_{\underline{\zeta}} \rho \\
=&\int_L \iota_{\underline{\zeta}} (f_{U_\mathfrak{r}})_!(h; \mathcal{S}_\mathfrak{r}^\epsilon) \wedge \rho.
\end{aligned}$$

Thus,

$$(f_{U_\mathfrak{r}})_!(\iota_{\underline{\zeta}} h; \mathcal{S}_\mathfrak{r}^\epsilon) = \iota_{\underline{\zeta}} (f_{U_\mathfrak{r}})_!(h; \mathcal{S}_\mathfrak{r}^\epsilon).$$

Similarly,

$$\begin{aligned}
\int_L (f_{U_\mathfrak{r}})_!(dh; \mathcal{S}_\mathfrak{r}^\epsilon) \wedge \rho &= \int_{(\mathfrak{s}_\mathfrak{r}^\epsilon)^{-1}(0)} \nu_\epsilon^* dh \wedge (f_{U_\mathfrak{r}} \circ \nu_\epsilon)^* \rho \wedge \tau_\mathfrak{r} \\
&= \int_{(\mathfrak{s}_\mathfrak{r}^\epsilon)^{-1}(0)} d\nu_\epsilon^* h \wedge (f_{U_\mathfrak{r}} \circ \nu_\epsilon)^* \rho \wedge \tau_\mathfrak{r} \\
&= (-1)^{\deg h+1} \int_{(\mathfrak{s}_\mathfrak{r}^\epsilon)^{-1}(0)} \nu_\epsilon^* h \wedge d(f_{U_\mathfrak{r}} \circ \nu_\epsilon)^* \rho \wedge \tau_\mathfrak{r} \\
&= (-1)^{\deg h+1} \int_{(\mathfrak{s}_\mathfrak{r}^\epsilon)^{-1}(0)} \nu_\epsilon^* h \wedge (f_{U_\mathfrak{r}} \circ \nu_\epsilon)^* d\rho \wedge \tau_\mathfrak{r} \\
&= (-1)^{\deg h+1} \int_L (f_{U_\mathfrak{r}})_!(h; \mathcal{S}_\mathfrak{r}^\epsilon) \wedge d\rho \\
&= \int_L d(f_{U_\mathfrak{r}})_!(h; \mathcal{S}_\mathfrak{r}^\epsilon) \wedge \rho.
\end{aligned}$$

Hence,

$$(f_{U_\mathfrak{r}})_!(dh; \mathcal{S}_\mathfrak{r}^\epsilon) = d(f_{U_\mathfrak{r}})_!(h; \mathcal{S}_\mathfrak{r}^\epsilon).$$

Moreover, we have

$$(f_{U_\mathfrak{r}})_!(\mathcal{L}_{\underline{\varsigma}}h;\mathcal{S}^\epsilon_\mathfrak{r}) = (f_{U_\mathfrak{r}})_!(\iota_{\underline{\varsigma}}dh + d\iota_{\underline{\varsigma}}h;\mathcal{S}^\epsilon_\mathfrak{r}) = (\iota_{\underline{\varsigma}}d + d\iota_{\underline{\varsigma}})(f_{U_\mathfrak{r}})_!(h;\mathcal{S}^\epsilon_\mathfrak{r}) = \mathcal{L}_{\underline{\varsigma}}(f_{U_\mathfrak{r}})_!(h;\mathcal{S}^\epsilon_\mathfrak{r})$$

by Cartan's magic formula. Thus, if h is a basic form on $U_\mathfrak{r}$, then $(f_{U_\mathfrak{r}})_!(h;\mathcal{S}^\epsilon_\mathfrak{r})$ is a basic form on L. □

The proof of Theorem 5.4 will be based on an induction on the monoid

$$\Gamma = \left\{ \left(\omega(\beta), \frac{I_\mu(\beta)}{2} \right) \in \mathbb{R}_{\geq 0} \times \mathbb{Z} \;\middle|\; \beta \in \pi_2(X, L), \mathcal{M}(L, J, \beta) \neq \emptyset \right\}. \tag{4.3.3}$$

Consider the lexicographic order on Γ given by the following. Let $(\lambda, n), (\lambda', n') \in \Gamma$.

1) $(\lambda, n) = (\lambda', n')$ if and only if $\lambda = \lambda', n = n'$.

2) $(\lambda, n) < (\lambda', n')$ if one of the following holds.

   a) $\lambda < \lambda'$

   b) $\lambda = \lambda'$, $n < n'$.

We may renumber the elements of Γ as follows.

$$\Gamma = \left\{ (\lambda_i, n_{i,j}) \in \mathbb{R}_{\geq 0} \times \mathbb{Z} \mid i = 0, 1, \ldots, 0 \leq j \leq l_i \right\}$$

so that $\lambda_i < \lambda_{i+1}$ for all $i \geq 0$ and $n_{i,j} < n_{i,j+1}$ for all $1 \leq j \leq l_{i-1}$.

Proof of Theorem 5.4. We want to construct

$$\mathfrak{b}^{(i)} = \sum_{i'=0}^{i} \sum_{j'=1}^{l_{i'}} \mathfrak{b}_{i',j'} T^{\lambda_{i'}} e^{n_{i',j'}}, \qquad b^{(i)} = \sum_{i'=0}^{i} \sum_{j'=1}^{l_{i'}} b_{i',j'} T^{\lambda_{i'}} e^{n_{i',j'}} \tag{4.3.4}$$

such that the $\mathfrak{b}_{i',j'}, b_{i',j'}$ are G-basic forms on X, L, respectively, and the terms of $\mathfrak{m}_0^{\mathfrak{b}^{(i)}, b^{(i)}}(1)$ with valuation less than or equal to λ_i vanish in the sense that

$$\mathfrak{m}_0^{\mathfrak{b}^{(i)}, b^{(i)}}(1) \equiv 0 \mod T^{\lambda_i} \Lambda^+_{0,nov}, \tag{4.3.5}$$

by induction on i.

Let

$$\mathfrak{b}^{(0)} = 0, \quad b^{(0)} = 0.$$

Then

$$\mathfrak{m}_0^{\mathfrak{b}^{(0)},b^{(0)}}(1) \equiv \mathfrak{q}_{l=0,k=0,\beta=0}(1) \equiv 0 \mod \Lambda_{0,nov}^+.$$

Assume that we have constructed

$$\mathfrak{b}^{(i)} = \sum_{i'=0}^{i}\sum_{j'=1}^{l_{i'}} \mathfrak{b}_{i',j'}\, T^{\lambda_{i'}} e^{n_{i',j'}}, \quad b^{(i)} = \sum_{i'=0}^{i}\sum_{j'=1}^{l_{i'}} b_{i',j'} T^{\lambda_{i'}} e^{n_{i',j'}}$$

such that the $\mathfrak{b}_{i',j'}, b_{i',j'}$ are G-basic forms and

$$\mathfrak{m}_0^{\mathfrak{b}^{(i)},b^{(i)}}(1) \equiv 0 \mod T^{\lambda_i}\Lambda_{0,nov}^+. \tag{4.3.6}$$

We want to construct

$$\mathfrak{b}^{(i+1)} = \sum_{i'=0}^{i+1}\sum_{j'=1}^{l_{i'}} \mathfrak{b}_{i',j'}\, T^{\lambda_{i'}} e^{n_{i',j'}}, \qquad b^{(i+1)} = \sum_{i'=0}^{i+1}\sum_{j'=1}^{l_{i'}} b_{i',j'} T^{\lambda_{i'}} e^{n_{i',j'}},$$

such that (4.3.6) holds with i replaced by $i+1$. We note that if $\mathfrak{q}_{l,k,\beta}\, T^{\omega(\beta)} e^{\frac{I_\mu(\beta)}{2}}$

- either takes a tensor product of more than one term with positive valuation, at least one of which takes the form $\mathfrak{b}_{i+1,\bullet}\, T^{\lambda_{i+1}} e^{n_{i+1,\bullet}}$ or $b_{i+1,\bullet} T^{\lambda_{i+1}} e^{n_{i+1,\bullet}}$
- or has $\beta \neq 0$ and takes exactly one element of the form $\mathfrak{b}_{i+1,\bullet}$ and $b_{i+1,\bullet}$,

then since the operators $\mathfrak{q}_{l,k,\beta}$ are filtration-preserving, the resulting term will have valuation strictly higher than λ_{i+1} and thus be $0 \mod T^{\lambda_{i+1}}\Lambda_{0,nov}^+$. Therefore, the contributions of $\mathfrak{b}^{(i+1)} - \mathfrak{b}^{(i)}$ and $b^{(i+1)} - b^{(i)}$ to $\mathfrak{m}_0^{\mathfrak{b}^{(i+1)},b^{(i+1)}}(1) \mod T^{\lambda_{i+1}}\Lambda_{0,nov}^+$ are of the form

$$\mathfrak{q}_{1,0,\beta=0}(\mathfrak{b}_{i+1,\bullet}) T^{\lambda_{i+1}} e^{n_{i+1,\bullet}}, \quad \mathfrak{q}_{0,1,\beta=0}(\mathfrak{b}_{i+1,\bullet}) T^{\lambda_{i+1}} e^{n_{i+1,\bullet}}.$$

The other contributions must come from $\mathfrak{b}^{(i)}$ and $b^{(i)}$. Since we know their contributions to the terms of $\mathfrak{m}_0^{\mathfrak{b}^{(i+1)},b^{(i+1)}}(1)$ with valuation less than or equal to λ_i vanish, their contributions to the terms in $\mathfrak{m}_0^{\mathfrak{b}^{(i+1)},b^{(i+1)}}(1) \mod T^{\lambda_{i+1}}\Lambda_{0,nov}^+$ have to be exactly of valuation $T^{\lambda_{i+1}}$.

Let $o_{i+1,j}$ be the coefficient of $T^{\lambda_{i+1}} e^{n_{i+1,j}}$ in $\mathfrak{m}_0^{\mathfrak{b}^{(i)}, b^{(i)}}(1)$. By the above argument,

$$\begin{aligned}
&\mathfrak{m}_0^{\mathfrak{b}^{(i+1)}, b^{(i+1)}}(1) \\
\equiv &\sum_{j=1}^{l_{i+1}} \Big(o_{i+1,j} + \mathfrak{q}_{1,0,\beta=0}(\mathfrak{b}_{i+1,j}) + \mathfrak{q}_{0,1,\beta=0}(b_{i+1,j}) \Big) T^{\lambda_{i+1}} e^{n_{i+1,j}} \\
\equiv &\sum_{j=1}^{l_{i+1}} \left(o_{i+1,j} \pm \iota^*(\mathfrak{b}_{i+1,j}) + d(b_{i+1,j}) \right) T^{\lambda_{i+1}} e^{n_{i+1,j}} \mod T^{\lambda_{i+1}} \Lambda_{0,nov}^+.
\end{aligned}$$

Therefore, we only need to find $\mathfrak{b}_{i+1,j}, b_{i+1,j}$ such that

$$o_{i+1,j} \pm \iota^*(\mathfrak{b}_{i+1,j}) + d(b_{i+1,j}) = 0. \tag{4.3.7}$$

By Lemma 4.1, we have

$$o_{i+1,j} \in \Omega^\bullet_{bas}(L, \Lambda_{0,nov}).$$

We consider the maps

$$\Omega^\bullet_{bas}(L, \Lambda_{0,nov}) \xrightarrow[1)]{(\pi^*_{L/G})^{-1}} \Omega^\bullet(L/G, \Lambda_{0,nov}) \xrightarrow[2)]{\Delta^*_{Y/\!/G}} \Omega^\bullet(Y /\!/ G, \Lambda_{0,nov}) \xrightarrow[3)]{\pi^*_{Y/\!/G}} \Omega^\bullet(Y^- \times Y /\!/ G, \Lambda_{0,nov})$$

given by the following.

1) Since G acts on L freely, $L \to L/G$ is a principal G-bundle. Thus, by Lemma 2.2, there exists an isomorphism

$$\pi^*_{L/G} : \Omega^\bullet(L/G, \Lambda_{0,nov}) \xrightarrow{\cong} \Omega^\bullet_{bas}(L, \Lambda_{0,nov}).$$

2) $\Delta^*_{Y/\!/G}$ is the pullback map induced by the diagonal map

$$\Delta_{Y/\!/G} : Y /\!/ G \to Y /\!/ G \times Y /\!/ G = L/G, \qquad [p] \mapsto ([p],[p]).$$

3) $\pi^*_{Y/\!/G}$ is the pullback map induced by the projection map $\pi_{Y/\!/G} : Y^- \times Y /\!/ G \to Y /\!/ G$.

Let

$$\mathfrak{b}_{i+1,j} = \mp \pi^*_{Y/\!/G} \circ \Delta^*_{Y/\!/G} \circ (\pi^*_{L/G})^{-1}(o_{i+1,j}), \qquad b_{i+1,j} = 0,$$

where we use $-$ if the sign in (4.3.7) is $+$, and we use $+$ if the sign in (4.3.7) is $-$. It satisfies

$$\begin{aligned} o_{i+1,j} \pm \iota^* \mathfrak{b}_{i+1,j} + d(b_{i+1,j}) &= o_{i+1,j} \pm \Big(\mp \iota^* \circ \pi^*_{Y/\!/G} \circ \Delta^*_{Y/\!/G} \circ (\pi^*_{L/G})^{-1}(o_{i+1,j}) \Big) + 0 \\ &= o_{i+1,j} \pm \Big(\mp (\Delta_{Y/\!/G} \circ \pi_{Y/\!/G} \circ \iota)^* \circ (\pi^*_{L/G})^{-1}(o_{i+1,j}) \Big) \\ &= o_{i+1,j} \pm \Big(\mp \pi^*_{L/G} \circ (\pi^*_{L/G})^{-1}(o_{i+1,j}) \Big) = 0. \end{aligned}$$

Let $l \in \mathbb{N}$ and $\alpha \in \Omega^l(Y /\!/ G)$. Let us denote $f := \pi_{Y/\!/G}$. Then for all $\zeta \in \mathfrak{g}$, $p \in Y^- \times Y /\!/ G$, $v_2 \dots, v_l \in T_{f(p)}(Y^- \times Y /\!/ G)$, we have

$$(\iota_{\underline{\zeta}}(\pi^*_{Y/\!/G}\alpha))_p(v_2, \dots, v_l) = \alpha_{f(p)}(df \circ \underline{\zeta}(x), df_p(v_2), \dots, df_p(v_l)) = 0,$$

since $\underline{\zeta}(p_1, p_2) = (\underline{\zeta}(p_1), 0)$ for all $(p_1, p_2) \in Y \times Y /\!/ G$. Then

$$\mathcal{L}_{\underline{\zeta}}(\pi^*_{Y/\!/G}\alpha) = d(\iota_{\underline{\zeta}}\pi^*_{Y/\!/G}\alpha) + \iota_{\underline{\zeta}}(d\pi^*_{Y/\!/G}\alpha) = \iota_{\underline{\zeta}}(\pi^*_{Y/\!/G}d\alpha) = 0.$$

This shows that $\mathfrak{b}_{i+1,j}$ is a basic form and completes the induction. By construction, if we let

$$\mathfrak{b} = \lim_{i\to\infty} \mathfrak{b}^{(i)}, \quad b = 0,$$

then

$$\mathfrak{m}_0^{\mathfrak{b},b}(1) = 0.$$

□

Chapter 5

The equivariant A_∞ algebra associated with a Lagrangian torus fiber

In this chapter, we define an equivariant Lagrangian Floer theory on compact symplectic toric manifolds for the Lagrangian torus fibers with respect to the subtorus actions. It is compatible with the ordinary Lagrangian Floer theory discussed in [23], [22], and [21]. After preparing the notations and the setup in Section 5.1, we define an equivariant A_∞ algebra associated to a Lagrangian torus fiber in Section 5.3. Using the spectral sequences defined in Section 5.4, we prove that the set of Lagrangian torus fibers (with weak bounding cochain data) that have nontrivial equivariant Lagrangian Floer cohomology can be identified with a subset of the algebraic torus over the Novikov field, with certain valuation restrictions. We will see in Chapter 6 that the latter is a rigid analytic space.

5.1 The setup

We will assume the following setup and notations for the rest of the paper unless otherwise stated.

Let (X, ω, T^n, μ) be a compact symplectic toric manifold. More specifically, X is a (real) $2n$-dimensional manifold with symplectic form ω such that there is an effective Hamiltonian torus action on (X, ω) by T^n and $\mu : X \to \mathfrak{t}^*$ is an associated moment map, where $\mathfrak{t}$ is the Lie algebra of T^n and $\mathfrak{t}^*$ is the dual of $\mathfrak{t}$. Let $\Delta = \mu(X)$ be the moment polytope for the T^n-action. For each $\boldsymbol{u} \in \operatorname{int}\Delta$, $L(\boldsymbol{u}) := \mu^{-1}(\boldsymbol{u})$ is a T^n-invariant Lagrangian torus with a T^n-invariant relatively spin structure induced by the T^n-action. Let J be a T^n-invariant almost complex structure on X compatible with ω.

Let $G = KT^r \subset T^n$ be a compact r-dimensional connected subtorus of T^n given by a $(n \times r)$-matrix K with integer coefficients of rank r. The G-action is induced from the T^n-action on X.

We identify

$$\mathfrak{t}^* \cong M_{\mathbb{R}} \cong (\mathbb{R}^n)^*,$$

where $M_{\mathbb{R}}$ is the dual vector space of an n-dimensional R-vector space $N_{\mathbb{R}}$. Let $N \cong \mathbb{Z}^n$ be the full-rank lattice in $N_{\mathbb{R}}$ and $M := \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be the dual lattice of N such that $M_{\mathbb{R}} \cong M \otimes_{\mathbb{Z}} \mathbb{R}$, $N_{\mathbb{R}} \cong N \otimes_{\mathbb{Z}} \mathbb{R}$, and $T^n \cong N_{\mathbb{R}}/N$. For every $\boldsymbol{u} \in \operatorname{int}\Delta$, $\mu^{-1}(\boldsymbol{u})$ is diffeomorphic to T^n. We identify

$$\mathfrak{t} \cong N_{\mathbb{R}} \cong \mathbb{R}^n \quad \text{and} \quad H_1(L(\boldsymbol{u}), \mathbb{Z}) \cong N \cong \mathbb{Z}^n, \quad H^1(L(\boldsymbol{u}), \mathbb{Z}) \cong M \cong \mathbb{Z}^n\,.$$

Fix an integral basis $\{e_1^*, \ldots, e_n^*\}$ for the lattice N. Let $\{e_1, \ldots, e_n\}$ be the dual basis of M. Let the $a_{i,j} \in \mathbb{Z}$ be such that

$$H^1(L(\boldsymbol{u}), \mathbb{Z}) = \operatorname{span}_{\mathbb{Z}} \left\{ \alpha_i = \sum_{j=1}^{n} a_{i,j} e_j \;\middle|\; 1 \le i \le n \right\},$$

where $H^1_G(L(\boldsymbol{u}), \mathbb{Z}) = \operatorname{span}_{\mathbb{Z}}\{\alpha_{r+1}, \ldots, \alpha_n\}$.

We consider the following coefficient rings. Define the **universal Novikov field** by

$$\Lambda_{\text{nov}} = \left\{ \sum_{i \in \mathbb{N}} a_i T^{\lambda_i} e^{n_i} \;\middle|\; \lambda_i \in \mathbb{R}, a_i \in \mathbb{C}, \text{ and } n_i \in \mathbb{Z}, \forall i \in \mathbb{N}, \quad \lim_{i \to \infty} \lambda_i = \infty \right\}. \tag{5.1.1}$$

A non-Archimedean **valuation** function $\operatorname{val} : \Lambda_{\text{nov}} \to \mathbb{R} \cup \{\infty\}$ on Λ_{nov} is defined as follows.

$$y = \sum_{i \in \mathbb{N}} a_i T^{\lambda_i} e^{n_i} \mapsto \operatorname{val}(y) := \begin{cases} \min\{\lambda_i \mid i \in \mathbb{N}, a_i \neq 0\} & \text{if } y \neq 0 \\ \infty & \text{if } y = 0 \end{cases} \tag{5.1.2}$$

The **universal Novikov ring** is given by

$$\Lambda_{0,\mathrm{nov}} = \{y \in \Lambda_{\mathrm{nov}} \mid \mathrm{val}(y) \geq 0\}. \tag{5.1.3}$$

We also consider the energy-zero parts of Λ_{nov} and $\Lambda_{0,\mathrm{nov}}$. Define

$$\Lambda = \left\{ \sum_{i\in\mathbb{N}} a_i T^{\lambda_i} \mid \lambda_i \in \mathbb{R} \text{ and } a_i \in \mathbb{C} \quad \forall i \in \mathbb{N}, \quad \lim_{i\to\infty} \lambda_i = \infty \right\} \tag{5.1.4}$$

to be the **Novikov field** and

$$\Lambda_0 = \left\{ \sum_{i\in\mathbb{N}} a_i T^{\lambda_i} \in \Lambda \;\middle|\; \lambda_i \geq 0 \quad \forall i \in \mathbb{N} \right\} \tag{5.1.5}$$

to be the Novikov ring. The rings Λ, Λ_0 carries a valuation function

$$y = \sum_{i\in\mathbb{N}} a_i T^{\lambda_i} \mapsto \mathrm{val}(y) := \begin{cases} \min\{\lambda_i \mid i \in \mathbb{N}, a_i \neq 0\} & \text{if } y \neq 0 \\ \infty & \text{if } y = 0 \end{cases}. \tag{5.1.6}$$

The valuations (5.1.2) and (5.1.6) induce a non-Archimedean norm $y \mapsto |y|_R := \exp(-\mathrm{val}(y))$ on R, where $R \in \{\Lambda_{\mathrm{nov}}, \Lambda_{0,\mathrm{nov}}, \Lambda, \Lambda_0\}$ and exp is the exponential map with Euler's number as the base. Moreover, we have an exponential map $\exp : \Lambda_0 \to \Lambda_0$ defined as follows. Every $b \in \Lambda_0$ can be decomposed as $b = b_0 + b_+$, where $b_0 \in \mathbb{C}$ and b_+ satisfies $\mathrm{val}(b_+) > 0$. We define $\exp(b) = e^{b_0} \sum_{n\in\mathbb{N}} \frac{b_+^n}{n!}$, where e^{b_0} is the usual exponential of complex numbers.

5.2 The moduli space of pseudoholomorphic discs

Since we will mostly use moduli space of pseudoholomorphic discs with no interior marked point in the rest of the paper, we simplify the notations as follows.

Definition 5.1 (Moduli space of pseudoholomorphic discs with boundary marked points)**.**

For any $\beta \in \pi_2(X, L)$ and $k \in \mathbb{N}$, let $\mathcal{M}_{k+1}(L, J, \beta)$ be the set

$$\left\{ (\Sigma, j, \vec{z}, u) \left| \begin{array}{l} \Sigma \text{ is a genus 0 nodal Riemann surface with} \\ \text{connected boundary and complex structure} j; \\ u : (\Sigma, \partial\Sigma) \to (X, L) \text{ is smooth;} \\ du \circ j = J \circ du; \quad [u] = \beta \in \pi_2(X, L); \\ (\Sigma, j, \vec{z}, u) \text{ is stable;} \quad E(u) < \infty \\ \vec{z} = (z_0, z_1, \ldots, z_k) \in (\partial\Sigma)^{k+1}, \\ \text{where the } z_i \text{ are distinct non-nodal} \\ \text{boundary marked points and the enumeration} \\ \text{is in counterclockwise order along } \partial\Sigma \end{array} \right. \right\} \Big/ \sim, \tag{5.2.1}$$

where $(\Sigma, j, \vec{z}, u) \sim (\Sigma', j', \vec{z}', u')$ if and only if there exists a biholomorphism $\varphi : (\Sigma, j) \to (\Sigma', j')$ such that $u' \circ \varphi = u$ and $\varphi(z_i) = z_i'$ for all $0 \leq i \leq k$. A biholomorphism φ satisfying these conditions is called an **isomorphism** between $(\Sigma, j, \vec{z}, u)$ and $(\Sigma', j', \vec{z}', u')$.

For any element $x = [\Sigma, j, \vec{z}, u] \in \mathcal{M}_{k+1}(L, J, \beta)$, we define its **automorphism group** by

$$\operatorname{Aut} x = \left\{ \varphi : (\Sigma, j) \to (\Sigma, j) \left| \begin{array}{l} \varphi \text{ is a biholomorphism} \\ u \circ \varphi = u \\ \varphi(z_i) = z_i \quad \forall 0 \leq i \leq k \end{array} \right. \right\}.$$

Remark 5.1. By an abuse of notation, we also use $\vec{z}$ to denote the ordered subset $\{z_0, \ldots, z_k\}$ of $\partial\Sigma$. And, whenever I is another set, the elements in $\vec{z} \cap I$ are ordered by the original enumeration in $\vec{z}$.

Denote the evaluation map on $\mathcal{M}_{k+1}(\beta)$ at the i-th marked point by $\mathrm{ev}_{i,\beta}$. We recall a result in [21] below. The definitions related to G-equivariant Kuranishi structures are introduced in Chapter 8.

Proposition 5.1 (G-equivariant Kuranishi structure on $\mathcal{M}_{k+1}(L, J, \beta)$)**.** Let L be a Lagrangian torus fiber of the toric moment map μ over an interior point of the moment polytope

Δ. For $k \geq 1, \beta \in \pi_2(X, L)$ with $\beta \neq 0$, the moduli space $\mathcal{M}_{k+1}(L, J, \beta)$ has a G-equivariant Kuranishi structure with corners:

$$\widehat{\mathcal{U}} = \begin{pmatrix} \{\mathcal{U}_p = (U_p, \mathcal{E}_p, \psi_p, s_p) \,|\, p = [\Sigma, j, \vec{z}, u] \in \mathcal{M}_{k+1}(L, J, \beta)\}\,, \\ \{\vec{\alpha}_{pq} \,|\, p \in \mathcal{M}_{k+1}(L, J, \beta), q \in \operatorname{im}\psi_p\}\,. \end{pmatrix} \tag{5.2.2}$$

The normalized boundary of $\mathcal{M}_{k+1}(L, J, \beta)$ is a union of the fiber products

$$\partial\, \mathcal{M}_{k+1}(L, J, \beta) = \bigcup_{\substack{k_1, k_2 \geq 0 \\ k_1 + k_2 = k+1}} \; \bigcup_{\substack{\beta_1, \beta_2 \in \pi_2(X,L) \\ \beta_1 + \beta_2 = \beta}} \; \bigcup_{j=1}^{k_2} \mathcal{M}_{k_1+1}(L, J, \beta_1)_{\mathrm{ev}_{0,\beta}} \times_{\mathrm{ev}_{j,\beta}} \mathcal{M}_{k_2+1}(L, J, \beta_2),$$

and the systems of G-equivariant Kuranishi structures are compatible with the fiber product description. Moreover, $\mathrm{ev}_{i,\beta} : \mathcal{M}_{k+1}(L, J, \beta) \to L$ is a T^n-equivariant strongly continuous weakly submersive map.

Proof sketch. A T^n-equivariant Kuranishi structure on the moduli space $\mathcal{M}_{k+1}(L, J, \beta)$ satisfying the properties is constructed in [21] Section 4.3.

We sketch the proof. Let $x = [\Sigma, j, \vec{z}, u] \in \mathcal{M}_{k+1}(L, J, \beta)$. We want to construct a G-equivariant Kuranishi chart $(U_x, \mathcal{E}_x, \psi_x, s_x)$ at x, where $U_x = V_x/\Gamma_x$ for some manifold V_x and some finite group Γ_x. Let $\Gamma_x = \operatorname{Aut} x$. Let $G(x) = \{(g, \gamma) \in T^n \times \Gamma_x \mid gu = u \circ \gamma\}$. We say z is a special point if z is a nodal singularity or boundary marked point of Σ.

Let $\Sigma = \bigcup_{\alpha \in A} \Sigma_\alpha$, where each extended disc component Σ_α consists of an irreducible disc component and all the spheres rooted on it. Let $\vec{z}_\alpha = \vec{z} \cap \Sigma_\alpha$ be the set of marked points on Σ_α. For each $\alpha \in A$, choose a non-empty $G(x)$-invariant open subset $K_\alpha \subset \Sigma \setminus \partial\Sigma$ with compact closure $\overline{K}_\alpha$ which does not intersect $\partial\Sigma$ or nodal singularities and let

$$D_{u,\alpha} : W^{1,p}\left(\Sigma_\alpha, \partial\Sigma_\alpha; u^*TX, u^*TL\right) =: W_\alpha \to L^p\left(\Sigma_\alpha; u^*TX \otimes \Lambda^{0,1}\Sigma_\alpha\right),$$

$$D_u : W^{1,p}\left(\Sigma, \partial\Sigma; u^*TX, u^*TL\right) \to L^p\left(\Sigma; u^*TX \otimes \Lambda^{0,1}\Sigma\right)$$

be the linearization of $\bar{\partial}$. Choose, for each α, a finite-dimensional vector subspace $E_\alpha \subset C_c^\infty(K_\alpha, u^*TX)$ consisting of compactly supported elements such that the following holds.

i) $D_u\left(\{\xi \in W_\alpha \,|\, \xi(z) = 0 \;\forall \text{ special point } z\}\right) + E_\alpha = L^p\left(\Sigma_\alpha; u^*TX \otimes \Lambda^{0,1}\Sigma_\alpha\right).$

ii) $\bigoplus_{\alpha \in A} E_\alpha$ is $G(x)$-invariant.

iii) For each $\forall z_0 \in \partial\Sigma$, the map $\mathrm{Ev}_{z_0} : D_u^{-1}(\bigoplus_{\alpha\in A} E_\alpha) \to T_{u(z_0)}L$ given by $v \mapsto v(z_0)$ is surjective.

iv) If $\gamma \in \Gamma_x$ and $\Sigma_{\alpha'} = \gamma\Sigma_\alpha$, then $\gamma_* E_\alpha = E_{\alpha'}$.

Let $E_x(x) = \bigoplus_{\alpha\in A} E_\alpha$. If $(g,\gamma) \in G(x)$, let $E_x((g,\gamma)\cdot x) = g_* E_x(x)$.

For each α, choose l_α many appropriate extra interior marked points $\vec{w}_\alpha^+$ on Σ_α away from nodes to stabilize the domain of x. Let $\vec{w}^+$ be the ordered set of all such extra marked points on Σ. Let $v(\vec{w}^+) = (\Sigma, j, \vec{z}, \vec{w}^+)$.

Suppose $(v' = (\Sigma(v'), j', \vec{z}', \vec{w}^{+\prime}), u')$ is a smooth curve with $k+1$ boundary marked points and l interior marked points such that the domain v' is close to v in $\mathcal{M}_{k+1,l}$ and (v', u') is close to $(hx, \vec{w}^+)$ for some $h \in T^n$. For each v', there is an embedding $i_{v'} : \Sigma \setminus S \to \Sigma(v')$, where S is a neighborhood of the set of marked points and singularities.

We decompose $\Sigma(v') = \bigcup_{r\in R} \Sigma'_r$ into extended disc components as well. Let $r \in R$. Let $A(r) = \{\alpha \in A \mid i_{v'}(\Sigma_\alpha \setminus S) \subset \Sigma'_r\}$. For each $\alpha \in A(r)$, we obtain a map $P_{r,\alpha} : E_\alpha \to h_* E_\alpha \to C^\infty(\Sigma'_r, (u')^*TX \otimes \Lambda^{0,1}\Sigma'_r)$ using the convexity of the square of the distance function (and an exponential decay estimate) which allows us to (choose a "closest" $h \in T^n$ and) define a suitable parallel transport map. Then we define $E_x(v', u') = \bigoplus_{r\in R} \bigoplus_{\alpha\in A(r)} \operatorname{im} P_{r,\alpha}$. Let

$$V_x = \{(v', u') \text{ close to } (T^n \cdot x, \vec{w}) \mid \bar{\partial}\, u' \equiv 0 \text{ mod } E_x(v', u')\}.$$

Let $s_x : (v', u') \mapsto \bar{\partial}\, u'$ and $\mathcal{E}_x \to V_x$ be the orbibundle whose fiber is $E_x(v', u')$ at $[(v', u')]$ and $E_x(x)$ at x. Thus, an equivariant Kuranishi chart at x is defined.

For $(k_1, \beta_1), (k_2, \beta_2) \in \mathbb{N} \times \pi_2(X, L)$, we say $(k_1, \beta_1) < (k_2, \beta_2)$ if either $\omega(\beta_1) < \omega(\beta_2)$ or $\omega(\beta_1) = \omega(\beta_2)$ and $k_1 < k_2$. For coordinate changes to be defined, we need to modify the obstruction bundles $\mathcal{E}_x$ inductively on (k', β'). Suppose for all $(k', \beta') < (k, \beta)$ we have a Kuranishi structure and, in particular, coordinate changes are defined on $\mathcal{M}_{k'+1}(L, J, \beta')$. More specifically, we have a finite cover $\{U(\mathfrak{c}) \mid [\mathfrak{c}] \in P_{k'}(\beta')\}$ of $\mathcal{M}_{k'+1}(L, J, \beta')$, where

$P(\beta') \subset \mathcal{M}_{k'+1}(L, J, \beta')/T^n$ and $K(\mathfrak{c}) \subset \mathcal{M}_{k'+1}(L, J, \beta')$ is a T^n-invariant closed subset of the Kuranishi neighborhood at $\mathfrak{c}$. The fiber of the obstruction bundle of a point in $\mathcal{M}_{k'+1}(L, J, \beta')$ is given by a direct sum of (perturbations of) fibers of (some of) the $\mathcal{E}_{\mathfrak{c}}$'s so that the coordinate changes on $\mathcal{M}_{k'+1}(L, J, \beta')$ is defined. Then we define the Kuranishi structure on $\mathcal{M}_{k+1}(L, J, \beta)$ by a downward induction on the number of disc components. For each $d > 1$, let $S_d \mathcal{M}_{k+1}(L, J, \beta)$ be the set of elements with at least d disc components.

Suppose on $S_{d+1} \mathcal{M}_{k+1}(L, J, \beta)$ we have a finite cover $\{K_{\mathfrak{p}} \mid \mathfrak{p} \in \mathfrak{P}\}$ such that each $K_{\mathfrak{p}}$ is a T^n-invariant compact subset of the Kuranishi neighborhood $\psi_{\mathfrak{p}}\left(s_{\mathfrak{p}}^{-1}(0)\right)$ and that

$$S_{d+1} \mathcal{M}_{k+1}(L, J, \beta) \subset \bigcup_{\mathfrak{p} \in \mathfrak{P}} \operatorname{int} K_{\mathfrak{p}}.$$

As before, we find a cover of

$$\mathcal{K}_d \mathcal{M}_{k+1}(L, J, \beta) = S_d \mathcal{M}_{k+1}(L, J, \beta) \setminus \bigcup_{\mathfrak{p} \in \mathfrak{P}} \operatorname{int} K_{\mathfrak{p}}$$

by T^n-invariant compact subsets $K_{x_1}, \ldots, K_{x_m}$ of Kuranishi neighborhoods of finitely many points $x_1, \ldots, x_m \in S_d \mathcal{M}_{k+1}(L, J, \beta) \setminus S_{d+1} \mathcal{M}_{k+1}(L, J, \beta)$ and define the fibers of the obstruction bundles by appropriate direct sums of the fibers of the obstruction bundles at the x_i's.

Then we glue the Kuranishi structures on $\mathcal{K}_d \mathcal{M}_{k+1}(L, J, \beta)$ and $\bigcup_{\mathfrak{p} \in \mathfrak{P}} \operatorname{int} K_{\mathfrak{p}}$. In particular, when a point $x \in \left(\bigcup_{\mathfrak{p} \in \mathfrak{P}} \operatorname{int} K_{\mathfrak{p}}\right) \cap \left(\bigcup_i \operatorname{int} K_{x_i}\right)$, we define the fibers of $\mathcal{E}_x$ by taking direct sums of (perturbations of) the relevant fibers of the obstruction bundles from the two types of Kuranishi structures. The induction construction then allows us to obtain coordinate changes. □

5.3 The equivariant A_∞ algebra associated to a Lagrangian submanifold

In this section, we define an equivariant A_∞ algebra associated to a Lagrangian torus fiber L of the toric moment map μ over an interior point of the moment polytope Δ.

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^*$ be its dual. Let $S(\mathfrak{g}^*)$ be the symmetric algebra on $\mathfrak{g}^*$. Let $\Omega(L)$ be the de Rham complex of L and $\Omega_G(L, \mathbb{R}) = (\Omega(L) \otimes S(\mathfrak{g}^*))^G$ be the Cartan model of L. Let d_G be the Cartan differential. Recall the definition of the universal Novikov ring in (5.1.3). Define

$$C_G(L, \Lambda_{0,\text{nov}}) := \Omega_G(L, \mathbb{R}) \widehat{\otimes}_{\mathbb{R}} \Lambda_{0,\text{nov}} . \tag{5.3.1}$$

Let $C = C_G(L, \Lambda_{0,\text{nov}})$. It is a graded $\Lambda_{0,\text{nov}}$-module: $C = \widehat{\bigoplus_{p \in \mathbb{N}}} C^p$, where

$$C^p = \widehat{\bigoplus_{i+2j+2n=p}} \left(\Omega^i(L) \otimes S^j(\mathfrak{g}^*)\right)^G \widehat{\otimes}_{\mathbb{R}} (\Lambda_0 \cdot e^n). \tag{5.3.2}$$

Define a degree on C such that $\deg h = \min \left\{ p \in \mathbb{N} \,\middle|\, h \in \bigoplus_{m=0}^{p} C^m \right\}$. Denote by $C[1]$ the module determined by $C[1]^p = C^{p+1}$. Let $B_0 C[1] = \Lambda_{0,nov}$ and

$$B_k(C[1]) = C[1] \otimes \cdots \otimes C[1] \qquad \forall k > 0.$$

Let $BC[1] = \widehat{\bigoplus_{k \in \mathbb{N}}} B_k C[1]$. For any $\beta \in \pi_2(X, L)$, we define $\mathfrak{m}^G_{k,\beta} : B_k C[1] \to C[1]$, the contribution of the moduli space $\mathcal{M}_{k+1}(L, J, \beta)$ by using the evaluation maps $\text{ev}_{j,\beta} : \mathcal{M}_{k+1}(L, J, \beta) \to L$ as follows. We denote by $(\text{ev}^G_{j,\beta})^*$ the G-equivariant pullback by the evaluation map at the k-th boundary marked point and $(\text{ev}^G_{0,\beta})_!$ the G-equivariant integration along the fiber by the evaluation map at the 0-th marked point, which we discuss in Section 8.29.

Let $x_1, \ldots, x_k \in \Omega(L, \Lambda_{0,nov})$. For $\beta = 0$,

$$\begin{cases} \mathfrak{m}^G_{0,\beta=0}(1) = 0 \\ \mathfrak{m}^G_{1,\beta=0}(x_1) = d_G x_1, \text{ where } d_G \text{ is the Cartan differential,} \\ \mathfrak{m}^G_{2,\beta=0}(x_1 \otimes x_2) = (-1)^{\deg x_1} x_1 \wedge x_2 \\ \mathfrak{m}^G_{k,\beta=0} = 0 \qquad \forall k \geq 3. \end{cases} \tag{5.3.3}$$

For $\beta \neq 0$, define

$$\mathfrak{m}^G_{0,\beta}(1) = (\text{ev}^G_{0,\beta})_!(1) \tag{5.3.4}$$

and

$$\mathfrak{m}^G_{k,\beta}(x_1 \otimes \cdots \otimes x_k) = (\text{ev}^G_{0,\beta})_! \left((\text{ev}^G_{1,\beta})^* x_1 \wedge \cdots \wedge (\text{ev}^G_{k,\beta})^* x_k\right) \quad \forall k \geq 1. \tag{5.3.5}$$

Then for $k \in \mathbb{N}$ we define $\mathfrak{m}_k^G : B_k(C[1]) \to C[1]$ by, $\forall x_1 \otimes \cdots x_k \in B_k C[1]$,

$$\mathfrak{m}_k^G(x_1 \otimes \cdots \otimes x_k) = \sum_{\beta \in \pi_2(X,L)} \mathfrak{m}_{k,\beta}^G(x_1 \otimes \cdots \otimes x_k) T^{\frac{\omega(\beta)}{2\pi}} e^{\frac{I_\mu(\beta)}{2}}. \tag{5.3.6}$$

Let L be a Lagrangian torus fiber of the compact symplectic toric manifold X and let

$$S = \{(\omega(\beta), I_\mu(\beta)) \mid \beta \in \pi_2(X, L)\}. \tag{5.3.7}$$

Definition 5.2 (S-gapped curved filtered G-equivariant A_∞ algebra)**.** An **S-gapped curved**[1] **filtered G-equivariant** A_∞ **algebra** is a tuple $(C, \{\mathfrak{m}_k^G\}_{k \in \mathbb{N}}, S, G)$ such that the following holds.

- $(C, \{\mathfrak{m}_k^G\}_{k \in \mathbb{N}}, S)$ is an S-gapped curved filtered A_∞ algebra as defined in Definition 4.4.

- G is a compact connected Lie group.

- C is an $H^*(BG)$-algebra and $BC[1]$ is an $H^*(BG)$-coalgebra and, $\forall k \in \mathbb{N}$, the operator $\mathfrak{m}_k^G$ is an $H^*(BG)$-algebra homomorphism.

Proposition 5.2. Let $(C_G(L, \Lambda_{0,\mathrm{nov}}), \{\mathfrak{m}_k^G\}_{k \in \mathbb{N}}, S, G)$ be the data defined in (5.3.1), (5.3.3), (5.3.4), (5.3.5), and (5.3.7). It is an S-gapped curved filtered G-equivariant A_∞ algebra.

Proof. For any $k \in \mathbb{N}$, $(\lambda, n) \in S$, let

$$\mathfrak{m}_{k,(\lambda,n)}^G = \sum_{\substack{\beta \in \pi_2(X,L) \\ \omega(\beta)=\lambda, I_\mu(\beta))=n}} \mathfrak{m}_{k,\beta}^G \,.$$

Then S-gappedness follows. For each $k \in \mathbb{N}, \beta \in \pi_2(X, L)$, we have

$$\sum_{\substack{k_1,k_2 \geq 0 \\ k_1+k_2=k+1}} \sum_{\substack{\beta_1,\beta_2 \in \pi_2(X,L) \\ \beta_1+\beta_2=\beta}} \sum_{j=1}^{k_1} (-1)^* \, \mathfrak{m}_{k_1,\beta_1}^G(x_1 \otimes \cdots x_{j-1} \otimes \mathfrak{m}_{k_2,\beta_2}^G(x_j \otimes \cdots x_{j+k_2-1}) \otimes \cdots \otimes x_k)$$

$$= \underbrace{\mathfrak{m}_{1,0}^G \, \mathfrak{m}_{k,\beta}^G(x_1 \otimes \cdots \otimes x_k)}_{\text{(I)}} + \underbrace{\sum_{i=1}^{k} (-1)^* \, \mathfrak{m}_{k,\beta}^G(x_1 \otimes \cdots \otimes \mathfrak{m}_{1,0}^G(x_i) \otimes \cdots \otimes x_k)}_{\text{(II)}}$$

[1] The word "curved" means an $\mathfrak{m}_0 : \Lambda_{0,nov} \to C[1]$ is included, in contrast to the classical A_∞ algebra.

$$+\underbrace{\sum_{\substack{k_1+k_2=k+1\\ \beta_1+\beta_2=\beta\\ (k_1,\beta_1)\neq(1,0)\\ (k_2,\beta_2)\neq(1,0)}}\sum_{j=1}^{k_1}(-1)^{\dagger}\,\mathfrak{m}^G_{k_1,\beta_1}(x_1\otimes\cdots x_{j-1}\otimes\mathfrak{m}^G_{k_2,\beta_2}(x_j\otimes\cdots x_{j+k_2-1})\otimes\cdots\otimes x_k),}_{\text{(III)}}$$

where $*=\sum_{l=1}^{i-1}(\deg x_l+1)$ and $\dagger=\sum_{l=1}^{j-1}(\deg x_l+1)$. To show the A_∞ relation, it suffices to show the sum is zero for all k,β. By Proposition 5.1 and Proposition 8.4, (III) corresponds to $\mathrm{Corr}^{G,\epsilon}_{\partial\mathcal{M}_{k+1}(L,J,\beta)}(x_1\otimes\cdots\otimes x_k)$. Moreover, (I) corresponds to $d^G\circ\mathrm{Corr}^{G,\epsilon}_{\mathcal{M}_{k+1}(L,J,\beta)}(x_1\otimes\cdots\otimes x_k)$ and (II) corresponds to $\mathrm{Corr}^{G,\epsilon}_{\mathcal{M}_{k+1}(L,J,\beta)}\circ d_G(x_1\otimes\cdots\otimes x_k)$. Thus, by Stokes' Theorem 8.2 (or Proposition 8.3), the sum is zero. Hence, by construction, $(C_G(L,\Lambda_{0,\mathrm{nov}}),\{\mathfrak{m}^G_k\}_{k\in\mathbb{N}},S,G)$ is an S-gapped curved G-equivariant filtered A_∞ algebra. □

Definition 5.3 (($(\mathfrak{m}^G_{k,\beta})^b$). Let $k\in\mathbb{N}$ and $\beta\in\pi_2(X,L)$. For any $b\in H^1(L,\Lambda_0)$, define $(\mathfrak{m}^G_{k,\beta})^b : B_k(C[1])\to C[1]$ by

$$(\mathfrak{m}^G_{k,\beta})^b(x_1\otimes\cdots\otimes x_k)=\exp(\partial\beta\cap b)\,\mathfrak{m}^G_{k,\beta}(x_1\otimes\cdots\otimes x_k). \tag{5.3.8}$$

And we define $(\mathfrak{m}^G_k)^b : B_k(C_G(L)[1])\to C_G(L)[1]$ by

$$(\mathfrak{m}^G_k)^b=\sum_{\beta\in\pi_2(X,L)}(\mathfrak{m}^G_{k,\beta})^bT^{\frac{\omega(\beta)}{2\pi}}e^{\frac{I\mu(\beta)}{2}}. \tag{5.3.9}$$

Proposition 5.3. Let $L=\mu^{-1}(u)$ for some $u\in\operatorname{int}\Delta$. Let $b\in H^1(L,\Lambda_0)$. Then $(C_G(L,\Lambda_{0,\mathrm{nov}}),\{(\mathfrak{m}^G_k)^b\}_{k\in\mathbb{N}})$ is a G-equivariant A_∞ algebra.

Proof.

$$\begin{aligned}&\sum_{\substack{k_1,k_2\geq 0\\ k_1+k_2=k+1\\ \beta_1,\beta_2\in\pi_2(X,L)\\ \beta_1+\beta_2=\beta}}\sum_{j=1}^{k_1}(-1)^*(\mathfrak{m}^G_{k_1,\beta_1})^b(x_1\otimes\cdots x_{j-1}\otimes(\mathfrak{m}^G_{k_2,\beta_2})^b(x_j\otimes\cdots\otimes x_{j+k_2-1})\otimes\cdots\otimes x_k)\\ =&e^{\partial\beta\cap b}\sum_{\substack{k_1,k_2\geq 0\\ k_1+k_2=k+1\\ \beta_1,\beta_2\in\pi_2(X,L)\\ \beta_1+\beta_2=\beta}}\sum_{j=1}^{k_1}(-1)^*\,\mathfrak{m}^G_{k_1,\beta_1}(x_1\otimes\cdots\otimes x_{j-1}\otimes\mathfrak{m}^G_{k_2,\beta_2}(x_j\otimes\cdots\otimes x_{j+k_2-1})\otimes\cdots\otimes x_k)\\ =&0\quad\text{by Proposition 5.2.}\end{aligned}$$

□

Definition 5.4 (Unit of an A_∞ algebra)**.** An element $\boldsymbol{e} \in C^0$ is called a **unit** of a G-equivariant A_∞ algebra $(C, \{\mathfrak{m}_k^G\}_{k\in\mathbb{N}}, S, G)$ if the following holds.

1) $\mathfrak{m}_k^G(x_1 \otimes \cdots \otimes \boldsymbol{e} \otimes \cdots \otimes x_k) = 0$ for all $x_1, \ldots, x_k \in C$ whenever $k \geq 2$ or $k = 1$.

2) $\mathfrak{m}_2^G(\boldsymbol{e}, x) = x = (-1)^{\deg x}\, \mathfrak{m}_2(x, \boldsymbol{e})$ for all $x \in C$.

The element $1 \in C_G(L, \Lambda_{0,\mathrm{nov}})$ is a unit of $(C_G(L, \Lambda_{0,\mathrm{nov}}), \{(\mathfrak{m}_k^G)^b\}_{k\in\mathbb{N}})$.

Definition 5.5 (Potential function)**.** Define the **potential function**

$$\mathfrak{PO}_G : \bigcup_{\boldsymbol{u}\in \operatorname{int}\Delta} \{\boldsymbol{u}\} \times H^1(\mu^{-1}(\boldsymbol{u}), \Lambda_0/(2\pi i\,\mathbb{Z})) \to \Lambda$$

by

$$(\mathfrak{m}_0^G)^b(1) = \mathfrak{PO}_G^{\boldsymbol{u}}(b)e.$$

By Theorem 5.2 ([23] Proposition 4.6), for all $\boldsymbol{u} \in \operatorname{int}\Delta$, there exists $\mathfrak{PO}^{\boldsymbol{u}}(b) \in \Lambda_0$ such that

$$\mathfrak{PO}^{\boldsymbol{u}}(b)PD[L(\boldsymbol{u})]e = \exp(\partial\beta \cap b)\,\mathfrak{m}_0(1) = \exp(\partial\beta \cap b)\,\mathfrak{m}_0^G(1) = (\mathfrak{m}_0^G)^b(1).$$

Thus, $\mathfrak{PO}_G(b)$ is defined and equal to $\mathfrak{PO}(b)$ for all $b \in H^1(\mu^{-1}(\boldsymbol{u}), \Lambda_0)$. For this reason, we will omit G in the notation of the potential function from now on.

Note that we have the inclusion

$$\bigcup_{\boldsymbol{u}\in \operatorname{int}\Delta} \{\boldsymbol{u}\} \times H^1(\mu^{-1}(\boldsymbol{u}), \Lambda_0/(2\pi i\,\mathbb{Z})) \to (\Lambda^*)^n$$

via

$$\left(u_1, \ldots, u_n, \sum_{i=1}^n x_i e_i\right) \mapsto (\exp(x_1)T^{u_1}, \ldots, \exp(x_n)T^{u_n}) =: (y_1, \ldots, y_n). \tag{5.3.10}$$

Then the potential function takes the form of a formal Laurent series in $y_1, \ldots, y_n$.

Corollary 5.1. For any $\boldsymbol{u} \in \operatorname{int}\Delta$, $\forall b \in H^1(\mu^{-1}(\boldsymbol{u}), \Lambda_0)$, we have $(\mathfrak{m}_1^G)^b \circ (\mathfrak{m}_1^G)^b = 0$.

Proof. For any $x \in C_G(L, \Lambda_{0,\mathrm{nov}})$, by the A_∞ relations (Proposition 5.3), we have

$$(\mathfrak{m}_1^G)^b \circ (\mathfrak{m}_1^G)^b(x) = -(\mathfrak{m}_2^G)^b \left((\mathfrak{m}_0^G)^b(1) \otimes x\right) + (-1)^{\deg x}(\mathfrak{m}_2^G)^b(x \otimes (\mathfrak{m}_0^G)^b(1)) = 0.$$

□

Therefore, we may define the equivariant Lagrangian Floer cohomology as follows.

Definition 5.6 (Equivariant Lagrangian Floer cohomology)**.** For any Lagrangian torus fiber $L(\boldsymbol{u}) = \mu^{-1}(\boldsymbol{u})$, $\boldsymbol{u} \in \operatorname{int}\Delta$, and any $b \in H^1(\mu^{-1}(\boldsymbol{u}), \Lambda_0)$, we define the **$G$-equivariant Lagrangian Floer cohomology** associated to the pair $(L(\boldsymbol{u}), b)$ by

$$HF_G((L(\boldsymbol{u}), b), (L(\boldsymbol{u}), b), \Lambda_{0,nov}).$$

In [23], the authors proved that one can express the potential function for a compact Fano toric manifold (X, ω) purely from the information of its moment polytope as follows.

Theorem 5.1 (Theorem 4.5 [23])**.** Let (X, ω, T^n, μ) be a compact symplectic toric Fano manifold with moment polytope

$$\Delta = \mu(X) = \bigcap_{i=1}^{m} \{\boldsymbol{u} \in \mathfrak{t}^* \mid \langle \boldsymbol{u}, v_i \rangle - \lambda_i \geq 0\}, \tag{5.3.11}$$

where m is the number of the facets of Δ, $v_i = (v_{i,1}, \ldots, v_{i,n})$ is the inner normal vector of the i-th facet. We denote the affine function $\langle \boldsymbol{u}, v_i \rangle - \lambda_i$ by $l_i(\boldsymbol{u})$. On

$$\bigcup_{\boldsymbol{u} \in \operatorname{int}\Delta} \{\boldsymbol{u}\} \times H^1\left(\mu^{-1}(\boldsymbol{u}), \frac{\Lambda_0}{2\pi i\,\mathbb{Z}}\right),$$

we have

$$\mathfrak{PO}\left(u_1, \ldots, u_n, \sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{m} \exp(\langle v_i, x \rangle) T^{l_i(u)},$$

where $x = (x_1, \ldots, x_n)$. In particular, if we use the coordinates (5.3.10), the potential function $\mathfrak{PO}$ defines a Laurent polynomial

$$\mathfrak{PO} = \sum_{i=1}^{m} y_1^{v_{i,1}} \cdots y_n^{v_{i,n}} T^{-\lambda_i} \in \Lambda[y_1^{\pm 1}, \ldots, y_n^{\pm 1}]. \tag{5.3.12}$$

When (X, ω, T^n, μ) is compact symplectic toric but not necessarily Fano, the same formula (5.3.12) computes the leading order potential function $\mathfrak{PO}_0$ of X.

Theorem 5.2 (Potential function of a compact symplectic toric manifold, [23] Theorem 4.6)**.** Let (X, ω, T^n, μ) be a compact symplectic toric manifold with moment polytope (5.3.11). Let $\boldsymbol{u} \in \operatorname{int}\Delta$ and $b \in H^1(L(\boldsymbol{u}), (\Lambda_0/2\pi i))$. Then there exists an index set $I \subset \mathbb{N}$ such that $\forall 1 \le i \le m, j \in I$, there exist $r_j \in \mathbb{Q}$, $e_j^i \in \mathbb{N}$, and $\rho_j > 0$ satisfying the following.

i) $\sum_{i=1}^{m} e_j^i > 0 \quad \forall 1 \le j \le n.$

ii) If we let $l_i(\boldsymbol{u}) = \langle \boldsymbol{u}, v_i \rangle - \lambda_i$,

$$v'_{j,k} = \sum_{i=1}^{m} e_j^i v_{i,k}, \quad l'_j = \sum_{i=1}^{m} e_j^i l_i, \quad \text{and} \quad v'_j = (v'_{j,1}, \dots, v'_{j,n}),$$

then the potential function is given by

$$\mathfrak{PO}(\boldsymbol{u}, \sum_{i=1}^{n} x_i e_i) - \sum_{i=1}^{m} \exp(\langle v_i, x \rangle) T^{l_i(\boldsymbol{u})} = \sum_{j \in I} r_j \exp\big(\langle v'_j, x \rangle\big) T^{l'_j(\boldsymbol{u}) + \rho_j}. \tag{5.3.13}$$

In particular, if we use the coordinates (5.3.10),

$$\mathfrak{PO} = \sum_{i=1}^{m} y_1^{v_{i,1}} \cdots y_n^{v_{i,n}} T^{-\lambda_i} + \sum_{j \in I} r_j \left(\prod_{i=1}^{m} (y_1^{v_{i,1}} \cdots y_n^{v_{i,n}} T^{-\lambda_i})^{e_j^i} \right) T^{\rho_j} \tag{5.3.14}$$

The rest of the section is devoted to the proof of Theorem 5.4.

Since $(\mathfrak{m}_1^G)^b$ commutes with d_G, it is defined on $H_G^1(L)$.

Lemma 5.1. Let $\boldsymbol{u} \in \operatorname{int}\Delta$ and let

$$b = \sum_{i=1}^{n} c_i \alpha_i \in H^1(L(\boldsymbol{u}), \Lambda_0), \quad \text{where}$$

where $\alpha_{r+1}, \dots, \alpha_n$ generates $H_G^1(L(\boldsymbol{u}), \mathbb{Z})$. Then for any $r+1 \le i \le n$, we have

$$(\mathfrak{m}_1^G)^b(\alpha_i) = \left(\frac{\partial\, \mathfrak{PO}^{\boldsymbol{u}}}{\partial c_i}(b) \right) PD[L(\boldsymbol{u})]e.$$

Proof. For each $1 \leq j \leq m$, let $\beta_j \in H_2(X, L(\boldsymbol{u}), \mathbb{Z}) \cong \pi_2(X, L(\boldsymbol{u}))$ be the class of the basic disc, which is a small disc transverse to μ^{-1} (j-th facet of Δ). Let $\mathfrak{m}_{k,\beta}$ be the ordinary A_∞ operator on the de Rham model. Note that

$$\mathfrak{PO}^{\boldsymbol{u}}(b)PD[L(\boldsymbol{u})]e = \sum_{\substack{\beta\in\pi_2(X,L(\boldsymbol{u}))\\ I_\mu(\beta)=2}} \sum_{k=0}^{\infty} (\mathfrak{m}_{k,\beta})(b \otimes \cdots \otimes b) T^{\frac{\omega(\beta)}{2\pi}} e$$

only involves moduli spaces $\mathcal{M}_{k+1}(L(\boldsymbol{u}), J, \beta)$ for $I_\mu(\beta) = 2$. On the one hand,

$$\deg(\mathfrak{m}_{k,\beta}(b^{\otimes k})) = k - ((n - 3 + I_\mu(\beta) + k + 1) - n) = 2 - I_\mu(\beta) \geq 0$$

implies $I_\mu(\beta) \leq 2$. On the other hand, for every $\beta \in \pi_2(X, L(\boldsymbol{u}))$, the evaluation map $\mathrm{ev}_0 : \mathcal{M}^{main}_{k+1}(L(\boldsymbol{u}), J, \beta) \to L$ is a submersion. In particular, whenever $\mathcal{M}^{main}_{k+1}(L(\boldsymbol{u}), J, \beta) \neq \emptyset$, its dimension is no less than $\dim L(\boldsymbol{u}) = n$:

$$n - 3 + I_\mu(\beta) + 1 \geq n \quad \Rightarrow \quad I_\mu(\beta) \geq 2.$$

This proves the claim. Therefore, for $r + 1 \leq i \leq n$, we have

$$\begin{aligned}
\left(\frac{\partial \mathfrak{PO}^{\boldsymbol{u}}}{\partial c_i}(b)\right) e =& \frac{\partial}{\partial c_i}\left(\sum_{\substack{\beta\in\pi_2(X,L(\boldsymbol{u}))\\ I_\mu(\beta)=2}} \sum_{k=0}^{\infty} (\mathfrak{m}_{k,\beta})(b \otimes \cdots \otimes b) T^{\frac{\omega(\beta)}{2\pi}} e \right) \\
=& \sum_{\substack{\beta\in\pi_2(X,L(\boldsymbol{u}))\\ I_\mu(\beta)=2}} \sum_{k=1}^{\infty} \sum_{l=1}^{k} \mathfrak{m}_{k,\beta}(b^{\otimes l-1} \otimes \alpha_i \otimes b^{\otimes k-l}) T^{\frac{\omega(\beta)}{2\pi}} e \\
=& \sum_{\substack{\beta\in\pi_2(X,L(\boldsymbol{u}))\\ I_\mu(\beta)=2}} \exp(\partial\beta \cap b)\, \mathfrak{m}_{1,\beta}(\alpha_i) T^{\frac{\omega(\beta)}{2\pi}} e \\
=& \sum_{\substack{\beta\in\pi_2(X,L(\boldsymbol{u}))\\ I_\mu(\beta)=2}} \exp(\partial\beta \cap b)\, \mathfrak{m}^G_{1,\beta}(\alpha_i) T^{\frac{\omega(\beta)}{2\pi}} e \\
=& (\mathfrak{m}^G_1)^b(\alpha_i).
\end{aligned}$$

□

Corollary 5.2. $(\mathfrak{m}^G_1)^b|_{H_G(L(\boldsymbol{u}),\Lambda_0)} = 0$ if and only if

$$\frac{\partial \mathfrak{PO}^{\boldsymbol{u}}}{\partial c_i}(b) = 0 \qquad \forall r + 1 \leq i \leq n.$$

Proof. Since the G-action on $L(\boldsymbol{u})$ is free, a set of generators $\{\alpha_{r+1}, \dots, \alpha_n\}$ for $H^1_G(L(\boldsymbol{u}), \mathbb{R})$ also generates $H_G(L(\boldsymbol{u}), \Lambda_{0,nov})$, where the multiplication on the latter is given by wedge product. Then $\{\alpha_{r+1}, \dots, \alpha_n\}$ generates $H_G(L(\boldsymbol{u}), \Lambda_{0,nov})$ with multiplication given by $(\mathfrak{m}_2^G)^b$ as well. The A_∞ relations imply that

$$\begin{aligned}
&(\mathfrak{m}_1^G)^b\left((\mathfrak{m}_2^G)^b(\eta \otimes \tau)\right) \pm (\mathfrak{m}_2^G)^b\left((\mathfrak{m}_1^G)^b(\eta) \otimes \tau\right) \pm (\mathfrak{m}_2^G)^b\left(\eta \otimes (\mathfrak{m}_1^G)^b(\tau)\right) \\
&\pm (\mathfrak{m}_3^G)^b\left((\mathfrak{m}_0^G)^b(1) \otimes \eta \otimes \tau\right) \pm (\mathfrak{m}_3^G)^b\left(\eta \otimes (\mathfrak{m}_0^G)^b(1) \otimes \tau\right) \pm (\mathfrak{m}_3^G)^b\left(\eta \otimes \tau \otimes (\mathfrak{m}_0^G)^b(1)\right) = 0.
\end{aligned}$$

In particular,

$$(\mathfrak{m}_1^G)^b\left((\mathfrak{m}_2^G)^b(\eta \otimes \tau)\right) \pm (\mathfrak{m}_2^G)^b\left((\mathfrak{m}_1^G)^b(\eta) \otimes \tau\right) \pm (\mathfrak{m}_2^G)^b\left(\eta \otimes (\mathfrak{m}_1^G)^b(\tau)\right) = 0.$$

Thus, $(\mathfrak{m}_1^G)^b = 0$ on $H_G(L(\boldsymbol{u}), \Lambda_{0,nov})$ if and only if $(\mathfrak{m}_1^G)^b = 0$ on $H^1_G(L(\boldsymbol{u}), \Lambda_{0,nov})$. By Proposition 5.1, this holds if and only if

$$\frac{\partial\, \mathfrak{PO}^{\boldsymbol{u}}}{\partial c_i}(b) = 0 \qquad \forall r+1 \le i \le n.$$

□

5.4 Spectral sequences

In this section, we define a spectral sequence similar to the one in [47] which may be used to compute the equivariant Lagrangian Floer cohomology. Using the properties of the spectral sequence, we conclude that the set of Lagrangian torus fibers (with weak bounding cochain data) that have nontrivial equivariant Lagrangian Floer cohomology can be identified with a subset of the algebraic torus over the Novikov field, with certain valuation restrictions.

Let $\boldsymbol{u} \in \operatorname{int} \Delta$, $L = \mu^{-1}(\boldsymbol{u})$, and $b \in H^1(L, \Lambda_0)$. Let

$$C = \Omega_G(L, \mathbb{R}) \widehat{\otimes} \Lambda_{0,\mathrm{nov}}, \quad \delta = (\mathfrak{m}_1^G)^b.$$

Then $(C[1], \delta)$ is a cochain complex.

Define

$$F^{\lambda}C^p = \widehat{\bigoplus_{\substack{m,n\in\mathbb{N}\\ m+2n=p}}} \Omega^m_G(L)\otimes(T^{\lambda}\Lambda_0\cdot e^n).$$

Define $E : C^p \to \mathbb{R}$ by $E(x) = \lambda$ if $x \in F^{\lambda}C^p$ but $x \notin F^{\lambda'}C^p$ for any $\lambda' > \lambda$.

Let $\delta_{1,0} = (\mathfrak{m}^G_{1,\beta=0})^b = \mathfrak{m}_{1,\beta=0} = d_G$. Since non-constant pseudoholomorphic discs with boundary on L have a universal energy lower bound, there exists $\lambda'' > 0$ such that, $\forall\lambda$, $\forall x \in F^{\lambda}C$, we have $(\delta-\delta_{1,0})x \in F^{\lambda+\lambda''}C$. Let $0 < \lambda_0 < \lambda''$. We use the λ_0 to define a decreasing integral filtration as follows. For any $q \in \mathbb{N}$, let

$$\mathcal{F}^q\, C^p = \widehat{\bigoplus_{\substack{m,n\in\mathbb{N}\\ m+2n=p}}} \Omega^m_G(L)\otimes(T^{q\lambda_0}\Lambda_0\cdot e^n)$$

and let $\mathcal{F}^{\infty}\, C^p = \{0\}$.

Definition 5.7. Define

$$\begin{aligned}
A^{p,q}_r &:= \mathcal{F}^q\, C^p \cap \delta^{-1}(\mathcal{F}^{q+r-1}\, C^{p+1})\\
Z^{p,q}_r &:= A^{p,q}_r + \mathcal{F}^{q+1}\, C^p = \mathcal{F}^q\, C^p \cap \delta^{-1}(\mathcal{F}^{q+r-1}\, C^{p+1}) + \mathcal{F}^{q+1}\, C^p\\
B^{p,q}_r &:= \mathcal{F}^q\, C^p \cap \delta(\mathcal{F}^{q-r+2}\, C^{p-1}) + \mathcal{F}^{q+1}\, C^p\\
E^{p,q}_r &:= \frac{A^{p,q}_r}{B^{p,q}_r \cap A^{p,q}_r}.
\end{aligned}$$

For any $r \geq 0$, we have

$$B^{p,q}_{r+1} = \mathcal{F}^q\, C^p \cap \delta(\mathcal{F}^{q-r+1}\, C^{p-1}) + \mathcal{F}^{q+1}\, C^p \subset B^{p,q}_r$$

and

$$Z^{p,q}_{r+1} = \mathcal{F}^q\, C^p \cap \delta^{-1}(\mathcal{F}^{q+r}\, C^{p+1}) + \mathcal{F}^{q+1}\, C^p \subset Z^{p,q}_r.$$

It is easy to check the following.

Lemma 5.2. Let R be a commutative ring. Suppose A, M, T are R-modules, where $M \subset A$ is a submodule. Then we have the following:

i) $\dfrac{A}{A\cap(M+T)} \cong \dfrac{A+T}{M+T}$;

ii) $A \cap (M + T) = M + A \cap T$.

Proposition 5.4.

i) $A_r^{p,q} \cap B_r^{p,q} = \mathcal{F}^q\, C^p \cap \delta(\mathcal{F}^{q-r+2}\, C^{p-1}) + \mathcal{F}^{q+1}\, C^p \cap \delta^{-1}(\mathcal{F}^{q+r-1}\, C^{p+1})$.

ii) $E_r^{p,q} \cong \dfrac{Z_r^{p,q}}{B_r^{p,q}}$

Proof. i) Note that $\mathcal{F}^q\, C^p \cap \delta(\mathcal{F}^{q-r+2}\, C^{p-1}) \subset A_r^{p,q}$. Thus, by Lemma 5.2 ii),

$$\begin{aligned} A_r^{p,q} \cap B_r^{p,q} &= \mathcal{F}^q\, C^p \cap \delta(\mathcal{F}^{q-r+2}\, C^{p-1}) + A_r^{p,q} \cap \mathcal{F}^{q+1}\, C^p \\ &= \mathcal{F}^q\, C^p \cap \delta(\mathcal{F}^{q-r+2}\, C^{p-1}) + \mathcal{F}^{q+1}\, C^p \cap \delta^{-1}(\mathcal{F}^{q+r-1}\, C^{p+1}) \end{aligned}$$

ii) Apply Lemma 5.2 i) to

$$A = A_r^{p,q}, \qquad M = \mathcal{F}^q\, C^p \cap \delta(\mathcal{F}^{q-r+2}\, C^{p-1}), \qquad T = \mathcal{F}^{q+1}\, C^p,$$

we have

$$E_r^{p,q} = \frac{A_r^{p,q}}{B_r^{p,q} \cap A_r^{p,q}} \cong \frac{A_r^{p,q} + \mathcal{F}^{q+1}\, C^p}{B_r^{p,q}} \cong \frac{Z_r^{p,q}}{B_r^{p,q}}.$$

□

Definition 5.8 ($E_\infty^{p,q}$)**.** For a fixed pair (p, q), if $r > q + 2$, then

$$\begin{aligned} B_r^{p,q} &= \mathcal{F}^q\, C^p \cap \delta(\mathcal{F}^{q-r+2}\, C^{p-1}) + \mathcal{F}^{q+1}\, C^p \\ &= \mathcal{F}^q\, C^p \cap \delta(C^{p-1}) + \mathcal{F}^{q+1}\, C^p \end{aligned}$$

is independent of r. Moreover, for $r > q + 2$, we have inclusions

$$\cdots \subset Z_{q+5}^{p,q} \subset Z_{q+4}^{p,q} \subset Z_{q+3}^{p,q},$$

and thus an inverse system

$$\cdots \subset E_{q+5}^{p,q} \subset E_{q+4}^{p,q} \subset E_{q+3}^{p,q}.$$

We define

$$E_\infty^{p,q} := \varprojlim E_r^{p,q}.$$

We will use the following lemma (proved in Proposition 6.3.9 [24]) to prove Theorem 5.3.

Lemma 5.3 (Proposition 6.3.9 in [24])**.** Let $C = \widehat{\bigoplus_{p\in\mathbb{N}}} C^p$ be a graded finitely generated free module over Λ_0 such that C and each C^p is complete with respected to the energy filtration. Let $\delta : C^* \to C^{*+1}$ be a degree 1 operator such that

$$\delta \circ \delta = 0 \quad \text{and} \quad \delta(F^\lambda C) \subset F^\lambda C.$$

Let $W \subset C^p$ be a finitely generated Λ_0-submodule. Then there exists a constant c depending on W but not on λ such that

$$\delta(W) \cap F^\lambda C^{p+1} \subset \delta\left(W \cap F^{\lambda - c} C^p\right). \tag{5.4.1}$$

Theorem 5.3. Let $\boldsymbol{u} \in \operatorname{int} \Delta$, and $L = \mu^{-1}(\boldsymbol{u})$. Let $b \in H^1(L, \Lambda_0)$. There exists a spectral sequence with the following properties.

i) The E_2-page is given by $H_G(L, \Lambda_{0,\text{nov}})$, where

$$E_2^{p,q} \cong \widehat{\bigoplus_{m\in\mathbb{N}}} H_G^{p-2m}(L, \mathbb{R}) \widehat{\otimes} \frac{\mathcal{F}^q(\Lambda_0 \cdot e^m)}{\mathcal{F}^{q+1}(\Lambda_0 \cdot e^m)} \tag{5.4.2}$$

ii) $\forall r, p \in \mathbb{N}, \forall q \in \mathbb{Z}$, there exists a well-defined map $\delta_r : E_r^{p,q} \to E_r^{p+1, q+r-1}$ satisfying:

a)

$$\delta_r^{p+1,q+r-1} \circ \delta_r^{p,q} = 0;$$

b)

$$E_{r+1}^{p,q} \cong \frac{\ker \delta_r^{p,q}}{\operatorname{im} \delta_r^{p-1,q-r+1}} \tag{5.4.3}$$

c)

$$e^{\pm 1} \circ \delta_r^{p,q} = \delta_r^{p\pm 2,q} \circ e^{\pm 1}$$

iii) There exists some $r_0 \geq 2$ with

$$E_2^{p,q} \quad \Longrightarrow \quad E_{r_0}^{p,q} \cong E_{r_0+1}^{p,q} \cong \cdots \cong E_\infty^{p,q} = \frac{\mathcal{F}^q \, HF_G^p\Big((L,b),(L,b),\Lambda_{0,\text{nov}}\Big)}{\mathcal{F}^{q+1} \, HF_G^p\Big((L,b),(L,b),\Lambda_{0,\text{nov}}\Big)}. \tag{5.4.4}$$

Proof. i) We compute

$$
\begin{aligned}
A_2^{p,q} &= \{x \in \mathcal{F}^q\, C^p \mid \delta x \in \mathcal{F}^{q+1}\, C^{p+1}\} = \ker d_G \cap \mathcal{F}^q\, C^p, \\
B_2^{p,q} &= \delta(\mathcal{F}^q\, C^{p-1}) \cap \mathcal{F}^q\, C^p + \mathcal{F}^{q+1}\, C^p = \operatorname{im} d_G \cap \mathcal{F}^q\, C^p + \mathcal{F}^{q+1}\, C^p. \\
\Rightarrow \quad E_2^{p,q} &= \frac{A_2^{p,q}}{B_2^{p,q} \cap A_2^{p,q}} \cong \frac{A_2^{p,q} + \mathcal{F}^{q+1}\, C^p}{B_2^{p,q}} \\
&\cong \frac{\Big(\ker d_G \cap \mathcal{F}^q\, C^p + \mathcal{F}^{q+1}\, C^p\Big) / \mathcal{F}^{q+1}\, C^p}{\Big(\operatorname{im} d_G \cap \mathcal{F}^q\, C^p + \mathcal{F}^{q+1}\, C^p\Big) / \mathcal{F}^{q+1}\, C^p} \\
&\cong \bigoplus_{m \in \mathbb{N}} H_G^{p-2m}(L;\mathbb{R}) \otimes (\mathcal{F}^q(\Lambda_0 \cdot e^m) / \mathcal{F}^{q+1}(\Lambda_0 \cdot e^m))
\end{aligned}
$$

ii) Define $\delta_r[x] = [\delta x] \in E_r^{p+1,q+r-1}$. Then

$$
\begin{aligned}
\delta(A_r^{p,q}) &= \mathcal{F}^{q+r-1}\, C^{p+1} \cap \delta(\mathcal{F}^q\, C^p) \\
&\subset \mathcal{F}^{q+r-1}\, C^{p+1} \cap \delta^{-1}(\{0\}) \\
&\subset \mathcal{F}^{q+r-1}\, C^{p+1} \cap \delta^{-1}(\mathcal{F}^{q+2r-2}\, C^{p+2}) = A_r^{p+1,q+r-1}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
\delta(A_r^{p,q} \cap B_r^{p,q}) &= \delta\Big(\delta(\mathcal{F}^{q-r+2}\, C^{p-1}) \cap \mathcal{F}^q\, C^p + \delta^{-1}(\mathcal{F}^{q+r-1}\, C^{p+1}) \cap \mathcal{F}^{q+1}\, C^p\Big) \\
&= \delta\Big(\, \mathcal{F}^{q+1}\, C^p \cap \delta^{-1}(\mathcal{F}^{q+r-1}\, C^{p+1})\Big) \\
&= \delta(\mathcal{F}^{q+1}\, C^p) \cap \mathcal{F}^{q+r-1}\, C^{p+1} \\
&\subset \delta(\mathcal{F}^{q+1}\, C^p) \cap \mathcal{F}^{q+r-1}\, C^{p+1} + \mathcal{F}^{q+r}\, C^{p+1} \cap \delta^{-1}\big(\mathcal{F}^{q+2r-2}\, C^{p+2}\big) \\
&= A_r^{p+1,q+r-1} \cap B_r^{p+1,q+r-1}.
\end{aligned}
$$

Therefore, δ_r is well-defined.

a) $\delta_r^{p+1,q+r-1} \circ \delta_r^{p,q} = 0$ follows from $\delta \circ \delta = 0$.

b) We have

$$\begin{aligned}
&\ker \delta_r^{p,q}\\
=&\frac{A_r^{p,q} \cap \delta^{-1}(A_r^{p+1,q+r-1} \cap B_r^{p+1,q+r-1})}{A_r^{p,q} \cap B_r^{p,q}}\\
=&\frac{A_r^{p,q} \cap \delta^{-1}(B_r^{p+1,q+r-1})}{A_r^{p,q} \cap B_r^{p,q}} \qquad \text{since } A_r^{p,q} \subset \delta^{-1}(A_r^{p+1,q+r-1})\\
=&\frac{\mathcal{F}^q\, C^p \cap \delta^{-1}(\mathcal{F}^{q+r-1}\, C^{p+1}) \cap \delta^{-1}\Big(\delta(\mathcal{F}^{q+1}\, C^p) \cap \mathcal{F}^{q+r-1}\, C^{p+1} + \mathcal{F}^{q+r}\, C^{p+1}\Big)}{A_r^{p,q} \cap B_r^{p,q}}\\
=&\frac{\mathcal{F}^q\, C^p \cap \delta^{-1}\Big(\delta(\mathcal{F}^{q+1}\, C^p) \cap \mathcal{F}^{q+r-1}\, C^{p+1} + \mathcal{F}^{q+r}\, C^{p+1}\Big)}{A_r^{p,q} \cap B_r^{p,q}}\\
&\qquad [\text{ since } \Big(\delta(\mathcal{F}^{q+1}\, C^p) \cap \mathcal{F}^{q+r-1}\, C^{p+1} + \mathcal{F}^{q+r}\, C^{p+1}\Big) \subset \mathcal{F}^{q+r-1}]\\
=&\frac{\mathcal{F}^{q+1}\, C^p \cap \delta^{-1}(\mathcal{F}^{q+r-1}\, C^{p+1}) + \mathcal{F}^q\, C^p \cap \delta^{-1}(\mathcal{F}^{q+r}\, C^{p+1})}{A_r^{p,q} \cap B_r^{p,q}}.
\end{aligned}$$

$$\begin{aligned}
&\operatorname{im} \delta_r^{p-1,q-r+1}\\
=&\frac{\delta(A_r^{p-1,q-r+1}) + A_r^{p,q} \cap B_r^{p,q}}{A_r^{p,q} \cap B_r^{p,q}}\\
=&\frac{\mathcal{F}^q\, C^p \cap \delta(\mathcal{F}^{q-r+1}\, C^{p-1}) + \mathcal{F}^q\, C^p \cap \delta(\mathcal{F}^{q-r+2}\, C^{p-1}) + \mathcal{F}^{q+1}\, C^p \cap \delta^{-1}(\mathcal{F}^{q+r-1}\, C^{p+1})}{A_r^{p,q} \cap B_r^{p,q}}\\
=&\frac{\mathcal{F}^q\, C^p \cap \delta(\mathcal{F}^{q-r+1}\, C^{p-1}) + \mathcal{F}^{q+1}\, C^p \cap \delta^{-1}(\mathcal{F}^{q+r-1}\, C^{p+1})}{A_r^{p,q} \cap B_r^{p,q}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{\ker \delta_r^{p,q}}{\operatorname{im} \delta_r^{p-1,q-r+1}}\\
=&\frac{\mathcal{F}^{q+1}\, C^p \cap \delta^{-1}(\mathcal{F}^{q+r-1}\, C^{p+1}) + \mathcal{F}^q\, C^p \cap \delta^{-1}(\mathcal{F}^{q+r}\, C^{p+1})}{\mathcal{F}^q\, C^p \cap \delta(\mathcal{F}^{q-r+1}\, C^{p-1}) + \mathcal{F}^{q+1}\, C^p \cap \delta^{-1}(\mathcal{F}^{q+r-1}\, C^{p+1})}\\
\cong&\frac{\mathcal{F}^q\, C^p \cap \delta^{-1}(\mathcal{F}^{q+r}\, C^{p+1})}{\mathcal{F}^q\, C^p \cap \delta(\mathcal{F}^{q-r+1}\, C^{p-1}) + \mathcal{F}^{q+1}\, C^p \cap \delta^{-1}(\mathcal{F}^{q+r}\, C^{p+1})} \qquad \text{by Lemma 5.2}\\
=&\frac{A_{r+1}^{p,q}}{A_{r+1}^{p,q} \cap B_{r+1}^{p,q}} = E_{r+1}^{p,q}.
\end{aligned}$$

c) follows from the fact that δ commutes with multiplying by $e^{\pm 1}$ and that $\deg e^{\pm 1} = \pm 2$.

iii) We consider the restriction of the original spectral sequence to pages starting from E_2. Take $C = E_2$ and $W = E_2^{p,q}$ in Lemma 5.3 to get a constant c such that (5.4.1) holds.

Let $r > 2$ be big enough such that $(r-1)\lambda_0 - c > \lambda_0$. Let $r \geq r_0$. Then

$$(q+r-1)\lambda_0 - c = q\lambda_0 + (r-1)\lambda_0 - c > (q+1)\lambda_0$$

and thus, by (5.4.1),

$$\delta(\mathcal{F}^q\, C^p) \cap \mathcal{F}^{q+r-1}\, C^{p+1} \subset \delta\left(\mathcal{F}^{q+1}\, C^p\right).$$

Consider any $x \in A_r^{p,q} = \mathcal{F}^q\, C^p \cap \delta^{-1}(\mathcal{F}^{q+r-1}\, C^{p+1})$. Then $\delta x \in \delta(\mathcal{F}^{q+1}\, C^p)$ and thus there exists

$$y \in \mathcal{F}^{q+1}\, C^p \cap \delta^{-1}(\mathcal{F}^{q+r-1}\, C^{p+1}) \subset A_r^{p,q} \cap B_r^{p,q} \tag{5.4.5}$$

such that $[\delta x] = [\delta y]$. Thus, $\delta_r[x] = \delta_r[y] = 0$. Therefore, there exists a $r_0 \gg 2$ such that we have

$$E_{r_0}^{p,q} \cong E_{r_0+1}^{p,q} \cong \cdots \cong E_\infty^{p,q}.$$

Consider the map

$$\pi_{p,q} : \mathcal{F}^q\, H^p(C,\delta) \to E_\infty^{p,q}$$

defined as follows. An element $[x] \in \mathcal{F}^q\, H^p(C,\delta)$ is represented by

$$x \in \mathcal{F}^q\, C^p \cap \delta^{-1}(0) \subset \mathcal{F}^q\, C^p \cap \delta^{-1}(\mathcal{F}^{q+r-1}\, C^{p+1}) = A_r^{p,q}$$

for any $r \geq \max\{r_0, q+2\}$. We define $\pi_{p,q}[x]$ to be the class represented by x in $E_\infty^{p,q} \cong \frac{A_r^{p,q}}{A_r^{p,q} \cap B_r^{p,q}}$.

Suppose $x, x' \in \mathcal{F}^q\, H^p(C,\delta)$ with $x - x' = \delta y$ for some $y \in \mathcal{F}^q\, C^{p-1}$. Let $r \geq \max\{r_0, q+2\}$. Then

$$\delta y \in \delta(\mathcal{F}^q\, C^{p-1}) = \delta(\mathcal{F}^{q-r+2}\, C^{p-1}) \cap \mathcal{F}^q\, C^p \subset A_r^{p,q} \cap B_r^{p,q}.$$

This implies $\pi_{p,q}[x] = \pi_{p,q}[x']$ in $E_r^{p,q}$ and hence the well-definedness of $\pi_{p,q}$.

Let $[x] \in E_\infty^{p,q}$. Then for r large we have that $[x] \in E_r^{p,q}$ and that there exists y satisfying (5.4.5).

$$[x-y] = [x] \in E_r^{p,q} \quad \text{and} \quad \delta(x-y) = 0.$$

Thus, $[x-y] \in \mathcal{F}^q H^p(C,\delta)$ with $\pi_{p,q}[x-y] = [x]$, showing the surjectivity of $\pi_{p,q}$.

Suppose $[x] \in \ker \pi_{p,q}$. Then, for r large enough, we have

$$x \in \delta(\mathcal{F}^{q-r+2}\, C^{p-1}) + \mathcal{F}^{q+1}\, C^p = \delta(C^{p-1}) + \mathcal{F}^{q+1}\, C^p.$$

Hence, $[x] \in \mathcal{F}^{q+1} H^p(C,\delta)$ and thus $\ker \pi_{p,q} \subset \mathcal{F}^{q+1} H^p(C,\delta)$. Moreover, for large enough r,

$$A_r^{p,q} \cap B_r^{p,q} = \delta(C^{p-1}) + \mathcal{F}^{q+1}\, C^p \Rightarrow \mathcal{F}^{q+1}\, H^p(C,\delta) \subset \ker \pi_{p,q}.$$

Therefore, $\ker \pi_{p,q} = \mathcal{F}^{q+1}\, H^p(C,\delta)$.

□

Theorem 5.4. Let $\boldsymbol{u} \in \operatorname{int}\Delta$ and $b \in H^1(L(\boldsymbol{u}),\Lambda_0)$. Then $b = \sum_{i=1}^{n} x_i e_i = \sum_{i=1}^{n} c_i \alpha_i$, where $e_1,\ldots,e_n$ generate $H^1(L(\boldsymbol{u}),\mathbb{Z}) \cong M$ as in Section 5.1 and $\alpha_{r+1},\ldots,\alpha_n$ generate $H^1_G(L(\boldsymbol{u}),\mathbb{Z})$. Let

$$\operatorname{Crit}_G^{\Delta}(\mathfrak{PO}) := \left\{ (y_1,\ldots,y_n) \in (\Lambda^*)^n \;\middle|\; \begin{array}{l} \dfrac{\partial \mathfrak{PO}}{\partial c_i} = 0 \quad \forall r+1 \le i \le n \\ (\operatorname{val}(y_1),\ldots,\operatorname{val}(y_n)) \in \operatorname{int}\Delta \end{array} \right\} \tag{5.4.6}$$

and let $MLag_G(X,\omega)$ be the set

$$\left\{ (\boldsymbol{u},b) \in \bigcup_{\boldsymbol{u}\in \operatorname{int}\Delta} \{\boldsymbol{u}\} \times H^1\left(L(\boldsymbol{u}), \frac{\Lambda_0}{2\pi i\,\mathbb{Z}}\right) \;\middle|\; HF_G((L(\boldsymbol{u}),b),(L(\boldsymbol{u}),b),\Lambda_{\mathrm{nov}}) \neq 0 \right\}. \tag{5.4.7}$$

Then the following are equivalent.

i) $(y_1,\ldots,y_n) = (e^{x_1}T^{u_1},\ldots,e^{x_n}T^{u_n}) \in \operatorname{Crit}_G^{\Delta}(\mathfrak{PO})$.

ii)

$$HF_G\left((\mu^{-1}(\boldsymbol{u}),b),(\mu^{-1}(\boldsymbol{u}),b),\Lambda_{0,\mathrm{nov}}\right) \cong H_G(\mu^{-1}(\boldsymbol{u}),\mathbb{R}) \otimes_{\mathbb{R}} \Lambda_{0,\mathrm{nov}}\,.$$

iii) $(\boldsymbol{u},b) \in MLag_G(X,\omega)$; i.e.

$$HF_G\left((\mu^{-1}(\boldsymbol{u}),b),(\mu^{-1}(\boldsymbol{u}),b),\Lambda_{0,\mathrm{nov}}\right) \neq 0.$$

Proof. The equivalence of i) and iii) follows from Proposition 5.2. ii) $\Rightarrow$ iii) is trivial. If ii) does not hold, then the spectral sequence in Theorem 5.3 does not collapse at E_2 page. This is equivalent to saying $(\mathfrak{m}_1^G)^b(\alpha) \neq 0$ for some element $\alpha \in H_G^1(L, \Lambda_{0,\mathrm{nov}})$. By degree count, $(\mathfrak{m}_1^G)^b(\alpha) \in \Lambda_{0,\mathrm{nov}}$, which implies $E_3 = 0$ and thus $HF_G\left((\mu^{-1}(\boldsymbol{u}), b), (\mu^{-1}(\boldsymbol{u}), b), \Lambda_{0,\mathrm{nov}}\right) = 0$. $\square$

Chapter 6

$\mathrm{Crit}_G^\Delta(\mathfrak{PO})$

In this chapter, we prove that the set of Lagrangian torus fibers (with weak bounding cochain data) with non-vanishing equivariant Lagrangian Floer cohomology can be identified with a rigid analytic space. After discussing the proof in Section 6.1, we will provide examples in Section 6.2. In particular, we can locate such Lagrangian torus fibers in the moment polytope using tropical geometry. When the compact symplectic toric manifold is Fano and the acting subtorus is trivial, the equivariant Lagrangian Floer theory agrees with the ordinary Lagrangian Floer theory. And, in this case, the barycentric Lagrangian torus fiber obtained by tropicalizing the non-archimedean rigid analytic space induced by the Jacobian ideal of the potential function is known to generate the Fukaya category ordinary Lagrangian Floer theory on compact toric Fano manifolds. (See [15] and [27].)

6.1 $\mathrm{Crit}_G^\Delta(\mathfrak{PO})$ is a rigid analytic space

We denote the coordinate valuation map by

$$\mathrm{trop} : \Lambda^n \to (\mathbb{R} \cup \{\infty\})^n, \qquad (y_1, \dots, y_n) \mapsto (\mathrm{val}(y_1), \dots, \mathrm{val}(y_n)).$$

Lemma 6.1. Suppose $\Delta \subset \mathbb{R}^n$ is a polytope of the form (5.3.11). If A is an affinoid algebra such that $\mathrm{trop}^{-1}(\Delta) = \mathrm{Sp}\, A$, then A is a Cohen-Macaulay ring of dimension n. Moreover,

$\mathrm{trop}^{-1}(\mathrm{int}\,\Delta)$ is a rigid analytic space.

Proof. Suppose the moment polytope Δ is defined by m affine inequalities.

$$\Delta = \bigcap_{i=1}^{m} \{u \in M_{\mathbb{R}} \,|\, \langle u, v_i \rangle - \lambda_i \geq 0\},$$

where $m > n$ is the number of facets of Δ and each $v_i = (v_{i,1} \dots, v_{i,n}) \in N_{\mathbb{Z}}$ is the inner normal vector of the i-th facet. Denote $y^{v_i} := y_1^{v_{i,1}} \cdots y_n^{v_{i,n}}$. Then

$$\begin{aligned}
\mathrm{trop}^{-1}(\Delta) &= \{(y_1, \dots, y_n) \in \Lambda^n \mid (\mathrm{val}(y_1), \dots, \mathrm{val}(y_n)) \in \Delta\} \\
&= \{(y_1, \dots, y_n) \in \Lambda^n \mid \mathrm{val}(y^{v_i}) - \mathrm{val}(T^{\lambda_i}) \geq 0 \quad \forall 1 \leq i \leq m\} \\
&= \left\{(y_1, \dots, y_n) \in \Lambda^n \,\middle|\, \left|\frac{y^{v_i}}{T^{\lambda_i}}\right| \leq 1 \quad \forall 1 \leq i \leq m\right\}.
\end{aligned}$$

By the smoothness of the Delzant polytope, the linear map

$$\tilde{\varphi} : \mathbb{Z}^m \twoheadrightarrow M_{\mathbb{Z}} \cong \mathbb{Z}^n, \qquad (c_1, \dots, c_m) \mapsto \begin{pmatrix} v_{1,1} & \cdots & v_{m,1} \\ \vdots & \ddots & \vdots \\ v_{1,n} & \cdots & v_{m,n} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix},$$

is surjective, and it induces a surjective ring homomorphism

$$\varphi : \Lambda[z_1^{\pm 1}, \dots, z_m^{\pm 1}] \twoheadrightarrow \Lambda[y_1^{\pm 1}, \dots, y_n^{\pm 1}], \qquad z^c \mapsto y^{\tilde{\varphi}(c)}. \tag{6.1.1}$$

Here $z^c := z_1^{c_1} \cdots z_m^{c_m}$ if $c = (c_1, \dots, c_m)$. In particular, $\varphi(z_i) = y^{v_i}$.

Let $\lambda = (\lambda_1, \dots, \lambda_m)$ and

$$T_{m,\lambda} = \Lambda \left\langle z_1 T^{-\lambda_1}, \dots, z_m T^{-\lambda_m} \right\rangle,$$

By [48] Proposition 6.9, $\mathrm{trop}^{-1}(\Delta) \cong \mathrm{Sp}\, A$, where

$$A = T_{m,\lambda} / (\ker \varphi) T_{m,\lambda} \tag{6.1.2}$$

is a Λ-affinoid algebra of dimension n and $\mathrm{trop}^{-1}(\Delta)$ is a Λ-affinoid space. Then

$$\mathrm{trop}^{-1}(\mathrm{int}\,\Delta) \cong \{z \in \mathrm{trop}^{-1}(\Delta) \mid |z_i T^{-\lambda_i}| < 1 \quad \forall 1 \leq i \leq m\}$$

is an admissible open subset of $\mathrm{trop}^{-1}(\Delta)$ by Proposition [5] 5.1/Proposition 7 (see Proposition B.2). Therefore, $\mathrm{trop}^{-1}(\mathrm{int}\,\Delta)$ is also a rigid analytic Λ-space. □

Moreover, by [48] Proposition 6.9 and [48] Remarks 6.5 and 6.6,

$$\Lambda\langle\Delta\rangle := \left\{ \sum_{c\in\mathbb{Z}^n} a_c y^c \,\middle|\, \mathrm{val}(a_c) + \langle u, c\rangle \to \infty \quad \text{as } |c|\to\infty \quad \forall u \in \Delta \right\} \tag{6.1.3}$$

is the completion of $\Lambda[y_1^{\pm 1}, \ldots, y_n^{\pm 1}]$ with respect to the norm $|\cdot|_\Delta : \Lambda[y_1^{\pm 1}, \ldots, y_n^{\pm 1}] \to \mathbb{R}_{\geq 0}$, which is defined by

$$\left|\sum_{c\in\mathbb{Z}^n} a_c y^c\right|_\Delta := \sup_{\substack{c\in\mathbb{Z}^n \\ u\in\Delta}} |a_c|_\Lambda \cdot \exp(-\langle u, c\rangle).$$

We can alternatively argue that $\Lambda\langle\Delta\rangle$ is an affinoid algebra as follows. Since $\Lambda\langle\Delta\rangle$ is a Λ-Banach algebra with this norm and $|y^{v_i}T^{-\lambda_i}|_\Delta = \sup_{u\in\Delta} e^{-(\langle u, v_i\rangle - \lambda_i)} \leq e^0 < 1$ for all $1 \leq i \leq m$, we have a continuous homomorphism $\Phi : \Lambda\langle z_1, \ldots, z_m\rangle \to \Lambda\langle\Delta\rangle$ prescribed by $z_i \mapsto y^{v_i}T^{-\lambda_i}$ according to [6] 6.1.1/Proposition 4. Moreover, by the smoothness of the polytope, since $v_1, \ldots, v_m$ contains a $\mathbb{Z}$-basis of $\mathbb{Z}^n$, every $c \in \mathbb{Z}^n$ is a $\mathbb{Z}$-linear combination $c = \sum_{i=1}^m c(v_i)v_i$ of $v_1, \ldots, v_m$. Thus, if $f = \sum_{c\in\mathbb{Z}^n} a_c y^c \in \Lambda\langle\Delta\rangle$, then

$$f = \Phi\left(\sum_{c\in\mathbb{Z}^n} a_c T^{\sum_{i=1}^m c(v_i)\lambda_i} z_1^{c(v_1)} \cdots z_m^{c(v_m)}\right).$$

As $|c(v_1)| + \cdots + |c(v_1)| \to \infty$, we have

$$\mathrm{val}\left(a_c T^{\sum_{i=1}^m c(v_i)\lambda_i}\right) = \mathrm{val}(a_c) + \sum_{i=1}^m c(v_i)\lambda_i \leq \mathrm{val}(a_c) + \sum_{i=1}^m c(v_i)\langle u, v_i\rangle \to \infty$$

for all $u \in \Delta$. Therefore, $\sum_{c\in\mathbb{Z}^n} a_c T^{\sum_{i=1}^m c(v_i)\lambda_i} z_1^{c(v_1)} \cdots z_m^{c(v_m)} \in \Lambda\langle z_1, \ldots, z_m\rangle$. This shows that Φ is a continuous epimorphism and thus $\Lambda\langle\Delta\rangle$ is a Λ-affinoid algebra ([6] 6.1.1/Definition 1), commonly denoted by $\Lambda\left\langle y^{v_1}T^{-\lambda_1}, \ldots, y^{v_m}T^{-\lambda_m}\right\rangle$.

Proposition 6.1. Let X be a compact symplectic toric manifold as in Section 5.1. Then $\mathrm{Crit}_G^\Delta(\mathfrak{PO})$ is a rigid analytic space. Moreover, the closure of the image of the map trop : $\mathrm{Crit}_G^\Delta(\mathfrak{PO}) \to \Delta, (y_1, \ldots, y_n) \mapsto (\mathrm{val}(y_1), \ldots, \mathrm{val}(y_n))$ is a polytopal set, i.e. a union of polytopes.

Proof. By Proposition 5.2, $\mathfrak{PO}$ takes the form of (5.3.14). For all $1 \leq i \leq n$, let

$$f_i = \sum_{j=1}^n a_{i,j}\left(y_j \frac{\partial \mathfrak{PO}}{\partial y_j}\right)$$

$$= \sum_{j=1}^{n} a_{i,j} \left(\sum_{k=1}^{m} v_{k,j} y_1^{v_{k,1}} \cdots y_n^{v_{k,n}} T^{-\lambda_k} + \sum_s \sum_{k=1}^{m} r_s v_{k,j} \prod_{k=1}^{m} (y_1^{v_{k,1}} \cdots y_n^{v_{k,n}} T^{-\lambda_k})^{e_s^k} \right). \quad (6.1.4)$$

Then

$$\begin{aligned} W &= V\left(f_{r+1}, \ldots, f_n\right) \cap \operatorname{trop}^{-1}(\Delta) \\ &= \{(y_1, \ldots, y_n) \in \operatorname{trop}^{-1}(\Delta) \mid f_i(y_1, \ldots, y_n) = 0 \quad \forall r+1 \le i \le n\} \\ &= \operatorname{Sp} \frac{\Lambda \langle \Delta \rangle}{(f_{r+1}, \ldots, f_n)} \end{aligned}$$

is a Λ-affinoid space.

By [5] 5.1/Proposition 7 (see Proposition B.2),

$$M := \operatorname{Crit}_G^{\Delta}(\mathfrak{PO}) = W \cap \operatorname{trop}^{-1}(\operatorname{int} \Delta) = \left\{ (y_1, \ldots, y_n) \in W \;\middle|\; \left| \frac{y^{v_1}}{T^{\lambda_1}} \right| < 1, \ldots \left| \frac{y^{v_m}}{T^{\lambda_m}} \right| < 1 \right\}$$

is a finite intersection of admissible open subsets and thus is an admissible open subset of W. Hence, $(M, \mathscr{O}_W|_M)$ is a rigid analytic space.

The last claim follows from [30] Proposition 5.2. □

Proposition 6.2 ($\operatorname{Crit}_G^{\Delta}(\mathfrak{PO})$ of $\mathbb{CP}^n$)**.** Suppose $G = \iota(T^r)$ is an r-dimensional subtorus of T^n acting on $(\mathbb{CP}^n, T^n, \omega, \mu)$, which has moment polytope

$$\Delta = \left\{ (u_1, \ldots, u_n) \in \mathbb{R}^n \;\middle|\; \begin{array}{l} u_i \ge 0 \quad \forall 1 \le i \le n, \\ 1 - \sum_{i=1}^{n} u_i \ge 0 \end{array} \right\}.$$

Then $\operatorname{Crit}_G^{\Delta}(\mathfrak{PO})$ is a rigid analytic space of dimension r.

Proof. By Theorem 5.1,

$$\mathfrak{PO} = y_1 + \cdots y_n + \frac{T}{y_1 \cdots y_n}.$$

Consider

$$\begin{aligned} Q &= \left\{ (y_1, \ldots, y_n) \in (\Lambda^*)^n \;\middle|\; \sum_{j=1}^{n} a_{i,j} \left(y_j \frac{\partial \mathfrak{PO}}{\partial y_j} \right) = 0 \quad \forall r+1 \le i \le n \right\} \\ &= \left\{ (y_1, \ldots, y_n) \in (\Lambda^*)^n \;\middle|\; \sum_{j=1}^{n} a_{i,j} \left(y_j - \frac{T}{y_1 \cdots y_n} \right) = 0 \quad \forall r+1 \le i \le n \right\}. \end{aligned}$$

Then

$$\mathrm{Crit}_G^{\Delta}(\mathfrak{PO}) = Q \cap \mathrm{trop}^{-1}(\mathrm{int}\,\Delta) = Q \cap \left\{ y \in \mathrm{trop}^{-1}(\Delta) \,\middle|\, \left| y^{v_i} T^{-\lambda_i} \right| < 1 \quad \forall 1 \le i \le m \right\}.$$

We first consider

$$\begin{aligned} W &= Q \cap \mathrm{trop}^{-1}(\Delta) \\ &\cong \mathrm{Sp}\, \frac{\Lambda\langle y_1, \ldots, y_n, z\rangle}{\left(y_1 \cdots y_n z - T, \sum_{j=1}^{n} a_{r+1,j}(y_j - z), \ldots, \sum_{j=1}^{n} a_{n,j}(y_j - z) \right)} \end{aligned}$$

Recall that $A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix}$ is an invertible matrix. Let

$$A^{-1} = \begin{pmatrix} b_{1,1} & \cdots & b_{1,n} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \cdots & b_{n,n} \end{pmatrix}.$$

Change variables by setting

$$\begin{pmatrix} X_1 \\ \vdots \\ X_n \\ X_{n+1} \end{pmatrix} = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} & 0 \\ \vdots & \ddots & \vdots & \\ a_{n,1} & \cdots & a_{n,n} & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 - z \\ \vdots \\ y_n - z \\ z \end{pmatrix}.$$

By [6] 6.1.1/Proposition 4, this defines a continuous epimorphism

$$\varphi : \Lambda \langle X_1, \ldots, X_n, X_{n+1} \rangle \to \frac{\Lambda\langle y_1, \ldots, y_n, z\rangle}{\left(y_1 \cdots y_n z - T, \sum_{j=1}^{n} a_{r+1,j}(y_j - z), \ldots, \sum_{j=1}^{n} a_{n,j}(y_j - z) \right)}.$$

Then

$$W \cong \mathrm{Sp}\, \frac{\Lambda\langle X_1, \ldots, X_n, X_{n+1}\rangle}{\left(X_{n+1} \prod_{i=1}^{n} (X_{n+1} + \sum_{j=1}^{n} b_{i,j} X_j) - T, X_{r+1}, \ldots, X_n \right)}$$

$$\cong \operatorname{Sp} \frac{\Lambda\langle X_1, \ldots, X_r, X_{n+1}\rangle}{\left(X_{n+1}\prod_{i=1}^n(X_{n+1} + \sum\limits_{j=1}^r b_{i,j}X_j) - T\right)}$$

By [48] Proposition 6.9 and [48] Theorem 4.6, the Tate algebra

$$T_{r+1} = \Lambda\langle X_1, \ldots, X_r, X_{n+1}\rangle$$

is a Cohen-Macaulay ring of dimension $r+1$, which is also an integral domain. Since $X_{n+1}\prod_{i=1}^n(X_{n+1} + \sum\limits_{j=1}^r b_{i,j}X_j) - T \neq 0$, it is not a zero divisor and thus is a regular sequence in T_{r+1}. Thus, by [8] Theorem 2.1.2, the Krull dimension of W is $r+1-1 = r$.

Since $\operatorname{Crit}_G^{\Delta}(\mathfrak{PO}) = Q \cap \operatorname{trop}^{-1}(\operatorname{int}\Delta) \subset W$, $\dim \operatorname{Crit}_G(\mathfrak{PO}) \leq r$. On the other hand, let $\epsilon > 0$ be sufficiently small. Consider

$$\Delta_\epsilon = \bigcap_{i=1}^m \{u \in \mathbb{R}^n \mid \langle u, v_i\rangle - \lambda_i \geq \epsilon\} \subset \Delta.$$

$$\begin{aligned}
Q \cap \operatorname{trop}^{-1}(\Delta) &\supset Q \cap \operatorname{trop}^{-1}(\Delta_\epsilon) \\
&\cong \operatorname{Sp} \frac{\Lambda\left\langle \frac{y_1}{T^\epsilon}, \ldots, \frac{y_n}{T^\epsilon}, \frac{T^{1-\epsilon}}{y_1\cdots y_n}\right\rangle}{\left(\sum\limits_{j=1}^n a_{r+1,j}(y_j - z), \ldots, \sum\limits_{j=1}^n a_{n,j}(y_j - z)\right)} \\
&\cong \operatorname{Sp} \frac{\Lambda\left\langle z_1, \ldots, z_n, z_{n+1}\right\rangle}{\left(z_1\cdots z_{n+1} - T^{1-(n+1)\epsilon}, \sum\limits_{j=1}^n a_{r+1,j}(z_j - z_{n+1}), \ldots, \sum\limits_{j=1}^n a_{n,j}(z_j - z_{n+1})\right)} \\
&=: U.
\end{aligned}$$

Change the variables by setting

$$\begin{pmatrix} X_1' \\ \vdots \\ X_n' \\ X_{n+1}' \end{pmatrix} = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} & 0 \\ \vdots & \ddots & \vdots & \\ a_{n,1} & \cdots & a_{n,n} & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 - z_{n+1} \\ \vdots \\ z_n - z_{n+1} \\ z_{n+1}' \end{pmatrix}.$$

Then

$$U \cong \operatorname{Sp} \frac{\Lambda\langle X_1', \ldots, X_n', X_{n+1}'\rangle}{\left(X_{n+1}'\prod_{i=1}^n(X_{n+1}' + \sum\limits_{j=1}^n b_{i,j}X_j') - T^{1-(n+1)\epsilon}\right)}$$

$$\cong \mathrm{Sp}\,\frac{\Lambda\langle X'_1,\ldots,X'_r,X'_{n+1}\rangle}{\left(X'_{n+1}\prod_{i=1}^{n}(X'_{n+1}+\sum\limits_{j=1}^{n}b_{i,j}X'_j)-T^{1-(n+1)\epsilon}\right)}$$

Since $X'_{n+1}\prod_{i=1}^{n}(X'_{n+1}+\sum\limits_{j=1}^{n}b_{i,j}X'_j)-T^{1-(n+1)\epsilon}$ is not a zero divisor in the integral domain $\Lambda\langle X'_1,\ldots,X'_r,X'_{n+1}\rangle$, which is Cohen-Macaulay of dimension $r+1$, we see that $\dim U = r$. Since $Q\cap \mathrm{trop}^{-1}(\Delta_\epsilon)\subset \mathrm{Crit}_G(\mathfrak{PO})\subset Q\cap\mathrm{trop}^{-1}(\Delta)$ and $\dim(Q\cap\mathrm{trop}^{-1}(\Delta_\epsilon)) = \dim(Q\cap\mathrm{trop}^{-1}(\Delta)) = r$, we conclude that $\dim \mathrm{Crit}_G^{\Delta}(\mathfrak{PO}) = r$. □

Proposition 6.3. Let X be a compact symplectic toric manifold of complex dimension n as in 5.2. If $G\cong T^{n-1}$ be an $(n-1)$-dimensional subtorus of T^n as in Section 5.1, then $\mathrm{Crit}_G(\mathfrak{PO})$ is a rigid analytic space of dimension $n-1$.

Proof. Consider

$$Q=\left\{(y_1,\ldots,y_n)\in(\Lambda^*)^n \,\middle|\, \sum_{j=1}^{n}a_{n,j}\left(y_j\frac{\partial\mathfrak{PO}}{\partial y_j}\right)=0\right\}.$$

Then

$$\dim\left(Q\cap\mathrm{trop}^{-1}(\Delta)\right)=\dim\frac{\Lambda\langle\Delta\rangle}{\left(\sum\limits_{j=1}^{n}a_{n,j}\left(y_j\frac{\partial\mathfrak{PO}}{\partial y_j}\right)\right)}.$$

Since $\Lambda\langle\Delta\rangle$ is a subring of the formal power series ring, $\Lambda\langle\Delta\rangle$ is also an integral domain. Since $\sum\limits_{j=1}^{n}a_{n,j}\left(y_j\frac{\partial\mathfrak{PO}}{\partial y_j}\right)\neq 0$, it is a regular sequence in $\Lambda\langle\Delta\rangle$, which is a Cohen-Macaulay ring of dimension n by [48] Proposition 6.9. Therefore, $\dim(Q\cap\mathrm{trop}^{-1}(\Delta)) = n-1$. Let

$$\Delta_\epsilon=\bigcap_{i=1}^{m}\{u\in\mathbb{R}^n \mid \langle u,v_i\rangle-\lambda_i\geq\epsilon\}.$$

Then $\dim(Q\cap\mathrm{trop}^{-1}(\Delta_\epsilon)) = n-1$ as well. Since

$$Q\cap\mathrm{trop}^{-1}(\Delta_\epsilon)\subset\mathrm{Crit}_G(\mathfrak{PO})\subset Q\cap\mathrm{trop}^{-1}(\Delta),$$

we conclude that $\dim\mathrm{Crit}_G(\mathfrak{PO}) = n-1$. □

Corollary 6.1. Let X be a compact symplectic toric manifold of complex dimension $n\leq 2$ as in Theorem 5.2. If $G\cong T^r$ be a subtorus of T^2 as in Section 5.1 for some $0\leq r\leq n$, then $\mathrm{Crit}_G(\mathfrak{PO})$ is a rigid analytic space of dimension r.

Proof. The case when $r = 0$ follows from [23]. The case when $r = 1, n = 2$ follows from Proposition 6.3. The case when $r = 2, n = 2$ follows from [48] Proposition 6.9. $\square$

6.2 Examples

Consider the action on a toric manifold (X^4, ω, T^2, μ) by a subtorus $G = \iota(S^1)$, where

$$S^1 \hookrightarrow T^2, \qquad \theta \mapsto (k_1\theta, k_2\theta),$$

for some $(k_1, k_2) \in \mathbb{Z}^2 \setminus \{(0,0)\}$. Since the action is free, $H^1_G(L(\boldsymbol{u}), \mathbb{R}) \cong H^1(L(\boldsymbol{u})/G, \mathbb{R}) \hookrightarrow H^1(L(\boldsymbol{u}), \mathbb{R})$ is generated by

$$\alpha_2 = -k_2 e_1 + k_1 e_2.$$

Complete it to a basis $\{\alpha_1, \alpha_2\}$ of $H^1(L(\boldsymbol{u}), \mathbb{R})$. Let $b = c_1\alpha_1 + c_2\alpha_2$. By Theorem 5.4, there is a bijection

$$MLag_G(\mathbb{CP}^2, \omega) \to V\left(\frac{\partial \mathfrak{PO}}{\partial c_2}\right) \cap \operatorname{trop}^{-1}(\operatorname{int}\Delta) =: \operatorname{Crit}^{\Delta}_G(\mathfrak{PO})$$
$$\left(u_1, u_2, b = \sum_{i=1}^{2} x_i e_i\right) \mapsto (y_1, y_2) = (e^{x_1}T^{u_1}, e^{x_2}T^{u_2}).$$

Then

$$\operatorname{val}(y_i) = \operatorname{val}(e^{x_i}T^{u_i}) = \operatorname{val}(e^{x_i}) + \operatorname{val}(T^{u_i}) = u_i.$$

In particular, given $(y_1, y_2) \in \operatorname{Crit}^{\Delta}_G(\mathfrak{PO})$, the Lagrangian associated with it is $\mu^{-1}(\operatorname{val}(y_1), \operatorname{val}(y_2))$.

6.2.1 S^1-action on $\mathbb{CP}^2$

Example 6.1 (S^1-action on $\mathbb{CP}^2$)**.** Consider $(\mathbb{CP}^2, \omega, T^2, \mu)$ associated with the moment polytope

$$\Delta = \left\{ (u_1, u_2) \in \mathbb{R}^2 \,\middle|\, \begin{aligned} & u_i \geq 0 \quad \forall 1 \leq i \leq 2, \\ & 1 - u_1 - u_2 \geq 0 \end{aligned} \right\}.$$

Its potential function is then given by

$$\mathfrak{PO} = y_1 + y_2 + \frac{T}{y_1 y_2}.$$

Denote by f the Laurent polynomial

$$\frac{\partial \mathfrak{PO}}{\partial c_2} = -k_2 y_1 \frac{\partial \mathfrak{PO}}{\partial y_1} + k_1 y_2 \frac{\partial \mathfrak{PO}}{\partial y_2} = -k_2(y_1 - \frac{T}{y_1 y_2}) + k_1(y_2 - \frac{T}{y_1 y_2}) = -k_2 y_1 + k_1 y_2 + (k_2 - k_1)\frac{T}{y_1 y_2}$$

and denote $Y = V(f) \cap (\Lambda^*)^2$. By Kapranov's theorem ([42] Theorem 3.1.3, also see Theorem C.1),

$$\overline{\mathrm{trop}(Y)} = V\left(\mathrm{trop}\left(f\right)\right).$$

i) Suppose $k_1, k_2, k_2 - k_1$ are all non-zero.

$$\begin{aligned} V(\mathrm{trop}\, f) =& \{u \in \mathbb{R}^2 \mid u_1 = 1 - u_1 - u_2 \le u_2\} \\ & \cup \{u \in \mathbb{R}^2 \mid u_2 = 1 - u_1 - u_2 \le u_1\} \\ & \cup \{u \in \mathbb{R}^2 \mid u_1 = u_2 \le 1 - u_1 - u_2\}. \end{aligned}$$

$\mathrm{trop}(\mathrm{Crit}_G^{\Delta}(\mathfrak{PO})) = \mathrm{trop}(Y) \cap \mathrm{int}\,\Delta$ is shown in Figure 6.1.

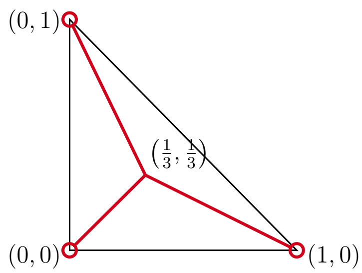


Figure 6.1: Case when $k_1, k_2, k_1 - k_2 \neq 0$

Moreover,

$$\begin{aligned} W :=& \, Y \cap \mathrm{trop}^{-1}(\Delta) \\ =& \, \mathrm{Sp}\, \frac{\Lambda\left\langle y_1, y_2, \frac{T}{y_1 y_2}\right\rangle}{\left(-k_2 y_1 + k_1 y_2 + (k_2 - k_1)\frac{T}{y_1 y_2}\right)} \\ \cong& \, \mathrm{Sp}\, \frac{\Lambda\left\langle y_1, y_2, z\right\rangle}{\left(-k_2 y_1 + k_1 y_2 + (k_2 - k_1) z, y_1 y_2 z - T\right)} \end{aligned}$$

is an affinoid space, and

$$\mathrm{Crit}_G^{\Delta}(\mathfrak{PO}) = W \setminus \mathrm{trop}^{-1}\left(\{(0,0),(0,1),(1,0)\}\right).$$

We can compute the genus of W as a rigid analytic curve as follows. We have a canonical reduction map

$$\rho : W \to \widetilde{W} = \operatorname{Spec} \frac{\mathbb{C}[y_1, y_2]}{((-k_2 y_1 + k_1 y_2) y_1 y_2)}.$$

By [17] Proposition 5.6.2, since W is a non-singular connected one-dimensional affinoid space, the genus of W equal to the arithmetic genus of the compactification of $\widetilde{W}$.

Let C_1, C_2, C_3 be the divisors corresponding to $-k_2 y_1 + k_1 y_2 = 0$, $y_1 = 0$, $y_2 = 0$, respectively. Then by the adjunction formula ([32] Chapter V, Exercise 1.3), the arithmetic genus of $\widetilde{W}$ is equal to

$$g_a(C_1 + C_2 + C_3) = \sum_{i=1}^{3} g_a(C_i) + C_1 \cdot C_2 + C_2 \cdot C_3 + C_1 \cdot C_3 - 2 = 1.$$

By the above argument, $g(W)$ is a rigid analytic curve of genus 1.

ii) If $k_1 = 0$ and $k_2 \neq 0$, then $f = -k_2 \left(y_1 - \frac{T}{y_1 y_2} \right)$.

$$V(\operatorname{trop} f) = \{ u \in \mathbb{R}^2 \mid u_1 = 1 - u_1 - u_2 \}.$$

$\operatorname{trop}\left(\operatorname{Crit}_G^{\Delta}(\mathfrak{PO})\right) = \operatorname{trop} V(f) \cap \operatorname{int} \Delta$ is shown in Figure 6.2.

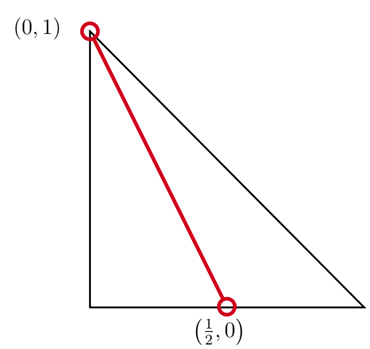


Figure 6.2: Case when $k_1 = 0$

Indeed,

$$\operatorname{Crit}_G^{\Delta}(\mathfrak{PO}) = Y \cap \operatorname{trop}^{-1}(\operatorname{int} \Delta)$$

$$= \left\{ (y_1, y_2) \in (\Lambda^*)^2 \;\middle|\; \begin{array}{l} -k_2 y_1 + k_2 \dfrac{T}{y_1 y_2} = 0 \\ \mathrm{val}(y_1) > 0, \mathrm{val}\left(\dfrac{T}{y_1^2}\right) > 0, \\ \mathrm{val}\left(\dfrac{T}{y_1 T / y_1^2}\right) > 0, \end{array} \right\}$$

$$= \left\{ \left(y_1, \frac{T}{y_1^2}\right) \in (B_\Lambda^1)^2 \;\middle|\; \begin{array}{l} \mathrm{val}(y_1) > 0, \mathrm{val}\left(\dfrac{T}{y_1^2}\right) > 0, \\ y_1 \neq 0, \dfrac{T}{y_1^2} \neq 0 \end{array} \right\}$$

$$= \left\{ (y_1, \frac{T}{y_1^2}) \in \Lambda^2 \;\middle|\; e^{-\frac{1}{2}} < |y_1| < 1 \right\}$$

$$\cong \left\{ y_1 \in B_\Lambda^1 \;\middle|\; e^{-\frac{1}{2}} < |y_1| < 1 \right\} \subset \mathrm{Sp}\,\Lambda \left\langle y_1, T y_1^{-2} \right\rangle$$

is an annulus.

iii) If $k_2 = 0$ and $k_1 \neq 0$, then $f = k_1 \left(y_2 - \frac{T}{y_1 y_2} \right)$.

$$V\left(\mathrm{trop}\left(f\right)\right) = \{u \in \mathbb{R}^2 \mid u_2 = 1 - u_1 - u_2\}.$$

$\mathrm{trop}\left(\mathrm{Crit}_G^\Delta(\mathfrak{PO})\right) = \mathrm{trop}\, V(f) \cap \mathrm{int}\,\Delta$ is shown in Figure 6.3.

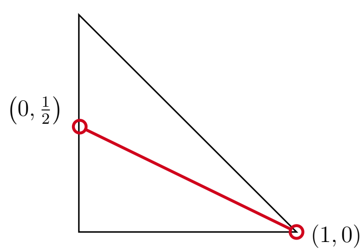


Figure 6.3: Case when $k_2 = 0$

Similar to the case $k_1 = 0$,

$$\begin{aligned} \mathrm{Crit}_G^\Delta(\mathfrak{PO}) =& Y \cap \mathrm{trop}^{-1}(\mathrm{int}\,\Delta) \\ &\cong \left\{ y_2 \in B_\Lambda^1 \;\middle|\; e^{-\frac{1}{2}} < |y_2| < 1 \right\} \subset \mathrm{Sp}\,\Lambda \left\langle y_2, T y_2^{-2} \right\rangle \end{aligned}$$

is an annulus.

iv) If $k_2 - k_1 = 0$ and $k_1, k_2 \neq 0$, then $f = -k_2 y_1 + k_1 y_2$.

$$V(\operatorname{trop}(f)) = \{u \in \mathbb{R}^2 \mid u_1 = u_2\}.$$

$\operatorname{trop}\left(\operatorname{Crit}_G^{\Delta}(\mathfrak{PO})\right) = \operatorname{trop} V(f) \cap \operatorname{int} \Delta$ is shown in Figure 6.4.

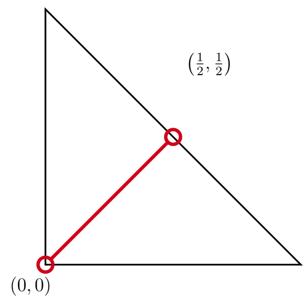


Figure 6.4: Case when $k_2 - k_1 = 0$

Moreover,

$$\begin{aligned}
\operatorname{Crit}_G^{\Delta}(\mathfrak{PO}) &= Y \cap \operatorname{trop}^{-1}(\operatorname{int}\Delta) \\
&= \left\{ (y_1, y_1) \in \Lambda^2 \,\middle|\, \operatorname{val}(y_1) > 0, \operatorname{val}\left(\frac{T}{y_1^2}\right) > 0, y_1 \neq 0 \right\} \\
&\cong \left\{ y_1 \in B_\Lambda^1 \,\middle|\, e^{-\frac{1}{2}} < |y_1| < 1 \right\} \subset \operatorname{Sp} \Lambda \left\langle y_1, T y_1^{-2} \right\rangle
\end{aligned}$$

is an annulus.

6.2.2 S^1-action on a one-point blowup of $\mathbb{CP}^2$

Example 6.2 (S^1-action on a one-point blowup of $\mathbb{CP}^2$)**.** Consider the one-point blowup $(\mathbb{CP}^2(1), \omega, T^2, \mu)$ of $\mathbb{CP}^2$ whose moment polytope is given by

$$\Delta = \left\{ (u_1, u_2) \in \mathbb{R}^2 \,\middle|\, \begin{array}{l} u_i \geq 0 \quad \forall 1 \leq i \leq 2, \\ 1 - u_1 - u_2 \geq 0 \\ 1 - \alpha - u_2 \geq 0 \end{array} \right\}.$$

Its potential function is

$$\mathfrak{PO} = y_1 + y_2 + \frac{T}{y_1 y_2} + \frac{T^{1-\alpha}}{y_2}.$$

$$\begin{aligned}\implies f := \frac{\partial \mathfrak{PO}}{\partial c_2} &= -k_2 y_1 \frac{\partial \mathfrak{PO}}{\partial y_1} + k_1 y_2 \frac{\partial \mathfrak{PO}}{\partial y_2} \\ &= -k_2 \left(y_1 - \frac{T}{y_1 y_2} \right) + k_1 \left(y_2 - \frac{T}{y_1 y_2} - \frac{T^{1-\alpha}}{y_2} \right) \\ &= -k_2 y_1 + k_1 y_2 + (k_2 - k_1) \frac{T}{y_1 y_2} - k_1 \frac{T^{1-\alpha}}{y_2}.\end{aligned}$$

1. **Suppose $k_1, k_2, k_2 - k_1$ are all non-zero.**

We now consider the tropicalization of $\mathrm{Crit}_G^{\Delta}(\mathfrak{PO})$.

$$\mathrm{trop}(f) : \mathbb{R}^2 \to \mathbb{R}, \quad u \mapsto \min\{u_1, u_2, 1 - u_1 - u_2, 1 - \alpha - u_2\}.$$

a) $u_1 = u_2 \le \min\{1 - u_1 - u_2, 1 - \alpha - u_2\}$

$$\implies \begin{cases} u_1 = u_2 \\ u_1 \le 1 - 2u_1 \\ u_1 \le 1 - \alpha - u_1 \end{cases} \Rightarrow \begin{cases} u_2 &= u_1, \\ u_1 &\le \min\{\frac{1}{3}, \frac{1-\alpha}{2}\} \\ &= \begin{cases} \frac{1}{3} & \text{if } \alpha \le \frac{1}{3} \\ \frac{1-\alpha}{2} & \text{if } \alpha \ge \frac{1}{3} \end{cases} \end{cases}$$

b) $u_1 = 1 - u_1 - u_2 \le \min\{u_2, 1 - \alpha - u_2\}$

$$\implies \begin{cases} u_2 = 1 - 2u_1 \\ u_1 \le 1 - 2u_1 \\ u_1 \le 1 - \alpha - (1 - 2u_1) \end{cases} \Rightarrow \begin{cases} u_2 = 1 - 2u_1 \\ \alpha \le u_1 \le \frac{1}{3}, \end{cases}$$

which can happen only if $\alpha \le \frac{1}{3}$.

c) $u_1 = 1 - \alpha - u_2 \le \min\{1 - u_1 - u_2, u_2\}$

$$\implies \begin{cases} u_2 = 1 - \alpha - u_1 \\ u_1 \le 1 - u_1 - (1 - \alpha - u_1) \\ u_1 \le 1 - \alpha - u_1 \end{cases} \Rightarrow \begin{cases} u_2 &= 1 - \alpha - u_1 \\ u_1 &\le \min\{\alpha, \frac{1-\alpha}{2}\} \\ &= \begin{cases} \alpha & \text{if } \alpha \le \frac{1}{3} \\ \frac{1-\alpha}{2} & \text{if } \alpha \ge \frac{1}{3} \end{cases} \end{cases}$$

d) $u_2 = 1 - u_1 - u_2 \le \min\{u_1, 1-\alpha-u_2\}$

$$\Longrightarrow \begin{cases} u_2 = \frac{1-u_1}{2} \\ \frac{1-u_1}{2} \le u_1 \\ \frac{1-u_1}{2} \le 1-\alpha-\frac{1-u_1}{2} \end{cases} \Rightarrow \begin{cases} u_2 & = \frac{1-u_1}{2} \\ u_1 & \ge \max\{\alpha, \frac{1}{3}\} \\ & = \begin{cases} \frac{1}{3} & \text{if } \alpha \le \frac{1}{3} \\ \alpha & \text{if } \alpha \ge \frac{1}{3} \end{cases} \end{cases}$$

e) $u_2 = 1 - \alpha - u_2 \le \min\{u_1, 1-u_1-u_2\}$

$$\Longrightarrow \begin{cases} u_2 = \frac{1-\alpha}{2} \\ \frac{1-\alpha}{2} \le u_1 \\ \frac{1-\alpha}{2} \le 1-u_1-\frac{1-\alpha}{2} \end{cases} \Rightarrow \begin{cases} u_2 = \frac{1-\alpha}{2} \\ \frac{1-\alpha}{2} \le u_1 \le \alpha, \end{cases}$$

which can happen only if $\alpha \ge \frac{1}{3}$.

f) $1 - u_1 - u_2 = 1 - \alpha - u_2 \le \min\{u_1, u_2\}$

$$\Longrightarrow \begin{cases} u_1 = \alpha \\ 1-\alpha-u_2 \le \alpha \\ 1-\alpha-u_2 \le u_2 \end{cases} \Rightarrow \begin{cases} u_1 & = \alpha \\ u_2 & \ge \max\{1-2\alpha, \frac{1-\alpha}{2}\} \\ & = \begin{cases} 1-2\alpha & \text{if } \alpha \le \frac{1}{3} \\ \frac{1-\alpha}{2} & \text{if } \alpha \ge \frac{1}{3} \end{cases} \end{cases}$$

i) The case when $0 < \alpha < \frac{1}{3}$ is shown in Figure 6.5.

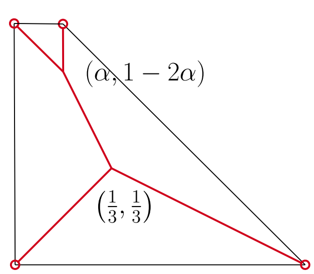


Figure 6.5: Case when $k_1, k_2, k_2 - k_1 \neq 0$ and $0 < \alpha < \frac{1}{3}$

ii) The case when $\alpha = \frac{1}{3}$ is shown in Figure 6.6.

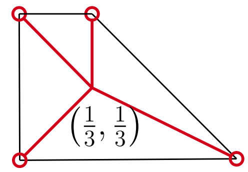


Figure 6.6: Case when $k_1, k_2, k_2 - k_1 \neq 0$ and $\alpha = \frac{1}{3}$

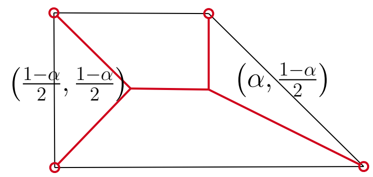


Figure 6.7: Case when $k_1, k_2, k_2 - k_1 \neq 0$, $\frac{1}{3} < \alpha < 1$

iii) The case when $1/3 < \alpha < 1$ is shown in Figure 6.7.

We have

$$
\begin{aligned}
W :=& V(f) \cap \operatorname{trop}^{-1}(\Delta) \\
=& \operatorname{Sp} \frac{\Lambda \left\langle y_1, y_2, \frac{T}{y_1 y_2}, \frac{T^{1-\alpha}}{y_2} \right\rangle}{\left(-k_2 y_1 + a y_2 + (k_2 - k_1)\frac{T}{y_1 y_2} - k_1 \frac{T^{1-\alpha}}{y_2}\right)} \\
\cong& \operatorname{Sp} \frac{\Lambda \langle y_1, y_2, z, x \rangle}{\begin{pmatrix} -k_2 y_1 + k_1 y_2 + (k_2 - k_1) z - k_1 x, \\ y_1 y_2 z - T, y_2 x - T^{1-\alpha} \end{pmatrix}},
\end{aligned}
$$

and

$$
\operatorname{Crit}_G^{\Delta}(\mathfrak{PO}) = W \setminus \operatorname{trop}^{-1}\left(\{(0,0), (1,0), (0, 1-\alpha), (\alpha, 1-\alpha)\}\right).
$$

2. **Suppose** $k_1 = 0$ **and** $k_2, k_2 - k_1 \neq 0$**.** Then $f = -k_2\left(y_1 - \frac{T}{y_1 y_2}\right)$ and

$$
\begin{aligned}
\operatorname{Crit}_G^{\Delta}(\mathfrak{PO}) :=& V(f) \cap \operatorname{trop}^{-1}(\operatorname{int} \Delta) \\
=& \left\{ (y_1, y_2) \in (\Lambda^*)^2 \;\middle|\; \begin{array}{l} -k_2 y_1 + k_2 \dfrac{T}{y_1 y_2} = 0 \\ \operatorname{val}(y_1) > 0, \operatorname{val}(y_2) > 0, \\ \operatorname{val}\left(\dfrac{T^{1-\alpha}}{y_2}\right) > 0, \operatorname{val}\left(\dfrac{T}{y_1 y_2}\right) > 0, \end{array} \right\}
\end{aligned}
$$

$$= \left\{ \left(y_1, \frac{T}{y_1^2}\right) \in (\Lambda^*)^2 \;\middle|\; \begin{array}{l} \operatorname{val}(y_1) > 0, \operatorname{val}\left(\frac{T}{y_1^2}\right) > 0, \\ \operatorname{val}\left(\frac{T}{y_1 T / y_1^2}\right) > 0, \operatorname{val}\left(\frac{T^{1-\alpha}}{T/y_1^2}\right) > 0 \end{array} \right\}$$

$$= \left\{ \left(y_1, \frac{T}{y_1^2}\right) \in (B^1_\Lambda)^2 \;\middle|\; \begin{array}{l} |y_1| < 1, |y_1| > e^{-\frac{1}{2}}, \\ |y_1| < e^{-\frac{\alpha}{2}} \end{array} \right\}$$

$$= \left\{ \left(y_1, \frac{T}{y_1^2}\right) \in \Lambda^2 \;\middle|\; e^{-\frac{1}{2}} < |y_1| < e^{-\frac{\alpha}{2}} \right\}$$

is an annulus.

Moreover, we have

$$\operatorname{trop} f : \mathbb{R}^2 \to \mathbb{R}, \quad u \mapsto \min\{u_1, 1 - u_1 - u_2\}.$$

$\operatorname{trop}(\operatorname{Crit}^\Delta_G(\mathfrak{PO})) = \operatorname{trop} V(f) \cap \operatorname{int} \Delta$ is shown in Figure 6.8.

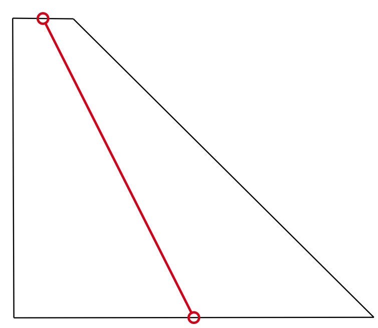


Figure 6.8: Case when $k_1 = 0$

3. **Suppose** $k_2 = 0$ **and** $k_1, k_2 - k_1 \neq 0$**.** Then

$$f = k_1 \left(y_2 - \frac{T}{y_1 y_2} - \frac{T^{1-\alpha}}{y_2} \right).$$

We have

$$\operatorname{trop}(f) = \operatorname{trop}\left(y_2 - \frac{T}{y_1 y_2} - \frac{T^{1-\alpha}}{y_2} \right) : \mathbb{R}^2 \to \mathbb{R}, \quad u \mapsto \min\{u_2, 1 - u_1 - u_2, 1 - \alpha - u_2\}.$$

Thus,

$$V(\operatorname{trop} f) = \{u_2 = 1 - u_1 - u_2 \leq 1 - \alpha - u_2\} \cup \{1 - u_1 - u_2 = 1 - \alpha - u_2 \leq u_2\}$$

$$\cup \{u_2 = 1-\alpha-u_2 \le 1-u_1-u_2\}$$
$$=\{u_2 = \frac{1-u_1}{2}, u_1 \ge \alpha\} \cup \{u_1 = \alpha, u_2 \ge \frac{1-\alpha}{2}\} \cup \{u_2 = \frac{1-\alpha}{2}, u_1 \le \alpha\}.$$

The set $\mathrm{trop}(\mathrm{Crit}_G^\Delta(\mathfrak{PO}))$ is shown in Figure 6.9.

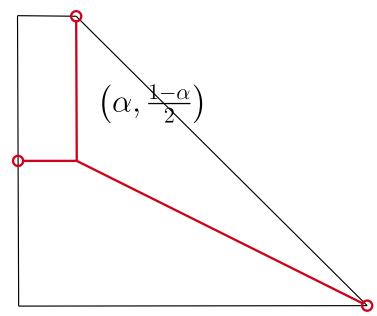


Figure 6.9: Case when $k_2 = 0$

Moreover,

$$\begin{aligned} W :=& V(f) \cap \mathrm{trop}^{-1}(\Delta) \\ =& \mathrm{Sp}\, \frac{\Lambda\left\langle y_1, y_2, \frac{T}{y_1y_2}, \frac{T^{1-\alpha}}{y_2}\right\rangle}{\left(a\left(y_2 - \frac{T}{y_1y_2} - \frac{T^{1-\alpha}}{y_2}\right)\right)} \\ \cong& \mathrm{Sp}\, \frac{\Lambda\langle y_1, y_2, z, x\rangle}{(y_2 - z - x, y_1y_2z - T, y_2x - T^{1-\alpha})}, \end{aligned}$$

and $\mathrm{Crit}_G^\Delta(\mathfrak{PO}) = W \setminus \mathrm{trop}^{-1}\left(\left\{\left(0, \frac{1-\alpha}{2}\right), (\alpha, 1-\alpha), (1,0)\right\}\right)$.

4. **Suppose $k_2 - k_1 = 0$ and $k_1, k_2 \neq 0$.**

We have

$$\mathrm{trop}(f) = \mathrm{trop}(-y_1 + y_2 - \frac{T^{1-\alpha}}{y_2}) : \mathbb{R}^2 \to \mathbb{R}, \quad u \mapsto \min\{u_1, u_2, 1-\alpha-u_2\}.$$

Thus,

$$\begin{aligned} V(\mathrm{trop}\, f) =& \{u_1 = u_2 \le 1-\alpha-u_2\} \cup \{u_1 = 1-\alpha-u_2 \le u_2\} \\ & \cup \{u_2 = 1-\alpha-u_2 \le u_1\} \\ =& \{u_1 = u_2, u_1 \le \frac{1-\alpha}{2}\} \cup \{u_2 = 1-\alpha-u_1, u_1 \le \frac{1-\alpha}{2}\} \end{aligned}$$

$$\cup \{u_2 = \frac{1-\alpha}{2}, u_1 \geq \frac{1-\alpha}{2}\}$$

The set $\operatorname{trop}(\operatorname{Crit}_G^\Delta(\mathfrak{PO})) = \operatorname{trop} V(f) \cap \operatorname{int} \Delta$ in Figure 6.10.

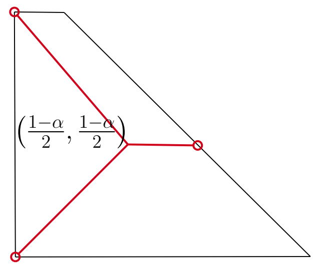


Figure 6.10: Case $k_2 - k_1 = 0$

Moreover,

$$\begin{aligned} W :=& V(f) \cap \operatorname{trop}^{-1}(\Delta) \\ =& \operatorname{Sp} \frac{\Lambda \left\langle y_1, y_2, \frac{T}{y_1 y_2}, \frac{T^{1-\alpha}}{y_2} \right\rangle}{(-y_1 + y_2 - \frac{T^{1-\alpha}}{y_2})} \\ \cong& \operatorname{Sp} \frac{\Lambda \langle y_1, y_2, z, x \rangle}{(-y_1 + y_2 - x, y_1 y_2 z - T, y_2 x - T^{1-\alpha})}, \end{aligned}$$

and $\operatorname{Crit}_G^\Delta(\mathfrak{PO}) = W \setminus \operatorname{trop}^{-1}\left(\left\{(0,0), (0, 1-\alpha), \left(\frac{1+\alpha}{2}, \frac{1-\alpha}{2}\right)\right\}\right)$.

6.2.3 S^1-action on a two-point blowup of $\mathbb{CP}^2$

Example 6.3 (S^1-action on a two-point blowup of $\mathbb{CP}^2$)**.** Consider the two-point blowup $(\mathbb{CP}^2(2), \omega, T^2, \mu)$ of $\mathbb{CP}^2$ whose moment polytope is given by

$$\Delta = \left\{(u_1, u_2) \in \mathbb{R}^2 \,\middle|\, -1 \leq u_1 \leq 1, -1 \leq u_2 \leq 1, u_1 + u_2 \leq 1 + \alpha\right\},$$

where $-1 < \alpha < 1$. Its potential function is

$$\mathfrak{PO} = Ty_1 + Ty_2 + \frac{T^{1+\alpha}}{y_1 y_2} + \frac{T}{y_1} + \frac{T}{y_2}.$$

$$\Longrightarrow 0 = f := \frac{\partial \mathfrak{PO}}{\partial c_2} = -k_2 y_1 \frac{\partial \mathfrak{PO}}{\partial y_1} + k_1 y_2 \frac{\partial \mathfrak{PO}}{\partial y_2}$$

$$= T\left(-k_2\left(y_1 - \frac{T^\alpha}{y_1y_2} - \frac{1}{y_1}\right) + k_1\left(y_2 - \frac{T^\alpha}{y_1y_2} - \frac{1}{y_2}\right)\right)$$
$$= T\left(-k_2y_1 + k_1y_2 + (k_2-k_1)\frac{T^\alpha}{y_1y_2} + \frac{k_2}{y_1} - \frac{k_1}{y_2}\right)$$

1. **Suppose $k_1, k_2, k_2 - k_1$ are all non-zero.**

   We have

   $$\operatorname{trop}(f): \mathbb{R}^2 \to \mathbb{R}, \quad u \mapsto \min\{1+u_1, 1+u_2, 1+\alpha-u_1-u_2, 1-u_1, 1-u_2\}.$$

   a) $u_1 = u_2 \le \min\{\alpha - u_1 - u_2, -u_1, -u_2\}$.

   $$\implies u_2 = u_1 \le \min\{\frac{\alpha}{3}, 0\} = \begin{cases} \frac{\alpha}{3} & \text{if } \alpha \le 0 \\ 0 & \text{if } \alpha \ge 0 \end{cases}.$$

   b) $u_1 = \alpha - u_1 - u_2 \le \min\{u_2, -u_1, -u_2\}$.

   $$\implies \begin{cases} u_2 = \alpha - 2u_1 \\ u_1 \le \alpha - 2u_1 \\ u_1 \le -u_1 \\ u_1 \le 2u_1 - \alpha \end{cases} \Rightarrow \begin{cases} u_2 = \alpha - 2u_1 \\ \alpha \le u_1 \le \min\{\frac{\alpha}{3}, 0\} \end{cases} \Rightarrow \begin{cases} u_2 = \alpha - 2u_1 \\ \alpha \le u_1 \le \frac{\alpha}{3} \\ \alpha \le 0 \end{cases}.$$

   c) $u_1 = -u_1 \le \min\{u_2, \alpha - u_1 - u_2, -u_2\}$.

   $$\implies \begin{cases} u_1 = 0 \\ u_2 \ge 0 \\ \alpha - u_2 \ge 0 \\ -u_2 \ge 0 \end{cases} \Rightarrow \begin{cases} u_1 = u_2 = 0 \\ \alpha \ge 0 \end{cases}.$$

   d) $u_1 = -u_2 \le \min\{\alpha - u_1 - u_2, -u_1, u_2\}$.

   $$\implies \begin{cases} u_2 = -u_1 \\ u_1 \le \alpha \\ u_1 \le -u_1 \end{cases} \Rightarrow \begin{cases} u_2 = -u_1 \\ u_1 \le \min\{0, \alpha\} = \begin{cases} \alpha & \text{if } \alpha \le 0 \\ 0 & \text{if } \alpha \ge 0 \end{cases} \end{cases}.$$

e) $u_2 = \alpha - u_1 - u_2 \leq \min\{u_1, -u_1, -u_2\}$.

$$\Longrightarrow \begin{cases} u_2 = \frac{\alpha - u_1}{2} \\ \frac{\alpha - u_1}{2} \leq u_1 \\ \frac{\alpha - u_1}{2} \leq -u_1 \\ \frac{\alpha - u_1}{2} \leq -\frac{\alpha - u_1}{2} \end{cases} \Rightarrow \begin{cases} u_2 = \frac{\alpha - u_1}{2} \\ \max\{\alpha, \frac{\alpha}{3}\} \leq u_1 \leq -\alpha \end{cases} \Rightarrow \begin{cases} u_2 = \frac{\alpha - u_1}{2} \\ \frac{\alpha}{3} \leq u_1 \leq -\alpha \\ \alpha \leq 0 \end{cases} \cdot$$

f) $u_2 = -u_1 \leq \min\{\alpha - u_1 - u_2, u_1, -u_2\}$.

$$\Longrightarrow \begin{cases} u_2 = -u_1 \\ -u_1 \leq \alpha \\ -u_1 \leq u_1 \end{cases} \Rightarrow \begin{cases} u_2 = -u_1 \\ u_1 \geq \max\{0, -\alpha\} = \begin{cases} -\alpha & \text{if } \alpha \leq 0 \\ 0 & \text{if } \alpha \geq 0 \end{cases} \cdot \end{cases}$$

g) $u_2 = -u_2 \leq \min\{\alpha - u_1 - u_2, u_1, -u_1\}$.

$$\Longrightarrow \begin{cases} u_2 = 0 \\ 0 \leq u_1 \\ 0 \leq \alpha - u_1 \\ 0 \leq -u_1 \end{cases} \Rightarrow \begin{cases} u_1 = u_2 = 0 \\ \alpha \geq 0 \end{cases}.$$

h) $\alpha - u_1 - u_2 = -u_1 \leq \min\{u_1, u_2, -u_2\}$.

$$\Longrightarrow \begin{cases} u_2 = \alpha \\ -u_1 \leq u_1 \\ -u_1 \leq \alpha \\ -u_1 \leq -\alpha \end{cases} \Rightarrow \begin{cases} u_2 = \alpha \\ u_1 \geq \max\{0, \alpha, -\alpha\} = \begin{cases} -\alpha & \text{if } \alpha \leq 0 \\ \alpha & \text{if } \alpha \geq 0 \end{cases} \cdot \end{cases}$$

i) $\alpha - u_1 - u_2 = -u_2 \le \min\{u_1, u_2, -u_1\}$.

$$\Longrightarrow \begin{cases} u_1 = \alpha \\ -u_2 \le \alpha \\ -u_2 \le u_2 \\ -u_2 \le -\alpha \end{cases} \Rightarrow \begin{cases} u_1 = \alpha \\ u_2 \ge \max\{0, \alpha, -\alpha\} = \begin{cases} -\alpha & \text{if } \alpha \le 0 \\ \alpha & \text{if } \alpha \ge 0 \end{cases} \end{cases}$$

j) $-u_1 = -u_2 \le \min\{u_1, u_2, \alpha - u_1 - u_2\}$.

$$\Longrightarrow \begin{cases} u_1 = u_2 \\ -u_1 \le u_1 \\ -u_1 \ \le \alpha - 2u_1 \end{cases} \Rightarrow \begin{cases} u_1 = u_2 \\ u_1 \ge 0 \\ u_1 \le \alpha \end{cases} \Rightarrow \begin{cases} u_1 = u_2 \\ 0 \le u_1 \le \alpha \cdot \\ \alpha \ge 0 \end{cases}$$

1) The case $-1 < \alpha < 0$ is shown in Figure 6.11.

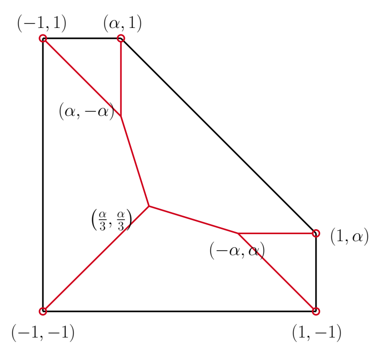


Figure 6.11: Case when $k_1, k_2, k_2 - k_1 \neq 0, -1 < \alpha < 0$

2) The case $\alpha = 0$ is shown in Figure 6.12.

3) The case $\alpha > 0$ is shown in Figure 6.13.

Then

$$\begin{aligned} W :=& V(f) \cap \operatorname{trop}^{-1}(\Delta) \\ =& \operatorname{Sp} \frac{\Lambda \left\langle Ty_1, Ty_2, \frac{T^{1+\alpha}}{y_1 y_2}, \frac{T}{y_1}, \frac{T}{y_2} \right\rangle}{\left(T \left(-k_2 y_1 + k_1 y_2 + (k_2 - k_1) \frac{T^\alpha}{y_1 y_2} + \frac{k_2}{y_1} - \frac{k_1}{y_2} \right) \right)} \end{aligned}$$

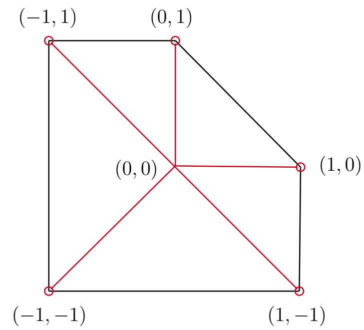


Figure 6.12: Case when $k_1, k_2, k_2 - k_1 \neq 0, \alpha = 0$

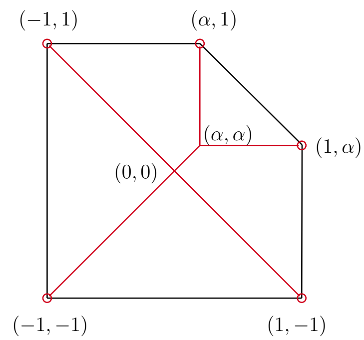


Figure 6.13: Case when $k_1, k_2, k_2 - k_1 \neq 0, 0 < \alpha < 1$

$$\cong \mathrm{Sp}\,\frac{\Lambda\langle z_1, z_2, z, x_1, x_2\rangle}{\begin{pmatrix} -k_2 z_1 + k_1 z_2 + (k_2 - k_1)z + k_2 x_1 - k_1 x_2, \\ z_1 z_2 z - T^{3+\alpha}, x_1 z_1 - T^2, x_2 z_2 - T^2 \end{pmatrix}},$$

and $\mathrm{Crit}_G^{\Delta}(\mathfrak{PO}) = W \cap \mathrm{trop}^{-1}(\{(-1,-1),(-1,1),(\alpha,1),(1,\alpha),(1,-1)\})$.

2. **Suppose** $k_1 = 0$ **and** $k_2, k_2 - k_1 \neq 0$**.** Then

$$f = -k_2 T\left(y_1 - \frac{T^\alpha}{y_1 y_2} - \frac{1}{y_1}\right)$$

and

$$\mathrm{trop}\, f : \mathbb{R}^2 \to \mathbb{R}, \qquad (u_1, u_2) \mapsto \min\{1 + u_1, 1 + \alpha - u_1 - u_2, 1 - u_1\}.$$

Thus,

$$\begin{aligned}\mathrm{trop}(V(f)) =& V(\mathrm{trop}\, f) \\ =& \{u_1 = \alpha - u_1 - u_2 \leq -u_1\} \cup \{u_1 = -u_1 \leq \alpha - u_1 - u_2\}\end{aligned}$$

$$\cup \{\alpha - u_1 - u_2 = -u_1 \le u_1\}$$

$$=\{u_2 = \alpha - 2u_1, u_1 \le 0\} \cup \{u_1 = 0, u_2 \le \alpha\}$$

$$\cup \{u_2 = \alpha, u_1 \ge 0\}.$$

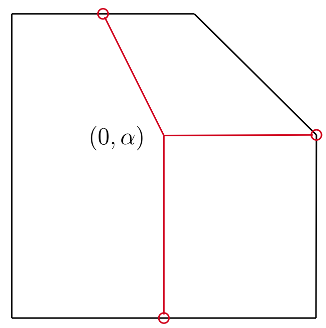


Figure 6.14: Case when $k_1 = 0, k_2, k_2 - k_1 \neq 0$

We have

$$W = V(f) \cap \operatorname{trop}^{-1}(\Delta) = \operatorname{Sp} \frac{\Lambda \left\langle Ty_1, Ty_2, \frac{T^{1+\alpha}}{y_1y_2}, \frac{T}{y_1}, \frac{T}{y_2} \right\rangle}{\left(-k_2T\left(y_1 - \frac{T^\alpha}{y_1y_2} - \frac{1}{y_1}\right)\right)}$$

$$\cong \operatorname{Sp} \frac{\Lambda \langle z_1, z_2, z, x_1, x_2 \rangle}{\begin{pmatrix} z_1 - z_2 - x_1, z_1x_1 - T^2, \\ z_2x_2 - T^2, z_1z_2z - T^{3+\alpha} \end{pmatrix}},$$

and

$$\operatorname{Crit}_G^\Delta(\mathfrak{PO}) = W \setminus \operatorname{trop}^{-1}\left\{\left(\frac{\alpha - 1}{2}, 1\right), (0, -1), (1, \alpha)\right\}.$$

3. **Suppose** $k_2 = 0$ **and** $k_1, k_2 - k_1 \neq 0$**.** Then

$$f = k_1T\left(y_2 - \frac{T^\alpha}{y_1y_2} - \frac{1}{y_2}\right)$$

and

$$\operatorname{trop} f : \mathbb{R}^2 \to \mathbb{R}, \qquad (u_1, u_2) \mapsto \min\{1 + u_2, 1 + \alpha - u_1 - u_2, 1 - u_2\}.$$

$$\operatorname{trop}(V(f)) = V(\operatorname{trop} f)$$

$$
\begin{aligned}
=&\{u_2 = \alpha - u_1 - u_2 \le -u_2\} \cup \{u_2 = -u_2 \le \alpha - u_1 - u_2\} \\
&\cup \{\alpha - u_1 - u_2 = -u_2 \le u_2\} \\
=&\{u_2 = \frac{\alpha - u_1}{2}, u_1 \ge \alpha\} \cup \{u_2 = 0, u_1 \le \alpha\} \\
&\cup \{u_1 = \alpha, u_2 \ge 0\}
\end{aligned}
$$

The case $k_2 = 0, k_1, k_2 - k_1 \neq 0$ is shown in Figure 6.15. We have

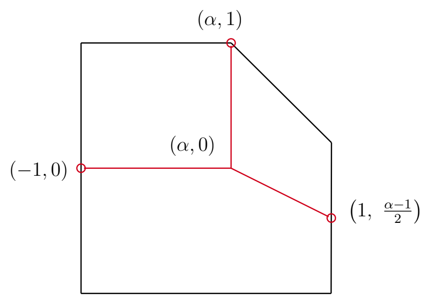


Figure 6.15: Case when $k_2 = 0, k_2 - k_1 \neq 0$

$$
\begin{aligned}
W = V(f) \cap \operatorname{trop}^{-1}(\Delta) &= \operatorname{Sp} \frac{\Lambda \left\langle Ty_1, Ty_2, \frac{T^{1+\alpha}}{y_1 y_2}, \frac{T}{y_1}, \frac{T}{y_2} \right\rangle}{\left(k_1 T \left(y_2 - \frac{T^\alpha}{y_1 y_2} - \frac{1}{y_2}\right)\right)} \\
&\cong \operatorname{Sp} \frac{\Lambda \langle z_1, z_2, z, x_1, x_2 \rangle}{(z_2 - z - x_2, z_2 x_2 - T^2,)},
\end{aligned}
$$

and

$$
\operatorname{Crit}^{\Delta}_{G}(\mathfrak{PO}) = W \setminus \operatorname{trop}^{-1} \left\{ \left(1, \frac{\alpha - 1}{2}\right), (-1, 0), (\alpha, 1) \right\}.
$$

4. **Suppose** $k_2 - k_1 = 0$ **and** $k_1, k_2 \neq 0$**.** Then

$$
f = T \left(-k_2 y_1 + k_1 y_2 + \frac{k_2}{y_1} - \frac{k_1}{y_2}\right)
$$

and

$$
\operatorname{trop} f : \mathbb{R}^2 \to \mathbb{R}, \qquad (u_1, u_2) \mapsto \min\{u_1, u_2, -u_1, -u_2\}.
$$

Thus,

$$
\operatorname{trop}(V(f)) = V(\operatorname{trop} f)
$$

$$
\begin{aligned}
=&\{u_1 = u_2 \leq \min\{-u_1, -u_2\}\} \cup \{u_1 = -u_1 \leq \min\{u_2, -u_2\}\} \\
&\cup \{u_1 = -u_2 \leq \min\{u_2, -u_1\}\} \cup \{u_2 = -u_1 \leq \min\{u_1, -u_2\}\} \\
&\cup \{u_2 = -u_2 \leq \min\{u_1, -u_1\}\} \cup \{-u_1 = -u_2 \leq \min\{u_1, u_2\}\} \\
=&\{u_1 = u_2 \leq 0\} \cup \{u_1 = u_2 = 0\} \\
&\cup \{u_2 = -u_1, u_1 \leq 0\} \cup \{u_2 = -u_1, u_1 \geq 0\} \\
&\cup \{u_2 = u_1 = 0\} \cup \{u_1 = u_2 \geq 0\}.
\end{aligned}
$$

The case $k_2 - k_1 = 0, k_1, k_2 \neq 0$ is shown in Figure 6.16.

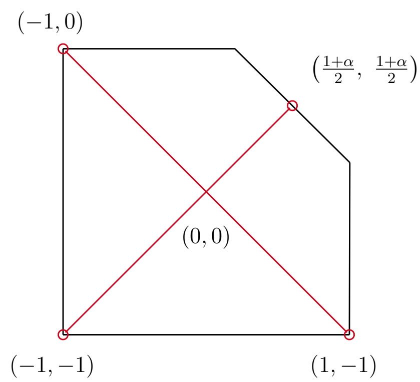


Figure 6.16: Case when $k_2 - k_1 = 0, k_1, k_2 \neq 0$

We have

$$
\begin{aligned}
W = V(f) \cap \operatorname{trop}^{-1}(\Delta) &\cong \operatorname{Sp} \frac{\Lambda \left\langle Ty_1, Ty_2, \frac{T^{1+\alpha}}{y_1 y_2}, \frac{T}{y_1}, \frac{T}{y_2} \right\rangle}{\left(T\left(-k_2 y_1 + k_1 y_2 + \frac{k_2}{y_1} - \frac{k_1}{y_2}\right)\right)} \\
&\cong \operatorname{Sp} \frac{\Lambda \langle z_1, z_2, z, x_1, x_2 \rangle}{\begin{pmatrix} -k_2 z_1 + k_1 z_2 + k_2 x_1 - k_1 x_2, \\ z_1 x_1 - T^2, z_2 x_2 - T^2, z_1 z_2 z - T^{3+\alpha} \end{pmatrix}},
\end{aligned}
$$

and

$$
\operatorname{Crit}_G^{\Delta}(\mathfrak{PO}) = W \setminus \operatorname{trop}^{-1}\left(\left\{\left(\frac{\alpha+1}{2}, \frac{\alpha+1}{2}\right), (-1,0), (-1,-1), (1,-1)\right\}\right).
$$

6.2.4 S^1-action on $S^2\left(\frac{c}{2}\right) \times S^2\left(\frac{d}{2}\right)$

Example 6.4 (S^1-action on $S^2\left(\frac{c}{2}\right) \times S^2\left(\frac{d}{2}\right)$, $c < d$)**.** Denote by $S^2(r)$ the 2-sphere with radius r. Consider $\left(S^2\left(\frac{c}{2}\right) \times S^2\left(\frac{d}{2}\right), \omega, T^2, \mu\right)$ whose moment polytope is given by

$$\Delta = \left\{(u_1, u_2) \in \mathbb{R}^2 \,\middle|\, 0 \leq u_1 \leq c, 0 \leq u_2 \leq d\right\}.$$

Its potential function is

$$\mathfrak{PO} = y_1 + y_2 + \frac{T^c}{y_1} + \frac{T^d}{y_2}.$$

$$\begin{aligned}\Longrightarrow 0 = f :=& \frac{\partial \mathfrak{PO}}{\partial c_2} = -k_2 y_1 \frac{\partial \mathfrak{PO}}{\partial y_1} + k_1 y_2 \frac{\partial \mathfrak{PO}}{\partial y_2} \\ =& -k_2\left(y_1 - \frac{T^c}{y_1}\right) + k_1\left(y_2 - \frac{T^d}{y_2}\right).\end{aligned}$$

1. **Suppose $k_1, k_2 \neq 0$.** We have

$$\operatorname{trop}(f) : \mathbb{R}^2 \to \mathbb{R}, \qquad (u_1, u_2) \mapsto \min\{u_1, c - u_1, u_2, d - u_2\}.$$

a) $u_1 = c - u_1 \leq \min\{u_2, d - u_2\}$

$$\Longrightarrow \begin{cases} u_1 = \frac{c}{2} \\ \frac{c}{2} \leq u_2 \\ \frac{c}{2} \leq d - u_2 \end{cases} \Rightarrow \begin{cases} u_1 = \frac{c}{2} \\ \frac{c}{2} \leq u_2 \leq d - \frac{c}{2} \end{cases}$$

b) $u_1 = u_2 \leq \min\{c - u_1, d - u_2\}$

$$\Longrightarrow \begin{cases} u_1 = u_2 \\ u_1 \leq c - u_1 \\ u_1 \leq d - u_1 \end{cases} \Rightarrow \begin{cases} u_1 = u_2 \\ u_1 \leq \frac{c}{2} \end{cases}$$

c) $u_1 = d - u_2 \leq \min\{c - u_1, u_2\}$

$$\Longrightarrow \begin{cases} u_2 = d - u_1 \\ u_1 \leq c - u_1 \\ u_1 \leq d - u_1 \end{cases} \Rightarrow \begin{cases} u_2 = d - u_1 \\ u_1 \leq \frac{c}{2} \end{cases}$$

d) $c - u_1 = u_2 \le \min\{u_1, d - u_2\}$

$$\Longrightarrow \begin{cases} u_2 = c - u_1 \\ c - u_1 \le u_1 \\ c - u_1 \le d - (c - u_1) \end{cases} \Rightarrow \begin{cases} u_2 = c - u_1 \\ u_1 \ge \frac{c}{2} \end{cases}$$

e) $c - u_1 = d - u_2 \le \min\{u_1, u_2\}$

$$\Longrightarrow \begin{cases} u_2 = d - c + u_1 \\ c - u_1 \le u_1 \\ c - u_1 \le d - c + u_1 \end{cases} \Rightarrow \begin{cases} u_2 = d - c + u_1 \\ u_1 \ge \frac{c}{2} \end{cases}$$

f) $u_2 = d - u_2 \le \min\{u_1, c - u_1\}$

$$\Longrightarrow \begin{cases} u_2 = \frac{d}{2} \\ \frac{d}{2} \le u_1 \\ \frac{d}{2} \le c - u_1 \end{cases} \Rightarrow \begin{cases} u_2 = \frac{d}{2} \\ \frac{d}{2} \le u_1 \le c - \frac{d}{2} \end{cases},$$

which cannot happen because $c < d$.

The set $\operatorname{trop}(V(f)) \cap \operatorname{int} \Delta$ is shown in Figure 6.17

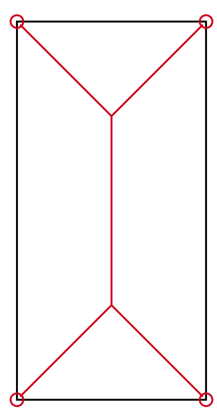


Figure 6.17: Case when $k_1, k_2 \neq 0$

We have

$$W = V(f) \cap \operatorname{trop}^{-1}(\Delta) = \operatorname{Sp} \frac{\Lambda \left\langle y_1, y_2, \frac{T^c}{y_1}, \frac{T^d}{y_2} \right\rangle}{\left(-k_2 \left(y_1 - \frac{T^c}{y_1}\right) + k_1 \left(y_2 - \frac{T^d}{y_2}\right)\right)}$$

$$\cong \mathrm{Sp} \frac{\Lambda\langle y_1, y_2, x_1, x_2\rangle}{\begin{pmatrix} -k_2(y_1 - x_1) + k_1(y_2 - x_2), \\ x_1y_1 - T^c, x_2y_2 - T^d \end{pmatrix}},$$

and

$$\mathrm{Crit}_G^{\Delta}(\Delta) = W \setminus \mathrm{trop}^{-1}\left(\{(c,0),(0,0),(0,d),(c,d)\}\right)$$

2. **Suppose $k_1 = 0$ and $k_2 \neq 0$.** Then

$$f = -k_2\left(y_1 - \frac{T^c}{y_1}\right).$$

Then

$$V(f) = \left\{\left(T^{\frac{c}{2}}, y_2\right) \mid y_2 \in \Lambda^*\right\} \cup \left\{\left(-T^{\frac{c}{2}}, y_2\right) \mid y_2 \in \Lambda^*\right\},$$

$$\mathrm{trop}\, V(f) = \left\{(u_1, u_2) \in \mathbb{R}^2 \,\middle|\, u_1 = \frac{c}{2}\right\},$$

and

$$\mathrm{trop}\left(\mathrm{Crit}_G^{\Delta}(\mathfrak{PO})\right) = \mathrm{int}\,\Delta \cap \mathrm{trop}\, V(f) = \left\{\frac{c}{2}\right\} \times (0, b).$$

It is shown in Figure 6.18.

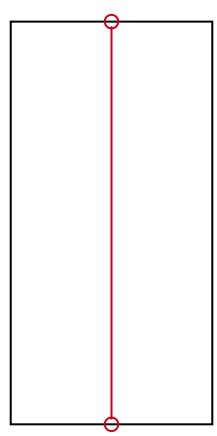


Figure 6.18: Case when $k_1 = 0$

Indeed,

$$\mathrm{Crit}_G^{\Delta}(\mathfrak{PO}) = \mathrm{trop}\, V(f) \cap \mathrm{int}\,\Delta$$

$$= \left\{ (y_1, y_2) \in \Lambda^2 \,\middle|\, \begin{array}{l} y_1 - \dfrac{T^c}{y_1} = 0 \\ 0 < \mathrm{val}(y_1) < c, 0 < \mathrm{val}(y_2) < d \end{array} \right\}$$

$$= \left\{ \pm \frac{c}{2} \right\} \times \left\{ y_2 \in \mathbb{B}^1_\Lambda \,\middle|\, 0 < \mathrm{val}(y_2) < d \right\}$$

is a union of two annuli.

3. **Suppose** $k_2 = 0$ **and** $k_1 \neq 0$**.** Then

$$f = k_1 \left(y_2 - \frac{T^d}{y_2} \right).$$

Then

$$V(f) = \left\{ \left(y_1, T^{\frac{d}{2}} \right) \mid y_1 \in \Lambda^* \right\} \cup \left\{ \left(y_1, -T^{\frac{d}{2}} \right) \mid y_2 \in \Lambda^* \right\},$$

$$\operatorname{trop} V(f) = \left\{ (u_1, u_2) \in \mathbb{R}^2 \,\middle|\, u_2 = \frac{d}{2} \right\}$$

and

$$\operatorname{int} \Delta \cap \operatorname{trop} V(f) = (0, a) \times \left\{ \frac{d}{2} \right\}.$$

It is shown in Figure 6.19.

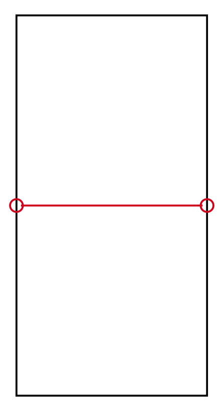


Figure 6.19: Case when $k_2 = 0$

Indeed,

$$\mathrm{Crit}^{\Delta}_{G}(\mathfrak{PO}) = \operatorname{trop} V(f) \cap \operatorname{int} \Delta$$

$$= \left\{ (y_1, y_2) \in \Lambda^2 \;\middle|\; \begin{array}{l} y_2 - \dfrac{T^d}{y_2} = 0 \\ \\ 0 < \mathrm{val}(y_1) < c, 0 < \mathrm{val}(y_2) < d \end{array} \right\}$$

$$= \left\{ y_1 \in \mathbb{B}^1_\Lambda \;\middle|\; 0 < \mathrm{val}(y_1) < d \right\} \times \left\{ \pm \frac{d}{2} \right\}$$

is a union of two annuli.

One can often assign Lagrangian submanifolds to certain tropical curves, which the tropicalization pictures that appear in Section 6.2 are examples of. (See [45], [43], [44], and [14].) In the case when the tropical curves are moment map images of the associated Lagrangians, we are interested in learning about the intersection of the "tropical" Lagrangians and the Lagrangian torus fibers that intersect them.

Chapter 7

Equivariant Hamiltonian isotopy invariance

In this chapter, we show that the equivariant Lagrangian Floer cohomology defined in Chapter 5 is invariant under equivariant Hamiltonian isotopy. As a result, a Lagrangian torus fiber with nontrivial Lagrangian Floer cohomology is not displaceable by the Hamiltonian diffeomorphisms invariant under the given subtorus action.

We first review the definition of an equivariant Hamiltonian isotopy.

Definition 7.1 (Hamiltonian isotopy)**.** Let $H : [0,1] \times X \to \mathbb{R}$ be a time-dependent Hamiltonian function on (X, ω). Denote by H_t the map $H(t, \cdot)$ and X_{H_t} the Hamiltonian vector field of H_t satisfying $\iota_{X_{H_t}} \omega = dH_t$. A **Hamiltonian isotopy** on X generated by H is a smooth map

$$\psi_H : [0,1] \times X \to X, \qquad (t, x) \mapsto \psi_H^t(x)$$

satisfying

$$\frac{d}{dt}\psi_H^t = X_{H_t} \circ \psi_H^t.$$

Definition 7.2 (G-equivariant Hamiltonian isotopy)**.** A Hamiltonian isotopy $\psi_H : [0,1] \times X \to X$ of a symplectic manifold (X, ω) with a symplectic action by a compact Lie group G is a

G-equivariant Hamiltonian isotopy if

$$\psi_H^t(g \cdot x) = g \cdot \psi_H^t(x) \qquad \forall x \in X, \quad \forall g \in G, \quad \forall t \in [0,1].$$

We will define the Floer cohomology $HF_G(L, b, H, \Lambda_{\mathrm{nov}})$ for a Hamiltonian function H satisfying the conditions of the following theorem in Section 7.1.

Theorem 7.1 (G-equivariant Hamiltonian isotopy invariance)**.** Let (X, ω, T^n, μ) be a compact symplectic toric manifold with moment polytope $\Delta = \mu(X) \subset \mathbb{R}^n$. Let $G \hookrightarrow T^n$ be a compact r-dimensional connected subtorus of T^n with the induced action on X. Let $\boldsymbol{u} \in \operatorname{int}\Delta$ and $L = \mu^{-1}(\boldsymbol{u})$. Let $H : [0,1] \times X \to \mathbb{R}$ be a G-invariant time-dependent smooth Hamiltonian function. Let

$$\psi_H : [0,1] \times X \to X, \qquad (t,x) \mapsto \psi_H^t(x)$$

be the G-equivariant Hamiltonian isotopy generated by H such that $\psi_H^0 = \mathrm{id}$ and that $\psi_H^1(L) \cap L$ is a finite union

$$\psi_H^1(L) \cap L = \bigsqcup_{a \in \pi_0\left(\psi_H^1(L) \cap L\right)} R_a,$$

where each $R_a = G \cdot q_a$ for some q_a in the component represented by $a \in \pi_0(\psi_H^1(L) \cap L)$. We fix our choice of q_a for each $a \in \pi_0(\psi_H^1(L) \cap L)$. Let $\boldsymbol{u} \in \operatorname{int}\Delta$. Let $L = \mu^{-1}(\boldsymbol{u})$ and $b \in H^1(L, \Lambda_0)$. Then, over the universal Novikov field, we have

$$HF_G((L,b)\,,\,(L,b), \Lambda_{\mathrm{nov}}) \cong HF_G(L, b, H, \Lambda_{\mathrm{nov}}), \tag{7.0.1}$$

where the right-hand-side is defined in (7.1.2).

Proof Outline. Before proving Theorem 7.1 in detail, we outline the proof below.

i) We first construct a cochain complex $(CF_G(L, b, H), \delta_H^G)$ for a Hamiltonian function H satisfying the conditions of Theorem 7.1.

ii) Let $C_G(L, \Lambda_{\mathrm{nov}}) = \Omega_G(L) \widehat{\otimes} \Lambda_{\mathrm{nov}}$ and $\delta^G = (\mathfrak{m}_1^G)^b$. We show the cochain maps

$$\left(CF_G(L, b, H), \delta_H^G\right) \underset{\mathfrak{g}}{\overset{\mathfrak{f}}{\rightleftharpoons}} \left(C_G(L, \Lambda_{\mathrm{nov}}), \delta^G = (\mathfrak{m}_1^G)^b\right)$$

define a cochain homotopy equivalence; i.e. $\mathfrak{f}\circ\mathfrak{g}\sim \mathrm{id}$ and $\mathfrak{g}\circ\mathfrak{f}\sim \mathrm{id}$.

$\square$

7.1 The cochain complex $\left(CF_G(L,b,H),\delta_H^G\right)$

Denote $\psi_H^1(L)$ by L_H. By our assumption,

$$L_H\cap L=\bigsqcup_{a\in\pi_0(\psi_H^1(L)\cap L)} R_a^H,$$

where each $R_a^H=G\cdot q_a$ is a G-orbit for some q_a in the component $a\in\pi_0(\psi_H^1(L)\cap L)$. Let ψ_H^{-1} be the inverse of ψ_H^1. We have

$$L\cap\psi_H^{-1}(L)=\psi_H^{-1}(\psi_H^1(L)\cap L)=\psi_H^{-1}\left(\bigcup_{a\in\pi_0(\psi_H^1(L)\cap L)} R_a^H\right)=\bigcup_{a\in\pi_0(\psi_H^1(L)\cap L)}\psi_H^{-1}(R_a^H).$$

Define

$$CF_G(L,b,H):=\bigoplus_{a\in\pi_0(L_H\cap L)}\Omega_G(R_a^H)\widehat{\otimes}_{\mathbb{R}}\Lambda_{\mathrm{nov}}\,.$$

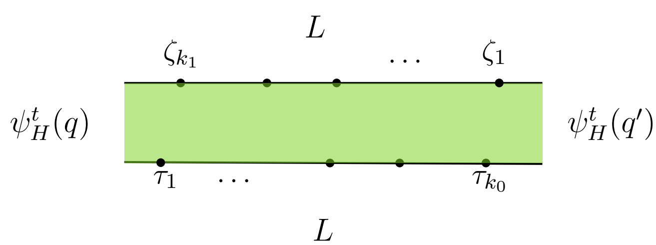
$\forall a,a'\in\pi_0(\psi_H^1(L)\cap L)$, define

$$\pi_2(R_a^H,R_{a'}^H)=\left\{u:\mathbb{R}\times[0,1]\to X\;\middle|\;\begin{array}{l} u \text{ is smooth,}\\ u(s,0)\in L,\quad u(s,1)\in L\quad \forall s\in\mathbb{R},\\ \exists\, q\in\psi_H^{-1}(R_a^H),\quad q'\in\psi_H^{-1}(R_{a'}^H)\quad \text{such that}\\ \lim_{s\to-\infty}u(s,t)=\psi_H^t(q)\quad\forall t\in[0,1],\\ \lim_{s\to\infty}u(s,t)=\psi_H^t(q')\quad\forall t\in[0,1]\end{array}\right\}\Big/\sim,$$

where $u_1\sim u_2$ if and only if u_1 and u_2 represent the same class in $\pi_2(X)$.

Let $k_1, k_0 \in \mathbb{N}$, $a, a' \in \pi_0(\psi_H^1(L) \cap L)$, and $B \in \pi_2(R_a^H, R_{a'}^H)$. Let $\mathcal{M}_{k_1,k_0}(R_a^H, R_{a'}^H, B, J)$ be the compactification of

$$\mathcal{M}^{\text{reg}}_{k_1,k_0}(R_a^H, R_{a'}^H, B, J) := \left\{ (u, \boldsymbol{\tau_1}, \boldsymbol{\tau_0}) \,\middle|\, \begin{array}{l} u : \mathbb{R} \times [0,1] \to X \text{ is smooth}, \quad [u] = B, \\ \dfrac{\partial u}{\partial s} + J\left(\dfrac{\partial u}{\partial t} - X_{H_t}(u)\right) = 0, \quad E(u) < \infty, \\ u(s,0) \in L \quad \text{and} \qquad u(s,1) \in L \quad \forall s \in \mathbb{R}, \\ \exists\, q \in \psi_H^{-1}(R_a^H), \quad q' \in \psi_H^{-1}(R_{a'}^H) \quad \text{such that} \\ \lim\limits_{s \to -\infty} u(s,t) = \psi_H^t(q) \quad \forall t \in [0,1] \quad \text{and} \\ \lim\limits_{s \to \infty} u(s,t) = \psi_H^t(q') \quad \forall t \in [0,1], \\ \boldsymbol{\tau_1} = ((\zeta_1, 1), \dots, (\zeta_{k_1}, 1)) \in (\mathbb{R} \times \{1\})^{k_1}, \\ \text{where } -\infty < \zeta_{k_1} < \cdots < \zeta_1 < +\infty, \\ \boldsymbol{\tau_0} = ((\tau_1, 0), \dots, (\tau_{k_0}, 0)) \in (\mathbb{R} \times \{0\})^{k_0}, \\ \text{where } -\infty < \tau_1 < \cdots < \tau_{k_0} < +\infty \end{array} \right\} / \sim .$$



$$\begin{array}{ccc} & \mathcal{M}_{k_1,k_0}(R_a^H, R_{a'}^H, B, J) & \\ \swarrow^{\text{ev}_{-\infty,B}} & & \searrow^{\text{ev}_{+\infty,B}} \\ R_a^H & & R_{a'}^H \end{array}$$

Define the evaluation maps as follows.

$$\forall 1 \le j \le k_1, \quad \text{ev}^{(1)}_{j,B} : \mathcal{M}^{\text{reg}}_{k_1,k_0}(R_a^H, R_{a'}^H, B, J) \to L, \quad (u, \boldsymbol{\tau_1}, \boldsymbol{\tau_0}) \mapsto u(\zeta_j, 1).$$

$$\forall 1 \le j \le k_0 \quad \text{ev}^{(0)}_{j,B} : \mathcal{M}^{\text{reg}}_{k_1,k_0}(R_a^H, R_{a'}^H, B, J) \to L, \quad (u, \boldsymbol{\tau_1}, \boldsymbol{\tau_0}) \mapsto u(\tau_j, 0).$$

$$\text{ev}_{-\infty,B} : \mathcal{M}_{k_1,k_0}(R_a^H, R_{a'}^H, B, J) \to R_a, \quad (u, \boldsymbol{\tau_1}, \boldsymbol{\tau_0}) \mapsto \psi_H^1\left(\lim_{s \to -\infty} u(s,0)\right).$$

$$\mathrm{ev}_{+\infty,B} : \mathcal{M}_{k_1,k_0}(R_a^H, R_{a'}^H, B, J) \to R_{a'}, \quad (u, \boldsymbol{\tau_1}, \boldsymbol{\tau_0}) \mapsto \psi_H^1\left(\lim_{s\to+\infty} u(s,0)\right).$$

The evaluation maps are G-equivariant.

Let $k_1, k_0 \in \mathbb{N}$, $a, a' \in \pi_0(\psi_H^1(L) \cap L)$, and $B \in \pi_2(R_a^H, R_{a'}^H)$. Define

$$\mathfrak{n}_B : \Omega_G(R_a^H)\widehat{\otimes}_{\mathbb{R}} \Lambda_{\mathrm{nov}}[1] \to \Omega_G(R_{a'}^H)\widehat{\otimes}_{\mathbb{R}} \Lambda_{\mathrm{nov}}[1]$$

by

$$\mathfrak{n}_B\,(\eta) = (\mathrm{ev}_{+\infty,B}^G)_!(\mathrm{ev}_{-\infty,B}^G)^*\eta.$$

Define $\delta_H^G : CF_G(L,b,H)[1] \to CF_G(L,b,H)[1]$ such that, for each $a \in \pi_0(L_H \cap L)$,

$$\delta_H^G : \Omega_G(R_a^H)\widehat{\otimes}\,\Lambda_{\mathrm{nov}}[1] \to CF_G(L,b,H)$$

is given by

$$\delta_H^G(\eta) = \sum_{a'\in\pi_0(L_H\cap L)} \sum_{B\in\pi_2(R_a^H,R_{a'}^H)} \exp(\partial B\cap b)\,\mathfrak{n}_B(\eta)\exp(\partial B\cap b)T^{\frac{\omega(B)}{2\pi}}e^{\frac{I_\mu(B)}{2}}. \tag{7.1.1}$$

Lemma 7.1. $\mathcal{M}_{k_1,k_0}(R_a^H, R_{a'}^H, B, J)$ has an oriented G-equivariant Kuranishi structure such that $\mathrm{ev}_{-\infty,B}, \mathrm{ev}_{+\infty,B}$ are strongly continuous and weakly submersive. Moreover, its normalized boundary is a union of the following types of fiber products below.

i) $\mathcal{M}_{k_1',k_0'}(R_a^H, R_c^H, B', J)_{\mathrm{ev}_{+\infty,B'}} \times_{\mathrm{ev}_{-\infty,B''}} \mathcal{M}_{k_1'',k_0''}(R_c^H, R_{a'}^H, B'', J)$, where

- $c \in \pi_0(L_H \cap L)$,
- $k_1', k_1'', k_0', k_0'' \in \mathbb{N}$ such that $k_1' + k_1'' = k_1$, $k_0' + k_0'' = k_0$,
- $B' \in \pi_2(R_a^H, R_c^H)$, $B'' \in \pi_2(R_c^H, R_{a'}^H)$ such that $B'\#B'' = B$.

ii) $\mathcal{M}_{k_1',k_0}(R_a^H, R_{a'}^H, B', J)_{\mathrm{ev}_{i,B'}^{(1)}} \times_{\mathrm{ev}_0} \mathcal{M}_{k_1''+1}(L, J, B'')$, where

- $1 \le i \le k_1'$,
- $k_1', k_1'' \in \mathbb{N}$ such that $k_1' + k_1'' = k_1 + 1$, and
- $B' \in \pi_2(R_a^H, R_{a'}^H)$, $B'' \in \pi_2(X, L)$ such that $B'\#B'' = B$.

iii) $\mathcal{M}_{k_1,k_0'}(R_a^H, R_{a'}^H, B', J)_{\mathrm{ev}^{(0)}_{i,B'}} \times_{\mathrm{ev}_0} \mathcal{M}_{k_0''+1}(L, J, B'')$, where

- $1 \le i \le k_0'$,
- $k_0', k_0'' \in \mathbb{N}$ such that $k_0' + k_0'' = k_0 + 1$,
- $B' \in \pi_2(R_a^H, R_{a'}^H)$, $B'' \in \pi_2(X, L)$ such that $B' \# B'' = B$.

Moreover, the G-equivariant maps $\mathrm{ev}_{-\infty,B}, \mathrm{ev}_{+\infty,B}$ are strongly smooth and weakly submersive.

Proof. The boundary decomposition follows from [19] Proposition 15.21. The construction of a G-invariant Kuranishi structure on $\mathcal{M}_{k_1,k_0}(R_a^H, R_{a'}^H, B)$ is similar to the proof of Proposition 5.1 (See [21] Section 4.3.). □

Proposition 7.1. δ_H^G is well-defined and $\delta_H^G \circ \delta_H^G = 0$.

Proof. The proposition follows from Theorem 8.2, Proposition 8.4, and Lemma 7.1. In particular, in Lemma 4.1, the contribution of i), ii) with $(k_1', k_0, B') \neq (1, 0, 0)$, and iii) with $(k_1, k_0', B') \neq (0, 1, 0)$ are trivial. And the contributions of ii) with $(k_1', k_0, B') = (1, 0, 0)$ and iii) with $(k_1', k_0, B') = (0, 1, 0)$ cancel with each other. □

Thus, we define

$$HF_G(L, b, H, J, \Lambda_{nov}) := \frac{\ker \delta_H^G}{\operatorname{im} \delta_H^G}. \tag{7.1.2}$$

7.2 The Floer continuation map $\mathfrak{f}$

Consider a smooth non-decreasing function $\chi : \mathbb{R} \to [0, 1]$ such that

$$\chi(s) = \begin{cases} 0 & \text{if } s \le -1 \\ 1 & \text{if } s \ge 1 \end{cases}. \tag{7.2.1}$$

Define $F : \mathbb{R} \times [0, 1] \times X \to \mathbb{R}$ by

$$F(s, t, x) = (1 - \chi(s))H(t, x) \quad \forall (s, t, x) \in \mathbb{R} \times [0, 1] \times X.$$

Then

$$F(s,t,x) = \begin{cases} H(t,x) & \text{if } s \leq -1 \\ 0 & \text{if } s \geq 1 \end{cases}.$$

Since H is G-invariant, F is also G-invariant.

Note that $F_s^t = F(s,t,\cdot) : X \to \mathbb{R}$ defines a Hamiltonian vector field $X_{F_s^t}$ via $dF_s^t = \iota_{X_{F_s t}}\omega$. Let $\psi_s^t : X \to X$ be the flow of $X_{F_s^t}$, namely it satisfies

$$\frac{d}{dt}\psi_s^t = X_{F_s^t} \circ \psi_s^t.$$

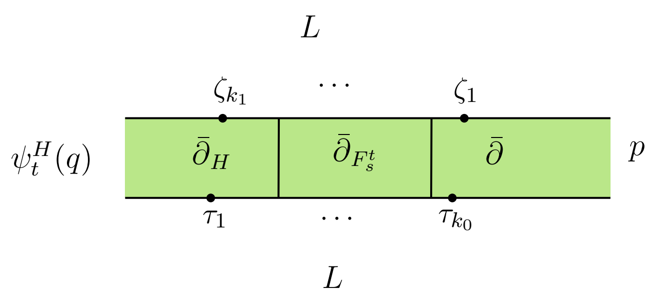
Recall we assumed that $L_H \cap L = \bigsqcup_{a \in \pi_0(L_H, L)} R_a^H$ is a finite union. $\forall a \in \pi_0(L_H \cap L)$, define

$$\pi_2(R_a^H, L) = \left\{ u : \mathbb{R} \times [0,1] \to X \;\middle|\; \begin{array}{l} u \text{ is smooth,} \\ u(s,0) \in L \quad \forall s \in \mathbb{R}, \\ u(s,1) \in L \quad \forall s \in \mathbb{R}, \\ \exists q \in \psi_H^{-1}(R_a^H) \text{ such that} \\ \lim_{s \to -\infty} u(s,t) = \psi_H^t(q) \quad \forall t \in [0,1], \\ \lim_{s \to \infty} u(s,t) = p \quad \forall t \in [0,1] \text{ for some } p \in L \end{array} \right\} \Big/ \sim,$$

where $u_1 \sim u_2$ if and only if u_1 and u_2 represent the same class in $\pi_2(X)$.

Let $k_1, k_0 \in \mathbb{N}$, $a \in \pi_0(L_H \cap L)$, and $B \in \pi_2(R_a^H, L)$. Let $\mathcal{M}_{k_1,k_0}(R_a^H, L, B, J)$ be the compactification of

$$\mathcal{M}^{\mathrm{reg}}_{k_1,k_0}(R^H_a, L, B, J) := \left\{ (u, \boldsymbol{\tau_1}, \boldsymbol{\tau_0}) \;\middle|\; \begin{array}{l} u : \mathbb{R} \times [0,1] \to X \text{ is smooth}, \quad [u] = B, \\ \dfrac{\partial u}{\partial s} + J\left(\dfrac{\partial u}{\partial t} - X_{F^t_s}(u)\right) = 0, \qquad E(u) < \infty, \\ u(s,0) \in L \quad \text{and} \quad u(s,1) \in L \quad \forall s \in \mathbb{R}, \\ \lim\limits_{s \to -\infty} u(s,t) = \psi^t_H(q) \quad \forall t \in [0,1] \\ \text{ for some } q \in \psi^{-1}_H(R^H_a), \\ \lim\limits_{s \to \infty} u(s,t) = p \quad \forall t \in [0,1] \text{ for some } p \in L, \\ \boldsymbol{\tau_1} = ((\zeta_1, 1), \dots, (\zeta_{k_1}, 1)) \in (\mathbb{R} \times \{1\})^{k_1}, \\ \text{ where } -\infty < \zeta_{k_1} < \cdots < \zeta_1 < +\infty, \\ \boldsymbol{\tau_0} = ((\tau_1, 0), \dots, (\tau_{k_0}, 0)) \in (\mathbb{R} \times \{0\})^{k_0}, \\ \text{ where } -\infty < \tau_1 < \cdots < \tau_{k_0} < +\infty \end{array} \right\}.$$



$$\begin{array}{ccc} & \mathcal{M}_{k_1,k_0}(R^H_a, p, B, J) & \\ {}^{\mathrm{ev}_{-\infty,B}} \swarrow & & \searrow^{\mathrm{ev}_{+\infty,B}} \\ R^H_a & & L \end{array}$$

Define the evaluation maps as follows.

$$\forall 1 \le j \le k_1, \quad \mathrm{ev}^{(1)}_{j,B} : \mathcal{M}^{\mathrm{reg}}_{k_1,k_0}(R^H_a, L, B, J) \to L, \quad (u, \boldsymbol{\tau_1}, \boldsymbol{\tau_0}) \mapsto u(\zeta_j, 1).$$

$$\forall 1 \le j \le k_0, \quad \mathrm{ev}^{(0)}_{j,B} : \mathcal{M}^{\mathrm{reg}}_{k_1,k_0}(R^H_a, L, B, J) \to L, \quad (u, \boldsymbol{\tau_1}, \boldsymbol{\tau_0}) \mapsto u(\tau_j, 0).$$

$$\mathrm{ev}_{-\infty,B} : \mathcal{M}^{\mathrm{reg}}_{k_1,k_0}(R^H_a, L, B, J) \to R^H_a, \quad (u, \boldsymbol{\tau_1}, \boldsymbol{\tau_0}) \mapsto \psi^1_H\left(\lim_{s \to -\infty} u(s,0)\right).$$

$$\mathrm{ev}_{+\infty,B}:\mathcal{M}^{\mathrm{reg}}_{k_1,k_0}(R^H_a,L,B,J)\to L,\quad (u,\boldsymbol{\tau_1},\boldsymbol{\tau_0})\mapsto \lim_{s\to\infty}u(s,1).$$

Let $k_1,k_0\in\mathbb{N}$, $a,a'\in\pi_0(L_H\cap L)$, and $B\in\pi_2(R^H_a,R^H_{a'})$. Define

$$\mathfrak{f}_B:\Omega_G(R^H_a)\widehat{\otimes}_{\mathbb{R}}\Lambda_{\mathrm{nov}}[1]\to C_G(L)[1],$$

by

$$\mathfrak{f}_B(\eta)=(\mathrm{ev}^G_{+\infty,B})_!(\mathrm{ev}^G_{-\infty,B})^*\eta.$$

Define

$$\mathfrak{f}:C_G(L,b,H,J)[1]\to C_G(L)[1]$$

such that, for each $a\in\pi_0(L_H\cap L)$,

$$\mathfrak{f}:\Omega_G(R^H_a)\widehat{\otimes}\Lambda_{\mathrm{nov}}[1]\to C_G(L)[1]$$

is given by

$$\mathfrak{f}(\eta)=\sum_{B\in\pi_2(R^H_a,L)}\exp(\partial B\cap b)\,\mathfrak{f}_B(\eta)\exp(\partial B\cap b)T^{\frac{\omega(B)}{2\pi}}e^{\frac{I_\mu(B)}{2}}.$$

Lemma 7.2. $\mathcal{M}_{k_1,k_0}(R^H_a,L,B,J)$ has an oriented G-equivariant Kuranishi structure such that $\mathrm{ev}_{-\infty,B},\mathrm{ev}_{+\infty,B}$ are strongly continuous and weakly submersive. Moreover, its normalized boundary is a union of the four types of fiber products below.

i) $\mathcal{M}_{k_1',k_0'}(R^H_a,R^H_c,B',J)\times\mathcal{M}_{k_1'',k_0''}(R^H_c,L,B'',J)$, where

- $c\in\pi_0(L_H\cap L)$
- $k_1',k_1'',k_0',k_0''\in\mathbb{N}$ such that $k_1'+k_1''=k_1$, $k_0'+k_0''=k_0$, and
- $B'\in\pi_2(R^H_a,R^H_c)$, $B''\in\pi_2(R^H_c,p)$ such that $B'\#B''=B$.

ii) $\mathcal{M}_{k_1',k_0'}(R^H_a,L,B',J)_{\mathrm{ev}_{+\infty}}\times_{\mathrm{ev}_0}\mathcal{M}_{k_1''+k_0''}(L,B'',J)$, where

- $k_1',k_1'',k_0',k_0''\in\mathbb{N}$ such that $k_1'+k_1''=k_1$, $k_0'+k_0''=k_0$, and
- $B'\in\pi_2(R^H_a,L)$, $B''\in\pi_2(X,L)$ such that $B'\#B''=B$.

iii) $\mathcal{M}_{k_1',k_0}(R^H_a,L,B',J)_{\mathrm{ev}^{(1)}_{i,B'}}\times_{\mathrm{ev}_0}\mathcal{M}_{k_1''+1}(L,J,B'')$, where

- $1 \leq i \leq k_1'$,
- $k_1', k_1'' \in \mathbb{N}$ such that $k_1' + k_1'' = k_1 + 1$, and
- $B' \in \pi_2(R_a^H, L)$, $B'' \in \pi_2(X, L)$ such that $B' \# B'' = B$.

iv) $\mathcal{M}_{k_1, k_0'}(R_a^H, L, B', J)_{\mathrm{ev}_{i,B'}^{(0)}} \times_{\mathrm{ev}_0} \mathcal{M}_{k_0''+1}(L, J, B'')$, where

- $1 \leq i \leq k_0'$, $1 \leq j \leq k_0''$,
- $k_0', k_0'' \in \mathbb{N}$ such that $k_0' + k_0'' = k_0 + 1$, and
- $B' \in \pi_2(R_a^H, p)$, $B'' \in \pi_2(X, L)$ such that $B' \# B'' = B$.

Proof. The boundary decomposition follows from [19] Proposition 15.22. The construction of such a G-equivariant Kuranishi structure is similar to the proof of Proposition 5.1 (See [21] Section 4.3.) □

Corollary 7.1. $\mathfrak{f}$ is a cochain map:

$$\mathfrak{f} \circ \delta_H^G - \delta^G \circ \mathfrak{f} = 0. \tag{7.2.2}$$

Proof. This follows from Lemma 7.2, Stokes' Theorem 8.2, and the composition formula (Proposition 8.4). □

7.3 The map $\mathfrak{g}$

Let χ be as in (7.2.1). Define $\overline{F} : \mathbb{R} \times [0,1] \times X \to \mathbb{R}$ by

$$\overline{F}(s,t,x) = \overline{F}_s^t(x) := \chi(s) H_t(x).$$

Then $\overline{F}_s^t(x) = 0$ if $s \leq -1$ and $\overline{F}_s^t(x) = H_t(x)$ if $s \geq 1$.

Note that $\overline{F}_s^t = \overline{F}(s,t,\cdot) : X \to \mathbb{R}$ defines a Hamiltonian vector field $X_{\overline{F}_s^t}$ via $\iota_{X_{\overline{F}_s^t}} \omega = d\overline{F}_s^t$.

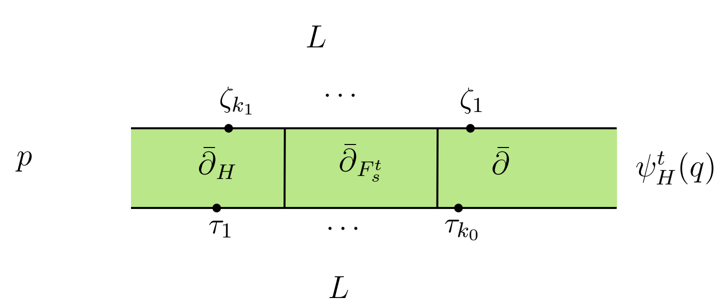
$\forall a \in \pi_0(\psi_H^1(L) \cap L)$, define

$$\pi_2(L, R_a^H) = \left\{ u : \mathbb{R} \times [0,1] \to X \;\middle|\; \begin{array}{l} u \text{ is smooth,} \\ u(s,0) \in L \quad \text{and} \quad u(s,1) \in L \quad \forall s \in \mathbb{R}, \\ \lim\limits_{s \to -\infty} u(s,t) = p \quad \forall t \in [0,1] \text{ for some } p \in L, \\ \exists q \in \psi_H^{-1}(R_a^H) \quad \text{such that} \\ \lim\limits_{s \to +\infty} u(s,t) = \phi_H^t(q) \quad \forall t \in [0,1] \end{array} \right\} \Big/ \sim,$$

where $u_1 \sim u_2$ if and only if u_1 and u_2 represent the same class in $\pi_2(X)$.

Let $k_1, k_0 \in \mathbb{N}$, $a \in \pi_0(L_H \cap L)$, and $B \in \pi_2(L, R_a^H)$. Let $\mathcal{M}_{k_1,k_0}(L, R_a^H, B, J)$ be the compactification of

$$\mathcal{M}_{k_1,k_0}^{\text{reg}}(L, R_a^H, B, J) := \left\{ (u, \boldsymbol{\tau_1}, \boldsymbol{\tau_0}) \;\middle|\; \begin{array}{l} u : \mathbb{R} \times [0,1] \to X \text{ is smooth,} \quad [u] = B, \\ \dfrac{\partial u}{\partial s} + J\left(\dfrac{\partial u}{\partial t} - X_{\overline{F}_s^t}(u)\right) = 0, \qquad E(u) < \infty, \\ u(s,0) \in L \quad \text{and} \quad u(s,1) \in L \quad \forall s \in \mathbb{R}, \\ \lim\limits_{s \to -\infty} u(s,t) = p \quad \forall t \in [0,1] \text{ for some } p \in L, \\ \lim\limits_{s \to +\infty} u(s,t) = \psi_H^t(q) \quad \forall t \in [0,1] \text{ for some } q \in \psi_H^{-1}(R_a^H), \\ \boldsymbol{\tau_1} = ((\zeta_1, 1), \dots, (\zeta_{k_1}, 1)) \in (\mathbb{R} \times \{1\})^{k_1}, \\ \quad \text{where } -\infty < \zeta_{k_1} < \cdots < \zeta_1 < +\infty, \\ \boldsymbol{\tau_0} = ((\tau_1, 0), \dots, (\tau_{k_0}, 0)) \in (\mathbb{R} \times \{0\})^{k_0}, \\ \quad \text{where } -\infty < \tau_1 < \cdots < \tau_{k_0} < +\infty \end{array} \right\}.$$



$$\begin{array}{ccc} & \mathcal{M}_{k_1,k_0}(L, R_a^H, B, J) & \\ {}^{\mathrm{ev}_{-\infty,B}} \swarrow & & \searrow {}^{\mathrm{ev}_{+\infty,B}} \\ L & & R_a^H \end{array}$$

Define the evaluation maps as follows.

$$\forall 1 \leq j \leq k_1, \quad \mathrm{ev}_{j,B}^{(1)} : \mathcal{M}_{k_1,k_0}^{\mathrm{reg}}(L, R_a^H, B, J) \to L, \quad (u, \boldsymbol{\tau_1}, \boldsymbol{\tau_0}) \mapsto u(\zeta_j, 1).$$

$$\forall 1 \leq j \leq k_0, \quad \mathrm{ev}_{j,B}^{(0)} : \mathcal{M}_{k_1,k_0}^{\mathrm{reg}}(L, R_a^H, B, J) \to L, \quad (u, \boldsymbol{\tau_1}, \boldsymbol{\tau_0}) \mapsto u(\tau_j, 0).$$

$$\mathrm{ev}_{-\infty,B} : \mathcal{M}_{k_1,k_0}^{\mathrm{reg}}(L, R_a^H, B, J) \to L, \quad (u, \boldsymbol{\tau_1}, \boldsymbol{\tau_0}) \mapsto \lim_{s \to -\infty} u(s, 1).$$

$$\mathrm{ev}_{+\infty,B} : \mathcal{M}_{k_1,k_0}^{\mathrm{reg}}(L, R_a^H, B, J) \to R_a^H, \quad (u, \boldsymbol{\tau_1}, \boldsymbol{\tau_0}) \mapsto \psi_H^1 \left(\lim_{s \to +\infty} u(s, 0) \right).$$

Let $a \in \pi_0(L_H \cap L)$, and $B \in \pi_2(L, R_a^H)$.

Define

$$\mathfrak{g}_B : C_G(L)[1] \to \Omega_G(R_a^H) \widehat{\otimes}_{\mathbb{R}} \Lambda_{\mathrm{nov}}[1]$$

by

$$\mathfrak{g}_B(\eta) = (\mathrm{ev}_{+\infty,B}^G)_! (\mathrm{ev}_{-\infty,B}^G)^* \eta.$$

Define

$$\mathfrak{g} : C_G(L)[1] \to C_G(L, b, H, J)[1]$$

such that, for each $a \in \pi_0(L_H \cap L)$,

$$\mathfrak{g} : C_G(L)[1] \to \Omega_G(R_a^H) \widehat{\otimes} \Lambda_{\mathrm{nov}}[1]$$

is given by

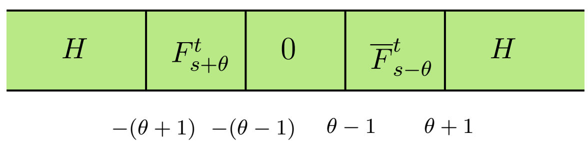
$$\mathfrak{g}(\eta) = \sum_{B \in \pi_2(L, R_a^H)} \exp(\partial B \cap b)\, \mathfrak{g}_B(\eta) \exp(\partial B \cap b) T^{\frac{\omega(B)}{2\pi}} e^{\frac{I_\mu(B)}{2}}.$$

7.4 Proof of Hamiltonian isotopy invariance

Define a smooth function $\tilde{F} : [0,\infty) \times \mathbb{R} \times [0,1] \times X \to \mathbb{R}$ as follows. Fix a large constant, say 10.

For $\theta \geq 10$, let

$$\tilde{F}(\theta, s, t, x) = \tilde{F}^t_{\theta,s}(x) = \begin{cases} F^t_{s+\theta}(x) & \text{if } s \leq 0 \\ \overline{F}^t_{s-\theta}(x) & \text{if } s \geq 0 \end{cases}.$$



Let $\tilde{\chi} : \mathbb{R} \to [0,1]$ be a smooth nondecreasing function such that

$$\tilde{\chi}(\theta) = \begin{cases} 0 & \text{if } \theta \leq 1 \\ 1 & \text{if } \theta \geq 9 \end{cases}.$$

For $0 \leq \theta < 10$, define

$$\tilde{F}(\theta, s, t, x) = (1 - \tilde{\chi}(\theta))H_t(x) + \tilde{\chi}(\theta)\tilde{F}^t_{10,s}(x).$$

Consider

$$\pi_2^\theta(R_a^H, R_{a'}^H) = \left\{ u : \mathbb{R} \times [0,1] \to X \;\middle|\; \begin{array}{l} u \text{ is smooth,} \\ u(s,0) \in L \quad \text{and} \quad u(s,1) \in \psi^1_{\theta,s}(L) \quad \forall s \in \mathbb{R}, \\ \lim_{s \to -\infty} u(s,t) = \psi^t_0(q_a) \quad \forall t \in [0,1], \\ \lim_{s \to \infty} u(s,t) = \psi^t_1(q_{a'}) \quad \forall t \in [0,1] \end{array} \right\} \Big/ \sim,$$

where $u_1 \sim u_2$ if and only if u_1 and u_2 represent the same class in $\pi_2(X)$.

Let $a, a' \in \pi_0(\psi^1_H(L) \cap L)$. Define

$$\pi_2^{[0,\infty]}(R_a^H, R_{a'}^H) = \left(\pi_2(R_a^H, R_{a'}^H) \cup \bigcup_{\theta \in [0,\infty)} \pi_2^\theta(R_a^H, R_{a'}^H) \right) \Big/ \sim,$$

where $[u_1] \sim [u_2]$ if and only if $[u_1] = [u_2]$ in $\pi_2(X)$.

Let $\theta \in [0, \infty)$, $k_1, k_0 \in \mathbb{N}$, $a, a' \in \pi_0(\psi_H^1(L) \cap L)$, and $B \in \pi_2^{[0,\infty]}(R_a^H, R_{a'}^H)$. Define

$$\mathcal{M}_{k_1,k_0}^{\theta,reg}(R_a^H, R_{a'}^H, B, J)$$
$$:= \left\{ (u, \tau_1, \tau_0) \;\middle|\; \begin{array}{l} u : \mathbb{R} \times [0,1] \to X \text{ is smooth}, \quad [u] = B, \\ \dfrac{\partial u}{\partial s} + J\left(\dfrac{\partial u}{\partial t} - X_{\tilde{F}^t_{\theta,s}}(u)\right) = 0, \\ E(u) = \left\|\dfrac{\partial u}{\partial s}\right\|^2 + \left\|\dfrac{\partial u}{\partial t} - X_{\tilde{F}^t_{\theta,s}}(u)\right\|^2 < \infty, \\ u(s,0) \in L \quad \text{and} \quad u(s,1) \in L \quad \forall s \in \mathbb{R}, \\ \exists\, q \in \psi_H^{-1}(R_a^H), \quad q' \in \psi_H^{-1}(R_{a'}^H) \quad \text{such that} \\ \lim_{s\to-\infty} u(s,t) = \psi_H^t(q) \quad \text{and} \quad \lim_{s\to\infty} u(s,t) = \psi_H^t(q') \quad \forall t \in [0,1], \\ \tau_0 = ((\tau_1, 0), \ldots, (\tau_{k_0}, 0)) \in (\mathbb{R} \times \{0\})^{k_0}, \\ \text{ where } -\infty < \tau_{0,1} < \cdots < \tau_{0,k_0} < +\infty, \\ \tau_1 = ((\zeta_1, 1), \ldots, (\zeta_{k_1}, 1)) \in (\mathbb{R} \times \{1\})^{k_1}, \\ \text{ where } -\infty < \zeta_{k_1} < \cdots < \zeta_1 < +\infty \end{array} \right\}.$$

When $\theta = \infty$, let

$$\mathcal{M}_{k_1,k_0}^{+\infty,reg}(R_a^H, R_{a'}^H, B, J) := \bigcup_{B' \# B'' = B} \; \bigcup_{\substack{k_1' + k_1'' = k_1 \\ k_0' + k_0'' = k_0}} \mathcal{M}_{k_1',k_0'}(R_a^H, L, B', J) \times \mathcal{M}_{k_1'',k_0''}(L, R_{a'}^H, B'', J),$$

where $\mathcal{M}_{k_1'',k_0''}(L, R_a^H, B, J)$ is the compactification of

$$\mathcal{M}^{\mathrm{reg}}_{k_1,k_0}(L, R_a^H, B, J) := \left\{ (u, \boldsymbol{\tau_1}, \boldsymbol{\tau_0}) \;\middle|\; \begin{array}{l} u : \mathbb{R} \times [0,1] \to X \text{ is smooth}, \quad [u] = B, \\ \dfrac{\partial u}{\partial s} + J\left(\dfrac{\partial u}{\partial t} - X_{F_s^t}(u)\right) = 0, \qquad E(u) < \infty, \\ u(s,0) \in L \quad \text{and} \qquad u(s,1) \in L \quad \forall s \in \mathbb{R}, \\ \lim\limits_{s\to-\infty} u(s,t) = p \quad \forall t \in [0,1] \text{ for some } p \in L \\ \lim\limits_{s\to\infty} u(s,t)\psi_H^t(q) \quad \forall t \in [0,1] \text{ for some } q \in \psi_H^{-1}(R_a^H), \\ \boldsymbol{\tau_1} = ((\zeta_1, 1), \ldots, (\zeta_{k_1}, 1)) \in (\mathbb{R} \times \{1\})^{k_1}, \\ \text{ where } -\infty < \zeta_{k_1} < \cdots < \zeta_1 < +\infty, \\ \boldsymbol{\tau_0} = ((\tau_1, 0), \ldots, (\tau_{k_0}, 0)) \in (\mathbb{R} \times \{0\})^{k_0}, \\ \text{ where } -\infty < \tau_1 < \cdots < \tau_{k_0} < +\infty \end{array} \right\}.$$

Let

$$\mathcal{M}^{[0,+\infty],reg}_{k_1,k_0}(R_a^H, R_{a'}^H, B, J) = \bigcup_{\theta \in [0,+\infty]} \left(\{\theta\} \times \mathcal{M}^{\theta,reg}_{k_1,k_0}(R_a^H, R_{a'}^H, B, J) \right)$$

and let $\mathcal{M}^{[0,+\infty]}_{k_1,k_0}(R_a^H, R_{a'}^H, B, J)$ be its compactification.

$$\begin{array}{ccc} & \mathcal{M}^{[0,+\infty]}_{k_1,k_0}(R_a^H, R_{a'}^H, B, J) & \\ \swarrow^{\mathrm{ev}_{-\infty,B}} & & \searrow^{\mathrm{ev}_{+\infty,B}} \\ R_a^H & & R_{a'}^H \end{array}$$

Define the evaluation maps as follows.

$$\forall 1 \le j \le k_1, \quad \mathrm{ev}^{(1)}_{j,B} : \mathcal{M}^{[0,+\infty),reg}_{k_1,k_0}(R_a^H, R_{a'}^H, B, J) \to L, \quad (\theta, (u, \boldsymbol{\tau_1}, \boldsymbol{\tau_0})) \mapsto u(\zeta_j, 1).$$

$$\forall 1 \le j \le k_0, \quad \mathrm{ev}^{(0)}_{j,B} : \mathcal{M}^{[0,+\infty),reg}_{k_1,k_0}(R_a^H, R_{a'}^H, B, J) \to L, \quad (\theta, (u, \boldsymbol{\tau_1}, \boldsymbol{\tau_0})) \mapsto u(\tau_j, 0).$$

$$\mathrm{ev}_{-\infty,B} : \mathcal{M}^{[0,+\infty),reg}_{k_1,k_0}(R_a^H, R_{a'}^H, B, J) \to R_a^H, \quad (\theta, (u, \boldsymbol{\tau_1}, \boldsymbol{\tau_0})) \mapsto \psi_H^1\left(\lim_{s\to-\infty} u(s,0)\right).$$

$$\mathrm{ev}_{+\infty,B} : \mathcal{M}^{[0,+\infty),reg}_{k_1,k_0}(R_a^H, R_{a'}^H, B, J) \to R_{a'}^H, \quad (\theta, (u, \boldsymbol{\tau_1}, \boldsymbol{\tau_0})) \mapsto \psi_H^1\left(\lim_{s\to+\infty} u(s,0)\right).$$

Let $a, a' \in \pi_0(L_H \cap L)$, and $B \in \pi_2^{[0,\infty]}(R_a^H, R_{a'}^H)$.

Define

$$\Theta_B : \Omega_G(R^H_a)\widehat{\otimes}_{\mathbb{R}} \Lambda_{\mathrm{nov}}[1] \to \Omega_G(R^H_{a'})\widehat{\otimes}_{\mathbb{R}} \Lambda_{\mathrm{nov}}[1],$$

by

$$\Theta_B\,(\eta) = (\mathrm{ev}^G_{+\infty,B})_!(\mathrm{ev}^G_{-\infty,B})^*\eta.$$

Define

$$\Theta : CF_G(L,b,H,J)[1] \to CF_G(L,b,H,J)[1]$$

such that, for each $a, a' \in \pi_0(\psi^1_H(L) \cap L)$,

$$\Theta : \Omega_G(R^H_a)\widehat{\otimes}_{\mathbb{R}} \Lambda_{\mathrm{nov}}[1] \to \Omega_G(R^H_{a'})\widehat{\otimes}_{\mathbb{R}} \Lambda_{\mathrm{nov}}[1]$$

is given by

$$\Theta(\eta) = \sum_{B\in\pi_2^{[0,\infty]}(R^H_a,R^H_{a'})} \exp(\partial B \cap b)\Theta_B(\eta)\exp(\partial B \cap b)T^{\frac{\omega(B)}{2\pi}}e^{\frac{I_\mu(B)}{2}}.$$

Lemma 7.3. $\mathcal{M}^{[0,+\infty]}_{k_1,k_0}(R^H_a, R^H_{a'}, B, J)$ has an oriented G-equivariant Kuranishi structure such that $\mathrm{ev}_{-\infty,B}, \mathrm{ev}_{+\infty,B}$ are strongly continuous and weakly submersive. Moreover, its normalized boundary is a union of the following types of fiber products below.

i) $\mathcal{M}_{k_1',k_0'}(R^H_a, R^H_c, B', J)_{\mathrm{ev}_\infty} \times_{\mathrm{ev}_{-\infty}} \mathcal{M}^{[0,+\infty]}_{k_1'',k_0''}(R^H_c, R^H_{a'}, B'', J)$, where

- $c \in \pi_0(L_H \cap L)$,
- $k_1', k_1'', k_0', k_0'' \in \mathbb{N}$ such that $k_1' + k_1'' = k_1$, $k_0' + k_0'' = k_0$, and
- $B' \in \pi_2(R^H_a, R^H_c)$, $B'' \in \pi_2(R^H_c, R^H_{a'})$ such that $B' \# B'' = B$.

ii) $\mathcal{M}^{[0,+\infty]}_{k_1',k_0'}(R^H_a, R^H_c, B', J)_{\mathrm{ev}_\infty} \times_{\mathrm{ev}_{-\infty}} \mathcal{M}_{k_1'',k_0''}(R^H_c, R^H_{a'}, B'', J)$, where

- $c \in \pi_0(L_H \cap L)$,
- $k_1', k_1'', k_0', k_0'' \in \mathbb{N}$ such that $k_1' + k_1'' = k_1$, $k_0' + k_0'' = k_0$, and
- $B' \in \pi_2(R^H_a, R^H_c)$, $B'' \in \pi_2(R^H_c, R^H_{a'})$ such that $B' \# B'' = B$.

iii) $\mathcal{M}^{[0,+\infty]}_{k_1,k_0'}(R^H_a, R^H_{a'}, B', J)_{\mathrm{ev}_{1,i,B'}} \times_{\mathrm{ev}_0} \mathcal{M}_{k_0''+1}(L, B'', J)$, where

- $1 \leq i \leq k_1'$,
- $k_1', k_1'' \in \mathbb{N}$ such that $k_1' + k_1'' = k_1 + 1$, and
- $B' \in \pi_2(R_a^H, R_{a'}^H)$, $B'' \in \pi_2(X, L)$ such that $B' \# B'' = B$.

iv) $\mathcal{M}^{[0,+\infty]}_{k_1',k_0}(R_a^H, R_{a'}^H, B', J)_{\mathrm{ev}_{0,i,B'}} \times_{\mathrm{ev}_0} \mathcal{M}_{k_0''+1}(L, B'', J)$, where

- $1 \leq i \leq k_0'$,
- $k_0', k_0'' \in \mathbb{N}$ such that $k_0' + k_0'' = k_0 + 1$, and
- $B' \in \pi_2(R_a^H, R_{a'}^H)$, $B'' \in \pi_2(X, L)$ such that $B' \# B'' = B$.

v) $\mathcal{M}_{k_1',k_0'}(R_a^H, L, B', J)_{\mathrm{ev}_\infty} \times_{\mathrm{ev}_{-\infty}} \mathcal{M}_{k_1'',k_0''}(L, R_{a'}^H, B'', J)$, where

- $k_1', k_1'', k_0', k_0'' \in \mathbb{N}$ such that $k_1' + k_1'' = k_1$, $k_0' + k_0'' = k_0$, and
- $B' \in \pi_2(R_a^H, L)$, $B'' \in \pi_2(L, R_{a'}^H)$ such that $B' \# B'' = B$.

vi) $\widetilde{\mathcal{M}}_{k_1,k_0}(R_a^H, R_{a'}^H, B, J)$, whose quotient space by the free $\mathbb{R}$-action is $\mathcal{M}_{k_1,k_0}(R_a^H, R_{a'}^H, B, J)$.

Proof. The cases where bubbling happens at $s = -\infty$ and $s = +\infty$ correspond to i) and ii), respectively. The cases where bubbling happens at $t = 1$ and $t = 0$ correspond to iii) and iv), respectively. The case $\theta = 0$ corresponds to vi). The case $\theta = \infty$ corresponds to v). The construction of a G-invariant Kuranishi structure is similar to the proof of Proposition 5.1 (See [21] Section 4.3.). The boundary decomposition is similar to the proof of [19] Proposition 15.22. □

Corollary 7.2.

$$\mathfrak{g} \circ \mathfrak{f} - \mathbb{1}^G = \Theta \circ \delta_H^G + \delta_H^G \circ \Theta.$$

Proof. In Lemma 7.3, the contributions of i) and ii) vanish. The terms iii) and iv) correspond to $(\Theta - d_G) \circ \delta_H^G$ and $\delta_H^G \circ (\Theta - d_G)$. And v) corresponds to $\mathfrak{g} \circ \mathfrak{f}$. The contribution of v) vanishes except the case when $B = 0$, which corresponds to $\mathbb{1}^G$. Then the corollary follows from Lemma 7.3, Proposition 8.4, and Theorem 8.2. □

Corollary 7.3. $\mathfrak{f} \circ \mathfrak{g}$ is cochain homotopic to the identity map.

Proof. The proof is similar to that of Corollary 7.2. □

Corollary 7.4. If $HF_G((L,b),(L,b),\Lambda_{\rm nov}) \neq 0$, then L is not displaceable by any G-equivariant Hamiltonian diffeomorphism that is the time-1 map of a G-equivariant Hamiltonian isotopy.

Proof. If L is displaceable by ψ_H^1, which is a G-equivariant Hamiltonian diffeomorphism that is the time-1 map of a G-equivariant Hamiltonian isotopy, then $\psi_H^1(L) \cap L = \emptyset$ implies $CF_G(L,b,H) = 0$. Thus,

$$0 = HF_G(L,b,H) \cong HF_G((L,b),(L,b),\Lambda_{\rm nov}).$$

□

Chapter 8

Equivariant Kuranishi structures

In this chapter, we discuss concepts related to equivariant Kuranishi structures which we use to define the A_∞ strctures. The definitions and the constructions of some of these equivariant Kuranishi data on the moduli spaces have been discussed in [23], [22], [21], and [18]. More specifically, in Section 8.1, we give the definitions of equivariant Kuranishi structures and equivariant good coordinate systems. In Section 8.2 and Section 8.3, we define CF-perturbations ("CF" stands for continuous family) and use it to define equivariant integration along the fiber on Kuranishi spaces. And in Section 8.4, we discuss the equivariant Stokes' theorem and equivariant smooth correspondences on Kuranishi spaces.

8.1 Equivariant Kuranishi structures

We review related concepts in the orbifold theory in Appendix A. Moreover, we assume G is a torus and G acts on $\mathcal{M}$ freely when necessary. For the general theory of Kuranishi structures, we refer the readers to the book [20].

Definition 8.1 (G-equivariant Kuranishi chart)**.** Let $\mathcal{M}$ be a separable metrizable topological space with a topological action by a compact connected Lie group G. A **G-equivariant Kuranishi chart** on $\mathcal{M}$ is a quadruple $\mathcal{U} = (U, \mathcal{E}, \psi, s)$ that satisfies the following.

(a) U and $\mathcal{E}$ are oriented smooth effective orbifolds, possibly with corners, each equipped with a smooth G-action.

(b) $\mathcal{E} \xrightarrow{\pi} U$ is a smooth G-equivariant orbibundle.

(c) $s : U \to \mathcal{E}$ is a smooth G-equivariant section of π.

(d) $\psi : s^{-1}(0) \to \mathcal{M}$ is a G-equivariant continuous map, which is a homeomorphism onto an open subset in $\mathcal{M}$.

We say U is a **Kuranishi neighborhood**, $\mathcal{E}$ is an **obstruction bundle**, ψ is a **parametrization** map, and s is a **Kuranishi map**.

Definition 8.2 (Restriction of a G-equivariant Kuranishi chart)**.** Let $\mathcal{U} = (U, \mathcal{E}, \psi, s)$ be a G-equivariant Kuranishi chart and $U' \subset U$ be a G-invariant open subset of U. Then the restriction of $\mathcal{U}$ to U' defines a Kuranishi chart $\mathcal{U}' = (U', \mathcal{E}|_{U'}, \psi|_{s^{-1}(0)\cap U'}, s|_{U'})$.

Definition 8.3 (Embedding of G-equivariant Kuranishi charts)**.** Let $\mathcal{M}$ be a separable metrizable topological space with a topological action by a compact Lie group G. An G-equivariant **embedding of G-equivariant Kuranishi charts**

$$\vec{\alpha} = (\alpha, \widehat{\alpha}) : (U, \mathcal{E}, \psi, s) \to (U', \mathcal{E}', \psi', s')$$

is a G-equivariant embedding of the orbibundles $\left(U \xrightarrow{\alpha} U', \mathcal{E} \xrightarrow{\widehat{\alpha}} \mathcal{E}'\right)$ satisfying the following.

i) $\widehat{\alpha} \circ s = s' \circ \alpha$.

ii) $\psi' \circ \alpha|_{s^{-1}(0)} = \psi$.

iii) $\forall x \in s^{-1}(0)$, the derivative $D_{\alpha(x)}s'$ induces an isomorphism

$$\frac{T_{\alpha(x)}U'}{(D_x\alpha)(T_xU)} \cong \frac{\mathcal{E}'_{\alpha(x)}}{\widehat{\alpha}(\mathcal{E}_x)}$$

induced by

$$\begin{array}{ccc} \mathcal{E}_x & \xrightarrow{\widehat{\alpha}} & \mathcal{E}'_{\alpha(x)} \\ {\scriptstyle D_x s}\uparrow & & \uparrow{\scriptstyle D_{\alpha(x)}s'} \\ T_xU & \xrightarrow[D_x\alpha]{} & T_{\alpha(x)}U' \end{array} .$$

Definition 8.4 (G-equivariant Kuranishi structure)**.** Let $\mathcal{M}$ be a separable metrizable topological space with a topological action by a compact Lie group G. A **G-equivariant Kuranishi structure**

$$\widehat{\mathcal{U}} = (\{\mathcal{U}_p \,|\, p \in \mathcal{M}\}, \{\vec{\alpha}_{pq} \,|\, p \in \mathcal{M}, q \in \operatorname{im}\psi_p\}) \tag{8.1.1}$$

on $\mathcal{M}$ consists of the following data.

1) $\forall p \in \mathcal{M}, \mathcal{U}_p = (U_p, \mathcal{E}_p, \psi_p, s_p)$ is a G-equivariant Kuranishi chart on $\mathcal{M}$ such that there exists a unique $o_p \in U_p$ satisfying $\psi_p(o_p) = p$.

2) $\forall p \in \mathcal{M}, \forall g \in G, \mathcal{U}_p = \mathcal{U}_{gp}$.

3) $\forall p \in \mathcal{M}$, $\forall q \in \operatorname{im}\psi_p$, we have a G-equivariant embedding of G-equivariant Kuranishi charts $\vec{\alpha}_{pq} = \left(U_{pq} \xrightarrow{\alpha_{pq}} U_p, \mathcal{E}_q|_{U_{pq}} \xrightarrow{\widehat{\alpha}_{pq}} \mathcal{E}_p\right)$ from the restriction of $\mathcal{U}_q$ to a G-invariant open subset U_{pq} of U_q with $q \in \psi_q(s_q^{-1}(0) \cap U_{pq})$. In particular, it satisfies the following.

   a) The following diagrams commute.

$$\begin{array}{ccc} \mathcal{E}_q|_{U_{pq}} & \xrightarrow{\widehat{\alpha}_{pq}} & \mathcal{E}_p \\ \pi_q \big\downarrow & & \big\downarrow \pi_p \\ U_{pq} & \xrightarrow[\alpha_{pq}]{} & U_p \end{array} \quad , \quad \begin{array}{ccc} \mathcal{E}_q|_{U_{pq}} & \xrightarrow{\widehat{\alpha}_{pq}} & \mathcal{E}_p \\ s_q \big\uparrow & & \big\uparrow s_p \\ U_{pq} & \xrightarrow[\alpha_{pq}]{} & U_p \end{array} \quad , \quad \begin{array}{ccc} s_q^{-1}(0) \cap U_{pq} & \xrightarrow{\alpha_{pq}} & s_p^{-1}(0) \cap U_p \\ & \searrow^{\psi_q} & \big\downarrow \psi_p \\ & & \mathcal{M} \end{array}$$

   b) If $x \in s_q^{-1}(0) \cap U_{pq}$ and $\alpha_{pq}(x) = y$, then $D_y s_p$ induces an isomorphism

$$\frac{T_y U_p}{(D_x \alpha_{pq})(T_x U_{pq})} \cong \frac{\mathcal{E}_p\big|_y}{\widehat{\alpha}_{pq}\left(\mathcal{E}_q\big|_x\right)} \tag{8.1.2}$$

   Such an embedding $\vec{\alpha}_{pq}$ is called a **G-equivariant Kuranishi coordinate change**.

4) For any $p \in \mathcal{M}, q \in \operatorname{im}\psi_p$ and any $g, g' \in G$, we have $\vec{\alpha}_{pq} = \vec{\alpha}_{(gp)(g'q)}$.

5) If $r \in \psi_q(s_q^{-1}(0) \cap U_{pq})$ and $U_{pqr} = \alpha_{qr}^{-1}(U_{pq}) \cap U_{qr}$, then the cocycle condition is satisfied in the following sense:

$$\alpha_{pr}(x) = \alpha_{pq} \circ \alpha_{qr}(x), \qquad \forall x \in U_{pqr},$$

and

$$\widehat{\alpha}_{pr}(v) = \widehat{\alpha}_{pq} \circ \widehat{\alpha}_{qr}(v), \qquad \forall x \in U_{pqr}, \quad \forall v \in (\mathcal{E}_r)_x.$$

A topological space $\mathcal{M}$ which satisfies the conditions above and has a G-equivariant Kuranishi structure is called a **G-equivariant Kuranishi space**.

Definition 8.5 (G-equivariant good coordinate system)**.** Let $\mathcal{M}$ be a separable metrizable topological space. A **G-equivariant good coordinate system**

$$\widehat{\mathcal{U}} = \left((\mathfrak{P}, \leq), \{\mathcal{U}_\mathfrak{p} \mid \mathfrak{p} \in \mathfrak{P}\}, \{\vec{\alpha}_{\mathfrak{pq}} \,|\, \mathfrak{p}, \mathfrak{q} \in \mathfrak{P}, \mathfrak{q} \leq \mathfrak{p}\}\right) \tag{8.1.3}$$

consists of

1) a finite partially ordered set $(\mathfrak{P}, \leq)$;

2) $\forall\, \mathfrak{p} \in \mathfrak{P}$, a G-equivariant Kuranishi chart $\mathcal{U}_\mathfrak{p} = (U_\mathfrak{p}, \mathcal{E}_\mathfrak{p}, \psi_\mathfrak{p}, s_\mathfrak{p})$ on $\mathcal{M}$ so that $\mathcal{M} \subset \bigcup_{\mathfrak{p} \in \mathfrak{P}} \operatorname{im} \psi_\mathfrak{p}$; and

3) $\forall\, \mathfrak{p}, \mathfrak{q} \in \mathfrak{P}$ with $\mathfrak{q} \leq \mathfrak{p}$, an G-equivariant embedding $(\alpha_{\mathfrak{pq}}, \widehat{\alpha}_{\mathfrak{pq}})$ of G-equivariant Kuranishi charts $\vec{\alpha}_{\mathfrak{pq}} = \left(U_{\mathfrak{pq}} \xrightarrow{\alpha_{\mathfrak{pq}}} U_\mathfrak{p}, \mathcal{E}_\mathfrak{q}|_{U_{\mathfrak{pq}}} \xrightarrow{\widehat{\alpha}_{\mathfrak{pq}}} \mathcal{E}_\mathfrak{p}\right)$ from the restriction of $\mathcal{U}_\mathfrak{q}$ to a G-invariant open subset $U_{\mathfrak{pq}}$ of $U_\mathfrak{q}$, called a **G-equivariant good coordinate change** from $\mathfrak{q}$ to $\mathfrak{p}$, satisfying

$$\operatorname{im} \psi_{\mathfrak{pq}} = \operatorname{im} \psi_\mathfrak{p} \cap \operatorname{im} \psi_\mathfrak{q},$$

such that the following holds.

i) $\vec{\alpha}_{\mathfrak{pp}} = \left(\mathrm{id}|_{U_\mathfrak{p}}, \mathrm{id}|_{\mathcal{E}_\mathfrak{p}}\right)$.

ii) If $\operatorname{im} \psi_\mathfrak{p} \cap \operatorname{im} \psi_\mathfrak{q} \neq \emptyset$, then either $\mathfrak{p} \leq \mathfrak{q}$ or $\mathfrak{q} \leq \mathfrak{p}$.

iii) If $\mathfrak{r} \leq \mathfrak{q} \leq \mathfrak{p}$ and $U_{\mathfrak{pqr}} = \alpha_{\mathfrak{qr}}^{-1}(U_{\mathfrak{pq}}) \cap U_{\mathfrak{qr}}$, then

$$\alpha_{\mathfrak{pr}}(x) = \alpha_{\mathfrak{pq}} \circ \alpha_{\mathfrak{qr}}(x), \qquad \forall x \in U_{\mathfrak{pqr}},$$

and,

$$\widehat{\alpha}_{\mathfrak{pr}}(v) = \widehat{\alpha}_{\mathfrak{pq}} \circ \widehat{\alpha}_{\mathfrak{qr}}(v), \qquad \forall x \in U_{\mathfrak{pqr}}, \quad \forall v \in (\mathcal{E}_\mathfrak{r})_x.$$

Definition 8.6 (Support system/support pair)**.** Let $\left(\mathcal{M}, \widehat{\mathcal{U}}\right)$ be a space with a G-equivariant good coordinate system

$$\widehat{\mathcal{U}} = \left((\mathfrak{P}, \leq), \{\mathcal{U}_{\mathfrak{p}} \mid \mathfrak{p} \in \mathfrak{P}\}, \{\vec{\alpha}_{\mathfrak{p}\mathfrak{q}} \mid \mathfrak{p}, \mathfrak{q} \in \mathfrak{P}, \mathfrak{q} \leq \mathfrak{p}\}\right).$$

A G**-equivariant support system** $\widehat{\mathcal{K}} = \{\mathcal{K}_{\mathfrak{p}} \mid \mathfrak{p} \in \mathfrak{P}\}$ on $\left(\mathcal{M}, \widehat{\mathcal{U}}\right)$ is a collection of sets satisfying the following.

- For each $\mathfrak{p} \in \mathfrak{P}$, $\mathcal{K}_{\mathfrak{p}} \subset \mathcal{U}_{\mathfrak{p}}$ is a nonempty G-invariant compact subset, which is the closure of some open set $\mathring{\mathcal{K}}_{\mathfrak{p}} \subset \mathcal{U}_{\mathfrak{p}}$.
- $\bigcup_{\mathfrak{p} \in \mathfrak{P}} \psi_{\mathfrak{p}} \left(\mathring{\mathcal{K}}_{\mathfrak{p}} \cap s_{\mathfrak{p}}^{-1}(0)\right) = \mathcal{M}$.

For any G-equivariant support system $\widehat{\mathcal{K}}$, we define

$$|\mathcal{K}| = \left(\bigsqcup_{\mathfrak{p} \in \mathfrak{P}} \mathcal{K}_{\mathfrak{p}}\right) / \sim, \tag{8.1.4}$$

where, for each $x \in \mathcal{K}_{\mathfrak{p}}, y \in \mathcal{K}_{\mathfrak{q}}$, we say $x \sim y$ if and only if either $y = \alpha_{\mathfrak{q}\mathfrak{p}}(x)$ or $x = \alpha_{\mathfrak{p}\mathfrak{q}}(y)$. The G-action on the charts induces a G-action on $|\mathcal{K}|$.

A G**-equivariant support pair** $\left(\widehat{\mathcal{K}}, \widehat{\mathcal{K}}^{++}\right)$ on $\left(\mathcal{M}, \widehat{\mathcal{U}}\right)$ consists of G-equivariant support systems $\widehat{\mathcal{K}} = \{\mathcal{K}_{\mathfrak{p}} \mid \mathfrak{p} \in \mathfrak{P}\}$, $\widehat{\mathcal{K}}^{++} = \{\mathcal{K}_{\mathfrak{p}}^{++} \mid \mathfrak{p} \in \mathfrak{P}\}$ such that for all $\mathfrak{p} \in \mathfrak{P}$ we have $\mathcal{K}_{\mathfrak{p}} \subset \mathring{\mathcal{K}}_{\mathfrak{p}}^{++}$. We write $\widehat{\mathcal{K}} < \widehat{\mathcal{K}}^{++}$ for such a pair.

Given a G-equivariant support pair $\left(\widehat{\mathcal{K}}, \widehat{\mathcal{K}}^{++}\right)$, a G-invariant metric on $|\mathcal{K}^{++}|$, and $\delta > 0$, we can define another support system

$$\widehat{\mathcal{K}}(2\delta) = \{\mathcal{K}_{\mathfrak{p}}(2\delta) \mid \mathfrak{p} \in \mathfrak{P}\},$$

where

$$\mathcal{K}_{\mathfrak{p}}(2\delta) = \{x \in \mathcal{K}_{\mathfrak{p}}^{++} \mid d(x, \mathcal{K}_{\mathfrak{p}}) < 2\delta\}.$$

Definition 8.7 (KG-embedding)**.** A G-equivariant **strict KG-embedding**

$$\{\vec{\alpha}_{\mathfrak{p}p} = (\alpha_{\mathfrak{p}p}, \widehat{\alpha}_{\mathfrak{p}p}) : \mathcal{U}_p \to \mathcal{U}_{\mathfrak{p}} \mid (p, \mathfrak{p}) \in \mathcal{M} \times \mathfrak{P}, p \in \operatorname{im} \psi_{\mathfrak{p}}\} : \widehat{\mathcal{U}} \longrightarrow \widehat{\mathcal{U}}$$

from a Kuranishi structure $\widehat{\mathcal{U}}$ on $\mathcal{M}$ to a good coordinate system $\widehat{\widehat{\mathcal{U}}}$ on $\mathcal{M}$ consists of one G-equivariant embedding of Kuranishi charts for each $(p, \mathfrak{p}) \in \mathcal{M} \times \mathfrak{P}, p \in \operatorname{im} \psi_{\mathfrak{p}}$ such that the following holds. If $\mathfrak{p}, \mathfrak{q} \in \mathfrak{P}, \mathfrak{q} \leq \mathfrak{p}$ and $p \in \operatorname{im} \psi_{\mathfrak{p}}$, $q \in \operatorname{im} \psi_p \cap \operatorname{im} \psi_{\mathfrak{q}}(U_{\mathfrak{pq}} \cap s_{\mathfrak{q}}^{-1}(0))$, then on $\mathcal{U}_q|_{U_{pq} \cap \alpha_{\mathfrak{q}q}^{-1}(U_{\mathfrak{pq}})}$ we have

$$\vec{\alpha}_{\mathfrak{p}p} \circ \vec{\alpha}_{pq} = \vec{\alpha}_{\mathfrak{pq}} \circ \vec{\alpha}_{\mathfrak{q}q}.$$

An **open substructure** of a G-equivariant Kuranishi structure $\widehat{\mathcal{U}}$ is a G-equivariant Kuranishi structure $\widehat{\mathcal{U}}'$ whose Kuranishi charts and coordinate changes are restrictions of those in $\widehat{\mathcal{U}}$ to G-invariant open subsets U_p' of Kuranishi neighborhoods U_p.

A G-equivariant **KG-embedding** $\widehat{\mathcal{U}} \longrightarrow \widehat{\widehat{\mathcal{U}}}$ is a G-equivariant strict KG-embedding $\widehat{\mathcal{U}}_0 \longrightarrow \widehat{\widehat{\mathcal{U}}}$ from an open substructure $\widehat{\mathcal{U}}_0$ of the Kuranishi structure $\widehat{\mathcal{U}}$.

Lemma 8.1. Suppose that $\mathcal{M}$ is a space with a G-equivariant Kuranishi structure with corners, where the G-action on each Kuranishi chart is free. Then $\mathcal{M}/G$ has a Kuranishi structure with corners.

Similarly, if $\mathcal{M}$ is a G-equivariant good coordinate system with corners, where the G-action on each Kuranishi chart is free, then $\mathcal{M}/G$ has a good coordinate system with corners.

Proof. Suppose the G-equivariant Kuranishi structure on $\mathcal{M}$ is given by

$$\widehat{\mathcal{U}} = \left(\{\mathcal{U}_p \,|\, p \in \mathcal{M}\}, \{\vec{\alpha}_{pq} \,|\, p \in \mathcal{M}, q \in \operatorname{im} \psi_p\} \right),$$

where the G-action on each chart $\mathcal{U}_p = (U_p, \mathcal{E}_p, \psi_p, s_p)$ is free.

By G-equivariance and the freeness of the G-action, we obtain Kuranishi charts of the form $\mathcal{U}_p/G = (U_p/G, \mathcal{E}_p/G, [\psi_p], [s_p])$. Suppose $\vec{\alpha}_{pq}$ is given by a pair of G-equivariant orbifold embeddings

$$\alpha_{pq} : U_{pq} \to U_p, \quad \widehat{\alpha}_{pq} : \mathcal{E}_q\big|_{U_{pq}} \to \mathcal{E}_p\,.$$

Then they induce maps $[\alpha_{pq}] : U_{pq}/G \to U_p/G$ and $[\widehat{\alpha}_{pq}] : (\mathcal{E}_q/G)|_{U_{pq}/G} \to \mathcal{E}_p/G$, which together define a Kuranishi coordinate change $[\vec{\alpha}_{pq}]$ from $\widehat{\mathcal{U}}_q/G$ to $\widehat{\mathcal{U}}_p/G$.

The data $\widehat{\mathcal{U}}/G = (\{\mathcal{U}_p/G \,|\, p \in \mathcal{M}/G\}, \{[\vec{\alpha}_{pq}] \,|\, p \in \mathcal{M}/G, q \in \operatorname{im}[\psi_p]\})$ satisfy the definition of a Kuranishi structure on $\mathcal{M}/G$. $\square$

Definition 8.8 (Dimension/Orientation)**.** Let $\mathcal{M}$ be a space with either a Kuranishi structure or a good coordinate system.

i) We define the dimension of a Kuranishi chart of the form $\mathcal{U} = (U, \mathcal{E}, \psi, s)$ by

$$\dim \mathcal{U} = \dim U - \operatorname{rank} \mathcal{E}.$$

We require the dimension of the Kuranishi charts in the Kuranishi structure (resp. good coordinate system) to be the same and define this common dimension to be the **dimension** of $\mathcal{M}$.

ii) An orientation of a Kuranishi chart $\mathcal{U} = (U, \mathcal{E}, \psi, s)$ is given by an orientation on U and an orientation on $\mathcal{E}$. An **orientation** on $\mathcal{M}$ is a choice of an orientation for each Kuranishi chart of the Kuranishi structure (resp. good coordinate system) such that the coordinate change maps are orientation-preserving.

Definition 8.9 (G-equivariant strongly smooth map: orbifold $\to$ manifold)**.** Let L be a smooth manifold with a G-action and U be a smooth effective orbifold with a smooth G-action. A G-equivariant continuous map $g : U \to L$ is a **strongly smooth map** if $g \circ \varphi : V \to L$ is smooth for all orbifold charts (V, Γ, φ) in the orbifold atlas of U.

Definition 8.10 (G-equivariant strongly smooth map: Kuranishi $\to$ manifold)**.** Let L be a smooth manifold with a G-action and $\left(\mathcal{M}, \widehat{\mathcal{U}}\right)$ be a G-equivariant Kuranishi space with Kuranishi structure

$$\widehat{\mathcal{U}} = \left(\{\mathcal{U}_p \,|\, p \in \mathcal{M}\}, \{\vec{\alpha}_{pq} \,|\, p \in \mathcal{M}, q \in \operatorname{im} \psi_p\}\right)$$

A G**-equivariant strongly smooth** map $\widehat{f} : (\mathcal{M}, \widehat{U}) \to L$ is a collection

$$\{f_p : U_p \to L \mid p \in \mathcal{M}\}$$

of G-equivariant strongly smooth maps satisfying the following. For all $p \in \mathcal{M}, q \in \operatorname{im} \psi_p$, the compatibility condition $f_p \circ \alpha_{pq} = f_q|_{U_{pq}}$ is satisfied. Define the map associated with the strongly smooth map $\widehat{f}$ by

$$f : \mathcal{M} \to L, \quad f(p) = f_p(o_p) \quad \forall p \in \mathcal{M},$$

where o_p is as in Definition 8.4 1). We can define a G-equivariant strongly smooth map $\widehat{\widehat{f}} = \{f_{\mathfrak{p}} : U_{\mathfrak{p}} \to L \mid \mathfrak{p} \in \mathfrak{P}\}$ from a space with good coordinate system to a manifold in a similar way.

Definition 8.11 (G-equivariant differential forms on a G-equivariant Kuranishi space)**.** Let $\mathcal{M}$ be a space with a G-equivariant Kuranishi structure as in Definition 8.4. A G**-equivariant differential** k**-form** on $\left(\mathcal{M}, \widehat{\mathcal{U}}\right)$ is given by a collection of differential forms

$$\widehat{\eta} = \left\{\eta_p \in \Omega^k_G(U_p) \,\middle|\, p \in \mathcal{M}\right\} \tag{8.1.5}$$

such that

$$(\alpha^G_{pq})^* \left(\eta_p\right)\big|_{U_{pq}} = \eta_q|_{U_{pq}}, \qquad \forall p \in \mathcal{M}, \quad \forall q \in \operatorname{im}\psi_p,$$

where

$$(\alpha^G_{pq})^* : \Omega^k_G(U_p) \to \Omega^k_G(U_q)$$

denotes the G-equivariant pullback via $U_q \xrightarrow{\text{restriction}} U_{pq} \xrightarrow{\alpha_{pq}} U_p$. We denote the set of G-equivariant differential k-forms on a G-equivariant Kuranishi space $\mathcal{M}$ by $\Omega^k_G\left(\mathcal{M}, \widehat{\mathcal{U}}\right)$ and denote $\Omega_G\left(\mathcal{M}, \widehat{\mathcal{U}}\right) = \bigoplus_{k \in \mathbb{N}} \Omega^k_G\left(\mathcal{M}, \widehat{\mathcal{U}}\right)$.

Definition 8.12 (G-equivariant differential forms on a good coordinate system)**.** Let $\mathcal{M}$ be a space with a G-equivariant good coordinate system as in (8.1.3). Let $\widehat{\widehat{\mathcal{K}}} = \{\mathcal{K}_{\mathfrak{p}} \mid \mathfrak{p} \in \mathfrak{P}\}$ be a support system on $\left(\mathcal{M}, \widehat{\widehat{\mathcal{U}}}\right)$. A G**-equivariant differential** k**-form** $\widehat{\widehat{\eta}}$ on $\left(\mathcal{M}, \widehat{\widehat{\mathcal{U}}}\right)$ assigns a G-equivariant differential k-form $\eta_{\mathfrak{p}}$ on $K_{\mathfrak{p}}$ for each $\mathfrak{p} \in \mathfrak{P}$ such that the following holds on a G-invariant open neighborhood of $\alpha^{-1}_{\mathfrak{pq}}(K_{\mathfrak{p}}) \cap K_{\mathfrak{q}}$:

$$(\alpha^G_{\mathfrak{pq}})^* \eta_{\mathfrak{p}} = \eta_{\mathfrak{q}}, \qquad \forall\, \mathfrak{p} \in \mathcal{M}, \quad \forall\, \mathfrak{q} \in \operatorname{im}\psi_{\mathfrak{p}}.$$

We denote the set of G-equivariant differential k-forms on a G-equivariant Kuranishi space $\mathcal{M}$ by $\Omega^k_G\left(\mathcal{M}, \widehat{\widehat{\mathcal{U}}}\right)$ and denote $\Omega_G\left(\mathcal{M}, \widehat{\widehat{\mathcal{U}}}\right) = \bigoplus_{k \in \mathbb{N}} \Omega^k_G\left(\mathcal{M}, \widehat{\widehat{\mathcal{U}}}\right)$.

Definition 8.13 (G-equivariant pullback map)**.** Let $\mathcal{M}$ be a space with a G-equivariant Kuranishi structure $\widehat{U}$ as in (8.1.1). Let $\widehat{f} = \{f_p : U_p \to L \mid p \in \mathcal{M}\} : (\mathcal{M}, \widehat{U}) \to L$ be a G-equivariant strongly smooth map and

$$\eta = \left\{\eta_p \in \Omega^k_G(U_p) \,\middle|\, p \in \mathcal{M}\right\}$$

be a G-equivariant differential k-form on $(\mathcal{M}, \widehat{U})$. Then the **$G$-equivariant pullback** of η via $\widehat{f}$ is given by

$$\widehat{f}^*\widehat{\eta} = \left\{f_p^*\eta_p \in \Omega^k_G(U_p) \,\middle|\, p \in \mathcal{M}\right\}.$$

We may also denote it by $f^*\widehat{\eta}$. Similarly, we can define the G-equivariant pullback $\widehat{\widehat{f}}^*\widehat{\eta}$ of a differential form $\widehat{\eta}$ on a good coordinate system via a G-equivariant strongly smooth map $\widehat{\widehat{f}}$.

Lemma 8.2. Let $\left(\mathcal{M}, \widehat{\mathcal{U}}\right)$ be a space with a G-equivariant Kuranishi structure

$$\widehat{\mathcal{U}} = \left(\{\mathcal{U}_p \,|\, p \in \mathcal{M}\}, \{\vec{\alpha}_{pq} \,|\, p \in \mathcal{M}, q \in \operatorname{im}\psi_p\}\right).$$

Then there exists a G-equivariant good coordinate system $\widehat{\widehat{\mathcal{U}}}$ on $\mathcal{M}$ and a G-equivariant KG-embedding $\widehat{\mathcal{U}} \longrightarrow \widehat{\widehat{\mathcal{U}}}$, given by a G-equivariant strict KG-embedding $\Phi : \widehat{\mathcal{U}}_0 \longrightarrow \widehat{\widehat{\mathcal{U}}}$ from an open substructure $\widehat{\mathcal{U}}_0$ of $\widehat{\mathcal{U}}$.

Moreover, the following holds.

i) Let $\widehat{h}$ be a G-equivariant differential form on $\left(\mathcal{M}, \widehat{\mathcal{U}}\right)$. Then there exists a G-equivariant differential form $\widehat{\widehat{h}}$ on $\left(\mathcal{M}, \widehat{\widehat{\mathcal{U}}}\right)$ such that $\Phi_G^*(\widehat{\widehat{h}}) = \widehat{h}\Big|_{\widehat{\mathcal{U}}_0}$.

ii) Let $\widehat{\mathcal{S}}$ be a G-equivariant CF-perturbation on $\left(\mathcal{M}, \widehat{\mathcal{U}}\right)$. Then there exists a G-equivariant CF-perturbation $\widehat{\widehat{\mathcal{S}}}$ on $\left(\mathcal{M}, \widehat{\widehat{\mathcal{U}}}\right)$ such that $\widehat{\mathcal{S}}\Big|_{\widehat{\mathcal{U}}_0}, \widehat{\widehat{\mathcal{S}}}$ are compatible with Φ and the following holds.

   a) If $\widehat{\mathcal{S}}$ is transverse to 0, then $\widehat{\widehat{\mathcal{S}}}$ is also transverse to 0.

   b) If $\widehat{f}$ is strongly submersive with respect to $\widehat{\mathcal{S}}$, then $\widehat{\widehat{f}}$ is strongly submersive with respect to $\widehat{\widehat{\mathcal{S}}}$.

c) If $\widehat{f}$ is weakly transverse to $g : M \to N$ with respect to $\widehat{\mathcal{S}}$, then $\widehat{f}$ is strongly transverse to $g : M \to N$ with respect to $\widehat{\widehat{\mathcal{S}}}$.

The proof of Lemma 8.2 is similar to that of [20] Theorem 3.35 and Lemma 9.10.

Proof sketch. For each $d \in \mathbb{N}$, let $S_d \mathcal{M} = \{p \in \mathcal{M} \mid \dim U_p \geq d\}$. The proof is based on a downward induction on d. Suppose a G-equivariant good coordinate system

$$\widehat{\mathcal{U}_{d+1}} = ((\mathfrak{P}, \leq), \{\mathcal{U}_\mathfrak{p} \mid \mathfrak{p} \in \mathfrak{P}\}, \{\vec{\alpha}_{\mathfrak{p}\mathfrak{q}} \mid \mathfrak{p}, \mathfrak{q} \in \mathfrak{P}, \mathfrak{q} \leq \mathfrak{p}\})$$

which covers $S_{d+1} \mathcal{M}$ is constructed. Pick a collection $\{K_\mathfrak{p} \mid \mathfrak{p} \in \mathfrak{P}\}$ of G-invariant compact subsets $K_\mathfrak{p} \subset U_\mathfrak{p}$ such that we have an open neighborhood of $S_{d+1} \mathcal{M}$:

$$S_{d+1} \mathcal{M} \subset \bigcup_{\mathfrak{p}} \psi_\mathfrak{p} \left(s_\mathfrak{p}^{-1}(0) \cap \operatorname{int} K_\mathfrak{p}\right).$$

Let

$$B = S_d \mathcal{M} \setminus \bigcup_{\mathfrak{p} \in \mathfrak{P}} \psi_\mathfrak{p} \left(s_\mathfrak{p}^{-1}(0) \cap \operatorname{int} K_\mathfrak{p}\right) \subset S_d \mathcal{M} \setminus S_{d+1} \mathcal{M}.$$

Also pick $x_1, \ldots, x_n \in S_d \mathcal{M} \setminus S_{d+1} \mathcal{M}$ and $\{K_i \mid 1 \leq i \leq n\}$ such that $K_i \subset U_{x_i}$ are G-invariant subsets and

$$\bigcup_{i=1}^n \psi_{x_i} \left(s_{x_i}^{-1}(0) \cap \operatorname{int} K_i\right) \supset B.$$

Then we can construct, as in the proof of [20] Theorem 3.35 and that of [20] Lemma 9.10, a G-equivariant good coordinate system that covers

$$\bigcup_{\mathfrak{p} \in \mathfrak{P}} \psi_\mathfrak{p} \left(s_\mathfrak{p}^{-1}(0) \cap \operatorname{int} K_\mathfrak{p}\right) \cup \bigcup_{i=1}^n \psi_{x_i} \left(s_{x_i}^{-1}(0)\right)$$

and satisfies the properties by induction on n. □

8.2 Equivariant CF-perturbations

Definition 8.14 (G-equivariant CF-perturbation representative on a G-invariant subset)**.** Let $\mathcal{U} = (U, \mathcal{E}, \psi, s)$ be a G-equivariant Kuranishi chart and $U_\mathfrak{r} \subset U$ be a G-invariant open subset of U. A G-**equivariant CF-perturbation representative**[1] of $\mathcal{U}$ on $U_\mathfrak{r}$ is a continuous family of data

$$\mathcal{S}_\mathfrak{r} = \{\mathcal{S}_\mathfrak{r}^\epsilon = (W_\mathfrak{r} \xrightarrow{\nu_\mathfrak{r}} U_\mathfrak{r}, \tau_\mathfrak{r}, \mathfrak{s}_\mathfrak{r}^\epsilon) \mid \epsilon \in (0,1]\} \tag{8.2.1}$$

such that the following holds.

i) $W_\mathfrak{r}$ is an effective orbifold with a smooth G-action.[2]

ii) $\nu_\mathfrak{r} : W_\mathfrak{r} \to U_\mathfrak{r}$ is a smooth oriented G-equivariant orbibundle.

iii) $\tau_\mathfrak{r} \in \Omega_G(W_\mathfrak{r})$ is a G-equivariant Thom form of $\nu_\mathfrak{r} : W_\mathfrak{r} \to U_\mathfrak{r}$.[3]

iv) Let $\nu_\mathfrak{r}^*(\mathcal{E}|_{U_\mathfrak{r}}) \to W_\mathfrak{r}$ be the pullback bundle of $\mathcal{E}|_{U_\mathfrak{r}} \to U_\mathfrak{r}$ via $\nu_\mathfrak{r}$ and let $pr_2 : \nu_\mathfrak{r}^*(\mathcal{E}|_{U_\mathfrak{r}}) \to \mathcal{E}|_{U_\mathfrak{r}}$ be the projection map. $\forall 0 < \epsilon \leq 1$, let $\tilde{\mathfrak{s}}_\mathfrak{r}^\epsilon : W_\mathfrak{r} \to \nu_\mathfrak{r}^*(\mathcal{E}|_{U_\mathfrak{r}})$ be a section of the bundle $\nu_\mathfrak{r}^*(\mathcal{E}|_{U_\mathfrak{r}}) \to W_\mathfrak{r}$ satisfying the following.

   a) $\mathfrak{s}_\mathfrak{r}^\epsilon = pr_2 \circ \tilde{\mathfrak{s}}_\mathfrak{r}^\epsilon : W_\mathfrak{r} \to \mathcal{E}|_{U_\mathfrak{r}}$ is a G-equivariant bundle map and the family $\{\mathfrak{s}_\mathfrak{r}^\epsilon\}_{\epsilon\in(0,1]}$ depends smoothly on ϵ.

   b) Moreover, $\lim\limits_{\epsilon\to 0} \mathfrak{s}_\mathfrak{r}^\epsilon = s \circ \nu_\mathfrak{r}$ in the compact C^1-topology.

$$\begin{array}{ccc}
\nu_\mathfrak{r}^*(\mathcal{E}|_{U_\mathfrak{r}}) & \xrightarrow{pr_2} & \mathcal{E}|_{U_\mathfrak{r}} \\
{\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle \pi|_{U_\mathfrak{r}}} \\
W_\mathfrak{r} & \xrightarrow[\nu_\mathfrak{r}]{} & U_\mathfrak{r}
\end{array}$$

Definition 8.15 (Equivalent G-equivariant CF-perturbation representatives on subsets)**.** Let $\mathcal{U} = (U, \mathcal{E}, \psi, s)$ be a G-equivariant Kuranishi chart and $U_\mathfrak{r} \subset U$ be a G-invariant open subset of U. Let

$$\mathcal{S}_\mathfrak{r} = \left\{\mathcal{S}_\mathfrak{r}^\epsilon = (W_\mathfrak{r} \xrightarrow{\nu_\mathfrak{r}} U_\mathfrak{r}, \mathfrak{s}_\mathfrak{r}^\epsilon, \tau_\mathfrak{r}) \,\middle|\, \epsilon \in (0,1]\right\}.$$

and

$$\mathcal{S}_\mathfrak{r}^i = \left\{\mathcal{S}_i^\epsilon = (W_i \xrightarrow{\nu_i} U_\mathfrak{r}, \mathfrak{s}_i^\epsilon, \tau_i) \,\middle|\, \epsilon \in (0,1]\right\} \quad \forall i \in \{1,2\}$$

[1]"CF" stands for "continuous family".

[2]Note that we require $W_\mathfrak{r}$ to be the total space of a G-orbibundle, unlike in the case of ordinary Kuranishi structures, W denotes an open subset of some vector space.

[3]The construction of a G-equivariant Thom form can be found in [31] Chapter 10.

be G-equivariant CF-perturbation representatives of $\mathcal{U} = (U, \mathcal{E}, \psi, s)$ on $U_{\mathfrak{r}}$.

- $\mathcal{S}^i_{\mathfrak{r}}$ is said to be a **projection** of $\mathcal{S}$ if there exists a G-equivariant bundle map $P : W_{\mathfrak{r}} \to W_i$ which fiberwise is a surjective linear map such that the following holds.

  a) $P_{G!}(\tau_{\mathfrak{r}}) = \tau_i$.

  b) For each $\epsilon \in (0, 1]$, $\mathfrak{s}^{\epsilon}_i \circ P = \mathfrak{s}^{\epsilon}_{\mathfrak{r}}$.

- $\mathcal{S}^1_{\mathfrak{r}}$ is said to be **equivalent** to $\mathcal{S}^2_{\mathfrak{r}}$ if there exist G-equivariant CF-perturbation representatives $\tilde{S}_j$ of $\mathcal{U}$ on $U_{\mathfrak{r}}$, $j = 0, \ldots, 2N$ such that

  a) $\forall 0 \leq k \leq N - 1$, $\tilde{\mathcal{S}}_{2k}$ and $\tilde{\mathcal{S}}_{2k+2}$ are both projections of $\tilde{\mathcal{S}}_{2k+1}$, and

  b) $\tilde{\mathcal{S}}_0 = \mathcal{S}^1_{\mathfrak{r}}$ and $\tilde{\mathcal{S}}_{2N} = \mathcal{S}^2_{\mathfrak{r}}$.

Definition 8.16 (G-equivariant CF-perturbation representative on a Kuranishi chart)**.** A **G-equivariant CF-perturbation representative** on a G-equivariant Kuranishi chart $\mathcal{U} = (U, \mathcal{E}, \psi, s)$ on $\mathcal{M}$ is a collection of data $\{\mathcal{S}_{\mathfrak{r}} \mid \mathfrak{r} \in \mathfrak{R}\}$, where

$$\mathcal{S}_{\mathfrak{r}} = \{\mathcal{S}^{\epsilon}_{\mathfrak{r}} = (W_{\mathfrak{r}} \xrightarrow{\nu_{\mathfrak{r}}} U_{\mathfrak{r}}, \mathfrak{s}^{\epsilon}_{\mathfrak{r}}, \tau_{\mathfrak{r}}) \mid \epsilon \in (0, 1]\},$$

such that the following holds.

i) $\forall\, \mathfrak{r} \in \mathfrak{R}$, $\mathcal{S}_{\mathfrak{r}}$ is a G-equivariant CF-perturbation representative of $\mathcal{U}$ on a G-invariant open subset $U_{\mathfrak{r}}$ of U.

ii) $\bigcup_{\mathfrak{r} \in \mathfrak{R}} U_{\mathfrak{r}} = U$.

iii) If $x \in U_{\mathfrak{r}_1} \cap U_{\mathfrak{r}_2}$ for some $\mathfrak{r}_1, \mathfrak{r}_2 \in \mathfrak{R}$, then there exists a G-invariant open subset $U_{\mathfrak{r}_{12}} \subset U_{\mathfrak{r}_1} \cap U_{\mathfrak{r}_2}$ such that the following holds. For each $k \in \{1, 2\}$, let $i_k : U_{\mathfrak{r}_{12}} \hookrightarrow U_{\mathfrak{r}_k}$ be the G-equivariant inclusion map and let

$$i_k^* \mathcal{S}_{\mathfrak{r}_k} = \left\{ \left(i_k^* W_{\mathfrak{r}_k} \to U_{\mathfrak{r}_{12}}, i_k^* \tau_{\mathfrak{r}_k}, i_k^* \, \mathfrak{s}^{\epsilon}_{\mathfrak{r}_k} \right) \,\middle|\, \epsilon \in (0, 1] \right\}$$

be the restriction of $\mathcal{S}_{\mathfrak{r}_k}$ to $U_{\mathfrak{r}_{12}}$. Then we require the CF-perturbation representatives $i_1^* \mathcal{S}_{\mathfrak{r}_1}, i_2^* \mathcal{S}_{\mathfrak{r}_2}$ of $\mathcal{U}$ on $U_{\mathfrak{r}_{12}}$ to be equivalent as in Definition 8.15.

Definition 8.17 (Equivalent G-equivariant CF-perturbations on a Kuranishi chart)**.** Let $\mathcal{S}^{(1)} = \{S_\mathfrak{r} \mid \mathfrak{r} \in \mathfrak{R}\}$ and $\mathcal{S}^{(2)} = \{S_\mathfrak{j} \mid \mathfrak{j} \in \mathfrak{J}\}$ be G-equivariant CF-perturbation representatives on a G-equivariant Kuranishi chart $\mathcal{U} = (U, \mathcal{E}, \psi, s)$. Then the G-equivariant CF-perturbations $\mathcal{S}^{(1)}, \mathcal{S}^{(2)}$ on $\mathcal{U}$ are said to be **equivalent** if, whenever $x \in U_\mathfrak{r} \cap U_\mathfrak{j}$ for some $\mathfrak{r} \in \mathfrak{R}, \mathfrak{j} \in \mathfrak{J}$, there exists a G-invariant suborbifold $U_{\mathfrak{r}\mathfrak{j}} \subset U$ such that $U_{\mathfrak{r}\mathfrak{j}} \subset U_\mathfrak{r} \cap U_\mathfrak{j}$ and the following holds. Let $i_\mathfrak{r} : U_{\mathfrak{r}\mathfrak{j}} \hookrightarrow U_\mathfrak{r}$ and $i_\mathfrak{j} : U_{\mathfrak{r}\mathfrak{j}} \hookrightarrow U_\mathfrak{j}$ be the G-equivariant inclusion maps. Then we require $i_\mathfrak{r}^* \mathcal{S}_\mathfrak{r}, i_\mathfrak{j}^* \mathcal{S}_\mathfrak{j}$ to be equivalent on $U_{\mathfrak{r}\mathfrak{j}}$.

Definition 8.18 (G-equivariant CF-perturbation on a Kuranishi chart)**.** Let $\mathcal{S}$ be a G-equivariant CF-perturbation representative on a G-equivariant Kuranishi chart $\mathcal{U}$. A **G-equivariant CF-perturbation** on $\mathcal{U}$ represented by $\mathcal{S}$ is the class

$$[\mathcal{S}] = \left\{ \mathcal{S}' \;\middle|\; \begin{array}{l} \mathcal{S}' \text{ is a CF-perturbation representative on } \mathcal{U}, \\ \mathcal{S}' \text{ is equivalent to } \mathcal{S} \end{array} \right\}$$

of G-equivariant CF-perturbation representatives on $\mathcal{U}$ that are equivalent to $\mathcal{S}$.

Definition 8.19. Let $\mathcal{U} = (U, \mathcal{E}, \psi, s)$ be a G-equivariant Kuranishi chart and let

$$\mathcal{S}_\mathfrak{r} = \{\mathcal{S}_\mathfrak{r}^\epsilon = (W_\mathfrak{r} \xrightarrow{\nu_\mathfrak{r}} U_\mathfrak{r}, \tau_\mathfrak{r}, \mathfrak{s}_\mathfrak{r}^\epsilon) \mid \epsilon \in (0, 1]\}$$

be a CF-perturbation on a G-invariant open subset $U_\mathfrak{r} \subset U$.

i) $\mathcal{S}_\mathfrak{r}$ is said to be **transverse to zero** if, $\forall 0 < \epsilon \leq 1$, the map $\mathfrak{s}_\mathfrak{r}^\epsilon|_{W_\mathfrak{r}^\epsilon}$ is transverse to the zero section on some G-invariant neighborhood $W_\mathfrak{r}^\epsilon \subset W_\mathfrak{r}$ of the support of $\tau_\mathfrak{r}$.

ii) Let L be a smooth manifold. A G-equivariant smooth map $f_\mathfrak{r} : U_\mathfrak{r} \to L$ is said to be **strongly submersive** with respect to $\mathcal{S}_\mathfrak{r}$ if $\mathcal{S}_\mathfrak{r}$ is transverse to zero and, $\forall 0 < \epsilon \leq 1$, , the map

$$f_\mathfrak{r} \circ \nu_\mathfrak{r}\big|_{(\mathfrak{s}_\mathfrak{r}^\epsilon)^{-1}(0)} : (\mathfrak{s}_\mathfrak{r}^\epsilon)^{-1}(0) \to L$$

is a submersion on some G-invariant neighborhood $W_\mathfrak{r}^\epsilon \subset W_\mathfrak{r}$ of the support of $\tau_\mathfrak{r}$.

iii) Let $f_{\mathfrak{r}} : U_{\mathfrak{r}} \to L$ be strongly submersive with respect to $\mathcal{S}_{\mathfrak{r}}$ and $g : N \to L$ be a smooth manifold between manifolds. We say $f_{\mathfrak{r}}$ is **strongly transverse to** g if $\mathcal{S}_{\mathfrak{r}}$ is transverse to zero and, for any $\epsilon \in (0,1]$ and any $x \in (\mathfrak{s}_{\mathfrak{r}}^{\epsilon})^{-1}(0)$, the map $f_{\mathfrak{r}} \circ \nu_{\mathfrak{r}}|_{(\mathfrak{s}_{\mathfrak{r}}^{\epsilon})^{-1}(0)}$ is transverse to g.

Definition 8.20. Let $\mathcal{U} = (U, \mathcal{E}, \psi, s)$ be a G-equivariant Kuranishi chart. Let $[\mathcal{S}]$ be a G-equivariant CF-perturbation on $\mathcal{U}$, where $\mathcal{S} = \{\mathcal{S}_{\mathfrak{r}} \mid \mathfrak{r} \in \mathfrak{R}\}$ and

$$\mathcal{S}_{\mathfrak{r}} = \{\mathcal{S}_{\mathfrak{r}}^{\epsilon} = (W_{\mathfrak{r}} \xrightarrow{\nu_{\mathfrak{r}}} U_{\mathfrak{r}}, \tau_{\mathfrak{r}}, \mathfrak{s}_{\mathfrak{r}}^{\epsilon}) \mid \epsilon \in (0,1]\}.$$

i) $[\mathcal{S}]$ is said to be **transverse to zero** if $\mathcal{S}_{\mathfrak{r}}$ is transverse to zero for all $\mathfrak{r} \in \mathfrak{R}$.

ii) Let L be a smooth manifold. Let $i_{\mathfrak{r}} : U_{\mathfrak{r}} \to U$ be the G-equivariant inclusion map. A G-equivariant smooth map $f : U \to L$ is said to be **strongly submersive** with respect to $[\mathcal{S}]$ if $[\mathcal{S}]$ is transverse to zero and $f \circ i_{\mathfrak{r}}$ is strongly submersive with respect to $S_{\mathfrak{r}}$ for all $\mathfrak{r} \in \mathfrak{R}$.

iii) We can define strong transversality between a strongly submersive map from a Kuranishi chart and a smooth map from a smooth manifold to the same target smooth manifold similarly.

Definition 8.21 (G-equivariant CF-perturbation on a Kuranishi space)**.** A G**-equivariant CF-perturbation** on a space $\mathcal{M}$ with a Kuranishi structure with corners

$$\widehat{\mathcal{U}} = (\{\mathcal{U}_p \,|\, p \in \mathcal{M}\}, \{\vec{\alpha}_{pq} \,|\, p \in \mathcal{M}, q \in \operatorname{im}\psi_p\})$$

is a collection

$$\widehat{\mathcal{S}} = \{[\mathcal{S}_p] \,|\, p \in \mathcal{M}\}$$

such that the following holds.

i) For each $p \in \mathcal{M}$, $[\mathcal{S}_p]$ is a G-equivariant CF-perturbation on $\mathcal{U}_p$ represented by

$$\mathcal{S}_p = \left\{\mathcal{S}_p^{\epsilon} = (W_p, \mathfrak{s}_p^{\epsilon}, \tau_p) \,\middle|\, \epsilon \in (0,1]\right\}$$

on $\mathcal{U}_p$.

ii) For any $p \in \mathcal{M}$, $q \in \operatorname{im} \psi_p$, the data $\left[\mathcal{S}_q|_{U_{pq}}\right]$ and $[\mathcal{S}_p]$ are compatible with the G-equivariant Kuranishi coordinate change

$$\vec{\alpha}_{pq} = \left(\alpha_{pq} : U_{pq} \to U_p,\, \widehat{\alpha}_{pq} : \mathcal{E}_q\,|_{U_{pq}} \to \mathcal{E}_p\right)$$

in the following sense.

a) For each $x \in U_{pq}$, $y = \alpha_{pq}(x)$, there exist a G-invariant open neighborhood $U_{q,x} \subset U_{pq}$ of x and a G-invariant open neighborhood $U_{p,y} = \alpha_{pq}(U_{q,x})$ of y such that there exist G-equivariant CF-perturbation representatives

$$\mathcal{S}_{q,x} = \left(W_{q,x}, \mathfrak{s}^{\epsilon}_{q,x}, \tau_{q,x}\right), \qquad \mathcal{S}_{p,y} = \left(W_{p,y}, \mathfrak{s}^{\epsilon}_{p,y}, \tau_{p,y}\right)$$

of $\left[\mathcal{S}_q|_{U_{q,x}}\right]$, $\left[\mathcal{S}_p|_{U_{p,y}}\right]$ satisfying the following.

- $W_{q,x} \subset W_q$, $W_{p,y} \subset W_p$ are G invariant suborbifolds such that $W_{q,x} \xrightarrow{h_{pq}} W_{p,y}$ is a G-equivariant diffeomorphism.
- $(h_{pq})_{G!}(\tau_{q,x}) = \tau_{p,y}$.
- $\mathfrak{s}^{\epsilon}_{q,x} = \mathfrak{s}^{\epsilon}_{q}\big|_{W_{q,x}}$, $\mathfrak{s}^{\epsilon}_{p,y} = \mathfrak{s}^{\epsilon}_{p}\big|_{W_{p,y}}$.
- For each $\epsilon \in (0,1]$, the following diagram commutes.

$$\begin{array}{ccc} \mathcal{E}_q|_{U_{q,x}} & \xrightarrow{\widehat{\alpha}_{pq}} & \mathcal{E}_p|_{U_{p,x}} \\ \mathfrak{s}^{\epsilon}_{q,x}\uparrow & & \uparrow\mathfrak{s}^{\epsilon}_{p,y} \\ W_{q,x} & \xrightarrow[h_{pq}]{} & W_{p,y} \end{array}$$

We can define a G-equivariant CF-perturbation

$$\widehat{\mathcal{U}} = \left((\mathfrak{P}, \leq)\,,\, \{\mathcal{U}_{\mathfrak{p}} \mid \mathfrak{p} \in \mathfrak{P}\}\,,\, \{\vec{\alpha}_{\mathfrak{pq}} \,|\, \mathfrak{p}, \mathfrak{q} \in \mathfrak{P}, \mathfrak{q} \leq \mathfrak{p}\}\right)$$

on a G-equivariant good coordinate system similarly.

Definition 8.22 (G-invariant partition of unity on a good coordinate system)**.** Let $\left(\mathcal{M}, \widehat{\mathcal{U}}\right)$ be a space with G-equivariant good coordinate system

$$\widehat{\mathcal{U}} = \left((\mathfrak{P}, \leq)\,,\, \{\mathcal{U}_{\mathfrak{p}} \mid \mathfrak{p} \in \mathfrak{P}\}\,,\, \{\vec{\alpha}_{\mathfrak{pq}} \,|\, \mathfrak{p}, \mathfrak{q} \in \mathfrak{P}, \mathfrak{q} \leq \mathfrak{p}\}\right).$$

Let $\left(\widehat{\mathcal{K}}, \widehat{\mathcal{K}}^{++}\right)$ be a G-equivariant support pair and let $\delta > 0$. Take a G-invariant metric on $|\mathcal{K}^{++}|$. A collection of functions $\{\chi_{\mathfrak{p}} \mid \mathfrak{p} \in \mathfrak{P}\}$ is said to be a G-invariant **partition of unity** on $\left(\mathcal{M}, \widehat{\mathcal{U}}\right)$ with respect to the data $\left(\widehat{\mathcal{K}}, \widehat{\mathcal{K}}^{++}, \delta\right)$ if it satisfies the following.

i) For each $\mathfrak{p}$, let

$$\Omega_{\mathfrak{p}}(\mathcal{K}, \delta) := \{x \in \mathcal{K}_{\mathfrak{p}}^{++} \mid d(x, \mathcal{K}_{\mathfrak{p}}) < \delta\}.$$

ii) For each $\mathfrak{p}$, $\chi_{\mathfrak{p}} : |\mathcal{K}^{++}| \to [0,1]$ is a G-invariant strongly smooth function in the following sense:

$$\chi_{\mathfrak{p}}\big|_{\mathcal{K}_{\mathfrak{p}}^{++} \cap \Omega_{\mathfrak{p}}(\mathcal{K},\delta)} : \mathcal{K}_{\mathfrak{p}}^{++} \cap \Omega_{\mathfrak{p}}(\mathcal{K}, \delta) \to [0,1]$$

is G-invariant and smooth.

iii) For each $\mathfrak{p}$, we require $\operatorname{supp} \chi_{\mathfrak{p}} \subset \Omega_{\mathfrak{p}}(\mathcal{K}, \delta)$.

iv) There exists an open neighborhood $\mathcal{N}$ of $\mathcal{M}$ in $|\mathcal{K}^{++}|$ such that

$$\sum_{\mathfrak{p}} \chi_{\mathfrak{p}}(x) = 1 \qquad \forall x \in \mathcal{N}.$$

Lemma 8.3. There exists a G-invariant partition of unity satisfying definition 8.22 subordinate to the G-equivariant good coordinate system associated with the Kuranishi structure on $\mathcal{M}_{k+1}(L, J, \beta)$ in Proposition 5.1.

Proof. Let $\left(\widehat{\mathcal{K}}, \widehat{\mathcal{K}}^{++}\right)$ be a G-equivariant support pair of the given good coordinate system. Take a G-invariant metric on $|\mathcal{K}^{++}|$. Then by [20] Proposition 7.68, if $\delta > 0$ is sufficiently small, a partition of unity $\tilde{\chi}_{\mathfrak{p}}$ associated with the data $\left(\mathcal{M}, \widehat{\mathcal{U}}, \widehat{\mathcal{K}}, \widehat{\mathcal{K}}^{++}, \delta\right)$ that may not be G-invariant exists. We average $\tilde{\chi}_{\mathfrak{p}}$ with respect to the G-action to obtain $\chi_{\mathfrak{p}}$, which is now G-invariant. □

We will define some sheaves of G-equivariant CF-perturbations via étale spaces.

Definition 8.23 (Étale space)**.** An **étale space** over a topological space Y is a pair $(\mathcal{A}, p)$ consisting of a topological space $\mathcal{A}$ and a continuous map $p : \mathcal{A} \to Y$ such that p is a local homeomorphism.

Given an étale space over Y, one can construct a sheaf as follows. Let Ω be any open subset of Y. We assign Ω the set of sections

$$\Gamma(\Omega, \mathcal{A}) = \{s : \Omega \to \mathcal{A} \mid s \text{ is continuous}, p \circ s = \mathrm{Id}_\Omega\}.$$

Definition 8.24 (Sheaf of G-equivariant CF-perturbations on a chart)**.** Let $\mathcal{U}_\mathfrak{p}$ be a G-equivariant Kuranishi chart. For any open subset $\Omega/G \subset U_\mathfrak{p}/G$, which is a quotient of a G-invariant open subset Ω of $U_\mathfrak{p}$, let

$$\mathcal{CF}^{G,\mathcal{U}_\mathfrak{p}}(\Omega)$$

be the set of G-equivariant CF-perturbations of $\mathcal{U}_\mathfrak{p}$ on Ω. Similar to [20] Proposition 7.22, $\mathcal{CF}^{G,\mathcal{U}_\mathfrak{p}}$ defines a sheaf on $U_\mathfrak{p}/G$. We define the **stalk** of $\mathcal{CF}^{G,\mathcal{U}_\mathfrak{p}}$ at a point $x \in \mathcal{U}_\mathfrak{p}$ by taking the direct limit

$$(\mathcal{CF}^{G,\mathcal{U}_\mathfrak{p}})_x = \varinjlim_{\Omega \ni x} \mathcal{CF}^{G,\mathcal{U}_\mathfrak{p}}(\Omega), \tag{8.2.2}$$

where Ω runs through all G-invariant open subsets of $U_\mathfrak{p}$ containing x. Indeed, the direct limit, up to isomorphism, can be constructed as

$$\bigsqcup_{\Omega \ni x} \mathcal{CF}^{G,\mathcal{U}_\mathfrak{p}}(\Omega)/\sim,$$

where, if $[\mathcal{S}_1] \in \mathcal{CF}^{G,\mathcal{U}_\mathfrak{p}}(\Omega)$ and $[\mathcal{S}_2] \in \mathcal{CF}^{G,\mathcal{U}_\mathfrak{p}}(\Omega')$, then $[\mathcal{S}_1] \sim [\mathcal{S}_2]$ if and only if there exists a G-invariant $\Omega'' \subset \mathcal{U}_\mathfrak{p}$, contained in both Ω and Ω', such that $x \in \Omega''$ and $[\mathcal{S}_1|_{\Omega''}] = [\mathcal{S}_2|_{\Omega''}]$.

For each G-invariant open subset $\Omega \subset U_\mathfrak{p}$ containing x, there is a map

$$\mathcal{CF}^{G,\mathcal{U}_\mathfrak{p}}(\Omega) \to (\mathcal{CF}^{G,\mathcal{U}_\mathfrak{p}})_x, \qquad [\mathcal{S}] \mapsto [\mathcal{S}_x]. \tag{8.2.3}$$

A member of $(\mathcal{CF}^{G,\mathcal{U}_\mathfrak{p}})_x$ is called a **germ**.

Definition 8.25. Let $\left(\mathcal{M}, \widehat{\mathcal{U}}\right)$ be a space with a G-equivariant good coordinate system. Suppose G acts on each Kuranishi chart freely. Let $\widehat{\mathcal{K}}$ be a support system on $\widehat{\mathcal{U}}$ and let $|\mathcal{K}|$ be as in (8.1.4). Let $x \in |\mathcal{K}|$ and $Q : \bigsqcup K_\mathfrak{p} \to |\mathcal{K}|$ be the map that identifies equivalent elements. Suppose

$$\mathfrak{P}(x) = \{\mathfrak{p} \in \mathfrak{P} \mid Q^{-1}(x) \cap K_\mathfrak{p} \neq \emptyset\} = \{\mathfrak{p}_1 \leq \cdots \leq \mathfrak{p}_k\}, \tag{8.2.4}$$

where $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ are distinct. We denote the maximal such $\mathfrak{p}_k$ by $\mathfrak{p}(x)$. We introduce the following notations.

i) Define

$$(\mathcal{CF}^{\mathrm{G}}_{\mathcal{K}})_x = \left\{[\mathcal{S}_x] \in (\mathcal{CF}^{G,\mathcal{U}_{\mathfrak{p}(x)}})_x \,\middle|\, \mathcal{S}_x \text{ is restrictable to } U_{\mathfrak{p}} \quad \forall\, \mathfrak{p} \in \mathfrak{P}(x)\right\}. \tag{8.2.5}$$

Let

$$|\,\mathcal{CF}^{\mathrm{G}}_{\mathcal{K}}\,| = \bigcup_{[x]\in|\mathcal{K}|/G} \{[x]\} \times (\mathcal{CF}^{\mathrm{G}}_{\mathcal{K}})_x$$

and $p^G_{\mathcal{K}} : |\,\mathcal{CF}^{\mathrm{G}}_{\mathcal{K}}\,| \to |\mathcal{K}|/G, \quad ([x], [\mathcal{S}_x]) \mapsto [x]$. We can topologize $|\,\mathcal{CF}^{\mathrm{G}}_{\mathcal{K}}\,|$ in a way similar to [20] Definition 12.20 such that $p^G_{\mathcal{K}}$ becomes a local homeomorphism. Then

$$\Omega/G \mapsto \mathcal{CF}^{\mathrm{G}}_{\mathcal{K}}(\Omega) = \left\{[\mathcal{S}] : \Omega \to |\,\mathcal{CF}^{\mathrm{G}}_{\mathcal{K}}\,| \,\middle|\, [\mathcal{S}] \text{ is continuous}, p^G_{\mathcal{K}} \circ [\mathcal{S}] = \mathrm{Id}_{\Omega/G}\right\}$$

defines a sheaf on $|\mathcal{K}|/G$.

ii) Define

$$(\mathcal{CF}^{\mathrm{G}}_{\pitchfork 0,\mathcal{K}})_x = \left\{[\mathcal{S}_x] \in (\mathcal{CF}^{\mathrm{G}}_{\mathcal{K}})_x \,\middle|\, \mathcal{S}_x \text{ is transverse to zero}\right\}. \tag{8.2.6}$$

One can similarly obtain a sheaf $\mathcal{CF}^{\mathrm{G}}_{\pitchfork 0,\mathcal{K}}$ on $|\mathcal{K}|/G$.

iii) Let $\widehat{f} : \left(\mathcal{M}, \widehat{\mathcal{U}}\right) \to L$ be a G-equivariant strongly smooth map. Define

$$(\mathcal{CF}^{G}_{\pitchfork f,\mathcal{K}})_x = \left\{[\mathcal{S}_x] \in (\mathcal{CF}^{\mathrm{G}}_{\mathcal{K}})_x \,\middle|\, \widehat{f} \text{ is strongly submersive with respect to } \mathcal{S}_x\right\}. \tag{8.2.7}$$

This defines a sheaf $\mathcal{CF}^{G}_{\pitchfork f,\mathcal{K}}$ on $|\mathcal{K}|/G$.

iv) Let $\widehat{f} : \left(\mathcal{M}, \widehat{\mathcal{U}}\right) \to L$ be a G-equivariant strongly smooth map and $g : N \to L$ be a smooth map between manifolds. Define

$$(\mathcal{CF}^{G}_{f\pitchfork g,\mathcal{K}})_x = \left\{[\mathcal{S}_x] \in (\mathcal{CF}^{\mathrm{G}}_{\mathcal{K}})_x \,\middle|\, \begin{array}{l} \widehat{f} \text{ is strongly transverse to } g \\ \text{with respect to } \mathcal{S}_x \end{array}\right\}. \tag{8.2.8}$$

This defines a sheaf $\mathcal{CF}^{G}_{f\pitchfork g,\mathcal{K}}$ on $|\mathcal{K}|/G$.

Definition 8.26 (Strongly transverse)**.** Let $\mathcal{U}$ be a G-equivariant Kuranishi chart on $\mathcal{M}$ and let $x \in \mathcal{U}$. Suppose $[\mathcal{S}_x]$ is a germ at x. Let $(W_x \xrightarrow{\nu_x} U_x, \tau_x, \{\mathfrak{s}_x^\epsilon\})$ be a representative of $[\mathcal{S}_x]$. Let $\left(V_x, \Gamma_x, F_x, \varphi_x, \widehat{\varphi}_x^W\right)$ and $\left(V_x, \Gamma_x, E_x, \varphi_x, \widehat{\varphi}_x^{\mathcal{E}}\right)$ be orbibundle charts of $W, \mathcal{E}$ at x, respectively. In particular, there exists a unique $o_x \in V_x$ such that $\varphi_x(o_x) = x$.

$$\begin{array}{ccc} V_x \times F_x & \xrightarrow{\widehat{\varphi}_x^W} & W_x \\ \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle \nu_x} \\ V_x & \xrightarrow{\varphi_x} & U_x \end{array} \quad , \quad \begin{array}{ccc} V_x \times E_x & \xrightarrow{\widehat{\varphi}_x^{\mathcal{E}}} & \mathcal{E}|_{U_x} \\ \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle \pi} \\ V_x & \xrightarrow{\varphi_x} & U_x \end{array} . \tag{8.2.9}$$

Let $\bar{\mathfrak{s}}_x^\epsilon = pr_2 \circ \left(\widehat{\varphi}_x^{\mathcal{E}}\right)^{-1} \circ \mathfrak{s}_x^\epsilon \circ \widehat{\varphi}_x^W : V_x \times F_x \to E_x$. We say $[\mathcal{S}_x]$ is **strongly transverse** if, $\forall \epsilon \in (0, 1]$, the derivative

$$\nabla_{(v,\xi)}^W \bar{\mathfrak{s}}_x^\epsilon : T_\xi F_x \to T_c E_x$$

in the F_x-direction is surjective for all $(v, \xi) \in \left(\widehat{\varphi}_x^{W_x}\right)^{-1} (\nu_x^{-1}(o_x) \cap \operatorname{supp}(\tau_x))$. Define

$$(\mathcal{CF}_{\pitchfork\pitchfork 0}^G)_x = \left\{[\mathcal{S}_x] \in (\mathcal{CF}_{\mathcal{K}}^{\mathrm{G}})_x \,\middle|\, \mathcal{S}_x \text{ is strongly transverse}\right\}. \tag{8.2.10}$$

Proposition 8.1. Let $\left(\mathcal{M}, \widehat{\mathcal{U}}\right)$ be a space with a G-equivariant good coordinate system. Suppose G acts on each chart freely. Let $\widehat{\mathcal{K}}$ be a support system on $\widehat{\mathcal{U}}$ and let $|\mathcal{K}|$ be as in (8.1.4).

i) The sheaves $\mathcal{CF}_{\mathcal{K}}^{\mathrm{G}}, \mathcal{CF}_{\pitchfork 0,\mathcal{K}}^{\mathrm{G}}$ are soft.

ii) If $\widehat{f}$ is weakly submersive then the sheaf $\mathcal{CF}_{\pitchfork f,\mathcal{K}}^G$ is soft. If $\widehat{f}$ is weakly transverse to g, then $\mathcal{CF}_{f\pitchfork g,\mathcal{K}}^G$ is soft.

Proof. The proof is similar to that of [20] Theorem 12.24. Let $\widehat{\mathcal{K}} = \{K_{\mathfrak{p}} \mid \mathfrak{p} \in \mathfrak{P}\}$ be a G-equivariant support system on $\mathcal{M}$ as in Definition 8.6. Let $|\mathcal{K}| = \bigsqcup K_{\mathfrak{p}}/\sim$ be the heterodimensional compactum as defined in (8.1.4).

Let $x \in |\mathcal{K}|$ and $Q : \bigsqcup K_{\mathfrak{p}} \to |\mathcal{K}|$ be the map that identifies equivalent elements. Then, by the definition of a good coordinate system, we have a totally ordered set

$$\mathfrak{P}(x) = \{\mathfrak{p} \in \mathfrak{P} \mid Q^{-1}(x) \cap K_{\mathfrak{p}} \neq \emptyset\} = \{\mathfrak{p}_1 \leq \ldots \leq \mathfrak{p}_k\} \tag{8.2.11}$$

for some positive integer k. We denote the maximal element $\mathfrak{p}_k$ by $\mathfrak{p}(x)$ and the minimal element $\mathfrak{p}_1$ by $\mathfrak{p}_-(x)$.

Lemma 8.4. There exists a germ $[\mathcal{S}_x] \in (\mathcal{CF}^G_{\mathcal{K}})_x$ such that $[\mathcal{S}_x|_{U_{\mathfrak{p}_1}}] \in (\mathcal{CF}^{G,\mathcal{U}_{\mathfrak{p}_1}}_{\pitchfork\pitchfork 0})_x$.

Proof of Lemma 8.4. We construct, for each j, a germ $[\mathcal{S}_{j,x}] \in (\mathcal{CF}^{G,\mathcal{U}_{\mathfrak{p}_j}}_{\pitchfork 0})_x$ such that, $\forall 1 \leq j \leq k$, $[\mathcal{S}_{j,x}]$ is restrictable to $\mathcal{U}_{\mathfrak{p}_i}$ for all $i < j$.

We induct on $1 \leq j \leq k$. Let $j = 1$. Take a G-invariant neighborhood $U_1 \subset U_{\mathfrak{p}_1}$ of x on which the coordinate change $\vec{\alpha}_{\mathfrak{p}(x),\mathfrak{p}_1}$ is defined. Let $W_1 = \mathcal{E}_{\mathfrak{p}_1}\big|_{U_1} \xrightarrow{\nu_1} U_1$ be the restriction of the obstruction bundle for $\mathfrak{p}_1$ and τ_1 be a G-equivariant Thom form. Define $\mathfrak{s}_1^\epsilon : W_1 \to \mathcal{E}_{\mathfrak{p}_1}\big|_{U_1}$ by

$$\mathfrak{s}_1^\epsilon(w) = s_{\mathfrak{p}_1} \circ \nu_1(w) + \epsilon w \qquad \forall w \in W_1. \tag{8.2.12}$$

Then

$$[\mathcal{S}_1] = \Big[\Big(W_1 \xrightarrow{\nu_1} U_1, \tau_1, \{\mathfrak{s}_1^\epsilon\}\Big)\Big]$$

is strongly transverse. Then $[\mathcal{S}_{1,x}] \in (\mathcal{CF}^{G,\mathcal{U}_{\mathfrak{p}_1}}_{\pitchfork\pitchfork 0})_x$. Note that the following holds.

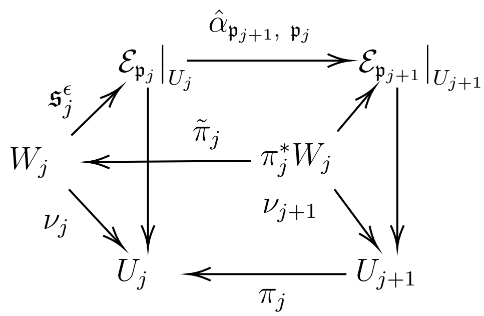
- $(\mathcal{CF}^{G,\mathcal{U}_{\mathfrak{p}_1}}_{\pitchfork\pitchfork 0})_x \subset (\mathcal{CF}^{G,\mathcal{U}_{\mathfrak{p}_1}}_{\pitchfork 0})_x$.
- If $\widehat{f}$ is weakly submersive, then $(\mathcal{CF}^{G,\mathcal{U}_{\mathfrak{p}_1}}_{\pitchfork\pitchfork 0})_x \subset (\mathcal{CF}^{G,\mathcal{U}_{\mathfrak{p}_1}}_{\pitchfork f})_x$.
- If $\widehat{f}$ is weakly transverse to g, then $(\mathcal{CF}^{G,\mathcal{U}_{\mathfrak{p}_1}}_{\pitchfork\pitchfork 0})_x \subset (\mathcal{CF}^{G,\mathcal{U}_{\mathfrak{p}_1}}_{f \pitchfork g})_x$.

Suppose a germ $[\mathcal{S}_{j,x}] \in (\mathcal{CF}^{G,\mathcal{U}_{\mathfrak{p}_j}}_{\pitchfork 0})_x$ is constructed. Then there exists some nonempty G-equivariant open subset $U_{j,0} \subset U_{\mathfrak{p}_j}$ such that the image of $[\mathcal{S}_j] = \Big[\Big(W_j \xrightarrow{\nu_j} U_{j,0}, \tau_j, \{\mathfrak{s}_j^\epsilon\}\Big)\Big]$ under the map $\mathcal{CF}^{G,\mathcal{U}_{\mathfrak{p}_j}}(U_j) \to (\mathcal{CF}^{G,\mathcal{U}_{\mathfrak{p}_j}})_x$ is $[\mathcal{S}_{j,x}]$.

We may take a nonempty G-invariant subset $U_j \subset U_{j,0}$ and a nonempty G-equivariant tubular neighborhood $U_{\mathfrak{p}_{j+1},x} \subset U_{\mathfrak{p}_{j+1}}$ of $\alpha_{\mathfrak{p}_{j+1},\mathfrak{p}_j}(U_j)$ such that the coordinate change map $\alpha_{\mathfrak{p}_{j+2},\mathfrak{p}_{j+1}}$ is defined on $U_{\mathfrak{p}_{j+1},x}$. Without loss of generality, we assume $U_{j,0} = U_j$

Then there is a projection map $\pi_j : U_{j+1} \to U_j$ such that $\pi_j \circ \alpha_{\mathfrak{p}_{j+1},\mathfrak{p}_j} = \mathrm{Id}$ on U_j.

Let $W_{j+1} = \pi_j^* W_j \xrightarrow{\nu_j} U_{j+1}$ be the pullback bundle and $\tau_{j+1} = \pi_j^* \tau_j$.



For any $w \in W_{j+1}$, define

$$\mathfrak{s}^{\epsilon}_{j+1}(w) = s_{\mathfrak{p}_{j+1}} \circ \nu_{j+1}(w) + \widehat{\alpha}_{\mathfrak{p}_{j+1},\mathfrak{p}_j}\left(\mathfrak{s}^{\epsilon}_j(\tilde{\pi}_j(w)) - s_{\mathfrak{p}_j} \circ \nu_j(\tilde{\pi}_j(w))\right). \tag{8.2.13}$$

Then

$$[\mathcal{S}_{j+1}] = \left[\left(W_{j+1} \xrightarrow{\nu_{j+1}} U_{j+1}, \tau_{j+1}, \{\mathfrak{s}^{\epsilon}_{j+1}\}\right)\right]$$

defines a germ $[\mathcal{S}_{j+1,x}] \in (\mathcal{CF}^{G,\mathcal{U}_{\mathfrak{p}_{j+1}}})_x$.

By this construction, $[\mathcal{S}_k]$ defines an element $[\mathcal{S}_x] \in (\mathcal{CF}^{\mathrm{G}}_{\mathcal{K}})_x$ satisfying Lemma 8.4. □

By forgetting the G-action, the construction coincides with the CF-perturbation constructed in [20] Lemma 12.12. In particular, the following holds.

i) $[\mathcal{S}_x] \in (\mathcal{CF}_{\mathcal{K}})_x$.

ii) If $\widehat{f}$ is weakly submersive, then $[\mathcal{S}_x] \in (\mathcal{CF}^G_{\pitchfork f,\mathcal{K}})_x$.

iii) If $\widehat{f}$ is weakly transverse to g, then $[\mathcal{S}_x] \in (\mathcal{CF}^G_{f\pitchfork g,\mathcal{K}})_x$.

Let $\bullet \in \{\pitchfork 0, \pitchfork f, f \pitchfork g\}$. Suppose $K \subset |\mathcal{K}|$ is a G-invariant closed subset and $[\mathcal{S}_K] \in \mathcal{CF}^G_{\bullet,\mathcal{K}}(K)$. Let $\mathcal{K}^{++}$ be another G-equivariant support system such that $\mathcal{K} < \mathcal{K}^{++}$. Then $[\mathcal{S}_K]$ is the restriction of some $[\mathcal{S}_{\mathfrak{r}_0}] \in \mathcal{CF}^G_{\bullet,\mathcal{K}}(U_{\mathfrak{r}_0} \cap |\mathcal{K}|)$ for some G-invariant subset $U_{\mathfrak{r}_0}$ of $|\mathcal{K}^{++}|$ containing K.

For each $x \in |\mathcal{K}| \setminus K$, let $[\mathcal{S}_x]$ and $[\mathcal{S}_{j,x}]$, $1 \le j \le \mathfrak{p}(x)$, be as in Lemma 8.4.

Since $|\mathcal{K}|$ is compact, we may take finitely many points $\{x_{\mathfrak{r}} \mid \mathfrak{r} \in \mathfrak{R}'\}$ and representatives $[(W_{\mathfrak{r}} \xrightarrow{\nu_{\mathfrak{r}}} U_{\mathfrak{r}}, \tau_{\mathfrak{r}}, \{\mathfrak{s}^{\epsilon}_{\mathfrak{r}}\})]$, $[(W_{x_{\mathfrak{r}},j} \xrightarrow{\nu_{x_{\mathfrak{r}},j}} U_{x_{\mathfrak{r}},j}, \tau_{x_{\mathfrak{r}},j}, \{\mathfrak{s}^{\epsilon}_{x_{\mathfrak{r}},j}\})]$ of $[\mathcal{S}_{x_{\mathfrak{r}}}]$ and $[\mathcal{S}_{j,x_{\mathfrak{r}}}]$ such that

$$\bigcup_{\mathfrak{r} \in \mathfrak{R}'} U_{\mathfrak{r}} \supset |\mathcal{K}|$$

and the following holds.

i) $\mathfrak{r}_0 \in \mathfrak{R}'$ and, for all $\mathfrak{r} \in \mathfrak{R}' \setminus \{\mathfrak{r}_0\}$, $U_{\mathfrak{r}} \cap K = \emptyset$.

ii) For any $\mathfrak{p}_{-}(x) \leq \mathfrak{p}_i \leq \mathfrak{p}(x)$, we consider local charts of the form (8.2.9). Let F_x be the fiber of ν_x and $E_{\mathfrak{p}_j}$ be the fiber of the orbibundle of $U_{\mathfrak{p}_j}$. By construction, we have a Γ_x-equivariant projection $\pi : V_{x,j} \to V_{x,1}$ and an embedding $I'_j : \pi^* \mathcal{E}_1 \to \mathcal{E}_j$. Then, for any $y \in V_{x,j}$ and $\xi \in F_x$, the following diagram is commutative.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_\xi F_x & \longrightarrow & T_{(y,\xi)}(V_{x,j} \times F_x) & \longrightarrow & T_y V_{x,j} & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle D_{(y,\xi)}\, \mathfrak{s}_1^{\epsilon}} & & \Big\downarrow {\scriptstyle D_{(y,\xi)}\, \mathfrak{s}_j^{\epsilon}} & & \Big\downarrow & & \\
0 & \longrightarrow & I'_j\left(\pi(y), E_{\mathfrak{p}_{-}(x)}\right) & \longrightarrow & T_c(E_{\mathfrak{p}_j}) & \longrightarrow & \dfrac{T_c(E_{\mathfrak{p}_j})}{I'_j\left(\pi(y), E_{\mathfrak{p}_{-}(x)}\right)} & \longrightarrow & 0
\end{array}$$

We may choose the CF-perturbations so that

a) $D_{(y,\xi)}\, \mathfrak{s}_1^{\epsilon}$ is surjective.

b) There exists a sufficiently small $\sigma_x > 0$ such that, if $|\, \mathfrak{s}_j^{\epsilon} - s_j|_{C^1} < \sigma_x$, then the third vertical map is surjective.

Let $\{\chi_{\mathfrak{r}} \mid \mathfrak{r} \in \mathfrak{R}'\}$ be a G-invariant partition of unity subordinate to this covering. Suppose $y \in |\mathcal{K}|$. Let

$$I(y) = \{\mathfrak{r} \in \mathfrak{R}' \mid \chi_{\mathfrak{r}}(y) \neq 0\}.$$

Then $\vec{\alpha}^{*}_{\mathfrak{p}(x_{\mathfrak{r}})\, \mathfrak{p}(y)}[\mathcal{S}_{\mathfrak{r}}]$ is defined. Let $(W_{\mathfrak{r}}, \tau_{\mathfrak{r}}, \{\mathfrak{s}^{\epsilon}_{\mathfrak{r}}\})$ be a representative of $\vec{\alpha}^{*}_{\mathfrak{p}(x_i)\, \mathfrak{p}(y)}[\mathcal{S}_{\mathfrak{r}}]$. Let $W_y = \prod_{\mathfrak{r} \in \mathfrak{R}'} W_{\mathfrak{r}}$, $\tau_y = \prod_{\mathfrak{r} \in \mathfrak{R}'} \tau_{\mathfrak{r}}$. Let $(w_{\mathfrak{r}})_{\mathfrak{r} \in I(y)} \in \prod_{\mathfrak{r} \in \mathfrak{R}'} (W_{\mathfrak{r}}|_z)$. Let $\sigma < \sigma_x$ and define

$$\mathfrak{s}^{\epsilon}_{y}\,(w) = s_y(z) + \chi_{\mathfrak{r}_0}(z)\left(\mathfrak{s}^{\epsilon}_{\mathfrak{r}_0}(w_{\mathfrak{r}_0}) - s_y(z)\right) + \sigma \sum_{\mathfrak{r} \in I(y)} \chi_{\mathfrak{r}}(z)\left(\mathfrak{s}^{\epsilon}_{\mathfrak{r}}(w_{\mathfrak{r}_0}) - s_y(z)\right).$$

If we forget the G-action, the construction is the same as that in [20] Theorem 12.24. Therefore, the transversality results follow from [20] Theorem 12.24. □

Proposition 8.1 implies the following.

Proposition 8.2 (Existence of a G-equivariant CF-perturbation on a good coordinate system)**.** Let $\left(\mathcal{M}, \widehat{\mathcal{U}}\right)$ be a space with G-equivariant good coordinate system. Let $\widehat{\mathcal{K}}$ be a support system on $\widehat{\mathcal{U}}$.

i) There exists a G-equivariant CF-perturbation $\widehat{\mathcal{S}}$ of $\widehat{\mathcal{U}}$ such that $\widehat{\mathcal{S}}$ is transverse to zero.

ii) If $\widehat{f} : (X, \widehat{\mathcal{U}}) \to L$ is a weakly submersive G-equivariant strongly smooth map, then $\widehat{\mathcal{S}}$ can be chosen so that $\widehat{f}$ is strongly submersive.

This induces a CF-perturbation on some Kuranishi structure on the same space in the same way as [20] Lemma 9.9.

8.3 Equivariant integration along the fiber

We review the G-equivariant integration along the fiber in Appendix **??** and refer the reader to [31] Chapter 10 for the detailed construction in the case of smooth G-manifolds. The case of ordinary Kuranishi structures is explained in [20] Chapter 7–9.

Definition 8.27 (G-equivariant integration along the fiber on a chart)**.** Let $[\mathcal{S}]$ be a G-equivariant CF-perturbation transverse to zero on a Kuranishi chart $\mathcal{U} = (U, \mathcal{E}, \psi, s)$, where $\mathcal{S} = \{\mathcal{S}_{\mathfrak{r}} \mid \mathfrak{r} \in \mathfrak{R}\}$ and

$$\mathcal{S}_{\mathfrak{r}} = \{\mathcal{S}_{\mathfrak{r}}^{\epsilon} = (W_{\mathfrak{r}} \xrightarrow{\nu_{\mathfrak{r}}} U_{\mathfrak{r}}, \tau_{\mathfrak{r}}, \mathfrak{s}_{\mathfrak{r}}^{\epsilon}) \mid \epsilon \in (0,1]\}.$$

Let $f : U \to L$ be a G-equivariant strongly smooth map on $\mathcal{U}$ such that $f : U \to L$ is strongly submersive with respect to $[\mathcal{S}]$. We define the **G-equivariant integration along the fiber of f via** $[\mathcal{S}]$ as follows.

Suppose $h \in \Omega^{l}_{G,c}(U)$ and $0 < \epsilon \leq \epsilon_0$. Let $\{\chi_{\mathfrak{r}} \mid \mathfrak{r} \in \mathfrak{R}\}$ be a partition of unity subordinate to the covering $\{U_{\mathfrak{r}} \mid \mathfrak{r} \in \mathfrak{R}\}$. Define $f_{G!}\left(h; \mathcal{S}^{\epsilon}\right) \in \Omega_{G,c}(L)$ as follows. Let

$$f_{G!}\left(h; \mathcal{S}^{\epsilon}\right)(\xi) = \sum_{\mathfrak{r} \in \mathfrak{R}} f_{!}\left(\chi_{\mathfrak{r}} h(\xi); \mathcal{S}_{\mathfrak{r}}^{\epsilon}\right), \qquad \forall \xi \in S(\mathfrak{g}^{*}),$$

where $f_{!}$ denotes the integration along the fiber in the case of ordinary Kuranishi structures.

Definition 8.28 (*G*-equivariant integration along the fiber on a good coordinate system)**.** Let $\left(\mathcal{M},\widehat{\mathcal{U}}\right)$ be a space with a good coordinate system

$$\widehat{\mathcal{U}} = ((\mathfrak{P},\leq)\,,\,\{\mathcal{U}_{\mathfrak{p}}\mid \mathfrak{p}\in\mathfrak{P}\}\,,\,\{\vec{\alpha}_{\mathfrak{pq}}\,|\,\mathfrak{p},\mathfrak{q}\in\mathfrak{P},\mathfrak{q}\leq\mathfrak{p}\})\,.$$

and let L be a smooth manifold. Let $\widehat{f}:\left(\mathcal{M},\widehat{\mathcal{U}}\right)\to L$ be a G-equivariant strongly smooth map. Let $\widehat{\mathcal{S}}$ be a CF-perturbation such that $\widehat{f}$ is strongly submersive with respect to $\widehat{\mathcal{S}}$. Let $\left(\widehat{\mathcal{K}},\widehat{\mathcal{K}}^{++}\right)$ be a G-equivariant support pair on $\left(\mathcal{M},\widehat{\mathcal{U}}\right)$.

Let $\widehat{h}=\{h_{\mathfrak{p}}\mid\mathfrak{p}\in\mathfrak{P}\}$ be a compactly supported G-equivariant differential form on $\mathcal{M}$. Define the ***G*-equivariant integration along the fiber** of h with respect to $\widehat{f},\widehat{\mathcal{S}}$ by

$$(\widehat{f}_G)_!\left(\widehat{h},\widehat{\mathcal{S}}^{\epsilon}\right)=\sum_{\mathfrak{p}\in\mathfrak{P}}(f_{\mathfrak{p}})_{G!}\left(\chi_{\mathfrak{p}}h_{\mathfrak{p}},\mathcal{S}^{\epsilon}_{\mathfrak{p}}\big|_{K_{\mathfrak{p}}(2\delta)\cap B_{\delta_2}(\mathcal{M})}\right).$$

This definition is independent of the choices of the support pair and the partition of unity [20] Proposition 7.81.

Definition 8.29 (*G*-equivariant integration along the fiber on a Kuranishi space)**.** Let $\left(\mathcal{M},\widehat{\mathcal{U}}\right)$ be a space with a G-equivariant Kuranishi structure $\widehat{\mathcal{U}}$ with corners and let N be a smooth manifold. Let $\widehat{f}:\left(\mathcal{M},\widehat{\mathcal{U}}\right)\to N$ be a G-equivariant strongly smooth map. Let $\widehat{\mathcal{S}}$ be a CF-perturbation such that $\widehat{f}$ is strongly submersive with respect to $\widehat{\mathcal{S}}$.

Let $\widehat{h}=\{h_p\mid p\in\mathcal{M}\}$ be a compactly supported G-equivariant differential form on $\mathcal{M}$.

Then, by Lemma 9.10 of [20], the Kuranishi data $\widehat{\mathcal{U}},\widehat{f},\widehat{\mathcal{S}},\widehat{h}$ induce (non-uniquely) some compatible good coordinate system data $\widehat{\widehat{\mathcal{U}}},\widehat{\widehat{f}},\widehat{\widehat{\mathcal{S}}},\widehat{\widehat{h}}$ on $\mathcal{M}$ such that the conditions in Definition 8.28 are satisfied.

Define the ***G*-equivariant integration along the fiber** of h with respect to $\widehat{f},\widehat{\mathcal{S}}$ by

$$\widehat{f}_{G!}\left(\widehat{h},\widehat{\mathcal{S}}^{\epsilon}\right)=\widehat{\widehat{f}}_{G!}\left(\widehat{\widehat{h}},\widehat{\widehat{\mathcal{S}}}^{\epsilon}\right).\tag{8.3.1}$$

This definition is independent of the choices by [20] Theorem 9.14.

8.4 Equivariant Stokes' theorem and the smooth correspondences

Definition 8.30 (Codimension-k corner of a manifold)**.** Let M be a manifold with corners and $k \in \mathbb{N}$. Define $S_k(M)$ to be the closure of the set

$$\left\{ x \in M \;\middle|\; \begin{array}{l} \text{there exists a neighborhood } V \text{ of } x \text{ such that} \\ V \text{ is diffeomorphic to } [0,\infty)^k \times \mathbb{R}^{n-k} \end{array} \right\}.$$

Lemma/Definition 8.1 (Normalized boundary of a manifold with corners, [20] Lemma 8.2)**.** For every manifold with corners V, there exists a manifold with corners ∂V and a smooth map $\pi : \partial V \to S_1(V)$ such that it induces a double covering map

$$\pi\big|_{S_1(V)\setminus S_2(V)} : S_1(V) \setminus S_2(V) \to S_1(V).$$

The Lemma is proved in [20] Lemma 8.2. We call ∂V the normalized boundary of V.

Definition 8.31 (Normalized boundary of an orbifold with corners)**.** Let U be an orbifold with corners and $\{(V_i, \Gamma_i, \varphi_i) \mid i \in I\}$ be an orbifold atlas on U. Then the **normalized boundary** of U is given by

$$\partial U = \bigcup_{i \in I} \varphi_i(\partial V_i / \Gamma_i).$$

Definition 8.32 (Normalized boundary of a Kuranishi space)**.** Let $(\mathcal{M}, \widehat{\mathcal{U}})$ be a G-equivariant Kuranishi space with corners as in Definition 8.4. The normalized boundary $\partial\left(\mathcal{M}, \widehat{\mathcal{U}}\right) := \left(\partial\,\mathcal{M}, \partial\widehat{\mathcal{U}}\right)$ of $\mathcal{M}$ is a Kuranishi space with corners, where

$$\partial\,\mathcal{M} = \bigcup_{p \in \mathcal{M}} \psi_p\left(s_p^{-1}(0) \cap \partial U_p\right)$$

and

$$\partial\widehat{\mathcal{U}} = \left(\left\{\partial\mathcal{U}_p \,|\, p \in \partial\,\mathcal{M}\right\}, \left\{\vec{\alpha}_{pq}\big|_{U_{pq}\cap\partial U_q} \,\middle|\, p \in \partial\,\mathcal{M}, q \in \psi_p(\partial U_p)\right\}\right),$$

which consists of G-equivariant Kuranishi charts

$$\partial\mathcal{U}_p = \left(\partial U_p, \mathcal{E}_p\big|_{\partial U_p}, \psi_p\big|_{\partial U_p}, s_p\big|_{\partial U_p}\right), \qquad \forall p \in \partial\,\mathcal{M}$$

and G-equivariant Kuranishi coordinate change data

$$\vec{\alpha}_{pq}\big|_{U_{pq}\cap\partial U_q} = \left(\alpha_{pq}\big|_{U_{pq}\cap\partial U_q}, \widehat{\alpha}_{pq}\big|_{\mathcal{E}_q|_{U_{pq}\cap\partial U_q}}\right), \qquad \forall p \in \partial\,\mathcal{M}, \quad q \in \psi_p(\partial U_p).$$

We can similarly define the normalized boundary of a good coordinate system.

Definition 8.33 (Normalized boundary of a good coordinate system)**.** Let $(\mathcal{M},\widehat{\mathcal{U}})$ be a G-equivariant space with good coordinate system with corners as in Definition 8.5. The normalized boundary $\partial\left(\mathcal{M},\widehat{\mathcal{U}}\right) := \left(\partial\,\mathcal{M},\partial\widehat{\mathcal{U}}\right)$ of $(\mathcal{M},\widehat{\mathcal{U}})$ is a good coordinate system with corners, where

$$\partial\,\mathcal{M} = \bigcup_{\mathfrak{p}\in\mathcal{M}} \psi_{\mathfrak{p}}\left(s_{\mathfrak{p}}^{-1}(0)\cap\partial U_{\mathfrak{p}}\right)$$

and

$$\partial\widehat{\mathcal{U}} = \left(\{\partial\mathcal{U}_{\mathfrak{p}} \,|\, \mathfrak{p}\in\partial\,\mathcal{M}\}, \left\{\vec{\alpha}_{\mathfrak{p}\mathfrak{q}}\big|_{U_{pq}\cap\partial U_{\mathfrak{q}}} \,\middle|\, \mathfrak{p}\in\partial\,\mathcal{M}, q\in\psi_{\mathfrak{p}}(\partial U_{\mathfrak{p}})\right\}\right),$$

which consists of G-equivariant Kuranishi charts

$$\partial\mathcal{U}_{\mathfrak{p}} = \left(\partial U_{\mathfrak{p}}, \mathcal{E}_{\mathfrak{p}}\,\big|_{\partial U_{\mathfrak{p}}}, \psi_{\mathfrak{p}}\big|_{\partial U_{\mathfrak{p}}}, s_{\mathfrak{p}}\big|_{\partial U_{\mathfrak{p}}}\right), \qquad \forall\,\mathfrak{p}\in\partial\,\mathcal{M}$$

and G-equivariant Kuranishi coordinate change data

$$\vec{\alpha}_{\mathfrak{p}\mathfrak{q}}\big|_{U_{\mathfrak{p}\mathfrak{q}}\cap\partial U_{\mathfrak{q}}} = \left(\alpha_{\mathfrak{p}\mathfrak{q}}\big|_{U_{\mathfrak{p}\mathfrak{q}}\cap\partial U_{\mathfrak{q}}}, \widehat{\alpha}_{\mathfrak{p}\mathfrak{q}}\big|_{\mathcal{E}_{\mathfrak{q}}|_{U_{\mathfrak{p}\mathfrak{q}}\cap\partial U_{\mathfrak{q}}}}\right), \qquad \forall\,\mathfrak{p}\in\partial\,\mathcal{M}, \quad \mathfrak{q}\in\psi_p(\partial U_{\mathfrak{p}}).$$

Theorem 8.1 (G-equivariant Stokes' Theorem on a good coordinate system with corners)**.** Let $(\mathcal{M},\widehat{\mathcal{U}})$ be a G-equivariant space with good coordinate system with corners as in Definition 8.5. Let N be a smooth G-manifold and $\widehat{f} : \left(\mathcal{M},\widehat{\mathcal{U}}\right) \to N$ be a G-equivariant strongly smooth map such that $\widehat{f}$ is strongly submersive with respect to some CF-perturbation $\widehat{\mathcal{S}}$ of $\widehat{\mathcal{U}}$. Then

$$\forall\widehat{\eta} = \{\eta_{\mathfrak{p}}\in\Omega^l_G(U_{\mathfrak{p}}) \mid \mathfrak{p}\in\mathfrak{P}\} \in \Omega^l_G\left(\mathcal{M},\widehat{\mathcal{U}}\right),$$

the following equality holds for sufficiently small $\epsilon > 0$:

$$d_G\left(\widehat{f}_{G!}\left(\widehat{\eta};\widehat{\mathcal{S}}^{\epsilon}\right)\right) = \widehat{f}_{G!}\left(d_G\widehat{\eta};\widehat{\mathcal{S}}^{\epsilon}\right) + (-1)^{\dim(\mathcal{M},\widehat{\mathcal{U}})+l}(\widehat{f}_{\partial})_{G!}\left(\widehat{\eta}_{\partial};\widehat{\mathcal{S}}^{\epsilon}_{\partial}\right),$$

where $\widehat{f}_{\partial}, \widehat{\eta}_{\partial}, \widehat{\mathcal{S}}^{\epsilon}_{\partial}$ are the restrictions of $\widehat{f}, \widehat{\eta}, \widehat{\mathcal{S}}^{\epsilon}$ to $\partial\left(\mathcal{M},\widehat{U}\right)$.

Proof of Theorem 8.1. For any $\xi \in \mathfrak{g}$, we have

$$d_G\left(\widehat{f}_{G!}\left(\widehat{\eta};\widehat{\mathcal{S}}^{\epsilon}\right)\right)(\xi) = d\left(\widehat{f}_!\left(\widehat{\eta}(\xi);\widehat{\mathcal{S}}^{\epsilon}\right)\right) \tag{8.4.1}$$

and

$$\begin{aligned}&\widehat{f}_{G!}\left(d_G\widehat{\eta};\widehat{\mathcal{S}}^{\epsilon}\right)(\xi) + (-1)^{\dim(\mathcal{M},\widehat{\mathcal{U}})+l}(\widehat{f}_{\partial})_{G!}\left(\widehat{\eta}_{\partial};\widehat{\mathcal{S}}^{\epsilon}_{\partial}\right)(\xi)\\ =&\widehat{f}_!\left(d(\widehat{\eta}(\xi));\widehat{\mathcal{S}}^{\epsilon}\right) + (-1)^{\dim(\mathcal{M},\widehat{\mathcal{U}})+l}(\widehat{f}_{\partial})_!\left(\widehat{\eta}_{\partial}(\xi);\widehat{\mathcal{S}}^{\epsilon}_{\partial}\right)\end{aligned} \tag{8.4.2}$$

Then (8.4.1) equals (8.4.2) by [20] Theorem 8.11 (the usual Stokes' Theorem on a good coordinate system with corners). □

We can prove the following theorem in a similar way by applying [20] Theorem 9.28 (the usual Stokes' Theorem on a Kuranishi space).

Theorem 8.2 (G-equivariant Stokes' Theorem on a Kuranishi space with corners)**.** Let $(\mathcal{M},\widehat{\mathcal{U}})$ be a G-equivariant Kuranishi space with corners as in 8.4 and $\widehat{\mathcal{S}}$ be a CF-perturbation of $\widehat{\mathcal{U}}$. Let L be a smooth G-manifold and $\widehat{f}:\left(\mathcal{M},\widehat{\mathcal{U}}\right)\to L$ be a G-equivariant map that is strongly submersive with respect to $\widehat{\mathcal{S}}$. Then $\forall\widehat{\eta}\in\Omega^l_G\left(\mathcal{M},\widehat{\mathcal{U}}\right)$, the following equality holds for sufficiently small $\epsilon > 0$:

$$d_G\left(\widehat{f}_{G!}\left(\widehat{\eta},\widehat{\mathcal{S}}^{\epsilon}\right)\right) = \widehat{f}_{G!}\left(d_G\widehat{\eta},\widehat{\mathcal{S}}^{\epsilon}\right) + (-1)^{\dim(\mathcal{M},\widehat{\mathcal{U}})+l}(\widehat{f}_{\partial})_{G!}\left(\widehat{\eta}_{\partial},\widehat{\mathcal{S}}^{\epsilon}_{\partial}\right),$$

where $\widehat{f}_{\partial},\widehat{\eta}_{\partial},\widehat{\mathcal{S}}^{\epsilon}_{\partial}$ are the restrictions of $\widehat{f},\widehat{\eta},\widehat{\mathcal{S}}^{\epsilon}$ to $\partial\left(\mathcal{M},\widehat{\mathcal{U}}\right)$.

Definition 8.34 (Weakly transverse to a smooth manifold map)**.** Let $\left(\mathcal{M},\widehat{\mathcal{U}}\right)$ be a G-equivariant Kuranishi space with Kuranishi structure

$$\widehat{\mathcal{U}} = \left(\{\mathcal{U}_p \,|\, p\in\mathcal{M}\},\{\vec{\alpha}_{pq}\,|\,p\in\mathcal{M}, q\in\operatorname{im}\psi_p\}\right)$$

Let

$$\widehat{f} = \{f_p : U_p\to L\,|\,p\in\mathcal{M}\}:\left(\mathcal{M},\widehat{\mathcal{U}}\right)\to L$$

be a G-equivariant strongly smooth map to a smooth G-manifold N. Let $g: N\to L$ be a G-equivariant smooth map between smooth manifolds. $\widehat{f}$ is said to be **weakly transverse to** g if f_p is transverse to g for each $p\in\mathcal{M}$.

Definition 8.35 (Weakly transverse strongly smooth maps)**.** Let $(\mathcal{M}_1,\widehat{\mathcal{U}_1})$, $(\mathcal{M}_2,\widehat{\mathcal{U}_2})$ be Kuranishi spaces and N be a smooth manifold. Let $\widehat{f_i} : (\mathcal{M}_i,\widehat{\mathcal{U}_i}) \to N$ be a strongly smooth map for $i \in \{1, 2\}$. Let $\Delta_L : L \to L \times L$ be the diagonal map. Then we say $\widehat{f_1}$ and $\widehat{f_2}$ are **weakly transverse** if the map

$$\widehat{f_1} \times \widehat{f_2} : (\mathcal{M}_1,\widehat{\mathcal{U}_1}) \times (\mathcal{M}_2,\widehat{\mathcal{U}_2}) \to L \times L$$

is weakly transverse to Δ_L as in Definition 8.34.

Definition 8.36 (Fiber product of a Kuranishi structure with a smooth manifold for maps in 8.34)**.** Let $\widehat{f}$ be weakly transverse to g as defined in Definition 8.34. Let f be the map associated with $\widehat{f}$ as in Definition 8.10. We can define a G**-equivariant Kuranishi structure on the fiber product**

$$\mathcal{M} \times_L N = \{(p, m) \in \mathcal{M} \times N \mid f(p) = g(m)\}.$$

Let $(p, m) \in \mathcal{M} \times_L N$ and $(U_p, \mathcal{E}_p, \psi_p, s_p)$ be the Kuranishi neighborhood assigned to p in $\widehat{\mathcal{U}}$. Let

$$U_p \times_L N = \{(x, m) \in U_p \times N \mid f_p(x) = g(m)\}.$$

Let $\pi_p : \mathcal{E}_p \to U_p$ be the obstruction bundle of $\mathcal{M}$ for p, $\mathrm{pr}_1 : U_p \times_L N \to U_p$ be the projection map to the first factor, and

$$\pi_{(p,m)} : \mathrm{pr}_1^* \mathcal{E}_p = \{((w, z), e) \in (U_p \times_L N) \times E_p \,|\, w = \pi_p(e)\} \to U_p \times_L N$$

be the pullback orbibundle. Then s_p induces a section of the pullback orbibundle by

$$s_{(p,m)}(w, z) = ((w, z), s_p(w)) \qquad \forall (w, z) \in U_p \times_L N. \tag{8.4.3}$$

Let

$$\mathcal{U}_{(p,m)} = \left(U_{(p,m)} := U_p \times_L N, \quad \mathcal{E}_{(p,m)} := pr_1^* \mathcal{E}_p, \quad s_{(p,m)}, \quad \psi_{(p,m)} := \psi_p \times \mathrm{Id}_N\right).$$

Let $(q, z) \in \psi_{(p,m)}(x, z)$ for some $(x, z) \in U_{(p,m)}$. Then we define

- $U_{(p,m),(q,z)} = U_{pq} \times_L N$;
- $\alpha_{(p,m),(q,z)} = \alpha_{pq} \times_N \mathrm{Id}_M : U_{pq} \times_L N \to U_p \times_L N$; and
- $\widehat{\alpha}_{(p,m),(q,z)} := \widehat{\alpha}_{pq} \times_L \mathrm{Id}_N : \mathcal{E}_{pq} \times_L N \to \mathcal{E}_p \times_L N$.

Then the fiber product $\mathcal{M} \times_L N$ induced by $\widehat{f}$ and g is a G-equivariant Kuranishi space with Kuranishi structure

$$\widehat{\mathcal{U}} \times_L N = \left(\begin{array}{l} \left\{ \mathcal{U}_{(p,m)} \,\middle|\, (p,m) \in \mathcal{M} \times_L N \right\}, \\ \left\{ \vec{\alpha}_{pq} = \left(\alpha_{(p,m),(q,z)}, \widehat{\alpha}_{(p,m),(q,z)}\right) \,\middle|\, \begin{array}{l} (p,m) \in \mathcal{M} \times_L N, \\ (q,z) \in \operatorname{im} \psi_{(p,m)} \end{array} \right\} \end{array} \right),$$

where

$$\mathcal{U}_{(p,m)} = \left(U_p \times_L N,\ \mathcal{E}_{(p,m)} := pr_1^* \mathcal{E}_p,\ s_{(p,m)} \text{is given by Eq. (8.4.3)}\ ,\ \psi_{(p,m)} := \psi_p \times \mathrm{Id}_M \right).$$

Definition 8.37 (Fiber product of Kuranishi structures)**.** Let $\widehat{f_1} : (\mathcal{M}_1, \widehat{\mathcal{U}_1}) \to L$ and $\widehat{f_2} : (\mathcal{M}_2, \widehat{\mathcal{U}_2}) \to L$ be G-equivariant weakly transverse strongly smooth maps as in Definition 8.35. We define

$$(\mathcal{M}_1, \widehat{\mathcal{U}_1})_{\widehat{f_1}} \times_{\widehat{f_2}} (\mathcal{M}_2, \widehat{\mathcal{U}_2})$$

to be the fiber product

$$\left((\mathcal{M}_1, \widehat{\mathcal{U}_1}) \times (\mathcal{M}_2, \widehat{\mathcal{U}_2}) \right) \times_L (L \times L)$$

induced by the weakly transverse maps $\widehat{f_1} \times \widehat{f_2}$ and Δ_L as in Definition 8.36.

Definition 8.38 (G-equivariant smooth correspondences)**.** Let N_s, N_t be oriented compact smooth G-manifolds without boundary. A G**-equivariant smooth correspondence** from N_s to N_t is a collection of data

$$\mathfrak{X} = \left(\mathcal{M}, \widehat{\mathcal{U}}, \widehat{f_s}, \widehat{f_t} \right),$$

where

- $\left(\mathcal{M}, \widehat{\mathcal{U}} \right)$ is an oriented G-equivariant Kuranishi space with corners,

- $\widehat{f}_s : \left(\mathcal{M}, \widehat{\mathcal{U}}\right) \to N_s$ is a G-equivariant strongly smooth map as in 8.10, and
- $\widehat{f}_t : \left(\mathcal{M}, \widehat{\mathcal{U}}\right) \to N_t$ is a G-equivariant strongly smooth and weakly submersive map.

A **perturbed G-equivariant smooth correspondence** from N_s to N_t is a pair $\left(\mathfrak{X}, \widehat{\mathcal{S}}\right)$, where $\mathfrak{X} = \left(\mathcal{M}, \widehat{\mathcal{U}}, \widehat{f}_s, \widehat{f}_t\right)$ is a smooth correspondence from N_s to N_t and $\widehat{\mathcal{S}}$ is a G-equivariant CF-perturbation of $\widehat{\mathcal{U}}$ with respect to which $\widehat{f}_t$ is strongly submersive.

Definition 8.39 (G-equivariant correspondence map)**.** Let $\left(\mathfrak{X} = (\mathcal{M}, \widehat{\mathcal{U}}, \widehat{f}_s, \widehat{f}_t), \widehat{\mathcal{S}}\right)$ be a perturbed smooth correspondence from N_s to N_t. For $\epsilon > 0$ sufficiently small, we define the **G-equivariant correspondence map** by

$$\mathrm{Corr}^{G,\epsilon}_{\left(\mathfrak{X},\widehat{\mathcal{S}}\right)} : \Omega^{\bullet}_G(N_s) \to \Omega^{\bullet + \dim N_t - \dim\left(\mathcal{M},\widehat{\mathcal{U}}\right)}_G(N_t)$$

associated with $\left(\mathfrak{X}, \widehat{\mathcal{S}}\right)$ by

$$\mathrm{Corr}^{G,\epsilon}_{\left(\mathfrak{X},\widehat{\mathcal{S}}\right)}(\eta) = (\widehat{f}_t)_{G!}\left((\widehat{f}_s)^*_G\eta; \mathcal{S}^{\epsilon}\right) \qquad \forall \eta \in \Omega^{\bullet}_G(N_s).$$

Then Stokes' Theorem 8.2 implies the following.

Proposition 8.3 (Compare with [20] Proposition 26.16)**.**

$$d_G \circ \mathrm{Corr}^{G,\epsilon}_{\left(\mathfrak{X},\widehat{\mathcal{S}}\right)} = \mathrm{Corr}^{G,\epsilon}_{\left(\mathfrak{X},\widehat{\mathcal{S}}\right)} \circ d_G + (-1)^{\dim \mathfrak{X} + \deg(\cdot)} \mathrm{Corr}^{G,\epsilon}_{\partial\left(\mathfrak{X},\widehat{\mathcal{S}}\right)}.$$

Definition 8.40 (Composition of smooth correspondences)**.** Let N_s, N_t be oriented compact smooth manifolds without boundary. For each bi-index $ji \in \{21, 32\}$, let $\mathfrak{X}_{ji} = \left(\mathcal{M}_{ji}, \widehat{\mathcal{U}}_{ji}, \widehat{f}_{s;ji}, \widehat{f}_{t;ji}\right)$ be a smooth correspondence from N_i to N_j. Assume $\widehat{f}_{t,21}$ and $\widehat{f}_{s,32}$ are weakly submersive. We define the **composition** $\mathfrak{X}_{31} = \left(\mathcal{M}_{31}, \widehat{\mathcal{U}}_{31}, \widehat{f}_{s,31}, \widehat{f}_{t,31}\right)$ of $\mathfrak{X}_{21}$ with $\mathfrak{X}_{32}$ by

$$\left(\mathcal{M}_{31}, \widehat{\mathcal{U}}_{31}\right) = \left(\mathcal{M}_{32}, \widehat{\mathcal{U}}_{32}\right) {}_{\widehat{f}_{s,32}} \times_{\widehat{f}_{t,21}} \left(\mathcal{M}_{21}, \widehat{\mathcal{U}}_{21}\right),$$

$$\widehat{f}_{s,31} : \left(\mathcal{M}_{31}, \widehat{\mathcal{U}}_{31}\right) \longrightarrow \mathcal{M}_{21} \xrightarrow{\widehat{f}_{s,21}} N_1,$$

$$\widehat{f}_{s,32} : \left(\mathcal{M}_{31}, \widehat{\mathcal{U}}_{31}\right) \longrightarrow \mathcal{M}_{32} \xrightarrow{\widehat{f}_{t,32}} N_2.$$

Then $\mathfrak{X}_{31}$ is again a smooth correspondence.

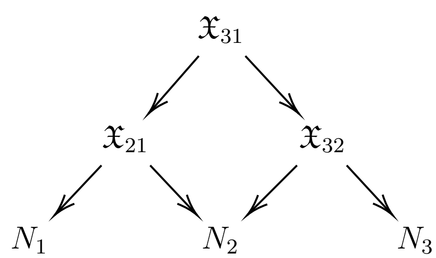
Definition 8.41 (Composition of perturbed G-equivariant smooth correspondences)**.** Let N_s, N_t be oriented compact smooth manifolds without boundary. For each bi-index $ji \in \{21, 32\}$, let $\left(\mathfrak{X}_{ji}, \widehat{\mathcal{S}}_{ji}\right)$ be a perturbed G-equivariant smooth correspondence from N_i to N_j, where $\mathfrak{X}_{ji} = \left(\mathcal{M}_{ji}, \widehat{\mathcal{U}}_{ji}, \widehat{f}_{s,ji}, \widehat{f}_{t,ji}\right)$ Assume $\widehat{f}_{t,21}$ and $\widehat{f}_{s,32}$ are weakly submersive. One can define the composition $\left(\mathfrak{X}_{31}, \widehat{\mathcal{S}}_{31}\right)$ of $\left(\mathfrak{X}_{32}, \widehat{\mathcal{S}}_{32}\right)$ with $\left(\mathfrak{X}_{21}, \widehat{\mathcal{S}}_{21}\right)$ so that

- $\mathfrak{X}_{31}$ is the composition of $\mathfrak{X}_{21}$ and $\mathfrak{X}_{32}$ as in Definition 8.40, and that
- $\widehat{f}_{t,31}$ is strongly submersive with respect to $\widehat{\mathcal{S}}_{31} = \widehat{\mathcal{S}}_{21\,\widehat{f}_{t,21}} \times_{\widehat{f}_{s,32}} \widehat{\mathcal{S}}_{32}$, whose construction we refer to Definition 10.13 of [20].

Then $\left(\mathfrak{X}_{31}, \widehat{\mathcal{S}}_{31}\right)$ is again a perturbed G-equivariant smooth correspondence.

Proposition 8.4 (Equivariant composition formula)**.** In the case of Definition 8.41, we have

$$\mathrm{Corr}^{G,\epsilon}_{\left(\mathfrak{X}_{32}, \widehat{\mathcal{S}}_{32}\right)} \circ \mathrm{Corr}^{G,\epsilon}_{\left(\mathfrak{X}_{21}, \widehat{\mathcal{S}}_{21}\right)} = \mathrm{Corr}^{G,\epsilon}_{\left(\mathfrak{X}_{31}, \widehat{\mathcal{S}}_{31}\right)}.$$



Proof. $\forall \eta \in \Omega_G(N_1)$, $\forall \xi \in \mathfrak{g}$,

$$\begin{aligned}
&\mathrm{Corr}^{G,\epsilon}_{\left(\mathfrak{X}_{32}, \widehat{\mathcal{S}}_{32}\right)} \circ \mathrm{Corr}^{G,\epsilon}_{\left(\mathfrak{X}_{21}, \widehat{\mathcal{S}}_{21}\right)}(\eta)(\xi) \\
=&\mathrm{Corr}^{\epsilon}_{\left(\mathfrak{X}_{32}, \widehat{\mathcal{S}}_{32}\right)} \circ \mathrm{Corr}^{\epsilon}_{\left(\mathfrak{X}_{21}, \widehat{\mathcal{S}}_{21}\right)}(\eta(\xi)) \\
=&\mathrm{Corr}^{\epsilon}_{\left(\mathfrak{X}_{31}, \widehat{\mathcal{S}}_{31}\right)}(\eta(\xi)) \qquad \text{by [20] Theorem 10.21} \\
=&\mathrm{Corr}^{G,\epsilon}_{\left(\mathfrak{X}_{31}, \widehat{\mathcal{S}}_{31}\right)}(\eta)(\xi).
\end{aligned}$$

□

Appendix A

Orbifolds

We review some orbifold theory in this appendix. The interested reader can read more about general orbifold theory in [2] and [20] Chapter 23 and equivariant orbifolds and equivariant Kuranishi charts in [18].

Definition A.1 (Orbifold chart)**.** Let U be a paracompact Hausdorff topological space. An n-dimensional effective **orbifold chart** of U is a triple (V, Γ, φ) such that

i) V is a smooth n-dimensional manifold (possibly with corners);

ii) Γ is a finite group acting smoothly and effectively on V;

iii) $\varphi : V \to U$ is a continuous map which induces a homeomorphism $\bar{\varphi} : V/\Gamma \to \varphi(V)$ onto an open subset $\varphi(V)$ of U.

Let $x \in U$. We say (V, Γ, φ) is an **orbifold chart at** x if there exists a point $o_x \in V$ such that $\varphi(o_x) = x$ and $\Gamma \cdot o_x = \{o_x\}$. Given an orbifold chart at x, the **tangent space** of the orbifold U at x is given by $T_xU = (T_{o_x}V)/\Gamma$.

Let (V, Γ, φ) be an orbifold chart and $p \in V$. An **orbifold subchart** of (V, Γ, φ) relative to p is an orbifold chart $(V_p, \Gamma_p, \varphi|_{V_p})$ such that Γ_p is the isotropy group of Γ at p, V_p is a Γ_p-invariant open neighborhood of p in V, and $\varphi|_{V_p}$ induces an injective map $V_p/\Gamma_p \to U$.

Definition A.2 (Embedding of orbifold charts)**.** Let $f : U_1 \to U_2$ be a continuous map between paracompact topological spaces. An **embedding** $(h, \lambda) : (V_1, \Gamma_1, \varphi_1) \to (V_2, \Gamma_2, \varphi_2)$ from an orbifold chart of U_1 to an orbifold chart of U_2 **relative to** f consists of

- a group isomorphism $h : \Gamma_1 \to \Gamma_2$ and
- an h-equivariant embedding of manifolds $\lambda : V_1 \to V_2$

such that $f \circ \varphi_1 = \varphi_2 \circ \lambda$.

An embedding of effective orbifold charts is an **isomorphism** if λ is also a diffeomorphism. If an isomorphism of two orbifold charts on the same topological space U is taken relative to the identity map, we may simply say it is an isomorphism of orbifold charts.

Definition A.3 (Orbifold)**.** Let U be a paracompact Hausdorff topological space. An n-dimensional (effective) **orbifold atlas** is a collection

$$\{(V_i, \Gamma_i, \varphi_i) \mid i \in I\}$$

of n-dimensional effective orbifold charts such that the following holds.

i) $\bigcup_{i \in I} \varphi_i(V_i) = U$.

ii) If $\varphi_i(p) = \varphi_j(q) = x$ for some $p \in V_i$, $q \in V_j$, then there exists an isomorphism of orbifold charts

$$(h_{qp}, \lambda_{qp}) : (V_{i,p}, (\Gamma_i)_p, \varphi_i\big|_{V_{i,p}}) \to (V_{j,q}, (\Gamma_j)_q, \varphi_j\big|_{V_{j,q}}),$$

called a **transition map**, from some orbifold subchart of $(V_i, \Gamma_i, \varphi_i)$ relative to p to some orbifold subchart of $(V_i, \Gamma_i, \varphi_i)$ relative to q.

A n-dimensional **maximal orbifold atlas** $\mathcal{A}$ on U is an n-dimensional orbifold atlas such that: $\mathcal{B} \subset \mathcal{A}$ whenever $\mathcal{B}$ and $\mathcal{A} \cup \mathcal{B}$ are both orbifold atlases on U. An n-dimensional effective **orbifold** $(U, \mathcal{A})$ is a paracompact Hausdorff topological space U equipped with an n-dimensional maximal orbifold atlas $\mathcal{A}$ on U.

Definition A.4 (Embedding of orbifolds)**.** A topological map $f : U_1 \to U_2$ between effective orbifolds is an **embedding of orbifolds** if, $\forall x \in U_1$, there exists an embedding of effective orbifold charts (h, λ) from an orbifold chart $(V_x, \Gamma_x, \varphi_x)$ of U_1 at x to an orbifold chart $(V'_y, \Gamma'_y, \varphi'_y)$ of U_2 at $y = f(x)$ relative to f. An embedding of orbifolds is a **diffeomorphism** of orbifolds if it is a homeomorphism. Let U be an effective orbifold. We denote the group of diffeomorphisms from U to itself by $\mathrm{Diff}(U)$, which is a topological group under the compact open topology.

Definition A.5 (Orbibundle chart)**.** Let $U, \mathcal{E}$ be orbifolds and $\pi : \mathcal{E} \to U$ be a continuous surjective map between the underlying topological spaces. An **orbibundle chart** is a quintuple $(V, E, \Gamma, \varphi, \widehat{\varphi})$, where

- (V, Γ, φ) is an orbifold chart on U,
- E is a finite-dimensional vector space with a linear Γ-action, and
- $(V \times E, \Gamma, \widehat{\varphi})$ is an orbifold chart on $\mathcal{E}$, where Γ acts on $V \times E$ diagonally

such that

i) $\pi \circ \widehat{\varphi} = \varphi \circ \mathrm{pr}_1$.

ii) $\widehat{\varphi}$ induces a homeomorphism on the quotients $\overline{\widehat{\varphi}} : (V \times E)/\Gamma \to \pi^{-1}(\overline{\varphi}(V/\Gamma))$ such that $\overline{\varphi}^{-1} \circ \pi \circ \overline{\widehat{\varphi}} = \overline{\mathrm{pr}}_1$.

$$\begin{CD} V \times E @>{\widehat{\varphi}}>> \mathcal{E} \\ @V{\mathrm{pr}_1}VV @VV{\pi}V \\ V @>>{\varphi}> U \end{CD} \qquad \begin{array}{ccc} (V \times E)/\Gamma & \xrightarrow{\quad \overline{\widehat{\varphi}} \quad} & \pi^{-1}(\overline{\varphi}(V/\Gamma)) \\ & {\scriptstyle \overline{\mathrm{pr}}_1} \searrow \quad \swarrow {\scriptstyle \overline{\varphi}^{-1} \circ \pi} & \\ & V/\Gamma & \end{array} .$$

Definition A.6 (Embedding of orbibundle charts)**.** Let $\mathcal{E}_1, U_1, \mathcal{E}_2, U_2$ be orbifolds and $\pi_1 : \mathcal{E}_1 \to U_1$, $\pi_2 : \mathcal{E}_2 \to U_2$ be continuous surjective maps between the underlying topological spaces. Let $f : U_1 \to U_2, \widehat{f} : \mathcal{E}_1 \to \mathcal{E}_2$ be continuous maps such that $f \circ \pi_1 = \pi_2 \circ \widehat{f}$. An **embedding of orbibundle charts** is a triple

$$(h, \lambda, \widehat{\lambda}) : (V_1, E_1, \Gamma_1, \varphi_1, \widehat{\varphi}_1) \to (V_2, E_2, \Gamma_2, \varphi_2, \widehat{\varphi}_2)$$

from an orbibundle chart on $\pi_1 : \mathcal{E}_1 \to U_1$ to an orbibundle chart on $\pi_2 : \mathcal{E}_2 \to U_2$ relative to $(f, \widehat{f})$ such that the following holds.

i) $(h, \lambda) : (V_1, \Gamma_1, \varphi_1) \to (V_2, \Gamma_2, \varphi_2)$ is an embedding of orbifold charts relative to f.

ii) $(h, \widehat{\lambda}) : (V_1 \times E_1, \Gamma_1, \widehat{\varphi}_1) \to (V_2 \times E_2, \Gamma_2, \widehat{\varphi}_2)$ is an embedding of orbifold charts relative to $\widehat{f}$.

iii) If $\pi_{V_1} : V_1 \times E_1 \to V_1$ and $\pi_{V_2} : V_2 \times E_2 \to V_2$ are projection maps to their first factors, then

   a) $\lambda \circ \pi_{V_1} = \pi_{V_2} \circ \widehat{\lambda}$ and

   b) for each $x \in V_1$, $\widehat{\lambda}\Big|_{\{x\}\times E_1} : \{x\} \times E_1 \to \{\lambda(x)\} \times E_2$ is a linear embedding.

An embedding of orbibundle charts is an **isomorphism** if (h, λ) and $(h, \widehat{\lambda})$ define isomorphisms of orbifold charts and, for each $x \in V$, $\widehat{\lambda}\Big|_{\{x\}\times E_1} : \{x\} \times E_1 \to \{\lambda(x)\} \times E_2$ is a linear isomorphism.

Definition A.7 (Orbibundle subchart at a point)**.** Let $(V, E, \Gamma, \varphi, \widehat{\varphi})$ be an orbibundle chart. If $(V_p, \Gamma_p, \varphi|_{V_p})$ is an orbifold subchart of (V, Γ, φ) relative to $p \in V$, then $(V_p, E, \Gamma_p, \varphi|_{V_p}, \widehat{\varphi}|_{V_p \times E})$ is also an orbibundle chart, called an **orbibundle subchart of** $(V, E, \Gamma, \varphi, \widehat{\varphi})$ **at** p.

Definition A.8 (Orbibundle atlas)**.** Let $U, \mathcal{E}$ be orbifolds and $\pi : \mathcal{E} \to U$ be a continuous surjective map between the underlying topological spaces. An **orbibundle atlas** is a locally finite collection of orbibundle charts

$$\{(V_i, E_i, \Gamma_i, \varphi_i, \widehat{\varphi}_i) \,|\, i \in I\}$$

such that

i) $\{(V_i, \Gamma_i, \varphi_i) \,|\, i \in I\}$ is an orbifold atlas on U;

ii) $\{(V_i \times E_i, \Gamma_i, \widehat{\varphi}_i) \,|\, i \in I\}$ is an orbifold atlas on $\mathcal{E}$;

iii) for any $i, j \in I$, if $p \in V_i$, $q \in V_j$ satisfy $\varphi_i(p) = \varphi_j(q)$, then there exist an isomorphism

$$(h_{qp}, \lambda_{qp}, \widehat{\lambda}_{qp}) : \left(V_{i,p}, E_i, \Gamma_{i,p}, \varphi\big|_{V_{i,p}}, \widehat{\varphi}_i\big|_{V_{i,p}\times E_i}\right) \to \left(V_{j,q}, E_j, \Gamma_{j,q}, \varphi\big|_{V_{j,q}}, \widehat{\varphi}_j\big|_{V_{j,q}\times E_j}\right)$$

between an orbibundle subchart of $(V_i, E_i, \Gamma_i, \varphi_i, \widehat{\varphi}_i)$ at p and an orbibundle subchart of $(V_j, E_j, \Gamma_j, \varphi_j, \widehat{\varphi}_j)$ at q.

Definition A.9 (Embedding of orbibundles)**.** For $\alpha \in \{1, 2\}$, let

$$\left(\mathcal{E}_\alpha \xrightarrow{\pi_\alpha} U_\alpha, \quad \widehat{\mathcal{V}}_\alpha = \{(V_i^\alpha, E_i^\alpha, \Gamma_i^\alpha, \varphi_i^\alpha, \widehat{\varphi}_i^\alpha) \,|\, i \in I_\alpha\}\right)$$

be a pair such that $\widehat{\mathcal{V}}_\alpha$ is an orbibundle atlas on $\mathcal{E}_\alpha \xrightarrow{\pi_\alpha} U_\alpha$. An **embedding of orbibundles** $\left(f, \widehat{f}\right) : \left(\mathcal{E}_1 \xrightarrow{\pi_1} U_1, \widehat{\mathcal{V}}_1\right) \to \left(\mathcal{E}_2 \xrightarrow{\pi_2} U_2, \widehat{\mathcal{U}}_2\right)$ consists of two orbifold embeddings $U_1 \xrightarrow{f} U_2$ and $\mathcal{E}_1 \xrightarrow{\widehat{f}} \mathcal{E}_2$ such that the follows holds.

i) For any $i \in I_1, j \in I_2$ and $p \in V_i^1$, $q \in V_j^2$ with $f(\varphi_i^1(p)) = \varphi_j^2(q)$, there exists an embedding $\left(h_{qp}, f_{qp}, \widehat{f}_{qp}\right)$ relative to $\left(f, \widehat{f}\right)$, from an orbibundle subchart of $(V_i^1, E_i^1, \Gamma_i^1, \varphi_i^1, \widehat{\varphi}_i^1)$ at p to an orbibundle subchart of $\left(V_j^2, E_j^2, \Gamma_j^2, \varphi_j^2, \widehat{\varphi}_j^2\right)$ at q.

ii) $\pi_2 \circ \widehat{f} = f \circ \pi_1$.

Two orbibundle atlases $\widehat{\mathcal{V}}_1, \widehat{\mathcal{V}}_2$ on $\mathcal{E} \xrightarrow{\pi} U$ are **equivalent** if the pair of identity maps $(\mathrm{Id}, \widehat{\mathrm{Id}}) : \left(\mathcal{E} \xrightarrow{\pi} U, \widehat{\mathcal{V}}_1\right) \to \left(\mathcal{E} \xrightarrow{\pi} U, \widehat{\mathcal{V}}_2\right)$ is an embedding of orbibundles and $\mathrm{Id}, \widehat{\mathrm{Id}}$ are diffeomorphisms of orbifolds.

Definition A.10 (Orbibundle)**.** An **orbibundle** $(\mathcal{E} \xrightarrow{\pi} U, [\widehat{\mathcal{V}}])$ consists of a continuous surjective map $\mathcal{E} \xrightarrow{\pi} U$ between the underlying topological spaces of two orbifolds and an equivalence class of orbibundle atlases on π.

Definition A.11 (G-action on an orbifold)**.** Let G be a compact connected Lie group and U be an effective orbifold. A continuous group homomorphism

$$\alpha : G \to \mathrm{Diff}(U), \qquad \alpha(g)(x) = g \cdot x \quad \forall x \in U,$$

is a **smooth action** of G on U, if $\forall g \in G, \forall x \in U$ there exist

- an open neighborhood R of g in G,
- orbifold charts $(V_x, \Gamma_x, \varphi_x)$ at x and $(V'_y, \Gamma'_y, \varphi'_y)$ at $y = g \cdot x$ of U,
- a group isomorphism $h_{g,x} : \Gamma_x \to \Gamma'_y$, and
- a smooth map $f_{g,x} : R \times V_x \to V'_y$

such that the following holds.

i) $f_{g,x}$ is $h_{g,x}$-equivariant:

$$f_{g,x}(\gamma \cdot p) = h_{g,x}(\gamma) f_{g,x}(p) \quad \forall p \in V_x.$$

ii) $\varphi'_y(g \cdot p) = g \cdot \varphi_x(p)$ for all $p \in V_x$.

An effective orbifold equipped with a smooth G-action is called a **G-orbifold**.

Definition A.12 (G-equivariant orbibundle)**.** Let $\pi : \mathcal{E} \to U$ be an orbibundle between G-orbifolds and

$$\{(V_i, E_i, \Gamma_i, \varphi_i, \widehat{\varphi}_i) \,|\, i \in I\}$$

be an orbibundle atlas on π. Then, in particular, $\forall g \in G, x \in \mathcal{E}$, the G-action on $\mathcal{E}$ induces some smooth map $f_{g,x} : R \times (V_x \times E_x) \to V_{g \cdot x} \times E_{g \cdot x}$. We say π is a **G-equivariant orbibundle** if the following holds.

i) π is G-equivariant: $\pi(g \cdot x) = g \cdot \pi(x) \qquad \forall g \in G, \quad \forall x \in U.$

ii) For each $g \in G, p \in V_1$, the map

$$E_x \to E_{g \cdot x}, \qquad v \mapsto \mathrm{pr}_2 \circ f_{g,x}(g, x, v)$$

is linear.

A **G-equivariant section** of a G-equivariant orbibundle $\pi : \mathcal{E} \to U$ is an orbifold embedding $s : U \to \mathcal{E}$ such that $\pi \circ s = \mathrm{Id}_U$ and $g \cdot s(x) = s(g \cdot x)$ for all $g \in G$, $x \in U$.

An **embedding of G-equivariant orbibundles** $\left(f, \widehat{f}\right) : \left(\mathcal{E}_1 \xrightarrow{\pi_1} U_1, \widehat{\mathcal{V}}_1\right) \to \left(\mathcal{E}_2 \xrightarrow{\pi_2} U_2, \widehat{\mathcal{U}}_2\right)$ is an embedding of orbibundles such that $f, \widehat{f}$ are both G-equivariant.

Definition A.13 (Differential form on an orbifold)**.** A **differential form** on an orbifold $(U, \{(V_i, \Gamma_i, \varphi_i) \,|\, i \in I\})$ is a collection

$$\eta = \{\eta_i \in \Omega(V)^{\Gamma_i} \mid i \in I\}$$

which associates each orbifold chart $(V_i, \Gamma_i, \varphi_i)$ with a Γ_i-invariant differential form $\eta_i \in \Omega(V_i)$ such that the following holds.

i) If $(h, \lambda) : (V_i, \Gamma_i, \varphi_i) \to (V_j, \Gamma_j, \varphi_j)$ is an isomorphism of orbifold charts, then $\lambda^* \eta_j = \eta_i$.

ii) If $\mathfrak{B}_j = (V_j, \Gamma_j, \varphi_j)$ is an orbifold subchart of $(V_i, \Gamma_i, \varphi_i)$, then $\eta_j = \eta_i|_{\mathfrak{B}_j}$.

Denote the set of differential forms on U by $\Omega(U)$. An **orientation** on an orbifold U is a choice of a differential form $\eta \in \Omega(U)$ such that η_i never vanishes.

Definition A.14 (Equivariant differential forms an orbifold)**.** Let U be a G-orbifold and $\Omega(U)$ be the space of differential forms on U. The set of G**-equivariant differential forms** on U is given by

$$\Omega_G(U) := \left(\Omega(U) \otimes S(\mathfrak{g}^*)\right)^G .$$

Appendix B

Rigid analytic geometry

In this appendix, we review some basic definitions and properties of rigid analytic geometry. We refer the interested reader to [6], [5], [17], [41], [46], [7], and [12]. For tropical analytic geometry and polyhedral domains (of the form $\mathrm{trop}^{-1}(\Delta)$, where Δ is a polyhedron), we refer the

We will work over an algebraically closed field Λ, which is a non-Archimedean field, namely, a field that is complete with respect to a non-Archimedean absolute value (see Definition B.2). Note that the general rigid analytic geometry concerns a non-Archimedean field K, which may not be algebraically closed.

Definition B.1 (Non-Archimedean valuation)**.** A function $\mathrm{val} : \Lambda \to \mathbb{R} \cup \{\infty\}$ is a **non-Archimedean valuation** on Λ if the following holds.

i) $\mathrm{val}(a) = \infty$ if and only if $a = 0$.

ii) $\mathrm{val}(ab) = \mathrm{val}(a) + \mathrm{val}(b)$ for all $a, b \in \Lambda$.

iii) $\mathrm{val}(a + b) \geq \min\{\mathrm{val}(a), \mathrm{val}(b)\}$ for all $a, b \in \Lambda$.

Definition B.2 (Non-Archimedean absolute value)**.** A function $|\cdot| : \Lambda \to \mathbb{R}_{\geq 0}$ is a **non-Archimedean absolute value** on Λ if the following holds.

i) $|a| = 0$ if and only if $a = 0$.

ii) $|ab| = |a| + |b|$ for all $a, b \in \Lambda$.

iii) $|a + b| \leq \max\{|a|, |b|\}$ for all $a, b \in \Lambda$.

From iii) one can see that $|n \cdot a| \leq |a|$ for any $n \in \mathbb{N}$, $|a| > 0$, which shows that the Archimedean property does not hold for a non-Archimedean absolute value. We can associate a non-Archimedean absolute value to a field with non-Archimedean valuation by defining

$$|a| = e^{-\operatorname{val}(a)},$$

where e is Euler's number.

We now introduce some major players of rigid analytic geometry.

Definition B.3 (Closed unit polydisc B^n_Λ)**.** The **closed unit polydisc** B^n_Λ is defined by

$$B^n_\Lambda = \{(x_1, \dots, x_n) \in \Lambda^n \mid |x_i| \leq 1 \quad \forall 1 \leq i \leq n\}.$$

The set of all power series that converge on B^n_Λ is called the Tate algebra.

Definition B.4 (Tate algebra)**.** Let $n \geq 1$. The **Tate algebra** in n variables is defined by

$$T_n = \left\{ \sum_{\boldsymbol{c} \in \mathbb{N}^n} a_{\boldsymbol{c}} x^{\boldsymbol{c}} \in \Lambda[\![x_1, \dots, x_n]\!] \,\middle|\, a_{\boldsymbol{c}} \in \Lambda, \lim_{|\boldsymbol{c}| \to \infty} |a_{\boldsymbol{c}}| = 0 \right\},$$

where if $\boldsymbol{c} = (c_1, \dots, c_n)$ is a multi-index, then $x^{\boldsymbol{c}} = x_1^{c_1} \cdots x_n^{c_n}$ and $|\boldsymbol{c}| = c_1 + \cdots + c_n$. Equivalently,

$$T_n = \left\{ \sum_{\boldsymbol{c} \in \mathbb{N}^n} a_{\boldsymbol{c}} x^{\boldsymbol{c}} \in \Lambda[\![x_1, \dots, x_n]\!] \,\middle|\, \lim_{|\boldsymbol{c}| \to \infty} \operatorname{val}(a_{\boldsymbol{c}}) = \infty \right\}.$$

We denote it by $\Lambda \langle x_1, \dots, x_n \rangle$. In particular, $T_0 = \Lambda$.

Proposition B.1 (Properties of the Tate algebra)**.** Let $n \geq 1$. The Tate algebra T_n is normal and is a Noetherian integral domain.

Definition B.5 (Λ-affinoid algebra and Λ-affinoid space)**.** A Λ-algebra that is isomorphic to T_n/I for some ideal I in T_n is called a **Λ-affinoid algebra** and the maximal spectrum of A, denoted by

$$\operatorname{Sp} A = \{\mathfrak{a} \subset A \mid \mathfrak{a} \text{ is a maximal ideal of } A\},$$

is called a **Λ-affinoid space**. A Λ-algebra morphism $f : A \to B$ between two Λ-affinoid algebras is called a **morphism of Λ-affinoid algebras**. If $f^{\#} : \operatorname{Sp} B \to \operatorname{Sp} A$ is induced by a morphism $f : A \to B$ of Λ-affinoid algebras such that

$$f^{\#}(\mathfrak{m}) = f^{-1}(\mathfrak{m}) \quad \forall\, \mathfrak{m} \in \operatorname{Sp} B,$$

then $f^{\#}$ is called a **morphism of Λ-affinoid spaces**.

Definition B.6 (Λ-affinoid subdomain)**.** Let $f : A \to B$ be a Λ-affinoid algebra homomorphism. Then $f^{\#}(\operatorname{Sp} B)$ is a **Λ-affinoid subdomain** of $\operatorname{Sp} A$ if the following universal property holds. Whenever $g : A \to C$ is a Λ-affinoid algebra homomorphism such that $g^{\#}(\operatorname{Sp} C) \subset f^{\#}(\operatorname{Sp} B)$, there exists a unique morphism $h : A \to C$ of Λ-affinoid algebras such that $g^{\#} = f^{\#} \circ h^{\#}$.

We call such a morphism $f^{\#}$ of Λ-affinoid spaces an **open immersion** if it satisfies the universal property as above.

By [6] Proposition 7.7.2/1, if $f^{\#} : \operatorname{Sp} B \to \operatorname{Sp} A$ satisfies the universal property, then it is injective. So the affinoid subdomain $f^{\#}(\operatorname{Sp} B) \subset \operatorname{Sp} A$ can be identified with the affinoid space $\operatorname{Sp} B$.

Definition B.7 (Grothendieck topology)**.** A **Grothendieck topology (G-topology)** consists of

- a category $\mathcal{C}$, called the admissible open subsets, and,
- for each $U \in \operatorname{ob} \mathcal{C}$, a set $\operatorname{Cov} U$, called the set of admissible coverings of U, which consists of families of the form $(U_i \xrightarrow{\Phi_i} U)_{i \in I}$, where each Φ_i is a morphism in $\mathcal{C}$,

such that the following holds.

i) If $\Phi : U' \to U$ is an isomorphism in $\operatorname{mor} \mathcal{C}$, then the family $(U' \xrightarrow{\Phi} U) \in \operatorname{Cov} U$.

ii) If $(U_i \xrightarrow{\Phi_i} U)_{i \in I} \in \operatorname{Cov} U$ and $(U_{ij} \xrightarrow{\Phi_{ij}} U_i)_{j \in J} \in \operatorname{Cov} U_i$, then $(U_{ij} \xrightarrow{\Phi_{ij}} U_i \xrightarrow{\Phi_i} U)_{i \in I, j \in J} \in \operatorname{Cov} U$.

iii) If $(U_i \xrightarrow{\Phi_i} U)_{i \in I} \in \operatorname{Cov} U$ and $V \to U$ belongs to $\operatorname{mor} \mathcal{C}$, then the fiber products $U_i \times_U V$ exist in $\mathcal{C}$ and $(U_i \times_U V \to V)_{i \in I}$ belongs to $\operatorname{Cov} V$.

A category with a Grothendieck topology is called a **site**. It's a generalization of a topological space. A topological space can be viewed as a site whose admissible open subsets are open subsets of X and the set of admissible coverings of an open subset U of X consists of the open covers of U.

Definition B.8 (Weak Grothendieck topology on a Λ-affinoid space)**.** The **weak G-topology** on a Λ-affinoid space X is defined as follows. The *admissible open* subsets are the affinoid subdomains of X and the *admissible coverings* of an affinoid subdomain $U \subset X$ are the coverings of U by finitely many affinoid subdomains of X.

Definition B.9 (Structure sheaf)**.** Let X be an affinoid space. We define a structure presheaf $\mathscr{O}_X$ on the site X as follows. Let $U \subset X$ be an affinoid subdomain of X which is the image of a morphism $f^\# : \operatorname{Sp} B \to \operatorname{Sp} A$ of affinoid spaces. $\mathscr{O}_X(U) = B$. By Tate's acyclicity theorem [5] 4.3/Theorem 1, $\mathscr{O}_X$ is a sheaf.

Definition B.10 (Strong Grothendieck topology on a Λ-affinoid space)**.** The **strong G-topology** on a Λ-affinoid space X is defined as follows.

- A subset $U \subset X$ is *admissible open* if there is a covering $U = \bigcup_{i \in I} U_i$ of U by (not necessarily finitely many) affinoid subdomains U_i of X such that, for any Λ-affinoid space morphism $\varphi : Y \to X$ with $\varphi(Y) \subset U$, the covering $\{\varphi^{-1}(U_i)\}_{i \in I}$ has a refinement which is a covering by finitely many affinoid subdomains of Y.

- A covering $U = \bigcup_{j \in J} U_j$ of an admissible open subset U of X is *admissible* if, for each affinoid space morphism $\varphi : Y \to X$ with $\varphi(Y) \subset U$, the covering $\{\varphi^{-1}(U_i)\}_{i \in I}$ has a refinement which is a covering by finitely many affinoid subdomains of Y.

Any sheaf defined with respect to the weak G-topology extends uniquely to a sheaf with respect to the strong G-topology. In particular, $\mathscr{O}_X$ extends to a sheaf with respect to the strong G-topology.

Definition B.11 (Λ-rigid analytic space)**.** A Λ-rigid analytic space is a pair $(X, \mathscr{O}_X)$ such that the following holds.

i) X can be endowed with a G-topology which satisfies the completeness conditions in the following sense.

   (G0) $\emptyset, X$ are admissible open.

   (G1) If $U = \bigcup_{i \in I} U_i$ is an admissible covering and $V \subset U$ is a subset such that $V \cap U_i$ is admissible open for all i, then V is admissible open in X.

   (G2) If U, U_i are admissible open for all $i \in I$ and $U = \bigcup_{i \in I} U_i$ is an admissible covering which admits an admissible refinement, then $(U_i)_{i \in I}$ is an admissible covering of U.

ii) $\mathscr{O}_X$ is a sheaf of Λ-algebras such that there exists an admissible covering $X = \bigcup_{i \in I} X_i$ where each $(X_i, \mathscr{O}_X|_{X_i})$ is a Λ-affinoid space.

Proposition B.2 ([5] 5.1/Proposition 7)**.** Suppose X is an affinoid space with the strong Grothendieck topology and $f \in \mathscr{O}_X(X)$. Let

$$U = \{x \in X \mid |f(x)| < 1\}, \quad U' = \{x \in X \mid |f(x)| > 1\}, \quad U'' = \{x \in X \mid |f(x)| > 0\}.$$

Then any finite union of the sets of these types is admissible open in the strong G-topology. Any finite covering by finite unions of such sets is admissible.

Definition B.12 (Dimension of a rigid analytic space)**.** Let X be a rigid analytic space and $x \in X$. Define

$$\mathscr{O}_{X,x} := \varinjlim_{U \ni x} \mathscr{O}_X(U),$$

where the direct limit is taken over all admissible open subsets U of X containing x.

The dimension $\dim_x X$ of a rigid analytic space X at a point $x \in X$ is defined to be the Krull dimension of $\mathscr{O}_{X,x}$. The **dimension** $\dim X$ of a rigid analytic space X is defined to be

$$\dim X := \sup_{x \in X} \dim_x X = \sup_{x \in X} \dim \mathscr{O}_{X,x} .$$

In particular, the dimension of a Λ-affinoid space $X = \operatorname{Sp} A$ is the Krull dimension of A.

For any Λ-scheme X of finite type, one can associate a Λ-rigid analytic space X^{an} with X such that the underlying set of X^{an} is the set of closed points of X.

Appendix C

Tropical geometry

We review some tropical geometry which we use in Chapter 6.

Let Λ be a field with a valuation and $\Lambda^* = \Lambda \setminus \{0\}$. Let

$$\Lambda_0 = \{y \in \Lambda \mid \mathrm{val}(y) \geq 0\}, \quad \Lambda_+ = \{y \in \Lambda \mid \mathrm{val}(y) > 0\},$$

and $k = \Lambda_0/\Lambda_+$ be the residue field.

Recall there is a tropicalization map defined on the algebraic n-torus $(\Lambda^*)^n$:

$$\mathrm{trop} : (\Lambda^*)^n \to \mathbb{R}^n, \qquad (y_1, \ldots, y_n) \mapsto (\mathrm{val}(y_1), \ldots, \mathrm{val}(y_n)).$$

Definition C.1 (Tropicalization of a Laurent polynomial)**.** For any Laurent polynomial

$$f = \sum_{c \in \mathbb{Z}^n} a_c y^c \in \Lambda[y_1^{\pm 1}, \ldots, y_n^{\pm 1}],$$

where $y^c := y_1^{c_1} \cdots y_n^{c_n}$, $a_c \in \Lambda$, define the **tropicalization** $\mathrm{trop}\, f : \mathbb{R}^n \to \mathbb{R}$ of f by

$$(\mathrm{trop}\, f)(u) = \min\{\mathrm{val}(a_c) + \langle u, c \rangle \mid c \in \mathbb{Z}^n\} \qquad \forall u \in \mathbb{R}^n\,.$$

Definition C.2 (Tropical variety)**.** We define the **tropical hypersurface** $V(\mathrm{trop}\, f)$ associated with a Laurent polynomial

$$f = \sum_{c \in \mathbb{Z}^n} a_c y^c \in \Lambda[y_1^{\pm 1}, \ldots, y_n^{\pm 1}]$$

such that $u \in \mathbb{R}^n$ is an element of $V(\operatorname{trop} f)$ if and only if there exist at least two $c', c'' \in \mathbb{Z}^n$ such that

$$\operatorname{trop}(f)(u) = \operatorname{val}(a_{c'}) + \langle u, c' \rangle = \operatorname{val}(a_{c''}) + \langle u, c'' \rangle .$$

Let $I \subset \Lambda[y_1^{\pm 1}, \ldots, y_n^{\pm 1}]$ be an ideal. The tropical variety associated with $V(I)$ is defined to be

$$V(\operatorname{trop}(I)) = \bigcap_{f \in I} \operatorname{trop}(V(f)).$$

Definition C.3 (Initial form/ideal)**.** Let

$$f = \sum_{c \in \mathbb{Z}^n} a_c y^c \in \Lambda[y_1^{\pm 1}, \ldots, y_n^{\pm 1}], \quad u \in \mathbb{R}^n .$$

The Laurent polynomial

$$in_u(f) = \sum_{\substack{c \in \mathbb{Z}^n \\ \operatorname{trop}(f)(u) = \operatorname{val}(a_c) + \langle u, c \rangle}} \overline{T^{-\operatorname{val}(a_c)} a_c} \cdot y^c \in k[y_1^{\pm 1}, \ldots, y_n^{\pm 1}],$$

where $T \in \Lambda$ is an element satisfying

$$\operatorname{val}(T^\lambda) = \lambda \qquad \forall \lambda \in \operatorname{val}(\Lambda^*)$$

and $\overline{} : \Lambda^* \to k^*$ is the reduction map, is called the **initial form** of the Laurent polynomial f at u. Similarly, if I is an ideal in $\Lambda[y_1^{\pm 1}, \ldots, y_n^{\pm 1}]$ and $u \in \mathbb{R}^n$, the **initial ideal** $in_u(I)$ is given by

$$in_u(I) = (in_u(f) \mid f \in I) .$$

Theorem C.1 (Kapranov's Theorem, [42] Theorem 3.1.3)**.** Let Λ be an algebraically closed field with a non-trivial valuation. Let

$$f = \sum_{c \in \mathbb{Z}^n} a_c y^c \in \Lambda[y_1^{\pm 1}, \ldots, y_n^{\pm 1}]$$

be a Laurent polynomial. Then

$$V(\operatorname{trop} f) = \{u \in \mathbb{R}^n \mid in_u(f) \text{ is not a monomial}\} = \overline{\operatorname{trop}(V(f))},$$

where the last set is the closure of the image $\operatorname{trop}(V(f))$ of $V(f) \subset (\Lambda^*)^n$ under the coordinate-wise valuation map in $\mathbb{R}^n$.

Theorem C.2 (Fundamental theorem of tropical algebraic geometry, [42] Theorem 3.2.3)**.** Let Λ be an algebraically closed field with a non-trivial valuation. Let $I \subset \Lambda[y_1^{\pm 1}, \ldots, y_n^{\pm 1}]$ be an ideal. Then

$$\bigcap_{f \in I} \mathrm{trop}(V(f)) = \{u \in \mathbb{R}^n \mid in_u(I) \neq \langle 1 \rangle\} = \overline{\mathrm{trop}(V(I))},$$

where the last set is the closure of the image $\mathrm{trop}(V(I))$ of $V(I) \subset (\Lambda^*)^n$ under the coordinate-wise valuation map in $\mathbb{R}^n$.

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