Revisiting Localization, Periodicity and Galois Symmetry

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Abstract of the Dissertation

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It is known that two complex algebraic varieties can be algebraically isomorphic but not homeomorphic. Such examples can be obtained by changing the coefficients of the defining equations by some automorphism of the ground field.

This dissertation aims to understand how the entire Galois group of $\overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$, changes the underlying manifold structures of smooth complex varieties defined by equations with coefficients in $\overline{\mathbb{Q}}$. It is known by the theory of finite covering spaces (étale theory) that the Galois action does not change that aspect of the homotopy type determined by finite group theory (the profinite homotopy type). Thus we can use the known theory of manifolds in a given homotopy type to study the Galois conjugates of algebraic varieties in a given étale homotopy type. We study three aspects of this problem: (1) what algebraic-topological data is sufficient to specify a topological manifold in a homotopy type; (2) what might be the étale construction for manifolds; (3) how might one express the Galois action in terms of the algebraic-topological data. We suggest an approach using the study in (2) in order to propose a geometric interpretation of the question in (3).

Dedication Page

Dedicated to a great work: 1970 MIT notes
"Geometric Topology: Localization, Periodicity and Galois Symmetry"

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Chapter 1

Introduction

1.1 Review of Problem

When one applies topology to study algebraic objects like zero loci of polynomials, one needs to be very careful about which topological data are algebraic and which are transcendental. One motivating example to consider is a complex variety defined by some polynomials with coefficients in the algebraic closure $\overline{\mathbb{Q}}$ of the rationals $\mathbb{Q}$. Any field automorphism of $\mathbb{C}$ produces an algebraic isomorphism from some new variety to the given variety. However, the algebraic isomorphism is usually not analytically continuous. Hence, one can expect that a field automorphism of $\mathbb{C}$ might change the homeomorphism type of the variety, or even worse, the homotopy type.

A famous example was given by Serre in 1964 ([Ser64]). He constructed a complex variety defined over some algebraic number field and an automorphism of this field so that the automorphism changes the fundamental group of the variety.

In other words, the usual homotopy invariants might be inappropriate to study algebraic objects, e.g., the fundamental group. There are two ways to define the fundamental group in topology: one is the group of homotopy classes of paths and the other is the group of automorphisms of the universal cover. The first definition is not good enough for varieties since it uses the transcendental topology of $\mathbb{C}$. However, the second one can be adapted to algebraic discussions. It has been known for a long time, dating back to Riemann, that a finite covering map over a variety is algebraic. Hence, one should think about the system of all finite coverings. The automorphism group of this system is akin to the universal cover definition for the fundamental group; in particular, this group is isomorphic to the system of all finite quotients of the fundamental group.

There is a way to extend this idea to homotopy theory. Namely, one may only keep the finite part of a homotopy type, which is called the profinite completion. It was proven by Artin-Mazur ([AM69]) that the profinite completion of a complex variety is indeed an algebraic invariant. One corollary is that the symmetry on varieties induced by the ground field automorphisms does not change its profinite completion.

Some people consider the transcendental data useless to study algebraic objects. On the contrary, they might be very useful for some algebraic problems. One can use the transcendental data to give representations of some algebraic automorphism group: namely, how the transcendental data of varieties are altered under algebraic automorphisms. For the purpose of this dissertation, the transcendental data we consider are the underlying manifold structures of smooth complex varieties over $\overline{\mathbb{Q}}$. More explicitly, we are interested in Sullivan's question in [Sul09, p. 271]:

Question 1.1.1. Analyze the action of the Galois group of the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ on the manifold structures in a profinite homotopy type associated to nonsingular algebraic varieties defined over $\overline{\mathbb{Q}}$.

Before introducing our studies, it is worth stressing one difficulty for this question. Most algebraic automorphisms of $\mathbb{C}$ are not continuous. Indeed, the only continuous ones are the identity and complex conjugation. Hence, we need some tools to avoid the usual continuity assumptions.

For this question, this dissertation splits into three parts.

1.1.1 Algebraic-topological Data to Specify a Manifold

An intuitive idea is to extract the information of manifold structures by some algebraic data and then formulate the Galois symmetry on manifold structures in terms of these algebraic data. Historically, topologists have had a powerful machinery known as the surgery theory to study manifolds.

A motivating question for understanding the surgery theory is when a space X is homotopy equivalent to a closed manifold. An obvious requirement is that the space X must have a Poincaré duality on its homology. It turns out that the Poincaré duality induces a canonical homotopy sphere bundle on X such that, when X is an actual manifold, this bundle carries the homotopy information of the normal bundle of X in a Euclidean space.

There are some obstructions for this homotopy sphere bundle to come from some geometric bundle. If the obstructions vanish, then there is a map $M \to X$ for some manifold M. The next step is to make this map a homotopy equivalence by some process called surgery. During this process, some other obstruction called the surgery obstruction appears. Once this surgery obstruction also vanishes, we get some manifold homotopy equivalent to the original space X.

For technical reasons, we restrict our discussions to topological manifolds and simply connected spaces. One reason for this is to avoid the discussions of the surgery obstruction by some techniques. A second reason is that in the simply connected case one can split the homotopy information into different primes, which is like splitting a number theory problem about integers into different primes. In our case, one splits the obstruction for a geometric bundle structure on a homotopy sphere bundle into different primes.

Historically, Sullivan reduces the obstruction at odd primes to some K-theory discussions. The obstruction at prime 2 is a bit more complicated. By the works of Levitt, Morgan-Sullivan, Madsen-Milgram, Brumfiel-Morgan and many others, the obstructions at prime 2 are reduced to some discussions about characteristic classes.

On the other hand, there is also a more conceptual description for this obstruction by Levitt-Ranicki using L-theory (which is an analogue of K-theory for quadratic forms).

We believe that people already know the equivalence of the obstruction at different primes and the L-theoretical obstruction because of some evidence of computations for L-theory by Taylor-Williams. However, we cannot find a formal proof in literature. So we provide a proof for this equivalence in this dissertation. (For a more detailed discussion, see Section 4.4 and 4.5.)

Theorem 1.1.2. *The L-theoretical obstruction for the existence of a manifold homotopy equivalent a given simply-connected Poincaré space is equivalent to the obstruction previously defined at different primes.*

One can also consider the uniqueness problem for manifolds in a homotopy type. This is equivalent to thinking about some 'moduli' of all manifolds in a given homotopy type. Sullivan also studies this 'moduli' at different primes. At odd primes, this 'moduli' is also reduced to some K-theory discussion; at prime 2, it is reduced to some cohomology classes (with Rourke and Morgan separately, and Milgram independently).

1.1.2 Étale Construction for Manifolds

As we said before, the fundamental group requires transcendental information on a variety but the system of all its finite quotients is algebraic. There is an approach to extend this idea to spaces.

As we know, a finite covering map over a variety is algebraic. So one can consider the system of all finite covering maps over open subsets of a variety X (étale morphisms). This is Grothendieck's idea to replace the usual notion of open subsets used to define a topology by a more categorical consideration. In his language, this system of all finite covering maps over open subsest gives a new 'topology' for the variety X. A formal construction by Artin-Mazur for such a topology, which is a generalization of the Cěch nerve construction, produces some new space $X_{\text{ét}}$.

The miracle of this new space $X_{\text{ét}}$ (étale homotopy type) is that it captures the finite information for the homotopy type of X, which is called the profinite completion, when X is a complex variety. For example, the finite part of the fundamental group of X is exactly the fundamental group of $X_{\text{ét}}$. Since every construction above is purely formal, the finite homotopy type of X is also algebraic information. In particular, the automorphisms of the ground field do not change the finite homotopy type.

The second part of this dissertation is to understand this phenomenon for more geometric objects, like manifolds. We just replace 'the system of all finite covering maps over open subsets of a variety X' by 'the system of branched coverings over a manifold M'. By applying Artin-Mazur's formal construction to this new topology for M, we get a space $M_{\text{ét}}$. Then we prove a similar result. (For more details, see 5.2.1)

Theorem 1.1.3. *For a manifold M, $M_{\text{ét}}$ captures the (pro-)finite information of the homotopy type of M.*

1.1.3 Galois Symmetry on the Algebraic-topological Data

In this part, we first generalize the idea in Part 1 of the 'moduli' of all manifolds in a given homotopy type to define the 'moduli' of all manifolds in a given finite homotopy type. More explicitly, we take out the finite data of the 'moduli' at each prime and then compile them for all primes.

Suppose that there are some elements in this generalized moduli which are represented by complex varieties defined over $\overline{\mathbb{Q}}$ (call such elements algebraic). From the discussions in Part 2, we know that the Galois automorphisms of $\overline{\mathbb{Q}}$ do not change the finite homotopy type of a complex variety defined over $\overline{\mathbb{Q}}$. Then the Galois automorphisms act on the algebraic elements in this moduli.

On the other hand, there is a way to define some abelianized Galois symmetry on this 'moduli' directly by Sullivan. Before doing this, we recall some of his other works.

He applies Galois symmetry on varieties to the example of the Grassmannian varieties, which are the finite classifying spaces of bundles. He stablizes the Galois automorphisms on the finite Grassmannians to get a Galois symmetry on the infinite Grassmannian BU. This Galois symmetry on BU is indeed an abelianized Galois symmetry, which is exactly the same as the Adams operations on bundles. As a consequence, he proves the Adams conjecture: namely, the Adams operations on bundles do not change the underlying homotopy sphere bundles in the finite homotopy sense.

Recall from Part 1 that the odd prime information of the moduli is reduced to some K-theory discussion. In this way the abelianized Galois symmetry on BU induces a symmetry on the moduli at odd primes. For prime 2, Sullivan sketches his idea and we put some efforts in realizing his approach.

Then one can prove that this newly defined abelianized Galois symmetry on the moduli is compatible with the Galois symmetry on the algebraic elements in the moduli. An obvious corollary from this is the following. (For a more precise statement, see 6.2.1 and 6.2.2)

Theorem 1.1.4. *The Galois symmetry on the algebraic elements (if they exist) of the 'moduli' of all manifolds in a given finite homotopy type is an abelianized symmetry and this symmetry extends to the whole 'moduli' in a canonical way.*

It is still very mysterious for us why this Galois symmetry is abelianized. We hope to get a more geometric explanation for this but unfortunately we have not yet completed this prior to the thesis defense date. Nevertheless, we write down our ideas and make some conjectures about this problem at the very end in order to record some plausible arguments and pictures for future studies. It seems that our ideas have a deep relation with Grothendieck's dessins d'enfants.

1.2 Organization

Briefly speaking, we give the preliminaries for this dissertation in Chapter 2. Chapters 3 and 4 study the first part of the problem, Chapter 5 is the second part and Chapter 6 is the third part.

Explicitly, in Chapter 2, we review the surgery theory, completions (or localizations) of spaces, Artin-Mazur's étale homotopy theory and the a priori invariant method for defining cohomology classes and K-theory elements. Chapter 3 develops the $\mathbb{Z}/n$ algebraic surgery theory as a generalization of $\mathbb{Z}/n$ manifolds and proves many product formulae which will be used in the next chapter. In Chapter 4 we use the method of a priori invariants to study the homotopy type of L-spectra and prove the known equivalence of different bundle lifting theories. Chapter 5 is devoted to generalizing Artin-Mazur's theorem on the étale homotopy theory of complex varieties to a more geometric setting of branched coverings over pseudomanifolds. In Chapter 6, we study Sullivan's result of the Galois symmetry on the underlying topological manifold structures of smooth complex varieties. In the end, we formulate some conjectures for a possible geometric interpretation of Sullivan's result.

Chapter 2

Preliminaries

This chapter includes a historical review for the problem which this dissertation aims to study and the preliminaries that will be used for later chapters.

2.1 Surgery Theory

It is impossible to cover all aspects of the surgery theory in this section. We only give a brief and quick review on the part used for this dissertation. In particular, we only focus on the simply connected case, though the non simply connected case is often more interesting. The subsection 1 is a historical review and the remaining two subsections are more technical.

2.1.1 Manifolds in a Homotopy Type

In this section, we review the existence and uniqueness problem of manifold structures in a given homotopy type. In history Browder and Novikov independently proposed the following way for the simply connected case (see [Bro72]).

Let X be a space representing the given homotopy type. Assume that X is simply connected finite CW complex. Further assume that X has a Poincaré duality, that is, there is a class $[X] \in H_n(X;\mathbb{Z})$ such that the induced cap product $-\cap[X] : H^{n-*}(X;\mathbb{Z}) \to H_*(X;\mathbb{Z})$

is an isomorphism. Spivak proved that the Poincaré duality on X induces a canonical spherical fibration $\nu_X \to X$ ([Spi67]), which is called the Spivak normal spherical fibration. His construction is like the following.

Embed X into an Euclidean space $\mathbb{R}^K$, for some K large enough. Take a regular neighborhood N of X. Then N is a smooth manifold with boundary ∂N. Note that the natural inclusion $X \to N$ is a homotopy equivalence. The homotopy fiber of $\partial N \to N \simeq X$ is a sphere because of the Poincaré duality on X.

For a spherical fibration $\gamma \to B$, the Thom space $Th(\gamma)$ is the mapping cone of the fibration map. Then the canonical spherical fibration ν_X on X has an extra data, i.e., a map $S^K \to Th(\nu_X)$ induced by the natural quotient map $S^K \to S^K/(S^K - N) \cong N/\partial N$.

Let G_l be the topological monoid of self homotopy equivalences of the sphere S^{l-1}. There is a natural inclusion $G_l \to G_{l+1}$ by taking suspension. Let G be the union of all G_l's. Then BG is the classifying space for stable spherical fibrations. The Splivak normal spherical fibration ν_X on X induces a natural map $\nu_X : X \to BG$.

The first obstruction to the existence of a manifold homotopy equivalent to X is the existence of a vector bundle structure on the Spivak spherical fibration ν_X, namely, the existence of a lifting for the map $\nu_X : X \to BG$ along $BO \to BG$. So there is a sequence of obstruction classes in $H^{*+1}(X; \pi_*(G/O))$.

Suppose that the obstruction classes vanish, then one can slightly alter the map $S^K \to Th(\nu_X)$ to be transversal to the zero section X. Then one gets a degree 1 map $M \to X$, where M is a smooth manifold. This map is covered by a vector bundle map $\nu_M \to E_X$, where ν_M is the stable normal bundle of M in a Euclidean space and the underlying spherical fibration E_X is ν_X. We call such a map a degree 1 normal map.

Apply the surgery process on M to try to make this degree 1 normal map map a homotopy equivalence. The obstruction in this step is isomorphic to the simply connected surgery group P_n if $n \geq 5$, which can be defined by the framed bordism group of framed manifolds with boundary PL homeomophic to S^{n-1} (see [Sul65, p. 10]). It is known that

$$P_n \cong \begin{cases} \mathbb{Z}, & \text{if } n \equiv 0 \pmod 4 \\ \mathbb{Z}/2, & \text{if } n \equiv 2 \pmod 4 \\ 0, & \text{otherwise} \end{cases}$$

The theory is analogous for PL (piecewise-linear) or TOP (topological) manifolds. The only difference is the first obstruction, namely, the existence of a PL or TOP bundle structure on ν_X. This is equivalent to the existence of a lifting $\nu_X : X \to BG$ along $BPL \to BG$ or $BTOP \to BG$. Technically, one can get rid of the surgery obstruction (for dimension at least 6, see [Bro72, Corollary 3.8]) for PL or TOP manifolds. The main reason is that there is no exotic PL sphere for dimension at least 5.

For some technical reason, let us focus on the TOP case.

Like the number theory, one can split a simply-connected topological problem into different primes, by localization or profinite completion (which we will review in the next section). For odd primes, Sullivan proved that the obstruction is equivalent to the existence of a real K theory orientation for ν_X, namely, the existence of a Thom class in $\widetilde{KO}_{(p)}(Th(\nu_X))$ ([Sul09, Theorem 6.5]).

For prime 2, Brumfiel-Morgan ([BM76]) and Madsen-Milgram ([MM75]) independently discovered the obstruction. The difference of their obstructions is subtle. Let us concentrate on Brumfiel-Morgan's result. There exist characteristic classes $k^G \in H^{4*+3}(X;\mathbb{Z}/2)$ and $l^G \in H^{4*}(X;\mathbb{Z}/8)$ for a spherical fibration ν_X. The obstruction for having a TOP bundle structure at prime 2 is the vanishing of k^G and an $\mathbb{Z}_{(2)}$-coefficient lifting of l^G. Similar results were known by Quinn ([Qui72]) and Jones ([Jon71]). We will sketch the construction in section 3.

Moreover, Ranicki defined Poincaré dualities on chain complexes and defined chain-level bordisms ([Ran80]). In fact, he defined two kinds of chain complexes that can have Poincaré duality, symmetric or quadratic. He then defined the symmetric L-group L^s_n and the quadratic

L-group L^q_n of bordism classes of such chain complexes. He further constructed two L-spectra $\mathbb{L}^s$ and $\mathbb{L}^q$ whose homotopy groups are the L-groups. We will give more details for the L-theory in the next two subsections.

Weiss ([Wei85]) also defined a chain-level analogue of spherical fibrations and Quinn's geometric normal spaces ([Qui72]), namely, chain bundles and normal chain complexes. It is known that normal chains and Poincaré symmetric-quadratic pairs are equivalent ([Ran92, Theorem 2.8]). There is also an L-group L^n_n of bordism classes of normal chain complexes and an L-spectrum $\mathbb{L}^n$ whose homotopy groups are L^n_n. We will also give the details in the next subsection.

There is a direct fibration of spectra $\mathbb{L}^q \to \mathbb{L}^s \to \mathbb{L}^n$ by the construction immediately.

The bundle lifting obstruction has an integral version by Levitt-Ranicki ([Ran92, Proposition 16.1], [LR87]), without splitting the problem into different primes. A spherical fibration ν_X has a canonical $\mathbb{L}^n$-orientation. ν_X has a TOP bundle structure if and only if the $\mathbb{L}^n$-orientation lifts to an $\mathbb{L}^s$-orientation. Though people know the homotopy types of $\mathbb{L}$-spectra (by Taylor-Williams [TW79])), we could not find a proof whether Levitt-Ranicki's theory is equivalent to the theories at different primes introduced above. We will give a proof for this equivalence in Chapter 4 although we believe that it is known to all experts.

As for the uniqueness problem, Sullivan defined the structure set to study all manifolds in a given homotopy type ([Sul96]). Let us reformulate his definition in the following way.

Definition 2.1.1. The homotopy manifold category **HMan** is a category whose objects are all closed topological manifolds and whose morphisms are homotopy classes of homotopy equivalences.

Remark 2.1.1. There might be a set-theoretic concern for the class of all manifolds. To avoid this, it is equivalent to consider all submanifolds of Euclidean spaces. Therefore, we may assume that **HMan** is a small category.

For any manifold M, consider the overcategory $\mathbf{HMan}_{/M}$, whose objects are homotopy classes of homotopy equivalences $N \to M$.

Definition 2.1.2. The homotopy manifold structure category $\mathbf{HS}(M)$ over a closed manifold M is a subcategory of $\mathbf{HMan}_{/M}$, whose objects are the same as $\mathbf{HMan}_{/M}$ and whose morphisms from $f_1 : N_1 \to M$ to $f_2 : N_2 \to M$ are homeomorphisms $g : N_1 \to N_2$ so that $f_2 \circ g$ and f_1 are homotopic.

Definition 2.1.3. The structure set $\mathbf{S}^{TOP}(M)$ is the isomorphism classes of objects of $\mathbf{HS}(M)$.

For any homotopy equivalence $f : M \to N$, one can transport the homotopy manifold structures over M to N. That is, there is a categorical equivalence $f_* : \mathbf{HS}(M) \to \mathbf{HS}(N)$ by composing with f and it induces a bijection $f_* : \mathbf{S}^{TOP}(M) \to \mathbf{S}^{TOP}(N)$.

Analogously, one can also define the structure set $\mathbf{S}^{TOP}(M, \partial M)$ over a manifold M with boundary ∂M, where a typical element is a homotopy equivalence $N \to M$ whose restriction $\partial N \to \partial M$ is also a homotopy equivalence.

Theorem 2.1.2. *([Sul96]) Let M^m be a simply connected manifold.*

If $\partial M \neq \emptyset$, assuming that ∂M is also simply connected and $m \geq 6$, then $\mathbf{S}^{TOP}(M, \partial M) \simeq [M, G/TOP]$.

If $\partial M = \emptyset$, assuming $m \geq 5$, then there is an exact sequence of based sets

$$0 \to \mathbf{S}^{TOP}(M) \to [M, G/TOP] \to P_m \to 0$$

Corollary 2.1.3. *If M^m is a simply connected closed manifold of dimension $m \geq 6$, then*

$$\mathbf{S}^{TOP}(M) \simeq [M - \mathrm{pt}, G/TOP]$$

In this way, one can reduce the problem about $\mathbf{S}^{TOP}(M)$ to the problem of determining the homotopy type of G/TOP. Fortunately, this was already known ([Sul96, p. 85, Theorem 4][KS77, p. 329, 15.3])

Theorem 2.1.4. *When localized at prime 2, $G/TOP \simeq \prod_{k>0}(K(\mathbb{Z}_{(2)}, 4k) \times K(\mathbb{Z}/2, 4k-2))$; when localized at odd primes, $G/TOP \simeq BSO_{(\text{odd})}$.*

We will review the techniques used for the proof of this theorem in Section 3.

2.1.2 L Groups

We review the definition of L-groups in this subsection. Throughout this article, by a chain complex we always mean a bounded chain complex whose underlying abelian group at each degree is a finitely generated free abelian group.

For a chain complex C_*, the tensor product $C \otimes C \cong \mathrm{Hom}_{\mathbb{Z}}(C^{-*}, C_*)$ has a natural $\mathbb{Z}/2$ action. An n-dimensional symmetric structure on a chain complex C is a degree n homotopy $\mathbb{Z}/2$ invariant ϕ. More explicitly, let $W_* \to \mathbb{Z} \to 0$ be a free resolution of the trivial $\mathbb{Z}[\mathbb{Z}/2]$-module $\mathbb{Z}$. An explicit form of W_* is

$$\cdots \to \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}/2] \to 0 \to \cdots$$

where T is the generator in $\mathbb{Z}/2$. Each degree n homotopy $\mathbb{Z}/2$ invariant ϕ is represented by an element $\phi \in Q^s_n(C) = H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(W, C \otimes C))$. Informally, ϕ consists of chain maps $\phi_i : C^{n+i-*} \to C_*$ for $i \geq 0$ so that each ϕ_i is a chain homotopy between $\phi_{i-1} : C^{n+i-1-*} \to C_*$ and its dual $D(\phi_{i-1}) : C^{n+i-1-*} \to C_*$.

A symmetric chain complex (C, ϕ) is Poincaré if the chain map $\phi_0 : C^{n-*} \to C_*$ is a chain homotopy equivalence. The notion of Poincaré symmetric chains is a derived generalization of nondegenerate symmetric bilinear forms of free abelian groups or nondegenerate linking forms on torsion abelian groups.

Recall that a homotopy coinvariant is an element in $Q^q_n(C) = H_n(W \otimes_{\mathbb{Z}[\mathbb{Z}/2]} (C \otimes C))$. Define a quadratic structure on a chain complex C by a homotopy coinvariant ψ. Similarly, ψ consists of chain maps $\psi_i : C^{n+i-*} \to C_*$ for $i \leq 0$ so that each ψ_i is a chain homotopy between ψ_{i-1} and its dual $D(\psi_{i-1}) : C^{n+i-1-*} \to C_*$. A quadratic chain complex (C, ψ) is called Poincaré if the chain map $\psi_0 : C^{n-*} \to C_*$ is a chain homotopy equivalence.

Let $\mathbf{\Delta}$ be the category of totally ordered nonempty finite sets with morphisms order-preserving inclusions. A Δ-set is a contravariant functor from $\mathbf{\Delta}$ to the category of sets. Indeed, a Δ-set has the same hierarchy and boundary maps like a simplicial set, but a Δ-set does not have degeneracy maps. Indeed, a Δ-set X can be viewed as a category with objects the simplices in X and with morphisms the inclusions between simplices. A presheaf of chain complexes over X is a functor $\mathcal{C}$ from the category X to the category of chain complexes. Define the Verdier dual presheaf $D(\mathcal{C})$ by $D(\mathcal{C})(\sigma) = \operatorname{hocolim}_{\tau\subset\sigma} \mathcal{C}(\tau)^{-*} = \oplus_{\tau\subset\sigma}\mathcal{C}(\tau)^{-*+|\tau|}$ for any simplex σ of X.

Now assume X is a finite Δ-set. Let $\mathbf{Hom}(D(\mathcal{C}),\mathcal{C})$ be the differential graded presheaf of homomorphisms. It also has a natural $\mathbb{Z}/2$-action , so we can define n-dimensional symmetric/quadratic structures on a presheaf. A symmetric/quadratic structure is (locally) Poincaré if the chain map $D(\mathcal{C})(\sigma) \to \mathcal{C}(\sigma)$ is a chain homotopy equivalence for each simplex σ of X.

Define the assembly (or equivalently, the set of global sections) of a presheaf $\mathcal{C}$ by $\mathcal{C}(X) = \operatorname{hocolim}_{\sigma\in X} \mathcal{C}(\sigma) = \oplus_{\sigma\in X}\mathcal{C}(\sigma)_{*-n+|\sigma|}$, where n is the dimension of X.

Remark 2.1.5. All the definitions above like presheaf, Verdier dual presheaf, assembly and symmetric/quadratic structures on a presheaf can be generalized to the case when X is a regular cell complex (for a definition see [CF67, p. I.1.1]).

Lemma 2.1.6. *Let X be an n-dimensional closed PL manifold with a PL triangulation. Let $\mathcal{C}$ be a presheaf of chain complexes over X. Then the assembly $D(\mathcal{C})(X)$ is canonically chain homotopy equivalent to $\mathcal{C}(X)^{n-*} = \Sigma^n \operatorname{Hom}(\mathcal{C}(X),\mathbb{Z})$.*

Proof. Firstly,

$$\begin{aligned}
\mathcal{C}(X)^{n-*} &= \Sigma^n \operatorname{Hom}(\mathcal{C}(X),\mathbb{Z}) = \Sigma^n \operatorname{Hom}(\operatorname*{hocolim}_{\sigma\in X} \mathcal{C}(\sigma),\mathbb{Z}) \\
&= \Sigma^n \operatorname{Hom}(\oplus_{\sigma\in X}\mathcal{C}(\sigma)_{*-n+|\sigma|},\mathbb{Z}) = \Sigma^n(\oplus_{\sigma\in X}\mathcal{C}(\sigma)^{-*-n+|\sigma|}) \\
&= \oplus_{\sigma\in X}\mathcal{C}(\sigma)^{-*+|\sigma|} = \operatorname*{hocolim}_{\sigma\in X} \mathcal{C}(\sigma)^{-*}
\end{aligned}$$

Notice that $\mathcal{C}(\sigma)^{-*}$ is a precosheaf over X. So it suffices to prove that given a precosheaf $\mathcal{D}$ over X, the canonical chain map $\operatorname{hocolim}_{\sigma\in X}\operatorname{hocolim}_{\tau\subset\sigma}\mathcal{D}(\tau)\to\operatorname{hocolim}_{\sigma\in X}\mathcal{D}(\sigma)$ is a chain homotopy equivalence.

Let us write down the chain complexes of two sides explicitly. $\operatorname{hocolim}_{\sigma\in X}\mathcal{D}(\sigma) = \oplus_{\sigma\in X}\mathcal{D}(\sigma)_{*-|\sigma|}$, with the differential decomposed like the following. For each σ, there is an obvious differential $\mathcal{D}(\sigma)_{r-|\sigma|}\to\mathcal{D}(\sigma)_{r-1-|\sigma|}$; for each pair $\tau\subset\sigma$ of codimension 1, there is a map $\mathcal{D}(\sigma)_{r-|\sigma|}\to\mathcal{D}(\tau)_{r-|\sigma|}=\mathcal{D}(\tau)_{r-1-|\tau|}$ induced by the cosheaf structure.

On the other hand, $\operatorname{hocolim}_{\sigma\in X}\operatorname{hocolim}_{\tau\subset\sigma}\mathcal{D}(\tau)=\oplus_{\sigma\in X}\oplus_{\tau\subset\sigma}\mathcal{D}(\tau)_{*-|\tau|-n+|\sigma|}$, with the differential decomposed into three parts. For each pair $\tau\subset\sigma$, there is an obvious differential $\mathcal{D}(\tau)_{r-|\tau|-n+|\sigma|}\to\mathcal{D}(\tau)_{r-1-|\tau|-n+|\sigma|}$; for each pair $\gamma\subset\tau$ of codimension 1, there are a map $\mathcal{D}(\tau)_{r-|\tau|-n+|\sigma|}\to\mathcal{D}(\gamma)_{r-|\tau|-n+|\sigma|}=\mathcal{D}(\gamma)_{r-1-|\gamma|-n+|\sigma|}$ induced by the cosheaf structure; for each pair $\sigma\subset\delta$ of codimension 1, there is an identity map $\mathcal{D}(\tau)_{r-|\tau|-n+|\sigma|}\to\mathcal{D}(\tau)_{r-|\tau|-n+|\sigma|}=\mathcal{D}(\tau)_{r-1-|\tau|-n+|\delta|}$.

Hence, $\operatorname{hocolim}_{\sigma\in X}\operatorname{hocolim}_{\tau\subset\sigma}\mathcal{D}(\tau)$ is also isomorphic to $\oplus_{\alpha\in X}\oplus_{\alpha\subset\beta}\mathcal{D}(\alpha)_{*-|\alpha|-n+|\beta|}$, with the differential decomposed into three parts like above. So we have proved that $\operatorname{hocolim}_{\sigma\in X}\operatorname{hocolim}_{\tau\subset\sigma}\mathcal{D}(\tau)=\operatorname{hocolim}_{\alpha\in X}\operatorname{hocolim}_{\alpha\subset\beta}\mathcal{D}(\alpha)$, where by $\operatorname{hocolim}_{\alpha\subset\beta}\mathcal{D}(\alpha)$ we mean the colimit over the constant diagram $\mathcal{D}(\alpha)$ indexed by all simplices β containing the fixed α. The map into $\operatorname{hocolim}_{\sigma\in}\mathcal{D}(\sigma)$ is induced by $\operatorname{hocolim}_{\alpha\subset\beta}\mathcal{D}(\alpha)\to\mathcal{D}(\alpha)$. We only need to prove that this map is a chain homotopy equivalence and the lemma follows from [RW90, Proposition 1.14].

Notice that a PL triangulation on X induces a dual cell decomposition on X since X is a PL manifold. The dual cell decomposition on X is a regular cell decomposition. Let $D(\alpha)$ be the regular cell dual to a simplex α. But $\operatorname{hocolim}_{\alpha\subset\beta}\mathcal{D}(\alpha)=\mathcal{D}(\alpha)\otimes C_*(D(\alpha))$, where $C_*(D(\alpha))$ is the cellular chain complex of $D(\alpha)$ (with respect to the dual cell decomposition). It is obvious that the chain map $\mathcal{D}(\alpha)\otimes C_*(D(\alpha))\to\mathcal{D}(\alpha)$ is a chain homotopy equivalence. □

Corollary 2.1.7. *Let X be a closed n-dimensional PL manifold with a PL triangulation. Let $\mathcal{C}$ be an m-dimensional Poincaré presheaf of symmetric/quadratic chain complexes over X.*

The assembly $\mathcal{C}(X)$ is a Poincaré symmetric/quadratic chain complex of dimension $m+n$.

Proof. The lemma showed that $D(\mathcal{C})(X) \to \mathcal{C}(X)^{n-*}$ is a chain homotopy equivalence, where n is the dimension of X. Hence, the symmetric/quadratic structure on the presheaf $\mathcal{C}$ induces a symmetric/quadratic structure on $\mathcal{C}(X)$. Moreover, since $\mathcal{C}$ is locally Poincaré, [RW90, Proposition 1.14] indicates that $D(\mathcal{C})(X)_{-m+*} \to C(X)$ is a chain homotopy equivalence and hence the induced map $\mathcal{C}(X)^{m+n-*} \to \mathcal{C}(X)$ is also a chain homotopy equivalence. □

Remark 2.1.8. The lemmas 2.1.6 and 2.1.7 both hold for a PL regular cell decomposition of a PL manifold, since the only fact we need to use in the proof is that the dual cone decomposition of a PL regular cell decomposition for a PL manifold is still a regular cell decomposition.

Definition 2.1.4. Let $X = \Delta^1$, the Δ-set of the unit interval. A presheaf $\mathcal{C}$ over X is indeed a chain map $C \oplus C' \to D$. Let ϕ (or ψ) be an n-dimensional symmetric (or quadratic) structure on $\mathcal{C}$. Call D a Poincaré symmetric (or quadratic) bordism between two Poincaré symmetric (or quadratic) chain complexes C and C' if the presheaf $(\mathcal{C}, \phi)$ (or $(\mathcal{C}, \psi)$) is locally Poincaré. Moreover, if $C' = 0$, call $C \to D$ a Poincaré symmetric (or quadratic) pair. □

Remark 2.1.9. A (Poincaré) symmetric (or quadratic) presheaf $\mathcal{C}$ over an n-dimensional simplex Δ^n is also called an n-ad of (Poincaré) symmetric (or quadratic) chain complexes. A (Poincaré) symmetric (or quadratic) presheaf over a finite Δ-set X is indeed a presheaf of ads of (Poincaré) symmetric (or quadratic) chain complexes.

A non-Poincaré symmetric (or quadratic) chain complexes is like a Poincaré chain with a 'singularity'. A more precise statement is the following, which will be used in many places.

Lemma 2.1.10. *(Ranicki's Miracle Lemma, [Ran80, Proposition 3.4]) The chain homotopy classes of n-dimensional symmetric (or quadratic) chain complexes are in one-to-one correspondence with the homotopy classes of n-dimensional Poincaré symmetric (or quadratic) pairs.*

Proof Sketch. The proof is based on two constructions. Let us focus on the symmetric case only. Given a Poincaré symmetric pair $D \xrightarrow{f} C$, apply the Thom construction, that is, the homotopy cofiber $\mathrm{Cofib}(f)$ of the chain map. Then there is a symmetric structure on $\mathrm{Cofib}(f)$ induced from the symmetric structure on $D \to C$.

Conversely, for a symmetric chain complex (C, ϕ), consider the the natural homotopy fiber $\mathrm{Fib}(\phi_0)$ of the symmetric map $\phi_0 : C^{n-*} \to C_*$. Then there exists a Poincaré symmetric structure on the pair $\mathrm{Fib}(\phi_0) \to C^{n-*}$. (In references, $\mathrm{Fib}(\phi_0)$ is sometimes written as ∂C). □

In [Wei85] (also see [Ran92, Definition 2.2]), Weiss defined a chain bundle on a chain complex C by a 0-dimensional cycle $\gamma \in \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(\widehat{W}, C^{-*} \times C^{-*})$, where $\widehat{W}$ is the Tate complex associated to the group ring $\mathbb{Z}[\mathbb{Z}/2]$, which is

$$\cdots \to \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}/2] \to \cdots$$

$\widehat{Q}_*(C) = H_*(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(\widehat{W}, C \otimes C))$ is like the 'K-theory of spherical fibrations' for chain complexes. $\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(\widehat{W}, C^{-*} \otimes C^{-*})$ has the homotopy invariance property, i.e., the chain maps $f^*, (f')^* : \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(\widehat{W}, D^{-*} \otimes D^{-*}) \to \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(\widehat{W}, C^{-*} \otimes C^{-*})$ induced by two homotopic chain maps $f, f' : C \to D$ is still chain homotopic, where the homotopy relies on a choice of a 'diagonal' element $\omega \in (C(\Delta^1) \otimes C(\Delta^1))_1$ (see [Ran80, Proposition 1.1]).

The identity map of ΣC represents a chain homotopy from the 0-map $C \xrightarrow{0} \Sigma C$ to itself. It induces a chain homotopy map $\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(\widehat{W}, C^{-1-*} \otimes C^{-1-*}) \to \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(\widehat{W}, C^{-*} \otimes C^{-*})$. By shifting the degree, we get a chain map

$$S : \Sigma \,\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(\widehat{W}, C^{-*} \otimes C^{-*}) \to \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(\widehat{W}, C^{1-*} \otimes C^{1-*})$$

Moreover, there is a natural long exact sequence connecting homotopy coinvariants $Q_n^q(C)$, homotopy invariants $Q_n^s(C)$ and Tate cohomology $\widehat{Q}_n(C)$ (see [Ran80, Proposition 1.2] for the construction and proof), i.e.,

$$\cdots \to Q_n^q(C) \xrightarrow{1+T} Q_n^s(C) \xrightarrow{J} \widehat{Q}_n(C) \xrightarrow{\partial} Q_{n-1}^q(C) \to \cdots$$

where $1+T$ corresponds to the polarization of a quadratic form into a symmetric form.

A normal structure on an m-dimensional symmetric chain complex (C, ϕ) is defined by a chain bundle γ over C and a homology $\zeta \in (\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(\widehat{W}, C^{1-*} \otimes C^{1-*}))_{n+1}$ between $J(\phi)$ and $\phi_*(S^m(\gamma))$.

There are some basic facts about normal chain complexes. Define an n-dimensional symmetric-quadratic pair $f : C \to D$ by a Poincaré symmetric pair structure on $C \to D$ and a Poincaré quadratic refinement of the symmetric structure on C. There are also notions of bordisms and k-ads of symmetric-quadratic pairs.

Proposition 2.1.11. *(1)([Ran92, Proposition 2.6(ii)]) Each Poincaré symmetric complex has a canonical normal structure.*

(2)([Ran92, Proposition 2.8(i)]) There is a natural one-to-one correspondence between the homotopy classes of n-dimensional Poincaré symmetric-quadratic pairs and those of n-dimensional normal chains.

Like before, we can define a normal structure on a presheaf of chain complexes over an arbitrary finite Δ-set. Consequently n-ads and bordisms of normal complexes can also be defined. There is a correspondence between presheaves of normal chains and presheaves of symmetric-quadratic pairs. The lemma 2.1.7 for the normal case or the case of symmetric-quadratic pairs is also true by the same argument.

Definition 2.1.5. ([Ran80, p. 137] and [Ran92, p. 40]) Define L^s_m, L^q_m and L^n_m by the sets of bordism classes of m-dimensional Poincaré symmetric chain complexes, Poincaré quadratic chain complexes and normal complexes, respectively. They are indeed abelian groups, where the additions are induced by the direct sums of chains.

Because of the equivalence between normal chain complexes and Poincaré symmetric-quadratic chain pairs, there is a natural long exact sequence ([Ran92, p. 45])

$$\cdots \to L^q_m \xrightarrow{1+T} L^s_m \xrightarrow{J} L^n_m \xrightarrow{\partial} L^q_{m-1} \to \cdots \tag{2.1.1}$$

For geometric intuitions, one can think of a Poincaré symmetric chain as a Poincaré space, a Poincaré quadratic chain as a degree one normal map between two Poincaré spaces and a normal chain as a normal space (see [Qui72] for a definition). One can use these intuitions to make algebraic constructions like geometric constructions. For example, we may define an algebraic gluing of two m-dimensional Poincaré symmetric pairs $D \to C$ and $(-D) \to C'$ along the common boundary D and get a m-dimensional Poincaré symmetric chain complex $C \bigcup_D C'$ ([Ran81, p. 77]), like the gluing of two manifolds along the common boundary.

Analogously, we can also construct the following chain-level bordism invariants.

Let (C, ϕ) be an m-dimensional Poincaré symmetric chain complex. ϕ induces a nondegenerate symmetric bilinear form on $H_{2k}(C)$ when $m = 4k$ and a nondegenerate linking form on the torsion subgroup $\mathrm{Tor}(H_{2k+1}(C))$ when $m = 4k + 1$. If $m = 4k$, the signature $\mathrm{Sign}(C) \in \mathbb{Z}$ is defined by the signature of the bilinear form over $H_{2k}(C) \otimes \mathbb{R}$; if $m = 4k + 1$, the de Rham invariant $\mathrm{dR}(C, \phi) \in \mathbb{Z}/2$ is defined by the $\mathbb{Z}/2$ rank of $\mathrm{Tor}(H_{2k+1}(C)) \otimes \mathbb{Z}/2$.

For a $4k$-dimensional Poincaré symmetric chain pair $D \to C$, we can also define the signature $\mathrm{Sign}(C)$ by the signature of the nondegenerate symmetric bilinear form on $\mathrm{Im}(H_{2k}(C) \to H_{2k}(C, D))$. Then we have the Novikov's additive formula of Poincaré symmetric pairs, namely, $\mathrm{Sign}(C \bigcup_D C') = \mathrm{Sign}(C) + \mathrm{Sign}(C')$, where $D \to C$ and $(-D) \to C'$ are two $4k$-dimensional Poincaré symmetric chain pairs.

Similarly, given an m-dimensional Poincaré quadratic chain complex (C, ψ), ψ induces an nondegenerate quadratic form on $H_{2k}(C)$ when $m = 4k$ and a nondegenerate skew-quadratic form on $H_{2k+1}(C)$ when $m = 4k+2$. If $m = 4k$, we can define the index $I(C) = \frac{1}{8}\,\mathrm{Sign}(C) \in \mathbb{Z}$, where $\mathrm{Sign}(C)$ is the signature of the quadratic form on $H_{2k}(C) \otimes \mathbb{R}$; if $m = 4k + 2$, the Kervaire invariant $K(C, \phi) \in \mathbb{Z}/2$ is the Kervaire-Arf invariant of the quadratic form on $H_{2k+1}(C) \otimes \mathbb{Z}/2$.

Let C' be an m-dimensional normal chain complex and let $D \to C$ be the corresponding Poincaré symmetric-quadratic pair.

If $m = 4k + 3$, define the Kervaire invariant $K(C') \in \mathbb{Z}/2$ by the Kervaire invariant of D.

If $m = 4k+1$, the index of D is 0 since D is a boundary. Then there exists a Poincaré quadratic pair $(-D) \to \widetilde{C}$. Define the de Rham invariant $\mathrm{dR}(C') \in \mathbb{Z}/2$ of C' by the de Rham invariant of the Poincaré symmetric complex $C \bigcup_D \widetilde{C}$.

The de Rham invariant is independent of the choice of the quadratic pair $(-D) \to \widetilde{C}$. Indeed, let $(-D) \to \widetilde{C}'$ be another pair. Let I be the chain complex of the unit interval. Due to its dimension, the quadratic complex $(-\widetilde{C}) \bigcup_{D\times 0} D \otimes I \bigcup_{D \times 1} \widetilde{C}'$ must be the boundary of some quadratic pair W. Then $C \otimes I \bigcup_{D \otimes I} (-W)$ is the symmetric bordism between $C \bigcup_D \widetilde{C}$ and $C \bigcup_D \widetilde{C}'$. Hence their de Rham invariants agree.

If $m = 4k$, the quadratic complex D is the boundary of some Poincaré quadratic pair $(-D) \to \widetilde{C}$. Then define the $\mathbb{Z}/8$-signature by $\mathrm{Sign}(C', D) = \mathrm{Sign}(C \bigcup_D \widetilde{C}) \in \mathbb{Z}/8$. It is also invariant under different choices of $\widetilde{C}$ by a similar argument like above.

With all the invariants defined above and the long exact sequence of L-groups, one can prove the following.

Proposition 2.1.12. *([Ran81, Proposition 4.3.1])*

(1)

$$L^q_m \cong \begin{cases} \mathbb{Z}, & \text{if } m \equiv 0 \pmod 4 \\ \mathbb{Z}/2, & \text{if } m \equiv 2 \pmod 4 \\ 0, & \text{otherwise} \end{cases}$$

(2)

$$L^s_m \cong \begin{cases} \mathbb{Z}, & \text{if } m \equiv 0 \pmod 4 \\ \mathbb{Z}/2, & \text{if } m \equiv 1 \pmod 4 \\ 0, & \text{otherwise} \end{cases}$$

(3)

$$L^n_m \cong \begin{cases} \mathbb{Z}/8, & \text{if } m \equiv 0 \pmod 4 \\ \mathbb{Z}/2, & \text{if } m \equiv 1 \text{ or } 3 \pmod 4 \\ 0, & \text{otherwise} \end{cases}$$

Moreover, all the isomorphisms are explicitly given by the invariants defined above.

2.1.3 L Spectra

As we know before, the L-groups are the homotopy groups of some $\mathbb{L}$-spectra. We review the construction of $\mathbb{L}$-spectra in this subsection.

Let $\mathbb{L}^s(m)$ be the pointed Δ-set whose k-simplices are all the k-ads of $(k-m)$-dimensional Poincaré symmetric chain complexes, with the obvious face maps. (Strictly speaking, there is a set-theoretic issue here but one can get rid of this). The base k-simplex is the k-ad of zero chains. Each $\mathbb{L}^s(m)$ satisfies the Kan condition and $\mathbb{L}^s(m) = \Omega\mathbb{L}^s(m+1)$. Then $(\mathbb{L}^s(m))$ forms an Ω-spectrum with homotopy groups isomorphic to L^s_k.

The spectra $\mathbb{L}^q(m)$ and $\mathbb{L}^n(m)$ are similarly constructed. There is a fibration sequence of spectra $\mathbb{L}^q \xrightarrow{1+T} \mathbb{L}^s \xrightarrow{J} \mathbb{L}^n$, whose associated long exact sequence of homotopy groups is exactly the long exact sequence of L-groups (2.1.1).

For manifolds and the bundle lifting problem stated in the previous subsection, the $\mathbb{L}$-spectra we want to use is the connective version.

The l-connective cover $\mathbb{L}^s\langle l\rangle$ of $\mathbb{L}^s$ is constructed like follows. Let the m-th space $\mathbb{L}^s\langle l\rangle(m)$ be the Δ-set with k-simplicies all k-ads of $(k-m)$-dimensional Poincaré symmetric chain complexes such that the presheaf restricted to each cell of dimension less than $(m+l)$ is an ad of acyclic chains. Again, $\mathbb{L}^s\langle l\rangle(m)$ is an $(l+m-1)$-connected Kan Δ-set and $\mathbb{L}^s\langle l\rangle(m) = \Omega\mathbb{L}^s\langle l\rangle(m+1)$.

The l-connective spectra $\mathbb{L}^q\langle l\rangle$ and $\mathbb{L}^n\langle l\rangle$ are defined likewise.

Warning 2.1.13. In the rest of this dissertation, we use the abbreviation $\mathbb{L}^s$ and $\mathbb{L}^q$ to represent the 0-connective symmetric L-spectrum $\mathbb{L}^s\langle 0\rangle$ and the 1-connective quadratic L-spectrum $\mathbb{L}^q\langle 1\rangle$ repectively. In particular, the homotopy group $\pi_i(\mathbb{L}^s)$ vanishes for $i<0$ and $\pi_i(\mathbb{L}^q)$ vanishes for $i<1$. We use the same symbol $\mathbb{L}^s$ and $\mathbb{L}^q$ to represent the 0-th space of the two connective spectra to avoid too many notations. The reader may easily differ the meaning of spectra or spaces by the context.

In [Ran92, Proposition 7.1], Ranicki proved the equivalence of Poincaré quadratic complexes for a surgery problem and Wall's definition of surgery obstructions. Hence, by Quinn's construction of G/TOP (which is a Δ-set consisting of ads of all surgery problems, [Qui70]), there is a canonical homotopy equivalence $G/TOP \to \mathbb{L}^q$.

Due to different connectiveness of $\mathbb{L}^s$ and $\mathbb{L}^q$, we recall the definition '1/2-connective' for normal L-spectrum in [Ran92, Deinition 15.14] in order to get the fibration sequence of three connective spectra.

A chain complex C is q-connective if the truncation at degree $\leq q$ of C is acyclic. A symmetic chain complex (C, ϕ) is q-Poincaré if ∂C is q-connective. We can also define a presheaf of q-Poincaré symmetric chain complexes over a finite Δ-set.

Remark 2.1.14. In general, one can define the spectra $\mathbb{L}^q(R), \mathbb{L}^s(R), \mathbb{L}^n(R)$ for any ring R. There is an analogous way to take connective covers $\mathbb{L}^a\langle l\rangle(R)$ like above, for $a = q, s, n$. In [Ran92, p. 157], there is an alternative way to make l-connective spectra $\mathbb{L}^a(\langle l\rangle, R)$, for $a = q, s, n$. Take $\mathbb{L}^q(\langle l\rangle, R)$ for example. Let the m-th space $\mathbb{L}^q(\langle l\rangle, R)(m)$ be the Δ-set with k-simplices all k-ads of $(k-m)$-dimensional Poincaré quadratic l-connective chain complexes (see the definition for l-connectiveness below). In general, $\mathbb{L}^q(\langle l\rangle, R)$ is canonically homotopy equivalent to $\mathbb{L}^q\langle l\rangle(R)$, but $\mathbb{L}^s(\langle l\rangle, R)$ is not homotopy equivalent $\mathbb{L}^s\langle l\rangle(R)$ unless the homotopy group $\pi_*(\mathbb{L}^s(R))$ has 4-periodicity, e.g., $R = \mathbb{Z}$ ([Ran92, Example 15.8]).

Let $\mathbb{L}^n\langle 1/2\rangle(m)$ be the Δ-set whose k-simplices are all k-ads of $(k-m)$-dimensional 0-connective 1-Poincaré normal chain complex such that the presheaf restricted to each cell of dimension less than m is an ad of contractible chains. For an alternative construction, one can use k-ads of symmetric-quadratic pairs.

The spaces $\mathbb{L}^n\langle 1/2\rangle(m)$ satisfy the Kan condition and form an Ω-spectrum.

Warning 2.1.15. Like above, for the rest of this dissertation, we use the symbol $\mathbb{L}^n$ for both the spectrum $\mathbb{L}^n\langle 1/2\rangle(m)$ and the 0-th space of this spectrum. In particular, the homotopy group $\pi_i\mathbb{L}^n$ vanishes for $i < 0$, is isomorphic to $\mathbb{Z}$ for $i = 0$ and is isomorphic to L^n_i for $i > 0$.

Therefore, there is a natural fiber sequence of connective spectra ([Ran92, Proposition 15.16(i)])

$$\mathbb{L}^q \xrightarrow{1+T} \mathbb{L}^s \xrightarrow{J} \mathbb{L}^n$$

Like the mock bundle picture to define the cobordism group over any space ([BRS76]), a cellular map from a finite simplicial complex (or a Δ-complex) X to any of the defined $\mathbb{L}$-spaces above can be considered as a 'mock bundle' over X with 'fibers' in chain complexes, which is exactly a presheaf $\mathcal{C}$ of ads of (Poincaré) symmetric/quadratic/normal complexes. So the $\mathbb{L}$-theory cohomology group $(\mathbb{L}^a)^k(X)$ actually consists of bordism classes of presheaves for $a = s, q, n$.

2.2 Profinite Completion and Galois Symmetry

We review the definition for the localization and the completion of a space. Then we revisit the definitions for understanding the statement Artin-Mazur's comparison theorem about étale homotopy type of complex varieties, including some lemmas we will use in Chapter 5. An application of these ideas is skeched in the last subsection, that is, Sullivan's proof of the Adams conjecture.

2.2.1 Localization and Completion

Let l be a set of primes in $\mathbb{Z}$. The localization of $\mathbb{Z}$ at l is the ring $\mathbb{Z}_{(l)}$ consisting of the rationals whose denominators are not divided by the primes in l.

Let X be a simple space, namely, $\pi_1(X)$ is an abelian group and the $\pi_1(X)$-action on each homotopy group $\pi_i(X)$ is trivial.

Definition 2.2.1. X is an l-local space if each homotopy group $\pi_i(X)$ is an $\mathbb{Z}_{(l)}$-module.

Theorem 2.2.1. *([Sul09, p. 33, Corollary]) There exists a left adjoint functor for the inclusion functor from the homotopy category of l-local simple spaces to the homotopy category* *of simple spaces.*

There are two proofs for this theorem. One is to define l-local cells and construct l-local CW complexes; the other one is to use the Postnikov tower of a space and localize the homotopy group at each stage. For more details, one can read [Sul09, Chapter 2].

Definition 2.2.2. The left adjoint functor in the last theorem is called the l-localization functor.

We will use $X_{(l)}$ for the l-localization of X and the pair of adjoint functors induce a natural map $X \to X_{(l)}$.

Besides localization of rings and spaces, we also have completions of rings and spaces. Let us first review Artin-Mazur's definition of completion.

Definition 2.2.3. A small category I is cofiltering if

(1) for any two objects i and j, there exists an object k with morphisms $k \to i$ and $k \to j$;

(2) for any two morphisms $i \rightrightarrows j$, there exists an object k with a morphism $k \to i$ such that the two compositions $k \to i \rightrightarrows j$ are equal.

Definition 2.2.4. Call a functor of small cofiltering categories $\phi : I \to J$ cofinal if

(1) for any object j of J, there is an object j of I with a morphism $\phi(i) \to j$;

(2) for any two morphisms $\phi(i) \rightrightarrows j$, where i and j are objects of I and J respectively, there exists a morphism $i_1 \to i$ in I so that the compositions $\phi(i_1) \to \phi(i) \rightrightarrows j$ are identical.

Definition 2.2.5. Let $\mathbf{D}$ be a category. A pro-object of $\mathbf{D}$ is a functor $I \to \mathbf{D}$ for some cofiltering small category I. The associated pro-category $\mathrm{Pro}(\mathbf{D})$ consists of pro-objects of $\mathbf{D}$ and the morphism sets are defined by

$$\mathrm{Hom}(\{X_i\}_{i\in I}, \{Y_j\}_{j\in J}) = \varprojlim_j \varinjlim_i \mathrm{Hom}(X_i, Y_j)$$

There is a more concrete way to represent a pro-morphism.

Proposition 2.2.2. *([AM69, Corollary 3.2]) Let $f : X \to Y$ be a pro-morphism, where X and Y are two pro-objects indexed by I and J respectively. Then there exists a cofiltering small category K with cofinal functors $\phi : K \to I$, $\psi : K \to J$ such that f is equivalent to a morphism $\{f_k : X_k \to Y_k\}_{k\in K}$. In addition, this representative for f is unique up to isomorphism in the cofinal sense.*

Definition 2.2.6. Let **Gr** be the category of groups. Call a full subcategory $\mathcal{C}$ of **Gr** complete if

(1) $\mathcal{C}$ contains the trivial group;

(2) for any exact sequence

$$1 \to G \to H \to K \to 1$$

$H \in \mathcal{C} \Rightarrow G \in \mathcal{C}$ and $G, K \in \mathcal{C} \Rightarrow H \in \mathcal{C}$;

(3) $G^{|H|} \in \mathcal{C}$ for any $G, H \in \mathcal{C}$.

Example 2.2.3. There are two useful examples for $\mathcal{C}$. One is the class of all finite groups and the other is the class of all p-groups for a fixed prime p. We assume $\mathcal{C}$ is one of them in the following context.

In practice, we can avoid the set theoretic issue by considering isomorphism classes of objects and hence we will assume $\mathcal{C}$ is small.

Lemma 2.2.4. *([AM69, Lemma 3.3]) The inclusion functor* $\mathrm{Pro}(\mathcal{C}) \to \mathrm{Pro}(\boldsymbol{Gr})$ *admits a left adjoint* $\widehat{\cdot} : \mathrm{Pro}(\boldsymbol{Gr}) \to \mathrm{Pro}(\mathcal{C})$.

Definition 2.2.7. The left adjoint functor $\widehat{\cdot} : \mathrm{Pro}(\mathbf{Gr}) \to \mathrm{Pro}(\mathcal{C})$ in the lemma is called the $\mathcal{C}$-completion of (pro-)groups.

Proof. ([AM69, p. 26]) Let $G = \{G_i\}$ be a pro-group. Consider the pro-system of all pro-homomorphisms $f : G \to A$ with $A \in \mathcal{C}$. We get a pro-group $\{A\}$ indexed by the homomorphisms f, where a morphism $f \to f'$ is defined by the diagram

$$\begin{array}{ccc} G & \xrightarrow{f} & A \\ & \searrow^{f'} & \downarrow \\ & & A' \end{array} .$$

Then we define $\widehat{G}$ by $\{A\}$ and check that it is left adjoint to the inclusion $\mathrm{Pro}(\mathcal{C}) \to \mathrm{Pro}(\mathbf{Gr})$. □

Let $\mathbf{W}_0$ be the category of based connected CW complexes and let $\mathbf{H}_0$ be its homotopy category. By a (pro) space we mean a (pro) object of $\mathbf{H}_0$.

Let A be an abelian group. The homology of a pro-space $X = \{(X_i, x_i)\}$ with coefficient A is defined by the pro-group $H_n(X; A) = \{H_n(X_i; A)\}$ and the cohomology is the group $H^n(X; A) = \varinjlim_i H^n(X_i; A)$. Similarly, define the homotopy group $\pi_n X$ by the pro-group $\{\pi_n(X_i, x_i)\}$.

Remark 2.2.5. One may hope to get rid of the basepoint issue in the discussion. There are examples of pro-objects of the space category which are not pro-objects of based spaces. [Isa01, Definition 5.1] suggested the following way. Let $X = \{X_i\}$ be a pro-object of the category of CW complexes. Define the fundamental groupoid of X by the pro-system of fundamental groupoids $\Pi X = \{\Pi X_i\}$. Let $\Pi_n X_i$ be a functor from ΠX_i to the category of (abelian) groups, which maps each point $x_i \in X_i$ to $\pi_n(X_i, x_i)$. Call the pro-system $\Pi_n X = \{\Pi_n X_i\}$ the pro local system of homotopy groups of X.

Let $\mathcal{C}\mathbf{W}_0$ be the full subcategory of $\mathbf{W}_0$ consisting of the CW complexes with all homotopy groups in $\mathcal{C}$. Let $\mathcal{C}\mathbf{H}_0$ be the corresponding homotopy category.

Definition 2.2.8. The $\mathcal{C}$-completion of a pro-space $X \in \mathrm{Pro}(\mathbf{H}_0)$ is an object $\widehat{X}$ of $\mathrm{Pro}(\mathcal{C}\mathbf{H}_0)$ together with a pro-morphism $X \to \widehat{X}$ such that any pro-morphism $X \to Y$ with $Y \in \mathrm{Pro}(\mathcal{C}\mathbf{H}_0)$ uniquely factors through some pro-morphism $\widehat{X} \to Y$.

Theorem 2.2.6. *([AM69, Theorem 3.4]) The $\mathcal{C}$-completion of a pro-space always exists. In other words, the natural inclusion $\mathrm{Pro}(\mathcal{C}\boldsymbol{H}_0) \to \mathrm{Pro}(\boldsymbol{H}_0)$ has a left adjoint $\widehat{\cdot} : \mathrm{Pro}(\boldsymbol{H}_0) \to \mathrm{Pro}(\mathcal{C}\boldsymbol{H}_0)$.*

The idea is analogous to the case of groups, namely, the pro-space $\widehat{X}$ is constructed out of all homotopy classes of pro-maps $f : X \to F$ with $F \in \mathcal{C}\mathbf{H}_0$.

Remark 2.2.7. Sullivan further defined a homotopy inverse limit $\widehat{X}^S$ of any pro-space $\widehat{X}$ when $\mathcal{C}$ is the class of finite groups or p-groups ([Sul09, Definition 3.1]) .

Remark 2.2.8. Let $\mathcal{C}$ be the class of p-groups. Then Artin-Mazur-Sullivan's $\mathcal{C}$-completion of a space is not homotopy equivalent to the Bousfield-Kan's $\mathbb{Z}/p$-localization, unless the space is nilpotent ([BK72, p. 8.3]).

It is not hard that

Proposition 2.2.9. *([AM69, Corollary 3.7])*

$$\pi_1(\widehat{X}) \simeq \widehat{\pi_1(X)}$$

Definition 2.2.9. A map $f : X \to Y$ of pro-spaces is a weak equivalence if f induces an isomorphism on (pro) homotopy groups.

Definition 2.2.10. ([AM69, Theorem 4.3]) A map of pro-spaces $f : X \to Y$ is called a $\mathcal{C}$-equivalence if

(1) $\widehat{\pi_1 X} \to \widehat{\pi_1 Y}$ is an isomorphism;

(2) for any $\mathcal{C}$ local system A over Y, $H^n(Y; A) \to H^n(X; A)$ is an isomorphism for every n.

Theorem 2.2.10. *([AM69, Theorem 4.3]) A map of pro-spaces $f : X \to Y$ is a $\mathcal{C}$-equivalence if and only if $\widehat{f} : \widehat{X} \to \widehat{Y}$ is a weak equivalence. In particular, the $\mathcal{C}$-completion $X \to \widehat{X}$ is a $\mathcal{C}$-equivalence.*

Now let us review Sullivan's completion. Let l be a set of primes. Let $\mathcal{C}$ be the class of finite groups whose orders are only divisible by the primes in l. Then Artin-Mazur's $\mathcal{C}$-completion of X is a pro space $\widehat{X}_l$.

Theorem 2.2.11. *([Sul09, Proposition 3.3]) The homotopy inverse limit of $\widehat{X}_l$ exists.*

Definition 2.2.11. This homotopy inverse limit is Sullivan's l-adic completion of X. We will use the same notation $\widehat{X}_l$.

The idea for the proof is the following. Let F be an arbitrary l-finite space, i.e., each homotopy group $\pi_i(X)$ is a finite group in the class $\mathcal{C}$ and $\pi_i(X)$ vanishes for sufficiently large i. For any space X, the set $[X, F]$ of homotopy classes of maps $X \to F$ is a finite set. So $[-, F]$ is a functor from the homotopy category of spaces to the category of finite sets. Then the pro-space $\widehat{X}_l$ gives a pro system of such functors. These functors have an inverse limit, namely, a functor from the homotopy category of spaces to the category of compact Hausdorff spaces. One can check that this limiting functor satisfies the Brown's representability axioms and hence there is a space representing this functor.

Here are some properties of Sullivan's completion.

Proposition 2.2.12. *([Sul09, Proposition 3.16]) If* $\widehat{\pi_1(X)}_l = 0$*, then there is a natural homotopy equivalence* $\widehat{X}_l \simeq \prod_{p \in l} \widehat{X}_p$.

Proposition 2.2.13. *([Sul09, Theorem 3.9]) Suppose X is simply connected and each homotopy group $\pi_i(X)$ is finitely generated. Then*

(1) $H^i(X; \mathbb{Z}/n) \cong H^i(\widehat{X}_l; \mathbb{Z}/n)$*, where n is a product of primes in l;*

(2) $\widehat{\pi_i(X)}_l \cong \pi_i(\widehat{X}_l)$*;*

(3) $\widehat{H^i(X; \mathbb{Z})}_l \cong H^i(X; \widehat{\mathbb{Z}}_l) \cong H^i(\widehat{X}_l; \widehat{\mathbb{Z}}_l)$.

2.2.2 Sites and Hypercoverings

In this subsection, we review Artin-Mazur's comparison theorem for complex varieties, what they called the generalized Riemann existence theorem. We only review all the definitions and the lemmas which are needed to prove this theorem.

Definition 2.2.12. A site is a category **C** with a distinguised set of families of morphisms $(U_i \to U)_{i \in I}$ for each object U, which is called the set of coverings of U, such that

(1) $(U \xrightarrow{1} U)$ is a covering for each object U;

(2) for each covering $(U_i \to U)_{i \in I}$ and each morphism $V \to U$, the pullback $U_i \times_U V$ exists and $(U_i \times_U V \to V)_{i \in I}$ is a covering of V;

(3) for any coverings $(U_i \to U)_{i\in I}$ and $(V_{ij} \to U_i)_{j\in J_i}$ for each $i \in I$, the induced family $(V_{ij} \to U)_{ij}$ is also a covering of U.

For simplicity, we always assume that the underlying category of a site admits finite limits and finite coproducts.

Example 2.2.14. The set category **Set** has a natural site structure, where, for each set S, its covering set consists of all families of maps $(S_i \to S)_{i\in I}$ such that the union of the images is S.

Example 2.2.15. Let X be a topological space. The category of its open subsets $\mathbf{Op}(X)$ is a site, with inclusions as its morphisms. Indeed, a covering in the site $\mathbf{Op}(X)$ is an open cover over an open subset.

Example 2.2.16. Let X be a scheme. The étale site $\mathbf{S}_{\text{ét}}(X)$ over X consists of all étale morphisms $U \to X$, where the coverings are the surjective families. If X is quasi-compact, one can take the coverings to be finite surjective families.

Example 2.2.17. Let G be a profinite group. The category $\mathrm{Fin}(G)$ of finite continuous G-sets is a site.

Definition 2.2.13. A morphism of sites $\widetilde{F} : \mathbf{C}_2 \to \mathbf{C}_1$ is a functor $F : \mathbf{C}_1 \to \mathbf{C}_2$ which preserves finite limits, arbitrary colimits and the covering set for each object.

Definition 2.2.14. A point of a site $\mathbf{C}$ is a morphism of sites $\widetilde{P} : \mathbf{Set} \to \mathbf{C}$. In particular, the image of the terminal object $1_{\mathbf{C}}$ of $\mathbf{C}$ is the one-point set.

A pointed simplicial object K_* of a pointed site $(\mathbf{C}, \widetilde{P})$ is a simplicial object so that $P(K_*)$ is a pointed simplicial set, that is, there is a choice of a point in $P(K_0)$.

There is a generalization of the Čech nerves, namely, the hypercoverings of a site.

Definition 2.2.15. ([AM69, Definition 8.4]) A hypercovering K_* of a (pointed) site $\mathbf{C}$ is a (pointed) simplicial object of $\mathbf{C}$ such that

(1) $K_0 \to 1_{\mathbf{C}}$ is a covering, where $1_{\mathbf{C}}$ is the terminal object of $\mathbf{C}$;

(2) the canonical morphism $K_{n+1} \to (\operatorname{Cosk}_n K)_{n+1}$ is a covering for any $n \geq 0$, where Cosk_n is the n-th coskeleton of a simplicial object (see [AM69, p. 6] for a definition of coskeleton).

A simplicial morphism of hypercoverings $K_* \to K'_*$ is called a refinement if for each level n the map $K_n \to K'_n$ is a covering.

Let X be an object of $\mathbf{C}$ and S be a finite set. Define the object $X \times S$ of $\mathbf{C}$ by $\bigsqcup_S X$.

Let I_* be the simplicial set of the unit interval. Then for any simplicial object K_* of $\mathbf{C}$ we define the simplicial object $K_* \times I_*$ by $(K_* \times I_*)_n = K_n \times I_n$.

Two simplicial maps $f, g : K_* \to K'_*$ of simplicial objects are strictly homotopic if there is a map $F : K_* \times I_* \to K'_*$ connecting them, i.e., $F \circ j_0 = f$ and $F \circ j_1 = g$, where $j_0, j_1 : K_* \to K_* \times I_*$ are induced by the maps $[0], [1] : \mathrm{pt} \to I_*$. Call f and g homotopic if they are connected by a finite chain of strict homotopies.

Let $HR(\mathbf{C})$ be the category of hypercoverings of the (pointed) site $\mathbf{C}$, whose morphisms are homotopy classes of simplicial morphisms.

Lemma 2.2.18. *([AM69, Corollary 8.13]) $HR(\boldsymbol{C})$ is a cofiltering category.*

Definition 2.2.16. An object X of a site $\mathbf{C}$ is connected if it is not a nontrivial coproduct in $\mathbf{C}$. A site $\mathbf{C}$ is locally connected if each object is a coproduct of some connected objects, where each connected object in the coproduct is called a connected component. Call a locally connected site $\mathbf{C}$ connected if its terminal object is connected.

There is a natural connected component functor $\pi : \mathbf{C} \to \mathbf{Set}$ for a locally connected site $\mathbf{C}$ defined by mapping an object to the index set of its connected components. Then for any hypercovering K_* of $\mathbf{C}$, $\pi(K_*)$ is a simplicial set. If $\mathbf{C}$ is pointed, then $\pi(K_*)$ is a pointed/based simplicial set; if $\mathbf{C}$ is connected, then the simplicial set $\pi(K_*)$ is connected.

Since $HR(\mathbf{C})$ is cofiltering, we get a pro-system of homotopy based simplicial sets $\pi C = \{\pi(K_*)\}_{K_* \in HR(\mathbf{C})}$.

Definition 2.2.17. Let $\mathbf{C}$ be a pointed connected site. The pro homotopy type of $\mathbf{C}$ is defined by the pro-object πC in $\mathbf{H}_0$. Define the homotopy group $\pi_n(\mathbf{C})$ by the pro-group $\{\pi_n \pi(K_*)\}_{K_* \in HR(\mathbf{C})}$.

Example 2.2.19. Suppose X is a pointed connected topological space. The ordinary topology site $\mathbf{Op}(X)$ is pointed and each hypercovering is canonically pointed. Suppose any open cover of X admits a refinement of good covers, i.e., any finite intersection of connected opens is contractible. Then the pro-space $\pi\mathbf{Op}(X)$ is weak equivalent to the singular simplicial space $\mathrm{Sing}(X)$.

Let us omit the definition of fibered categories and only define the descent data in the form for our later use.

For each object X of a site $\mathbf{C}$, let $\mathbf{F}(X)$ be the small category of objects $\{X \times S | S \text{ is a finite set}\}$, where morphisms are of the form $X \times S \to X \times S'$ so that the following diagram commutes.

$$\begin{array}{ccc} X \times S & \longrightarrow & X \times S' \\ & \searrow & \downarrow \\ & & X \end{array}$$

Any morphism $f : X \to Y$ in $\mathbf{C}$ induces a natural functor $f^* : \mathbf{F}(Y) \to \mathbf{F}(X)$.

Definition 2.2.18. Let K_* be a hypercovering of a site $\mathbf{C}$. A locally constant covering (or a descent data) on K_* is an object α of $\mathbf{F}(K_0)$ together with an isomorphism $\phi : \partial_0^* \alpha \xrightarrow{\simeq} \partial_1^* \alpha$ in $\mathbf{F}(K_1)$ such that $\partial_1^* \phi = \partial_2^* \phi \circ \partial_0^* \phi$ in $\mathbf{F}(K_2)$.

If $\mathbf{C}$ is locally connected, then a locally constant covering consists of a hypercovering K_*, a finite set S and a 1-cocycle σ of $\pi(K_*)$ with values in the symmetric group $\mathrm{Sym}(S)$. Two locally constant coverings (K_*, S, σ) and (K'_*, S', σ') are isomorphic if $S = S'$ and there exists a common refinement K'' with $\phi_0 : K'' \to K$ and $\phi_1 : K'' \to K'$ such that $\phi_0^* \sigma$ and $\phi_1^* \sigma'$ are cohomologous.

Lemma 2.2.20. *([AM69, Corollary 10.6]) Let $\boldsymbol{C}$ be a locally connected site and let K_* be a hypercovering. The set of isomorphism classes of locally constant coverings (K_*, S, σ) is bijective to the set of isomorphism classes of simplicial covering sets of $\pi(K_*)$.*

Let G be a finite group. Define a principal bundle over $\mathbf{C}$ with fiber G by a locally constant sheaf of a site $\mathbf{C}$ with stalk $|G|$ left acted by G, namely, a locally constant covering $(K_*, |G|, \sigma)$ with $\sigma \in G \subset \mathrm{Sym}(|G|)$. Hence,

Lemma 2.2.21. *([AM69, Corollary 10.7]) Let $\boldsymbol{C}$ be a locally connected site and let G be a finite group. The set of isomorphism classes of principal bundles over $\boldsymbol{C}$ with fiber G is bijective to $\mathrm{Hom}(\pi_1 \boldsymbol{C}, G)$.*

Lemma 2.2.22. *([AM69, Corollary 10.8]) Let $\boldsymbol{C}$ be a connected pointed site and let A be a finite abelian group. Any locally constant sheaf $\mathfrak{A}$ with stalk A induces a local system $\widetilde{A}$ on the pro-space $\pi \boldsymbol{C}$. Moreover, there is a canonical isomorphism*

$$H^*(\boldsymbol{C}; \mathfrak{A}) \simeq H^*(\pi \boldsymbol{C}; \widetilde{A})$$

Finally, let us review the results of étale homotopy theory for schemes. Let $\mathcal{C}$ be the class of finite groups.

Theorem 2.2.23. *([AM69, Theorem 11.2]) Let X be a pointed connected geometrically unibranched Noetherian scheme. Then the pro homotopy type $X_{\text{ét}}$ of the étale site $\boldsymbol{S}_{\text{ét}}(X)$ is a pro $\mathcal{C}$-space, i.e., an object of $\mathcal{C}\boldsymbol{H}_0$.*

The proof is based on the following key lemma.

Lemma 2.2.24. *([AM69, Proposition 11.3]) Let G be a profinite group. Let K_* be a hypercovering of the site $\mathrm{Fin}(G)$. Then $\pi_n \pi(K_*)$ is a finite group for each n.*

For a scheme X over $\mathbb{C}$ of finite type, let X_{cl} be the underlying complex algebraic set with analytic topology.

Theorem 2.2.25. *(Generalized Riemann Existence Theorem, [AM69, Theorem 12.9])*

Let X be a pointed connected finite type scheme over $\mathbb{C}$. Then there is a canonical map $X_{\mathrm{cl}} \to X_{\text{ét}}$, which induces an isomorphism in the category $\mathrm{Pro}(\boldsymbol{H}_0)$ after profinite completion.

Artin-Mazur first proved that $X_{\mathrm{cl}} \to X_{\text{ét}}$ is a $\mathbf{C}$-equivalence. To show that the map is indeed a profinite homotopy equivalence, the followings are needed.

Definition 2.2.19. Let $\mathcal{C}$ be a class of group and let $\mathbf{C}$ be a site. An object X of $\mathbf{C}$ has $\mathcal{C}$-dimension at most d if for any locally constant sheaf $\mathfrak{A}$ of abelian groups over $\mathbf{C}$, $H^n(X;\mathfrak{A}) = 0$ for each $n > d$. $\mathbf{C}$ has $\mathcal{C}$-dimension at most d if for each object X there is a covering Y over X having $\mathcal{C}$-dimension at most d.

Let f be a pointed morphism of pointed connected sites $\mathbf{C} \to \mathbf{C}'$ induced by a functor $F : \mathbf{C}' \to \mathbf{C}$. For any hypercovering K_* of $\mathbf{C}'$, consider the maps $L_* \to F(K_*)$ of hypercoverings in $\mathbf{C}$. They induce a morphism of pro-spaces

$$\pi(f) : \pi\mathbf{C} = \{\pi(L_*)\}_{L_* \in HR(\mathbf{C})} \to \{\pi(K_*)\}_{K_* \in HR(\mathbf{C}')} = \pi\mathbf{C}'$$

Lemma 2.2.26. *([AM69, Theorem 12.5]) Let $\mathcal{C}$ be a complete class of groups. Let $\boldsymbol{C}$ and $\boldsymbol{C}'$ be pointed connected sites of dimension at most d for some d. Let f be a pointed morphism of pointed connected sites $\boldsymbol{C} \to \boldsymbol{C}'$. If $\pi(f)$ is a $\mathcal{C}$-equivalence, then the map $\widehat{\pi(f)}$ induced by $\mathcal{C}$-completion is an isomorphism in the category $\boldsymbol{H}_0$.*

2.2.3 Adams Conjecture and Sullivan's Proof

Adams studied the image of the Hopf-Whitehead J-homomorphism $J : \pi_i(GL(n,\mathbb{C})) \to \pi_{2n+i}S^{2n}$ in his series of papers [Ada63][Ada65a][Ada65b][Ada66]. The J-homomorphism is indeed the map on homotopy groups induced by $BU \to BG$. In this way, the J-homomorphism induces a natural map $K(X) \to J(X)$, where $J(X)$ is the Grothendieck group of spherical fibrations over X.

The Adams operation ψ^k on $K(X)$ is used in Adams' series of papers. The natural operation ψ^k is essentially determined by the following axioms, due to the splitting principle of vector bundles:

(1) ψ^k is a ring homomorphism on $K(X)$;

(2) $\psi^k(\gamma) = \gamma^k$ if γ is a line bundle.

Adams conjectured the following.

Conjecture 2.2.27. (Adams Conjecture, [Ada63, Conjecture 1.2]) Let X be a finite CW complex. For any $\eta \in K(X)$ and any k, there is an N large enough such that $k^N(\psi^k\eta - \eta)$ is a trivial element in $J(X)$.

He proved the following.

Theorem 2.2.28. *([Ada65b, Theorem 1.1]) If the Adams conjecture is true, then the kernel of $K(X) \to J(X)$ is a linear combination of such forms $k^N(\psi^k\eta - \eta)$.*

Define the profinite K-theory by $\widehat{K}(X) = \widehat{K(X)}$. The classifying space of $\widehat{K}(X)$ is the profinite completion $\widehat{BU}$ of BU, which follows directly from Sullivan's construction of profinite completion ([Sul74, p. 8]).

Then we modify the Adams operation ψ^k on $\widehat{K}(X) \cong \prod_p \widehat{K(X)}_p$ via replacing ψ^k by identity only when p divides k. Then the modified Adams operations ψ^k on $\widehat{K}(X)$ induces an action $\widehat{\mathbb{Z}}^\times$ on $\widehat{K}(X)$.

The Adams conjecture is restated and then proven by Sullivan.

Theorem 2.2.29. *([Sul74, p. 9, Theorem]) The modified Adams operation does not change the underlying stable spherical fibration type for any element of $\widehat{K}(X)$. This is also true for the real K-theory.*

Remark 2.2.30. Sullivan not only proved the Adams conjecture, but also provided a canonical factorization of $\widehat{BU}_p \xrightarrow{\psi^k} \widehat{BU}_p$ into $\widehat{BU}_p \to \widehat{G/U}_p \to \widehat{BU}_p$ for p not dividing k, where $\widehat{G/U}_p \to \widehat{BU}_p$ is part of the full fibration $G/U \to BU \to BG$.

Let us sketch Sullivan's key idea for his proof here, which uses the étale theory in the algebraic geometry.

Let X be the complex Grassmannian $Gr_s(\mathbb{C}^N)$. Notice that it is a variety over $\mathbb{Q}$. Then any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ induces a self étale homotopy equivalence $\sigma : Gr_s(\mathbb{C}^N)_{\text{ét}} \to Gr_s(\mathbb{C}^N)_{\text{ét}}$. The equivalence map is compatible with the stablization of N. Then we have a self profinite complete homotopy equivalence $\sigma : \widehat{BU(s)} \to \widehat{BU(s)}$. The map is also compatible with respect to s. Note that $BU(s-1) \to BU(s)$ is the universal spherical fibration with fiber S^{2s-1} that is induced from a vector bundle. After passing to stable range, $\sigma : \widehat{BU} \to \widehat{BU} \to \widehat{BG}$ factors through $\widehat{G/U}$ canonically.

On the other hand, the modified Adams operation induces an abelianized Galois action $\widehat{\mathbb{Z}}^{\times}$ on $\widehat{BU}$. The self homotopy equivalence on $\widehat{BU}$ is, uniquely up to homotopy, determined by the induced map on $H^*(\widehat{BU}; \widehat{\mathbb{Z}}) \otimes \mathbb{Q}$. After checking the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and the abelianzied Galois group $\widehat{\mathbb{Z}}^{\times}$ actions on the cohomology of $\widehat{BU}$, then one can prove the following, which implies the Adams conjecture.

Theorem 2.2.31. *(Sullivan, [Sul74])*

The natural action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *on* $\widehat{BU}$ *is equivalent to the abelianzied Galois group* $\widehat{\mathbb{Z}}^{\times}$ *action induced by modified Adams operations. The action does not change the underlying profinite spherical fibration.*

The method can also be applied to BO or BSO by considering $BO(n, \mathbb{C})$ or $BSO(n, \mathbb{C})$.

2.3 A Priori Invariants by Periods

In this section, we review the method used by Thom and Sullivan for defining cohomology classes of a space X by periods. We review the a priori K-theory invariants by periods too.

2.3.1 $\mathbb{Z}/n$-Manifolds

We follow [MS74, Section 1] to review the discussions of $\mathbb{Z}/n$ manifolds. A $\mathbb{Z}/n$ manifold M^m is defined by an oriented manifold with boundary ∂M together with an identification of ∂M to a disjoint union of n copies (labelled by $1, \dots, n$) of some oriented $(m-1)$-manifold δM. δM is also called the Bockstein of M. Conventionally, when $n = 0$, a $\mathbb{Z}/n$ manifold means a closed manifold.

A $\mathbb{Z}/n$ manifold with boundary is an oriented manifold N with boundary ∂N and an embedding of a disjoint union of n (labelled) copies of an oriented manifold M' (with boundary $\partial M'$) into ∂N. Call the manifold with boundary M' the Bockstein of N and call the $\mathbb{Z}/n$ manifold $\partial N - \bigsqcup_n \mathrm{Int}(M')$ the boundary of N.

Then one can define a bordism between two $\mathbb{Z}/n$ manifolds. $\Omega_*^{SO}(X; \mathbb{Z}/n)$ is the bordism group of singular $\mathbb{Z}/n$ manifolds in X so that the restricted map on the Bocksteins are identical.

Remark 2.3.1. We can similarly define $\mathbb{Z}/n$ manifolds (with or without boundary) with other structures like $STOP, SPL, U, Spin$ and then get the corresponding bordism groups with coefficient $\mathbb{Z}/n$.

Like the algebraic map $i : \mathbb{Z}/n \to \mathbb{Z}/nm$ by multiplication with m, one can construct $\Omega_*^{SO}(X; \mathbb{Z}/n) \to \Omega_*^{SO}(X; \mathbb{Z}/nm)$ by taking a disjoint union of m copies of a $\mathbb{Z}/n$ manifold M and view it as a $\mathbb{Z}/nm$ manifold. Define $\Omega_*^{SO}(X; \mathbb{Z}/p^\infty) = \varinjlim_k \Omega_*^{SO}(X; \mathbb{Z}/p^k)$ for any prime number p.

On the other hand, the natural quotient map $\mathbb{Z}/nm \to \mathbb{Z}/n$ also induces a map $\Omega_*^{SO}(X; \mathbb{Z}/nm) \to \Omega_*^{SO}(X; \mathbb{Z}/n)$ by regrouping mn copies of Bocksteins into n copies.

The product of any two $\mathbb{Z}/n$ manifolds $M \times N$ might not be a $\mathbb{Z}/n$ manifold. The problem is that the neighborhood $\delta M \times \delta N$ does not look like n-sheets coming into the boundary. [MS74] made a modification like follows.

A canonical neighborhood of each point in $\delta M \times \delta N$ is a product of a Euclidean space and a cone over $n*n$, the join between two sets of n points. It is the boundary of some 2-dimensional $\mathbb{Z}/n$ manifolds W, since $\Omega_1^{SO}(\mathrm{pt};\mathbb{Z}/n)=0$. Then replace the tubular neighborhood of $\delta M\times\delta N$ by $\delta M\times\delta N\times W$. The resulting manifold $M\otimes N$ is a $\mathbb{Z}/n$-manifold. The $\mathbb{Z}/n$-bordism class of $M\otimes N$ is independent of the choice of W, since $\Omega_2^{SO}(\mathrm{pt};\mathbb{Z}/n)=0$. The modified multiplication is associative up to $\mathbb{Z}/n$-bordism since $\Omega_3^{SO}(\mathrm{pt};\mathbb{Z}/n)=0$.

There is a natural map $\rho: M\otimes N\to M\times N$ which is the identity both away from some neighborhood of $\delta M\times\delta N$ and near $\delta M\times\delta N$.

Lemma 2.3.2. *([MS74, Proposition 1.5]) The tangent bundle $T(M\otimes N)$ is stably equivalent to $\rho^*(TM\times TN)\oplus\pi^*E$ for some vector bundle E over $W/(n*n)$, where π is the quotient map $M\otimes N\to\delta M\times\delta N\times W/(n*n)\to W/(n*n)$.*

2.3.2 Cohomological and K-Theoretical A Priori Invariants

Let $L_M^{\mathbb{Q}}=1+L_4^{\mathbb{Q}}(M)+L_8^{\mathbb{Q}}(M)+\cdots$ be the rational Hirzebruch L-genus of the tangent bundle TM a manifold M. The rational class has a $\mathbb{Z}_{(2)}$-lifting L_M, which is proved by the transversality of $\mathbb{Z}/2^k$-manifolds ([MS74, Theorem 3.3]). Furthermore, the $\mathbb{Z}/2$-reduction of L_M is the square of even Wu classes, namely, $V_M^2=1+v_2^2(M)+v_4^2(M)+\cdots$ ([MS74, Corollary 3.2]).

Remark 2.3.3. The class L in [MS74] might cause some confusions. To be clear for the readers, in [MS74] the class $L\in H^*(BSO;\mathbb{Z}_{(2)})$ is the inverse of the Hirzebruch L-genus as stated in the [MS74, Corollary 3.2]. But the paragraph above the [MS74, Theorem 3.3] clarifies that when applied to manifold, the notation L_M means the class L for the normal bundle ν_M, which is equivalent to what we used above.

The natural Hurewicz homomorphism $\Omega_*(X;\mathbb{Z}/2)\to H_*(X;\mathbb{Z}/2)$ is surjective, whose kernel is generated by elements like $[M,f]\cdot[N]$ with N of positive dimension. Moreover, $H^*(X;\mathbb{Z}/2)\cong\mathrm{Hom}(H_*(X;\mathbb{Z}/2),\mathbb{Z}/2)$. Hence, we can define a cohomology class

$x \in H^n(X;\mathbb{Z}/2)$ by a homomorphism $\phi : \Omega_n(X;\mathbb{Z}/2) \to \mathbb{Z}/2$ such that $\phi([M,f]\cdot[N]) = 0$ if the dimension of N is positive.

However, in application, we perturb the Hurewicz homomorphism to $h' : \Omega_*^{SO}(X;\mathbb{Z}/2) \to H_*(X;\mathbb{Z}/2)$ by $h'([M,f]) = f_*(V_M^2 \cap [M])$. Then h' is still surjective and the kernel of h' is generated by the elements like $[M,f]\cdot[N] - [M,f]\cdot\chi_2(N)$, where $\chi_2(N) = \langle V_N^2, [N]\rangle$ is the mod 2 Euler characteristic of N.

Theorem 2.3.4. *([BM76, Proposition A.3]) Each graded class $x_* \in H^*(X;\mathbb{Z}/2)$ is uniquely determined by a homomorphism $\sigma : \Omega_*^{SO}(X;\mathbb{Z}/2) \to \mathbb{Z}/2$ such that*

$$\sigma((M,f)\cdot N) = \sigma(M,f)\cdot\chi_2(N) \in \mathbb{Z}/2$$

where χ_2 is the mod 2 Euler charateristic, $(M,f) \in \Omega_*^{SO}(X;\mathbb{Z}/2)$ and $N \in \Omega_*^{SO}(\mathrm{pt};\mathbb{Z}/2)$.

The defining equation for x_* is

$$\sigma(M^m,f) = \langle V_M^2 \cdot f^*(\sum_{i\geq 0} x_{m-4i}), [M^m]\rangle \in \mathbb{Z}/2$$

where $(M,f) \in \Omega_m^{SO}(X;\mathbb{Z}/2)$.

The previous discussion can be generalized to $\mathbb{Z}/2^r$. That is, the natural Hurewicz homomorphism $h : \Omega_*(X;\mathbb{Z}/2^r) \to H_*(X;\mathbb{Z}/2^r)$ is surjective; $H^*(X;\mathbb{Z}/2^r) \cong \mathrm{Hom}(H_*(X;\mathbb{Z}/2^r),\mathbb{Z}/2^r)$. The kernel of the Hurewicz homomorphism is a bit more complicated.

Lemma 2.3.5. *([BM76, Lemma A.10]) The map $\Omega_*^{SO}(X;\mathbb{Z}/2^r) \to H_*(X;\mathbb{Z}/2^r)$ is surjective for any r, whose kernel is generated by three types of elements.*

(1) $[M,f]\cdot[N]$, where $[M,f] \in \Omega_*^{SO}(X;\mathbb{Z}/2^r)$ and $[N] \in \Omega_{>0}^{SO}(\mathrm{pt})$.

(2) $j_q([M,f]\cdot[N])$, where $[M,f] \in \Omega_*^{SO}(X;\mathbb{Z}/2)$ and $[N] \in \Omega_{>0}^{SO}(\mathrm{pt};\mathbb{Z}/2)$ and $j_q : \mathbb{Z}/2 \to \mathbb{Z}/2^r$.

(3) $\rho_r\delta([M,f]\cdot[N])$, where $[M,f] \in \Omega_*^{SO}(X;\mathbb{Z}/2)$, $[N] \in \Omega_{>0}^{SO}(\mathrm{pt};\mathbb{Z}/2)$ and $\rho_r : \mathbb{Z} \to \mathbb{Z}/2^r$.

By modifying the Hurewicz homomorphism, one gets that

Theorem 2.3.6. *([BM76, Proposition A.11]) Each graded class $x_* \in H^*(X;\mathbb{Z}/2^r)$ is uniquely determined by a homomorphism $\sigma : \Omega_*^{SO}(X;\mathbb{Z}/2^r) \to \mathbb{Z}/2^r$ such that*

(1)

$$\sigma((M,f)\cdot N) = \sigma_{\mathbb{Q}}(M,f)\cdot \mathrm{Sign}(N) \in \mathbb{Z}/2^r$$

where $(M,f) \in \Omega_*^{SO}(X;\mathbb{Z}/2^r)$ and $N \in \Omega_*^{SO}(\mathrm{pt})$;

(2)

$$\sigma(j_r((M,f)\cdot N)) = \sigma(j_r(M,f))\cdot \mathrm{Sign}(N) \in \mathbb{Z}/2 \xrightarrow{j_r} \mathbb{Z}/2^r$$

where $j_r : \mathbb{Z}/2 \to \mathbb{Z}/2^r$, $(M,f) \in \Omega_*^{SO}(X;\mathbb{Z}/2)$ and $N \in \Omega_*^{SO}(\mathrm{pt};\mathbb{Z}/2)$;

(3)

$$\sigma(\delta((M,f)\cdot N)) = \sigma(\delta(M,f))\cdot \mathrm{Sign}(N) \in \mathbb{Z}/2 \xrightarrow{j_r} \mathbb{Z}/2^r$$

where $j_r : \mathbb{Z}/2 \to \mathbb{Z}/2^r$, $(M,f) \in \Omega_*^{SO}(X;\mathbb{Z}/2)$ and $N \in \Omega_*^{SO}(\mathrm{pt};\mathbb{Z}/2)$

The defining equation for x_* is

$$\sigma(M^m,f) = \langle L_M \cdot f^*(\sum_{i\geq 0} x_{m-4i}), [M^m]\rangle \in \mathbb{Z}/2^r$$

where $(M,f) \in \Omega_m^{SO}(X;\mathbb{Z}/2^r)$.

If we take the colimit of the previous discussion, then we get the Hurewicz homomorphism $\Omega_*(X;\mathbb{Z}/2^\infty) \to H_*(X;\mathbb{Z}/2^\infty)$ whose kernel is generated by $[M,f]\cdot[N]$. So one can prove the following.

Theorem 2.3.7. *([MS74, Theorem 4.1]) Each graded class $x_* \in H^*(X;\mathbb{Z}_{(2)})$ is uniquely determined by a commutative diagram*

$$\begin{array}{ccc} \Omega_*^{SO}(X)\otimes\mathbb{Q} & \xrightarrow{\sigma_{\mathbb{Q}}} & \mathbb{Q} \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ \Omega_*^{SO}(X;\mathbb{Z}/2^\infty) & \xrightarrow{\sigma_2} & \mathbb{Z}/2^\infty \end{array}$$

where $\pi : \mathbb{Q} \to \mathbb{Q}/(\mathbb{Z}[\frac{1}{3},\frac{1}{5},\ldots]) \cong \mathbb{Z}/2^\infty$, such that

(1)

$$\sigma_{\mathbb{Q}}((M,f)\cdot N) = \sigma_{\mathbb{Q}}(M,f)\cdot \mathrm{Sign}(N) \in \mathbb{Q}$$

where $(M,f) \in \Omega_*^{SO}(X)$ *and* $N \in \Omega_*^{SO}(\mathrm{pt})$*;*

(2)

$$\sigma_2((M,f)\cdot N) = \sigma_2(M,f)\cdot \mathrm{Sign}(N) \in \mathbb{Z}/2^r \xrightarrow{j_\infty} \mathbb{Z}/2^\infty$$

where $(M,f) \in \Omega_*^{SO}(X;\mathbb{Z}/2^r)$*,* $N \in \Omega_*^{SO}(\mathrm{pt};\mathbb{Z}/2^r)$ *and* $j_\infty : \mathbb{Z}/2^r \to \mathbb{Z}/2^\infty$*.*

The defining equations for x_* *are*

$$\sigma_{\mathbb{Q}}(M^m,f) = \langle L_M \cdot f^*(\sum_{i\geq 0} x_{m-4i}), [M^m]\rangle \in \mathbb{Q}$$

and

$$\sigma_2(N^m,g) = \langle L_N \cdot g^*(\sum_{i\geq 0} x_{m-4i}), [N^m]\rangle \in \mathbb{Z}/2^r \xrightarrow{j_\infty} \mathbb{Z}/2^\infty$$

where $(M,f) \in \Omega_m^{SO}(X)$ *and* $(N,g) \in \Omega_m^{SO}(X;\mathbb{Z}/2^r)$*.*

For odd primes, the periods can be used to define elements of real K-theory. It is based on the Conner-Floyd theory and the Anderson duality for real K-theory.

Theorem 2.3.8. *(Conner-Floyd, [CF66]) There is an isomorphism*

$$\delta : \Omega_{4*+i}^{SO}(X) \otimes_{\Omega_*^{SO}} \mathbb{Z}_{(\mathrm{odd})} \to KO_i(X)_{(\mathrm{odd})}$$

where the homomorphism $\Omega_*^{SO} \to \mathbb{Z}_{(\mathrm{odd})}$ *is given by the signature of manifolds.*

The isomorphism δ indeed corresponds to the element $\Delta_{4n} \in \widetilde{KO}_{(\mathrm{odd})}(MSO(4n))$ defined by $\Delta_{4n} = \frac{\Lambda_+ - \Lambda_-}{\Lambda_+ + \Lambda_-}$, where Λ_+, Λ_- are the ± 1 eigenspaces of the $*$-operator on the exterior algebra of the universal bundle over $BSO(4n)$. By calculations, $\mathrm{ph}(\Delta) = \frac{1}{L^{\mathbb{Q}}} \cdot U$, where ph is the rational Pontryagin character, $L^{\mathbb{Q}}$ is the rational Hirzebruch L-genus and U is the universal Thom class of MSO.

Let $\mathbb{Z}/(\mathrm{odd})$ be the colimit of $\mathbb{Z}/n$ for all odd integers n. Let $\widehat{KO}(X)_{\mathrm{odd}}$ be the inverse limit of $KO(X;\mathbb{Z}/n)$ for all n odd. Hence,

Theorem 2.3.9. *(Anderson Duality, cf. [MM79, p. 85]) Let* X *be a finite complex. There is a natural exact sequence given by the evaluation homomorphism*

$$0 \to \mathrm{Ext}(KO_0(SX), \mathbb{Z}_{(\mathrm{odd})}) \to KO^0(X)_{(\mathrm{odd})} \to \mathrm{Hom}(KO_0(X)_{(\mathrm{odd})}, \mathbb{Z}_{(\mathrm{odd})}) \to 0$$

Remark 2.3.10. By an argument of the Atiyah-Hirzebruch spectral sequence, if $H_*(X;\mathbb{Z}_{(\mathrm{odd})})$ is concentrated in even degrees, then the map $KO^0(X)_{(\mathrm{odd})} \to \mathrm{Hom}(KO_0(X)_{(\mathrm{odd})}, \mathbb{Z}_{(odd)})$ is an isomorphism.

Indeed, for any cohomology theory h_* of finite type, i.e., $h^i(\mathrm{pt})$ is a finitely generated abelian group for any i, there is an Anderson dual homology theory Dh_* ([And69]) such that for any finite CW complex X there is an exact sequence for any i

$$0 \to \mathrm{Ext}(h_i(SX), \mathbb{Z}_{(\mathrm{odd})}) \to h^i(X)_{(\mathrm{odd})} \to \mathrm{Hom}(h_i(X)_{(\mathrm{odd})}, \mathbb{Z}_{(\mathrm{odd})}) \to 0$$

$KO_{(\mathrm{odd})}$ is Anderson dual to itself. Hence,

Theorem 2.3.11. *([Sul96, p. 87 Theorem 6]) Let X be a finite complex. Any commutative diagram like the following with some conditions uniquely determines an element in $KO_{(\mathrm{odd})}(X)$.*

$$\begin{array}{ccc} \Omega_*^{SO}(X)\otimes\mathbb{Q} & \xrightarrow{\sigma_{\mathbb{Q}}} & \mathbb{Q} \\ \downarrow \pi & & \downarrow \pi \\ \Omega_*^{SO}(X;\mathbb{Z}/(\mathrm{odd})) & \xrightarrow{\sigma_{\mathrm{odd}}} & \mathbb{Z}/(\mathrm{odd}) \end{array}$$

where $\pi : \mathbb{Q} \to \mathbb{Q}/(\mathbb{Z}[\frac{1}{2}]) \cong \mathbb{Z}/(\mathrm{odd})$ *satisfying that*

(1)

$$\sigma_{\mathbb{Q}}((M,f)\cdot N) = \sigma_{\mathbb{Q}}(M,f)\cdot \mathrm{Sign}(N) \in \mathbb{Q}$$

where $(M,f) \in \Omega_*^{SO}(X)$ *and* $N \in \Omega_*^{SO}(\mathrm{pt})$*;*

(2)

$$\sigma_{\mathrm{odd}}((M,f)\cdot N) = \sigma_{\mathrm{odd}}(M,f)\cdot \mathrm{Sign}(N) \in \mathbb{Z}/n \xrightarrow{j_\infty} \mathbb{Z}/(\mathrm{odd})$$

where n is odd, $(M,f)\in\Omega_*^{SO}(X;\mathbb{Z}/n)$, $N \in \Omega_*^{SO}(\mathrm{pt};\mathbb{Z}/n)$ *and* $j_\infty : \mathbb{Z}/n \to \mathbb{Z}/(\mathrm{odd})$.

Since $\mathrm{Hom}(\cdot,\mathbb{Z}/n)$ of Z/n-modules is an exact functor, the Pontryagin duality for the $\mathbb{Z}/n$ real K-theory holds, namely, the evaluation map $KO(X;\mathbb{Z}/n) \to \mathrm{Hom}(KO(X;\mathbb{Z}/n),\mathbb{Z}/n)$ is an isomorphism ([Sul09, p. 206]). On the other hand, $\widehat{KO(X)}_{\mathrm{odd}}$ is isomorphic to $\widehat{KO}(X)_{\mathrm{odd}}$ ([Sul09, p. 207]) if X is a finite complex. Hence, one gets that

Corollary 2.3.12. *([Sul09, Theorem 6.3]) Let X be a finite complex. Any element of $\widehat{KO}(X)_{\text{odd}}$ is uniquely determined by a homomorphism $\sigma : \Omega_*^{SO}(X; \mathbb{Z}/(\text{odd})) \to \mathbb{Z}/(\text{odd})$ such that*

$$\sigma((M,f)\cdot N) = \sigma(M,f)\cdot \text{Sign}(N) \in \mathbb{Z}/n \xrightarrow{j_\infty} \mathbb{Z}/(\text{odd})$$

where n is odd, $(M,f) \in \Omega_*^{SO}(X;\mathbb{Z}/n)$, $N \in \Omega_*^{SO}(\text{pt};\mathbb{Z}/n)$ and $j_\infty : \mathbb{Z}/n \to \mathbb{Z}/(\text{odd})$.

2.3.3 Applications to Surgery Theory

We sketch the proof regarding the homotopy type of G/PL and the construction of Brumfiel-Morgan's characteristic classes for spherical fibrations.

Sullivan proved that the PL-surgery space G/PL localized at prime 2 splits as a product of Eilenberg-Maclane spaces twisted by a cohomology operation at degree 4 ([Sul65][Sul96, Theorem 4(i)]). The homotopy equivalence map is induced by some characteristic classes for PL surgeries [Sul96, p. 88]. Later [RS71][MS74] constructed more elaborate classes.

For any $\mathbb{Z}/2$ manifold M and any map $M \to G/PL$, there is induced a degree 1 normal map $N \to M$. The Kervaire invariant of the map $N \to M$ induces a homomorphism $\Omega_*^{SO}(G/PL;\mathbb{Z}/2) \to \mathbb{Z}/2$. It is not hard to see that this homomorphism satisfies the product formulae in Subsection 1. Hence,

Theorem 2.3.13. *([Sul96, p. 88, Corollary 1][RS71, Theorem 4.6]) There is a graded class $k = k_2 + k_6 \cdots \in H^{4*+2}(G/PL;\mathbb{Z}/2)$ such that for any $\mathbb{Z}/2$ manifold M and any map $f : M \to G/PL$, the Kervaire invariant for the surgery problem induced by f is $\langle f^*k \cdot V_M^2, [M]\rangle$.*

For any $4l$-dimensional manifold $\mathbb{Z}/2^r$ manifold M and any map $f : M \to G/PL$, there is a surgery obstruction $\sigma(M,f)$ defined in $\mathbb{Z}/2^r$ for the surgery problem induced by f ([MS74, p. 9.5]). Define $\sigma'(M,f) = \sigma(M,f) - \langle \beta(f^*k \cdot V_M Sq^1 V_M), [M]\rangle$ ([MS74, p. 540]), where β is the $\mathbb{Z}/2 \to \mathbb{Z}/2^r$ Bockstein. The induced map $\Omega_*^{SO}(G/PL;\mathbb{Z}/2^\infty) \to \mathbb{Z}/2^\infty$ satisfies the product formula ([MS74, Theorem 8.6]). Hence,

Theorem 2.3.14. *([MS74, Theorem 8.7]) There is a graded class* $l = l_4 + l_8 \cdots \in H^{4*}(G/PL; \mathbb{Z}_{(2)})$ *such that for any* $\mathbb{Z}/2^r$ *manifold* M^{4m} *and any map* $f : M \to G/PL$*, the surgery obstruction for the surgery problem induced by* f *is* $\langle f^* l \cdot L_M, [M]\rangle + \langle \beta(f^* k \cdot V_M Sq^1 V_M), [M]\rangle$.

To prove the splitting of G/PL at prime 2, one needs the following two lemmas.

Lemma 2.3.15. *([Coo68, p. 170 (3.7)]) The k-invariant* $X_{i-1} \to K(\pi_i X, i+1)$ *of the Postnikov system of a simple space* X *vanishes if and only if the Hurewicz map* $h_i : \pi_i(X) \to H_i(X; \mathbb{Z})$ *is a splitting monomorphism.*

Lemma 2.3.16. *([Coo68, p. 172 (3.8)]) Let* X *be a simple space with* $\pi_i(X) \cong \mathbb{Z}$*. The order* d *of the k-invariant* $X_{i-1} \to K(\pi_i X, i+1)$ *of the Postnikov system is the least positive integer* d *such that there is a cohomology class in* $H^i(X)$ *with value* d *on the generator of* $\pi_i(X)$.

Hence, one can prove the following. For more details, we recommend [Coo68].

Theorem 2.3.17. *([Sul96, Theorem 4]) The classes* k *and* l *above induce a homotopy equivalence* $(G/PL)_{(2)} \simeq K(\mathbb{Z}/2, 2) \times_{\beta \cdot Sq^2} K(\mathbb{Z}_{(2)}, 4) \times \prod_{i \geq 2} (K(\mathbb{Z}/2, 4i-2) \times K(\mathbb{Z}_{(2)}, 4i))$*, where* β *is the* $\mathbb{Z}/2 \to \mathbb{Z}_{(2)}$ *Bockstein.*

Remark 2.3.18. In fact, $(G/TOP)_{(2)} \simeq \prod_{i \geq 1} (K(\mathbb{Z}/2, 4i-2) \times K(\mathbb{Z}_{(2)}, 4i))$ ([KS77, p. 329]). In this sense, the manifold theory for TOP is simpler than PL.

If we apply the odd-prime a priori invariant method to the space G/PL, then we can get the following. For more details, we recommend [MM79, Chapter 4].

Theorem 2.3.19. *([Sul96, Theorem 4][MM79, Corollary 4.31]) There is a canonical* H*-space equivalence* $(G/PL)_{(\text{odd})} \simeq BSO^{\otimes}_{(\text{odd})}$*, where the* H*-space structure on* G/PL *is induced by Whitney sums and the superscript* $\otimes$ *means that the* H*-space structure on* BSO *is induced by tensor products of vector bundles.*

Remark 2.3.20. $(G/TOP)_{(\text{odd})} \simeq BSO^{\otimes}_{(\text{odd})}$ since G/TOP and G/PL are only differed by a $\mathbb{Z}/2$ invariant ([KS77, p. 246]).

Now let us sketch the construction of Brumfiel-Morgan's characteristic classes for spherical fibrations.

Let ν^k be a spherical fibration $S(\nu) \to X$. Recall that the Thom space $Th(\nu)$ is the union of the mapping cylinder $D(\nu)$ of ν and a cone $C(\nu)$ along $S(\nu)$. Let M^{m+k} be a manifold with a map $f : M \to Th(\nu)$. Call f Poincaré transversal if f is transversal to $S(\nu)$, $f^{-1}(S(\nu)) \to f^{-1}(D(\nu))$ is a spherical fibration and f induces a map of spherical fibrations. One can directly deduce that if f is Poincaré transversal then $f^{-1}(D(\nu))$ is a Poincaré duality space of dimension m with the fundamental class $[M] \cap f^*U_\nu$, where U_ν is the Thom class for ν.

There is a sequence of obstructions $O_{k+n+1} \in H^{k+n+1}(M; P_n)$ for making f Poincaré transversal. The construction is like the following.

There is an open cover $\{U_i\}$ mon X such that ν over each U_i has a PL bundle structure. Then there is some triangulation on M such that each simplex σ^i is mapped into some $Th(\nu|_{U_i})$. Suppose for each simplex of dimension at most $k+n$, the restriction of f is already Poincaré transversal.

Then for each $(k+n+1)$-dimensional simplex σ^{k+n+1}, we hope to get a value $O_{k+n+1}(\sigma^{k+n+1}) \in P_n$. So the problem is reduced to the case of a map $f : D^{k+n+1} \to Th(\nu)$ such that $f|_{S^{k+n}}$ is Poincaré transversal and ν has some PL bundle structure. Then we can slightly shift f without changing the restriction on $f^{-1}(C(\nu))$ to some map f' such that f' is PL transversal to the zero-section. Let A be the preimage of the zero section under f' and let $B = f^{-1}(D(\nu))$. Then A is a PL manifold with boundary and the inclusion $A \cap S^{k+n} \to B \cap S^{k+n}$ is a degree 1 normal map. This is an element in P_n. We can slightly homotope f' so that the union C of a collar of $B \cap S^{k+n}$ and a tubular of A becomes $(f')^{-1}(D(\nu))$. If $n \geq 5$ and the surgery obstruction vanishes, then we can homotope f' such that the inclusion $A \cap S^{k+n} \to B \cap S^{k+n}$ is a homotopy equivalence and C becomes a Poincaré pair in D^{n+k+1}. So f' on D^{n+k+1} is Poincaré transversal. If $n \leq 4$, one can see [BM76, p. 18].

Come back to the map $f : M^{m+k} \to Th(\nu^k)$. We can embed M into some sphere S^{N+k+m}

for some $N > 2(k+m)$ and consider the Pontryagin-Thom construction, namely, an induced map $F : S^{N+k+m} \to Th(\nu) \wedge Th(EPL(N))$, where $EPL(N) \to BPL(N)$ is the universal block PL bundle of dimension N. The obstruction theory above gives us only one obstruction class $O \in H^{N+k+m}(S^{N+k+m}; P_{m-1}) \cong P_{m-1}$ for Poincaré transversality. The class O vanishes if and only if we can cobord (M, f) to some Poincaré transversal (M', f'). For more details see [BM76, Section 3].

Then for a $\mathbb{Z}/2$ manifold M^{4m+3+k} and a map $f : M \to Th(\nu^k)$, since $P_{4m+1} = 0$, we can always assume f is homotopy transversal on δM. Then the obstruction for cobording (M, f) to some Poincaré transversal (M', f') is an element in $P_{4m+2} = \mathbb{Z}/2$. This is a homomorphism $\Omega_{4m+3}(Th(\nu); \mathbb{Z}/2) \to \mathbb{Z}/2$. One can check that it satisfies the product formula. By the Thom isomorphism for the spherical fibration ν^k one gets the following.

Theorem 2.3.21. *([BM76, Theorem 5.4]) There exists a graded characteristic class $k^G \in H^{4*+3}(X; \mathbb{Z}/2)$ for a spherical fibration ν^k on X such that for any $\mathbb{Z}/2$ manifold M^{4m+3+k} and any map $f : M \to Th(\nu)$ the obstruction for f being Poincaré transversal by a bordism is $\langle f^*(k^G \cdot U_\nu) \cdot V_M^2, [M] \rangle \in \mathbb{Z}/2$.*

To define the $\mathbb{Z}/8$ characteristic class for ν^k, consider a $\mathbb{Z}/8$ manifold M^{4m+k} and a map $f : M \to Th(\nu)$. Since $P_{4m-1} = 0$, we may assume f is Poincaré transversal on δM.

Let us first consider the case that f is Poincaré transversal. Define the valuation on (M, f) by the signature of the Poincaré space $f^{-1}(D(\nu))$ in $\mathbb{Z}/8$. For the general case, [BM76, p. 61] proved that there exists a map $a : K^{k+4} \to MSG(k)$ for a $\mathbb{Z}/2$ manifold

$$K^{k+4} = S^{k+3} \times I/(x, 0) \sim (-x, 1)$$

such that

(1) the Kervaire obstruction to the Poincaré transversality of $a|_{\delta K}$ is $1 \in \mathbb{Z}/2$;

(2) $\langle a^*(V^2 \cdot U), [K] \rangle = 0 \in \mathbb{Z}/2$.

Then consider the bordism class $[M, f] - j_8[K, a]$, where $j_8 : \mathbb{Z}/2 \to \mathbb{Z}/8$, it is Poincaré transversal by some bordism. So we can apply the previously defined valuation on $[M, f] -$

$j_8[K, a]$. Modify this valuation by minus $j_8\langle f^*(k^G \cdot U) \cdot V_M Sq^1 V_M, [M]\rangle$. Chenck the product formulae for this valuation ([BM76, Theorem 8.4]). Hence,

Theorem 2.3.22. *([BM76, p. 8.1]) There exists a graded characteristic class $l^G \in H^{4*}(X; \mathbb{Z}/8)$ for a spherical fibration ν^k on X satisfying the following properties:*

(1) $\rho_2(l^G) = V^2$, where $\rho_2 : \mathbb{Z}/8 \to \mathbb{Z}/2$;

(2) βl^G is the obstruction class for Poincaré transversalities, where β is the $\mathbb{Z}/8 \to \mathbb{Z}_{(2)}$, namely, for any $\mathbb{Z}/2^r$ manifold M^{4m+1+k} and any map $f : M \to Th(\nu)$ the obstruction f being Poincaré transversal by a bordism is $\langle f^*(\beta l^G \cdot U_\nu) \cdot L_M, [M]\rangle + j_r\langle f^*(k^G \cdot U_\nu) \cdot V_{\delta M} Sq^1 V_{\delta M}, [\delta M]\rangle \in \mathbb{Z}/2^r$, where $j_r : \mathbb{Z}/2 \to \mathbb{Z}/2^r$.

We have to mention that the following theorem is essentially due to Morgan-Sullivan, though their result only states for PL bundles. However, if one knows some transversality theorem for TOP manifolds and TOP bundles, then their result generalizes to TOP bundles.

Theorem 2.3.23. *([MS74, p. 530]) The $\mathbb{Z}_{(2)}$-class L for vector bundles defined above, whose rationalization is the inverse Hirzebruch L-genus, can be lifted a $\mathbb{Z}_{(2)}$-class l^{TOP} for TOP bundles.*

The Winkelnkemper's axiom I states that transversality unlocks the secret of manifolds. One can define a TOP transversality theory on a spherical fibration ν by an assignment to each singular simplex $f : \sigma^i \to Th(\nu)$ a deformation of f till f is Poincaré tranversal in the interior and in each face. Then one can define a concordance equivalence relation between TOP transversality theories. Indeed, the equivalence classes of all TOP bundle structures on ν is in one-to-one correspondence to the concordance classes of transversality theories on ν ([LM72], [BM76, Theorem E], [LR87, Theorem 1.11]). Since k^G and βl^G are the obstructions for existence of TOP transversality theory on ν in the 2-local sense, so we have that

Theorem 2.3.24. *k^G and βl^G are the obstructions for a spherical fibration ν to have a TOP bundle structure in the 2-local sense.*

Remark 2.3.25. If ν has a TOP bundle structure, the $\mathbb{Z}/8$ reduction of $l^{TOP}(\nu)$ is exactly $l^G(\nu)$.

By applying the odd-prime period method and the correspondence between transversalities and bundle structures, one can also get that

Theorem 2.3.26. *([Sul09, Theorem 6.5]) The existence of a $KO_{(\mathrm{odd})}$-orientation is the obstruction for a spherical fibration ν to have a TOP bundle structure in the odd-prime local sense.*

Remark 2.3.27. This theorem is also true if 'a TOP bundle structure' by 'a PL bundle structure', since they are only differed a $\mathbb{Z}/2$ invariant.

Chapter 3

L-theory with $\mathbb{Z}/n$ coefficient

We generalize the idea of $\mathbb{Z}/n$ manifolds to chain-level. We put many efforts to construct the bordism invariants for $\mathbb{Z}/n$ and prove the product formulae for these invariants.

3.1 $\mathbb{Z}/n$ Chains and their bordisms

Like $\mathbb{Z}/n$ manifolds, we can define $\mathbb{Z}/n$ symmetric, quadratic or normal chains. The definitions for three cases are similar so we only discuss the symmetric case here. We use the notation $n \cdot C$ for the direct sum of n (labelled) copies of a chain complex C.

Definition 3.1.1. An m-dimensional $\mathbb{Z}/n$ symmetric chain complex C is an m-dimensional symmetric pair $n\delta C \to C$, $n\delta C$ is labelled by $1, \ldots, n$.

δC is also called the Bockstein of C.

Definition 3.1.2. A $\mathbb{Z}/n$ symmetric complex C is Poincaré if both the chain δC and the pair $n\delta C \to C$ are Poincaré.

Definition 3.1.3. A $\mathbb{Z}/n$ symmetric chain pair $D \to C$ consists of symmetric pairs $(-\delta D) \to \delta C$, $n\delta D \to D$ and $D \bigcup_{n\delta D} n\delta C \to C$. It is a $\mathbb{Z}/n$ Poincaré pair if $n\delta D \to D$ is a $\mathbb{Z}/n$ Poincaré symmetric chain and $(-\delta D) \to \delta C$, $D \bigcup_{n\delta D} n\delta C \to C$ are both Poincaré pairs.

Remark 3.1.1. We only use the symmetric-quadratic pair description for $\mathbb{Z}/n$ normal complexes. A $\mathbb{Z}/n$ Poincaré symmetric-quadratic chain pair of dimension m consists of a $\mathbb{Z}/n$ Poincaré quadratic chain complex $n\delta E \to E$ of dimension $m-1$, a Poincaré symmetric chain pair $(-\delta E) \to \delta D$ of dimension $m-1$ and a Poincaré symmetric chain pair $E \bigcup_{n\delta E} n\delta D \to D$ of dimension m. In abbreviation, we use the symbol (D, E) instead of writing all the maps above.

Then one can define a bordism between two $\mathbb{Z}/n$ (Poincaré) symmetric chains. One can also define a k-ad of $\mathbb{Z}/n$ (Poincaré) symmetric chains.

Definition 3.1.4. Let $L^s_m(\mathbb{Z}, \mathbb{Z}/n)$ be the set of m-dimensional bordism classes of $\mathbb{Z}/n$ Poincaré symmetric chain complexes.

Like L^s_m, the $\mathbb{Z}/n$ L-group $L^s_m(\mathbb{Z}, \mathbb{Z}/n)$ is also an abelian group with the additive structure by the direct sum of chains. Analogously, one can also define the abelian groups $L^q_m(\mathbb{Z}, \mathbb{Z}/n)$ and $L^n_m(\mathbb{Z}, \mathbb{Z}/n)$.

Like the long exact sequence of L-groups in the last chapter, we also have

Proposition 3.1.2. *There is a long exact sequence*

$$\cdots \to L^q_m(\mathbb{Z}, \mathbb{Z}/n) \to L^s_m(\mathbb{Z}, \mathbb{Z}/n) \to L^n_m(\mathbb{Z}, \mathbb{Z}/n) \to L^q_{m-1}(\mathbb{Z}, \mathbb{Z}/n) \to \cdots$$

Like the $\mathbb{Z}/n$ manifold case, there are also natural maps $L^s(\mathbb{Z}, \mathbb{Z}/n) \to L^s(\mathbb{Z}, \mathbb{Z}/nm)$ and $L^s(\mathbb{Z}, \mathbb{Z}/nm) \to L^s(\mathbb{Z}, \mathbb{Z}/n)$.

The Bockstein of $\mathbb{Z}/n$ chains induces a natural map $L^s_m(\mathbb{Z}, \mathbb{Z}/n) \xrightarrow{\delta} L^s_{m-1}$.

Proposition 3.1.3. *There is a long exact sequence*

$$\cdots \to L^s_m \xrightarrow{\times n} L^s_m \xrightarrow{\iota} L^s_m(\mathbb{Z}, \mathbb{Z}/n) \xrightarrow{\delta} L^s_{m-1} \to \cdots$$

Proof. $\mathrm{Im}(\delta) = \mathrm{Ker}(\times n)$, $\mathrm{Ker}(\iota) \subset \mathrm{Im}(\times n)$ and $\mathrm{Im}(\iota) \subset \mathrm{Ker}(\delta)$ are obvious. Let us check the rest.

$\mathrm{Ker}(\delta) \subset \mathrm{Im}(\iota)$: Take $C \in \mathrm{Ker}(\delta)$. Then there is a Poincaré symmetric chain pair $-\delta C \to D$. Then $\iota(C \bigcup_{n\delta_C} nD) = C$ since the $C \bigcup_{n\delta_C} nD \times I$ gives the $\mathbb{Z}/n$ bordism between C and $C \bigcup_{n\delta_C} nD$, where I is the cellular chain of a unit interval.

$\mathrm{Im}(\times n) \subset \mathrm{Ker}(\iota)$: Take a $C \in L^s_m$. $nC \times I$ is a $\mathbb{Z}/n$ chain with boundary nC. So $\iota(nC) = 0$. □

By a direct calculation, the followings are immediate.

Proposition 3.1.4. *(1)*

$$L^s_m(\mathbb{Z}, \mathbb{Z}/2^k) \cong \begin{cases} \mathbb{Z}/2^k, & \text{if } m \equiv 0 \pmod 4 \\ \mathbb{Z}/2, & \text{if } m \equiv 1, 2 \pmod 4 \\ 0, & \text{otherwise} \end{cases}$$

(2) Let p be an odd prime.

$$L^s_m(\mathbb{Z}, \mathbb{Z}/p^k) \cong \begin{cases} \mathbb{Z}/p^k, & \text{if } m \equiv 0 \pmod 4 \\ 0, & \text{otherwise} \end{cases}$$

(3) The isomorphisms are given by the bordism invariants defined below.

To define the bordism invariants, let us only focus on the case when $n = 2^k$, since the odd n case is almost the same and much easier.

The signature of a $\mathbb{Z}/n$ manifold is defined to be the signature modulo n. It is well defined because of Novikov's additivity theorem. It is also a bordism invariant for the same reason. Likewise, define the signature invariant $\sigma^s_0 = \mathrm{Sign} \in \mathbb{Z}/n$ for $\mathbb{Z}/n$ Poincaré symmetric chains.

For a $4l+2$ dimensional $\mathbb{Z}/2^k$ Poincaré symmetric chain complex C, define the bordism invariant $\sigma^s_2(C) = \mathrm{dR}(\delta C) \in \mathbb{Z}/2$.

The dimension $4l+1$ case is a bit more complicated. [MS74] defined the de Rham invariant for $4l+1$ dimensional $\mathbb{Z}/n$ oriented manifolds. Here we use the same idea .

Let C be a $4l+1$ dimensional $\mathbb{Z}/2^k$ Poincaré symmetric chain complex. Let A be the self-annihilating subspace of $H_{2l}(\delta C)$, which has half of the total rank. Let T be the torsion of $H_{2l}(\delta C)$. Let $T_{C,A}$ be the torsion of $H_{2l}(C)/f(2^k \cdot (A+T))$. Define the de Rham invariant by $\mathrm{dR}(C) = \mathrm{rank}_{\mathbb{Z}/2}(T_{C,A} \otimes \mathbb{Z}/2) \in \mathbb{Z}/2$.

There is an alternative way to define de Rham invariant. Since 2^k copies of δC is a boundary, the signature of δC vanishes. So it is the boundary of some Poincaré symmetric chain pair, namely, $(-\delta C) \to C'$. Then define $\sigma_1^s(C) = \mathrm{dR}(C \bigcup_{2^k \delta C} 2^k C') \in \mathbb{Z}/2$.

The second definition is independent of the choice of C'. Indeed, let C'' be another choice and then $2^k((-C') \bigcup_{\delta C \otimes 0} \delta C \otimes I \bigcup_{\delta C \otimes 1} C'')$ must be the boundary of some Poincaré symmetric pair W since its de Rham invariant must vanish. Then $C \otimes I \bigcup_{2^k \delta C \otimes I} (-W)$ is a bordism between $C \bigcup_{2^k \delta C} 2^k C'$ and $C \bigcup_{2^k \delta C} 2^k C''$. So they have the same de Rham invariant.

σ_1^s is a bordism invariant by a similar argument.

Warning 3.1.5. Extend σ_m^s by 0 when the chain complex does not have the corresponding dimension. This rule applies to other bordism invariants.

Analogously, we have the followings.

Proposition 3.1.6. *There is a long exact sequence*

$$\cdots \to L_m^q \xrightarrow{\times n} L_m^q \xrightarrow{\iota} L_m^q(\mathbb{Z}, \mathbb{Z}/n) \xrightarrow{\delta} L_{m-1}^q \to \cdots$$

Proposition 3.1.7. *There is a long exact sequence*

$$\cdots \to L_m^n \xrightarrow{\times n} L_m^n \xrightarrow{\iota} L_m^n(\mathbb{Z}, \mathbb{Z}/n) \xrightarrow{\delta} L_{m-1}^n \to \cdots$$

Again by direct computations, we have

Proposition 3.1.8. *(1)*

$$L_m^q(\mathbb{Z}, \mathbb{Z}/2^k) \cong \begin{cases} \mathbb{Z}/2^k, & \text{if } m \equiv 0 \pmod 4 \\ \mathbb{Z}/2, & \text{if } m \equiv 2, 3 \pmod 4 \\ 0, & \text{otherwise} \end{cases}$$

(2) Let p be an odd prime.

$$L^q_m(\mathbb{Z}, \mathbb{Z}/p^k) \cong \begin{cases} \mathbb{Z}/p^k, & \text{if } m \equiv 0 \pmod 4 \\ 0, & \text{otherwise} \end{cases}$$

(3) The isomorphisms are given by the bordism invariants defined below.

The bordism invariants for the quadratic case is like the symmetric case. Let C be a $\mathbb{Z}/2^k$ Poincaré quadratic chain complex.

If C has dimension $4l$, then δC has dimension $4l-1$ and must be the boundary of some Poincaré quadratic chain pair, namely, $(-\delta C) \to C'$. Then $C \bigcup_{2^k\delta C} 2^k C'$ is a Poincaré quadratic chain and hence its signature is divisible by 8. Define $\sigma^q_0(C) = \frac{1}{8} \cdot \mathrm{Sign}(C \bigcup_{2^k\delta C} 2^k C') \in \mathbb{Z}/2^k$.

If C has dimension $4l+2$, then the dimension argument also shows that δC must be the boundary of some quadratic Poincaré pair, namely, $(-\delta C) \to C'$. Define $\sigma^q_2(C) = K(C \bigcup_{2^k\delta C} 2^k C') \in \mathbb{Z}/2$.

If C has dimension $4l+3$, define $\sigma^q_3(C) = K(\delta C) \in \mathbb{Z}/2$.

Proposition 3.1.9. *(1) If $k = 1, 2$, then*

$$L^n_m(\mathbb{Z}, \mathbb{Z}/2^k) \cong \begin{cases} \mathbb{Z}/2^k \oplus \mathbb{Z}/2, & \text{if } m \equiv 0 \pmod 4 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2^k, & \text{if } m \equiv 1 \pmod 4 \\ \mathbb{Z}/2, & \text{if } m \equiv 2, 3 \pmod 4 \end{cases}$$

If $k \geq 3$, then

$$L^n_m(\mathbb{Z}, \mathbb{Z}/2^k) \cong \begin{cases} \mathbb{Z}/8 \oplus \mathbb{Z}/2, & \text{if } m \equiv 0 \pmod 4 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2^k, & \text{if } m \equiv 1 \pmod 4 \\ \mathbb{Z}/2, & \text{if } m \equiv 2, 3 \pmod 4 \end{cases}$$

(2) Let p be an odd prime. Then $L^n_m(\mathbb{Z}, \mathbb{Z}/p^k) = 0$.

(3) The isomorphisms are induced by the bordism invariants defined in the next section.

Proof. The result follows from the diagram, where each horizontal and vertical line is an exact sequence.

$$
\begin{array}{ccccccccc}
 & & \cdots & & \cdots & & \cdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & L^s_m & \longrightarrow & L^s_m & \longrightarrow & L^s_m(\mathbb{Z},\mathbb{Z}/n) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & L^q_m & \longrightarrow & L^q_m & \longrightarrow & L^q_m(\mathbb{Z},\mathbb{Z}/n) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & L^n_m & \longrightarrow & L^n_m & \longrightarrow & L^n_m(\mathbb{Z},\mathbb{Z}/n) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \cdots & & \cdots & & \cdots & &
\end{array}
$$

□

Remark 3.1.10. All the direct sum decompositions above are unnatural. However, we will find a way for an explicit decomposition in the last part of this chapter.

A $\mathbb{Z}/n$ (Poincaré) symmetric presheaf on a finite Δ-set is a (Poincaré) presheaf of ads of $\mathbb{Z}/n$ symmetric chain complexes, like presheaf of ads of symmetric chain complexes. Later on, we might discard the word 'ads' for simplicity and only use the phrase 'a presheaf of $\mathbb{Z}/n$ (Poincaré) symmetric chain complexes'.

Like the lemma 2.1.7, the following two lemmas are immediate.

Lemma 3.1.11. *Let X be a closed n-dimensional PL manifold with a PL triangulation. Let $\mathcal{C}$ be an m-dimensional (Poincaré) presheaf of $\mathbb{Z}/k$ symmetric/quadratic/normal chain complexes over X. Then the assembly $\mathcal{C}(X)$ is also a $\mathbb{Z}/k$ (Poincaré) $(n+m)$-dimensional symmetric/quadratic/normal chain complex.*

Lemma 3.1.12. *Let X be an oriented n-dimensional $\mathbb{Z}/k$ PL manifold with a PL triangulation. Let $\mathcal{C}$ be an m-dimensional (Poincaré) presheaf of symmetric/quadratic/normal chain complexes over X. Then the assembly $\mathcal{C}(X)$ is also a $\mathbb{Z}/k$ (Poincaré) $(n+m)$-dimensional symmetric/quadratic/normal chain complex.*

3.2 Product of $\mathbb{Z}/n$ Chains

We can define the product structure for $\mathbb{Z}/n$ Poincaré symmetric/quadratic/normal chains with the same idea for defining products of $\mathbb{Z}/n$ manifolds. Like the previous section, we only foucs on the symmetric case. The quadratic and normal cases are analogous.

Let C and D be two $\mathbb{Z}/n$ Poincaré symmetric complexes.

Let I be the chain complex of the unit interval. First, take $(C \otimes n\delta D \otimes I)\bigcup(C \otimes D)\bigcup(n\delta C \otimes D \otimes I)$, where the union is via an identification of $C \otimes n\delta D \otimes 0$ in $C \otimes n\delta D \otimes I$ with $C \otimes n\delta D$ in $C \otimes D$ and by a similar identification for another part. Then the 'singularity' part in the boundary is $n\delta C \otimes n\delta D \otimes I \bigcup n\delta C \otimes n\delta D \otimes I \cong \delta C \otimes \delta D \otimes (n^2 I \bigcup n^2 I)$, where $n^2 I \bigcup n^2 I$ is the chain complex of the $\mathbb{Z}/n$ manifold $n * n$. Since $n * n$ bounds a $\mathbb{Z}/n$ 2-dimensional manifold. Let W be the corresponding cellular chain complex. Define the product of C and D by

$$C \otimes n\delta D \otimes I \bigcup C \otimes D \bigcup n\delta C \otimes D \otimes I \bigcup \delta C \otimes \delta D \otimes W \tag{3.2.1}$$

By abuse of notations, we still use $C \otimes D$ to represent their modified product. The existence of such a modification is due to the fact $\Omega_1^{SO}(\mathrm{pt}; \mathbb{Z}/n) = 0$. The $\mathbb{Z}/n$ bordism class of $C \otimes D$ does not depend on the choice of W, since $\Omega_2^{SO}(\mathrm{pt}; \mathbb{Z}/n) = 0$. The modified product is associative up to bordism since $\Omega_3^{SO}(\mathrm{pt}; \mathbb{Z}/n) = 0$.

Remark 3.2.1. In fact, the modified product of two $\mathbb{Z}/n$ manifold is like our modified chain-level products. That is, the modified product $M \otimes N$ is diffeomorphic to the union

$$(M \otimes \bigsqcup_n \delta N \times I)\bigcup(M \times N)\bigcup(\bigsqcup_n \delta M \otimes N \times I)\bigcup(\delta M \times \delta N \times W)$$

Consider the graph $n*n$, namely, there are two sets of n points $\{i_1, \cdots, i_n\}$ and $\{j_1, \cdots, j_n\}$ and there is a path (s,t) connecting each pair of vertices i_s, j_t. A choice of W gives a permutation γ of $\{1, \ldots, n\}$. It adds for each s a path p_s into $n * n$ connecting the vertices $i_s, j_{\gamma(s)}$. Call the new graph γ'. W also regroup the original paths (s,t) of γ into n classes $\{(i_{s_k}, j_{t_k})\}_{1 \leq k \leq n}$, where each vertex i_t and j_s appear only once in the set of vertices of each group of paths, so that each group of paths together with $\{p_s\}_{1\leq s\leq n}$ gives a loop in the new graph. In this way, we csee what W does for $\delta(C\otimes D)$. W connects the t-th copy $\delta C\otimes D$ with $\gamma(t)$-th copy $C\otimes\delta D$ for $1\leq t\leq n$. Considering $\delta(C\otimes D)\otimes I$ we get that

Lemma 3.2.2. *$\delta(C\otimes D)$ is bordant to $(\delta C\otimes D)\oplus(C\otimes\delta D)$, as $\mathbb{Z}/n$ Poincaré symmetric chains.*

For later uses, let us further consider a product of presheaves over two $\mathbb{Z}/n$ manifolds. Let $\mathcal{C}$ and $\mathcal{D}$ be presheaves of symmetric chain complexes over two oriented $\mathbb{Z}/n$ PL manifolds M and N respectively.

By adding a collar neighborhood of Bockstein δM to M, we may assume the presheaf $\mathcal{C}$ on a collar neighborhood of a cell $\delta\sigma$ of δM is $\mathcal{C}(\delta\sigma)\otimes C_*(c(n))$, where $c(n)$ is the cone over n points. It is similar for $\mathcal{D}$ over N.

$M\times N$ has a natural regular cell decomposition inherited from the cell decompositions of M and N, namely, each cell in $M\times N$ has the form $\sigma\times\tau$ for some cell σ of M and some cell τ of N. Then it is natural to define the presheaf $\mathcal{C}\times\mathcal{D}$ over $M\times N$ by $(\mathcal{C}\times\mathcal{D})(\sigma\times\tau)=\mathcal{C}(\sigma)\otimes\mathcal{D}(\tau)$.

Consider the modified product $M\otimes N$. For any cell $\delta\sigma\times\delta\tau$ of $\delta M\times\delta N$, we just replace its tubular neighborhood $\delta\sigma\times\delta\tau$ in $M\times N$ by $\delta\sigma\times\delta\tau\times c(n*n)$ by $\delta\sigma\times\delta\tau\times W$, where $c(n*n)$ is the cone over $n*n$. On the chain level, we define $\mathcal{C}\otimes\mathcal{D}$ by the same value as $\mathcal{C}\times\mathcal{D}$ except $(\mathcal{C}\otimes\mathcal{D})(\delta\sigma\times\delta\tau\times W)=\mathcal{C}(\delta\sigma)\otimes\mathcal{D}(\delta\tau)\otimes C_*(W)$.

Comparing products of presheaves over modified products of manifolds and modified products of two symmetric chains, it is obvious that

Proposition 3.2.3. *The chain-level modified product of the assemblies $\mathcal{C}(M)\otimes\mathcal{D}(N)$ is isomorphic to the assembly of the presheaf over the modified product of manifolds $(\mathcal{C}\otimes\mathcal{D})(M\otimes N)$.*

Proposition 3.2.4. *The Bockstein of the assembly of the modified product presheaf $\delta(\mathcal{C}\otimes\mathcal{D})(M\otimes N)$ is bordant to $(\delta(\mathcal{C}(M))\otimes\mathcal{D}(N))\oplus(\mathcal{C}(M)\otimes\delta(\mathcal{D}(N)))$, as $\mathbb{Z}/n$ Poincaré symmetric chains.*

3.3 Product Structure of L-Theory with $\mathbb{Z}/n$ coefficient

We only consider the case $n = 2^k$ in this section, since the odd n case is much easier.

Recall that tensor products of chains induce the ring structures on L^s_*, L^n_* and the L^s_*-module structure on L^q_*. The map $L^s_* \to L^n_*$ also induces an L^s_*-algebra structure on L^n_*.

The product structures on the L-groups can be lifted to the spectra level. We recall the construction in [Ran92, Appendix B] here.

The simplicial approximation of the diagonal map of Δ^n gives a Δ-set structure on Δ^n, that is, a $(k+l)$-cell $\sigma_{t_0,t_1,\cdots,t_k,s_0,s_1,\cdots,s_l}$ is indexed by each sequence $0 \leq t_0 < t_1 < \cdots t_k \leq s_0 < s_1 < \cdots s_l \leq n$.

Let $\mathcal{C}$ and $\mathcal{D}$ be two n-ads of chains. Then define a presheaf $\mathcal{C} \otimes \mathcal{D}$ of chains over Δ^n by $(\mathcal{C} \otimes \mathcal{D})(\sigma_{t_0,\cdots,t_k,s_0,\cdots,s_l}) = \mathcal{C}(\Delta^{t_0,\cdots,t_k}) \otimes \mathcal{D}(\Delta^{s_0,\cdots,s_l})$. After taking a union, $\mathcal{C} \otimes \mathcal{D}$ is indeed an n-ad of chains.

All these algebraic structures can be defined for the L-groups with $\mathbb{Z}/n$-coefficient, by the modified products constructed in the previous section. Moreover, the natural map $L^a_* \to L^a_*(\mathbb{Z}, \mathbb{Z}/n)$ is a ring map when $a = s, n$ and an L^s-module map when $a = q$.

We will calculate the explicit product structures for these L-groups with $\mathbb{Z}/n$-coefficient.

3.3.1 Symmetric and Quadratic Case

Firstly, like manifolds, the signature of chains is multiplicative.

Lemma 3.3.1. *Let C and D be two Poincaré symmetric pairs, then*

$$\mathrm{Sign}(C \otimes D) = \mathrm{Sign}(C) \cdot \mathrm{Sign}(D)$$

Hence, the multiplicative structure on L^s_* induces the following isomorphisms.

Proposition 3.3.2.

$$L^s_{4s} \otimes L^s_{4t} \xrightarrow{\cong} L^s_{4(s+t)}$$

$$L^s_{4s} \otimes L^q_{4t} \xrightarrow{\cong} L^q_{4(s+t)}$$

$$L^n_{4s} \otimes L^n_{4t} \xrightarrow{\cong} L^n_{4(s+t)}$$

Using the long exact sequence 3.1.3 and the calculation of $L^s_*(\mathbb{Z}, \mathbb{Z}/2^k)$, we get that

Proposition 3.3.3.

$$L^s_{4s}(\mathbb{Z}, \mathbb{Z}/2^k) \otimes L^s_{4t}(\mathbb{Z}, \mathbb{Z}/2^k) \xrightarrow{\cong} L^s_{4(s+t)}(\mathbb{Z}, \mathbb{Z}/2^k)$$

$$L^s_{4s}(\mathbb{Z}, \mathbb{Z}/2^k) \otimes L^q_{4t}(\mathbb{Z}, \mathbb{Z}/2^k) \xrightarrow{\cong} L^q_{4(s+t)}(\mathbb{Z}, \mathbb{Z}/2^k)$$

Lemma 3.3.4. *Let C and D be $\mathbb{Z}/2^k$ Poincaré symmetric chain complexes. Then*

$$\sigma^s_0(C \otimes D) = \sigma^s_0(C) \cdot \sigma^s_0(D) \in \mathbb{Z}/2^k$$

The argument in [MS74, Chapter 6] also proves that

Proposition 3.3.5.

$$L^s_{4s+1} \otimes L^s_{4t} \xrightarrow{\cong} L^s_{4(s+t)+1}$$

$$L^s_{4s+1}(\mathbb{Z}, \mathbb{Z}/2^k) \otimes L^s_{4t}(\mathbb{Z}, \mathbb{Z}/2^k) \xrightarrow{\cong} L^s_{4(s+t)+1}(\mathbb{Z}, \mathbb{Z}/2^k)$$

It follows that

Lemma 3.3.6. *Let C and D be $\mathbb{Z}/2^k$ Poincaré symmetric chain complexes. Then*

$$\sigma^s_1(C \otimes D) = \sigma^s_1(C) \cdot \sigma^s_0(D) + \sigma^s_0(C) \cdot \sigma^s_1(D) \in \mathbb{Z}/2$$

Analogously,

Proposition 3.3.7.

$$L^q_{4s+2} \otimes L^s_{4t} \xrightarrow{\cong} L^q_{4(s+t)+2}$$

$$L^q_{4s+2}(\mathbb{Z}, \mathbb{Z}/2^k) \otimes L^s_{4t}(\mathbb{Z}, \mathbb{Z}/2^k) \xrightarrow{\cong} L^q_{4(s+t)+2}(\mathbb{Z}, \mathbb{Z}/2^k)$$

Lemma 3.3.8. *Let C be a $\mathbb{Z}/2^k$ Poincaré symmetric chain complex and let D be a $\mathbb{Z}/2^k$ Poincaré quadratic chain complex. Then*

$$\sigma^q_2(C \otimes D) = \sigma^s_0(C) \cdot \sigma^q_2(D) \in \mathbb{Z}/2$$

The following technical point is essentially [MS74, Theorem 6.1], which is crucial for the rest discussions.

Lemma 3.3.9. *Let S and Q be generators of L^s_{4s+1} and L^q_{4t+2}. Suppose $S \otimes Q$ is the boundary of some Poincaré quadratic pair $S \otimes Q \to W$. Then* $\mathrm{Sign}(W) = 4 \in \mathbb{Z}/8$*.*

The proof is based on the following lemma, which is essentially equivalent to [MS74, Theorem 5.8]. Let C be a $(4k-1)$ dimensional Poincaré quadratic chain complex. Then C is equivalent to a linking form q (valued in $\mathbb{Q}/\mathbb{Z}$) on the torsion group $TH_{2k-1}(C)$. Suppose C is the boundary of some Poincaré quadratic pair $C \to D$.

Lemma 3.3.10. $\sqrt{|TH_{2k-1}(C)|} \cdot e^{\frac{\pi i}{4}\mathrm{Sign}(D)} = \sum_{x \in TH_{2k-1}(C)} e^{2\pi i q(x)}$.

Sketch Proof of Lemma 3.3.9. There is a correspondence between bordism classes of Poincaré symmetric or quadratic chains and the Witt group of nonsingular (skew-)symmetric or (skew-)quadratic forms on abelian groups (which are free finitely generated or finite abelian group respectively) ([Ran81, Proposition 4.2.1]).

Hence, S is equivalent to a symmetric linking form $l_1 : \mathbb{Z}/2 \times \mathbb{Z}/2 \to \mathbb{Q}/\mathbb{Z}$ and Q is equivalent to a symplectic form l_2 on $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ together with a quadratic form q_2 so that $q_2(1,0) = q_2(0,1) = q_2(1,1) = \frac{1}{2} \in \mathbb{Z}/2 \subset \mathbb{Q}/\mathbb{Z}$. Then $S \otimes Q$ is equivalent to the quadratic form q on $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ with $q = q_2$.

Applying the previous lemma, we get that $\mathrm{Sign}(W) = 4 \in \mathbb{Z}/8$ by a direct calulation. □

Remark 3.3.11. If one wants to see a more elementary proof, we recommend the [MS74, Theorem 5.9] and the example in [MS74, p. 508].

Hence, we also have the followings.

Corollary 3.3.12. *Let S and Q be the generators of L^s_{4s+1} and L^q_{4t+2} respectively. Suppose 2^k copies of $S \otimes Q$ is the boundary of the Poincaré quadratic pair $S \otimes Q \to W_k$. Then $\sigma^q_0(W_k) = 2^{k-1} \in \mathbb{Z}/2^k$.*

Lemma 3.3.13. *Let C be a $\mathbb{Z}/2^k$ Poincaré quadratic chain complex and D be a $\mathbb{Z}/2^k$ Poincaré symmetric chain complex. Then*

$$\sigma_0^q(C \otimes D) = \sigma_0^q(C) \cdot \sigma_0^s(D) + j_{2^k}(\sigma_2^q(\delta C) \cdot \sigma_1^s(D) + \sigma_2^q(C) \cdot \sigma_1^s(\delta D)) \in \mathbb{Z}/2^k$$

where $j_{2^k} : \mathbb{Z}/2 \to \mathbb{Z}/2^k$.

Proposition 3.3.14. *The homomorphisms*

$$L^s_{4s+1} \otimes L^q_{4t+3}(\mathbb{Z}, \mathbb{Z}/2^k) \xrightarrow{\cong} L^s_{4s+1}(\mathbb{Z}, \mathbb{Z}/2^k) \otimes L^q_{4t+3}(\mathbb{Z}, \mathbb{Z}/2^k) \to L^q_{4(s+t)+4}(\mathbb{Z}, \mathbb{Z}/2^k)$$

$$L^s_{4s+2}(\mathbb{Z}, \mathbb{Z}/2^k) \otimes L^q_{4t+2} \xrightarrow{\cong} L^s_{4s+2}(\mathbb{Z}, \mathbb{Z}/2^k) \otimes L^q_{4t+2}(\mathbb{Z}, \mathbb{Z}/2^k) \to L^q_{4(s+t)+4}(\mathbb{Z}, \mathbb{Z}/2^k)$$

are isomorphic to the nontrivial homomorphism $j_{2^k} : \mathbb{Z}/2 \to \mathbb{Z}/2^k$.

3.3.2 Normal Case

Now we are left with the normal case only. Let (D, E) be a $\mathbb{Z}/2^k$ Poincaré symmetric-quadratic chain pair of dimension m. For the use in the next chapter, we only consider the case of dimension $m \geq 2$.

(1) $m \equiv 3 \pmod 4$**.**

Define $\sigma_3^n(D, E) = \sigma_2^q(E) \in \mathbb{Z}/2$. 3.3.8 also proves the product formula as follows.

Lemma 3.3.15. *Let H be a $\mathbb{Z}/2^k$ Poincaré symmetric chain complex. Then*

$$\sigma_3^n(D \otimes H, E \otimes H) = \sigma_3^n(D, E) \cdot \sigma_0^s(H) \in \mathbb{Z}/2$$

Due to the surjectivity of $L^s_{4s} \to L^n_{4s}$, then

Proposition 3.3.16.

$$L^s_{4s} \otimes L^n_{4t+3}(\mathbb{Z}, \mathbb{Z}/2^k) \xrightarrow{\cong} L^n_{4s} \otimes L^n_{4t+3}(\mathbb{Z}, \mathbb{Z}/2^k) \xrightarrow{\cong} L^n_{4(s+t)+3}(\mathbb{Z}, \mathbb{Z}/2^k)$$

(2) $m \equiv 1 \pmod 4$

The first summand $\mathbb{Z}/2$ in the unnatural decomposition $L^n_m(\mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ is the image of the homomorphism $L^n_m \to L^n_m(\mathbb{Z}/2)$, which is canonically given.

Let us construct the generator of the second summand as follows. Let $(\widetilde{D}, \widetilde{E})$ be a Poincaré symmetric-quadratic pair of dimension 3 such that the Kervaire invariant $\sigma^q_2(\widetilde{E}) = 1 \in \mathbb{Z}/2$. Let $\widetilde{G}$ be a $\mathbb{Z}/2$ Poincaré symmetric complex of dimension 2 such that $\sigma^s_1(\delta G) = 1$.

Lemma 3.3.17. *Each element (D, E) of the complement of the first summand of $L^n_5(\mathbb{Z}/2)$ has $\sigma^q_0(E) = 1 \in \mathbb{Z}/2$.*

Proof. The nontrivial class in the first summand has vanishing σ^q_0 because it can be represented by a Poincaré symmetric chain. Therefore, once we find a class (D, E) such that $\sigma^q_0(E) = 1$, the other element in the complement also has $\sigma^q_0(E) = 1$. Indeed, for $(\widetilde{D}, \widetilde{E}) \otimes \widetilde{G}$,

$$\sigma^q_0((\widetilde{D}, \widetilde{E}) \otimes \widetilde{G}) = \sigma^q_2(\widetilde{E}) \cdot \sigma^s_1(\delta\widetilde{G}) = 1 \in \mathbb{Z}/2$$

□

We use the class $(\overline{D}_0, \overline{E}_0) = (\widetilde{D}, \widetilde{E}) \otimes \widetilde{G} \otimes G$ to generate the second summand in $L_m(\mathbb{Z}/2)$, where G is a $(m-5)$-dimensional Poincaré symmetric chain with signature 1.

The previous argument also proves that

Proposition 3.3.18.

$$L^n_{4s+1}(\mathbb{Z}, \mathbb{Z}/2^k) \otimes L^s_{4t}(\mathbb{Z}, \mathbb{Z}/2^k) \xrightarrow{\cong} L^n_{4s+1}(\mathbb{Z}, \mathbb{Z}/2^k) \otimes L^n_{4t}(\mathbb{Z}, \mathbb{Z}/2^k) \xrightarrow{\cong} L^n_{4s+t+1}(\mathbb{Z}, \mathbb{Z}/2^k)$$

Proposition 3.3.19.

$$L^n_{4s+3} \otimes L^s_{4t+2}(\mathbb{Z}, \mathbb{Z}/2^k) \xrightarrow{\cong} L^n_{4s+3}(\mathbb{Z}, \mathbb{Z}/2^k) \otimes L^s_{4t+2}(\mathbb{Z}, \mathbb{Z}/2^k) \to L^n_{4(s+t+1)+1}(\mathbb{Z}, \mathbb{Z}/2^k) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

is an isomorphism onto the second summand.

Due to the dimension, δE must be the boundary of some Poincaré quadratic pair δF. Then $E \bigcup_{2^k \delta E} 2^k \delta F$ is a Poincaré quadratic chain.

If $\sigma_0^q(E \bigcup_{2^k \delta E} 2^k \delta F) = 0 \in \mathbb{Z}/2^k$, then there exists another $\delta F'$ such that $\mathrm{Sign}(E \bigcup_{2^k \delta E} 2^k \delta F') = 0 \in \mathbb{Z}$. $E \bigcup_{2^k \delta E} 2^k \delta F'$ must be the boundary of some Poincaré quadratic pair F'. Then $D \bigcup_E F'$ is a $\mathbb{Z}/2^k$ Poincaré symmetric complex. Now define $\sigma_1^n(D, E) = \sigma_1^s(D \bigcup_E F') \in \mathbb{Z}/2$

If $\sigma_0^q(E \bigcup_{2^k \delta E} 2^k \delta F) = 1$, then define $\sigma_1^n(D, E) = \sigma_1^n((D, E) - j_k(\overline{D}_0, \overline{E}_0))$, where $j_k : \mathbb{Z}/2 \to \mathbb{Z}/2^k$.

(3) $m \equiv 0 \pmod 4$

First let $k \geq 3$. Assume that the Kervaire invariant $\sigma_2^q(\delta E)$ vanishes. Then δE is the boundary of some Poincaré quadratic pair δF. Due to the dimension, the quadratic complex $2^k \delta F \bigcup_{2^k \delta E} E$ must be the boundary of some Poincaré quadratic pair F. Now $2^k(\delta F \bigcup_{\delta E} \delta D) \to D \bigcup_E F$ can be thought of as a $\mathbb{Z}/2^k$ Poincaré symmetric complex and (D, E) is bordant to $(D \bigcup_E F, 0)$ as $\mathbb{Z}/2^k$ Poincaré symmetric-quadratic pairs. Define $\sigma_0^n(D, E) = \mathrm{Sign}(D \bigcup_E F) \in \mathbb{Z}/8$. The value is independent of the choices of δF and F. Indeed, a replacement of δF changes the signature by a multiple of 2^k but a change of F only alters the signature by a multiple of 8.

The first summand in the unnatural decomposition $L_m^n(\mathbb{Z}, \mathbb{Z}/2^k) \cong \mathbb{Z}/8 \oplus \mathbb{Z}/2$ is also canonical, i.e., it is the image under the homomorphism $L_m^n \to L_m^n(\mathbb{Z}, \mathbb{Z}/2^k)$. We construct the generator of the second summand $\mathbb{Z}/2$ as follows.

Lemma 3.3.20. *The multiplication*

$$L_4^n(\mathbb{Z}, \mathbb{Z}/2) \otimes L_1^s \cong (\mathbb{Z}/2 \oplus \mathbb{Z}/2) \otimes \mathbb{Z}/2 \to L_5^n(\mathbb{Z}, \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

is an isomorphism.

Proof. The bijectivity on the first summands of both sides follows from $L_4^n \otimes L_1^s \xrightarrow{\cong} L_5^n$.

There exists a 4-dimensional Poincaré symmetric-quadratic pair (D, E) with nonvanishing $\sigma_2^q(E)$. Indeed, take a Poincaré quadratic chain δE with nonvanishing Kervaire invariant. $2\delta E$ must be the boundary of some Poincaré quadratic pair E. Because of the dimension, as a Poincaré symmetric chain, δE must be the boundary of some Poincaré symmetric pair δD. Then $2\delta D \bigcup_{2\delta E} E$ is also a boundary.

Let G be a 1-dimensional Poincaré symmetric chain with $\sigma_1^s(G) = 1$. Then

$$\sigma_0^q(\delta E \otimes G) = \sigma_2^q(\delta E) \cdot \sigma_1^s(G) = 1 \in \mathbb{Z}/2$$

Hence, (D, E) is in the complement of the first summand of $L_5^n(\mathbb{Z}, \mathbb{Z}/2)$ as well. $\square$

Applying the 4-periodicity, it also proves that

Proposition 3.3.21.

$$L_{4s}^n(\mathbb{Z}, \mathbb{Z}/2) \otimes L_{4t+1}^s \xrightarrow{\cong} L_{4s}^n(\mathbb{Z}, \mathbb{Z}/2) \otimes L_{4t+1}^s(\mathbb{Z}, \mathbb{Z}/2) \xrightarrow{\cong} L_{4(s+t)+1}^n(\mathbb{Z}, \mathbb{Z}/2)$$

Thus, there must be a $\mathbb{Z}/2$ Poincaré symmetric-quadratic pair (D_0', E_0') of dimension 4, unique up to bordism, such that $(D_0', E_0') \otimes G'$ is bordant to $(\overline{D}_0, \overline{E}_0)$, where G' is a 1-dimensional Poincaré symmetric complex of de Rham invariant 1.

We fix the $\mathbb{Z}/2$ Poincaré symmetric-quadratic pair $(D_0, E_0) = (D_0', E_0') \otimes G$ of dimension m, where G is a $(m-4)$-dimensional Poincaré symmetric complex with signature 1. Then $j_k(D_0, E_0)$ is the element to produce the decomposition $L_m^n(\mathbb{Z}, \mathbb{Z}/2^k) \cong \mathbb{Z}/8 \oplus \mathbb{Z}/2$, where $j_k : \mathbb{Z}/2 \to \mathbb{Z}/2^k$.

For a general (D, E), let $(D', E') = (D, E) - j_k(\sigma_2^q(\delta E) \cdot (D_0, E_0))$. Define $\sigma_0^n(D, E) = \sigma_0^n(D', E') \in \mathbb{Z}/8$.

It is immediate that

Lemma 3.3.22. *$\sigma_0^n : L_{4k}^n(\mathbb{Z}, \mathbb{Z}/2^k) \cong \mathbb{Z}/8 \oplus \mathbb{Z}/2 \to \mathbb{Z}/8$ is a projection.*

Proposition 3.3.23.

$$L_{4s}^n(\mathbb{Z}, \mathbb{Z}/2^k) \otimes L_{4t}^s \xrightarrow{\cong} L_{4s}^n(\mathbb{Z}, \mathbb{Z}/2^k) \otimes L_{4t}^n \xrightarrow{\cong} L_{4(s+t)}^n(\mathbb{Z}, \mathbb{Z}/2^k)$$

We also have the following.

Proposition 3.3.24.

$$L^n_{4s+3} \otimes L^s_{4t+1} \xrightarrow{\cong} L^n_{4s+3}(\mathbb{Z}, \mathbb{Z}/2^k) \otimes L^s_{4t+1} \to L^n_{4(s+t+1)}(\mathbb{Z}, \mathbb{Z}/2^k) \cong \mathbb{Z}/8 \oplus \mathbb{Z}/2$$

is an injection into the first summand.

Proof. It is irrelevant of the second summand due to the following diagram.

$$\begin{array}{ccc} L^n_{4s+3} \otimes L^s_{4t+1} & \longrightarrow & L^n_{4(s+t+1)} \\ \downarrow & & \downarrow \\ L^n_{4s+3}(\mathbb{Z}, \mathbb{Z}/2^k) \otimes L^n_{4t+1} & \longrightarrow & L^n_{4(s+t+1)}(\mathbb{Z}, \mathbb{Z}/2^k) \end{array}$$

On the other hand, because of the 4-periodicity, it reduces to consider the following diagram.

$$\begin{array}{ccc} L^n_4(\mathbb{Z}, \mathbb{Z}/2) \otimes L^s_1 & \longrightarrow & L^n_3 \otimes L^s_1 \\ \downarrow \cong & & \downarrow \\ L^n_5(\mathbb{Z}, \mathbb{Z}/2) & \longrightarrow & L^n_4 \end{array}$$

The generator of $L^n_3 \otimes L^s_1$ is mapped to $(\widetilde{D}, \widetilde{E}) \otimes \widetilde{G} \in L^n_5(\mathbb{Z}, \mathbb{Z}/2)$, with the notation introduced in the proof of 3.3.17. □

We have also proved the following.

Lemma 3.3.25. *Let (D, E) be a $\mathbb{Z}/2^k$ Poincaré symmetric-quadratic pair with $k \geq 3$ and let G be a Poincaré symmetric complex. Then*

$$\sigma^n_0((D, E) \otimes G) = \sigma^n_0(D, E) \cdot \sigma^s_0(G) + j_8(\sigma^n_3(D, E) \cdot \sigma^s_1(G)) \in \mathbb{Z}/8$$

where $j_8 : \mathbb{Z}/2 \to \mathbb{Z}/8$.

Now consider the case for $k = 1, 2$.

If $k = 1$, (D, E) is a $\mathbb{Z}/2$ Poincaré symmetric-quadratic pair and $4(D, E)$ is a $\mathbb{Z}/8$ pair. Define

$$\sigma^n_{0,2}(D, E) = \sigma^n_0(4(D, E)) \in 4 \cdot \mathbb{Z}/8 \cong \mathbb{Z}/2$$

Alternatively, there is a similar way to define $\sigma^n_{0,2}$ like the case of $k \geq 3$. We do not repeat the tedious process to describe it and check the agreement of two definitions.

Proposition 3.3.26.

$$L^n_{4s+2}(\mathbb{Z}, \mathbb{Z}/2^k) \otimes L^n_{4t+2}(\mathbb{Z}, \mathbb{Z}/2^k) \xrightarrow{0} L^n_{4(s+t+1)}(\mathbb{Z}, \mathbb{Z}/2^k)$$

Proof. We only consider the case when $k = 1$ and the case for general k is analogous.

$L^n_{4(s+t+1)}(\mathbb{Z}/2)$ has a decomposition $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ and we want to show that the projection of the image to each direct summand of $L^n_{4(s+t+1)}$ is 0.

For the first summand, consider the diagram

$$\begin{array}{ccc} L^s_{4s+2}(\mathbb{Z}, \mathbb{Z}/2) \otimes L^s_{4t+2}(\mathbb{Z}, \mathbb{Z}/2) & \xrightarrow{0} & L^s_{4(s+t+1)}(\mathbb{Z}, \mathbb{Z}/2) \\ \downarrow \cong & & \downarrow \cong \\ L^n_{4s+2}(\mathbb{Z}, \mathbb{Z}/2) \otimes L^n_{4t+2}(\mathbb{Z}, \mathbb{Z}/2) & \longrightarrow & L^n_{4(s+t+1)}(\mathbb{Z}, \mathbb{Z}/2) \end{array}$$

For the second summand, remember that the direct sum decomposition follows from the isomorphism

$$L^n_4(\mathbb{Z}, \mathbb{Z}/2) \otimes L^s_1 \xrightarrow{\cong} L^n_5(\mathbb{Z}, \mathbb{Z}/2)$$

However, $L^n_2(\mathbb{Z}, \mathbb{Z}/2) \otimes L^s_1 \to L^n_3(\mathbb{Z}, \mathbb{Z}/2)$ is a zero map because of the diagram

$$\begin{array}{ccc} L^s_2(\mathbb{Z}, \mathbb{Z}/2) \otimes L^s_1 & \longrightarrow & L^s_3(\mathbb{Z}, \mathbb{Z}/2) = 0 \\ \downarrow \cong & & \downarrow \\ L^n_2(\mathbb{Z}, \mathbb{Z}/2) \otimes L^s_1 & \longrightarrow & L^n_3(\mathbb{Z}, \mathbb{Z}/2) \end{array}$$

□

Then we also get that

Lemma 3.3.27. *Let (D, E) be a $\mathbb{Z}/2$ Poincaré symmetric-quadratic pair and let G be a $\mathbb{Z}/2$ Poincaré symmetric complex. Then*

$$\sigma^n_{0,2}((D, E) \otimes G) = \sigma^n_{0,2}(D, E) \cdot \sigma^s_0(G) + \sigma^n_3(D, E) \cdot \sigma^s_1(G) \in \mathbb{Z}/2$$

Lemma 3.3.28. *Let (D, E) be a $\mathbb{Z}/2$ Poincaré symmetric-quadratic pair and G be a $\mathbb{Z}/2$ Poincaré symmetric complex . Then*

$$\sigma^n_{0,2}(\delta((D, E) \otimes G)) = \sigma^n_{0,2}(\delta(D, E)) \cdot \sigma^s_0(G) + \sigma^n_{0,2}(D, E) \cdot \sigma^s_0(\delta G) \in \mathbb{Z}/2$$

Proof. $\delta((D, E) \otimes G)$ is bordant to $(\delta D, \delta E) \otimes G \oplus (D, E) \otimes \delta G$ as $\mathbb{Z}/2$ Poincaré symmetric-quadratic pairs. □

Lemma 3.3.29. *Let (D, E) be a $\mathbb{Z}/2$ Poincaré symmetric-quadratic pair and let G be a $\mathbb{Z}/2$ Poincaré symmetric complex. Then*

$$\sigma^n_1((D, E) \otimes G) = \sigma^n_1(D, E) \cdot \sigma^s_0(G) + \sigma^n_{0,2}(D, E) \cdot \sigma^s_1(G) \in \mathbb{Z}/2$$

Similarly, if $k = 2$, (D, E) is a $\mathbb{Z}/4$ Poincaré symmetric-quadratic pair and define

$$\sigma^n_{0,4}(D, E) = \sigma^n_0(2(D, E)) \in 2 \cdot \mathbb{Z}/8 \cong \mathbb{Z}/4$$

There are also some multiplicative formulae of bordism invariants. They are pretty similar to the $k = 1$ case, so let us skip stating them.

(4) Product Structure on $L^n_*(\mathbb{Z}, \mathbb{Z}/2^k)$

We already proved the product formula $L^n_{4s+2}(\mathbb{Z}, \mathbb{Z}/2^k) \otimes L^n_{4t+2}(\mathbb{Z}, \mathbb{Z}/2^k) \xrightarrow{0} L^n_{4(s+t+1)}(\mathbb{Z}, \mathbb{Z}/2^k)$.

Proposition 3.3.30.

$$L^n_{4s+1}(\mathbb{Z}, \mathbb{Z}/2^k) \otimes L^n_{4t+2}(\mathbb{Z}, \mathbb{Z}/2^k) \xrightarrow{0} L^n_{4(s+t)+3}(\mathbb{Z}, \mathbb{Z}/2^k)$$

Proof. It follows from the diagram

$$\begin{array}{ccc} L^s_{4s+1}(\mathbb{Z}, \mathbb{Z}/2^k) \otimes L^s_{4t+2}(\mathbb{Z}, \mathbb{Z}/2^k) & \xrightarrow{0} & L^s_{4(s+t)+3}(\mathbb{Z}, \mathbb{Z}/2^k) \\ \downarrow & & \downarrow \cong \\ L^n_{4s+1}(\mathbb{Z}, \mathbb{Z}/2^k) \otimes L^n_{4t+2}(\mathbb{Z}, \mathbb{Z}/2^k) & \longrightarrow & L^n_{4(s+t)+3}(\mathbb{Z}, \mathbb{Z}/2^k) \end{array}$$

and the diagram

$$\begin{array}{ccc} L^n_{4s-2}(\mathbb{Z},\mathbb{Z}/2^k)\otimes L^n_3\otimes L^s_{4t+2}(\mathbb{Z},\mathbb{Z}/2^k) & \xrightarrow{0} & L^n_{4(s+t)}(\mathbb{Z},\mathbb{Z}/2^k)\otimes L^n_3 \\ \downarrow & & \downarrow \\ L^n_{4s+1}(\mathbb{Z},\mathbb{Z}/2^k)\otimes L^n_{4t+2}(\mathbb{Z},\mathbb{Z}/2^k) & \longrightarrow & L^n_{4(s+t)+3}(\mathbb{Z},\mathbb{Z}/2^k) \end{array}$$

□

Proposition 3.3.31.

$$L^n_{4s}(\mathbb{Z},\mathbb{Z}/2^k)\otimes L^n_{4t+3}(\mathbb{Z},\mathbb{Z}/2^k)\to L^n_{4(s+t)+3}(\mathbb{Z},\mathbb{Z}/2^k)$$

is isomorphic to the map

$$(\mathbb{Z}/2^l\oplus\mathbb{Z}/2)\otimes\mathbb{Z}/2\xrightarrow{\pi_1}\mathbb{Z}/2^l\otimes\mathbb{Z}/2\to\mathbb{Z}/2$$

where the first arrow is the projection onto the first factor. $l=3$ *if* $k\geq 3$ *and* $l=k$ *if* $k=1,2$.

Proof. The map on the first direct summand $\mathbb{Z}/2^l$ is the 4-periodicity. For the second direct summand $\mathbb{Z}/2$, it can be deduced by the diagram

$$\begin{array}{ccc} L^n_{4s}(\mathbb{Z},\mathbb{Z}/2^k)\otimes L^n_{4t+3}\otimes L^s_1 & \longrightarrow & L^n_{4s}(\mathbb{Z},\mathbb{Z}/2^k)\otimes L^n_{4(t+1)} \\ \downarrow & & \downarrow \\ L^n_{4(s+t)+3}(\mathbb{Z},\mathbb{Z}/2^k)\otimes L^s_1 & \longrightarrow & L^n_{4(s+t+1)}(\mathbb{Z},\mathbb{Z}/2^k) \end{array}$$

and the fact that $(4\mathbb{Z}/8)\otimes\mathbb{Z}/2$ is 0 in $\mathbb{Z}/8\otimes\mathbb{Z}/2$.

□

So we have also proved the following.

Lemma 3.3.32. *Let* (D,E) *and* (D',E') *be two* $\mathbb{Z}/2$ *Poincaré symmetric-quadratic pairs. Then*

$$\sigma^n_3((D,E)\otimes(D',E'))=\sigma^n_{0,2}(D,E)\cdot\sigma^n_3(D',E')+\sigma^n_3(D,E)\cdot\sigma^n_{0,2}(D',E')\in\mathbb{Z}/2$$

Proposition 3.3.33.

$$L^n_{4s+1}(\mathbb{Z},\mathbb{Z}/2^k)\otimes L^n_{4t+3}(\mathbb{Z},\mathbb{Z}/2^k)\to L^n_{4(s+t+1)}(\mathbb{Z},\mathbb{Z}/2^k)$$

is isomorphic to the map

$$(\mathbb{Z}/2\oplus\mathbb{Z}/2^l)\otimes\mathbb{Z}/2\xrightarrow{\pi_1}\mathbb{Z}/2\otimes\mathbb{Z}/2\xrightarrow{\cong}\mathbb{Z}/2\xrightarrow{j_l}\mathbb{Z}/2^l\xrightarrow{i}\mathbb{Z}/2^l\oplus\mathbb{Z}/2$$

where the first arrow is the projection onto the first direct summand and i is the natural inclusion. $l=3$ if $k\geq 3$ and $l=k$ if $k=1,2$.

Proof. The map on the first direct summand $\mathbb{Z}/2\otimes\mathbb{Z}/2$ is calculated before. The map on the second direct summand can be deduced from the fact that $(4\mathbb{Z}/8)\otimes\mathbb{Z}/2$ is 0 in $\mathbb{Z}/8\otimes\mathbb{Z}/2$ and the diagram chasing on

$$\begin{array}{ccc} L^n_{4s+1}(\mathbb{Z},\mathbb{Z}/2^k)\otimes L^n_{4t+3}\otimes L^s_1 & \longrightarrow & L^n_{4s+1}(\mathbb{Z},\mathbb{Z}/2^k)\otimes L^n_{4(t+1)} \\ \downarrow & & \downarrow \\ L^n_{4(s+t+1)}(\mathbb{Z},\mathbb{Z}/2^k)\otimes L^s_1 & \longrightarrow & L^n_{4(s+t+1)+1}(\mathbb{Z},\mathbb{Z}/2^k) \end{array}$$

□

Proposition 3.3.34.

$$L^n_{4s}(\mathbb{Z},\mathbb{Z}/2^k)\otimes L^n_{4t}(\mathbb{Z},\mathbb{Z}/2^k)\to L^n_{4(s+t)}(\mathbb{Z},\mathbb{Z}/2^k)$$

is isomorphic to the map

$$(\mathbb{Z}/2^l\oplus\mathbb{Z}/2)\otimes(\mathbb{Z}/2^l\oplus\mathbb{Z}/2)\to(\mathbb{Z}/2^l\otimes\mathbb{Z}/2^l)\oplus(\mathbb{Z}/2^l\otimes\mathbb{Z}/2)\oplus(\mathbb{Z}/2\otimes\mathbb{Z}/2^l)$$
$$\xrightarrow{(1+0+0,0+i+i)}\mathbb{Z}/2^l\oplus\mathbb{Z}/2$$

where the first arrow is the projection and the second arrow means the identity on $\mathbb{Z}/2^l$ and the sum of two $\mathbb{Z}/2$'s onto $\mathbb{Z}/2$. $l=3$ if $k\geq 3$ and $l=k$ if $k=1,2$.

Proof. All the nontrivial maps are 4-periodicity. The map follows from the diagram

$$\begin{array}{ccc} L^n_{4s}(\mathbb{Z},\mathbb{Z}/2^k)\otimes L^n_{4t}(\mathbb{Z},\mathbb{Z}/2^k)\otimes L^s_1 & \longrightarrow & L^n_{4s}(\mathbb{Z},\mathbb{Z}/2^k)\otimes L^n_{4t+1}(\mathbb{Z},\mathbb{Z}/2^k) \\ \downarrow & & \downarrow \\ L^n_{4(s+t)}(\mathbb{Z},\mathbb{Z}/2^k)\otimes L^s_1 & \longrightarrow & L^n_{4(s+t)+1}(\mathbb{Z},\mathbb{Z}/2^k) \end{array}$$

□

We have also proved the following.

Lemma 3.3.35. *Let (D,E) and (D',E') be two $\mathbb{Z}/2^k$ Poincaré symmetric-quadratic pairs with $k \geq 3$. Then*

$$\begin{aligned}\sigma_0^n((D,E)\otimes(D',E')) &= \sigma_0^n(D,E)\cdot\sigma_0^n(D',E') \\ &\quad + j_8(\sigma_1^n(D,E)\cdot\sigma_3^n(D',E') + \sigma_3^n(D,E)\cdot\sigma_1^n(D',E')) \in \mathbb{Z}/8\end{aligned}$$

where $j_8 : \mathbb{Z}/2 \to \mathbb{Z}/8$.

For the case when $k = 1, 2$, the result is similar.

Proposition 3.3.36. *Let (D,E) and (D',E') be two $\mathbb{Z}/2^k$ Poincaré symmetric-quadratic pairs with $k \geq 3$. Then*

$$\begin{aligned}\sigma_0^n(\delta((D,E)\otimes(D',E'))) &= \sigma_0^n(\delta(D,E))\cdot\sigma_0^n(D',E') + \sigma_0^n(D,E)\cdot\sigma_0^n(\delta(D',E')) \\ &\quad + j_8(\sigma_3^n(\delta(D,E))\cdot\sigma_1^n(D',E') + \sigma_1^n(D,E)\cdot\sigma_3^n(\delta(D',E'))) \\ &\quad + j_8(\sigma_3^n(D,E)\cdot\sigma_1^n(\delta(D',E')) + \sigma_1^n(\delta(D,E))\cdot\sigma_3^n(D',E'))\end{aligned}$$

Proposition 3.3.37.

$$L_{4s}^n(\mathbb{Z},\mathbb{Z}/2^k)\otimes L_{4t+1}^n(\mathbb{Z},\mathbb{Z}/2^k) \to L_{4(s+t)+1}^n(\mathbb{Z},\mathbb{Z}/2^k)$$

is the map isomorphic to the map

$$\begin{aligned}(\mathbb{Z}/2^l \oplus \mathbb{Z}/2)\otimes(\mathbb{Z}/2\oplus\mathbb{Z}/2^l) &\to (\mathbb{Z}/2^l\otimes\mathbb{Z}/2)\oplus(\mathbb{Z}/2^l\otimes\mathbb{Z}/2^l)\oplus(\mathbb{Z}/2\otimes\mathbb{Z}/2) \\ &\xrightarrow{(1+0+0,0+1+j_l)} \mathbb{Z}/2\oplus\mathbb{Z}/2^l\end{aligned}$$

where the first arrow is the projection. $l = 3$ if $k \geq 3$ and $l = k$ if $k = 1, 2$.

Proof. The map onto the second direct summand $\mathbb{Z}/2^l$ is either already proven. The map onto the first direct summand $\mathbb{Z}/2$ follows from the diagram

$$\begin{array}{ccc} L_{4s}^n(\mathbb{Z},\mathbb{Z}/2^k)\otimes L_{4t-2}^s(\mathbb{Z},\mathbb{Z}/2^k)\otimes L_3^n & \longrightarrow & L_{4s+3}^n(\mathbb{Z},\mathbb{Z}/2^k)\otimes L_{4t-2}^n(\mathbb{Z},\mathbb{Z}/2^k) \\ \downarrow & & \downarrow \\ L_{4(s+t)}^n(\mathbb{Z},\mathbb{Z}/2^k)\otimes L_{4t+1}^n(\mathbb{Z},\mathbb{Z}/2^k) & \longrightarrow & L_{4(s+t)+1}^n(\mathbb{Z},\mathbb{Z}/2^k) \end{array}$$

□

The product $L^n_{4s+2}(\mathbb{Z}, \mathbb{Z}/2^k) \otimes L^n_{4t+3}(\mathbb{Z}, \mathbb{Z}/2^k) \to L^n_{4(s+t+1)+1}(\mathbb{Z}, \mathbb{Z}/2^k)$ is essentially proven above. Hence, we also get that

Lemma 3.3.38. *Let (D, E) and (D', E') be $\mathbb{Z}/2$ Poincaré symmetric-quadratic pairs. Then*

$$\sigma^n_1((D, E) \otimes (D', E')) = \sigma^n_{0,2}(D, E) \cdot \sigma^n_1(D', E') + \sigma^n_1(D, E) \cdot \sigma^n_{0,2}(D', E') \in \mathbb{Z}/2$$

Proposition 3.3.39. *The only nontrivial part in the following maps*

$$L^n_{4s}(\mathbb{Z}, \mathbb{Z}/2^k) \otimes L^n_{4t+2}(\mathbb{Z}, \mathbb{Z}/2^k) \to L^n_{4(s+t)+2}(\mathbb{Z}, \mathbb{Z}/2^k)$$

$$L^n_{4s+1}(\mathbb{Z}, \mathbb{Z}/2^k) \otimes L^n_{4t+1}(\mathbb{Z}, \mathbb{Z}/2^k) \to L^n_{4(s+t)+2}(\mathbb{Z}, \mathbb{Z}/2^k)$$

$$L^n_{4s+3}(\mathbb{Z}, \mathbb{Z}/2^k) \otimes L^n_{4t+3}(\mathbb{Z}, \mathbb{Z}/2^k) \to L^n_{4(s+t+1)+2}(\mathbb{Z}, \mathbb{Z}/2^k)$$

is the 4-periodicity.

Proof. The second and the third map can be essentially deduced from the fact $L^n_{4s+2} = 0$ and the product structure on L^n_*.

The rest follows from the diagram

$$\begin{array}{ccc} L^n_{4s}(\mathbb{Z}, \mathbb{Z}/2^k) \otimes L^s_{4t+2}(\mathbb{Z}, \mathbb{Z}/2^k) \otimes L^n_3 & \longrightarrow & L^n_{4s+3}(\mathbb{Z}, \mathbb{Z}/2^k) \otimes L^n_{4t+2}(\mathbb{Z}, \mathbb{Z}/2^k) \\ \downarrow & & \downarrow \\ L^n_{4(s+t)+2}(\mathbb{Z}, \mathbb{Z}/2^k) \otimes L^n_3 & \longrightarrow & L^n_{4(s+t+1)+2}(\mathbb{Z}, \mathbb{Z}/2^k) \end{array}$$

□

Chapter 4

L-theory Characteristic Classes and Bundle Lifting Problem

In this chapter, we first construct some cohomology classes for three L-spectra and then prove that these classes induce splittings of the spectra localized at 2. The method we use is the a priori invariant method for cohomolgy classes introduced in the Chapter 2.

When we have the characteristic classes, we determine the relationship among characteristic classes of different L-theories under the fibration $\mathbb{L}^q \to \mathbb{L}^s \to \mathbb{L}^n$.

Moreover, there are ring structures and module structures over $\mathbb{L}$-spectra and we will also calculate the induced coproducts of these classes. Note that, for the surgery theory, the ring structure on $\mathbb{L}^q$ is induced from the Whitney sum structure on G/TOP. This ring structure can be induced from $\mathbb{L}^s$-module structure on $\mathbb{L}^q$. We show that the coproduct of the characteristic classes of $\mathbb{L}^q$ induced by surgery theory and the coproduct induced from the module structure are essentially the same.

Levitt-Ranicki's L-theory orientations of bundles ([Ran92, Proposition 16.1]) imply that the cohomology classes of $\mathbb{L}$-spectra we construct in this chapter induce some characteristic classes for spherical fibrations and for TOP-bundles. We prove that these classes are exactly the same classes constructed in [MS74] and [BM76].

Throughout this chapter, the symmetric spectrum $\mathbb{L}^s$ is 0-connective, the quadratic spectrum $\mathbb{L}^q$ is 1-connective and the normal spectrum $\mathbb{L}^n$ is 1/2-connective, as we clarified in Chapter 2. By abuse of notations, we still use $\mathbb{L}^a$ to represent the 0-th space for each spectrum ($a = s, q, n$) and a reader can easily distinguish the meanings by the context.

We have to point out that the results in this chapter about the homotopy types of three L-spectra at prime 2 and at odd primes are all proven in [TW79] already. However, they don't whether their results are equivalent to the bundle lifting theory. We made the homotopy equivalences more explicit and proved that our constructions are essentially the same as the constructions by Brumfiel-Morgan, Morgan-Sullivan, Rourke-Sullivan at prime 2 and by Sullivan at odd primes.

4.1 L-theory Characteristic Classes and Splittings of L-spectra at Prime 2

4.1.1 Quadratic L-spectrum

This subsection is a reproof of the splitting of $\mathbb{L}^q(\simeq G/TOP)$ at prime 2, with the technique of $\mathbb{Z}/n$ quadratic chains discussed in the previous chapter.

Lemma 4.1.1. *The Hurewicz map $h : \pi_{4k+2}(\mathbb{L}^q) \to H_{4k+2}(\mathbb{L}^q;\mathbb{Z})$ is an injection onto a direct summand.*

Proof. It suffices to prove that the mod 2 Hurewicz map h_2

$$h_2 : L^q_{4k+2} \cong \pi_{4k+2}(\mathbb{L}^q) \xrightarrow{i} \Omega^{SO}_{4k+2}(\mathbb{L}^q;\mathbb{Z}/2) \xrightarrow{j} H_{4k+2}(\mathbb{L}^q;\mathbb{Z}/2)$$

is injective.

Let us construct a splitting inverse of i and then i is obviously an injection. For any $(M^{4k+2}, f) \in \Omega^{SO}_{4k+2}(\mathbb{L}^q;\mathbb{Z}/2)$, it corresponds to a presheaf $\mathcal{C}_f$ of Poincaré quadratic chains over M. The assembly $\mathcal{C}_f(M)$ gives a splitting

$$\Omega^{SO}_{4k+2}(\mathbb{L}^q;\mathbb{Z}/2)\to L^q_{4k+2}(\mathbb{Z},\mathbb{Z}/2)\overset{\cong}{\leftarrow} L^q_{4k+2}$$

To prove h_2 is injective, let $g : S^{4k+2}\to\mathbb{L}^q$ be the generator of the homotopy group. Then the injectivity of i implies that (S^{4k+2},g) does not bound any $\mathbb{Z}/2$ singular manifold in $\mathbb{L}^q$. Hence, there exists some nonvanishing generalized Stiefel-Whitney number of (S^{4k+2},g). Since all the Stiefel-Whitney classes of a sphere vanish, it means that some cohomology class $H^{4k+2}(\mathbb{L}^q;\mathbb{Z}_2)$ evaluates nontrivially on $h_2(S^{4k+2},g)$. □

Proposition 4.1.2. *There exists a graded class*

$$k^q=k^q_2+k^q_6+\cdots\in H^{4*+2}(\mathbb{L}^q;\mathbb{Z}/2)$$

which agrees with the Kervaire class of G/TOP constructed in [RS71, Theorem 4.6].

Proof. Let M be a $\mathbb{Z}/2$ manifold with a map $f:M\to\mathbb{L}^q$. Let $\mathcal{C}_f$ be the associated presheaf of quadratic chains over M. Define a map $\sigma^q_2:\Omega^{SO}(\mathbb{L}^q;\mathbb{Z}/2)\to\mathbb{Z}/2$ by $\sigma^q_2(M,f)=\sigma^q_2(\mathcal{C}_f(M))$.

Let N be another $\mathbb{Z}/2$ manifold. The product property follows from the chain-level product formula

$$\sigma^q_2((M,f)\cdot N)=\sigma^q_2(\mathcal{C}_f(M)\otimes C_*(N))=\sigma^q_2(\mathcal{C}_f(M))\cdot\sigma^s_0(C_*(N))=\sigma^q_2(M,f)\cdot\chi_2(N)$$

□

Since $\pi_{4k}(\mathbb{L}^q)\simeq L^q_{4k}\simeq\mathbb{Z}$, to prove the splitting injectivity of the Hurewicz map localized at prime 2, it suffices to construct some characteristic class $l^q_{4k}\in H^{4k}(\mathbb{L}^q_{4k},\mathbb{Z}_{(2)})$ such that its evaluation on the generator of L^q_{4k} is an odd number.

Let M be a $\mathbb{Z}$ or $\mathbb{Z}/2^k$ manifold with a map $f:M\to\mathbb{L}^q$ and let $\mathcal{C}_f$ be the associated presheaf of quadratic chain complexes over M. Define

$$\sigma^q_0(M,f)=\sigma^q_0(\mathcal{C}_f(M))-j_k\langle\beta(V_M Sq^1V_M\cdot f^*k^q),[M]\rangle\in\mathbb{Z}\text{ or }\mathbb{Z}/2^k$$

Recall that the de Rham invariant of a $\mathbb{Z}/2^k$ manifold M satisfies the equation ([MS74, Lemma 8.2])

$$\mathrm{dR}(M) = \langle VSq^1V, [M]\rangle \in \mathbb{Z}/2$$

where $VSq^1V = (1 + v_2 + v_4 + \cdots) \cdot Sq^1(1 + v_2 + v_4 + \cdots)$ and $\{v_i\}$ is the Wu class.

Then

$$\mathrm{dR}(\delta M) = \langle \beta(VSq^1V), [M]\rangle \in \mathbb{Z}/2 \simeq 2^{k-1} \cdot \mathbb{Z}/2^k$$

where β is the $\mathbb{Z}/2 \to \mathbb{Z}/2^k$ Bockstein homomorphism.

The chain-level product formula implies the following.

Lemma 4.1.3. *Let M and N be $\mathbb{Z}/2^k$ manifolds with a map $f : M \to \mathbb{L}^q$. Then*

$$\sigma_0^q((M, f) \cdot N) = \sigma_0^q(M, f) \cdot \mathrm{Sign}(N) \in \mathbb{Z} \text{ or } \mathbb{Z}/2^k$$

Therefore,

Proposition 4.1.4. *There exists a graded class*

$$l^q = l_4^q + l_8^q + \cdots \in H^{4*}(\mathbb{L}^q; \mathbb{Z}_{(2)})$$

such that for any map $f : M \to \mathbb{L}^q$,

$$\sigma_0^q(M, f) = \langle L_M \cdot f^*l^q, [M]\rangle \in \mathbb{Z} \text{ or } \mathbb{Z}/2^k$$

where M is a $\mathbb{Z}$ or $\mathbb{Z}/2^k$ manifold.

Recall that the set $[X, \mathbb{L}^q]$ classifies the bordism classes of presheaves of Poincaré quadratic chains over X. The previous results can be restated as follows.

Proposition 4.1.5. *For any presheaf $\mathcal{Q}$ of 0-connective Poincaré quadratic chains over X, there exist graded characteristic classes*

$$k^q(\mathcal{Q}) = k_2^q + k_6^q + \cdots \in H^{4*+2}(X; \mathbb{Z}/2)$$

$$l^q(\mathcal{Q}) = l_4^q + l_8^q + \cdots \in H^{4*}(X; \mathbb{Z}_{(2)})$$

which are invariant under presheaf bordism.

Now let $f : S^{4k} \to \mathbb{L}^q$ be a generator of $\pi_{4k}(\mathbb{L}^q) \simeq \mathbb{Z}$. Then the associated presheaf of quadratic chains $\mathcal{C}_f$ has $\sigma_0^q(\mathcal{C}_f(S^{4k})) = 1$. Hence,

$$\langle l_{4k}^q, f_*[S^{4k}]\rangle = \langle L_{S^{4k}} \cdot f^* l_{4k}^q, [S^{4k}]\rangle = \sigma_0^q(\mathcal{C}_f(S^{4k})) = 1$$

Then we have reproved that

Theorem 4.1.6. *Localized at prime 2,*

$$\mathbb{L}^q \simeq \prod_{k>0}(K(\mathbb{Z}_{(2)}, 4k) \times K(\mathbb{Z}/2, 4k-2))$$

where the homotopy equivalence is given by the classes l^q and k^q.

4.1.2 Symmetric L-spectrum

Applying the argument of 4.1.1, we can prove an analogous statement about $\mathbb{L}^s$.

Lemma 4.1.7. *The Hurewicz map $h : \pi_{4k+1}(\mathbb{L}^s) \to H_{4k+1}(\mathbb{L}^s, \mathbb{Z})$ is an injection onto a direct summand.*

We will define the $4*$-degree integral class before the $(4*+1)$-degree $\mathbb{Z}/2$ class.

It is different from $\mathbb{L}^q$ that the space $\mathbb{L}^s$ is not connected. Denote the t-th component by $\mathbb{L}_t^s$. Notice that the 0-cells of $\mathbb{L}_t^s$ are just 0-dimensional Poincaré symmetric chains with the 0-th homology of rank t.

Let M be a $\mathbb{Z}$ or $\mathbb{Z}/2^l$ manifold with a map $f : M \to \mathbb{L}^s$ and let $\mathcal{C}_f$ be the associated presheaf of symmetric chains. Define

$$\sigma_0^s(M, f) = \sigma_0^s(\mathcal{C}_f(M)) \in \mathbb{Z} \text{ or } \mathbb{Z}/2^l$$

It is immediate that the product formula holds, i.e.,

$$\sigma_0^s((M, f) \cdot N) = \sigma_0^s(M, f) \cdot \mathrm{Sign}(N)$$

where N is another $\mathbb{Z}$ or $\mathbb{Z}/2^l$ manifold. Hence, we have

Proposition 4.1.8. *There exists a graded class*

$$l_t^s = l_{t,0}^s + l_{t,4}^s + l_{t,8}^s + \cdots \in H^{4*}(\mathbb{L}_t^s; \mathbb{Z}_{(2)})$$

such that for any map $f : M \to \mathbb{L}_t^s$,

$$\sigma_0^s(M, f) = \langle L_M \cdot f^* l_t^s, [M] \rangle \in \mathbb{Z} \text{ or } \mathbb{Z}/2^l$$

where M *is a closed manifold or a* $\mathbb{Z}/2^l$ *manifold.*

$\mathbb{L}^s$ also has an additional structure due to the infinite loop space structure, i.e.,

$$a_{t,t'} : \mathbb{L}_t^s \times \mathbb{L}_{t'}^s \to \mathbb{L}_{t+t'}^s$$

The homotopy equivalence between different components is indeed the composition of maps

$$a_t : \mathbb{L}_0^s = \mathbb{L}_0^s \times \mathrm{pt} \to \mathbb{L}_0^s \times \mathbb{L}_t^s \xrightarrow{a_{0,t}} \mathbb{L}_t^s$$

Let us write down the map explicitly, let $f : \sigma^m \to \mathbb{L}_t^s$ and $g : \sigma^n \to \mathbb{L}_{t'}^s$ be cellular maps from simplices of dimension m and n respectively. They correspond to presheaves $\mathcal{C}_f$ and $\mathcal{C}_g$ respectively. Now we consider the map $a \circ (f \times g) : \sigma^m \times \sigma^n \to \mathbb{L}_{t+t'}$ and its associated presheaf $\mathcal{C}_{a\circ(f\times g)}$. $\sigma^m \times \sigma^n$ has a natural regular cell decomposition of the form $\tau^{m'} \times \tau^{n'}$, where $\tau^{m'}$ and $\tau^{n'}$ are faces of σ^m and σ^n respectively. Then $\mathcal{C}_{a\circ(f\times g)}(\tau^{m'} \times \tau^{n'}) \cong (\mathcal{C}_f(\tau^{m'}) \otimes C_*(\tau^{n'})) \oplus (C_*(\tau^{m'}) \otimes \mathcal{C}_f(\tau^{n'}))$, where $C_*(\tau^{m'})$ and $C_*(\tau^{n'})$ are cellular chain complexes. Hence, we have proved that

Lemma 4.1.9. *Let* M *and* N *be two* $\mathbb{Z}/2^l$ *manifolds with maps* $f : M \to \mathbb{L}^s$ *and* $g : N \to \mathbb{L}^s$. *Let* $\mathcal{C}_f$ *and* $\mathcal{C}_g$ *be the associated presheves of Poincaré symmetric chains over* M *and* N *respectively. Let* $\mathcal{C}_{a\circ(f\otimes g)}$ *be the presheaf associated to* $a \circ (f \otimes g)$. *Then* $\mathcal{C}_{a\circ(f\otimes g)}(M \otimes N)$ *is bordant to* $(\mathcal{C}_f(M) \otimes C_*(N)) \oplus (C_*(M) \otimes \mathcal{C}_g(N))$

The l^s-classes in different components are connected by the additional structure.

Proposition 4.1.10.

$$a_{t,t'}^* l_{t+t'}^s = l_t^s \times 1 + 1 \times l_{t'}^s$$

Proof. Let M and N be $\mathbb{Z}/2^l$ manifolds with maps $f : M \to \mathbb{L}_t^s$ and $g : N \to \mathbb{L}_{t'}^s$. By the lemma above we know that $\mathcal{C}_{a\circ(f\times g)}(M \times N)$ is bordant to $(\mathcal{C}_f(M) \otimes C_*(N)) \oplus (C_*(M) \otimes \mathcal{C}_g(N))$.

On one hand,

$$\sigma_0^s(M \otimes N, a \circ (f \otimes g)) = \langle L_{M\otimes N} \cdot (f \otimes g)^* a^*(l_{t+t'}^s), [M \otimes N]\rangle$$

On the other hand,

$$\begin{aligned}\sigma_0^s((\mathcal{C}_f(M) \otimes C_*(N)) \oplus (C_*(M) \otimes \mathcal{C}_g(N))) \quad &= \quad \langle L_M \cdot f^* l_t^s, [M]\rangle \cdot \langle L_N, [N]\rangle \\ &\quad + \langle L_M, [M]\rangle \cdot \langle L_N \cdot g^* l_{t'}^s, [N]\rangle\end{aligned}$$

□

By a similar argument we also have

Proposition 4.1.11.

$$a_t^* l_t^s = t + l_0^s$$

In particular,

Proposition 4.1.12.

$$l_{t,0}^s = t \in H^0(\mathbb{L}_t^s; \mathbb{Z}_{(2)}) \simeq \mathbb{Z}_{(2)}$$

Now let M be a $\mathbb{Z}/2$ manifold with a map $f : M \to \mathbb{L}_t^s$ and let $\mathcal{C}_f$ be the associated presheaf. Define

$$\sigma_1^s(M, f) = \sigma_1^s(\mathcal{C}_f(M)) - \langle V_M Sq^1 V_M \cdot f^* l_t^s, [M]\rangle \in \mathbb{Z}/2$$

Then the product formula follows from the chain-level formula. That is,

Lemma 4.1.13. *Let M and N be $\mathbb{Z}/2$ manifolds with a map $f : M \to \mathbb{L}_t^s$. Then*

$$\sigma_1^s((M, f) \cdot N) = \sigma_1^s(M, f) \cdot \chi_2(N)$$

Therefore, we have

Proposition 4.1.14. *There exists a graded class*

$$r_t^s = r_{t,1}^s + r_{t,5}^s + \cdots \in H^{4*+1}(\mathbb{L}_t^s; \mathbb{Z}/2)$$

such that for any $f : M \to \mathbb{L}_t^s$

$$\sigma_1^s(M, f) = \langle V_M^2 \cdot f^* r_t^s, [M] \rangle \in \mathbb{Z}/2$$

where M *is a* $\mathbb{Z}/2$ *manifold.*

With the same argument above, we have

Proposition 4.1.15.

$$a_t^* r_t^s = r_0^s$$

Similarly to the quadratic case, we can pull back the classes to the characteristic classes of presheaves of Poincaré symmetric chains.

Proposition 4.1.16. *For any presheaf* $\mathcal{S}$ *of* 0*-connective Poincaré symmetric chains over* X*, there exist graded characteristic classes*

$$r^s(\mathcal{S}) = r_1^s + r_5^s + \cdots \in H^{4*+1}(X; \mathbb{Z}/2)$$

$$l^s(\mathcal{S}) = l_0^s + l_4^s + l_8^s + \cdots \in H^{4*}(X; \mathbb{Z}_{(2)})$$

which are invariant under bordism.

With the same proof as the quadratic case, we have that

Theorem 4.1.17. *Localized at prime* 2*,*

$$\mathbb{L}^s \simeq K(\mathbb{Z}, 0) \times \prod_{k>0} (K(\mathbb{Z}_{(2)}, 4k) \times K(\mathbb{Z}/2, 4k-3))$$

where the homotopy equivalence is given by the l^s *and* r^s *classes.*

4.1.3 Normal L-spectrum

We have to notice that $\pi_*(\mathbb{L}^n)$ has 4-periodicity for $* \geq 1$ and for the lower degrees, $\pi_0(\mathbb{L}^n) \cong \mathbb{Z}$ and $\pi_i(\mathbb{L}^n) = 0$ for $i < 0$.

Since $\mathbb{L}^n$ is not connected, we let $\mathbb{L}^n_t$ be the t-th component.

Again, we apply the argument of 4.1.1 and get that

Lemma 4.1.18. *The Hurewicz map $h : \pi_{2k+1}(\mathbb{L}^n) \to H_{2k+1}(\mathbb{L}^n; \mathbb{Z})$ is an injection onto a direct summand.*

We will define the cohomology classes of $\mathbb{L}^n$ in the order of $(4*+3)$-degree, $4*$-degree and $(4*+1)$-degree.

Let M be a $\mathbb{Z}/2$ manifold with $f : M \to \mathbb{L}^n$ and let $(\mathcal{D}_f, \mathcal{E}_f)$ be the associated presheaf of symmetric-quadratic chain pairs. Define

$$\sigma_3^n(M, f) = \sigma_3^n(\mathcal{D}_f(M), \mathcal{E}_f(M))$$

The product formula is immediate, i.e.,

$$\sigma_3^n((M, f) \cdot N) = \sigma_3^n(M, f) \cdot \chi_2(N)$$

where N is a $\mathbb{Z}/2$ manifold.

Theorem 4.1.19. *There exists a graded class*

$$k_t^n = k_{t,3}^n + k_{t,7}^n + \cdots \in H^{4*+3}(\mathbb{L}_t^n; \mathbb{Z}/2)$$

such that for any map $f : M \to \mathbb{L}_t^n$

$$\sigma_3^n(M, f) = \langle V_M^2 \cdot f^* k_t^n, [M] \rangle \in \mathbb{Z}/2$$

where M is a $\mathbb{Z}/2$ manifold.

Next, let M be a $\mathbb{Z}/8$ manifold with a map $f : M \to \mathbb{L}_t^n$ and let $(\mathcal{D}_f, \mathcal{E}_f)$ be the associated presheaf. Define

$$\sigma_0^n(M, f) = \sigma_0^n((\mathcal{D}_f(M), \mathcal{E}_f(M))) - j_8 \langle V_M Sq^1 V_M \cdot f^* k_t^n, [M] \rangle \in \mathbb{Z}/8$$

where $j_8 : \mathbb{Z}/2 \to \mathbb{Z}/8$.

The following product formulae are directly proven by the chain-level formulae.

Lemma 4.1.20. *Let M be a $\mathbb{Z}/p$ manifold and N be a $\mathbb{Z}/q$ manifold with a map $f : M \to \mathbb{L}^n_k$.*

(1) If $p = 8$ and $q = 0$, then

$$\sigma^n_0((M, f) \cdot N) = \sigma^n_0(M, f) \cdot \mathrm{Sign}(N) \in \mathbb{Z}/8$$

(2) If $p = 2$ and $q = 2$, then

$$\sigma^n_0(j_8((M, f) \cdot N)) = \sigma^n_0(j_8(M, f)) \cdot \mathrm{Sign}(N) \in 4 \cdot \mathbb{Z}/8$$

where $j_8 : \mathbb{Z}/2 \to \mathbb{Z}/8$.

(3) If $p = 2$ and $q = 2$, then

$$\sigma^n_0(\delta((M, f) \cdot N)) = \sigma^n_0(\delta(M, f)) \cdot \mathrm{Sign}(N) \in 4 \cdot \mathbb{Z}/8$$

Then we can deduce the existence of a $\mathbb{Z}/8$ class.

Proposition 4.1.21. *There exists a graded class*

$$l^n_t = l^n_{t,0} + l^n_{t,4} + l^n_{t,8} + \cdots \in H^{4*}(\mathbb{L}^n_k; \mathbb{Z}/8)$$

such that for any map $f : M \to \mathbb{L}^n_t$ we have

$$\sigma^n_0(M, f) = \langle L_M \cdot f^* l^n_t, [M] \rangle \in \mathbb{Z}/8$$

where M is a $\mathbb{Z}/8$ manifold.

Now let M be a $\mathbb{Z}/2$ manifold again with a map $f : M \to \mathbb{L}^n_t$ and let $(\mathcal{D}_f, \mathcal{E}_f)$ be the associated preheaf. Define

$$\sigma^n_1(M, f) = \sigma^n_1((\mathcal{D}_f(M), \mathcal{E}_f(M))) - \langle V_M Sq^1 V_M \cdot f^* \rho_2 l^n_t, [M] \rangle \in \mathbb{Z}/2$$

where $\rho_2 : \mathbb{Z}/8 \to \mathbb{Z}/2$.

Because of the chain-level product formula, we have

Lemma 4.1.22. *Let M and N be $\mathbb{Z}/2$ manifolds with a map $f : M \to \mathbb{L}^n_k$. Then*

$$\sigma^n_1((M, f) \cdot N) = \sigma^n_1(M, f) \cdot \chi_2(N) \in \mathbb{Z}/2$$

Consequently,

Proposition 4.1.23. *There exists a graded class $r^n_t = r^n_{t,1} + r^n_{t,5} + \cdots \in H^{4*+1}(\mathbb{L}^n_t; \mathbb{Z}/2)$, such that for any $f : M \to \mathbb{L}^n_t$*

$$\sigma^n_1(M, f) = \langle V^2_M \cdot f^* r^n_t, [M]\rangle \in \mathbb{Z}/2$$

where M is a $\mathbb{Z}/2$ manifold.

$\mathbb{L}^n$ has an additional structure like $\mathbb{L}^s$, namely,

$$b_{t,t'} : \mathbb{L}^n_t \times \mathbb{L}^n_{t'} \to \mathbb{L}^n_{t+t'}$$

It is like the symmetric case that the homotopy equivalence between different components is also given by a composition of maps

$$b_t : \mathbb{L}^n_0 = \mathbb{L}^n_0 \times \text{pt} \to \mathbb{L}^n_0 \times \mathbb{L}^n_t \xrightarrow{b_{0,t}} \mathbb{L}^n_t$$

Then we have that

Proposition 4.1.24.

$$b^*_t k^n_t = k^n_0$$

$$b^*_t l^n_t = \rho_8(t) + l^n_0$$

$$b^*_t r^n_t = r^n_0$$

where $\rho_8 : \mathbb{Z}_{(2)} \to \mathbb{Z}/8$.

In particular,

Proposition 4.1.25.

$$l^n_{t,0} = \rho_8(t) \in H^0(\mathbb{L}^n_t; \mathbb{Z}/8) \simeq \mathbb{Z}/8$$

We can also formulate these classes in terms of characteristic classes for presheaves.

Proposition 4.1.26. *For any presheaf $\mathcal{N}$ of 0-connective, 1-Poincaré normal chains over X, there exist graded characteristic classes*

$$r^n(\mathcal{N}) = r_1^n + r_5^n + \cdots \in H^{4*+1}(X;\mathbb{Z}/2)$$

$$k^n(\mathcal{N}) = k_3^n + k_7^n + \cdots \in H^{4*+3}(X;\mathbb{Z}/2)$$

$$l^n(\mathcal{N}) = l_0^n + l_4^n + l_8^n + \cdots \in H^{4*}(X;\mathbb{Z}/8)$$

which are invariant under bordism.

Like the case of quadratic and symmetric case, we also have that

Theorem 4.1.27. *Localized at prime 2,*

$$\mathbb{L}^n \simeq K(\mathbb{Z},0) \times \prod_{k>0}(K(\mathbb{Z}/8,4k) \times K(\mathbb{Z}/2,4k-3) \times K(\mathbb{Z}/2,4k-1))$$

where the homotopy equivalence is given by the classes l^n, r^n, k^n.

4.2 Relations of Characteristic Classes

We show the relations of characteristic classes among different L-theories by the natural fibration

$$\mathbb{L}^q \xrightarrow{i} \mathbb{L}^s \xrightarrow{p} \mathbb{L}^n$$

One needs to be careful that the natural image of i is contained in $\mathbb{L}_0^s$.

Proposition 4.2.1.

$$i^* l_0^s = 8 l^q$$

$$i^* r_0^s = 0$$

$$p^* k_t^n = 0$$

$$p^* l_t^n = \rho_8 l_t^s$$

$$p^* r_t^n = r_t^s$$

where $\rho_8 : \mathbb{Z}_{(2)} \to \mathbb{Z}/8$.

Proof. Let M be a $\mathbb{Z}$ or $\mathbb{Z}/2^k$ manifold.

First let $f : M \to \mathbb{L}^q$ be a map and let $\mathcal{C}^q$ be the associated presheaf of quadratic chains. Then

$$\sigma_0^s(M, i \circ f) = \mathrm{Sign}(\mathcal{C}^q(M)) = 8\sigma_0^q(M, f) + 8\langle \beta(V_M Sq^1 V_M \cdot f^* k^q), [M] \rangle$$

The term $8\langle \beta(f^* k^q V_M Sq^1 V_M), [M] \rangle$ vanishes since $\langle \beta(V_M Sq^1 V_M \cdot f^* k^q), [M] \rangle$ is a 2-torsion. Thus we proved the first equation.

The second equality is obvious since the de Rham invariant of a Poincaré quadratic chain always vanishes.

Now let $g : M \to \mathbb{L}_t^s$ be a map and let $\mathcal{D}^s$ be the associated presheaf of symmetric chains. Notice that $\mathcal{D}^s(M)$ does not have the quadratic part as a symmetric-quadratic pair. Then the third equation is trivial.

By definition, when $k = 3$,

$$\sigma_0^n(M, p \circ g) = \mathrm{Sign}(\mathcal{D}^s(M)) = \sigma_0^s(M, g) \in \mathbb{Z}/8$$

which proves the fourth equation.

When $k = 1$.

$$\begin{aligned} \sigma_1^n(M, p \circ g) &= \sigma_1^n(\mathcal{D}^s(M), 0) - \langle V_M Sq^1 V_m \cdot g^* p^* \rho_2 l_t^n, [M] \rangle \\ &= \sigma_1^s(\mathcal{D}^s(M)) - \langle V_M Sq^1 V_m \cdot g^* \rho_2 l_t^s, [M] \rangle = \sigma_0^s(M, g) \end{aligned}$$

where ρ_2 means either $\mathbb{Z}_{(2)} \to \mathbb{Z}/2$ or $\mathbb{Z}/8 \to \mathbb{Z}/2$.

Thus the fifth equation holds.

□

It comes to the relation of the classes of the quadratic and normal theories.

Let $\mathbb{L}^q(1)$ be the first space in the spectrum $\mathbb{L}^q$. $\mathbb{L}^q(1)$ is also the delooping of $\mathbb{L}^q$.

Mimic the discussion of the space $\mathbb{L}^q$. Let M^m be a $\mathbb{Z}$ or $\mathbb{Z}/2^k$ manifold with a map $f : M \to \mathbb{L}^q(1)$ and let $\mathcal{C}_f$ the associated presheaf. Then $\mathcal{C}_f(M)$ is a $\mathbb{Z}/2^k$ Poincaré quadratic chain of dimension $m-1$.

If $k=1$, then define

$$\widetilde{\sigma}_3^q = \sigma_2^q(\mathcal{C}_f(M)) \in \mathbb{Z}/2$$

Like the previous section, we have that

Proposition 4.2.2. *There exists a graded class*

$$\widetilde{k}^q = \widetilde{k}_3^q + \widetilde{k}_7^q + \cdots \in H^{4*+3}(\mathbb{L}^q(1); \mathbb{Z}/2)$$

such that for any map $f : M \to \mathbb{L}^q(1)$,

$$\widetilde{\sigma}_3^q(M, f) = \langle V_M^2 \cdot f^* \widetilde{k}^q, [M] \rangle \in \mathbb{Z}/2$$

where M *is a* $\mathbb{Z}/2$ *manifold.*

Next, define

$$\widetilde{\sigma}_1^q(M, f) = \sigma_0^q(\mathcal{C}_f(M)) - \langle \beta(V_M Sq^1 V_M \cdot f^* \widetilde{k}^q), [M] \rangle \in \mathbb{Z} \text{ or } \mathbb{Z}/2^k$$

Proposition 4.2.3. *There exists a graded class*

$$\widetilde{l}^q = \widetilde{l}_5^q + \widetilde{l}_9^q + \cdots \in H^{4*+1}(\mathbb{L}^q(1); \mathbb{Z}_{(2)})$$

such that for any map $f : M \to \mathbb{L}^q(1)$,

$$\widetilde{\sigma}_1^q(M, f) = \langle L_M \cdot f^* \widetilde{l}^q, [M] \rangle \in \mathbb{Z} \textit{ or } \mathbb{Z}/2^k$$

where M *is a* $\mathbb{Z}$ *or* $\mathbb{Z}/2^k$ *manifold.*

With the same argument as before,

Theorem 4.2.4. *Localized at prime* 2,

$$\mathbb{L}^q(1) \simeq \prod_{k>0} (K(\mathbb{Z}_{(2)}, 4k+1) \times K(\mathbb{Z}/2, 4k-1))$$

where the homotopy equivalence is given by the $\widetilde{l}^q$ *and* $\widetilde{k}^q$ *classes.*

Consider the connecting map $\partial_t : \mathbb{L}^n_t \to \mathbb{L}^q(1)$.

Proposition 4.2.5.

$$\partial_t^* \widetilde{k}^q = k^n_t$$

$$\partial_t^* \widetilde{l}^q = -\beta l^n_t$$

where β is the $\mathbb{Z}/8 \to \mathbb{Z}_{(2)}$ Bockstein.

Proof. Let M^m be $\mathbb{Z}/2^p$ manifold with a map $f : M \to \mathbb{L}^n_t$ and let $(\mathcal{D}, \mathcal{E})$ be the associated presheaf of symmetric-quadratic pairs.

If $p = 1$, then $\sigma^n_3(M, f) = \sigma^q_2(\mathcal{E}(M)) = \widetilde{\sigma}^q_3(M, \partial \circ f)$. Hence, the first equation holds.

If $p = 0$, then $\widetilde{\sigma}^q_1(M, \partial \circ f) = \sigma^q_0(\mathcal{E}(M)) = 0$, since $\mathrm{Sign}(\mathcal{E}(M)) = 0$.

If $p = 3$, then $\delta\mathcal{E}(M)$ as a Poincaré quadratic chain must be the boundary of some F when $m \equiv 1 \pmod 4$, namely $-\delta\mathcal{E}(M) \to F$. Thus, $\sigma^q_0(\mathcal{E}(M)) = \frac{1}{8}\mathrm{Sign}(\mathcal{E}(M) \bigcup_{8\delta\mathcal{E}(M)} 8F) \in \mathbb{Z}/2$.

To define βl^n_t, we consider

$$\sigma^n_0(\delta\mathcal{D}(M), -\delta\mathcal{E}(M)) = \mathrm{Sign}(\delta\mathcal{D}(M) \bigcup_{-\delta\mathcal{E}(M)} (-F)) \in \mathbb{Z}/2$$

Since $\mathcal{E}(M) \bigcup_{8\delta\mathcal{E}(M)} 8\delta\mathcal{D}(M)$ is the boundary of a Poincaré symmetic pair,

$$8\,\mathrm{Sign}(\delta\mathcal{D}(M)) + \mathrm{Sign}(\mathcal{E}(M)) = 0$$

Then

$$\sigma^n_0(\delta\mathcal{D}(M), -\delta\mathcal{E}(M)) + \sigma^q_0(\mathcal{E}(M)) = 0$$

It is not hard to check that the modified terms in defining l-classes are the same. The case for a general p follows from the $p = 3$ case. Therefore, the second equation also holds. □

4.3 Coproducts of Characteristic Classes

In this section, we calculate the coproducts of the characteristic classes induced by the ring structure or the module structure of the L-theories, that is,

$$m^s_{t,t'} : \mathbb{L}^s_t \times \mathbb{L}^s_{t'} \to \mathbb{L}^s_{tt'}$$

$$m_t^{q,s} : \mathbb{L}^q \times \mathbb{L}_t^s \to \mathbb{L}^q$$

$$m_{t,t'}^n : \mathbb{L}_t^n \times \mathbb{L}_{t'}^n \to \mathbb{L}_{tt'}^n$$

Recall that for any two $\mathbb{Z}/n$ manifolds M and N there is a natural map $\rho : M \otimes N \to M \times N$. For any cohomology classes $a \in H^*(M; R), b \in H^*(N; R)$, where R is a commutative ring, define $a \otimes b = \rho^*(a \times b)$. We use the notation $f \otimes g : M \otimes N \xrightarrow{\rho} M \times N \xrightarrow{f \times g} X \times Y$ for any two maps $f : M \to X, g : N \to Y$.

Recall that

Proposition 4.3.1. *([MS74, Proposition 3.1, Proposition 8.3])*

$$L_{M \otimes N} = L_M \otimes L_N$$

$$V_{M \otimes N} Sq^1 V_{M \otimes N} = V_M^2 \otimes V_N Sq^1 V_N + V_M Sq^1 V_M \otimes V_N^2$$

In particular,

$$V_{M \otimes N}^2 = V_M^2 \otimes V_N^2$$

Proposition 4.3.2.

$$(m_{t,t'}^s)^* l_{tt'}^s = l_t^s \times l_{t'}^s$$

$$(m_{t,t'}^s)^* r_{tt'}^s = r_t^s \times l_{t'}^s + l_t^s \times r_{t'}^s$$

Proof. The methods to prove the equations are the same. So we only give the proof for the first one.

Let M and N be $\mathbb{Z}/2^q$ manifolds with maps $f : M \to \mathbb{L}_t^s$ and $g : N \to \mathbb{L}_{t'}^s$. Let $\mathcal{C}_f$ and $\mathcal{C}_g$ be the associated presheaves. Let $\mathcal{C}_{f \otimes g}$ be the presheaf associated to $m_{t,t'}^s \circ (f \otimes g) : M \otimes N \to \mathbb{L}_{tt'}^s$. By 3.2.3, $\mathcal{C}_{f \otimes g}(M \otimes N)$ is bordant to $\mathcal{C}_f(M) \otimes \mathcal{C}_g(N)$.

On the one hand,

$$\sigma_0^s(M \otimes N, m_{t,t'}^s \circ (f \otimes g)) = \mathrm{Sign}(\mathcal{C}_{f \otimes g}(M \otimes N)) = \langle L_{M \otimes N} \cdot (f \otimes g)^* (m_{t,t'}^s)^* l_{tt'}^s, [M \otimes N] \rangle$$

On the other hand,

$$\mathrm{Sign}(\mathcal{C}_{f \otimes g}(M \otimes N)) = \mathrm{Sign}(\mathcal{C}_f(M)) \cdot \mathrm{Sign}(\mathcal{C}_g(N)) = \langle L_M \cdot f^* l_t^s, [M] \rangle \cdot \langle L_N \cdot g^* l_{t'}^s, [N] \rangle$$

Hence, the first equation follows. $\square$

Proposition 4.3.3.

$$(m_t^{q,s})^* k^q = k^q \times l_t^s$$

$$(m_t^{q,s})^* l^q = l^q \times l_t^s + \beta(k^q \times r_t^s)$$

where β is the $\mathbb{Z}/2 \to \mathbb{Z}_{(2)}$ Bockstein.

Proof. The proof is like the previous proposition. We only focus on the second one with the assumption that the first equation holds.

Let M and N be $\mathbb{Z}/2^q$ manifolds with maps $f : M \to \mathbb{L}^q$ and $g : N \to \mathbb{L}_t^s$ and let $\mathcal{C}_f$ and $\mathcal{C}_g$ be the associated presheaves. Again we also have the presheaf $\mathcal{C}_{f\otimes g}$.

On one hand,

$$\sigma_0^q(M \otimes N, m_t^{s,q} \circ (f \otimes g)) = \langle L_{M\otimes N} \cdot (f \otimes g)^* (m_t^{q,s})^* k^q, M \otimes N\rangle$$

On the other hand,

$$\begin{aligned}
\sigma_0^q(M \otimes N, m_t^{s,q} \circ (f \otimes g)) &= \sigma_0^q(\mathcal{C}_f(M) \otimes \mathcal{C}_g(N)) \\
&\quad -\langle \beta(V_{M\otimes N} Sq^1 V_{M\otimes N} \cdot (f \otimes g)^*(m_t^{q,s})^* k^q), [M \otimes N]\rangle \\
&= \sigma_0^q(\mathcal{C}_f(M)) \cdot \sigma_0^s(\mathcal{C}_g(N)) \\
&\quad + j_q(\sigma_2^q(\delta\mathcal{C}_f(M)) \cdot \sigma_1^s(\mathcal{C}_g(N)) + \sigma_2^q(\mathcal{C}_f(M)) \cdot \sigma_1^s(\delta\mathcal{C}_g(N))) \\
&\quad -\langle \beta(V_M Sq^1 V_M \cdot f^* k^q), [M]\rangle \cdot \langle L_N \cdot g^* l_t^s, [N]\rangle \\
&\quad -\langle L_M \cdot \beta f^* k^q, [M]\rangle \cdot \langle V_N Sq^1 V_N \cdot g^* l_t^s, [N]\rangle \\
&\quad -\langle L_M \cdot f^* k^q, [M]\rangle \cdot \langle \beta(V_N Sq^1 V_N) \cdot g^* l_t^s, [N]\rangle
\end{aligned}$$

where $j_q : \mathbb{Z}/2 \to \mathbb{Z}/2^q$.

Furthermore,

$$\sigma_0^q(\mathcal{C}_f(M)) \cdot \sigma_0^s(\mathcal{C}_g(N)) = \langle L_M \cdot f^* l^q + \beta(V_M Sq^1 V_M \cdot f^* k^q), [M]\rangle \cdot \langle L_N \cdot g^* l_t^s, [N]\rangle$$

$$j_q(\sigma_2^q(\delta\mathcal{C}_f(M)) \cdot \sigma_1^s(\mathcal{C}_g(N))) = \langle L_M \cdot \beta f^* k^q, [M]\rangle \cdot \langle L_N \cdot g^* r_t^s + V_N Sq^1 V_N \cdot g^* l_t^s, [N]\rangle$$

$$\sigma_2^q(\mathcal{C}_f(M)) \cdot \sigma_1^s(\delta\mathcal{C}_g(N)) = \langle L_M \cdot f^* k^q, [M]\rangle \cdot \langle L_N \cdot \beta g^* r_t^s + \beta(V_N Sq^1 V_N) \cdot g^* l_t^s, [N]\rangle$$

Carefully comparing all the terms, then the second equation holds. □

There is a product structure on $\mathbb{L}^q$ defined by the tensor product of chains

$$m^q : \mathbb{L}^q \times \mathbb{L}^q \xrightarrow{1\times i} \mathbb{L}^q \times \mathbb{L}_0^s \xrightarrow{m_0^{s,q}} \mathbb{L}^q$$

Then

Lemma 4.3.4. *(1)* $(m^q)^* k^q = 0$

(2) $(m^q)^* l^q = 8 \cdot l^q \times l^q$

The infinite loop structure on $\mathbb{L}^q$ induces an addition $a^q : \mathbb{L}^q \times \mathbb{L}^q \to \mathbb{L}^q$ and an inversion $\tau : \mathbb{L}^q \to \mathbb{L}^q$. Like the symmetric case, we have

Lemma 4.3.5.

$$a^* k^q = 1 \times k^q + k^q \times 1$$

$$a^* l^q = 1 \times l^q + l^q \times 1$$

$$\tau^* k^q = k^q$$

$$\tau^* l^q = -l^q$$

However, historically the product structure $\widetilde{m}^q : \mathbb{L}^q \times \mathbb{L}^q \to \mathbb{L}^q$ people used is another one, namely, the Whitney sum of trivializations of bundles on $G/TOP \simeq \mathbb{L}^q$, or equivalently, the product of surgery problems $M \times N \to K \times L$.

The the product structures $\widetilde{m}^q$ and m^q are not the same, but we can reproduce $\widetilde{m}^q$ by the module structure $m^{q,s}$.

First, define the map

$$i_1 : \mathbb{L}^q \simeq \mathbb{L}^q \times \mathrm{pt} \xrightarrow{i} \mathbb{L}_0^s \times \mathrm{pt} \to \mathbb{L}_0^s \times \mathbb{L}_1^s \xrightarrow{a} \mathbb{L}_1^s$$

Then $\widetilde{m}^q$ is indeed a composition of maps

$$\mathbb{L}^q \times \mathbb{L}^q \xrightarrow{(i_1\times 1)\times(1\times i_1)\times(1\times 1)} (\mathbb{L}_1^s \times \mathbb{L}^q) \times (\mathbb{L}^q \times \mathbb{L}_1^s) \times (\mathbb{L}^q \times \mathbb{L}^q)$$
$$\xrightarrow{m^{s,q}\times m^{q,s}\times(\tau\circ m^q)} \mathbb{L}^q \times \mathbb{L}^q \times \mathbb{L}^q \xrightarrow{a} \mathbb{L}^q$$

Then we reprove the coproducts of the characteristic classes of $\mathbb{L}^q$ in [RS71][MS74].

Corollary 4.3.6. *([RS71, p. 407];[MS74, p. 539 and Theorem 8.8])*

(1)$(\widetilde{m}^q)^* k^q = 1 \times k^q + k^q \times 1$

(2)$(\widetilde{m}^q)^* l^q = 1 \times l^q + l^q \times 1 + 8 \cdot l^q \times l^q$

Lastly, let us calculate coproducts of the characteristic classes of $\mathbb{L}^n$. Before we prove the coproduct formulae, we need two lemmas.

Lemma 4.3.7. *Let M be a $\mathbb{Z}/2$ manifold with a map $f : M \to \mathbb{L}_t^n$ and let $(\mathcal{D}_f, \mathcal{E}_f)$ be the associated presheaf of symmetric-quadratic pairs. Then*

$$\sigma_{0,2}^n(M, f) = \langle V_M^2 \cdot f^* \rho_2 l_t^n, [M] \rangle$$

where $\rho_2 : \mathbb{Z}/8 \to \mathbb{Z}/2$.

Proof. Recall the definition of $\sigma_{0,2}^n$. Four copies $4M$ is a $\mathbb{Z}/8$ manifold and

$$\sigma_{0,2}^n(\mathcal{D}_f(M), \mathcal{E}_f(M)) = \sigma_0^n(4(\mathcal{D}_f(M), \mathcal{E}_f(M)))$$

Also,

$$\sigma_0^n(4M, 4f) = \sigma_0^n(4(\mathcal{D}(M), \mathcal{E}(M))) - j_8 \langle V_{4M} Sq^1 V_{4M} \cdot f^* k_t^n, 4[M] \rangle \in 4\mathbb{Z}/8 \simeq \mathbb{Z}/2$$

Obviously,

$$\langle V_{4M} Sq^1 V_{4M} \cdot f^* k_t^n, 4[M] \rangle = 4 \cdot \langle V_M Sq^1 V_M \cdot f^* k_t^n, [M] \rangle = 0 \in \mathbb{Z}/2$$

Hence, the equation holds. □

For the same reason, we also have

Lemma 4.3.8. *Let M be a $\mathbb{Z}/4$ manifold with a map $f : M \to \mathbb{L}_t^n$ and let $(\mathcal{D}_f, \mathcal{E}_f)$ be the associated presheaf of symmetric-quadratic pairs. Then*

$$\sigma_{0,4}^n(M, f) = \langle \rho_4 L_M \cdot f^* \rho_4 l_t^n, [M] \rangle$$

where $\rho_4 : \mathbb{Z}/8 \to \mathbb{Z}/4$.

Proposition 4.3.9. $(m_{t,t'}^n)^* k_{tt'}^n = k_t^n \times \rho_2 l_{t'}^n + \rho_2 l_t^n \times k_{t'}^n$*, where $\rho_2 : \mathbb{Z}/8 \to \mathbb{Z}/2$*

Proof. Let M and N be $\mathbb{Z}/2$ manifolds with maps $f : M \to \mathbb{L}_t^n$ and $g : N \to \mathbb{L}_{t'}^n$. Let $(\mathcal{D}_f, \mathcal{E}_f)$ and $(\mathcal{D}_g, \mathcal{E}_g)$ be the associated presheaves. There is also a presheaf $(\mathcal{D}_{f\otimes g}, \mathcal{E}_{f\otimes g})$ associated to $m_{t,t'}^n \circ (f \otimes g)$.

Then

$$(\mathcal{D}_{f\otimes g}(M), \mathcal{E}_{f\otimes g}(N)) = (\mathcal{D}_f(M), \mathcal{E}_f(M)) \otimes (\mathcal{D}_g(N), \mathcal{E}_g(N))$$

On one hand,

$$\sigma_3^n(M \otimes N, m_{t,t'}^n \circ (f \otimes g)) = \langle V_{M\otimes N}^2 \cdot (f \otimes g)^* (m_{t,t'}^n)^* k_{tt'}^n, [M \otimes N] \rangle$$

On the other hand,

$$\begin{aligned} \sigma_3^n(M \otimes N, m_{t,t'}^n \circ (f \otimes g)) &= \sigma_{0,2}^n(\mathcal{D}_f(M), \mathcal{E}_f(M)) \cdot \sigma_3^n(\mathcal{D}_g(N), \mathcal{E}_g(N)) \\ &\quad + \sigma_3^n(\mathcal{D}_f(M), \mathcal{E}_f(M)) \cdot \sigma_{0,2}^n(\mathcal{D}_g(N), \mathcal{E}_g(N)) \\ &= \langle V_M^2 \cdot f_2^\rho l_t^n, [M] \rangle \cdot \langle V_N^2 \cdot g^* k_{t'}^n, [N] \rangle \\ &\quad + \langle V_M^2 \cdot f^* k_t^n, [M] \rangle \cdot \langle V_N^2 \cdot g^* \rho_2 l_{t'}^n, [N] \rangle \end{aligned}$$

□

To show the coproduct of the $\mathbb{Z}/8$ class l^n, we need the following lemma in [BM76].

Lemma 4.3.10. *([BM76, Lemma 9.3]) $H_*(X \times Y, \mathbb{Z}/2^k)$ is generated by the Hurewicz image of the followings.*

(1) $j_{2^k}(f_*[M] \times g_*[N])$, where M and N are $\mathbb{Z}/2^l$ manifolds with $l \leq k$, $f : M \to X$, $g : N \to Y$ and $j_{2^k} : \mathbb{Z}/2^l \to \mathbb{Z}/2^k$.

(2) $\rho_{2^k}\delta(f_*[P]\times g_*[Q])$*, where* P *and* Q *are* $\mathbb{Z}/2^l$ *manifolds with* $l<k$*,* $f:P\to X$*,* $g:Q\to Y$*,* $\rho_{2^k}:\mathbb{Z}\to\mathbb{Z}/2^k$*.*

Lemma 4.3.11. *Let* M *and* N *be two* $\mathbb{Z}/8$ *manifolds with maps* $f:M\to\mathbb{L}^n_t$ *and* $g:N\to\mathbb{L}^n_{t'}$*. Then*

$$\begin{aligned}&\langle L_{M\otimes N}\cdot(f\otimes g)^*(m^n_{t,t'})^*l^n_{tt'},[M\otimes N]\rangle=\\&\quad\langle L_{M\otimes N}\cdot(f^*l^n_t\otimes g^*l^n_{t'}),[M\otimes N]\rangle+j_8\langle V^2_{M\otimes N}\cdot(f^*r^n_t\otimes g^*k^n_{t'}+f^*k^n_t\otimes g^*r^n_{t'})),[M\otimes N]\rangle\end{aligned}$$

where $j_8:\mathbb{Z}/2\to\mathbb{Z}/8$*.*

Proof. Let $(\mathcal{D}_f,\mathcal{E}_f)$ and $(\mathcal{D}_g,\mathcal{E}_g)$ be the associated presheaves. Let $(\mathcal{D}_{f\otimes g},\mathcal{E}_{f\otimes g})$ the presheaf associated to $m^n_{t,t'}\circ(f\otimes g)$.

On one hand,

$$\begin{aligned}&\sigma^n_0(\mathcal{D}_{f\otimes g}(M\otimes N),\mathcal{E}_{f\otimes g}(M\otimes N))\\=\;&\langle L_{M\otimes N}\cdot(f\otimes g)^*(m^n_{t,t'})^*l^n_{tt'},[M\otimes N]\rangle\\&+j_8\langle V_{M\otimes N}Sq^1V_{M\otimes N}\cdot(f\otimes g)^*(m^n_{t,t'})^*k^n_{tt'},[M\otimes N]\rangle\\=\;&\langle L_{M\otimes N}\cdot(f\otimes g)^*(m^n_{t,t'})^*l^n_{tt'},[M\otimes N]\rangle\\&+j_8\langle(V_MSq^1V_M\otimes L_N+L_M\otimes V_NSq^1V_N)\cdot(f^*k^n_t\otimes g^*l^n_{t'}+f^*l^n_t\otimes g^*k^n_{t'}),[M\otimes N]\rangle\end{aligned}$$

On the other hand,

$$\begin{aligned}&\sigma^n_0(\mathcal{D}_{f\otimes g}(M\otimes N),\mathcal{E}_{f\otimes g}(M\otimes N))\\=\;&\sigma^n_0(\mathcal{D}_f(M),\mathcal{E}_f(M))\cdot\sigma^n_0(\mathcal{D}_g(N),\mathcal{E}_g(N))+j_8(\sigma^n_1(\mathcal{D}_f(M),\mathcal{E}_f(M))\cdot\sigma^n_3(\mathcal{D}_g(N),\mathcal{E}_g(N)))\\&+j_8(\sigma^n_3(\mathcal{D}_f(M),\mathcal{E}_f(M))\cdot\sigma^n_1(\mathcal{D}_g(N),\mathcal{E}_g(N)))\\=\;&(\langle L_M\cdot f^*l^n_t,[M]\rangle+j_8\langle V_MSq^1V_M\cdot f^*k^n_t,[M]\rangle)\\&\cdot(\langle L_N\cdot g^*l^n_{t'},[N]\rangle+j_8\langle V_NSq^1V_N\cdot g^*k^n_{t'},[N]\rangle)\\&+j_8(\langle V^2_M\cdot f^*r^n_t,[M]\rangle+\langle V_MSq^1V_M\cdot f^*l^n_t,[M]\rangle)\cdot\langle V^2_N\cdot g^*k^n_{t'},[N]\rangle\\&+j_8\langle V^2_M\cdot f^*k^n_t,[M]\rangle\cdot(\langle V^2_N\cdot g^*r^n_{t'},[N]\rangle+\langle V_NSq^1V_N\cdot g^*l^n_{t'},[N]\rangle)\end{aligned}$$

Compare the terms carefully. Also notice that

$$j_8\langle V_M Sq^1 V_M f^* k_t^n, [M]\rangle \cdot j_8\langle V_N Sq^1 V_N g^* k_{t'}^n, [N]\rangle = 0 \in \mathbb{Z}/8$$

since $4 \times 4 = 0 \in \mathbb{Z}/8$. Then the equation follows. □

We can prove the $\mathbb{Z}/2$ and $\mathbb{Z}/4$ cases by reducing to the $\mathbb{Z}/8$ case. Notice that the j_8-term vanishes in these two cases.

Lemma 4.3.12. *Let M and N be two $\mathbb{Z}/2$ manifolds of dimension m and n with maps $f : M \to \mathbb{L}_t^n$ and $g : N \to \mathbb{L}_{t'}^n$. Then*

$$\langle L_{M\otimes N} \cdot (f \otimes g)^*(m_{t,t'}^n)^* \rho_2 l_{tt'}^n, [M \otimes N]\rangle = \langle L_{M\otimes N} \cdot (f^* \rho_2 l_t^n \otimes g^* \rho_2 l_{t'}^n), [M \otimes N]\rangle$$

where $\rho_2 : \mathbb{Z}/8 \to \mathbb{Z}/2$.

Lemma 4.3.13. *Let M and N be two $\mathbb{Z}/4$ manifolds of dimension m and n with maps $f : M \to \mathbb{L}_k^n$ and $g : N \to \mathbb{L}_l^n$. Then*

$$\langle L_{M\otimes N} \cdot (f \otimes g)^*(m_{t,t'}^n)^* \rho_4 l_{tt'}^n, [M \otimes N]\rangle = \langle L_{M\otimes N} \cdot (f^* \rho_4 l_t^n \otimes g^* \rho_4 l_{t'}^n), [M \otimes N]\rangle$$

where $\rho_4 : \mathbb{Z}/8 \to \mathbb{Z}/4$.

For the Bockstein case, we have

Lemma 4.3.14. *Let M and N be two $\mathbb{Z}/8$ manifolds with maps $f : M \to \mathbb{L}_t^n$ and $g : N \to \mathbb{L}_{t'}^n$. Then*

$$\begin{aligned}&\langle L_{M\otimes N} \cdot (f \otimes g)^*(m_{t,t'}^n)^* l_{tt'}^n, [\delta(M \otimes N)]\rangle \\ &= \langle L_{M\otimes N} \cdot f^* l_t^n \otimes g^* l_{t'}^n, [\delta(M \otimes N)]\rangle + j_8\langle V_{M\otimes N}^2 \cdot (f^* r_t^n \otimes g^* k_{t'}^n + f^* k_t^n \otimes g^* r_{t'}^n), [\delta(M \otimes N)]\rangle\end{aligned}$$

where $j_8 : \mathbb{Z}/2 \to \mathbb{Z}/8$.

Proof. As before, let $(\mathcal{D}_f, \mathcal{E}_f)$ and $(\mathcal{D}_g, \mathcal{E}_g)$ be the associated presheaves. Let $(\mathcal{D}_{f\otimes g}, \mathcal{E}_{f\otimes g})$ the presheaf associated to $m_{t,t'}^n \circ (f \otimes g)$.

On one hand,

$$
\begin{aligned}
& \sigma_0^n(\mathcal{D}_{\delta(f\otimes g)}(\delta(M\otimes N)), \mathcal{E}_{\delta(f\otimes g)}(\delta(M\otimes N))) \\
= \; & \langle L_{M\otimes N} \cdot (f\otimes g)^*(m_{t,t'}^n)^* l_{tt'}^n, [\delta(M\otimes N)]\rangle \\
& + j_8\langle V_{M\otimes N} Sq^1 V_{M\otimes N} \cdot (f\otimes g)^*(m_{t,t'}^n)^* k_{tt'}^n, [\delta(M\otimes N)]\rangle \\
= \; & \langle L_{M\otimes N} \cdot (f\otimes g)^*(m_{t,t'}^n)^* l_{tt'}^n, [\delta(M\otimes N)]\rangle \\
& + j_8\langle (V_M Sq^1 V_M \otimes V_N^2 + V_M^2 \otimes V_N Sq^1 V_N) \cdot (f^* k_t^n \otimes g^* l_{t'}^n + f^* l_t^n \otimes g^* k_{t'}^n), [\delta(M\otimes N)]\rangle
\end{aligned}
$$

Notice that

$$[\delta(M\otimes N)] = [\delta M]\otimes[N] + [M]\otimes[\delta N] \in H_{m+n-1}(M\otimes N; \mathbb{Z}/8)$$

On the other hand, by the chain-level product formula,

$$
\begin{aligned}
& \sigma_0^n(\mathcal{D}_{\delta(f\otimes g)}(\delta(M\otimes N)), \mathcal{E}_{\delta(f\otimes g)}(\delta(M\otimes N))) \\
= \; & \sigma_0^n(\delta(\mathcal{D}_f(M), \mathcal{E}_f(M))) \cdot \sigma_0^n(\mathcal{D}_g(N), \mathcal{E}_g(N)) \\
& + \sigma_0^n(\mathcal{D}_f(M), \mathcal{E}_f(M)) \cdot \sigma_0^n(\delta(\mathcal{D}_g(N), \mathcal{E}_g(N))) \\
& + j_8\sigma_3^n(\delta(\mathcal{D}_f(M), \mathcal{E}_f(M))) \cdot \sigma_1^n(\rho_2(\mathcal{D}_g(N), \mathcal{E}_g(N))) \\
& + j_8\sigma_1^n(\rho_2(\mathcal{D}_f(M), \mathcal{E}_f(M))) \cdot \sigma_3^n(\delta(\mathcal{D}_g(N), \mathcal{E}_g(N))) \\
& + j_8\sigma_3^n(\mathcal{D}_f(M), \mathcal{E}_f(M)) \cdot \sigma_1^n(\rho_2\delta(\mathcal{D}_g(N), \mathcal{E}_g(N))) \\
& + j_8\sigma_1^n(\rho_2\delta(\mathcal{D}_f(M), \mathcal{E}_f(M))) \cdot \sigma_3^n(\mathcal{D}_g(N), \mathcal{E}_g(N))
\end{aligned}
$$

Carefully write each term in the form of cohomology classes and we can check that the equation holds. □

The $\mathbb{Z}/2$ and $\mathbb{Z}/4$ Bockstein cases reduce to the $\mathbb{Z}/8$ Bockstein case as well. It follows that

Proposition 4.3.15.

$$(m_{t,t'}^n)^* l_{tt'}^n = l_k^n \times l_{t'}^n + j_8(r_t^n \times k_{t'}^n + k_t^n \times r_{t'}^n)$$

where $j_8 : \mathbb{Z}/2 \to \mathbb{Z}/8$*.*

In particular,

Proposition 4.3.16.

$$(m^n_{t,t'})^* \rho_4 l^n_{tt'} = \rho_4 l^n_t \times \rho_4 l^n_{t'}$$

$$(m^n_{t,t'})^* \rho_2 l^n_{tt'} = \rho_2 l^n_t \times \rho_2 l^n_{t'}$$

where $\rho_4 : \mathbb{Z}/8 \to \mathbb{Z}/4$ *and* $\rho_2 : \mathbb{Z}/8 \to \mathbb{Z}/2$*.*

Proposition 4.3.17.

$$(m^n_{t,t'})^* r^n_{tt'} = \rho_2 l^n_t \times r^n_{t'} + r^n_t \times \rho_2 l^n_{t'}$$

where $\rho_2 : \mathbb{Z}/8 \to \mathbb{Z}/2$*.*

Proof. Let M and N be $\mathbb{Z}/2$ manifolds with maps $f : M \to \mathbb{L}^n_t$ and $g : N \to \mathbb{L}^n_{t'}$. Let $(\mathcal{D}_f, \mathcal{E}_f)$ and $(\mathcal{D}_g, \mathcal{E}_g)$ be the associated presheaves. Let $(\mathcal{D}_{f\otimes g}, \mathcal{E}_{f\otimes g})$ be the presheaf associated to $m^n_{t,t'} \circ (f \otimes g)$.

On one hand,

$$\begin{aligned}
& \sigma^n_1(\mathcal{D}_{f\otimes g}(M \otimes N), \mathcal{E}_{f\otimes g}(M \otimes N)) \\
= \; & \langle V^2_{M\otimes N} \cdot (f \otimes g)^* (m^n_{t,t'})^* r^n_{tt'}, [M \otimes N]\rangle \\
& + \langle V_{M\otimes N} Sq^1 V_{M\otimes N} \cdot (f \otimes g)^* (m^n_{k,l})^* \rho_2 l^n_{tt'}, [M \otimes N]\rangle \\
= \; & \langle V^2_{M\otimes N} \cdot (f \otimes g)^* (m^n_{k,l})^* r^n_{tt'}, [M \otimes N]\rangle \\
& + \langle (V_M Sq^1 V_M \otimes V^2_N + V^2_M \otimes V_N Sq^1 V_N) \cdot (f^* \rho_2 l^n_t \otimes g^* \rho_2 l^n_{t'}), [M \otimes N]\rangle
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \sigma^n_1(\mathcal{D}_{f\otimes g}(M \otimes N), \mathcal{E}_{f\otimes g}(M \otimes N)) \\
= \; & \sigma^n_{0,2}(\mathcal{D}_f(M), \mathcal{E}_f(M)) \cdot \sigma^n_1(\mathcal{D}_g(N), \mathcal{E}_g(N)) \\
& + \sigma^n_1(\mathcal{D}_f(M), \mathcal{E}_f(M)) \cdot \sigma^n_{0,2}(\mathcal{D}_g(N), \mathcal{E}_g(N)) \\
= \; & \langle V^2_M \cdot f^* \rho_2 l^n_t, [M]\rangle \langle V^2_N \cdot g^* r^n_{t'} + V_N Sq^1 V_N \cdot g^* \rho_2 l^n_{t'}, [N]\rangle) \\
& + \langle V^2_M \cdot f^* r^n_t + V_M Sq^1 V_M \cdot f^* \rho_2 l^n_t, [M]\rangle \cdot \langle V^2_N \cdot g^* \rho_2 l^n_{t'}, [N]\rangle
\end{aligned}$$

□

4.4 Characteristic Classes of Bundle Theory

Because of the equivalence of Ranicki's formulation and Wall's formulation of L-groups, it is quite obvious that the classes k^q and l^q of $\mathbb{L}^q$ are equivalent to the Kervaire class and the l-class for the surgery space G/TOP.

Recall in the Chapter 2, there is a graded characteristic class $l^{TOP} \in H^{4*}(B; \mathbb{Z}_{(2)})$ for any TOP bundle over B. Moreover, there are graded characteristic classes $l^G \in H^{4*}(B; \mathbb{Z}/8)$ and $k^G \in H^{4*+3}(B; \mathbb{Z}/2)$ for any spherical fibration over B. In the 2-local sense, the spherical fibration has a TOP bundle structure if and only if the k^G-class vanishes and the $\mathbb{Z}/8$ class l^G has a $\mathbb{Z}_{(2)}$ lifting. For 2-local TOP bundles, the $\mathbb{Z}/8$ reduction of l^{TOP} is l^G.

In Chapter 2, we said that the bundle theory also has an integral description.

Theorem 4.4.1. *([LR87, Proposition 16.1]) Suppose X is a finite simplicial complex.*

(1) A (k-1)-spherical fibration $\nu : X \to BSG(k)$ has a canonical $\mathbb{L}^n$-orientation $U^n(\nu) : T(\nu) \to \Sigma^k \mathbb{L}^n$, where $T(\nu)$ is the Thom space of ν.

(2) A topological block bundle $\mu : X \to B\widetilde{STOP}(k)$ has a canonical $\mathbb{L}^s$-orientation $U^s(\mu) : T(\mu) \to \Sigma^k \mathbb{L}^s$ so that its $\mathbb{L}^n$-reduction is $U^n(\mu)$.

(3) A difference between any two stable topological bundle liftings μ, μ' of the same spherical fibration ν is represented by an element $d(\mu, \mu') \in (\mathbb{L}^q)^0(X)$.

Remark 4.4.2. The universal $\mathbb{L}^s$-orientation $U^s : M\widetilde{STOP}(k) \to \Sigma^k \mathbb{L}^s$ of the block bundle theory induces a universal $\mathbb{L}^s$-orientation $U^s : MSTOP(k) \to \Sigma^k \mathbb{L}^s$ for the (micro-)TOP bundle theory, by the natural inclusion $STOP(k) \to \widetilde{STOP}(k)$. Moreover, in the stable range $STOP \to \widetilde{STOP}$ is a homotopy equivalence ([RS70, Corollary 4.11]).

In this section, we prove that the localization at prime 2 of Levitt-Ranicki's theory is equivalent to Brumfiel-Morgan and Morgan-Sullivan's theories [MS74][BM76]. That is, we will prove that the classes of $\mathbb{L}^s, \mathbb{L}^n$ we constructed before correspond to the characteristic classes defined in [MS74][BM76] under Levitt-Ranicki' $\mathbb{L}$-theory orientations.

Before doing that, let us briefly review how to construct the $\mathbb{L}$-orientations U^n and U^s.

Let $\xi \to B$ be an oriented spherical fibration. Let $D(\xi)$ be the corresponding disc bundle (the mapping cylinder) and let $Th(\xi)$ be the Thom space. In the singular simplicial complex $S(Th(\xi))$, there is a subcomplex $N(Th(\xi))$ which consists of maps $f : \Delta^n \to Th(\xi)$ such that $f^{-1}(D(\xi))$ is an n-ad of normal spaces of dimension $n-k$ with respect to the pullback bundle. Notice that the inclusion map $N(Th(\xi)) \to S(Th(\xi))$ is a homotopy equivalence. Then the singular chains of n-ads of normal spaces induce a simplicial map $N(Th(\xi)) \to \Sigma^k \mathbb{L}^n_1$, which is the canonical $\mathbb{L}^n$-orientation.

In $N(Th(\xi))$, let $T(Th(\xi))$ be the subcomplex consisting of maps $f : \Delta^n \to Th(\xi)$ such that f is Poincaré transversal, namely, $f^{-1}(D(\xi))$ is an n-ad of $\mathbb{Z}$-coefficient homology Poincaré space of dimension $n-k$ with the fundamental class induced from the normal structure. Similarly, the singular chains of n-ads of homology Poincaré spaces induce a simplicial map $T(Th(\xi)) \to \mathbb{L}^s_1$.

There exists a TOP structure of ξ if and only ξ has a theory of transversality (see Chapter 2 or [LM72]), if and only if there is a canonical homotopy inverse of the inclusion $T(Th(\xi)) \to N(Th(\xi))$ ([LR87, Theorem 1.11]). Furthermore, $T(Th(\xi)) \to \mathbb{L}^s_1$ is the canonical $\mathbb{L}^s$-orientation.

Now it comes to the proof.

Let M be a $\mathbb{Z}/2^q$ PL manifold with a map $f : M \to M\widetilde{STOP}(h)$. Due to the transversality theorem of topological manifolds ([FQ90, (9.6C)]), we can homotope f so that it is transversal to the zero section over each simplex, i.e., for each simplex Δ^n of M, $f^{-1}(B\widetilde{STOP}(h))$ is an n-ad of $\mathbb{Z}/2^q$ topological manifolds of dimension $n-h$. The assembly of the n-ads is a $\mathbb{Z}/2$ topological submanifold L of M.

By our construction,

$$\mathrm{Sign}(L) = \sigma^s_0(M, U^s \circ f) = \langle L_M \cdot f^*(U^s)^* \Sigma^h l^s_1, [M] \rangle$$

Also,

$$\mathrm{Sign}(L) = \langle L_M \cdot f^*(l^{TOP} \cdot U), [M]\rangle$$

where $U \in \widetilde{H}^h(M\widetilde{STOP}(h))$ is the universal Thom class and $l^{TOP} \in H^{4*}(B\widetilde{STOP}; \mathbb{Z}_{(2)})$ is the class defined in[MS74].

After passage to the stable range, we have

Proposition 4.4.3.

$$(U^s)^* l_1^s = l^{TOP} \cdot U \in \widetilde{H}^{4*}(M\widetilde{STOP}; \mathbb{Z}_{(2)})$$

Proposition 4.4.4.

$$(U^s)^* l_1^s = l^{TOP} \cdot U \in \widetilde{H}^{4*}(M\widetilde{STOP}; \mathbb{Z}_{(2)})$$

Now let M be a $\mathbb{Z}/2$ PL manifold. Then

$$\begin{aligned} \mathrm{dR}(L) &= \sigma_1^s(M, U^s \circ f) \\ &= \langle V_M^2 \cdot f^*(U^s)^*\Sigma^h r_1^s, [M]\rangle + \langle V_M Sq^1 V_M \cdot f^*(U^s)^*\rho_2 \Sigma^h l_1^s, [M]\rangle \\ &= \langle V_M^2 \cdot f^*(U^s)^*\Sigma^h r_1^s, [M]\rangle + \langle V_M Sq^1 V_M \cdot f^*\rho_2(l^{TOP} \cdot U), [M]\rangle \end{aligned}$$

Let τ_M be the tangent bundle of M and ν be the normal bundle of $L \subset M$. Then

$$\begin{aligned} \mathrm{dR}(L) &= \langle V_L Sq^1 V_L, [L]\rangle \\ &= \langle V_{\tau_M|_L}^2 \cdot f^*(V_\nu Sq^1 V_\nu), [L]\rangle + \langle V_{\tau_M|_L} Sq^1 V_{\tau_M|_L} \cdot f^* L_\nu, [L]\rangle \\ &= \langle L_M \cdot f^*(V Sq^1 V \cdot U), [M]\rangle + \langle V_M Sq^1 V_M \cdot f^*\rho_2(l^{TOP} \cdot U), [M]\rangle \end{aligned}$$

Hence,

Proposition 4.4.5.

$$(U^s)^* r_1^s = V Sq^1 V \cdot U \in \widetilde{H}^{4*+1}(M\widetilde{STOP}; \mathbb{Z}/2)$$

In particular, $(U^s)^* r_{1,1}^s = 0$.

Next, let us get to the characteristic classes of spherical fibrations.

Now let M be a $\mathbb{Z}/2^q$ manifold of dimension $m+h$ with a map $f: M \to MSG(h)$. By a slight homotopy we can assume that f is transversal to the spherical fibration $S(ESG(h)) \subset MSG(h)$ and let $I = f^{-1}(D(ESG(h)))$.

Embed M in a sphere D^{N+m+h}, where there is a $\mathbb{Z}/2^q$-action on the boundary S^{N+m+h} so that $\partial M = \bigcup_{2^q} \delta M$ is equivariantly embedded in S^{N+m+h}. Consider the corresponding Pontryagin-Thom construction $F: D^{N+m+h} \to MSG(h) \wedge MSPL(N)$. Let $N(M)$ be the tubular neighborhood of M in D^{N+m+h} such that the preimage of the disc bundle in the $MSPL(N)$ is $N(M)$ under F. Since $MSG(h) \wedge MSPL(N) \simeq M(D(ESG(h)) \times ESPL(N))$ is again the Thom space of some spherical fibration, the preimage of the total disc bundle in $MSG(h) \wedge MSPL(N)$ is the restriction $N(M)|_I$.

Let $U_{MSG(h)} \in \widetilde{H}^h(MSG(h);\mathbb{Z})$ and $U_{MSG(h)\wedge MSPL(N)} \in \widetilde{H}^{h+N}(MSG(h) \wedge MSPL(N);\mathbb{Z})$ both be the Thom class. Let $x = [M] \cap U_{MSG(h)} \in H_m(I, I \bigcap \partial M;\mathbb{Z})$ and $y = [D^{N+m+h}] \cap U_{MSG(h)\wedge MSPL(N)} \in H_m(N(M)|_I, N(M)|_{I \bigcap \partial M};\mathbb{Z})$. Then x and y induce $\mathbb{Z}/2^q$ symmetric structures on $C_*(I)$ and $C_*(N(M)|_I)$ respectively. But the natural inclusion $I \to N(M)|_I$ induces a chain homotopy equivalence between the two $\mathbb{Z}/2^q$ symmetric chains.

Brumfiel-Morgan's obstruction for cobording f to be Poincaré transversal is their obstruction class for $F: S^{N+m+h} \to MSG(h) \wedge MSPL(N)$ (see Chapter 2). The obstruction is equivalent to whether $C_*(N(M)|_I) \simeq C_*(I)$ satisfies $\mathbb{Z}/2^q$ Poincaré duality. By Ranicki's miracle lemma 2.1.10, it is equivalent to the bordism class of $\partial C_*(I)$ in $L^q_{m-1}(\mathbb{Z}, \mathbb{Z}/2^q)$ (for the quadratic structure on $\partial C_*(I)$, see [Ran81, Proposition 7.4.1]).

On the other hand, associated to the composition map $U^n \circ f$, there is a presheaf $(\mathcal{D}_f, \mathcal{E}_f)$ of Poincaré symmetric-quadratic pair over M, whose assembly is exactly $(C^{m-*}(I), \partial C_*(I))$.

When $m \equiv 2 \pmod 2$, the bordism class of $\partial C_*(I)$ in L^q_{m-1} is determined by the Kervaire invariant $\sigma^n_3(\mathcal{D}_f, \mathcal{E}_f)$. Hence, we may assume that M is a $\mathbb{Z}/2$ manifold.

Also the construction of $k^G \in H^{4*+3}(BSG;\mathbb{Z}/2)$ in [BM76, Theorem 5.4]. Then

Proposition 4.4.6.

$$(U^n)^*k_1^n = k^G \cdot U \in \widetilde{H}^{4*+3}(MSG;\mathbb{Z}/2)$$

Recall the construction of l^G in [BM76, Section 8]. Let M be a $\mathbb{Z}/8$ manifold. When the map $f : M \to MSG(h)$ is Poincaré transversal, i.e., the preimage I is a presheaf of ads of $\mathbb{Z}/8$ homology Poincaré spaces over M. Then

$$\mathrm{Sign}(I) = \langle L_M \cdot f^*(l^G \cdot U), [M]\rangle + j_8\langle V_M Sq^1 V_M \cdot f^*(k^G \cdot U), [M]\rangle \in \mathbb{Z}/8$$

where $j_8 : \mathbb{Z}/2 \to \mathbb{Z}/8$.

We also know that

$$\begin{aligned}\mathrm{Sign}(I) &= \sigma_0^n(\mathcal{D}_f(M), \mathcal{E}_f(M)) + j_8\langle V_M Sq^1 V_M \cdot f^*(U^n)^*\Sigma^h k_1^n, [M]\rangle \\ &= \langle L_M \cdot f^*(U^n)^*\Sigma^h l_1^n, [M]\rangle + j_8\langle V_M Sq^1 V_M \cdot f^*(k^G \cdot U), [M]\rangle\end{aligned}$$

Hence, under the assumption of Poincaré transversality for f,

$$\langle L_M f^*(l^G \cdot U), [M]\rangle = \langle L_M f^*(U^n)^*\Sigma^h l_1^n, [M]\rangle$$

Recall from Chapter 2 that [BM76, p. 61] constructed a map $a : K^{h+4} \to MSG(h)$ for a $\mathbb{Z}/2$ manifold $K^{h+4} = S^{h+3} \times I/(x,0) \sim (-x,1)$ such that

(1) the Kervaire obstruction to the Poincaré transversality of $a|_{\delta K}$ is $1 \in \mathbb{Z}/2$;

(2) $\langle a^*(V^2 \cdot U), [K]\rangle = 0 \in \mathbb{Z}/2$, where V is the sum of even Wu classes.

We are left to prove that the assembled $\mathbb{Z}/2$ symmetric-quadratic Poincaré chain pair $(\mathcal{D}_a(K), \mathcal{E}_a(K))$ is bordant to the chosen (D_0', E_0') in the previous chapter. It suffices to prove that $\sigma_0^n(K, a) = 0 = \sigma_0^n(D_0', E_0')$ since $\sigma_2^q(\mathcal{E}_a(K)) = \sigma_2^q(E_0') = 1$.

Let N be a $\mathbb{Z}/2$ manifold of dimension congruent to $h' + 3$ modulo 4 together with a map $b : N \to MSG(h')$ such that it has nonvanishing Kervaire obstruction to Poincaré transversality, i.e., $\sigma_3^n(N, b) = 1 \in \mathbb{Z}/2$. Due to the product formulae of $L_*^n(\mathbb{Z}, \mathbb{Z}/2)$, it suffices that the Kervaire obstruction for cobording the following map to Poincaré transversality vanishes

$$K \otimes N \xrightarrow{a\otimes b} MSG(h) \times MSG(h') \xrightarrow{\Delta} MSG(h + h')$$

Recall the coproduct formula for k^G in [BM76, p. 9.2]

$$\Delta^*(k^G \cdot U) = (k^G \cdot U) \times (V^2 \cdot U) + (V^2 \cdot U) \times (k^G \cdot U)$$

Then

$$\begin{aligned} \sigma_3^n(K \otimes N, \Delta \circ (a \otimes b)) &= \langle V^2_{K \otimes N} \cdot (a \otimes b)^*(\Delta)^*(k^G \cdot U), [K \otimes N] \rangle \\ &= \langle (V_K^2 \otimes V_N^2) \cdot (a^*(V^2 \cdot U) \otimes b^*(k^G \cdot U)), [K \otimes N] \rangle \\ &= \langle a^*(V^2 \cdot U), [K] \rangle \cdot \langle V_N^2 \cdot b^*(k^G \cdot U), [N] \rangle = 0 \end{aligned}$$

But

$$\sigma_3^n(K \otimes N, \Delta \circ (a \otimes b)) = \sigma_0^n(K, a) \cdot \sigma_3^n(N, b)$$

Hence, $\sigma_0^n(K, a) = 0 = \sigma_0^n(D_0', E_0')$.

Recall from Chapter 2 and [BM76, p. 61] that, for a non Poincaré transversal f, the key step to define $l^G \in H^{4*}(BSG; \mathbb{Z}/8)$ is to subtract the bordism class $[K, a] \cdot [\mathbb{C}P^{\frac{m-4}{2}}]$ from $[M, f]$ as a modification, which is exactly the same modification as what we did in the chain level. Then

Proposition 4.4.7.

$$(U^n)^* l_1^n = l^G \cdot U \in \widetilde{H}^{4*}(MSG; \mathbb{Z}/8)$$

Recall from [BM76, 8.1(ii)1] that $\rho_2 l^G = V^2$. Hence,

Corollary 4.4.8.

$$\rho_2 (U^n)^* l_1^n = V^2 \cdot U \in \widetilde{H}^{4*}(MSG; \mathbb{Z}/2)$$

Finally, We prove that r_1^n corresponds to $V Sq^1 V \in H^{4*+1}(BSG; \mathbb{Z}/2)$.

Let M be a $\mathbb{Z}/2$ manifold. First assume that $(M, f : M \to MSG(h))$ is Poincaré transversal. Then I is a presheaf of $\mathbb{Z}/2$ homology Poincaré spaces over M. The presheaf is assembled to a $\mathbb{Z}/2$ Poincaré space L. The preimage of $ESG(h)$ is a spherical fibration of L which induces a normal structure compatible with the Poincaré duality.

Then

$$\begin{aligned}\mathrm{dR}(L) &= \langle V_L Sq^1 V_L, [L]\rangle \\ &= \langle V_M^2 \cdot f^*(VSq^1V \cdot U), [M]\rangle + \langle V_M Sq^1 V_M \cdot f^*(V^2 \cdot U), [M]\rangle\end{aligned}$$

So, under the assumption of Poincaré transversality for f,

$$\langle V_M^2 \cdot f^*(VSq^1V \cdot U), [M]\rangle = \langle V_M^2 \cdot f^*(U^n)^* r_1^n, [M]\rangle$$

For the general case, there exists a $\mathbb{Z}/2$ manifold P of dimension $h+5$ with a map $b : P \to MSG(h)$ such that

$$\langle b^*\beta(l^G \cdot U), [P]\rangle = 1 \in \mathbb{Z}/2$$

where β is the $\mathbb{Z}/2 \to \mathbb{Z}_{(2)}$ Bockstein. It means that the map b does not admit Poincaré transversality.

We can assume that the associated presheaf $(\mathcal{D}_b, \mathcal{E}_b)$ has vanishing σ_1^n. Otherwise, note that the de Rham invariant of the Wu manifold $SU(3)/SO(3)$ is 1. Then we may subtract $[P, b]$ by a map $S^{h+5} \to MSTOP(h)$ which is the Thom-Pontryagin construction associated to the Wu manifold $SU(3)/SO(3)$.

Lemma 4.4.9.

$$\langle V_P^2 \cdot b^*(VSq^1V \cdot U), [P]\rangle = \langle V_P^2 \cdot f^*(U^n)^* r_1^n, [P]\rangle \in \mathbb{Z}/2$$

Proof. Considering the definition of σ_1^n and the assumption that $\sigma_1^n(\mathcal{D}_b, \mathcal{E}_b) = 0$, it suffices to prove that

$$\langle V_P^2 \cdot b^*(VSq^1V \cdot U), [P]\rangle + \langle V_P Sq^1 V_P \cdot b^*(V^2 \cdot U), [P]\rangle = 0$$

According to [BM76], there exists a map $c : S^{h'+3} \to MSG(h')$ such that

$$\langle c^*(k^G \cdot U), [S^{h'+3}]\rangle = 1$$

Also recall the coproduct formula of l^G ([BM76, p. 9.1]), i.e.,

$$\Delta^*(l^G \cdot U) = (l^G \cdot U) \times (l^G \cdot U) + j_8((k^G \cdot U) \times (VSq^1V \cdot U) + (VSq^1V \cdot U) \times (k^G \cdot U))$$

where $j_8 : \mathbb{Z}/2 \to \mathbb{Z}/8$.

Considering the product formulae for $L^n_*(\mathbb{Z}, \mathbb{Z}/8)$, we have

$$\begin{aligned}
0 &= \sigma^n_0(\mathcal{D}_{b\otimes c}(P \otimes S^{h'+3}), \mathcal{E}_{b\otimes c}(P \otimes S^{h'+3})) \\
&= \langle L_{P\otimes S^{h'+3}} \cdot (b \otimes c)^*(l^G \cdot U), [P \otimes S^{h'+3}]\rangle \\
&\quad + j_8\langle V_{P\otimes S^{h'+3}} Sq^1 V_{P\otimes S^{h'+3}} \cdot (b \otimes c)^*(k^G \cdot U), P \otimes S^{h'+3}]\rangle \\
&= \langle (L_P \otimes 1) \cdot (b^*(l^G \cdot U) \otimes c^*(l^G \cdot U)), [P \otimes S^{h'+3}]\rangle \\
&\quad + j_8\langle (V_P^2 \otimes 1) \cdot (b^*(k^G \cdot U) \otimes c^*(V Sq^1 V \cdot U)), , [P \otimes S^{h'+3}]\rangle \\
&\quad + j_8\langle (V_P^2 \otimes 1) \cdot (b^*(V Sq^1 V \cdot U) \otimes c^*(k^G \cdot U)), [P \otimes S^{h'+3}]\rangle \\
&\quad + j_8\langle (V_P^2 \otimes 1 + V_P Sq^1 V_P \otimes 1) \\
&\quad \cdot (b^*(k^G \cdot U) \otimes c^*(V^2 \cdot U) + b^*(V^2 \cdot U) \otimes c^*(k^G \cdot U)), [P \otimes S^{h'+3}]\rangle \\
&= j_8(\langle V_P^2 \cdot b^*(V Sq^1 V \cdot U), [P]\rangle + \langle V_P Sq^1 V_P \cdot b^*(V^2 \cdot U), [P]\rangle)
\end{aligned}$$

□

If $f : M \to MSG(h)$ is not Poincaré transversal, modify it by the map

$$b \otimes \mathrm{pt} : P \otimes \mathbb{C}P^{2i} \to MSG(h)$$

when the dimension of M is $m = h + 4i + 1$, so that the new map is coborded to be Poincaré transversal. Therefore,

Proposition 4.4.10.

$$(U^n)^* r^n_1 = V Sq^1 V \cdot U \in \widetilde{H}^{4*+1}(MSG; \mathbb{Z}/2)$$

In particular, $(U^n)^* r^n_{1,1} = 0$.

All the above propositions complete the proof of the prime 2 part of 1.1.2.

4.5 L-theory at Odd Primes

To complete the story, let us briefly consider the odd-prime localization of L-theories and the bundle theory at odd primes.

Since the homotopy groups of $\mathbb{L}_0^n$ have only 2-primary torsion, the odd-localization of $\mathbb{L}^n$ is simply homotopy equivalent to $\mathbb{Z}$.

Sullivan ([Sul09, p. 218]) proved that, localized at odd primes, G/PL is homotopy equivalent to BSO. Since G/PL and G/TOP are differed by a $\mathbb{Z}/2$ twisting, $\mathbb{L}^q \simeq G/TOP$ is also homotopy equivalent to BSO. Therefore, the same is true for $\mathbb{L}^s$.

However, we want to use the Poincaré chain description of $\mathbb{L}$-theory to reprove the same result. Logically, the reproof is essentially the same as Sullivan's proof, which is based on the a priori invariant method for K-theory.

In the rest of this section, we only consider $\mathbb{L}^s$. The first goal is to construct an H-space map $\sigma^s_{\text{odd}} : \mathbb{L}^s_1 \to BSO^{\otimes}_{(\text{odd})}$, where the superscript $\otimes$ means that the product structure on $BSO^{\otimes}$ is induced by the tensor product of vector bundles.

Let M be a $\mathbb{Z}$ or $\mathbb{Z}/n$ manifold with a map $f : M \to \mathbb{L}^s_1$. Let $\mathcal{C}_f$ be the associated presheaf. Define

$$\sigma^s_{\text{odd}}(M, f) = \sigma^s_0(\mathcal{C}_f(M)) \in \mathbb{Z} \text{ or } \mathbb{Z}/n$$

Then the product formula of σ^s_{odd} holds, i.e.,

$$\sigma^s_{\text{odd}}((M, f) \cdot (N, g)) = \sigma^s_{\text{odd}}(M, f) \cdot \sigma^s_{\text{odd}}(N, g)$$

where N is another $\mathbb{Z}$ or $\mathbb{Z}/n$ manifold with a map $g : N \to \mathbb{L}^s_1$

In particular, we get a map

$$\sigma^s_{\text{odd}} : \Omega^{SO}_*(\mathbb{L}^s_1) \otimes_{\Omega^{SO}_*} \mathbb{Z}_{(\text{odd})} \to \mathbb{Z}_{(\text{odd})}$$

where $\Omega^{SO}(*) \to \mathbb{Z}_{(\text{odd})}$ is the signature map.

Under the same argument as [MM79, Lemma 4.26], σ^s_{odd} induces a map $\sigma^s_{\text{odd}} : \mathbb{L}^s_1 \to BSO^{\otimes}_{(\text{odd})}$. The proof of the following lemma is essentially the same as that of [MM79, Lemma 4.27]

Lemma 4.5.1.

$$\mathrm{ph}(\sigma^s_{\mathrm{odd}}) = l^s_1 \otimes \mathbb{Q} \in H^*(\mathbb{L}^s_1; \mathbb{Q})$$

Proof. Let M^m be a closed manifold with a map $f : M^{4m} \to \mathbb{L}^s_1$. Suppose M is embedded in some sphere $S^{4(m+N)}$. Then consider the diagram

$$\begin{array}{ccccccc}
M & \xrightarrow{f \times \nu_M} & \mathbb{L}^s_1 \times BSO(4N) & & & & \\
\downarrow & & \downarrow & & & & \\
S^{4(m+N)} & \longrightarrow & (\mathbb{L}^s_1)^+ \wedge MSO(4N) & \xrightarrow{\sigma^s_{\mathrm{odd}} \wedge \Delta} & BSO_{(\mathrm{odd})} \wedge BSO_{(\mathrm{odd})} & \xrightarrow{\otimes} & BSO_{(\mathrm{odd})}
\end{array}$$

Let $g : S^{4(m+N)} \to BSO_{(\mathrm{odd})}$ be the composition of the lower horizontal arrows. Then

$$\begin{aligned}
\sigma^s_{\mathrm{odd}}(M, f) = \langle g^* \, \mathrm{ph}, [S^{4(m+N)}] \rangle &= \langle f^* \, \mathrm{ph}(\sigma^s_{\mathrm{odd}}) \cdot L_M \cdot U_{\nu_M}, [S^{4(m+N)}] \rangle \\
&= \langle f^* \, \mathrm{ph}(\sigma^s_{\mathrm{odd}}) \cdot L_M, [M] \rangle
\end{aligned}$$

On the other hand,

$$\sigma^s_{\mathrm{odd}}(M, F) = \sigma^s_0(\mathcal{C}_f(M)) = \langle L_M \cdot f^* l^s_1, [M] \rangle$$

$\square$

It remains to show that $\sigma^s_{(\mathrm{odd})}$ induces an isomorphism of homotopy group. Take a generator $f : S^{4n} \to \mathbb{L}^s_0$ of $\pi_{4n}(\mathbb{L}^s_1)$ so that the signature of the assembly of the associated presheaf is 1. Then

$$1 = \sigma^s_{\mathrm{odd}}(S^{4n}) = \langle L_{S^{4n}} \cdot f^* l^s_1, [S^{4n}] \rangle = \langle f^* \, \mathrm{ph}(\sigma^s_{\mathrm{odd}}), [S^{4n}] \rangle$$

So $f \circ \sigma^s_{\mathrm{odd}} : S^{4n} \to BSO_{(\mathrm{odd})}$ is also a generator of $\pi_{4n}(BSO_{(\mathrm{odd})})$

Theorem 4.5.2. *Localized at odd primes, there is an H-space homotopy equivalence*

$$\sigma^s_{\mathrm{odd}} : \mathbb{L}^s_1 \to BSO^{\otimes}_{(\mathrm{odd})}$$

Proof. We already proved the homotopy equivalence. Note that the chain-level tensor product makes the restriction of the multiplicative structure of $\mathbb{L}^s$ to $\mathbb{L}^s_1$. The map σ^s_{odd} is an H-space map since $\mathrm{ph}(\sigma^s_{\mathrm{odd}}) = l^s_1 \otimes \mathbb{Q}$ is multiplicative. $\square$

Remark 4.5.3. The result was also proven by Taylor and Williams ([TW79, p. 192])

We can also restate the theorem in terms of presheaves.

Proposition 4.5.4. *For any presheaf $\mathcal{S}$ of* 0*-connective Poincaré symmetric chains over X, there exists an odd-prime real K-theory characteristic class $\gamma^s(\mathcal{S}) \in \widetilde{KO}(X)_{(\mathrm{odd})}$, which is invariant under bordism.*

Analogously,

Proposition 4.5.5. *For any presheaf $\mathcal{Q}$ of* 0*-connective Poincaré quadratic chains over X, there exists an odd-prime real K-theory characteristic class $\gamma^q(\mathcal{Q}) \in \widetilde{KO}(X)_{(\mathrm{odd})}$, which is invariant under bordism.*

With considerations of the L-theory orientations for spherical fibrations and TOP bundles, we reprove the following, which is also the odd-prime part of 1.1.2.

Corollary 4.5.6. *([Sul09, Theorem 6.5]) Localized at odd primes, the obstruction for lifting a spherical fibration to a TOP bundle is the existence of a real K-theory orientation.*

Chapter 5

Branched Covering and Profinite Completion

We generalize Artin-Mazur's comparison theorem (which they called the generalized Riemann existence theorem) in this chapter. Vaguely speaking, Artin-Mazur proved that the profinite completion of a complex variety can be built from all of its étale morphisms (for a precise statement, see Chapter 2). We want to see what is a good geometric analogue of this theorem.

We need to find appropriate replacements of varieties and étale morphisms, for which we use pseudomanifolds and branched coverings. Then we prove an anagolous statement like Artin-Mazur: the profinite completion of a pseudomanifold can be built from all of its branched coverings.

5.1 Pseudomanifold and Branched Covering

5.1.1 Simplicial Pseudomanifold

Definition 5.1.1. A piecewise linear pseudomanifold X of dimension n is a compact polyhedron such that it has some finite triangulation satisfying that

(1) each simplex is a face of some n-simplex;

(2) each $(n-1)$-simplex is the face of precisely two n-simplices.

Call a pseudomanifold X normal if the link of each i-simplex is connected for $i < n-1$.

Let X be a pseudomanifold of dimension n with a given triangulation T. Let T' be the barycentric subdivision of T. Each simplex of T' is uniquely represented by a sequence $(\sigma_0, \sigma_1, \cdots, \sigma_k)$, where each σ_i is a simplex of T and each σ_{i+1} is a face of σ_i.

An i-dimensional dual cone $C^i(\sigma^{n-i})$ of a simplex $\sigma^{n-i} \in T$ is defined by the union of all the closed simplices in T' like $(\sigma_0, \sigma_1, \cdots, \sigma_k = \sigma^{n-i})$.

The link $L(\sigma^{n-i})$ of σ^{n-i} is the subcomplex consisting of the simplices $(\sigma_0, \sigma_1, \cdots, \sigma_k \neq \sigma^{n-i})$ in $C^i(\sigma^{n-i})$.

The i-skeleton of X is the union of all simplices of dimension at most i and the i-coskeleton of X is the union of all dual cones of dimension at most i.

Notice that the i-skeleton intersects the $(n-i)$-coskeleton transversally.

It is obvious that the link of each codimension 1 simplex is S^0 and the link of each codimension 2 simplex is a finite number of S^1.

Proposition 5.1.1. *The link of a simplex in a pseudomanifold is also a pseudomanifold.*

Proof. Let τ^{n-i} be a simplex of the triangulation T of X. The proposition is obvious for $i = 1$. So let us assume $i \geq 2$.

Obviously, each simplex in the link of $L(\tau)$ is contained in some $(i-1)$-dimensional simplex of T' like $(\sigma_0^n, \sigma_1^{n-1}, \cdots, \sigma_{i-1}^{n-i+1})$, where τ is a face of σ_{i-1}^{n-i+1}.

Consider an $(i-2)$-simplex $(\sigma_0, \sigma_1, \cdots, \sigma_{i-2})$ of $L(\tau)$. Either σ_0 is some n-simplex α^n of X or σ_0 is some $(n-1)$-simplex β^{n-1}.

For the first case, $(\sigma_0, \sigma_1, \cdots, \sigma_{i-2}) = (\alpha^n, \cdots, \alpha^{n-j}, \alpha^{n-j-2}, \cdots, \alpha^{n-i+1})$ for some j. Since α^{n-j-2} is contained in only two faces of α^{n-j}, $(\sigma_0, \sigma_1, \cdots, \sigma_{i-2})$ is also contained in two simplices of $L(\tau)$.

For the second case, $(\sigma_0, \sigma_1, \cdots, \sigma_{i-2}) = (\beta^{n-1}, \cdots, \beta^{n-i+1})$. Because X is a pseudomanifold, β^{n-1} is the face of two simplicies in T. So $(\sigma_0, \sigma_1, \cdots, \sigma_{i-2})$ is also contained in two simplices of $L(\tau)$. □

Let B be a subpolyhedron of a pseudomanifold X. Let $C(B,X)$ be the union of simplices in T' disjoint from B. Notice that $C(B,X)$ is also the union of dual cones which are disjoint from B.

Lemma 5.1.2. *$X - B$ contracts to $C(B,X)$.*

Proof. Do induction on the dimension of dual cones, as follows. Let $CT^{(i)}$ be the i-coskeleton of T. Suppose that $X - B$ contracts to $CT^{(i-1)} - B$ already. Consider an i-dimensional dual cone $C^i(\sigma)$. If it does not intersect B, then $C^i \subset C(B,X)$. Otherwise, σ is a simplex of B and the intersection $C^i(\sigma) \bigcap B$ consists of the intersections like $C^i(\sigma) \bigcap \tau$, where τ ranges over simplices of B that contains σ. Then $C^i(\sigma) \bigcap B$ is a cone of $L(\sigma) \bigcap B$. Contract $C^i(\sigma)$ minus the cone point along the radial direction of the cone. Then $C^i(\sigma) \bigcap B$ minus the cone point contracts to $L(\sigma) \bigcap B$ and the complement of $C^i(\sigma) \bigcap B$ in $C^i(\sigma)$ contracts to $L(\sigma) - B$. □

In particular,

Corollary 5.1.3. *The complement of i-skeleton contracts to $(n-i-1)$-coskeleton.*

On the other hand, let D be a full subcomplex of T' consisting of dual cones. Let $S(D,X)$ be the union of simplicies of T that do not intersect D. Notice that each simplex is a cone over its boundary with the cone point its barycenter and the barycentric subdivision respects the cone structure. Apply the same argument for the skeletons and we get

Lemma 5.1.4. *$X - D$ contracts to $S(D,X)$.*

Corollary 5.1.5. *The complement of i-coskeleton contracts to $(n-i-1)$-skeleton.*

Definition 5.1.2. A compact polyhedron X is a pseudomanifold of dimension n with boundary ∂X if there exists a finite triangulation such that

(1) each simplex is a face of some n-simplex;

(2) each $(n-1)$-simplex is the face of at most two n-simplices.

(3) ∂X is the union of all $(n-1)$-simplicies that are not common faces of two n-simplices;

(4) ∂X is a pseudomanifold of dimension $(n-1)$.

Call $X - \partial X$ the interior of X.

Similarly, we can also define dual cones and links for a pseudomanifold with boundary.

Proposition 5.1.6. *The link of a simplex σ of codimension at least 2 in a pseudomanifold X with boundary is either a pseudomanifold with boundary if $\sigma \subset \partial X$, or a pseudomanifold otherwise.*

The proof is analogous to that of 5.1.1.

Likewise, 5.1.4 and 5.1.2 also hold for pseudomanifolds with boundary.

Recall the definition of regular neighborhoods. Let B be a full subcomplex of some triangulation T of X^n. Define a regular neighborhood $N(B, X)$ of B by $\overline{X - C(X, B)}$. Indeed, $N(B, X)$ consists of all closed dual cones that have nonempty intersetion with B.

Lemma 5.1.7. *$N(B, X)$ is a pseudomanifold with boundary of dimension n, where $\partial N(B, X)$ consists of the dual cones in $N(B, X)$ that do not intersect B, namely, $\partial N(B, X) = N(B, X) \bigcap C(B, X)$. In particular, $\partial N(B, X) = \emptyset$ iff $B = X$.*

Proof. Each simplex of $N(B, X)$ is contained in some n-simplex, since $N(B, X)$ is made of dual cones and each of them is contained in the dual cone of a vertex of B.

Each $(n-1)$-simplex in $N(B, X)$ is contained in either an n dual cone or an $(n-1)$ dual cone in $N(B, X)$. For the first case, the $(n-1)$-simplex must be contained in two n-simplices and it intersects B at the cone point. For the second case, it suffices that each $(n-1)$ dual cone of $N(B, X)$ is contained in one or two n-dual cones. A $(n-1)$ dual cone must be the dual cone of some 1-simplex of T. This 1-simplex is either in B or not in B. If it is in B, then the $(n-1)$ dual cone is contained in the two n-dual cones of the boundary of the 1-simplex and the $(n-1)$ dual cone is not in $\partial N(B, X)$. Otherwise, one vertex of the 1-simplex is in B

and the other vertex is not because B is full in T. Then the $(n-1)$ dual cone is contained in one n dual cone in B and it is in $\partial N(B, X)$. □

The argument in the proof also shows that

Lemma 5.1.8. *If $B \neq X$ and B is full, then $C(B, X)$ is also a pseudomanifold with boundary $\partial N(B, X)$.*

Lemma 5.1.9. *If B is full in X and $B \neq X$, then $\partial N(B, X)$ is collared both in $N(B, X)$ and $C(B, X)$.*

Proof. Due to the compactness, it suffices that $\partial N(B, X)$ is locally collared in both $N(B, X)$ and $C(B, X)$ ([RS72, Theorem 2.25]). Any point $x \in \partial N(B, X)$ is contained in the interior of some simplex σ^i of T. Since B is full, σ^i has some vertices in B and the other vertices are not in B. Let α be the maximal face of σ^i contained in B and let β be the maximal face of σ^i that does not intersect B. Then σ^i is the join $\alpha * \beta$. Hence a neighborhood of x in $\partial N(B, X)$ is isomorphic to $\alpha \times \beta \times C(\sigma^i)$ and the join product gives the collaring of $\alpha \times \beta \times C(\sigma^i)$ in both $N(B, X)$ and $C(B, X)$. □

5.1.2 Branched Covering

Definition 5.1.3. Let X and Y be pseudomanifolds. A k-fold branched covering map consists of a piecewise linear map $f : Y \to X$ and a closed sub-polyhedron $B \subset Y$ of codimension at least 2 such that the restriction map $Y - B \to X - f(B)$ is a k-fold covering map.

The sub-polyhedron $B \subset Y$ is called the branched locus and $Y - B$ is called the unbranched part for the branched covering.

Example 5.1.10. The identity map $X \to X$ with an arbitrary sub-polyhedron $B \subset X$ of codimension at least 2 is a branched covering map.

Proposition 5.1.11. *Let X be a pseudomanifold. Let V be a closed sub-polyhedron of codimension at least 2. Let $f_1 : Y_1 \to X - V$ be a finite covering map. Then there exists a branched covering map $f : Y \to X$ extending f_1 with the branched locus $f^{-1}(V)$.*

Moreover, among all the branched covering maps satisfying the property above, there is an initial one $f : Y \to X$ extending f_1, in the sense that for any other branched covering map $f' : Y' \to X$ extending f_1 there is a 1-fold branched covering map $g : Y \to Y'$ such that $f = g \circ f'$.

Proof. Give X a triangulation such that V is a subcomplex. We may assume the codimension of V is 2 because for other cases the argument is the same. We will inductively construct the initial branched covering $f : Y \to X$.

Let V_k be the $(n-k)$-skeleton of V, where n is the dimension of X. Suppose we already construct the 'initial' branched covering $f_{k-1} : Y_{k-1} \to X - V_{k-1}$ for for some $k > 1$.

Let $\{\sigma_i^{n-k}\}$ be the set of all $(n-k)$-simplices of V. For each σ_i^{n-k}, its link is a disjoint union of connected polyhedra $\bigsqcup_l L_{il}$. For each L_{il}, the preimage $f_{k-1}^{-1}(L_{il})$ is a disjoint union of connected polyhedra $\bigsqcup_j L'_{ilj}$. Then take the union of Y_{k-1} with $\mathrm{Int}(\sigma_{ilj}^{n-k}) \times C(L'_{ilj})$ for all i, l, j, where $C(L'_{ilj})$ is the cone of L'_{ilj}. Let Y_k be the union and extend f_{k-1} to $f_k : Y_k \to X - V_k$ by mapping each $\mathrm{Int}(\sigma_{ilj}^{n-k})$ onto $\mathrm{Int}(\sigma_i^{n-k})$. □

Definition 5.1.4. Call such an initial branched covering map $f : Y \to X$ in 5.1.11 a normal branched covering map.

Remark 5.1.12. With the same notation as in the proof, notice that the restricted map $L'_{ilj} \to L_{il}$ between components of links is also a normal branched covering. In particular, when $k = 2$, each link component L_{il} is piecewise linear homeomorphic to S^1 and the restricted map $L'_{ilj} \to L_{il}$ is a finite covering map of S^1.

Definition 5.1.5. Let X be a pseudomanifold. Define the étale site $\mathbf{S}_{\text{ét}}(X)$ of X as follows. On the category level, the objects are normal branched covering maps (f, B), while a morphism $(f, B) \to (f', B')$ consists of a commutative diagram such that $B' \subset \phi(B)$ and (ϕ, B) is also a normal branched covering map

$$\begin{array}{ccc} Y & \xrightarrow{\phi} & Y' \\ & \searrow^{f} & \downarrow{\scriptstyle f'} \\ & & X \end{array}$$

A covering of $(f : Y \to X, B)$ is a finite collection of morphisms $\{\phi_i : (Y_i, f_i, B_i) \to (Y, f, B)\}_{i \in I}$ such that $\bigcup_i \phi_i(Y_i - B_i) = Y - B$.

Remark 5.1.13. 5.1.11 implies that $\mathbf{S}_{\text{ét}}(X)$ admits all finite limits. Obviously, $\mathbf{S}_{\text{ét}}(X)$ also has all finite coproducts and $\mathbf{S}_{\text{ét}}(X)$ is connected if X is connected. $\mathbf{S}_{\text{ét}}(X)$ is also locally connected.

5.2 Riemann Existence Theorem for Pseudomanifold

In this section, we generalize Artin-Mazur's comparison theorem (or generalized Riemann existence theorem) for complex varieties to the case for pseudomanifolds. More explicitly, the Theorem 1.1.3 is the following.

Theorem 5.2.1. *(Generalized Riemann existence theorem for pseudomanifolds)*

If X is a pointed connected pseudomanifold, then $X_{\text{ét}}$ is isomorphic to the profinite completion of X in the category $\mathrm{Pro}(\boldsymbol{H}_0)$.

We will complete the proof in this section.

Lemma 5.2.2. *Let K_* be a hypercovering of $\boldsymbol{S}_{\text{ét}}(X)$. Then $\pi(K_*)$ is an object of $\mathcal{C}\boldsymbol{H}_0$, i.e., its homotopy group $\pi_n\pi(K_*)$ is a finite group for each n.*

Proof. The homotopy group $\pi_n\pi(K_*)$ for some fixed n is not affected by a change of skeletons of degree above $n+2$. Hence, we may assume K_* is isomorphic to its $n+2$-skeleton $\mathrm{Sk}_{n+2}(K_*)$.

Then there are only finitely many connected normal branched coverings over X in K_*. Let us remove the union of all branched loci from X. Let X' be the complement. Consider the restriction of branched covering maps on the preimages of X' and they are all finite covering maps of X'.

Let G be the fundamental groups of X' and let $\widehat{G}$ be the profinite completion of G. Consider the preimage of the basepoint in X'. We get a hypercovering H_* of the site $\mathrm{Fin}(\widehat{G})$ of finite continuous $\widehat{G}$-sets. Notice that each connected component of K_i corresponds to a connected component of $\mathrm{Fin}(\widehat{G})$. Hence, $\pi(K_*) = \pi(H_*)$. Then the lemma follows from 2.2.24. □

Let $\mathbf{S}_{\mathrm{cl}}(X)$ be the site of open subsets of X. To establish a map between $\mathbf{S}_{\mathrm{cl}}(X)$ and $\mathbf{S}_{\text{ét}}(X)$, we need a common refinement. Let $\mathbf{S}'(X)$ be the site consisting of finite covering maps onto some open subset of X. Then we have morphisms of pointed sites

$$\mathbf{S}_{\mathrm{cl}}(X) \leftarrow \mathbf{S}'(X) \xrightarrow{f} \mathbf{S}_{\text{ét}}(X)$$

Notice that each covering of $S'(X)$ is dominated by a covering of $S_{\mathrm{cl}}(X)$, i.e., we can find an open cover $\{U_i\}$ on V for a finite covering map $f : V \to \widetilde{V}$ such that the restriction f to each U_i is a homeomorphism. Hence, $\pi(\mathbf{S}'(X)) \to \pi(\mathbf{S}_{\mathrm{cl}}(X)) \to \mathrm{Sing}(X)$ is a weak equivalence of pro-spaces.

Let G be a finite group. Any principal bundle $\mathfrak{G}$ with fiber G over $\pi(\mathbf{S}_{\text{ét}}(X))$ is also a principal bundle over $\mathbf{S}'(X)$. So it induces a homomorphism $\pi_1(X) \to G$. On the other hand, the kernel of any homomorphism $\pi_1(X) \to G$ corresponds to a finite covering space X' of X. So the pullback principal G-bundle on X' his trivial bundle. It gives a principal bundle $\mathfrak{G}'$ with fiber G over $\pi(\mathbf{S}_{\text{ét}}(X))$. Due to 2.2.21, we get that

Lemma 5.2.3. *There is a natural bijection between* $\mathrm{Hom}(\pi_1 X, G)$ *and* $\mathrm{Hom}(\pi_1\pi(\boldsymbol{S}_{\text{ét}}(X)), G)$ *for any finite group* G.

Thus,

Corollary 5.2.4. *(Comparison of Fundamental Groups)*

$$\widehat{\pi_1 X} \simeq \pi_1\pi(\boldsymbol{S}_{\text{ét}}(X))$$

Let A be a finite abelian group. Let $\mathfrak{A}$ be a locally constant sheaf with stalk A on $\mathbf{S}_{\text{ét}}(X)$, which induces a local system $\widetilde{A}$ on X.

Proposition 5.2.5. *(Comparison of Cohomologies)*

For any q, $H^q(X;\widetilde{A}) \simeq H^q(\boldsymbol{S}_{\text{ét}}(X);\mathfrak{A})$.

Proof. We use the same symbol $\widetilde{A}$ to represent the induced locally constant sheaf on $\mathbf{S}'(X)$. Consider the Leray spectral sequence

$$H^q(\mathbf{S}_{\text{ét}}(X), R^r f_*\widetilde{A}) \Rightarrow H^{q+r}(\mathbf{S}'(X), \widetilde{A})$$

Then the goal is to show that $R^r f_*\widetilde{A} = 0$ for all $r \geq 1$.

Notice that the sheaf $R^r f_*\widetilde{A}$ over $\mathbf{S}_{\text{ét}}(X)$ is induced by the presheaf which associates to each normal branched covering map $(Y \to X, B)$ the abelian group $H^r(Y - B; \widetilde{A})$. Thus, we are left with the following statement. □

Lemma 5.2.6. *For any normal branched covering $(Y \to X, B)$ and for any element t of $H^r(Y - B; \widetilde{A})$ with $r \geq 1$, there exists a covering $\{Y_i \to Y, B_i\}$ in $\boldsymbol{S}_{\text{ét}}(X)$ such that the pullback of t vanishes in $H^r(Y_i - B_i; \widetilde{A})$ for each i.*

Proof. Since $\widetilde{A}$ is a sheaf of finite abelian groups, we can pass to a finite covering space of X such that $\widetilde{A}$ is a constant sheaf. So we will assume that $\widetilde{A}$ is a constant sheaf over $\mathbf{S}'(X)$.

When $r = 1$, an element $t \in H^1(Y - B; \widetilde{A})$ represents a homomorphism $\pi_1(Y - B) \to A$. Take the covering space of $Y - B$ corresponding to the kernel of the homomorphism. It is a finite covering space and the pullback of t to the covering space is zero. We use 5.1.11 to complete it to be a normal branched covering over Y.

When $r > 1$, it suffices to find a finite open cover $\{V_i\}$ of $Y - B$ satisfying that

(1) each V_i is a $K(G_i, 1)$ space, where each G_i has enough finite index subgroups, i.e., for any $d > 0$ there is some subgroup N of G_i such that d divides the index of N;

(2) $Y - B - V_i$ is a codimension at least 2 subpolyhedron of Y.

There is some finite covering space $\widetilde{V_i} \to V_i$ of degree n_i for each i so that the order of t divides n_i. Then $t = 0$ when pulled back to $\widetilde{V_i}$. Use 5.1.11 again to complete it to a branched covering map $Y_i \to Y$.

The claim can be deduced from the following two lemmas.

□

Lemma 5.2.7. *Any pseudomanifold Y admits a finite open cover $\{V_i\}$ such that each $Y - V_i$ is a codimension 2 closed subpolyhedron of Y and each V_i is homotopy equivalent to a finite graph.*

Proof. Give Y a finite triangulation. Let B_0 be the codimension 2 skeleton in Y and let $V_0 = Y - B_0$.

Now consider the barycentric subdivision of the triangulation of Y with respect to a choice of barycenters. Let B_1 be the codimension 2 coskeleton in Y and $V_1 = Y - B_1$. Then B_0 and B_1 intersects transversally and their intersection is a codimension 4 subpolyheadron.

Choose a second set of barycenters, disjoint from the first set, such that the dual cones of the second subdivision intersects the dual cones of the first subdivision transversally. Set B_2 to be codimension 2 coskeleton of the second barycentric subdivision in Y and let $V_2 = Y - B_2$. Then the intersection of B_0, B_1, B_2 is a codimension 6 subpolyheadron.

Keep doing this. After finite steps the intersection of B_i's is empty.

5.1.4 and 5.1.2 shows that each V_i is either homotopy equivalent to the 1-skeleton or 1-coskeleton of Y. □

Lemma 5.2.8. *Let Y be a pseudomanifold of dimension n and let B be a codimension at least 2 closed subpolyhedron of Y. Then there exists a finite open cover $\{V_i\}$ of $Y - B$ such that each $Y - B - V_i$ is a codimension 2 closed subpolyhedron of Y and each V_i is homotopy equivalent to a finite graph.*

Proof. Assume that $B \neq \emptyset$ and give Y a triangulation so that B is a full subcomplex.

With the same notation as before, we have proved that both $C(B, Y)$ and $N(B, Y)$ are pseudomanifolds of dimension n with a common boundary $\partial N(B, Y)$.

Let B_0 be the union of $\partial N(B,Y)$ and the codimension 2 skeleton of $C(B,Y)$. Let $V_0 = C(B,Y) - B_0$. 5.1.2 implies that V_0 contracts to the union of 1-dimensional coskeletons whose interiors have empty intersection with $\partial N(B,Y)$.

Let A_1 be the codimension 2 coskeleton of $C(Y,B)$. Let $B_1 = A_1 \bigcup \partial N(B,Y)$ and let $V_1 = C(B,Y) - B_1$. We have shown that $\partial N(B,Y)$ is collared in $C(B,Y)$. With the same argument, the pair $A_1 \bigcap \partial N(B,Y)$ is also collared in A_1. Hence the pair $(C(B,Y), V_1)$ contracts, along the collar, to a subspace pair which is piecewise linear homeomorphic to $(C(B,Y), C(B,Y) - A_1)$. 5.1.4 implies that $C(B,Y) - A_1$ contracts to the 1 skeleton of $C(B,Y)$.

Now apply the same argument as the previous proof. We can vary the barycenters of Y' to create finitely many B_i's so that the intersection of B_i's is empty. Let $V_i = Y' - B_i$.

It suffices to prove that $Y - B$ is piecewise-linear isomorphic to $C(B,Y) - \partial N(B,Y)$ and then the proof is completed. One can keep taking barycentric subdivision and get a sequence of regular neighborhoods $N(B,Y) = N_0(B,Y) \supset N_1(B,Y) \supset \dots$ and a sequence of the closure of the complement $C(B,Y) = C_0(B,Y) \subset C_1(B,Y) \subset \dots$. Let $\partial N_i(B,Y) \times [-1,0]$ be a collar in $C_i(B,Y)$ and $\partial N_i(B,Y) \times [0,1]$ be a collar in $N_i(B,Y)$. We may assume any pair of collars in i and $i+1$ is disjoint. Using [RS72, Theorem 3.8], there is a sequence of piecewise linear isomorphism $f_i : Y \to Y$, such that f_i fixes $N_{i+1}(Y,B) - \partial N_{i+1}(B,Y) \times [0,1]$ and $C_i(Y,B) - \partial N_i(B,Y) \times [-1,0]$ and f_i maps $N_i(B,Y) \times [-1,0]$ onto $C_{i+1}(Y,B) - (C_i(Y,B) - \partial N_i(B,Y) \times [-1,0])$. One can further modify inductively on each f_i such that the composition f_i fixes $f_{i-1} \circ \cdots \circ f_0 : \partial N(B,Y) \times [-1, -\frac{1}{2^i}]$. Then the map $\cdots \circ f_i \circ \cdots \circ f_0$ gives the piecewise-linear isomorphism $C(B,Y) - \partial N(B,Y) \to Y - B$, since the image any point becomes stable after some finite compositions. □

Finally, let us finish the proof of the main theorem.

Proof of Theorem 5.2.1. With the previous lemmas and 2.2.22, we have proved that the morphism of sites $\mathbf{S}_{\text{cl}}(X) \leftarrow \mathbf{S}'(X) \xrightarrow{f} \mathbf{S}_{\text{ét}}(X)$ induces a $\mathcal{C}$-equivalence between $X_{\text{ét}}$ and X, i.e., a weak equivalence between $X_{\text{ét}}$ and $\widehat{X}$. It remains to prove that the sites $\mathbf{S}_{\text{cl}}(X)$ and $\mathbf{S}_{\text{ét}}(X)$ are both of local dimension at most n, where n is the dimension of X. This is obvious for the site $\mathbf{S}_{\text{cl}}(X)$.

Now let $(f : Y \to X, B)$ be any normal branched covering map and let $\mathfrak{A}$ be a locally constant sheaf on $\mathbf{S}_{\text{ét}}(X)$ with stalk an finite abelian group A. It is equivalent to a local system $\widetilde{A}$ on X. We have shown that $R^r f_* \widetilde{A} = 0$ for r positive. The Leray spectral sequence implies that $H^*_{\mathbf{S}_{\text{ét}}(X)}((Y, B); \mathfrak{A}) \cong H^*_{\mathbf{S}'(X)}((Y, B); \widetilde{A})$. But $H^*_{\mathbf{S}'(X)}((Y, B); \widetilde{A}) \cong H^*(Y - B; \widetilde{A})$. So it is 0 for degree larger than n. $\square$

Chapter 6

Galois Symmetry on Simply Connected Topological Manifolds

In this chapter, we study the Galois symmetry on the underlying topological manifold structures of smooth complex varieties defined over $\overline{\mathbb{Q}}$. Although the result was stated and known by Sullivan before 1970 ([Sul71][Sul09, p. 271]), it does no harm to write down the full proof for a precise statement carefully. In the last section, we would like to share some ideas of our ongoing project on giving a geometric interpretation of this known result for the Galois symmetry.

6.1 Simply Connected Profinite Manifold Structure

6.1.1 Simply Connected p-adic Poincaré Space

In this section, we define simply connected Poincaré spaces in the p-adic profinite complete sense.

Definition 6.1.1. A connected space X is p-adic simply connected if $\widehat{\pi_1(X)}_p = 0$. A p-adic simply connected space X is p-complete and of finite p-type if $\widehat{\pi_1(X)}_p = 0$ and each homotopy group $\pi_i(X)$ is a finitely generated $\widehat{\mathbb{Z}}_p$-module.

Recall that there is a canonical spherical fibration on each Poincaré finite complex, which is also called the Spivak normal fibration ([Spi67, Theorem A]). We will use the normal spherical fibrations to define p-adic Poincaré Spaces.

Also recall that the classifying space for the oriented $\widehat{S^{n-1}}_p$-fibration theory is the p-adic profinite completion $\widehat{BSG(n)}_p$ of $BSG(n)$ ([Sul09, Theorem 4.2]), that is, any p-adic spherical fibration $\widehat{S^{n-1}}_p \to S(\gamma) \xrightarrow{X}$ corresponds to a map $X \xrightarrow{\gamma} \widehat{BSG(n)}_p$ unique up to homotopy. We still call the mapping cone $Th(\gamma)$ of γ the Thom space of the p-adic spherical fibration.

Definition 6.1.2. A connected, p-adic simply connected, p-complete, of finite p-type space X is p-adic normal of dimension m if it has a p-adic stable spherical fibration $\gamma : X \to \widehat{BSG(n)}_p$ and a map $f : \widehat{S^{N+m}}_p \to Th(\gamma)$, where $Th(\gamma)$ is the Thom space of γ.

The map f induces a class $[X]_p \in H_m(X; \widehat{\mathbb{Z}}_p)$ by the Thom isomorphism $[X]_p = f_*[\widehat{S^{N+m}}_p] \cap U_\gamma$, where U_γ is the Thom class of γ (with coefficient $\widehat{\mathbb{Z}}_p$).

Definition 6.1.3. A p-adic normal space X is p-adic Poincaré if

$$- \cap [X]_p : H^{m-*}(X; \widehat{\mathbb{Z}}_p) \xrightarrow{\cong} H_*(X; \widehat{\mathbb{Z}}_p)$$

In this case the p-adic normal structure is also called the p-adic Spivak normal fibration over X.

For the p-adic case, there is an analogous relation between the existence of a normal spherical fibration and homological Poincaré duality, like [Spi67].

Let X be a p-adic simply connected, finite simplicial complex. Then the p-adic completion $\widehat{X}_p$ is obviously p-complete and of finite p-type.

Proposition 6.1.1. *With the same assumption like above. If there is a class* $[X]_p \in H_m(X; \widehat{\mathbb{Z}}_p)$ *such that* $- \cap [X]_p : H^{m-*}(X; \widehat{\mathbb{Z}}_p) \xrightarrow{\cong} H_*(X; \widehat{\mathbb{Z}}_p)$*, then* $\widehat{X}_p$ *has a p-adic Spivak normal fibration, which is compatible with the cap product.*

Proof. X can be embedded into an Euclidean space $\mathbb{R}^{m+N}$ with N large enough. Let W be a regular neighborhood of X and then W is a smooth manifold with boundary ∂W. Then the inclusion $X \to W$ is a homotopy equivalence. By general position arguments, if N is large enough ($N \geq 3$), then $\pi_1(\partial W) \cong \pi_1(W) = 0$.

W is homotopy equivalent to X. Notice two isomorphisms

$$- \cap [W, \partial W] : H^{m+N-*}(W, \partial W; \widehat{\mathbb{Z}}_p) \to H_*(W; \widehat{\mathbb{Z}}_p)$$

$$- \cap [X]_p : H^{m-*}(X; \widehat{\mathbb{Z}}_p) \xrightarrow{\simeq} H_*(X; \widehat{\mathbb{Z}}_p)$$

There exists $U \in H^N(W; \widehat{\mathbb{Z}}_p)$ such that $U \cap [W, \partial W] = [X]_p$ and

$$- \cup U : H^*(W; \widehat{\mathbb{Z}}_p) \to H^{*+N}(W, \partial W; \widehat{\mathbb{Z}}_p)$$

is an isomorphism.

With the same argument of [Bro72, Lemma I.4.3], the homotopy fiber of the map $\widehat{\partial W}_p \to \widehat{W}_p$ is a p-adic simply connected homology sphere with coefficient $\widehat{\mathbb{Z}}_p$. Hence, the homotopy fiber is homotopy equivalent to $\widehat{S}_p^{N-1}$. The p-adic spherical fibration $\widehat{\partial W}_p \to \widehat{W}_p \simeq \widehat{X}_p$ and the natural quotient map $\widehat{S^{m+N}}_p \to \widehat{W}_p/\widehat{\partial W}_p$ form the data of the p-adic Spivak normal fibration. □

Proposition 6.1.2. *If X is a simply connected finite simplicial complex with a $\mathbb{Z}/p$-coefficient Poincaré duality induced by some class $[X]_{\mathbb{Z}/p} \in H_m(X; \mathbb{Z}/p)$, then $\widehat{X}_p$ is a p-adic Poincaré space.*

Proof. It is similar to the previous one with the replacement of coefficient $\widehat{\mathbb{Z}}_p$ by $\mathbb{Z}/p$. Again embed X in an Euclidean space and take a regular neighborhood W of X. Then the homotopy fiber of the map $\widehat{\partial W}_p \to \widehat{W}_p$ is a simply connected homology sphere with coefficient $\mathbb{Z}/p$, hence it is also a p-adic sphere. Then the p-adic spherical fibration $\widehat{\partial W}_p \to \widehat{W}_p \simeq \widehat{X}_p$ and $\widehat{S^{m+N}}_p \to \widehat{W}_p/\widehat{\partial W}_p$ induce a class $[X]_p \in H_m(X; \widehat{\mathbb{Z}}_p)$ so that its $\mathbb{Z}/p$ reduction is $[X]_{\mathbb{Z}/p}$. Finally, the $\mathbb{Z}/p$-coefficient Poincaré duality implies the $\widehat{\mathbb{Z}}_p$-coefficient Poincaré duality by the Bockstein spectral sequence, namely, the Poincaré duality map $C^{m-*}(X; \mathbb{Z}/p) \to C_*(X; \mathbb{Z}/p)$ induces an isomorphism of the Bockstein spectral sequences of two chain complexes $C^{m-*}(X; \widehat{\mathbb{Z}}_p)$ and $C_*(X; \widehat{\mathbb{Z}}_p)$. □

6.1.2 Manifold Structure at Odd Primes

The definition of p-adic completion of the structure set is based on the ideas of Sullivan in [Sul09][Sul71].

Let p be an odd prime. Recall from the previous chapters that $\widehat{G/TOP}_p \simeq \widehat{BSO^{\otimes}}_p$. We propose the following definition.

Definition 6.1.4. Let X be a p-adic simply connected, p-adic Poincaré space of dimension $m \geq 5$. Call X a p-adic formal manifold if its p-adic Spivak normal fibration γ has a $\widehat{KO}_p$ orientation

$$\Delta_X : Th(\gamma) \to \widehat{BSO^{\otimes}}_p$$

The $\mathbb{Q}_p$-coefficient Pontryagin character of Δ_X induces the $\mathbb{Q}_p$-coefficient L-genus of X by $L_X \cdot U_\gamma = \mathrm{ph}(\Delta_X)$, where U_γ is the Thom class of γ.

Definition 6.1.5. Let X be a p-adic simply connected p-adic formal manifold of dimension $m \geq 5$. A p-adic homotopy manifold structure over X is a pair (ϕ, β) consisting of a map $\phi : X \to \widehat{BSO^{\otimes}}_p$ and an element $\beta \in \widehat{\mathbb{Z}}_p^{\times}$ such that

$$\langle \mathrm{ph}(\phi) \cdot \frac{\mathrm{ph}(\Delta_X)}{U_\gamma}, [X]_p \rangle = \beta^{\frac{m}{2}} \langle \frac{\mathrm{ph}(\Delta_X)}{U_\gamma}, [X]_p \rangle$$

if dimension m is divisible by 4.

Remark 6.1.3. The appearance of $\beta \in \widehat{\mathbb{Z}}_p^{\times}$ in the definition is due to the ambiguity of the fundamental class $[X]_p$ induced by some multiplicative unit in $\widehat{\mathbb{Z}}_p$. For the future construction of abelianized Galois action on p-adic manifold structures, the term β is necessary.

Definition 6.1.6. Define the p-adic structure set $\mathbf{S}^{TOP}(X)_p^{\wedge}$ over a simply connected p-adic formal manifold X of dimension $m \geq 5$ by the set of all p-adic homotopy manifold structures over X.

Example 6.1.4. Let M be a p-adic simply connected topological manifold of dimension $m \geq 5$. Then M is obviously a p-adic formal manifold, namely, p-adic completion of M

together with the p-adic information for why M is a manifold specify a p-adic formal manifold. Furthermore, any p-adic homotopy equivalence from another topological manifold $N \to M$ is a p-adic homotopy manifold structure over M. $\square$

Next, we construct the abelianized Galois action $\widehat{\mathbb{Z}}_p^\times$ on $\mathbf{S}^{TOP}(X)_p^\wedge$.

Definition 6.1.7. Let X be a p-adic simply connected, p-adic formal manifold of dimension $m \geq 5$. Define the abelianized Galois action $\widehat{\mathbb{Z}}_p^\times$ at prime p on the p-adic structure set $\mathbf{S}^{TOP}(X)_p^\wedge$ by

$$(\sigma_p)_* : \mathbf{S}^{TOP}(X)_p^\wedge \to \mathbf{S}^{TOP}(X)_p^\wedge$$

$$(\phi, \beta) \to (\psi^{\sigma_p^{-1}} \phi \cdot \frac{\psi^{\sigma_p^{-1}} \Delta_X}{\Delta_X}, \beta \sigma_p^{-1})$$

where $\sigma_p \in \widehat{Z}_p^\times$.

6.1.3 Manifold Structure at Prime 2

In the footnote of [Sul09, p. 258], Sullivan had some discussions of the 2-adic case. We complete his discussions in this section.

From the previous chapters, there are characteristic classes $k^G \in H^{4*+3}(X; \mathbb{Z}/2)$ and $l^G \in H^{4*}(X; \mathbb{Z}/8)$ for a 2-adic spherical fibration $\gamma : X \to \widehat{BSG(N)}_2$, such that k^G and βl^G obstruct the existence of 2-adic topological bundle structure on γ, where β is the $\mathbb{Z}/8 \to \widehat{\mathbb{Z}}_2$ Bockstein.

Definition 6.1.8. Let X be a 2-adic simply connected, 2-adic Poincaré space of dimension $m \geq 5$. Call X a 2-adic formal manifold if the characteristic class k^G of its 2-adic Spivak normal fibration vanishes and the $\mathbb{Z}/8$-class l^G has a lifting $L_X \in H^{4*}(X; \widehat{\mathbb{Z}}_2)$.

Definition 6.1.9. Let X be a 2-adic simply connected, 2-adic formal manifold of dimension $m \geq 5$. A 2-adic homotopy manifold structure over X is a pair (l, k) consisting of graded classes $l \in H^{4*}(X; \widehat{\mathbb{Z}}_2)$ and $k \in H^{4*+2}(X; \mathbb{Z}/2)$, where we exclude the degree m class.

Remark 6.1.5. The surgery obstruction for any homotopy equivalence maps between manifolds $M \to N$ vanishes, which indicates the necessity to exclude the degree m class. One can also view 2.1.3 as an alternatively equivalent reason.

Definition 6.1.10. Define the 2-adic structure set $\mathbf{S}^{TOP}(X)^\wedge_2$ over a simply connected 2-adic formal manifold X of dimension $m \geq 5$ by the set of all 2-adic homotopy manifold structures over X.

Example 6.1.6. Like the odd prime case. If M is a 2-adic simply connected topological manifold of dimension $m \geq 5$, then M is a 2-adic formal manifold. Moreover, any 2-adic homotopy equivalence from another topological manifold $N \to M$ is a 2-adic homotopy manifold structure over M. □

Next, we construct the abelianized Galois action on $\mathbf{S}^{TOP}(X)^\wedge_2$.

Notice that there is a compatible Adams operation on the ordinary cohomology. Namely, let $(\sigma_p)_p \in \prod_p \widehat{\mathbb{Z}}_p^\times$ represent the abelianization of $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Define the cohomological Adams operation ψ_H^σ on $H^{2n}(-;\widehat{\mathbb{Z}}_p)$ by multiplication with $(\sigma_p)^n$.

Definition 6.1.11. Let X be a 2-adic simply connected 2-adic formal manifold of dimension $m \geq 5$. Define the abelianized Galois action $\widehat{\mathbb{Z}}_2^\times$ at prime 2 on the 2-adic structure set $\mathbf{S}^{TOP}(X)^\wedge_2$ by a map

$$(\sigma_2)_* : \mathbf{S}^{TOP}(X)^\wedge_2 \to \mathbf{S}^{TOP}(X)^\wedge_2$$

so that

$$(1 + 8 \cdot (\sigma_2)_* l) \cdot L_X = (1 + 8 \cdot \psi_H^{\sigma_2^{-1}} l) \cdot \psi_H^{\sigma_2^{-1}} L_X$$

$$(\sigma_2)_* k = k + k_X^{\sigma_2^{-1}}$$

where $k_X^{\sigma_2^{-1}}$ is defined as follows.

Remark 6.1.7. The change of l was already suggested by Sullivan in [Sul71], which agrees with our definition.

One consequence of the Adams conjecture is that the map $\widehat{BU}_2 \xrightarrow{\psi^{\sigma_2}-1} \widehat{BU}_2$ factors through $\widehat{G/U}_2$ naturally. Hence we have a map

$$f_{\sigma_2} : \widehat{BU}_2 \to \widehat{G/U}_2 \to \widehat{G/TOP}_2$$

Define $k_2^\sigma = f_{\sigma_2}^* k^q$, where $k^q \in H^{4*+2}((G/TOP)_2^\wedge; \mathbb{Z}/2) \cong H^{4*+2}((\mathbb{L}^q))_2^\wedge; \mathbb{Z}/2)$ is the Kervaire class defined previously.

Since k^{σ_2} is a combination of Stiefel-Whitney classes, it is also defined for spherical fibrations, i.e., $k^{\sigma_2} \in H^{4*+2}(\widehat{BSG(N)}_2; \mathbb{Z}/2)$.

Let $\gamma : X \to \widehat{BSG(N)}_2$ be the Spivak normal fibration and define $k_X^{\sigma_2} = \gamma^* k^{\sigma_2} \in H^{4*+2}(X; \mathbb{Z}/2)$.

In fact, there is an algorithm to calculate the graded class k^{σ_2}.

Lemma 6.1.8. *(Additivity of k^{σ_2})*
Let $\Delta : \widehat{BU}_2 \times \widehat{BU}_2 \to \widehat{BU}_2$ be the H-space product induced by the Whitney sum. Then $\Delta^* k^{\sigma_2} = k^{\sigma_2} \times 1 + 1 \times k^{\sigma_2}$.

Proof. Let $\alpha \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be the element so that its abelianization is $\sigma = \prod_p \sigma_p \in \widehat{\mathbb{Z}}^\times \cong \prod_p \widehat{\mathbb{Z}}_p^\times$.

We only consider the case when $\sigma_p = 1$ except for $p \neq 2$ and σ_2 is an actual integer. It suffices that the induced map $f_\sigma : \widehat{BU} \to \widehat{G/U}$ induced by the Adams conjecture is an H-space map.

Recall that $\psi^\sigma : \widehat{BU}_2 \to \widehat{BU}_2$ is the stablization of the étale homotopy equivalence induced the algebraic isomorphism $\alpha : Gr_n(\mathbb{C}^N) \to Gr_n(\mathbb{C}^N)$ for some Galois automorphism $\alpha \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. After passing N to ∞, we have $\alpha : \widehat{BU(n)} \to \widehat{BU(n)}$.

Notice that the unstable Whitney sum $\Delta : \widehat{BU(n)} \times \widehat{BU(m)} \to \widehat{BU(n+m)}$ respects the Galois action, since it is induced from the algebraic map $Gr_n(\mathbb{C}^N) \times Gr_m(\mathbb{C}^M) r \to Gr_{n+m}(\mathbb{C}^{N+M})$.

The proof of the Adams conjecture in [Sul09, p. 158] is deduced from two facts (indeed, one needs to unravel the mathematical diagrams in terms of the inertia lemma [Sul09, p. 99]).

The first fact is that $BU(n-1) \to BU(n)$ is the universal spherical fibration of a rank n vector bundle. The second is that the following diagram commutes.

$$\begin{array}{ccc}
\widehat{BU(n-1)} & \xrightarrow{\alpha} & \widehat{BU(n-1)} \\
\downarrow & & \downarrow \\
\widehat{BU(n)} & \xrightarrow{\alpha} & \widehat{BU(n)}
\end{array}$$

Now consider the diagram

$$\begin{array}{ccc}
\widehat{BU(n+m-1)} & \xrightarrow{\alpha} & \widehat{BU(n+m-1)} \\
\downarrow & & \downarrow \\
\widehat{BU(n+m)} & \xrightarrow{\alpha} & \widehat{BU(n+m)}
\end{array}$$

It suffices that the pullback of this diagram along the H-space map $\Delta : \widehat{BU(n)} \times \widehat{BU(m)} \to \widehat{BU(n+m)}$ is equivalent, up to homotopy, to the following diagram, where p_1, p_2 are the projection of $\widehat{BU(n)} \times \widehat{BU(m)}$ onto the two factors and $*$ is the fiberwise join product.

$$\begin{array}{ccc}
p_1^*\widehat{BU(n-1)} * p_2^*\widehat{BU(m-1)} & \xrightarrow{\alpha * \alpha} & p_1^*\widehat{BU(n-1)} * p_2^*\widehat{BU(m-1)} \\
\downarrow & & \downarrow \\
\widehat{BU(n)} \times \widehat{BU(m)} & \xrightarrow{\alpha} & \widehat{BU(n)} \times \widehat{BU(m)}
\end{array}$$

It is left to prove that the following diagram commutes

$$\begin{array}{ccc}
p_1^*\widehat{BU(n-1)} * p_2^*\widehat{BU(m-1)} & \xrightarrow{\alpha * \alpha} & p_1^*\widehat{BU(n-1)} * p_2^*\widehat{BU(m-1)} \\
\downarrow & & \downarrow \\
\widehat{BU(n+m-1)} & \xrightarrow{\alpha} & \widehat{BU(n+m-1)}
\end{array} \tag{6.1.1}$$

Indeed, the map $p_1^*\widehat{BU(n-1)} * p_2^*\widehat{BU(m-1)} \to \widehat{BU(n+m-1)}$ is realized by a map $p_1^*BU(n-1) * p_2^*BU(m-1) \to BU(n+m-1)$ as follows. Each element of $BU(n-1)$ can be uniquely written as a pair of subspaces $V_1^{n-1} \subset V_2^n$ in $\mathbb{C}^\infty$ and the map $BU(n-1) \to BU(n)$ takes $V_1^{n-1} \subset V_2^n$ to V_2. Now take an element $W_1^{m-1} \subset W_2^m$ in $BU(m-1)$. Let $V^\perp$ be the perpendicular 1-dimensional complementary of $V_1^{n-1} \subset V_2^n$ and the same for $W^\perp$. There is a unit circle $\{(e^{i\phi}, 0) \in V^\perp \oplus W^\perp\}$ in $V^\perp$. Similarly $\{(0, e^{i\phi})\}$ in $W^\perp$. There is a family of 1-dimensional subspaces $\{\mathbb{C}_t\}_{t\in I}$ in $V^\perp \oplus W^\perp$ whose unit circles are $\{(te^{i\phi}, \sqrt{1-t^2}e^{i\phi})\}$. Notice that $p_1^*BU(n-1) * p_2^*BU(m-1)$ (over $BU(n)\times BU(m)$) is a quotient of $BU(n-1)\times BU(m-1)\times I$. So the map $p_1^*BU(n-1)*p_2^*BU(m-1)\to BU(n+m-1)$ is induced by $BU(n-1)\times BU(m-1)\times I \to BU(n+m-1)$, which maps $(V_1^{n-1}\subset V_2^n, W_1^{m-1}\subset W_2^m)$ to $(V_1\oplus W_1\oplus \mathbb{C}_t)\subset (V_2\oplus W_2)$.

Moreover, the map $BU(n-1)\times BU(m-1)\times I\to BU(n+m-1)$ is homotopic to the stablization of a map $f : Gr_{n-1}(\mathbb{C}^N)\times Gr_{m-1}(\mathbb{C}^M)\times I \to Gr_{m+n-1}(\mathbb{C}^{N+M+2})$ with a similar construction like above. Notice that $\mathbb{C}^{N+M+2} = \mathbb{C}^2\oplus\mathbb{C}^N\oplus\mathbb{C}^M$. There are two unit circles in the axes of $\mathbb{C}^2$, namely $\{(e^{i\phi},0)\}$ and $\{(0,e^{i\phi})\}$. Then there is a family of 1-dimensional subspaces $\{\mathbb{C}'_t\}_{t\in I}$ of $\mathbb{C}^2$, whose unit circles are $\{(te^{i\phi},\sqrt{1-t^2}e^{i\phi})\}$. Given a subspace V^{n-1} in $\mathbb{C}^N$ and a subspace W^{m-1} in $\mathbb{C}^M$, $f_t(V,W) = \mathbb{C}'_t\oplus V\oplus W$ in $\mathbb{C}^{N+M+2}$.

Indeed, the map $f : Gr_{n-1}(\mathbb{C}^N)\times Gr_{m-1}(\mathbb{C}^M)\times I\to Gr_{m+n-1}(\mathbb{C}^{N+M+2})$ can be extended to a map $Gr_{n-1}(\mathbb{C}^N)\times Gr_{m-1}(\mathbb{C}^M)\times Gr_1(\mathbb{C}^2)\to Gr_{m+n-1}(\mathbb{C}^{N+M+2})$ induced by the direct sum of subspaces. Under the $Gr_1(\mathbb{C}^2)\cong \mathbb{C}P^1$, we embed I as the half real line $[0,\infty]$ in $\mathbb{C}P^1$.

Since $Gr_{n-1}(\mathbb{C}^N)\times Gr_{m-1}(\mathbb{C}^M)\times \mathbb{C}P^1\to Gr_{m+n-1}(\mathbb{C}^{N+M+2})$ is an algebraic map defined over $\mathbb{Z}$, we have the following commutative diagram

$$\begin{array}{ccc}
\widehat{BU(n-1)}\times\widehat{BU(m-1)}\times\widehat{\mathbb{C}P^1} & \xrightarrow{\alpha\times\alpha\times\alpha} & \widehat{BU(n-1)}\times\widehat{BU(m-1)}\times\widehat{\mathbb{C}P^1} \\
\downarrow & & \downarrow \\
\widehat{BU(n+m-1)} & \xrightarrow{\alpha} & \widehat{BU(n+m-1)}
\end{array}$$

However, notice that the map α on $\widehat{\mathbb{C}P^1}$ is in fact homotopic to the completion of the map $z\to z^\sigma$ on $\mathbb{C}P^1$, since the homotopy classes of self homotopy equivalences of $\widehat{\mathbb{C}P^1}$ is determined by the induced group homomorphism on $H^2(\widehat{\mathbb{C}P^1};\widehat{\mathbb{Z}})$.

But under the embedding $[0,\infty]\subset\mathbb{C}P^1$, the restriction of the map $z\to z^\sigma$ to $[0,\infty]$ is homotopy equivalent the identity map. Hence, we get the following commutative diagram

$$\begin{array}{ccc}
\widehat{BU(n-1)}\times\widehat{BU(m-1)}\times I & \xrightarrow{\alpha\times\alpha\times 1} & \widehat{BU(n-1)}\times\widehat{BU(m-1)}\times I \\
\downarrow & & \downarrow \\
\widehat{BU(n+m-1)} & \xrightarrow{\alpha} & \widehat{BU(n+m-1)}
\end{array}$$

Passing to the quotient of the spaces in the upper horizontal arrow, it is exactly the diagram 6.1.1. □

Example 6.1.9. The calculation of the class k^{σ_2} on the stable normal bundle $\nu_{\mathbb{C}P^N}$ of $\mathbb{C}P^N$ with N even was also suggested by Sullivan in the footnote of [Sul09, p. 258]. The additivity of k^{σ_2} implies that $k^{\sigma_2}_{\mathbb{C}P^N} = k^{\sigma_2}(T\mathbb{C}P^N) = k^{\sigma_2}(\nu_{\mathbb{C}P^N})$.

Let $\alpha \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be the element with its abelianization $\sigma \in \widehat{\mathbb{Z}}^\times$. Further assume that σ is represented by an integer.

The Galois automprhism α on $\mathbb{C}P^1$ is profinitely homotopy equivalent to the map $[x_0, x_1] \to [x_0^\sigma, x_1^\sigma]$ ([Sul09, Proposition 5.3]). Any homotopy class of self étale homotopy equivalence of $\mathbb{C}P^N$ is determined by a map on $H^2((\widehat{\mathbb{C}P^N}; \widehat{\mathbb{Z}})$. Hence, the Galois automorphism α on $\widehat{\mathbb{C}P^N}$ is homotopic to the map $f_\sigma([x_0, \cdots, x_N]) = [x_0^\sigma, \cdots, x_N^\sigma]$ (also see [Sul09, Corollary 5.4]). Let f_{σ_2} be the 2-adic part of f_σ, namely, $f_{\sigma_2}([x_0, \cdots, x_N]) = [x_0^{\sigma_2}, \cdots, x_N^{\sigma_2}]$

As in [Sul96, Theorem 9], the element in $\mathbf{S}^{TOP}(\mathbb{C}P^N)$ is determined by the 'splitting invariants' on the submanifolds $\mathbb{C}P^n$ for $n = 1, 2, \cdots, N-1$. As a result, the associated Kervaire class k^{σ_2} is determined by the 2-adic Kervaire invariant of f_{σ_2} on $\mathbb{C}P^n$ for n odd and $n \geq 3$, namely, the Kervaire invariant is $\langle k^{\sigma_2}, \mathbb{C}P^n \rangle$.

Since the transversal preimage of $\mathbb{C}P^n$ can be made into a complete intersection of several degree σ_2 hypersurfaces, by Lefschetz's theorem we know that $H_i(f_{\sigma_2}^{-1}(\mathbb{C}P^n); \mathbb{Z}/2) \to H_i(\mathbb{C}P^n; \mathbb{Z}/2)$ is an isomorphism for $i \neq n$. [Woo75][Woo79][Bro79] show that the Kervaire invariant of a complete intersection V^k in a complex projective space obstructs to finding a symplectic basis α_i for $H_k(V; \mathbb{Z}/2)$ such that V can be written as the connected sum of a manifold with the same homology like $\mathbb{C}P^k$ and several $S^k \times S^k$ indexed by α_i. So their Kervaire invariant of $f_{\sigma_2}^{-1}(\mathbb{C}P^n)$ is exactly the Kervaire invariant for the map f_{σ_2}, namely, the obstruction to finding some surgery process on $f_{\sigma_2}^{-1}(\mathbb{C}P^n)$ such that its $\mathbb{Z}/2$-homology is isomorphic to that of $\mathbb{C}P^n$.

When $n \neq 3, 7$, the Kervaire invariant of $f_{\sigma_2}^{-1}(\mathbb{C}P^n)$ is the modified Legendre symbol valued in $\mathbb{Z}/2$, i.e.,

$$\langle k^{\sigma_2}_{\mathbb{C}P^N}, [\mathbb{C}P^n]\rangle = K(f_{\sigma_2}|_{f_{\sigma_2}^{-1}(\mathbb{C}P^n)}) = \left(\frac{2}{\sigma_2}\right) = \begin{cases} 0 & \text{if } \sigma_2 \equiv \pm 1 \pmod 8 \\ 1 & \text{if } \sigma_2 \equiv \pm 3 \pmod 8 \end{cases}$$

Hence, if $\omega \in H^2(\mathbb{C}P^N; \mathbb{Z}/2)$ is the generator, then the n-th component of $k^{\sigma_2}_{\mathbb{C}P^N}$ is $\left(\frac{2}{\sigma_2}\right)\omega^n$.

When $n = 3, 7$, the Kervaire invariant vanishes.

For $n = 1$, we need to use an alternative definition for the Kervaire invariant. One can homotope the map f_{σ_2} such that $f_{\sigma_2}^{-1}(\mathbb{C}P^1 - \text{pt}) = f_{\sigma_2}^{-1}(\mathbb{C}P^1) - \text{pt}$. Let ν be the normal bundle of $\mathbb{C}P^1$ in $\mathbb{C}P^N$. Choose a framing on $\nu|_{\mathbb{C}P^1 - \text{pt}}$, namely, a map $\mathbb{C}P^1 - \text{pt} \to SO(2N-2)$. It induces a framing on $f_{\sigma_2}^{-1}(\mathbb{C}P^1) - \text{pt}$, namely, $f_{\sigma_2}^{-1}(\mathbb{C}P^1) - \text{pt} \to \mathbb{C}P^1 - \text{pt} \to SO(2N-2)$. We need to check whether the framed manifold $f_{\sigma_2}^{-1}(\mathbb{C}P^1) - \text{pt}$ is 0 or not in the almost framed bordism group P_2. Notice that ν has a complex structure, so we can choose a framing which factors through $SU(N-1)$, i.e., $\mathbb{C}P^1 - \text{pt} \to SU(N-1) \to SO(2N-2)$. So the framing on $f_{\sigma_2}^{-1}(\mathbb{C}P^1)$ also factors through $SU(N-1)$. However, $\pi_1(SU(N-1)) = 0$, so the framing on $f_{\sigma_2}^{-1}(\mathbb{C}P^1)$ has no twisting. That is, the Kervaire invariant on $f_{\sigma_2}^{-1}(\mathbb{C}P^1)$ is 0. So we proved that $\langle k^{\sigma_2}_{\mathbb{C}P^N}, [\mathbb{C}P^1]\rangle = 0$. $\square$

Let γ be the universal complex line bundle on $\mathbb{C}P^{2N}$. Notice that the normal bundle $\nu_{\mathbb{C}P^{2N}}$ is isomorphic to $(2N+1)\gamma^*$, where γ^* is the complex dual of γ. The additivity of k^{σ_2} implies that $k^{\sigma_2}_{\mathbb{C}P^{2N}} = k^{\sigma_2}(\gamma)$. In particular, $k^{\sigma_2}(\gamma)$ is irrelavant to N and we can let N tend to infinity.

Let $x_1, x_2, \cdots$ (of degree 2) be the roots of the Stiefel-Whitney classes induced from $BU(1) \times BU(1) \times \cdots \to BU$. Again, by the additivity of k^{σ_2} class, we can write

$$k^{\sigma_2} = k^{\sigma_2}_1(x_1 + x_2 + \cdots) + k^{\sigma_2}_3(x_1^3 + x_2^3 + \dots) + \dots$$

where each $k^{\sigma_2}_i \in \mathbb{Z}/2$ can be calculated by the previous example. That is,

$$k^{\sigma_2}_{2i+1} = \begin{cases} \left(\frac{2}{\sigma_2}\right) & \text{if } 2i+1 \neq 1, 3, 7 \\ 0 & \text{if } 2i+1 = 1, 3, 7 \end{cases}$$

6.1.4 Étale Manifold Structure Set

We define étale formal manifolds and their structure sets by splitting the information to different primes.

Definition 6.1.12. A connected space X is profinite simply connected if $\widehat{\pi_1(X)} = 0$. A profinitely simply connected space X is profinite complete and of étale finite type if each homotopy group $\pi_i(X)$ is isomorphic to the direct sum of a finitely generated free $\widehat{\mathbb{Z}}$-module and a finite abelian group.

Definition 6.1.13. A profinite simply connected, profinite complete and of étale finite type space X is étale normal of dimension m if it has a profinite complete stable spherical fibration $\gamma : X \to \widehat{BG(N)}$ and a map $f : \widehat{S^{N+m}} \to Th(\gamma)$.

Analogously, the map f induces a class $[\widehat{X}] \in H_m(X; \widehat{\mathbb{Z}})$ by $[\widehat{X}] = f_*[\widehat{S^{N+m}}] \cap U_\gamma$, where U_γ is the Thom class.

Definition 6.1.14. A profinite simply connected, étale normal space X is étale Poincaré if

$$- \cap [\widehat{X}] : H^{m-*}(X; \widehat{\mathbb{Z}}) \xrightarrow{\cong} H_*(X; \widehat{\mathbb{Z}})$$

In this case the étale normal structure is also called the étale Spivak normal fibration over X.

Proposition 6.1.10. *A profinite simply connected, étale Poincaré space X is p-adic Poincaré for each prime p.*

Proof. X has Poincaré duality with coefficient $\widehat{\mathbb{Z}}/p\widehat{\mathbb{Z}} \simeq \mathbb{Z}/p$. Hence it is a $\mathbb{Z}/p$-cofficient Poincaré space whose $\mathbb{Z}/p$ fundamental class is induced from $[\widehat{X}]$. Then apply 6.1.2. □

Definition 6.1.15. Let X be a profinite simply connected étale Poincaré space of dimension $m \geq 5$. Call X an étale formal manifold if it is a p-adic formal manifold for each prime p.

Remark 6.1.11. Recall from the previous chapters, the condition for an étale homotopy manifold is equivalent to the lifting of the $\widehat{\mathbb{L}^n}$-orientation of γ to some $\widehat{\mathbb{L}^s}$-orientation.

Definition 6.1.16. Let X be a profinite simply connected, étale formal manifold of dimension $m \geq 5$. An étale homotopy manifold structure over X consists of a p-adic homotopy manifold structure over X for each prime p.

Remark 6.1.12. An étale homotopy manifold structure over X is equivalent to an element of $[X, \widehat{\mathbb{L}^q}] \simeq \prod_p [X, \widehat{\mathbb{L}^q_p}]$.

Then we can define the étale structure set $\mathbf{S}^{TOP}(X)^\wedge$ over a profinite simply connected, étale formal manifold X of dimension $m \geq 5$ by $\prod_p \mathbf{S}^{TOP}(X)^\wedge_p$. There is an abelianized Galois group $\widehat{\mathbb{Z}}^\times \simeq \prod_p \widehat{\mathbb{Z}}_p^\times$ action on $\mathbf{S}^{TOP}(X)^\wedge$ by acting on the corresponding prime factor.

6.2 Galois Symmetry on Varieties

In the previous section, we have defined the p-adic structure set in the simply connected case. In this section, we apply the previous discussions to study the Galois symmetry on varieties.

Let X be a smooth complete variety defined over $\overline{\mathbb{Q}}$. Let us fix a choice of transcendental basis of $\mathbb{C}$ over $\mathbb{Q}$. Then the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ has an embedding into $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$.

In this way, the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the underlying topological manifold structures of $\{X^\sigma_{\mathbb{C}}\}_{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}$. In this section we study how to formulate the result in the language of étale structure set and then give a proof .

Let Y be a smooth complete variety defined over $\overline{\mathbb{Q}}$. Let $f : X \to Y$ be an algebraic map over $\overline{\mathbb{Q}}$ which induces an étale homotopy equivalence. For simplicity, we assume $Y_{\mathbb{C}}$ is profinite simply connected. Then f represents an element in both $\mathbf{S}^{TOP}(Y_{\mathbb{C}})^\wedge_p$ and $\mathbf{S}^{TOP}(Y_{\mathbb{C}})^\wedge$. Call such an element algebraic.

An element $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ takes $f : X \to Y$ to $X^\sigma \xrightarrow{\sigma} X \xrightarrow{f} Y$ in the structure set. This gives a Galois action on the algebraic elements of $\mathbf{S}^{TOP}(Y_{\mathbb{C}})^\wedge_p$ and $\mathbf{S}^{TOP}(Y_{\mathbb{C}})^\wedge$. We will prove the following theorem for the rest of this section.

Theorem 6.2.1. *Let Y be a smooth complete connected variety defined over $\overline{\mathbb{Q}}$ such that $\pi_1^{\text{ét}}(Y) = 0$ and Y has complex dimension at least 3. Then the Galois action $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on* *the algebraic elements in* $\mathbf{S}^{TOP}(Y_{\mathbb{C}})^\wedge_p$ *agrees with the abelianized Galois action we defined in the previous section, for any prime* p.

As a corollary, we get the Theorem 1.1.4, which is also stated in [Sul09, p. 271].

Corollary 6.2.2. *The Galois action* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *on the algebraic elements in* $\mathbf{S}^{TOP}(Y_{\mathbb{C}})^\wedge$ *is an abelianized action and this action can be extended to the whole structure set.*

The proof is based on the following lemma.

Lemma 6.2.3. *The Galois action on a smooth complex variety defined over* $\overline{\mathbb{Q}}$ *is compatible with the abelianized Galois action on the tangent bundle. That is, the following diagram commutes up to homotopy*

$$\begin{array}{ccc} \widehat{X_{\mathbb{C}}} & \xrightarrow{TX_{\mathbb{C}}} & \widehat{BU} \\ \big\downarrow{\scriptstyle \sigma^{-1}} & & \big\downarrow{\scriptstyle \psi^{\sigma^{-1}}} \\ \widehat{X^{\sigma}_{\mathbb{C}}} & \xrightarrow{TX^{\sigma}_{\mathbb{C}}} & \widehat{BU} \end{array}$$

where $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, X *is a smooth variety over* $\overline{\mathbb{Q}}$ *and the tangent bundle map is the tangent bundle map for the analytification of the complex varieties.*

Proof. We use the idea in [DS75] to give a proof. Consider the associated Stiefel bundle STX of TX over X. Namely, let the total space $STX_{\mathbb{C}} = \mathrm{Emb}_{\mathbb{C}}(TX_{\mathbb{C}}, X \times \mathbb{C}^N)$ consisting of fiberwise embeddings $TX_{\mathbb{C}} \to X \times \mathbb{C}^N$ for some large N. Indeed, one can make this construction over $\overline{\mathbb{Q}}$, namely, let $STX = \mathrm{Emb}_{\overline{\mathbb{Q}}}(TX, X \times \overline{\mathbb{Q}}^N)$. The bundle STX is also a $\overline{\mathbb{Q}}$-variety and the map $STX \to X$ is an algebraic bundle. Its fiber is indeed the Stiefel variety $V_{n,N}$ consisting of all n-frames in $\overline{\mathbb{Q}}^N$.

Then we have a map $STX \to Gr_n(\overline{\mathbb{Q}}^N)$, which maps an embedding $T_xX \to \overline{\mathbb{Q}}^N$ to its image. This is also an algebraic map over $\overline{\mathbb{Q}}$.

Then we get a zig-zag of algebraic maps of varieties $X \leftarrow STX \to Gr_n(\overline{\mathbb{Q}}^N)$. When passing to $\mathbb{C}$, the left arrow is homotopically highly connected when N is large enough. When N tends to infinity, the zig-zag gives an analytically continuous map $X_{\mathbb{C}} \to BGL(n, \mathbb{C})$, which is indeed the classifying map of the tangent bundle $TX_{\mathbb{C}}$. Since each map is defined over $\overline{\mathbb{Q}}$, the ziz-zag of maps commutes with the Galois action. Let $n \to \infty$ and the lemma is proven, since passing to profinite completion and the stable range σ becomes the modified Adams operation ψ^σ. $\square$

Corollary 6.2.4. *With the same notation as above, the following diagram also commutes up to homotopy*

$$\begin{array}{ccc} \widehat{X_{\mathbb{C}}} & \xrightarrow{\nu_{X_{\mathbb{C}}}} & \widehat{BU} \\ \downarrow{\scriptstyle \sigma^{-1}} & & \downarrow{\scriptstyle \psi^{\sigma^{-1}}} \\ \widehat{X^\sigma_{\mathbb{C}}} & \xrightarrow{\nu_{X^\sigma_{\mathbb{C}}}} & \widehat{BU} \end{array}$$

where $\nu_{X_{\mathbb{C}}}$ and $\nu_{X^\sigma_{\mathbb{C}}}$ both mean the stable inverse bundles of the corresponding tangent bundles.

Proof of Theorem 6.2.1. Let $f : X \to Y$ be an algebraic element in $\mathbf{S}^{TOP}(Y_{\mathbb{C}})^\wedge_p$.

Since $Y_{\mathbb{C}}$ is simply connected, $\widehat{Y_{\mathbb{C}}} \simeq \prod_p \widehat{Y_{\mathbb{C}}}_p$ and we can split $\widehat{f} : \widehat{X_{\mathbb{C}}} \to \widehat{Y_{\mathbb{C}}}$ to a product of maps $\widehat{f}_p : \widehat{X_{\mathbb{C}}}_p \to \widehat{Y_{\mathbb{C}}}_p$ and split the diagram in the previous lemma to different primes.

Assume the abelianization of σ is $\prod_p \sigma_p \in \prod_p \widehat{\mathbb{Z}}_p^\times \simeq \widehat{\mathbb{Z}}^\times$.

Let γ_Y be the Spivak normal spherical fibration $\gamma_Y : Y \to BSG$ and let $\widehat{\gamma_Y}_p$ be its p-adic completion. Similarly, we also have γ_X and $\widehat{\gamma_X}_p$. Notice that $\widehat{\gamma_X}_p = \widehat{f}_p \circ \widehat{\gamma_Y}_p$.

(1) Case I: p is an odd prime.

Let $\Delta_X : Th(\widehat{\gamma_X}_p) \to \widehat{BSO^\otimes}_p$ be the p-adic manifold structure underlying $X_{\mathbb{C}}$. Similarly, we also have Δ_Y. Then the element (ϕ, β) in the structure set of Y representing $\widehat{f}_p$ is deduced from $(\widehat{f}_p^{-1})^* \Delta_X = \phi \cdot \Delta_Y$.

Recall from [Sul09, Theorem 6.5] that $\widehat{BSPL}_p$ is equivalent to the classifying space $B_{\widehat{KO}_p} SG$ of spherical fibrations with $\widehat{KO}_p$ orientations. Since $TOP/PL \simeq K(\mathbb{Z}/2, 3)$ ([KS77, p. 251]), $\widehat{BSTOP}_p$ is also homotopy equivalent to $B_{\widehat{KO}_p} SG$. The Adams conjecture says that the Galois action on $\widehat{BU}_p$ fixes the underlying $\widehat{BSG}_p$. The Galois action on $\widehat{BU}_p$ also extends to $\widehat{BSTOP}_p$ via the Galois action on the orientation $\widehat{KO}_p$ ([Sul09, Theorem 6.7]).

Consider the diagram

$$\begin{array}{ccccc}
\widehat{X}_{\mathbb{C}p} & \xrightarrow{\nu_{X_{\mathbb{C}}}} & \widehat{BU}_p & \longrightarrow & \widehat{BSTOP}_p \\
\downarrow{\scriptstyle \sigma^{-1}} & & \downarrow{\scriptstyle \psi^{\sigma_p^{-1}}} & & \downarrow{\scriptstyle \sigma_p^{-1}} \\
\widehat{X}^{\sigma}_{\mathbb{C}p} & \xrightarrow{\nu_{X^{\sigma}_{\mathbb{C}}}} & \widehat{BU}_p & \longrightarrow & \widehat{BSTOP}_p
\end{array}$$

Then the underlying p-adic manifold structure data Δ_{X^σ} of $X^\sigma_{\mathbb{C}}$ is pulled back along σ^{-1} to $\psi^{\sigma_p^{-1}}\Delta_X$ over $X_{\mathbb{C}}$.

An implication of the Adams conjecture is that $\widehat{BSTOP}_p \xrightarrow{\sigma_p^{-1}-1} \widehat{BSTOP}_p$ canonically factors as the following composition of maps $\widehat{BSTOP}_p \xrightarrow{g_{\sigma_p^{-1}}} \widehat{G/TOP}_p \to \widehat{BSTOP}_p$.

Then the representation $(\phi', \beta') \in \mathbf{S}^{TOP}(Y_{\mathbb{C}})^\wedge_p$ of $\widehat{f}_p \circ \sigma : \widehat{X}^\sigma_{\mathbb{C}p} \to \widehat{Y}_{\mathbb{C}p}$ is calculated by the following equation and the Pontryagin character.

$$\phi' \cdot \Delta_Y = ((\widehat{f}_p \circ \sigma)^{-1})^* \Delta_{X^\sigma} = (\widehat{f}_p^{-1})^* \psi^{\sigma_p^{-1}} \Delta_X = \psi^{\sigma_p^{-1}} (\phi \cdot \Delta_Y)$$

It is exactly the definition of abelianized Galois action on $\mathbf{S}^{TOP}(Y_{\mathbb{C}})^\wedge_p$.

(2) Case II: $p = 2$.

Let $(l, k) \in \mathbf{S}^{TOP}(Y_{\mathbb{C}})^\wedge_2$ represent $\widehat{f}_2$. Let L_X, L_Y be the 2-adic L-genus of $X_{\mathbb{C}}, Y_{\mathbb{C}}$ respectively. Then by definition $(\widehat{f}_2^{-1})^* L_X = (1 + 8l) \cdot L_Y$.

By the odd prime case, we know that

$$(\sigma^{-1})^* L_{X^\sigma} = \mathrm{ph}(\sigma^{-1} \Delta_{X^\sigma}) = \mathrm{ph}(\psi^{\sigma^{-1}} \Delta_X) = \psi_H^{\sigma^{-1}} L_X$$

We lift it to the 2-adic case then we get that $(\sigma^{-1})^* L_{X^\sigma} = \psi_H^{\sigma_2^{-1}} L_X$.

Also by the Adams conjecture, the map $\widehat{X}_{\mathbb{C}2} \to \widehat{BU}_2 \xrightarrow{\psi^{\sigma^{-1}}} \widehat{BU}_2$ gives a map $\widehat{X}_{\mathbb{C}2} \to \widehat{BU}_2 \xrightarrow{g_{\sigma_2^{-1}}} \widehat{G/U}_2 \to \widehat{G/TOP}_2$, which precisely corresponds to $\widehat{X}^\sigma_{\mathbb{C}} \to \widehat{X}_{\mathbb{C}}$.

The representation $(l', k') \in \mathbf{S}^{TOP}(Y_{\mathbb{C}})^\wedge_2$ of $\widehat{f}_2 \circ \sigma$ is calculated as follows.

$$(1 + 8l') L_Y = ((\widehat{f}_2 \circ \sigma)^{-1})^* L_{X^\sigma_{\mathbb{C}}} = (\widehat{f}_2^{-1})^* \psi_H^{\sigma_2^{-1}} L_X = \psi_H^{\sigma_2^{-1}} ((\widehat{f}_2^{-1})^* L_X) = \psi_H^{\sigma_2^{-1}} ((1 + 8l) \cdot L_Y)$$

This is exactly the abelianized Galois action on the l-classes.

For k', the k-class $k_Y^{\sigma^{-1}}$ for the map $\widehat{Y}^\sigma_{\mathbb{C}} \xrightarrow{\sigma} \widehat{Y}_{\mathbb{C}}$ is given by $\widehat{Y}_{\mathbb{C}2} \to \widehat{BU}_2 \xrightarrow{g_{\sigma_2^{-1}}} \widehat{G/U}_2 \to \widehat{G/TOP}_2$.

Now consider the commutative diagram

$$\begin{array}{ccc} X^\sigma & \xrightarrow{\sigma} & X \\ \big\downarrow{\scriptstyle f^\sigma} & & \big\downarrow{\scriptstyle f} \\ Y^\sigma & \xrightarrow{\sigma} & Y \end{array}$$

Let (l^σ, k^σ) be the representation of f^σ in $\mathbf{S}^{TOP}(Y^\sigma_{\mathbb{C}})^\wedge_2$. Then $k^\sigma = \sigma^* k$.

Hence, k' is also the k-class for the map $\sigma \circ \widehat{f^\sigma} : \widehat{X^\sigma_{\mathbb{C}}} \to \widehat{Y_{\mathbb{C}}}$. Then

$$k' = k_Y^{\sigma^{-1}} + (\sigma^{-1})^* k^\sigma = k_Y^{\sigma^{-1}} + k$$

which agrees with the abelianized Galois action on the k-class. □

6.3 Conjectures and Questions

It seems very mysterious that the Galois action discussed above is abelianized. We propose the following question.

Question 6.3.1. What is a geometric reason for the Galois action on the algebraic elements of a profinite structure set to be an abelianized action?

In this section, we suggest an approach together with some conjectures for this question. We hope to complete this part of discussions in future studies.

Recall that an étale morphism $f : U \to X$ for a smooth complex variety X can always be completed to an algebraic branched covering map $f' : U' \to X$.

For simplicity, let us assume that the branched locus B_f of f' has complex codimension 1. B_f has finitely many components $B_1, \ldots, B_k$. Around each component B_i, pick a base point x_i in the complement of B_f. Take a small loop γ_i in the complement based at x_i around B_i. The conjugate class of β_i is independent of the choice of x_i. Choose a basepoint x in $X - B_f$. For each i choose a path δ_i in $X - B_f$ connecting x and x_i. Conjugate γ_i by δ_i and then we get a set of elements $\{\gamma'_i\}$ in $\pi_1(X - B_f, x)$.

Assume that $\pi_1(X) = 0$. By the Seifert-van Kampen theorem, the conjugate classes of $\gamma'_1, \ldots, \gamma'_k$ actually generate $\pi_1(X - B_f)$. Then the image $S(f)$ of $\pi_1(X - B_f) \to S_d$ induced by the étale morphism $f : U \to X$ is generated by the classes of $\alpha_i = f_*(\gamma_i')$ conjugated by $\pi_1(X - B_f)$. The classes of $\alpha_1, \ldots, \alpha_k$ are independent of the choice of basepoints and the choice of paths δ_i.

Note that the permutation information is not algebraic, since it uses the transcendental topology of X. However, choosing basepoints in this discussion makes the statement more complicated.

Question 6.3.2. Is there a basepoint-free discussion of the permutation information for a branched covering?

Conjecture 6.3.3. The fundamental groupoid language will give an answer.

Hence, for the étale site $\mathbf{S}_{\text{ét}}(X)$, each object $f_1 : U \to X$ is not only an étale morphism, but also contains the permutation data $\{\alpha_1, \ldots, \alpha_k\}$. Given two étale morphisms $f_1 : U_1 \to X$ and $f_2 : U_2 \to X$, a morphism between them in the category $\mathbf{S}_{\text{ét}}(X)$ is realized by some 'etale morphism $g : U_2 \to U_1$. How does g induce an morphism between the permutation data $\{\alpha_{1,1}, \ldots, \alpha_{1,k}\}$ of f_1 and $\{\alpha_{2,1}, \ldots, \alpha_{2,l}\}$ of f_2?

Notice that the branched locus B_{f_1} is a subvariety of the branched locus B_{f_2}. So the components $B_1, \ldots, B_k$ of B_{f_1} form a subset of those of B_{f_2}. The components of B_{f_2} must be $B_1, \ldots, B_k, \ldots, B_l$ for some $l \geq k$. Assume that f_1 has a degree d and f_2 has degree nd, where n is the degree of g. Then the permutation group has a natural quotient map $\psi_{d,nd} : S_{nd} \to S_d$ induced by the modulo d quotient map $\{1, \ldots, nd\} \to \{1, \ldots d\}$.

Then g induces the correspondence $\psi_{d,nd}(\alpha_{2,1}) = \alpha_{1,1}, \ldots, \psi_{d,nd}(\alpha_{2,k}) = \alpha_{1,k}$. In this way, we get a category $\mathbf{S}(X)$ which is isomorphic to $\mathbf{S}_{\text{ét}}(X)$, where the objects are

$$(U \to X, \text{the conjugate classes of } \{\alpha_1, \ldots, \alpha_k\})$$

where $U \to X$ is an étale morphism and $\alpha_1, \ldots, \alpha_k$ are the induced permutation data. The morphisms are defined like above.

Notice that for a Galois morphism $\sigma : X^\sigma \to X$, each étale morphism $f : U \to X$ is pulled back to an étale morphism $f^\sigma : U^\sigma \to X^\sigma$. However, the conjugate class of the pullback of permutations $\{\alpha_1, \ldots, \alpha_k\}$ for f might be different from the permutations $\{\alpha_1^\sigma, \ldots, \alpha_k^\sigma\}$ for f^σ, since permutations are transcendental information rather than algebraic information.

Define an abstract category $\mathbf{S}_P(X)$, whose objects are

$$(B, \pi_1(X-B) \to S_d, \text{the conjugate classes of } \{\alpha_1, \ldots, \alpha_k\})$$

where B is a codimension 2 subvariety of X with components $B_1, \ldots, B_k$, $\{\alpha_1, \ldots \alpha_k\}$ are permutations in S_d and the conjugation is induced by the image of $\pi_1(X-B) \to S_d$, such that the conjugate classes of $\{\alpha_1, \ldots \alpha_k\}$ generate the image of $\pi_1(X-B) \to S_d$. The morphisms in $\mathbf{S}_P(X)$ are defined like those in $\mathbf{S}(X)$. $\mathbf{S}_P(X)$ is a category larger than $\mathbf{S}(X)$ since only part of the permutations can be realized by an actual étale morphism over X.

Define an automophism of $\mathbf{S}_P(X)$ by an isomorphic functor $F : \mathbf{S}_P(X) \to \mathbf{S}_P(X)$ such that an object $(B, \pi_1(X-B) \to S_d, \text{the conjugate classes of } \{\alpha_1, \ldots, \alpha_k\})$ is mapped to an object with the same B and the same $\pi_1(X-B) \to S_d$.

Conjecture 6.3.4. There is a natural homomorphism from $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ to the automophisms of $\mathbf{S}_P(X)$.

Remark 6.3.5. This conjecture seems not possible. So we suggest to find an appropriated subcategory of $\mathbf{S}_P(X)$ such that to realize this homomorphism in the conjecture.

Recall the definition of Grothendieck's dessins d'enfants (e.g., see [Gro85][Sch94]).

Conjecture 6.3.6. Grothendieck's dessins d'enfants is equivalent to part of our homomorphism for $X = P^1$.

Grothendieck and many other mathematicians proved that the homomorphism from $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ to the automorphism group of dessins d'enfants is injective. It is natural to ask whether the same result is true in our formulation.

Question 6.3.7. Is the homomorphism from $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ to the automophism group of $\mathbf{S}_P(X)$ injective for any simply connected smooth variety X defined over $\overline{\mathbb{Q}}$?

Finally, we conjecture the following.

Conjecture 6.3.8. There is a concrete context showing that the homomorphism we defined above agrees with the abelianized Galois action on the profinite structure sets.

We do not know a concrete approach for this conjecture yet, but it seems that the class field theory and Sullivan's proof of Adams conjecture should help.

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