Tunnels, Wells, and Scalar Curvature

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Abstract of the Dissertation

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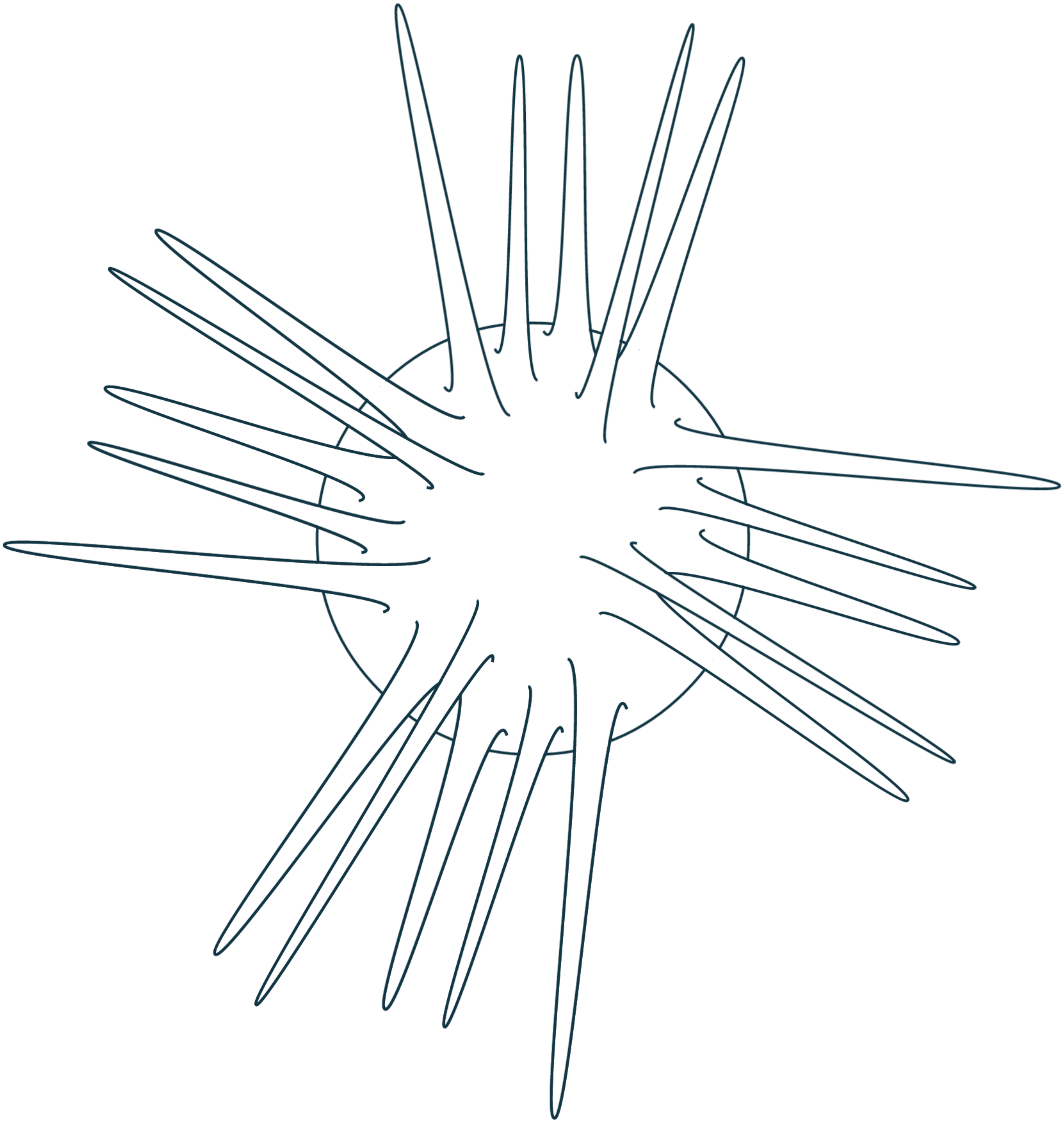
The rigidity theorems of Llarull and of Marques–Neves show two different ways scalar curvature can characterize the sphere and have associated stability conjectures. Here we produce the first examples related to these stability conjectures. The first set of examples demonstrate the necessity of including a condition on the minimum area of all minimal surfaces to prevent bubbling along the sequence. The second set of examples constructs sequences that do not converge in the Gromov–Hausdorff sense but do converge in the volume-preserving intrinsic flat sense. In order to construct such sequences, we improve the Gromov–Lawson Schoen–Yau tunnel construction so that one can attach tunnels or wells to a manifold with scalar curvature bounded below and only decrease the scalar curvature by an arbitrarily small amount. Moreover, we are able to generalize both the sewing construction of Basilio, Dodziuk, and Sormani and the construction due to Basilio, Kazaras, and Sormani of an intrinsic flat limit with no geodesics. Furthermore, using this technique, we build upon the perturbative counterexamples of Brendle–Marques–Neves to Min-Oo's Conjecture in order to construct counterexamples that make advances on the theme expressed in a question asked by Carlotto in 2021. These new counterexamples are non-perturbative in nature; moreover, we also produce examples with more complicated topology.



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Chapter 1

Introduction

Riemannian geometry provides the mathematical language to discuss the shape and size of a space. In particular, a Riemannian manifold (M^n, g) is a smooth n-manifold equipped with a Riemannian metric g. Using the Riemannian metric g, one can define geometric concepts such as length, area, and curvature. Over the decades, a main goal of Riemannian geometry has been to understand how curvature controls the shape and size of a Riemannian manifold. There are many different, but related, curvatures. The weakest of the well-known curvature invariants is scalar curvature. For example, as opposed to other curvatures such as sectional curvature and Ricci curvature, if the scalar curvature is bounded below by a positive constant then the diameter of the Riemannian manifold is not necessarily bounded. Furthermore, a fundamental result concerning positive scalar curvature is the tunnel construction of Gromov–Lawson [28] and Schoen–Yau [52]. The result states if M is a smooth manifold that admits a Riemannian metric with positive scalar curvature and N is a smooth manifold that can be obtained from M by performing surgeries in codimension at least three then there exists a metric on N with positive scalar curvature. In particular, this means if (M_i^n, g_i), $i = 1, 2$, are two Riemannian manifolds with positive scalar curvature then there exists a metric on the connected sum $M_1 \# M_2$ with positive scalar curvature.

Having scalar curvature lower bounds appears to be a flexible condition; nonetheless, lower bounds on scalar curvature have been shown to control the geometries that admit such bounds. These types of results are usually called rigidity theorems and are often used to characterize manifolds in Riemannian geometry. A typical rigidity theorem says that if a Riemannian manifold satisfies some conditions, usually including a bound on curvature, then it must be isometric to a specific model geometry. One can naturally formulate a stability theorem from a rigidity theorem. A stability theorem says if the hypotheses of a rigidity theorem are perturbed, then the manifolds that satisfy these hypotheses are quantitatively close to the manifold characterized by the rigidity theorem. In this thesis, we are concerned with rigidity theorems for spheres and hemispheres with lower bounds on scalar curvature.

1.1 Spheres with Larger Scalar Curvature

Two results that show how scalar curvature can characterize the sphere are the rigidity theorems due to Llarull [43] and due to Marques and Neves [44]. These two rigidity theorems naturally give rise to stability conjectures.

First, let us recall Llarull's theorem [43] which says that if there is a degree non-zero, smooth, distance non-increasing map from a closed, smooth, connected, Riemannian, spin, n-manifold, M^n, to the standard unit round n-sphere and the scalar curvature of M^n is greater than or equal to $n(n-1)$, then the map is a Riemannian isometry. Gromov in [30] proposed studying the stability question related to Llarull's rigidity theorem by investigating sequences of Riemannian manifolds $M_j^n = (M_j^n, g_j)$ with $\inf R^j \to n(n-1)$ and $Rad_{\mathbb{S}^n}(M_j) \to 1$. $Rad_{\mathbb{S}^n}(M^n)$ is the maximal radius r of the n-sphere, $\mathbb{S}^n(r)$, such that M^n admits a distance non-increasing map from M^n to $\mathbb{S}^n(r)$ of non-zero degree. Based on this, Sormani [54] proposed the following stability conjecture. Before stating the conjecture, we recall the following definition:

$$\text{MinA}(M, g) = \inf\{|\Sigma|_g : \Sigma \text{ is a closed minimal hypersurface in } M\}.$$

Also, we will condense notation and set $M_j^n = (M_j^n, g_j)$ when we have a sequence of Riemannian manifolds.

Conjecture 1.1.1. *Suppose $M_j^n = (M_j^n, g_j)$, $n \geq 3$, are closed smooth connected spin Riemannian manifolds such that*

$$R^j \geq n(n-1) - \frac{1}{j}, \ \text{MinA}(M_j^n) \geq A, \ \text{diam}\left(M_j^n\right) \leq D, \ \text{vol}\left(M_j^n\right) \leq V$$

where R^j is the scalar curvature of M_j^n. Furthermore, suppose there are smooth maps to the standard unit round n-sphere

$$f_j : M_j^n \to \mathbb{S}^n$$

which are 1-Lipschitz and $\deg f_j \neq 0$. Then M_j^n converges in the $\mathcal{VF}$-sense to the standard unit round n-sphere.

We construct the first examples related to Conjecture 1.1.1. We demonstrate why a condition preventing bubbling is required, and we investigate different modes of convergence. In order to construct these examples, we prove an enhancement of the Gromov–Lawson tunnel construction [28] (see also Schoen–Yau [53]) which retains control over the scalar curvature. The example we produce is a sequence of manifolds each of which is two spheres connected by a thin tunnel, which is related to Conjecture 1.1.1. This sequence converges in the volume-preserving intrinsic flat ($\mathcal{VF}$) sense to a disjoint union of two n-spheres (see Figure 1.1). Moreover, the sequence shows without the lower bound on MinA then the conclusion of Conjecture 1.1.1 fails to hold.

Theorem A. *There exists a convergent sequence of Riemannian manifolds $M_j^n = (\mathbb{S}^n, g_j)$, $n \geq 3$, with $M_j^n \xrightarrow{\mathcal{VF}} M_\infty$ such that*

$$R^j \geq n(n-1) - \frac{1}{j}, \ \text{diam}\left(M_j\right) \leq D, \ \textit{and} \ \text{vol}\left(M_j\right) \leq V,$$

for some constants $D, V > 0$. Furthermore, there are smooth degree one, 1-Lipschitz maps $f_j : M_j^n \to (\mathbb{S}^n, g_{rd})$ which converge to a 1-Lipschitz map $f_\infty : M_\infty \to (\mathbb{S}^n, g_{rd})$, and M_∞ is the disjoint union of two n-spheres.

Figure 1.1: A sequence of spheres that converge in $\mathcal{VF}$-sense to the disjoint union of two spheres.

Sormani proposed the MinA condition in [56] to prevent bad limiting behavior, such as bubbling and pinching, along the sequence. The motivation for such a condition comes from the sewing construction of Basilio, Dodziuk, and Sormani [7]. This construction shows the existence of a sequence of manifolds with positive scalar curvature, which has an intrinsic flat ($\mathcal{F}$) limit that does not have positive scalar curvature in some generalized sense. Other sequences of positive scalar curvature manifolds have also been constructed ([8], [9]) whose $\mathcal{F}$-limits have undesirable properties. The key to the construction of these examples is the ability to glue in tunnels with controlled geometry. In those examples, it is unknown if the scalar curvature of the tunnel and of the resulting manifold can be kept close to the scalar curvature of the manifold to which the tunnel is being glued. Therefore, these examples may not satisfy the curvature condition in Conjecture 1.1.1. In Chapter 2, we prove two main technical propositions, which are of independent interest. One of which allows us to get quantitative control over the scalar curvature of the tunnel and of the resulting manifold. In particular, we prove (Proposition 2.1.2):

Proposition. *Let (M^n, g), $n \geq 3$, be a Riemannian manifold with scalar curvature R^M. Let $\delta > 0$ be small enough, $j \in \mathbb{N}$, and $d \geq 0$. If $R^M \geq \kappa$ on two balls $B_g(p, 2\delta)$ and $B_g(p', 2\delta)$ in (M^n, g), then we can construct a new complete Riemannian manifold P^n, where we remove two balls and glue cylindrical region (T_j, g_j) diffeomorphic to $\mathbb{S}^{n-1} \times [0, 1]$,*

$$P^n = M^n \setminus (B_g(p, 2\delta) \cup B_g(p', 2\delta)) \sqcup T_j.$$

Furthermore, the following properties are satisfied:

i. The scalar curvature, R^j, of T_j satisfies $R^j > \kappa - \frac{1}{j}$.

ii. $g_j|_E = g|_E$ and $g_j|_{E'} = g|_{E'}$ where $E = B_g(p, 2\delta) \setminus B_g(p, \delta)$ and $E' = B_g(p', 2\delta) \setminus B_g(p', \delta)$ are identified with subsets of P.

iii. There exists constant $C > 0$ independent of j and d such that

$$d \leq \operatorname{diam}(T_j) < C(\delta + d) \quad \textit{and} \quad \operatorname{vol}(T_j) < C(\delta^n + d\delta^{n-1}).$$

iv. P has scalar curvature $R^P > \kappa - \frac{1}{j}$.

We use this new way of attaching tunnels to manifolds that maintains control over the scalar curvature to construct the sequence in Theorem A. Moreover, we can make a similar example related to Marques–Neves' rigidity theorem. The theorem of Marques–Neves pertains to the three dimensional sphere and the min-max quantity width. Let us recall the definition of width. Let g be a Riemannian metric on the 3-sphere and $x_4 : \mathbb{S}^3 \subseteq \mathbb{R}^4 \to \mathbb{R}$ be the height function. For each $t \in [-1, 1]$, let $\Sigma'_t = \{x \in \mathbb{S}^3 : x_4 = t\}$ and Λ' be the collection of all families $\{\Sigma_t\}$ such that $\Sigma_t = F_t(\Sigma'_t)$ for some smooth one-parameter family of diffeomorphisms F_t of the three sphere all of which are isotopic to the identity. The width of $(\mathbb{S}^3, g)$ is the following min-max quantity

$$\text{width}(\mathbb{S}^3, g) = \inf_{\{\Sigma_t\} \in \Lambda'} \sup_{t \in [-1,1]} |\Sigma_t|_g,$$

where $|\Sigma|_g$ is the two dimensional Hausdorff measure of Σ.

The rigidity theorem of Marques–Neves [44] says if there is a Riemannian metric on the 3-sphere with positive Ricci curvature, scalar curvature greater than or equal to 6, and $\text{width}(\mathbb{S}^3, g) \geq 4\pi$, then it is isometric to the standard unit round 3-sphere. This leads to the following naive stability conjecture.

Conjecture 1.1.2. *Suppose $M_j^3 = (\mathbb{S}^3, g_j)$ are homeomorphic spheres satisfying*

$$R^j \geq 6 - \frac{1}{j}, \ \text{width}(M_j^3) \geq 4\pi, \ \text{diam}\left(M_j^3\right) \leq D, \ \textit{and} \ \text{vol}\left(M_j^3\right) \leq V$$

where R^j is the scalar curvature of M_j^3. Then M_j^3 converges in the $\mathcal{VF}$-sense to $(\mathbb{S}^3, g_{rd})$ where g_{rd} is the Riemannian metric for the standard unit round 3-sphere.

In [49], Montezuma constructs Riemannian metrics g_w, $w > 0$, on the 3-spheres such that the scalar curvature is greater than or equal to 6 and the $\text{width}(\mathbb{S}^3, g_w) \geq w$. These manifolds look like a tree of spheres. In particular, they are constructed based on a finite full binary tree where the nodes are replaced with standard unit round 3-spheres and the edges are replaced with Gromov–Lawson tunnels of positive scalar curvature. The width is shown to be proportional to the depth of the tree. Finally, by taking one of these manifolds with large enough width and scaling it, one achieves scalar curvature greater than or equal to 6 is achieved. This example shows the failure of the rigidity statement of Marques–Neves and Conjecture 1.1.2 when positive Ricci curvature is not assumed.

Below we construct another counterexample that refutes Conjecture 1.1.2 which is similar to the example construct in Theorem A. By allowing the scalar curvature to be greater than or equal to $6 - \frac{1}{j}$, we are able to construct an example that is the connected sum of just two 3-spheres. Moreover, we are able to give explicit bounds on the volume and diameter of each manifold in the sequence. This counterexample is a sequence of spheres $M_j^3 = (\mathbb{S}^3, g_j)$ that converges in the volume preserving intrinsic flat ($\mathcal{VF}$) sense to the disjoint union of two spheres. The M_j^3 are two spheres connected by a thin tunnel (see Figure 1.1), and the tunnel gets increasingly thin along the sequence.

Theorem A′ (Counterexample to Conjecture 1.1.2)**.** *There exists a convergent sequence of Riemannian manifolds $M_j^3 = (\mathbb{S}^3, g_j)$, with $M_j^3 \xrightarrow{\mathcal{VF}} M_\infty$ such that*

$$R^j \geq 6 - \frac{1}{j}, \ \text{width}(M_j^3) \geq 4\pi, \ \text{diam}\left(M_j^3\right) \leq D, \ \textit{and} \ \text{vol}\left(M_j^3\right) \leq V,$$

for some constants $D, V > 0$, and M_∞ is the disjoint union of two 3-spheres.

Therefore, something stronger than width is required for a stability conjecture related to the rigidity theorem of Marques–Neves. A conjecture in [57] attributed to Marques and Neves does hypothesize a stronger condition (see Conjecture 1.1.3 below). In particular, it replaces the uniform lower bound on width with a uniform lower bound on MinA.

Since width is achieved by a minimal surface, we have that $\mathrm{width}(M^3, g) \geq \mathrm{MinA}(M^3, g)$. Moreover, Marques–Neves show in [44] that if $(\mathbb{S}^3, g)$ contains no stable minimal surfaces, then we have that $\mathrm{MinA}(\mathbb{S}^3, g) = \mathrm{width}(\mathbb{S}^3, g)$. Moreover, in the proof of Marques–Neves rigidity theorem, they only use the hypothesis of positive Ricci curvature to ensure the manifold contains no stable minimal embedded spheres. By [44, Appendix A], we see that if both the scalar curvature of a 3-manifold is sufficiently close to 6 and MinA is sufficiently close to 4π then the manifold contains no stable minimal embedded surfaces.

Conjecture 1.1.3. *Suppose $M_j^3 = (\mathbb{S}^3, g_j)$ are homeomorphic spheres satisfying*

$$R^j \geq 6 - \frac{1}{j}, \ \mathrm{MinA}(M_j^3) \geq 4\pi - \frac{1}{j}, \ \mathrm{diam}\left(M_j^3\right) \leq D, \ \textit{and} \ \mathrm{vol}\left(M_j^3\right) \leq V$$

where R^j is the scalar curvature of M_j^3. Then M_j^3 converges in the $\mathcal{VF}$-sense to $(\mathbb{S}^3, g_{rd})$ the standard unit round sphere.

The sequence of Riemannian manifolds constructed in Theorem A′ has $\mathrm{MinA}(M_j^n) \to 0$ and so does not satisfy the hypotheses of Conjecture 1.1.3. Theorems A and A′ show the necessity of including a hypothesis like the bound on MinA to prevent bubbling along the sequence.

When studying a stability conjecture related to scalar curvature, one also often considers examples similar to the example described by Ilmanen. Ilmanen first described this example to demonstrate that a sequence of manifolds of positive scalar curvature need not converge in the Gromov–Hausdorff (GH) sense. The example is a sequence of spheres with increasingly many arbitrarily thin wells attached to them (see Figure 1.2). Sormani and Wenger [58, Example A.7] showed, using their $\mathcal{F}$-convergence for integral currents, that the Ilmanen example converges in the $\mathcal{F}$-sense. Over the past decade, Ilmanen-like examples have been constructed in varying settings to demonstrate that GH-convergence is not the appropriate convergence in which to ask stability conjectures related to scalar curvature ([2], [3], [37], [38], [40], [41], [39], [50]). In these examples, it is unknown if one can attach a well and only decrease the scalar curvature by a small amount; consequently, it was unknown if Ilmanen-like examples could exist for Conjecture 1.1.1 and Conjecture 1.1.3.

Our other main technical proposition shows that one can attach a well to a manifold with scalar curvature bounded below and only decrease the scalar curvature by an arbitrarily small amount. Specifically, we show (Proposition 2.1.1):

Proposition. *Let (M^n, g), $n \geq 3$, be a Riemannian manifold with scalar curvature R^M. Let $\delta > 0$ be small enough, $j \in \mathbb{N}$, and $d > 0$. If $R^M \geq \kappa$ on $B_g(p, 2\delta)$ a ball in (M, g), then we can construct a well $W_j = (B_g(p, 2\delta), g_j)$ and a new complete Riemannian manifold (N^n, h),*

$$N^n = M^n, \qquad h|_{M \setminus B_g(p,2\delta)} = g|_{M \setminus B_g(p,2\delta)}, \qquad h|_{B_g(p,2\delta)} = g_j|_{B_g(p,2\delta)}.$$

Furthermore, the following properties are satisfied:

i. The scalar curvature, R^j, *of* W_j *satisfies* $R^j > \kappa - \frac{1}{j}$.

ii. $g_j|_E = g|_E$ *where* $E = B_g(p, 2\delta) \setminus B_g(p, \delta)$ *is identified with a subset of* W_j.

iii. There exists constant $C > 0$ *independent of* j *and* d *such that*

$$d \leq \mathrm{diam}\,(W_j) < C(\delta + d) \quad \textit{and} \quad \mathrm{vol}\,(W_j) < C(\delta^n + d\delta^{n-1}).$$

iv. N *has scalar curvature* $R^N > \kappa - \frac{1}{j}$.

Using this, we are able to construct Ilmanen-like examples related to Conjecture 1.1.1 and Conjecture 1.1.3. In particular, we construct a sequence of spheres with scalar curvature larger than $n(n-1) - \frac{1}{j}$, volumes and diameters bounded, and smooth maps to the unit round n-sphere which are 1-Lipschitz and $\deg f_j \neq 0$. This sequence does not converge in the GH-sense but does converge in the volume above distance below ($VADB$) sense and the $\mathcal{VF}$-sense. Likewise, we are able to construct a sequence of spheres M_j^3 with scalar curvature larger than $6 - \frac{1}{j}$, width larger than 4π, and volumes and diameters bounded that does not converge in the GH-sense but does converge in the VADB-sense and the $\mathcal{VF}$-sense. Therefore, we can construct Ilmanen-like examples related to Conjecture 1.1.1 and Conjecture 1.1.3. We, however, cannot verify that MinA stays uniformly bounded from below even though we expect that it does.

Theorem B. *There exists a convergent sequence of Riemannian manifolds* $M_j^n = (\mathbb{S}^n, g_j)$, *with* $M_j^n \xrightarrow{\mathrm{VADB}} M_\infty$ *and* $M_j^n \xrightarrow{\mathcal{VF}} M_\infty$ *such that*

$$R^j \geq n(n-1) - \frac{1}{j}, \;\; \mathrm{diam}\left(M_j^n\right) \leq D, \;\; \textit{and} \;\; \mathrm{vol}\left(M_j^n\right) \leq V,$$

for some constants $D, V > 0$, *and* M_∞ *is the* n*-sphere. Furthermore, there are smooth degree non-zero,* 1*-Lipschitz maps* $f_j : M_j^n \to (\mathbb{S}^n, g_{rd})$, *and* M_∞ *is the standard unit round* n*-sphere. However, the sequence has no subsequence that converges in the* GH*-sense.*

Theorem B′. *There exists a convergent sequence of Riemannian manifolds* $M_j^3 = (\mathbb{S}^3, g_j)$, *with* $M_j^3 \xrightarrow{\mathrm{VADB}} M_\infty$ *and* $M_j^3 \xrightarrow{\mathcal{VF}} M_\infty$ *such that*

$$R^j \geq 6 - \frac{1}{j}, \;\; \mathrm{width}(M_j^3) \geq 4\pi, \;\; \mathrm{diam}\left(M_j^3\right) \leq D, \;\; \textit{and} \;\; \mathrm{vol}\left(M_j^3\right) \leq V,$$

for some constants $D, V > 0$, *and* M_∞ *is the standard unit round* 3*-sphere. However, the sequence has no convergent subsequence in the* GH*-topology.*

In [57, Remark 9.4], Sormani suggests that it is believable that someone can construct a sequence of spheres with increasingly many increasingly thin wells which satisfy the hypothesis of Conjecture 1.1.3. Theorem B′ partially answers this question by constructing such a sequence that satisfies all the hypotheses of Conjecture 1.1.3 except the bound on MinA.

Figure 1.2: A sequence of spheres with increasingly many thin wells that converges in the VADB-sense and $\mathcal{VF}$-sense to a sphere but has no convergent subsequence in the GH-topology

The main tools to prove the above theorems are new construction propositions which are proved in Chapter 2. We adapt the bending argument of Gromov and Lawson in [28]. Originally, the construction in [28] was used to make tunnels of positive scalar curvature to show, for example, that the connected sum of two manifolds with positive scalar curvature carries a metric of positive scalar curvature. For 3-manifolds with constant positive sectional curvature, Dodziuk, Basilio, and Sormani in [7] refined the construction to give control over the volume and diameter of the tunnel while maintaining positive scalar curvature. Dodziuk in [26] further refined the construction by replacing the positive sectional curvature condition with positive scalar curvature and allowing for any dimension greater than or equal to three. In Chapter 2, we construct wells and tunnels such that, if the scalar curvature of a manifold is bounded below, then one can attach a well or tunnel and only decrease the lower bound by an arbitrarily small amount while maintaining bounds on the diameter and volume.

The new well construction allows us to generalize the construction of Sormani and Wenger [58, Example A.11] of a sequence of manifolds that converge in the $\mathcal{F}$-sense to space that is not precompact. In particular, we are able to construct a sequence of spheres with scalar curvatures greater than $\kappa \geq 0$, uniformly bounded diameters, and uniformly bounded volumes such that the sequence converges in the $\mathcal{VF}$-sense to a limit that is not precompact. To construct the sequence we attach a sequence of increasingly thin wells to a sphere (see Figure 1.3).

Theorem C. *There exists a convergent sequence of Riemannian manifolds $M_j^n = (\mathbb{S}^n, g_j)$, $n \geq 3$, with $M_j^n \xrightarrow{\mathcal{VF}} M_\infty$ such that*

$$R^j \geq \kappa, \ \ \operatorname{diam}\left(M_j^n\right) \leq D, \ \ \textit{and} \ \operatorname{vol}\left(M_j^n\right) \leq V,$$

for some nonnegative constants κ, D, V, and M_∞ is not precompact.

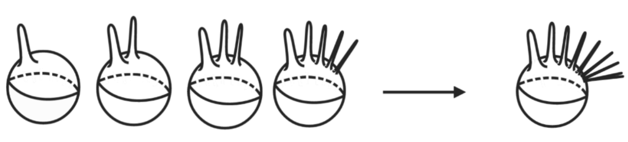

Figure 1.3: A sequence of spheres with increasingly many thin wells that converges in the $\mathcal{VF}$-sense to a limit which is not precompact.

The new tunnel construction allows us to extend the sewing construction in [7] and [9] to a more general setting. Basilio, Dodziuk, and Sormani [7] used sewing manifolds to investigate the following question of Gromov which asks: What is the weakest notion of convergence such that a sequence of Riemannian manifolds, M_j^n with scalar curvature $R^j \geq \kappa$ subconverges to a limit M_∞ which may not be a manifold but has scalar curvature greater than κ in some suitably generalized sense? They were able to show that when $\kappa = 0$ there is a sequence of Riemannian manifolds with non-negative scalar curvature whose limit fails to have non-negative generalized scalar curvature defined as

$$wR(p_0) := \lim_{r \to 0} 6(n+2) \frac{\mathrm{vol}_{\mathbb{E}^n} B(0,r) - \mathcal{H}^n(B(p_0, r))}{r^2 \cdot \mathrm{vol}_{\mathbb{E}^n} B(0,r)} \geq 0 \tag{1.1.1}$$

for the limit space.

Remark 1.1.4. *For a Riemannian manifold (M^n, g) with scalar curvature R, $wR(p) = R(p)$ for all $p \in M$.*

We are able to provide a similar answer to Gromov's question for any κ. In particular, for any κ, there exists a sequence of increasingly tightly sewn manifolds all of which have scalar curvature greater than κ. Furthermore, this sequence of increasingly tightly sewn manifolds will converge in the $\mathcal{F}$-sense to a pulled metric space (see [9, Section 2] for discussion of such spaces) which fail to have generalized scalar curvature greater than or equal to κ at the pulled point.

Theorem D. *There exists a sequence of manifolds $M_j^n = (M^n, g_j)$ with scalar curvature $R^j \geq \kappa - \frac{1}{j}$ which converges in the $\mathcal{F}$-sense to a metric space M_∞. Moreover, there is a point $p_0 \in M_\infty$ such that*

$$wR(p_0) := \lim_{r \to 0} 6(n+2) \frac{\mathrm{vol}_{\mathbb{E}^n} B(0,r) - \mathcal{H}^n(B(p_0, r))}{r^2 \cdot \mathrm{vol}_{\mathbb{E}^n} B(0,r)} = -\infty \tag{1.1.2}$$

Lastly, the new tunnel construction allows us to generalize the construction of Basilio, Kazaras, and Sormani [8]. They use long thin tunnels with positive scalar curvature to construct a sequence of manifolds that converges in the $\mathcal{F}$-sense to a space with no geodesics. Similarly, for any $\kappa > 0$, we are able to construct a sequence of manifolds with scalar curvature bounded below by κ whose limit is not a geodesic space.

Theorem E. *There is a sequence of closed, oriented, Riemannian manifolds (M_j^n, g_j), $n \geq 3$, with scalar curvature $R^j > \kappa > 0$ such that the corresponding integral current spaces converge in the intrinsic flat sense to*

$$M_\infty = \left(N, d_{\mathbb{E}^{n+1}}, \int_N \right),$$

where N is the round n-sphere of sectional curvature $\frac{2\kappa}{n(n-1)}$ and $d_{\mathbb{E}^{n+1}}$ is the Euclidean distance induced from the standard embedding of N into $\mathbb{E}^{n+1}$. Furthermore, N is not locally geodesic.

	Properties of M_j	Property of the limit, M_∞	Type of convergence	Does $\mathrm{MinA}_j \to 0$?
Theorem A	$R^j > n(n-1) - \frac{1}{j}$	Shows necessity of MinA lower bound in Conjecture 1.1.1.	$\mathcal{VF}, \mathcal{F}$	Yes
Theorem A′	$R^j > 6 - \frac{1}{j}$ width$(M_j) \geq 4\pi$	Counterexample to Conjecture 1.1.2.	$\mathcal{VF}, \mathcal{F}$	Yes
Theorem B	$R^j > n(n-1) - \frac{1}{j}$	No GH-convergent subsequence.	VADB, $\mathcal{VF}, \mathcal{F}$	?
Theorem B′	$R^j > 6 - \frac{1}{j}$ width$(M_j) \geq 4\pi$	No GH-convergent subsequence.	VADB, $\mathcal{VF}, \mathcal{F}$	?
Theorem C	$R^j > \kappa > 0$	Not precompact.	$\mathcal{VF}, \mathcal{F}$	?
Theorem D	$R^j > \kappa$ ($R^j > 0$, [7])	Generalized Scalar curvature is negative infinity at a point.	$\mathcal{F}$	Yes
Theorem E	$R^j > \kappa > 0$ ($R^j > 0$, [8])	No two points are connected by a geodesic.	$\mathcal{F}$	Yes

Table 1.1: Here we summarize the examples related to Conjecture 1.1.1 and Conjecture 1.1.3.

1.2 Hemispheres with Large Scalar Curvature

The positive mass theorem is a landmark result about the rigidity of certain metrics on $\mathbb{R}^n$. It has an analogous statement called Min-Oo's conjecture which pertains to uniformly positive scalar curvature and the hemisphere. First, let us recall the positive mass theorem. It says: if an asymptotically flat Riemannian manifold has nonnegative scalar curvature, then the mass of the manifold is nonnegative. Moreover, the mass is equal to zero if and only if the manifold is isometric to Euclidean space. The positive mass theorem was first proven for dimensions less than or equal to seven by Schoen and Yau [52] via minimal surface techniques. Using spinors, Witten [61] proved the positive mass theorem in all dimensions for spin manifolds. Over the years, there have been new proofs of the positive mass theorem using varying techniques. In 2001, Huisken and Ilmanen [34] gave a proof of the positive mass theorem by studying inverse mean curvature flow. More recently, Li [42] used Ricci flow to prove the positive mass theorem. In dimension three, there are several recent proofs of the positive mass theorem: Bray, Kazaras, Khuri and Stern [11] via levels sets harmonic functions, Miao [45] via positive harmonic functions and capacity of sets, and Agostiniani, Mazzieri, and Oronzio [1] via the Green's function.

Other related rigidity phenomena, involving positive scalar curvature and minimal surfaces, have been studied by Carlotto, Chodosh, and Eichmair [15]. For example, if (M^3, g) is an asymptotically flat three-dimensional manifold with nonnegative scalar curvature that contains a complete non-compact embedded surface S which is a (component of the) boundary of some properly embedded full-dimensional submanifold and is area-minimizing under compactly supported deformations, then M is flat Euclidean space and S is a flat plane. Now, this result should be compared with a special case of a localized gluing construction of Carlotto and

Schoen [16]. For a nice discussion of the topic see the article of Chruściel [17]. The general construction allows for localized solutions to the vacuum Einstein constraint equations and, roughly, states that one can construct asymptotically flat initial data sets that have positive ADM mass but are trivial outside a cone of a given angle. In particular, one can construct a metric on $\mathbb{R}^3$ that is asymptotically flat, scalar flat, and is Euclidean on $\mathbb{R}^2 \times (0, \infty)$, but is not the Euclidean metric on $\mathbb{R}^3$.

One is naturally led to wonder: Can something analogous be said for negative scalar curvature? The answer for negative curvature is that there are parallel results. For spin manifolds, Min-Oo [47] gave the first a characterization of hyperbolic space in the context of a positive mass theorem, which was later refined by Andersson and Dahl [6]. By combining Chruściel and Herzlich [21] and Wang [60] the positive mass theorem for n-dimensional asymptotically flat spin manifolds was established in the negative curvature setting. Without the spin assumption, Andersson, Cai, and Galloway [5] prove a positive mass theorem for n-dimensional manifolds with $3 \leq n \leq 7$ with an additional hypothesis on the mass aspect function. Also, Sakovich [51] used the Jang equation to prove the hyperbolic positive mass theorem in dimension three. In [20], Chruściel and Delay showed the positive of mass for asymptotically hyperbolic manifolds in any dimension via a gluing argument. Lastly, Huang, Jang, and Martin [33] proved the rigidity case for the hyperbolic positive mass theorem. For a history and a systematic presentation of these results see [14, Appendix C]. Moreover, there is also a localized gluing construction in the negative curvature setting due to Chruściel and Delay [18].

The story in the positive curvature setting differs from the other two. Recall that the analogous statement for the positive mass theorem is known as Min-Oo's Conjecture [48]. The conjecture states: if g is a smooth metric on the hemisphere $\mathbb{S}^n_+$ such that the scalar curvature, R_g, satisfies $R_g \geq n(n-1)$, the induced metric on the boundary $\partial\mathbb{S}^n_+$ agrees with the standard unit round metric on $\mathbb{S}^{n-1}$, and the boundary $\partial\mathbb{S}^n_+$ is totally geodesic with respect to g, then g is isometric to the standard unit round metric on $\mathbb{S}^n_+$. Many special cases of Min-Oo's Conjecture are known to be true ([12], [31], [32]).

However, in [13], Brendle, Marques, and Neves construct counterexamples to Min-Oo's Conjecture using perturbative techniques. They perform two perturbations to the standard unit round metric on $\mathbb{S}^n_+$. First, they perturb so that scalar curvature is strictly larger than $n(n-1)$ and the mean curvature of the boundary is positive. The second perturbation makes the boundary totally geodesic while preserving the scalar curvature lower bound.

In light of the above gluing constructions for zero and negative curvature, Carlotto asks in [14, Open Problem 3.16]: Can one design a novel class of counterexamples to Min-Oo's Conjecture based on a gluing scheme? More specifically, can one remove a neighborhood of a point on the boundary of $\mathbb{S}^n_+$ and then use a gluing scheme analogous to Corvino [23] or Corvino-Schoen [25] (also see [16] [19]) to produce counterexamples to Min-Oo's conjecture.

In [24], Corvino, Eichmair, and Miao produce different counterexamples to Min-Oo's conjecture. Their gluing scheme says that given two smooth compact Riemannian manifolds (M_i, g_i), $i = 1, 2$ of constant scalar curvature, κ, which contain two non-empty domains $U_i \subset M_i$ where g_i are *not V-static* then one can construct a smooth metric on $M_1 \# M_2$ with constant scalar curvature κ. Using this gluing theorem, they glue 3-spheres near the boundary of the perturbative counterexample constructed by Brendle, Marques, and Neves in [13, Theorem 7]. The resulting manifold satisfies the hypotheses of Min-Oo's Conjecture but can have arbitrarily large volume. Their gluing method, which is related to the gluing methods in [22], [35], [36], has two parts. First they do a conformal deformation to construct a neck connecting the two manifolds, and then they deform out of the conformal class to preserve the initial metrics away from the gluing region.

The quantitative version of Gromov–Lawson Schoen–Yau tunnels ([28], [53]), proved below Chapter 2, can be used to build upon the Brendle–Marques–Neves counterexamples [13] to construct new and more extreme counterexamples to Min-Oo's Conjecture. This construction allows for the removal of some of the assumptions required in [24]. In particular, we show the following:

Theorem F. *Let $D > 0$, $n \geq 3$, and let (M^n, g) be a Riemannian manifold such that the scalar curvature, R_g, satisfies $R_g > n(n-1)$. Define $H = \mathbb{S}^n_+$ and $\partial H = \Sigma$. Then $N = M\#H$ admits a metric $\tilde{g}$ such that:*

- *The scalar curvature, $\tilde{g}$, satisfies $R_{\tilde{g}} > n(n-1)$ everywhere.*
- *The induced metric on Σ agrees with the standard unit round metric on $\mathbb{S}^{n-1}$.*
- *Σ is totally geodesic with respect to $\tilde{g}$.*
- *$\mathrm{diam}_{\tilde{g}}(N) \geq D$.*

Remark 1.2.1. *We note, as opposed to the construction in [24], the gluing scheme employed here does not need the existence of non-V-static metrics on the manifolds that we glue together and we do not need an open subset of constant scalar curvature to which to glue.*

The following corollaries of Theorem F provide examples that make strides on the theme present in Carlotto's question by exploring the shape of manifolds that satisfy the geometric hypothesis of Min-Oo's Conjecture. Specifically, they show that these counterexamples can look geometrically very different than the standard unit round hemisphere. The first corollary shows we can have a counterexample with an arbitrarily large diameter.

Corollary 1. *For any $D > 0$, there exists a metric g on $\mathbb{S}^n_+$, $n \geq 3$, such that the following hold:*

- *The diameter of $M = (\mathbb{S}^n_+, g)$ satisfies $\mathrm{diam}_g(\mathbb{S}^n_+) \geq D$.*
- *The scalar curvature satisfies $R_g > n(n-1)$ everywhere.*
- *The induced metric on $\partial M = \Sigma$ agrees with the standard unit round metric on $\mathbb{S}^{n-1}$.*
- *Σ is totally geodesic with respect to g.*

We are also are able to construct examples of arbitrarily large volumes.

Corollary 2. *For any $V > 0$, there exists a metric g on $\mathbb{S}^n_+$, $n \geq 3$, such that the following hold:*

- *The volume of $M = (\mathbb{S}^n_+, g)$ satisfies $\mathrm{vol}_g(\mathbb{S}^n_+) \geq V$.*
- *The scalar curvature satisfies $R_g > n(n-1)$ everywhere.*

- *The induced metric on Σ agrees with the standard unit round metric on $\mathbb{S}^{n-1}$.*
- *Σ is totally geodesic with respect to g.*

Remark 1.2.2. *In [24, Remark 6.4], Corvino, Eichmair, and Miao suggest one should be able to construct similar examples using a gluing technique akin to the Gromov–Lawson tunnel construction [28]. Corollary 2 rigorously proves this remark.*

Remark 1.2.3. *We note that for our construction we cannot glue near the boundary of the counterexample in [13, Theorem 7] as they do in [24]; however, we can glue around any point where the scalar curvature is strictly larger than $n(n-1)$ in [13, Theorem 7]. Moreover, we can glue around any point in the counterexample constructed in [13, Corollary 6], in particular, near the boundary.*

We want to emphasize that in Theorem F that the only restriction on M is that it admits a metric with scalar curvature strictly larger than $n(n-1)$. In particular, any closed Riemannian manifold that admits positive scalar curvature can be connected to a counterexample of Min-Oo's Conjecture. Therefore, we obtain new counterexamples with non-trivial topology. For example:

Corollary 3. *Let $p, q, n \in \mathbb{N}$ such that $n \geq 3$ and $p + q = n$. Then there exists a metric g on $M = \mathbb{S}^n_+ \# (\mathbb{S}^p \times \mathbb{S}^q)$ with the following properties:*

- *The scalar curvature satisfies $R_g > n(n-1)$ everywhere.*
- *The induced metric on $\partial M = \Sigma$ agrees with the standard unit round metric on $\mathbb{S}^{n-1}$.*
- *Σ is totally geodesic with respect to g.*

Corvino, Eichmair, and Miao are able to obtain counterexamples to Min-Oo's Conjecture with a non-trivial fundamental group by gluing a counterexample of Min-Oo's Conjecture to $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ or $\mathbb{S}^n/\Gamma$ where Γ is a finite subgroup of $SO(n+1)$. Corollary 3 highlights that many more topologies can be produced.

We note that the result of Corollary 3 can be viewed as a 0-surgery on $\mathbb{S}^n_+ \cup (\mathbb{S}^p \times \mathbb{S}^q)$. Moreover, the original construction of Gromov–Lawson [28] works for surgeries in codimension greater than or equal to three. Therefore, one may wonder if the connect sum procedure in [57] can be extended to surgeries in codimension greater than or equal to three. We prove an analogous statement in Chapter 2, which may be of independent interest. As a result, we get the following statement about more topologies:

Theorem G. *Let M^n be a manifold obtained by performing a surgery in codimension greater than or equal to three on $\mathbb{S}^n_+$. There exists a metric g on M such that*

- *The scalar curvature, g, satisfies $R_g > n(n-1)$ everywhere.*
- *The induced metric on ∂M agrees with the standard unit round metric on $\mathbb{S}^{n-1}$.*
- *∂M is totally geodesic with respect to g.*

In [13, Theorem 7], Brendle, Marques, and Neves construct a metric g on the n-hemisphere, $n \geq 3$, with the following properties: the scalar curvature is at least $n(n-1)$ everywhere, there exists a point where the scalar curvature is strictly larger than $n(n-1)$, and the metric agrees with the standard unit round metric on $\mathbb{S}^n_+$ in a neighborhood of the boundary. They showed that this metric can be used to construct a metric on $\mathbb{RP}^n$, $n \geq 3$, that has analogous properties as the one on the hemisphere. We would like to point out that if we view g as a metric on an n-ball we can produce analogous metrics on any lens space by making the appropriate identifications on the boundary.

Theorem H. *Let $n \geq 3$. Then for any n-dimensional lens space L^n there exists a metric g such that (L^n, g) has the following properties:*

- *The scalar curvature satisfies $R_g \geq n(n-1)$.*
- *There exists a point $p \in L$ such that $R_g(p) > n(n-1)$.*
- *The metric g agrees with the standard unit round metric in a neighborhood of the equator in L.*

Related to the work of Brendle, Marques, and Neves is the following rigidity result of Miao and Tam [46] for hemispheres. Let g_{rd} denote the standard unit round metric on $\mathbb{S}^n_+$. Their theorem states: if g is a metric on $\mathbb{S}^n_+$ such that the scalar curvature satisfies $R_g \geq R_{g_{rd}}$, the mean curvature of the boundary satisfies $H_g \geq H_{g_{rd}}$, $g = g_{rd}$ on the boundary, the volume satisfies $V_g \geq V_{g_{rd}}$, and g is sufficiently close to g_{rd} in the C^2 norm, then g is isometric to g_{rd}. We note that the theorem of Miao and Tam is false without the perturbative hypothesis. The first example showing this was constructed in [24]. We note that Corollary 2 gives an alternative construction showing the need for the C^2-closeness in [46].

Moreover, we construct the following example which should be compared with Corollary 2 and the rigidity result of Miao and Tam.

Theorem I. *For all $0 < \epsilon < \frac{1}{100}$ and $D > 0$. There exists a metric g on $\mathbb{S}^n_+$ with the following properties:*

- *The scalar curvature satisfies $R_g > n(n-1)$.*
- *The mean curvature on $\partial\mathbb{S}^n_+$ satisfies $H_g > 0$.*
- *The induced metric on $\partial\mathbb{S}^n_+$ is the standard unit round metric on $\mathbb{S}^{n-1}$.*
- *The volume satisfies*
$$\frac{1}{2}\omega_n \leq \mathrm{vol}_g(\mathbb{S}^n_+) \leq \frac{1}{2}\omega_n + \epsilon,$$
where ω_n is the volume of the standard unit round n-sphere.
- *The diameter satisfies* $\mathrm{diam}_g(\mathbb{S}^n_+) > D$.

By relaxing the condition on the curvature on the boundary from totally geodesic to $H_g > 0$, we construct examples that are related to the result of Miao and Tam while keeping the volume arbitrarily close to the volume of the standard unit round $\mathbb{S}^n_+$. Moreover, from the proof of Theorem I one can see that outside a set of arbitrarily small volume the metric g is a small perturbation of the standard unit round metric on $\mathbb{S}^n_+$.

1.3 Background

In this section, we will review different types of convergences for Riemannian manifolds.

1.3.1 Gromov–Hausdorff Convergence

Here we will review the Gromov–Hausdorff distance between two metric spaces. Gromov defined this distance between two metric spaces by generalizing the concept of Hausdorff distance between two subsets of a metric space. We refer the reader to [29] for further details.

The Gromov–Hausdorff distance between two metric spaces (X_1, d_1) and (X_2, d_2) is

$$d_{\mathrm{GH}}((X_1, d_1), (X_2, d_2)) = \inf_Z \{d_H^Z(\phi_1(X_1), \phi_2(X_2))\}$$

where the infimum is taken over all complete metric spaces (Z, d^Z) and all distance preserving maps $\phi_i : X_i \to Z$. We say that a metric spaces (X_j, d_j) converge in the GH-sense to a metric space (X_∞, d_∞) if

$$d_{\mathrm{GH}}((X_j, d_j), (X_\infty, d_\infty)) \to 0$$

We note that the GH-distance naturally defines a distance between Riemannian manifolds since one can naturally assign a distance function to a Riemannian manifold via the metric.

Gromov, in the following theorem, characterizes when a sequence of compact metric spaces contains a subsequence that converges in the GH-sense.

Theorem 1.3.1. *For a sequence of compact metric spaces* (X_j, d_j) *such that* $\mathrm{diam}\,(X_j) < D < \infty$*, the following are equivalent:*

i. There exists a convergent subsequence.

ii. There is a function $N_1 : (0, \alpha) \to (0, \infty)$ *such that* $Cap_j(\epsilon) \le N_1(\epsilon)$

iii. There is a function $N_2 : (0, \alpha) \to (0, \infty)$ *such that* $Cov_j(\epsilon) \le N_2(\epsilon)$*, where*

$$Cap_j(\epsilon) = \text{ maximum number of disjoint } \frac{\epsilon}{2}\text{-balls in } X_j,$$

$$Cov_j(\epsilon) = \text{ minimum number of } \epsilon\text{-balls it takes to cover } X_j.$$

1.3.2 Intrinsic Flat Convergence

In this section we will review Sormani-Wenger intrinsic flat distance between two integral current spaces. Sormani and Wenger [58] defined intrinsic flat distance, which generalizes the notion of flat distance for currents in Euclidean space. To do so they used Ambrosio and Kirchheim's generalization of Federer and Fleming's integral currents to metric spaces. We refer the reader to [4] for further details about currents in arbitrary metric spaces and to [58] for further details about integral current spaces and intrinsic flat distance.

Let (Z, d^Z) be a complete metric space. Denote by $\mathrm{Lip}(Z)$ and $\mathrm{Lip}_b(Z)$ the set of real-valued Lipschitz functions on Z and the set of bounded real-valued Lipschitz functions on Z.

Definition 1.3.2 ([4], Definition 3.1)**.** We say a multilinear functional

$$T : \mathrm{Lip}_b(Z) \times [\mathrm{Lip}(Z)]^m \to \mathbb{R}$$

on a complete metric space (Z, d) is an m-dimensional current if it satisfies the following properties.

i. Locality: $T(f, \pi_1, \ldots, \pi_m) = 0$ if there exists and i such that π_i is constant on a neighborhood of $\{f \neq 0\}$.

ii. Continuity: T is continuous with respect to pointwise convergence of π_i such that $\mathrm{Lip}(\pi_i) \leq 1$.

iii. Finite mass: there exists a finite Borel measure μ on X such that

$$|T(f, \pi_1, \ldots, \pi_m)| \leq \prod_{i=1}^{m} \mathrm{Lip}(\pi_i) \int_Z |f| d\mu \tag{1.3.1}$$

for any $(f, \pi_1, \ldots, \pi_m)$.

We call the minimal measure satisfying (1.3.1) the mass measure of T and denote it $||T||$. We can now define many concepts related to a current. $\mathbf{M}(T) = ||T||(Z)$ is defined to be the mass of T and the canonical set of a m-current T on Z is

$$\mathrm{set}(T) = \left\{ p \in Z \;\middle|\; \liminf_{r \to 0} \frac{||T||(B(p, r))}{r^m} > 0 \right\}.$$

The boundary of a current T is defined as $\partial T : \mathrm{Lip}_b(X) \times [\mathrm{Lip}(X)]^{m-1} \to \mathbb{R}$, where

$$\partial T(f, \pi_1, \ldots, \pi_{m-1}) = T(1, f, \pi_1, \ldots, \pi_{m-1}).$$

Given a Lipschitz map $\phi : Z \to Z'$, we can pushforward a current T on Z to a current $\phi_\# T$ on Z' by defining

$$\phi_\# T(f, \pi_1, \ldots, \pi_m) = T(f \circ \phi, f \circ \pi_1, \ldots, f \circ \pi_m).$$

A standard example of an m-current on Z is given by

$$\phi_\# [[\theta]](f, \pi_1, \ldots, \pi_m) = \int_A (\theta \circ \phi)(f \circ \phi) d(\pi_1 \circ \phi) \wedge \cdots \wedge d(\pi_m \circ \phi),$$

where $\phi : \mathbb{R}^m \to Z$ is bi-Lipschitz and $\theta \in L^1(A, \mathbb{Z})$. We say that an m-current on Z is integer rectifiable if there is a countable collection of bi-Lipschitz maps $\phi_i : A_i \to X$ where $A_i \subset \mathbb{R}^m$ is precompact Borel measurable with pairwise disjoint images and weight functions $\theta_i \in L^1(A_i, \mathbb{Z})$ such that

$$T = \sum_{i=1}^{\infty} \phi_{i\#}[[\theta_i]].$$

Moreover, we say an integer rectifiable current whose boundary is also integer rectifiable is an integral current. We denote the space of integral m-currents on Z as $\mathbf{I}_m(Z)$. The flat distance between two integral currents T_1, $T_2 \in \mathbf{I}(Z)$ is

$$d_F^Z(T_1, T_2) = \inf\{\mathbf{M}(U) + \mathbf{M}(V) \mid U \in \mathbf{I}_m(X), V \in \mathbf{I}_{m+1}(X), T_2 - T_1 = U + \partial V\}.$$

We say that the triple (X, d, T) is an integral current space if (X, d) is a metric space, $T \in \mathbf{I}_m(\bar{X})$ where $\bar{X}$ is the completion of X, and $\mathrm{set}(T) = X$. The intrinsic flat ($\mathcal{F}$) distance between two integral current spaces (X_1, d_1, T_1) and (X_2, d_2, T_2) is

$$d_{\mathcal{F}}((X_1, d_1, T_1), (X_2, d_2, T_2)) = \inf_Z \{d_F^Z(\phi_{1\#}T_1, \phi_{2\#}T_2)\}$$

where the infimum is taken over all complete metric spaces (Z, d^Z) and isometric embeddings $\phi_1 : (\bar{X}_1, d_1) \to (Z, d^Z)$ and $\phi_2 : (\bar{X}_2, d_2) \to (Z, d^Z)$. We note that if (X_1, d_1, T_1) and (X_2, d_2, T_2) are precompact integral current spaces such that

$$d_{\mathcal{F}}((X_1, d_1, T_1), (X_2, d_2, T_2)) = 0$$

then there is a current preserving isometry between (X_1, d_1, T_1) and (X_2, d_2, T_2), i.e., there exists an isometry $f : X_1 \to X_2$ whose extension $\bar{f} : \bar{X}_1 \to \bar{X}_2$ pushes forward the current: $\bar{f}_\# T_1 = T_2$. We say a sequence of (X_j, d_j, T_j) precompact integral current spaces converges to $(X_\infty, d_\infty, T_\infty)$ in the $\mathcal{F}$-sense if

$$d_{\mathcal{F}}((X_j, d_j, T_j), (X_\infty, d_\infty, T_\infty)) \to 0.$$

If, in addition, $\mathbf{M}(T_i) \to \mathbf{M}(T_\infty)$, then we say (X_j, d_j, T_j) converges to $(X_\infty, d_\infty, T_\infty)$ in the voulme preserving intrinsic flat ($\mathcal{VF}$) sense. We note that we can view compact Riemannian manifolds (M^n, g) as precompact integral current spaces $(M^n, d_g, \int_{M^n} dvol_g)$, where d_g is the natural distance function on the Riemannian manifold and integration over the manifold, $\int_{M^n} dvol_g$, can be viewed as an integral current. Moreover, $\mathbf{M}(M^n) = \mathrm{vol}\,(M^n)$. Lakzian and Sormani in [38] were able to estimate the intrinsic distance between two diffeomorphic manifolds:

Theorem 1.3.3. *Suppose $M_1^n = (M^n, g_1)$ and $M_2^n = (M^n, g_2)$ are oriented precompact Riemannian manifolds with diffeomorphic subregions $U_j \subset M_j^n$ and diffeomorphisms $\psi_j : U \to U_j$ such that for all $v \in TU$ we have*

$$\frac{1}{(1+\epsilon)^2}\psi_1^* g_1(v,v) < \psi_2^* g_2(v,v) < (1+\epsilon)^2 \psi_1^* g_1(v,v).$$

We define the following quantities

i. $D_{U_j} = \sup\{\mathrm{diam}_{M_j}(W) : W \textit{ is a component of } U_j\}$.

ii. Define a to be a number such that $a > \frac{\arccos(1+\epsilon)^{-1}}{\pi} \max\{D_{U_1}, D_{U_2}\}$.

iii. $\lambda = \sup_{x,y \in U} |d_{M_1}(\psi_1(x), \psi_1(y)) - d_{M_2}(\psi_2(x), \psi_2(y))|$.

iv. $h = \sqrt{\lambda\left(\max\{D_{U_1}, D_{U_2}\} + \frac{\lambda}{4}\right)}$.

v. $\bar{h} = \max\left\{h, \sqrt{\epsilon^2 + 2\epsilon}D_{U_1}, \sqrt{\epsilon^2 + 2\epsilon}D_{U_2}\right\}$.

Then the intrinsic flat distance between M_1^n and M_2^n is bounded:

$$d_{\mathcal{F}}(M_1, M_2) \leq \left(2\bar{h} + a\right)\left(\mathrm{vol}_m(U_1) + \mathrm{vol}_m(U_2) + \mathrm{vol}_{m-1}(\partial U_1) + \mathrm{vol}_{m-1}(\partial U_2)\right) + \mathrm{vol}_m(M_1 \setminus U_1) + \mathrm{vol}_m(M_2 \setminus U_2).$$

Moreover, Sormani [55] proves the following Arzela-Ascoli theorem in the setting of $\mathcal{F}$-convergence.

Theorem 1.3.4. *Fix $L > 0$. Suppose $M_j = (X_j, d_j, T_j)$ are integral current spaces for $j \in \{1, 2, \ldots, \infty\}$ and $M_j \xrightarrow{\mathcal{F}} M_\infty$ and $F_j : X_j \to W$ are L-Lipschitz maps into a compact metric space W, then a subsequence converges to an L-Lipschitz map $F_\infty : X_\infty \to W$. Specifically, there exists isometric embeddings of the subsequence $\phi_j : X_j \to Z$, such that $d_F^Z(\phi_{j\#}T_j, \phi_{\infty\#}T_\infty) \to 0$ and for any sequence $p_j \in X_j$ converging to $p \in X_\infty$,*

$$d_Z(\phi_j(p_j), \phi_\infty(p)) \to 0,$$

one has converging images

$$d_W(F_j(p_j), F_\infty(p)) \to 0.$$

1.3.3 Volume Above Distance Below Convergence

Allen, Perales, and Sormani in [3] introduced a new notion of convergence of manifolds called volume above distance below (VADB) convergence. It is based on the volume-distance rigidity theorem which states that if there is a C^1-diffeomorphism $F : M \to N$ between two Riemannian manifolds which is also distance non-increasing then $\mathrm{vol}(N) \leq \mathrm{vol}(M)$; moreover, in case of equality the manifolds are isometric.

Definition 1.3.5. A sequence of Riemannian manifolds without boundary $M_j^n = (M^n, g_j)$ converge in the VADB-sense to a Riemannian manifold $M_\infty^n = (M^n, g_\infty)$ if

i. $\mathrm{vol}(M_j^n) \to \mathrm{vol}(M_\infty^n)$.

ii. $\mathrm{diam}(M_j^n) \leq D$.

iii. There exists a C^1-diffeomorphisms $\Psi_j : M_\infty^n \to M_j^n$ such that for all $p, q \in M_\infty^n$ we have

$$d_j(\Psi_j(p), \Psi_j(q)) \geq d_\infty(p, q).$$

We also record the following lemma from [3] which says that the above condition on the distance functions in the definition of VADB-convergence can be converted into a condition on Riemannian metrics.

Lemma 1.3.6. *Let $M_1^n = (M^n, g_1)$ and $M_0^n = (M^n, g_0)$ be Riemannian manifolds and $F : M_1^n \to M_0^n$ be a C^1-diffeomorphism. Then*

$$g_0(dF(v), dF(v)) \le g_1(v, v) \qquad \text{for all } v \in TM_1^n$$

if and only if

$$d_0(F(p), F(q)) \le d_1(p, q) \qquad \text{for all } p, q \in M_1^n.$$

Finally, we record the following theorem from [3] which describes the relationship between VADB-convergence and $\mathcal{VF}$-convergence.

Theorem 1.3.7. *If $M_j^n = (M^n, g_j)$ and $M_\infty^n = (M^n, g_\infty)$ are compact oriented Riemannian manifolds such that $M_j^n \xrightarrow{\text{VADB}} M_\infty^n$ then $M_j^n \xrightarrow{\mathcal{VF}} M_\infty^n$.*

Chapter 2

Quantitative Surgery

In this chapter, we prove the main new technical propositions: Proposition 2.1.1 (Wells and Tunnels), Proposition 2.1.2 (Wells and Tunnels), Proposition 2.2.1 (Surgery in Higher Codimensions). These are an improvement of the constructions of Gromov–Lawson [28], Basilio, Dodziuk, Sormani [7], and Dodziuk [26]. We construct wells and tunnels and get control over the volume and diameter while keeping the scalar curvature close to the scalar curvature of the manifold to which we are attaching the well. Proposition 2.1.1 (Wells and Tunnels) allows us to remove a ball from a Riemannian manifold M with scalar curvature $R^M \geq \kappa$ and glue in a well to create a new Riemannian manifold N; moreover, M and N will be isometric away from the gluing and the scalar curvature R^N of N will satisfy $R^N \geq \kappa - \epsilon$ for arbitrarily small ϵ. Proposition 2.1.2 (Wells and Tunnels) allows the analogous construction for connecting two manifolds with a tunnel. Therefore, given a Riemannian manifold M with $R^M \geq \kappa$ we can remove two balls and glue in a tunnel to create a Riemannian manifold P with $R^P \geq \kappa - \epsilon$ for arbitrarily small ϵ. As expressed by Gromov–Lawson in [28] the higher surgery result is very similar to connect sum construction and so we are able to show in Proposition 2.2.1 (Surgery in Higher Codimensions) that if (M, g) is a Riemannian manifold with scalar curvature at least κ then any manifold obtain from M by a codimension at least three surgery admits a metric with scalar curvature greater than $\kappa - \epsilon$ for any ϵ small enough.

2.1 Wells and Tunnels

In this section, we prove the main new technical propositions: Proposition 2.1.1 (Wells and Tunnels) and Proposition 2.1.2 (Wells and Tunnels). In particular we show,

Proposition 2.1.1 (Constructing Wells)**.** *Let (M^n, g), $n \geq 3$, be a Riemannian manifold with scalar curvature R^M. Let $\delta > 0$ be small enough, $j \in \mathbb{N}$, and $d > 0$. If $R^M \geq \kappa$ on $B_g(p, 2\delta)$ a ball in (M, g), then we can construct a well $W_j = (B_g(p, 2\delta), g_j)$ and a new complete Riemannian manifold (N^n, h),*

$$N^n = M^n, \qquad h|_{M \setminus B_g(p,2\delta)} = g|_{M \setminus B_g(p,2\delta)}, \qquad h|_{B_g(p,2\delta)} = g_j|_{B_g(p,2\delta)}.$$

Furthermore, the following properties are satisfied:

i. The scalar curvature, R^j, of W_j satisfies $R^j > \kappa - \frac{1}{j}$.

ii. $g_j|_E = g|_E$ *where* $E = B_g(p, 2\delta) \setminus B_g(p, \delta)$ *is identified with a subset of* W_j.

iii. There exists constant $C > 0$ *independent of* j *and* d *such that*

$$d \leq \operatorname{diam}(W_j) < C(\delta + d) \quad \textit{and} \quad \operatorname{vol}(W_j) < C(\delta^n + d\delta^{n-1}).$$

iv. N *has scalar curvature* $R^N > \kappa - \frac{1}{j}$.

Proposition 2.1.2 (Constructing Tunnels)**.** *Let* (M^n, g), $n \geq 3$, *be a Riemannian manifold with scalar curvature* R^M. *Let* $j \in \mathbb{N}$ *and* $d \geq 0$ *and let* $\delta > 0$ *be small enough depending only on* g. *If* $R^M \geq \kappa$ *on two balls* $B_g(p, 2\delta)$ *and* $B_g(p', 2\delta)$ *in* (M^n, g), *then we can construct a new complete Riemannian manifold* P^n, *where we remove two balls and glue cylindrical region* (T_j, g_j) *diffeomorphic to* $\mathbb{S}^{n-1} \times [0, 1]$,

$$P^n = M^n \setminus (B_g(p, 2\delta) \cup B_g(p', 2\delta)) \sqcup T_j.$$

Furthermore, the following properties are satisfied:

i. The scalar curvature, R^j, *of* T_j *satisfies* $R^j > \kappa - \frac{1}{j}$.

ii. $g_j|_E = g|_E$ *and* $g_j|_{E'} = g|_{E'}$ *where* $E = B_g(p, 2\delta) \setminus B_g(p, \delta)$ *and* $E' = B_g(p', 2\delta) \setminus B_g(p', \delta)$ *are identified with subsets of* P.

iii. There exists constant $C > 0$ *independent of* j *and* d *such that*

$$d \leq \operatorname{diam}(T_j) < C(\delta + d) \quad \textit{and} \quad \operatorname{vol}(T_j) < C(\delta^n + d\delta^{n-1}).$$

iv. P *has scalar curvature* $R^P > \kappa - \frac{1}{j}$.

We adapt the proof from [26]. The well and tunnel will be constructed as a codimension one submanifold. The submanifold will be defined by a curve, and this curve will control the geometry of the submanifold. First, we show how the curve defines the submanifold and how it affects its geometry. Second, we carefully construct the curve so that the submanifold will inherit the desired properties.

In particular, the construction will follow the following outline. First, we will describe how, given a curve, we can define a submanifold and write the scalar curvature in terms of quantities related to the curve. Second, we carefully construct a C^1-curve, γ, which will be used to define a submanifold that is the precursor to a well or a tunnel. Third, we adjust the construction of γ so the resulting manifold will be a well. Fourth, we describe the smoothing procedure to make γ a C^∞-curve. Fifth, we construct a well and check it has the desired properties. Sixth, we perform the analogous steps to construct a tunnel with the desired properties.

The main difficulty of this construction is the second step where one constructs the curve. Let's quickly recall how the previous constructions ([7], [26], and [28]) of the curve were shown. Gromov and Lawson construct a curve that bends from a vertical line segment to a horizontal line and they ensure the way the bending happens preserves positive scalar curvature. In dimension three and positive sectional curvature, Basilio, Dodziuk, and Sormani refine this construction of the curve to be more explicit than the proof given in [28] and in doing so obtain control over the volume and diameter of the curve they construct. Dodziuk then further demonstrates that the explicit construction in [7] can be extended to any dimension and positive scalar curvature. Here, we take the next natural step on this theme and demonstrate that a curve can be constructed while only decreasing the lower bound on the scalar curvature by an arbitrarily small amount and maintaining control over the volume and diameter. In addition, we also round off the end of the curve to construct a well.

In order to show this improvement on these constructions we make key observations about the scalar curvature calculation (2.1.1) (cf. [26, Equation (4)], [28, Equation (1)]) and the corresponding Lemma 2.1.5 (cf. [26, Lemma 3.1], [28, pg. 428]). We use these observations to construct tunnels whose scalar curvature is decreased by an arbitrarily small amount. Unlike previous constructions, we additionally construct wells. Altogether, this leads to technical differences in the construction near the beginning and end of the corresponding curve. Moreover, we are able to ensure that wells do not contain any closed minimal surfaces. The observations, in addition to the well construction, are vital for our ability to produce the manifolds that pertain to the main results of this work.

2.1.1 A Submanifold Defined by a Curve

Let (M^n, g) be a compact Riemannian manifold with scalar curvature $R^M \geq \kappa$. Let $\delta > 0$ and $B = B(p, 2\delta)$ be a geodesic ball in M. Consider the Riemannian product $(X, g_X) = (\mathbb{R} \times B, dt^2 + dr^2 + g_r)$. Let $\rho \in B$ be a geodesic radius from p to ∂B and define $S = \mathbb{R} \times \rho$, which is a total geodesic submanifold of $\mathbb{R} \times B$ with coordinates (t, r). Let γ be a smooth curve in S to be determined later. Finally, let $\Sigma = \{(y, q) \in X : (y, ||q||_g) \in \gamma\}$ be a submanifold of (X, g_X) with the induced metric, where $|| \cdot ||_g$ is the distance from p to q with respect to g. Note that we can identify S with a strip in $\mathbb{R}^2$. Now we want to calculate the scalar curvature of Σ. To do so we will need the following lemma from [26]:

Lemma 2.1.3. *The principal curvatures of the hypersurface $\mathbb{S}^{n-1}(\epsilon)$ in B are each of the form $\frac{1}{-\epsilon} + O(\epsilon)$ for small ϵ. Furthermore, let g_ϵ be the induced metric on $\mathbb{S}^{n-1}(\epsilon)$ and let $g_{rd,\epsilon}$ be the round metric of curvature $\frac{1}{\epsilon^2}$. Then, as $\epsilon \to 0$, $\frac{1}{\epsilon^2} g_\epsilon \to \frac{1}{\epsilon^2} g_{rd,\epsilon} = g_{rd}$ in the C^2 topology, moreover, $||g_{rd} - \frac{1}{\epsilon^2} g_\epsilon|| \leq \epsilon^2$.*

Remark 2.1.4. *The above lemma is written with the convention that the second fundamental form is $A(X, Y) = g(\nabla_X Y, N)$ and N is the outward pointing normal.*

Now to calculate the scalar curvature of Σ, fix $q \in \Sigma \cap S$. Let $e_1, \ldots, e_n$ be an orthonormal basis of of $T_q(\Sigma)$ where e_1 is tangent to γ. Note that the for points in $\Sigma \cap S$ the normal ν to Σ in X is the same as the normal to γ in S.

From the Gauss equations:

$$R^X(X, Y, Z, U) = R^\Sigma(X, Y, Z, U) - A(X, U)A(Y, Z) + A(X, Z)A(Y, U)$$

we see

$$K_{ij}^\Sigma = K_{ij}^X + \lambda_i \lambda_j.$$

where λ_i are principal curvatures corresponding to e_i and K_{ij}^{Σ} and K_{ij}^{X} are the respective sectional curvatures. We note that $\lambda_1 = k$ where k is the geodesic curvature of γ. For $i = 2, \ldots, n$ we see by Lemma 2.1.3

$$\begin{aligned}\lambda_i &= \langle \nabla_{e_i} e_i, \nu \rangle \\ &= \langle \nabla_{e_i} e_i, \cos\theta \partial_t + \sin\theta \partial_r \rangle \\ &= \cos\theta \langle \nabla_{e_i} e_i, \partial_t \rangle + \sin\theta \langle \nabla_{e_i} e_i, \partial_r \rangle \\ &= \sin\theta \langle \nabla_{e_i} e_i, \partial_r \rangle \\ &= \left(\frac{1}{-r} + O(r) \right) \sin\theta,\end{aligned}$$

where θ is the angle that between ν and the t-axis. Now note that

$$K_{1j}^{X} = R^X(e_j, e_1, e_1, e_j) = R^X(e_j, \cos\theta\partial_r, \cos\theta\partial_r, e_j) = \cos^2\theta K_{\partial_r, j}^{M}.$$

For $i \neq 1$ and $j \neq 1$

$$K_{ij}^{X} = R^X(e_j, e_i, e_i, e_j) = \; K_{i,j}^{M}.$$

Since

$$R^{\Sigma} = \sum_{i \neq j} K_{ij}^{\Sigma}$$

we see

$$\begin{aligned}R^{\Sigma} = R^M &- 2\mathrm{Ric}^M(\partial_r, \partial_r)\sin^2\theta \\ &+ (n-2)(n-1)\left(\frac{1}{r^2} + O(1)\right)\sin^2\theta \\ &- (n-1)\left(\frac{1}{r} + O(r)\right) k \sin\theta.\end{aligned} \tag{2.1.1}$$

2.1.2 Constructing the Curve

The construction of the curve that will define the well W and the construction of the curve that will define the tunnel T are very similar. First, we will construct a curve that will define a submanifold Σ, which can be thought of as the precursor to a well or a tunnel.

We want to construct a curve γ so that the resulting manifold Σ has $R^{\Sigma} > \kappa - \frac{1}{j}$ for any $j \in \mathbb{N}$. We will first construct γ as a piecewise curve of circular arcs and then smooth the curve. To do this, we will prescribe the geodesic curvature $k(s)$ of γ, and by Theorem 6.7 in [27], we know that $k(s)$ determines γ. The unit tangent vector to γ and the curvature are given by

$$\frac{d\gamma}{ds} = (\sin\theta, -\cos\theta) \qquad \text{and} \qquad k = \frac{d\theta}{ds}.$$

Therefore, if $\gamma(s)$ is defined for $s \leq s'$ and $k(s)$ is given for $s \geq s'$ we have $\gamma(s) = (t(s), r(s))$ where

$$
\begin{aligned}
\theta(s) &= \theta(s') + \int_{s'}^{s} k(u)du \\
t(s) &= t(s') + \int_{s'}^{s} \sin\theta(u)du \\
r(s) &= r(s') - \int_{s'}^{s} \cos\theta(u)du.
\end{aligned}
\tag{2.1.2}
$$

Now, we begin the construction of γ. Fix $j \in \mathbb{N}$. Let $\delta_0 < \delta$ and let $(0, \delta_0)$ be a point in the (t, r)-plane. Next, define the initial segment of γ as the line segment from $(0, 2\delta + \delta_0)$ to $(0, \delta_0)$ for $s \in [-2\delta, 0]$. Define the next segment to be an arc of a circle of curvature $k_0 = 1$ that is tangent to r-axis at $(0, \delta_0)$ and let γ run from 0 to $s_0 \leq \frac{\delta_0}{2}$ where s_0 is chosen so that $R^{\Sigma} > \kappa - \frac{1}{j}$ and that $\frac{\sin\theta(s_0)}{8r(s_0)} < 1$ for all $s \leq s_0$. We note that s_0 exists since $\theta(0) = 0$ and by the scalar curvature formula (2.1.1). Next, we prove a lemma that gives a condition on γ that controls the scalar curvature.

Lemma 2.1.5. *If δ_0 is small enough and if*

$$
\frac{\sin\theta(s)}{4r(s)} > k(s) \text{ for } s \geq s_0,
\tag{2.1.3}
$$

then $R^{\Sigma} > \kappa$.

Proof. By (2.1.1) we see if $k \leq 0$ then

$$
\begin{aligned}
R^{\Sigma} = {} & R^M - 2\mathrm{Ric}^M(\partial_r, \partial_r)\sin^2\theta \\
& + (n-2)(n-1)\left(\frac{1}{r^2} + O(1)\right)\sin^2\theta \\
& - (n-1)\left(\frac{1}{r} + O(r)\right)k\sin\theta.
\end{aligned}
\tag{2.1.4}
$$

and so the third and fourth terms will be nonnegative. By taking $\delta_0 > r$ small enough, the third and fourth terms will dominate the second term so $R^{\Sigma} > \kappa$.

Now, if $k > 0$, then by rewriting the right-hand side of (2.1.1) we get

$$
\begin{aligned}
R^{\Sigma} = {} & \frac{(n-2)(n-1)}{2r^2}\sin^2\theta + \left(\frac{(n-2)(n-1)}{2r^2} - 2\mathrm{Ric}^M(\partial_r, \partial_r) + O(1)\right)\sin^2\theta \\
& + \frac{-2(n-1)k}{r}\sin\theta + \left(\frac{(n-1)}{r} - O(r)\right)k\sin\theta \\
& + R^M,
\end{aligned}
\tag{2.1.5}
$$

so second and fourth terms will be positive by taking $\delta_0 > r$ is small enough and by assumption we have

$$
\frac{\sin\theta}{4r} > k
$$

which implies

$$\frac{(n-2)(n-1)}{2r^2}\sin^2\theta + \frac{-2(n-1)k}{r}\sin\theta > 0,$$

and so $R^\Sigma > \kappa$. □

Thus, as we continue to construct γ, we will ensure that (2.1.3) is satisfied. We will now extend γ by a circular arc of curvature $k_1 = \frac{\sin\theta(s_0)}{8r(s_0)}$ on $[s_0, s_1]$ where $s_1 - s_0 = \frac{r_0}{2}$, where $r(s_0) = r_0$. Let $\theta(s_0) = \theta_0$. By (2.1.2), we have first that $\sin\theta(s)$ is increasing and $r(s)$ is decreasing and so on $[s_0, s_1]$

$$\frac{\sin\theta(s)}{4r(s)} > \frac{\sin\theta_0}{4r_0} > \frac{\sin\theta_0}{8r_0} = k_1.$$

Second, we see that γ does not cross the t-axis because $s_1 - s_0 = \frac{r_0}{2}$, and third we have

$$\theta(s_1) - \theta_0 = k_1(s_1 - s_0) = \frac{\sin\theta_0}{8r_0}\frac{r_0}{2} = \frac{\sin\theta_0}{16}.$$

Now we proceed inductively. Define:

$$s_i = s_{i-1} + \Delta s_i, \qquad \Delta s_i = \frac{r_{i-1}}{2}, \qquad r_i = r(s_i), \qquad \theta_i = \theta(s_i), \qquad k_i = \frac{\sin\theta_{i-1}}{8r_{i-1}}.$$

As $\theta(s)$ is increasing we have that $\theta_i - \theta_{i-1} = \frac{\sin\theta_{i-1}}{16} > \frac{\sin\theta_0}{16}$ and so

$$\theta_i \geq \theta_0 + i\frac{\sin\theta_0}{16}.$$

Therefore, θ_i grows without bound so define m to be such that $\theta_{m-1} < \sin^{-1}\left(\frac{12}{13}\right) \leq \theta_m$. Redefine s_m so that $\theta_m := \sin^{-1}\left(\frac{12}{13}\right) = \bar{\theta}$. Note that $\Delta s_m \leq \frac{r_{m-1}}{2}$.

Now extend again by one circular arc. To do this we need to define $k_{m+1} > 0$ and $s_{m+1} = s_m + \Delta s_{m+1}$. We add a circular arc until $\theta_{m+1} = \frac{\pi}{2}$. By the definition of $\bar{\theta}$, there exists a k_{m+1} such that $1 - \sin\bar{\theta} < k_{m+1}\frac{17r_m}{18} < \frac{\sin\bar{\theta}}{9}$ and by (2.1.2) we know

$$\begin{aligned}
r_{m+1} &= r_m - \int_{s_m}^{s_{m+1}} \cos\theta(u)du \\
&= r_m - \int_{s_m}^{s_{m+1}} \cos\left(s_m + k_{m+1}(u - s_m)\right)du \\
&= r_m - \frac{1}{k_{m+1}}\left(\sin\theta_{m+1} - \sin\theta_m\right) \\
&= r_m - \frac{1}{k_{m+1}}\left(1 - \sin\bar{\theta}\right) \\
&> \frac{r_m}{18}.
\end{aligned}$$

and $k_{m+1} < \frac{\sin\bar{\theta}}{4r_m}$.

Remark 2.1.6. *Up until this point the curve γ works for both the construction of a well and a tunnel. However, from here on the construction of γ differs slightly for the well and the tunnel. We will continue now with the construction of the well and discuss the tunnel construction later in Subsection 2.1.6.*

2.1.3 Adjusting the Curve to Construct a Well

Now we will refine our construction of γ in order to construct a well. We want to extend by a line with a negative slope of length $d > 0$ and not have γ cross the t-axis. By the intermediate value theorem there exists an $\hat{s} \in (s_m, s_{m+1})$ such that $\theta(\hat{s}) = \widehat{\theta}$ where

$$\begin{cases} \max\left\{\bar{\theta},\, \cos^{-1}\left(\frac{r_{m+1}}{2}\right)\right\} < \widehat{\theta} < \frac{\pi}{2} & \text{if } d \leq 1 \\ \max\left\{\bar{\theta},\, \cos^{-1}\left(\frac{r_{m+1}}{2d}\right)\right\} < \widehat{\theta} < \frac{\pi}{2} & \text{if } d > 1 \end{cases} \tag{2.1.6}$$

since $\theta_{m+1} = \frac{\pi}{2}$.

Redefine s_{m+1} such that $s_{m+1} = \hat{s}$ and $\theta_{m+1} = \widehat{\theta}$. Extend γ to $[s_{m+1}, s_{m+1} + d]$ by setting $k = 0$ on $[s_{m+1}, s_{m+1} + d]$. Furthermore, note that by (2.1.2) we have $\theta(u) \equiv \theta_{m+1}$ on that interval and

$$\begin{aligned} r(s_{m+1} + d) &= r_{m+1} - \int_{r_{m+1}}^{r_{m+1}+d} \cos\theta(u)du \\ &= r_{m+1} - d\cos\theta_{m+1} \\ &\geq r_{m+1} - \frac{r_{m+1}}{2} \\ &> 0. \end{aligned}$$

Let $s_{m+1} + d = s_{m+2}$ and $\theta(s_{m+1} + d) = \theta_{m+2}$. We now extend on $[s_{m+2}, s_{m+3}]$ by a small circular arc of negative geodesic curvature such that $\theta(s_{m+3}) = 0$. Take

$$k_{m+3} < \frac{-2\sin\theta_{m+2}}{r_{m+2}}.$$

Since,

$$\theta(s) = \theta_{m+2} + \int_{s_{m+2}}^{s} k_{m+3} du = \theta_{m+2} + k_{m+3}(s - s_{m+2}).$$

we have

$$\begin{aligned} r(s_{m+3}) &= r_{m+2} - \int_{s_{m+2}}^{s_{m+3}} \cos\theta(u)du \\ r_{m+3} &= r_{m+2} - \frac{1}{k_{m+3}}\left(\sin\theta_{m+3} - \sin\theta_{m+2}\right) \\ r_{m+3} &= r_{m+2} + \frac{1}{k_{m+3}}\sin\theta_{m+2} \\ &> 0. \end{aligned}$$

We can extend γ on $[s_{m+3}, s_{m+4}]$ by a vertical straight line by setting $k_{m+4} = 0$, where s_{m+4} is chosen so that $r(s_{m+4}) = 0$.

Since γ is parameterized by arclength, we note that a bound on s_{m+4} is a bound on arclength. In following lemmas, we prove an upper bound for s_{m+4}.

Lemma 2.1.7. *There exists a constant $0 < C_1 < 1$ independent of j and d such that*

$$\frac{r_i}{r_{i-1}} \leq C_1$$

for $1 \leq i \leq m-1$

Proof. By (2.1.2) and by the mean value theorem we have

$$r_i = r_{i-1} - \int_{s_{i-1}}^{s_i} \cos\theta(u)du = r_{i-1} - \Delta s_i \cos\xi_i = r_{i-1}\left(1 - \frac{\cos\xi_i}{2}\right)$$

for some $\xi_i \in [s_{i-1}, s_i]$. Recalling that $\bar{\theta} \geq \xi_i$ for $1 \leq i \leq m-1$ we see that

$$\frac{r_i}{r_{i-1}} \leq 1 - \frac{\cos\bar{\theta}}{2} = \frac{21}{26}.$$

□

Lemma 2.1.8. *There is a constant C_2 independent of j and d such that $s_{m+4} \leq C_2\delta_0 + d$ which implies that the length of γ is bounded by $C_2\delta_0 + d$.*

Proof. We recall for $1 \leq i \leq m$, $\Delta s_i \leq \frac{r_{i-1}}{2}$ so

$$\begin{aligned} s_m &= 2\delta - \delta_0 + s_0 + \Delta s_1 + \cdots + \Delta s_m \\ &\leq 2\delta + s_0 + \frac{1}{2}\left(r_0 + r_1 + \cdots + r_{m-1}\right) \\ &\leq 2\delta + s_0 + \frac{r_0}{2}\left(1 + C_1 + \cdots + C_1^{m-1}\right) \\ &\leq 3\delta + \frac{\delta_0}{2}\left(\frac{1}{1-C_1}\right) \\ &\leq \frac{28}{5}\delta \end{aligned} \tag{2.1.7}$$

Now, we note that by (2.1.2) and (2.1.6):

$$\Delta s_{m+1} \leq \frac{1}{k_{m+1}}\left(\frac{\pi}{2} - \bar{\theta}\right) < r_m \frac{\frac{\pi}{2} - \bar{\theta}}{1 - \sin\bar{\theta}} \leq \frac{\frac{\pi}{2} - \bar{\theta}}{1 - \sin\bar{\theta}}\delta_0 = 13\left(\frac{\pi}{2} - \sin^{-1}\left(\frac{12}{13}\right)\right)\delta_0.$$

By (2.1.2), we have that

$$\theta_{m+3} = \theta_{m+2} + k_{m+3}\Delta s_{m+3}.$$

Therefore,

$$\Delta s_{m+3} = \frac{\theta_{m+2}}{-k_{m+3}} \leq \frac{2r_{m+2}\theta_{m+2}}{\sin\theta_{m+2}} \leq \frac{\pi}{\sin\bar{\theta}}\delta_0 = \frac{13\pi}{12}\delta_0 \tag{2.1.8}$$

because $\theta_{m+2} \leq \frac{\pi}{2}$, $r_{m+2} < \delta_0$, and $\bar{\theta} < \theta_{m+2}$.

By construction,

$$\Delta s_{m+2} = d \qquad \text{and} \qquad \Delta s_{m+4} \leq \delta_0.$$

Thus,

$$\begin{aligned} s_{m+4} &= s_0 + \Delta s_1 + \cdots + \Delta s_m + \Delta s_{m+1} + \Delta s_{m+2} + \Delta s_{m+3} + \Delta s_{m+4} \\ &\leq C_2\delta + d. \end{aligned}$$

□

2.1.4 Smoothing the Curve that Defines the Well

So far we have constructed $k(s)$ as a piecewise constant function, $k\big|_{(s_i,s_{i+1}]} = k_{i+1}$ for $i = 1, \ldots, m-1$ and $k_i < k_{i+1}$ and $k_{m+1} < k_m$. The resulting curve γ is C^1 and piecewise C^∞.

We begin the smoothing of γ by first smoothing out $k(s)$ on $[0, s_{m+3}]$. Let $g \in C^\infty(\mathbb{R})$ be a smooth function so that g is 0 if $s < 0$, 1 if $s > 1$, and strictly increasing on $[0,1]$. Let $h(x) = g(1-x)$ and $H = \int_0^1 h(x)dx$. Let $\tilde{k}(s)$ be the smooth function defined by

$$\widetilde{k}(s) = \begin{cases} 0 & s \in [-2\delta, 0] \\ g\left(\frac{s}{\alpha}\right) & s \in [0, \alpha] \\ 1 & s \in [\alpha, s_0 - \alpha] \\ (1-k_1)h\left(\frac{s-(s_0-\alpha))}{\alpha}\right) + k_1 & s \in [s_0 - \alpha, s_0] \\ k_1 & s \in [s_0, s_1] \\ (k_{i+1} - k_i)\, g\left(\frac{s-s_i}{\alpha}\right) + k_i & s \in [s_i, s_i + \alpha] \\ k_{i+1} & s \in [s_i + \alpha, s_{i+1}] \\ h\left(\frac{s-s_m}{\alpha}\right) & s \in [s_m, s_m + \alpha] \\ k_{m+1} & s \in [s_m + \alpha, s_{m+1}] \\ k_{m+1} h\left(\frac{s-s_{m+1}}{\alpha}\right) & s \in [s_{m+1}, s_{m+1} + \alpha] \\ 0 & s \in [s_{m+1} + \alpha, s_{m+2}] \\ -k_{m+3} h\left(\frac{s-s_{m+2}}{\alpha}\right) + k_{m+3} & s \in [s_{m+2}, s_{m+2} + \alpha] \\ k_{m+3} & s \in [s_{m+2} + \alpha, s_{m+3}], \end{cases}$$

where $1 \le i \le m$.

We note that α is the same for each i and that its value will be determined later. Now let $\tilde{\theta}$ be the angle function associated to $\tilde{k}$. By (2.1.2) we see that the smooth curve $\tilde{\gamma}(s) = (\tilde{t}(s), \tilde{r}(s))$ defined by $\tilde{k}(s)$ will converge uniformly to γ on $\left[\frac{\delta_0}{2}, s_{m+3}\right]$ as α goes to zero. Also, $\tilde{\theta}$ will converge uniformly to θ as α goes to zero. Therefore, take α small enough such that $\tilde{\theta}(s_{m+1})$ satisfies (2.1.6); therefore, we will still extend by a line with a negative slope.

Note

$$\begin{aligned} \tilde{\theta}(s_{m+2} + \alpha) &= \tilde{\theta}(s_{m+2}) + \int_{s_{m+2}}^{s_{m+2}+\alpha} -k_{m+3} h\left(\frac{u - s_{m+2}}{\alpha}\right) + k_{m+3} du \\ &= \tilde{\theta}(s_{m+2}) + \alpha k_{m+3}(1 - H) \\ &> 0. \end{aligned}$$

By the smoothing process, $\tilde{\theta}(s_{m+3})$ may no longer be greater than 0. We will now fix that. If $\tilde{\theta}(s_{m+3}) \le 0$ pick a $s^* \in (s_{m+2} + \alpha, s_{m+3}]$ such that $0 < \tilde{\theta}(s^*) < \alpha$ which exists by the intermediate value theorem.

If $\tilde{\theta}(s_{m+3}) > 0$, we can redefine s_{m+3} as $s_{m+3} + \frac{\tilde{\theta}(s_{m+2})}{-k_{m+3}}$ so that $\tilde{\theta}(s_{m+3}) = 0$. By the intermediate value theorem, pick a $s^* \in (s_{m+2} + \alpha, s_{m+3}]$ such that $0 < \tilde{\theta}(s^*) < \alpha$.

Redefine s_{m+3} in either case as $s_{m+3} = s^*$ and note $0 < \tilde{\theta}(s^*) < \alpha$. On $[s_{m+3}, s_{m+3} + 2\beta]$ define

$$\tilde{k}(s) = \begin{cases} -k_{m+3} g\left(\frac{s-s_{m+3}}{\beta}\right) + k_{m+3} & s \in [s_{m+3}, s_{m+3} + \beta] \\ 0 & s \in [s_{m+3} + \beta, s_{m+3} + 2\beta], \end{cases}$$

where $\beta = \frac{\tilde{\theta}(s_{m+3})}{-k_{m+3}(1-H)}$ so that

$$\int_{s_{m+3}}^{s_{m+3}+\beta} \tilde{k}(s) ds = -\tilde{\theta}(s_{m+3}).$$

This makes $\tilde{\theta}(s_{m+3} + 2\beta) = 0$.

By (2.1.2) we see that the smooth curve $\tilde{\gamma}(s) = (\tilde{t}(s), \tilde{r}(s))$ defined by $\tilde{k}(s)$ will converge uniformly to γ on $[-2\delta, s_{m+3}]$ as α goes to zero. Also, $\tilde{\theta}$ will converge uniformly to θ as α goes to zero; moreover, as α goes to zero so does β. Finally, take α small enough so that $\tilde{r}(s_{m+3} + 2\beta) > 0$. Extend the line segment at the end of $\tilde{\gamma}$ on $[s_{m+3} + 2\beta, L]$ where L is defined so that $\tilde{r}(L) = 0$. Note that $|L - (s_{m+3} + 2\beta)| < \delta_0$.

2.1.5 Attaching the Well

We have constructed a smooth curve $\tilde{\gamma}$ on $[-2\delta, L]$ that begins and ends as a vertical line segment. Define

$$\tilde{W}_j = \{(y, q) \in X : (y, ||q||_M) \in \tilde{\gamma}\}$$

and let g_j be the induced metric, i.e., $\tilde{g}_j = \tilde{\iota}_j^*(dt^2 + g)$ where $\tilde{\iota}_j : \bar{W}_j \to \mathbb{R} \times B$ is the inclusion map.

Lemma 2.1.9. *For small enough α we will show that $\tilde{\gamma}$ satisfies $R^{\tilde{W}_j} \geq \kappa - \frac{1}{j}$ on $[-2\delta, L]$. Also the length $\tilde{\gamma}$ is bounded by $C_3\delta + d$.*

Proof. By construction $\tilde{\gamma}$ is parameterized by arclength so by Lemma 2.1.8 length of $\tilde{\gamma}$ is bounded by $C_3\delta_0 + d$.

On $[-2\delta, 0]$, we have that

$$\begin{aligned} R^{\tilde{W}_j} &= R^M - 2\mathrm{Ric}^M(\partial_{\tilde{r}}, \partial_{\tilde{r}}) \sin^2 \tilde{\theta} + (n-2)(n-1)\left(\frac{1}{\tilde{r}^2} + O(1)\right) \sin^2 \tilde{\theta} \\ &\quad - (n-1)\left(\frac{1}{\tilde{r}} + O(r)\right) \tilde{k} \sin \tilde{\theta} \\ &= R^M \\ &> \kappa - \frac{1}{j}. \end{aligned}$$

On $[0, s_0]$, we have that $k_1 < 1$ because of our choice of s_0 and the construction.

$$\begin{aligned}
R^{\tilde{W}_j} &= R^M - 2\mathrm{Ric}^M(\partial_{\tilde{r}}, \partial_{\tilde{r}}) \sin^2 \tilde{\theta} + (n-2)(n-1)\left(\frac{1}{\tilde{r}^2} + O(1)\right) \sin^2 \tilde{\theta} \\
&\quad - (n-1)\left(\frac{1}{\tilde{r}} + O(r)\right) \tilde{k} \sin \tilde{\theta} \\
&\geq \kappa - 2\mathrm{Ric}^M(\partial_{\tilde{r}}, \partial_{\tilde{r}}) \sin^2 \tilde{\theta} + (n-2)(n-1)\left(\frac{1}{\tilde{r}^2} + O(1)\right) \sin^2 \tilde{\theta} \\
&\quad - (n-1)\left(\frac{1}{\tilde{r}} + O(r)\right) \sin \tilde{\theta} \\
&> \kappa - \frac{1}{j}.
\end{aligned}$$

since for small enough α we have that $\tilde{\theta}$ is uniformly close to θ and $\tilde{r}$ is uniformly close to r.

On $[s_0, s_1]$ we have that $\tilde{k}(s) = k_1$ and so

$$\frac{\sin \tilde{\theta}(s)}{4\tilde{r}(s)} - \tilde{k}(s) > 0$$

for small enough α.

On $[s_i, s_{i+1}]$ for $1 \leq i \leq m-1$, we have that

$$\frac{\sin \tilde{\theta}(s)}{4\tilde{r}(s)} - \tilde{k}(s) = \left(\frac{\sin \tilde{\theta}(s)}{4\tilde{r}(s)} - k_{i+1}\right) + \left(k_{i+1} - \tilde{k}(s)\right)$$

and so for small enough α we have the the first term is positive since $\frac{\sin \theta(s)}{4r(s)} > k(s)$ and the second term is positive by construction.

On $[s_m, s_{m+1}]$, recall $k_m > k_{m+1}$ and that $\frac{\sin \tilde{\theta}(s)}{4\tilde{r}(s)}$ is non-decreasing therefore we have that

$$\frac{\sin \tilde{\theta}(s)}{4\tilde{r}(s)} \geq \frac{\sin \tilde{\theta}(s_m)}{4\tilde{r}(s_m)} > k_m \geq \tilde{k}(s).$$

On $[s_{m+1}, s_{m+2}]$, we have that

$$\frac{\sin \tilde{\theta}(s)}{4\tilde{r}(s)} \geq \frac{\sin \tilde{\theta}(s_{m+1})}{4\tilde{r}(s_{m+1})} > k_{m+1} > \tilde{k}(s).$$

We have the first inequality since $\frac{\sin \tilde{\theta}(s)}{4\tilde{r}(s)}$ is non-decreasing. The second inequality was already verified above. The third inequality holds since by construction $k_{m+1} > \tilde{k}(s)$.

On $[s_{m+2}, L]$, we have by construction that $\tilde{k}(s)$ is non-positive so

$$\frac{\sin \tilde{\theta}(s)}{4\tilde{r}(s)} \geq \tilde{k}(s).$$

Therefore, by Lemma 2.1.5 we have shown $R^{\tilde{W}_j} > \kappa - \frac{1}{j}$ on $[-2\delta, L]$. □

Next we will prove the diameter and volume bounds for the well, but before we prove those bounds, we need to recall the following fact.

Proposition 2.1.10. *Let $B(p,r)$ be a geodesic ball of radius r in a closed Riemannian manifold (M^n, g). Then there exists constants C, r_0 depending on g such that for any p and for all $r \leq r_0$ we have*

$$\begin{aligned} \mathrm{vol}_g(B(p,r)) &\leq Cr^n \\ \mathrm{vol}_g(\partial B(p,r)) &\leq Cr^{n-1}. \end{aligned} \tag{2.1.9}$$

Lemma 2.1.11. *There is a constant $C(g)$ independent of j, d such that diameter* $\mathrm{diam}\,(\tilde{W}_j)$ *and volume* $\mathrm{vol}\,(\tilde{W}_j)$ *of $\tilde{W}_j$ satisfy*

$$d \leq \mathrm{diam}\,(\tilde{W}_j) < C(\delta + d) \ \text{ and } \ \mathrm{vol}\,(\tilde{W}_j) < C(\delta^n + d\delta^{n-1}).$$

Proof. Let $p, q \in \tilde{W}_j$ be two points and let x be the point at the tip of $\tilde{W}_j$, i.e., corresponding to $\tilde{\gamma}(L)$. By the triangle inequality and Lemma 2.1.3 we have

$$d_{g_j}(p,q) \leq d_{g_j}(p,x) + d_{g_j}(x,q) \leq \mathrm{length}(\tilde{\gamma}) + \mathrm{length}(\tilde{\gamma}) \leq C(\delta + d).$$

By construction we have $d \leq \mathrm{diam}\,(W)$. Therefore, $d \leq \mathrm{diam}\,(W) < C(\delta + d)$.

By possibly taking δ smaller, we have by Lemma 2.1.9 and Proposition 2.1.10 that

$$\mathrm{vol}\,(\tilde{W}_j) = \int_{\frac{-\delta_0}{2}}^{L} |\partial B(p, r(s))|_{\tilde{g}_j} ds \leq \int_{\frac{-\delta_0}{2}}^{L} C\delta^{n-1} ds \leq C(\delta^n + \delta^{n-1} d).$$

□

Lemma 2.1.12. *$(\tilde{W}_j, \tilde{g}_j)$ is isometric to $W_j = (B_g(p, 2\delta), g_j = dF_j^2 + g)$ and W_j attaches smoothly to M.*

Proof. Let $B = B_g(p, 2\delta)$ and recall that $||q||_g$ is the distance from q to p in B. Consider the function $F_j : B \to \mathbb{R}$, $F_j(q) = \tilde{t}\,(\tilde{r}^{-1}(||q||_g))$. By construction $\tilde{t}$ is smooth and $\tilde{r}'(s) < 0$ so $\tilde{r}^{-1}$ is smooth. Moreover, $||q||$ is smooth away from p. Thus, away from p, F is smooth. In a neighborhood of p we have by construction that $(\tilde{t}(s), \tilde{r}(s))$ is a vertical line segment so in that neighborhood $\tilde{t} \circ \tilde{r}^{-1} \equiv const$ and so F_j is smooth everywhere. Furthermore, by construction, we have that

$$\tilde{g}_j|_E = g|_E \text{ where } E = B_g(p, 2\delta) \setminus B_g(p, \delta).$$

Let $\Gamma_j = \{(t,p) \in X : F_j(p) = t\}$. Note that $\Gamma_j \subset X$ and that $\Gamma_j = \tilde{W}_j$. Let $g_j' = (\iota_j')^*(dt^2 + g)$ where $\iota_j' : \tilde{W}_j \to \mathbb{R} \times B$ is the inclusion map $\iota_j'(t,p) = (t,p)$. Let $id_j : \Gamma_j \to \tilde{W}_j$ be the identity map and conclude that $g_j' = \tilde{g}_j$. Consider the diffeomorphism $\Phi_j : B \to \Gamma_j$ where $\Phi_j(q) \mapsto (F_j(q), q)$. And so

$$\Phi_j^* g_j' = \Phi^*((\iota_j')^* g_j') = dF_j^2 + g.$$

□

And this completes the construction of N from Proposition 2.1.1 (Constructing Wells). Before moving to the tunnel construction we would like to record a lemma which shows there are no minimal surfaces in $\tilde{W}_j$:

Lemma 2.1.13. *There exist no closed minimal surfaces in* $(\tilde{W}_j, \tilde{g}_j)$.

Proof. We will show that $\tilde{W}_j$ is foliated by negative mean curvature hypersurfaces and so by the maximum principle does not contain any closed minimal surfaces. Recall

$$\tilde{W}_j = \left\{(y, q) \in X : (y, ||q||_M) \in \tilde{\gamma} = (\tilde{t}(s), \tilde{r}(s))\right\}$$

and that the metric on X is $g_X = dt^2 + dr^2 + g_r$ and $\tilde{W}_j$ has the induced metric. We note that $\tilde{W}_j$ is foliated by $\tilde{r}(s) = const$ hypersurfaces. Note that the second fundamental form is $A(X, Y) = g(\nabla_X Y, N)$ where N is the outward normal.

In particular, we note that for g_r the outward normal is ∂_r and $-\partial_s$ is the outward normal for $g_{r(s)}$. Therefore, we have the second fundamental form of these hypersurfaces, by Lemma 2.1.3, is

$$\partial_s g_{\tilde{r}(s)} = \left(\frac{1}{\tilde{r}(s)} + O(\tilde{r}(s))\right)(\tilde{r}'(s)) < 0.$$

We conclude that the mean curvature $H_{\tilde{r}(s)} < 0$ since by the construction $\frac{1}{\tilde{r}(s)}$ is the dominating term and $\tilde{r}'(s) < 0$ with respect to the outward normal. Therefore, $\tilde{W}_j$ is foliated by hypersurfaces with strictly negative mean curvature. ☐

2.1.6 Constructing a Tunnel

We will pick up the construction of the tunnel from Remark 2.1.6. Let γ be as it is before Remark 2.1.6. The same smoothing procedure as above can be used to smooth γ into a smooth curve. We will abuse notation and call this smoothed-out curve $\tilde{\gamma}$ as well.

Let $g \in C^\infty(\mathbb{R})$ be the smooth function so that g is 0 if $s < 0$, 1 if $s > 1$, and strictly increasing on $[0, 1]$. Let $h(x) = g(1-x)$ and $H = \int_0^1 h(x)dx$. Let $\tilde{k}(s)$ be the smooth function defined by

$$\widetilde{k}(s) = \begin{cases} g\left(\frac{s}{\alpha}\right) & s \in \left[-\frac{\delta_0}{2}, \alpha\right] \\ 1 & s \in [\alpha, s_0 - \alpha] \\ (1-k_1)h\left(\frac{s-s_0}{\alpha}\right) + k_1 & s \in [s_0 - \alpha, s_0] \\ k_1 & s \in [s_0, s_1] \\ (k_{i+1} - k_i)\, g\left(\frac{s-s_i}{\alpha}\right) + k_i & s \in [s_i, s_i + \alpha] \\ k_{i+1} & s \in [s_i + \alpha, s_{i+1}] \end{cases}$$

where $1 \leq i \leq m$.

Note that $\tilde{\theta}(s_{m+1})$ could no longer equal $\frac{\pi}{2}$ by the smoothing process. We fix that now. We note that $\tilde{\theta}(s)$ converges uniformly to $\theta(s)$ as α goes to zero. Take α be small enough such that $\tilde{\theta}(s_m + \alpha) < \frac{\pi}{2}$.

We want $\tilde{\theta}(s_{m+1}) < \frac{\pi}{2}$. Therefore, if not, then $\tilde{\theta}(s_{m+1}) \geq \frac{\pi}{2}$. Pick a $s^* \in (s_m + \alpha, s_{m+1}]$ such that $\frac{\pi}{2} - \alpha < \tilde{\theta}(s^*) < \frac{\pi}{2}$ which exists by the intermediate value theorem and redefine $s_{m+1} = s^*$.

Let $s_{m+2} = s_{m+1} + 2\beta$. On $[s_{m+1}, s_{m+2}]$, define

$$\tilde{k}(s) = \begin{cases} -k_{m+1} g\left(\frac{s - s_{m+1}}{\beta}\right) + k_{m+1} & s \in [s_{m+1}, s_{m+1} + \beta] \\ 0 & s \in [s_{m+1} + \beta, s_{m+2}], \end{cases}$$

where $\beta = \frac{\frac{\pi}{2} - \tilde{\theta}(s_{m+1})}{-k_{m+1}(1-H)}$ so that

$$\int_{s_{m+1}}^{s_{m+1}+\beta} \tilde{k}(s) ds = \frac{\pi}{2} - \tilde{\theta}(s_{m+1}).$$

Thus, $\tilde{\theta}(s) = \frac{\pi}{2}$ for all $s \in [s_{m+1} + \beta, s_{m+2}]$. Moreover, we have finished smoothing γ to $\tilde{\gamma}$.

Define a half tunnel $A_j = \{(y, q) \in X : (y, ||q||_g) \in \tilde{\gamma}\}$ with the induced metric. Later, we will glue two half tunnels together to make a tunnel T_j. In the following lemma, we record properties of A_j whose proofs are analogous to the ones above.

Lemma 2.1.14. *There is a constant C independent of j such that (A_j, h_j) satisfies the following*

i. *The scalar curvature R^j of A_j satisfies $R^j > \kappa - \frac{1}{j}$.*

ii. $\operatorname{diam}(A_j) < C(\delta)$.

iii. $\operatorname{vol}(A_j) < C(\delta^n)$.

iv. *A_j smoothly attaches to $M \setminus B_g(p, 2\delta)$*

v. *The new manifold $(M \setminus B_g(p, 2\delta)) \sqcup A_j$ is a manifold with boundary.*

We have constructed half of a tunnel, A_j. We now wish to modify the metric at the end of A_j so that it is a product metric of a round sphere and an interval. We follow the same procedure as [26]. Let $a = t(s_{m+1} + \beta)$, $b = t(s_{m+2})$, and $c = r(s_{m+2})$. We note that, by construction, the induced metric on $\{(q, y) \in X : a \leq t \leq b\}$ is $h_0 = g_c + dt^2$, where g_c is the induced metric on $\mathbb{S}^{n-1}(c)$. Let $h_1 = c^2 g_{rd} + dt^2$ where g_{rd} is the round metric on the unit round sphere. Let $\phi(t) = \psi\left(\frac{t-a}{\eta}\right)$ where $\psi(u)$ is a smooth function on $[0, 1]$ vanishing near zero, increasing to 1 at $u = \frac{3}{4}$ and equal to 1 for $u > \frac{3}{4}$. Define the metric h for $t \in [a, b]$ as

$$h(q, y) = g_c(q, y) + \phi(t)\left(c^2 g_{rd} - g_c\right) + dt^2.$$

This metric transitions smoothly between h_0 and h_1. Note

$$h - h_0 = \phi(t)\left(c^2 g_{rd} - g_c\right) = \phi(t) c^2 \left(g_{rd} - \frac{1}{c^2} g_c\right)$$

and that the first and second derivatives of $\phi(t)$ are $O(\eta^{-1})$ and $O(\eta^{-2})$, respectively. So by Lemma 2.1.3, we have that the second derivatives of $h - h_0$ are $O(\eta^2)$. Therefore, for η small enough, the scalar curvature of h is close to the scalar curvature of h_0 which, again by Lemma 2.1.3, has scalar curvature larger than $\kappa - \frac{1}{j}$ for small enough η. Therefore, we have changed the metric at the end of A_j so that it looks like $c^2 g_{rd} + dt^2$. Thus, given another ball $B_g(p', 2\delta)$ on M we can construct A'_j with a metric at the one end that it looks like $c^2 g_{rd} + dt^2$ with the same c by making the same choices in the construction as we did for A_j. Now we can immediately glue a cylinder, $([0, d] \times \mathbb{S}^{n-1}, dt^2 + c^2 g_{rd})$, connecting A'_j to A_j and so construct the tunnel T_j between $\partial B_g(p', 2\delta)$ and $\partial B_g(p, 2\delta)$.

We note that the diameter and volume of the cylinder $([0, d] \times \mathbb{S}^{n-1}, dt^2 + g_{S^{n-1}})$ are bounded by d and $C(n)d\delta^{n-1}$, respectively, where $C(n)$ is a constant that only depends on the dimension. Therefore, we can conclude that $\mathrm{diam}(T_j)$ and $\mathrm{vol}(T_j)$ satisfy the bounds in Proposition 2.1.2. Therefore, this completes the construction for Proposition 2.1.2.

2.2 Surgery in Higher Codimensions

Now, we prove a technical proposition, which is a Gromov–Lawson type surgery construction in codimension three or larger with a quantitative lower bound on the scalar curvature. As expressed by Gromov–Lawson in [28] the higher surgery result is very similar to connect sum construction. Therefore, the proof will combine the improved connect sum (tunnel) construction above and the proof in [28] (cf. [59]). In particular, we show

Proposition 2.2.1 (Surgery construction). *Let $\kappa \in \mathbb{R}$. Let (M^n, g), $n \geq 3$, be a Riemannian manifold with scalar curvature satisfying $R^M \geq \kappa$. Let N^n be a smooth manifold that can be obtained from M by performing surgeries in codimension greater than or equal to three. Then for any $\delta > 0$ small enough, there exists a metric $\bar{g}$ on N such that the scalar curvature satisfies $R^N > \kappa - \delta$.*

We will construct a curve γ in S, which we recall can be identified with a strip in $\mathbb{R}^2$, in order to prove Proposition 2.2.1 in a manner similar to Proposition 2.1.2. Let us describe the analogous set up we have here.

Let (M^n, g) be a Riemannian manifold with scalar curvature $R^M \geq \kappa$. Let $p = n - q$ and let $\mathbb{S}^p$ be an embedded sphere in M with a trivial normal bundle N. Let $\nu_1, \ldots, \nu_q$ be a global orthonormal sections of N. Note N is diffeomorphic to $\mathbb{S}^p \times \mathbb{R}^q$ via the map $(u, v) \mapsto (u, x_1, \ldots, x_q)$ where $v = \sum x_j(\nu_j)_u$. Now define $r : \mathbb{S}^p \times \mathbb{R}^q$, $r(u, v) = ||x||$ and $T(s) := \mathbb{S}^p \times \mathbb{B}^q(s) = \{(u, v) \in \mathbb{S}^p \times \mathbb{R}^q : r(v) \leq s\}$. Choose $\delta > 0$ small enough so that the exponential map $\exp : N \to M$ is an embedding on $T(\delta) \subset N$. Lift the metric via the exponential map to $T(\delta)$ and call this metric g. Therefore, r is then the distance function to $\mathbb{S}^p \times \{0\}$ in $T(\delta)$. Also, curves of the form $\{u\} \times \ell$ where ℓ is a geodesic ray in $\mathbb{B}^q(\delta)$ emanating form the origin, are geodesics in $T(\delta)$.

Let γ be a smooth curve in S and consider the submanifold

$$\Sigma = \{t, u, v) : (t, r(v)) \in \gamma\}$$

of $(\mathbb{R} \times T(\delta), dt^2 + g)$. Endow Σ with the induced Riemannian metric g_γ. For brevity we will suppress the δ and refer to $T(\delta)$ as T from here on out. Now we want to calculate the scalar curvature of Σ.

We note that submanifolds of the form $S = \mathbb{R} \times (\{u\} \times \ell)$ of $\mathbb{R} \times T$ are totally geodesic. Now to calculate the scalar curvature of Σ, fix $w \in \Sigma \cap S$. Let $e_1, \ldots, e_n$ be an orthonormal basis of of $T_w(\Sigma)$ where e_1 is tangent to γ and $e_2, \dots, e_q$ is tangent to $\partial\mathbb{B}^q$. Note that the for points in $\Sigma \cap S$ the normal ν to Σ in $\mathbb{R} \times T(\delta)$ is the same as the normal to γ in S.

By similar calculations as Section 2.1.1 we have that $\lambda_1 = k$ where k is the geodesic curvature of γ. For $i = 2, \dots, q$ we see by Lemma 2.1.3

$$\begin{aligned}\lambda_i &= \langle \nabla_{e_i} e_i, \nu \rangle = \langle \nabla_{e_i} e_i, \cos\theta \partial_t + \sin\theta \partial_r \rangle = \cos\theta \langle \nabla_{e_i} e_i, \partial_t \rangle + \sin\theta \langle \nabla_{e_i} e_i, \partial_r \rangle \\ &= \sin\theta \langle \nabla_{e_i} e_i, \partial_r \rangle = \left(\frac{1}{-r} + O(r) \right) \sin\theta,\end{aligned}$$

where θ is the angle that between ν and the t-axis. And for $(q+1), \dots, n$ we have

$$\begin{aligned}\lambda_i &= \langle \nabla_{e_i} e_i, \nu, \rangle = \langle \nabla_{e_i} e_i, \cos\theta \partial_t + \sin\theta \partial_r \rangle = \cos\theta \langle \nabla_{e_i} e_i, \partial_t \rangle + \sin\theta \langle \nabla_{e_i} e_i, \partial_r \rangle \\ &= \sin\theta \langle \nabla_{e_i} e_i, \partial_r \rangle = O(1) \sin\theta.\end{aligned}$$

Therefore, we see analogous to Section 2.1.1 we see,

$$\begin{aligned}R^\Sigma = R^M &- 2\mathrm{Ric}(\partial_r, \partial_r) \sin^2\theta + (q-2)(q-1) \left(\frac{1}{r^2} + O(1) \right) \sin^2\theta \\ &- (q-1) \left(\frac{1}{r} + O(r) \right) k \sin\theta.\end{aligned} \tag{2.2.1}$$

Now we can take the desired curve γ to be the curve from Section 2.1.6, in particular this is the curve that was used to construct half tunnels.

Therefore, we have now constructed a Riemannian metric, g_γ, on Σ which smoothly attaches to $M \setminus T$. Let P be the Riemannian manifold defined as $P = (M \setminus T) \sqcup \Sigma$ and with a slight abuse of notation we will let g_γ be the metric on all of P. Note that by construction P is a smooth manifold with boundary, which satisfies $R^P > \kappa - \delta$. Now we want to find a homotopy, $\{h_t\}$, through metrics with scalar curvature strictly greater than $\kappa - \delta$, from the induced metric on ∂P to $g_1^p + g_a^{q-1}$ on $\mathbb{S}^p \times \mathbb{S}^{q-1}$ where g_τ^m is the standard round metric on $\mathbb{S}^m$ of radius τ. Moreover, take $a < \delta$ small enough so the scalar curvature is strictly larger than $\kappa - \delta$. Then one can construct a metric $ds^2 + h_s$ on the collar, $C = [0,1] \times \mathbb{S}^p \times \mathbb{S}^{q-1}$, such that at h_0 is the induced metric on ∂P and at h_1 is a product metric $g_1^p + g_a^{q-1}$ on $\mathbb{S}^p \times \mathbb{S}^{q-1}$. Finally, we can glue in $\mathbb{B}^{p+1} \times \mathbb{S}^{q-1}$ with the product metric of the metric on the round hemisphere and the round metric on the sphere to complete the surgery. We handle these two steps in the next lemmas.

The following two lemmas were first proved in [28] (cf. [59]) in the setting of positive scalar curvature. Here we upgrade these lemmas to our setting.

Lemma 2.2.2. *Let $g_{\partial P}$ be the induced metric on ∂P from g_γ. There exists a homotopy through Riemannian metrics with scalar curvature strictly larger than $\kappa - \delta$ to the product metric $g_1^p + g_a^{q-1}$.*

Proof. We follow the proof of [59]. Let $g_{\partial P}$ be the induced metric on ∂P from g_γ, recall $r(L) < \delta$, and set $\eta = r(L)$. Let $\pi : \mathcal{N} \to \mathbb{S}^p$ be the normal bundle to the embedded $\mathbb{S}^p$ in M. The Levi-Civita connection on M, with respect to g_γ, gives rise to a normal connection on the total space of the normal bundle and so also a horizontal distribution $\mathcal{H}$ on the total space of $\mathcal{N}$. Equip the fibers of $\mathcal{N}$ with the metric $\hat{g} = h_\eta$, where h_η is the Riemannian metric on the q-ball that arises from the upper hemisphere of a round q-sphere of radius η. Now consider the Riemannian submersion $\pi : (\mathcal{N}, \tilde{g}) \to (\mathbb{S}^p, \check{g})$ where $\check{g} = g_{\partial P}$ and $\tilde{g}$ is the unique submersion metric arising from $\hat{g}$, $\check{g}$, and $\mathcal{H}$. Now we want to restrict this metric $\tilde{g}$ to the sphere bundle, $\mathbb{S}^p \times \mathbb{S}^{q-1}(\eta)$, and by a slight abuse of notation we will call this restricted metric $\tilde{g}$. We note that $\tilde{g}$ restricted to $\mathbb{S}^{q-1}_\eta$ is g^{q-1}_η, i.e., the round metric on the $(q-1)$-sphere of radius η.

Now by Lemma 2.1.3 (cf. [59, Lemma 3.9]) we have that $g_{\partial P}$ converges to $\tilde{g}$ as $\eta \to 0$. Therefore for small η, there is a homotopy from $g_{\partial P}$ to $\tilde{g}$ through metrics with scalar curvature strictly greater than $\kappa - \delta$. Now we want to construct a homotopy from $\tilde{g}$ to the product of two round spheres. Viewing $\tilde{g}$ as a submersion metric and applying O'Neill's formula [10, Chapter 9] we have the following equation for scalar curvature:

$$\tilde{R} = \check{R} \circ \pi + \hat{R} - |A|^2 - |T|^2 - |n|^2 - 2\check{\delta}(n).$$

T is a tensor that measures the obstruction to the bundle having totally geodesic fibers. By construction, the fibers are totally geodesic so $T = 0$ and n is the mean curvature vector and so vanishes when T vanishes. A is O'Neill's integrability tensor and it measures the integrability of the horizontal distribution, i.e., when A vanishes the horizontal distribution is integrable. Therefore, we have

$$\tilde{R} = \check{R} \circ \pi + \hat{R} - |A|^2$$

We can deform $\tilde{g}$ through Riemannian submersions to one with a base metric g^p_1 while keeping scalar curvature larger than $\kappa - \delta$. This can be done because the deformation occurs on a compact interval and we can shrink η to make $\hat{R}$ arbitrarily large. We just need to ensure that when we shrink η that the $|A|$ term does not grow. This follows from the canonical variation formula [10, Chapter 9] which states that if shrinks the fiber metric by t then the scalar curvature of the new submersion metric $\tilde{R}_t$ satisfies

$$\tilde{R}_t = \check{R} \circ \pi + \frac{1}{t}\hat{R} - t|A|^2$$

Finally, we can perform another linear homotopy through Riemannian submersions to the standard product metric $g^p_1 + g^{q-1}_\eta$, i.e., where $|A| = 0$. Again we can shrink η if necessary in order to preserve the scalar curvature bound. ∎

Lemma 2.2.3. *Let $\{h_t\}$ be a family of Riemannian metrics on a closed manifold M. Consider the Riemannian manifold $([0, c] \times M, ds^2 + h_{f(s)})$ where $f : \mathbb{R} \to \mathbb{R}$ is a smooth function. Then at $p = (t, q) \in \mathbb{R} \times M$ we have*

$$R^{[0,c]\times M} = R^{f(t)} + O((f'(t))^2 + f''(t))$$

where $R^{[0,c]\times M}$ and $R^{f(t)}$ are the scalar curvatures of $([0, c] \times M, dt^2 + h_{f(t)})$ and $(M, h_{f(t)})$, respectively.

Proof. Let $e_1, \ldots, e_{n+1}$ be an orthonormal basis of the tangent space $T_p(\mathbb{R} \times M)$ where e_1 is tangent to the $\mathbb{R}$ factor. We now compute the sectional curvatures. Let $2 \le i, j \le n + 1$ and note that $A_{ij} = \langle \nabla_{e_i} e_1, e_j \rangle = O(f'(t))$ so by the Gauss-Codazzi equations we have

$$K^{[0,c]\times M}_{i,j} = R^{[0,c]\times M}_{jiij} = R^{f(t)}_{ijji} - A_{ii}A_{jj} + A^2_{ij} = R^M_{ijji} + O((f'(t))^2).$$

Now we must compute $K_{1,j}^{[0,c]\times M}$ for $2 \leq j \leq n+1$.

$$\begin{aligned} K_{1,j}^{[0,c]\times M} &= R_{j11j}^{[0,c]\times M} = \partial_j \Gamma_{11,j} - \partial_1 \Gamma_{j1,j} + h_{f(t)}^{ml} \Gamma_{j1,m} \Gamma_{1j,l} - h_{f(t)}^{ml} \Gamma_{11,m} \Gamma_{jj,l} \\ &= O((f'(t))^2 + f''(t)) \end{aligned}$$

And so we can now compute

$$R^{[0,c]\times M} = 2 \sum_{i<j} K_{ij}^{[0,c]\times M} = 2K_{1j}^{[0,c]\times M} + 2 \sum_{1<i<j} K_{ij}^{[0,c]\times M} = R^{f(t)} + O((f'(t))^2 + f''(t))$$

□

If we take $f(s) = \frac{s}{c}$ in Lemma 2.2.3 and h_t to be the homotopy from Lemma 2.2.2 then by compactness we can find a large enough c such that the metric $dt^2 + h_{\frac{t}{c}}$ on the collar C has scalar curvature larger than $\kappa - \delta$ and transitions from the induced metric on ∂P to $g_1^p + g_a^{q-1}$. Thus completing the construction of $(N, \bar{g})$.

Chapter 3

Constructing Manifolds

3.1 Manifolds with Shrinking Tunnels

In this section, we will use Proposition 2.1.2 (Wells and Tunnels) to construct sequences of manifolds with thinner and thinner long tunnels. Furthermore, we will prove Theorems A and A′.

We will need first the following preliminary results.

Proposition 3.1.1. *There exists a sequence of rotationally symmetric manifolds* $M_j = (\mathbb{S}^n, g_j)$*,* $n \geq 3$*, such that* M_j *satisfies*

$$R^j \geq n(n-1) - \frac{1}{j}, \ \operatorname{diam}(M_j) \leq D, \ \textit{and} \ \operatorname{vol}(M_j) \leq V,$$

for some constants $0 < D, V$ *and converges to* M_∞ *which is the disjoint union of two* n*-spheres.*

Proof. We will construct the M_j as the connected sum of two standard unit round n-spheres for which the tunnel that connects the two spheres gets skinnier as j increases. By Proposition 2.1.2, we can remove a geodesic ball from both of the spheres and then construct a tunnel T_j connecting the two spheres. Let $(N, h) = (N', h') = (\mathbb{S}^n, g_{rd})$. Let $j \in \mathbb{N}$, $j \geq 10$, $d = 30$. Define

$$B_j := B_h\left(p, \frac{2}{j}\right) \subset N, \text{ and } B' := B_{h'}\left(p', \frac{2}{j}\right) \subset N'$$

where B_j and B'_j are geodesic balls in N, N' respectively. By Proposition 2.1.2, we can construct a tunnel T_j connecting ∂B_j to $\partial B'_j$ and the resulting manifold M_j will have the following properties:

i. $M_j = \big((N \sqcup N') \setminus \big(B_j \cup B'_j\big)\big) \sqcup T_j$

ii. $R^j \geq n(n-1) - \frac{1}{j}$.

iii. $M_j \setminus T_j$ is isometric to $(N \setminus B_j) \sqcup (N' \setminus B'_j)$.

iv. $\operatorname{diam}(M_j) \leq 4\pi + 30$,

$$2\operatorname{vol}_{g_{rd}}(\mathbb{S}^n) - \operatorname{vol}_h(B_j) - \operatorname{vol}_{h'}(B'_j) \leq \operatorname{vol}(M_j) \leq 2\operatorname{vol}_{g_{rd}}(\mathbb{S}^n) + \operatorname{vol}_{g_j}(T_j),$$

and

$$\lim_{j\to\infty} \operatorname{vol}_h(B_j) = \lim_{j\to\infty} \operatorname{vol}_{h'}(B'_j) = \lim_{j\to\infty} \operatorname{vol}_{g_j}(T_j) = 0.$$

In particular, $\lim_{j\to\infty} \operatorname{vol}(M_j) = 2\operatorname{vol}_{g_{rd}}(\mathbb{S}^n)$.

By Theorem 1.3.3, we have the intrinsic flat distance between M_j and $N \sqcup N'$ is

$$d_{\mathcal{F}}(M_j, N_1 \sqcup N_2) \lesssim \frac{1}{j}\left(\operatorname{vol}_{grd}(\mathbb{S}^n) + \operatorname{vol}_{g_{rd}}(\mathbb{S}^{n-1})\right) + \operatorname{vol}_h(B_j) + \operatorname{vol}_{h'}(B'_j) + \operatorname{vol}_{g_j}(T_j).$$

As $j \to \infty$, we that $\operatorname{vol}_h(B_j)$, $\operatorname{vol}_{h'}(B'_j)$, and $\operatorname{vol}_{g_j}(T_j)$ go to zero. Therefore, we conclude that M_j converges to $N \sqcup N'$ in the $\mathcal{VF}$ sense. □

Remark 3.1.2. *We have explicit bounds for D and V in Proposition 3.1.1 (and so also Theorems A and A′). From above we see $D \leq 2\pi + 30$ and $V \leq 4\pi^2 + \frac{1}{10}$.*

Remark 3.1.3. *From the construction in Proposition 2.1.2 (Wells and Tunnels) we see that $M_j^n = ([0, D_j] \times \mathbb{S}^{n-1}, g_j)$ defined above is rotationally symmetric. Moreover, near $\{0\} \times \mathbb{S}^{n-1}$ and $\{D_j\} \times \mathbb{S}^{n-1}$, we have that M_j^n is isometric to the standard unit round n-sphere. In particular, the metric takes the form $g_j = dt^2 + \sin^2(\rho_j(t))g_{\mathbb{S}^{n-1}}$ where D_j is the diameter of M_j and for $\rho_j : [0, D_j] \to [0, \infty)$ is a smooth function with the following properties. Recall $\tilde{\gamma}_j = (\tilde{t}_j(s), \tilde{r}_j(s))$ to be the curve define in Lemma 2.1.14 that defines the half tunnel A_j. Then*

$$\rho(t) = \begin{cases} \hat{r}(t), & t \in \left[0, \frac{1}{2}D_j\right] \\ \hat{r}(D - t), & t \in \left[\frac{1}{2}D_j, D_j\right] \end{cases} \quad \textit{and} \quad \hat{r}(t) = \begin{cases} \pi - t, & t \in \left[0, \pi - \frac{2}{j}\right] \\ \tilde{r}(t + (\delta - \pi)), & t \in \left[\pi - \frac{2}{j}, \frac{1}{2}D\right]. \end{cases}$$

We will now construct smooth 1-Lipschitz maps $F_j : M_j^n \to (\mathbb{S}^n, g_{rd})$. But first, we need the following result based on the mollification in [48, Section 3]. Since our lemma varies slightly from what is stated in [48] we provide an analogous proof.

Lemma 3.1.4. *Let $h : \mathbb{R} \to \mathbb{R}$ be an L-Lipschitz continuous function such that*

$$h(t) = \begin{cases} h_+(t), & t \in (0, \infty) \\ h_-(t), & t \in (-\infty, 0), \end{cases}$$

where h_+ and h_- are smooth functions. Then for small enough $\epsilon > 0$ there exists a function $h_\epsilon : \mathbb{R} \to \mathbb{R}$ such that

$$||h_\epsilon(t) - h(t)||_{C^2} \lesssim \epsilon^2, \ \ h'_\epsilon(t) \leq \sup\{h'(t) : t \in \mathbb{R} \setminus \{0\}\}, \ \ \textit{and} \ \ |h'_\epsilon(t)| \leq L.$$

Proof. Let $0 < \epsilon_0 < 1$. We will restrict our attention to $(-\epsilon_0, \epsilon_0)$. Let $\varphi \in C_c^\infty([-1,1])$ be the standard mollifier in $\mathbb{R}$ such that

$$0 \leq \varphi \leq 1 \qquad \text{and} \qquad \int_{-1}^{1} \varphi(t)dt = 1.$$

Let $\sigma(t) \in C_c^\infty\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right)$ be another bump function such that

$$\begin{aligned} &0 \leq \sigma(t) \leq \frac{1}{100} \text{ for } t \in \mathbb{R}, \\ &\sigma(t) = \frac{1}{100} \text{ for } |t| < \frac{1}{4}, \\ &0 < \sigma(t) \leq \frac{1}{100} \text{ for } \frac{1}{4} < |t| < \frac{1}{2}. \end{aligned}$$

Let $0 < \epsilon < \frac{1}{10}\epsilon_0$. Define $\sigma_\epsilon(t) = \epsilon^3 \sigma\left(\frac{t}{\epsilon}\right)$. Moreover, define

$$\begin{aligned} h_\delta(t) &= \int_{\mathbb{R}} h(t - \sigma_\delta(t)s)\varphi(s)ds, \qquad t \in (-\epsilon_0, \epsilon_0) \\ &= \begin{cases} \int_{\mathbb{R}} h(s) \cdot \frac{1}{\sigma_\delta(t)} \varphi\left(\frac{t-s}{\sigma_\delta(t)}\right) ds, & \sigma_\delta(t) > 0 \\ h(t), & \sigma_\delta(t) = 0. \end{cases} \end{aligned} \tag{3.1.1}$$

Now we want to compute $h'_\delta(s)$. For $|t| > \frac{\epsilon^3}{100}$,

$$\begin{aligned} h'_\epsilon(t) &= \frac{d}{dt}\int_{\mathbb{R}} h(t - \sigma_\epsilon(t)s)\varphi(s)ds \\ &= \int_{\mathbb{R}} h'(t - \sigma_\epsilon(t)s)\left(1 - s\epsilon^2\sigma'\left(\frac{t}{\epsilon}\right)\right)\varphi(s)ds. \end{aligned}$$

For $|t| < \frac{\epsilon}{4}$,

$$\begin{aligned}
h'_\delta(t) &= \frac{d}{dt}\int_{\mathbb{R}} h(s)\cdot \frac{1}{\sigma_\epsilon(t)}\varphi\left(\frac{t-s}{\sigma_\epsilon(t)}\right) ds \\
&= \int_{\mathbb{R}} h(s)\cdot \frac{d}{dt}\left(\frac{1}{\sigma_\epsilon(t)}\varphi\left(\frac{t-s}{\sigma_\epsilon(t)}\right)\right) ds \\
&= \int_{\mathbb{R}} h(s)\cdot \frac{d}{dt}\left(\frac{100}{\epsilon^3}\varphi\left(\frac{100(t-s)}{\epsilon^3}\right)\right) ds \\
&= (-1)\cdot \int_{\mathbb{R}} h(s)\cdot \frac{d}{ds}\left(\frac{100}{\epsilon^3}\varphi\left(\frac{100(t-s)}{\epsilon^3}\right)\right) ds \\
&= (-1)\cdot \int_{-\infty}^{0} h_-(s)\cdot \frac{d}{ds}\left(\frac{100}{\epsilon^3}\varphi\left(\frac{100(t-s)}{\epsilon^3}\right)\right) ds \\
&\quad + (-1)\cdot \int_{0}^{\infty} h_+(s)\cdot \frac{d}{ds}\left(\frac{100}{\epsilon^3}\varphi\left(\frac{100(t-s)}{\epsilon^3}\right)\right) ds \\
&= \int_{-\infty}^{0} h'_-(s)\cdot \left(\frac{100}{\epsilon^3}\varphi\left(\frac{100(t-s)}{\epsilon^3}\right)\right) ds \\
&\quad + \int_{0}^{\infty} h'_+(s)\cdot \left(\frac{100}{\epsilon^3}\varphi\left(\frac{100(t-s)}{\epsilon^3}\right)\right) ds \\
&= \int_{\mathbb{R}} h'(s)\cdot \left(\frac{100}{\epsilon^3}\varphi\left(\frac{100(t-s)}{\epsilon^3}\right)\right) ds \\
&= \int_{\mathbb{R}} h'(t-\sigma_\epsilon(t)s)\varphi(s)ds.
\end{aligned}$$

Now note for $|t| < \frac{\epsilon}{4}$ that σ_ϵ is a constant function; therefore, for all $t \in (-\epsilon_0, \epsilon_0)$

$$h'_\epsilon(t) = \int_{\mathbb{R}} h'(t-\sigma_\epsilon(t)s)\left(1 - s\epsilon^2\sigma'\left(\frac{t}{\epsilon}\right)\right)\varphi(s)ds. \tag{3.1.2}$$

By (3.1.1) and (3.1.2) we have

$$\begin{aligned}
||h_\epsilon(t) - h(t)||_u &\leq \int_{\mathbb{R}} ||h(t-\sigma_\epsilon(t)s) - h(t)||_u \varphi(s)ds \\
&\lesssim \epsilon^3.
\end{aligned}$$

and

$$\begin{aligned}
||h'_\epsilon(t) - h'(t)||_u &\leq \int_{\mathbb{R}} ||h'(t-\sigma_\epsilon(t)s) - h'(t)||_u \varphi(s)ds \\
&\quad + \int_{\mathbb{R}} \left|\left| h'(t-\sigma_\epsilon(t)s)s\epsilon^2\sigma'\left(\frac{t}{\epsilon}\right)\right|\right|_u \varphi(s)ds \\
&\lesssim \epsilon^3 + \epsilon^2\int_{\mathbb{R}} \left|\left| h'(t-\sigma_\epsilon(t)s)\sigma'\left(\frac{t}{\epsilon}\right)\right|\right|_u \varphi(s)ds \\
&\lesssim \epsilon^2.
\end{aligned}$$

Lastly, note that

$$\begin{aligned}
|h'_\epsilon(t)| &= \int_{\mathbb{R}} |h'(t-\sigma_\epsilon(t)s)| \left| \left(1 - s\epsilon^2\sigma'\left(\frac{t}{\epsilon}\right)\right)\right| |\varphi(s)| ds \\
&\le L \int_{\mathbb{R}} \left(1 - s\epsilon^2\sigma'\left(\frac{t}{\epsilon}\right)\right)(\varphi(s)) ds \\
&= L\left(1 - \epsilon^2\sigma'\left(\frac{t}{\epsilon}\right)\int_{\mathbb{R}} s\varphi(s)ds\right) \\
&\le L,
\end{aligned}$$

where the first inequality follows if ϵ is small enough and the last inequality follows since $s\varphi(s)$ is an odd function. Moreover, redoing this computation without the absolute values shows that $h'_\epsilon(t) \le \sup\{h'(t) : t \in \mathbb{R} \setminus \{0\}\}$. □

Now we are ready to construct smooth 1-Lipschitz maps $F_j : M_j^n \to (\mathbb{S}^n, g_{rd})$.

Lemma 3.1.5. *There exists a function $F_j : M_j^n \to \mathbb{S}^n$ that is a 1-Lipschitz diffeomorphism with* $\deg F_j \neq 0$

Proof. First define a decreasing 1-Lipschitz function $f_j : [0, D_j] \to [0, \pi]$.

$$f_j(t) = \begin{cases} \pi - t, & t \in [0, t_j] \\ a_j(t - t_j) + b_j, & t \in [t_j, D_j], \end{cases}$$

where $a_j = \frac{-\pi + t_j}{D_j - t_j}$, $b_j = \pi - t_j$, and t_j is chosen so that $f_j(t_j) = \frac{1}{10}\rho\left(\frac{1}{2}D_j\right)$. Note $\rho\left(\frac{1}{2}D_j\right)$ is the radius of the cylindrical part of the tunnel which is also the minimum that $\rho_j(t)$ attains on $\left[\frac{\pi}{2}, D_j - \frac{\pi}{2}\right]$.

By Lemma 3.1.4, we can smooth f_j to $f_{j,\epsilon}$ by choosing ϵ small enough. And so define $F_{j,\epsilon}(t, \theta) = (f_{j,\epsilon}(t), \theta)$. Since $f'_{j,\epsilon}(t) < 0$ and $f_{j,\epsilon}$ is a bijection, we have that $F_{j,\epsilon}$ is a diffeomorphism. We want to show that for all $v \in TM_j$

$$F^*_{j,\epsilon} g_{rd}(v, v) \le g_j(v, v).$$

Note that

$$F^*_{j,\epsilon} g_{rd} = \left(f'_{j,\epsilon}(t)\right)^2 dt^2 + \sin^2(f_{j,\epsilon}(t)) g_{\mathbb{S}^{n-1}}.$$

and

$$g_j = dt^2 + \sin^2(\rho_j(t)) g_{\mathbb{S}^{n-1}}.$$

First by (2.1.2) and Lemma 3.1.4 we know that $|f'_{j,\epsilon}(t)| \le 1$ for all t. Now we will show that $\sin^2(f_{j,\epsilon}(t)) \le \sin^2(\rho_j(t))$.

On $[0, \pi - t_j - 20\epsilon]$ we have by (2.1.2) that

$$\rho_j(t) = \pi - \int_0^t \cos\left(\theta_j(u)\right) du \ge \pi - t = f_j(t) = f_{j,\epsilon}(t).$$

On $[\pi - t_j - 20\epsilon, \pi - t_j]$ we have

$$\rho_j(t) = \pi - \int_0^t \cos(\theta_j(u))\, du > \pi - t = f_j(t)$$

and so for small enough ϵ, we have that $f_{j,\epsilon}(t)$ will also satisfy this inequality.

On $\left[\pi - t_j, D_j - \frac{\pi}{2}\right]$, we have that $f_j(t) \leq \frac{1}{10}\rho\left(\frac{1}{2}D_j\right)$ and that $\frac{1}{10}\rho\left(\frac{1}{2}D_j\right) < \rho_j(t) \leq \frac{\pi}{2}$. Therefore, $\sin^2(f_j(t)) \leq \sin^2(\rho_j(t))$ on $\left[0, D_j - \frac{\pi}{2}\right]$.

Lastly on $\left[D_j - \frac{\pi}{2}, D_j\right]$ we have the following: $\rho_j(t) = \pi - D_j + t$ and $f_{j,\epsilon}(t) = f_j(t) = a_j(t - t_j) + b_j$ by the construction. Moreover,

$$-f_j(t) + \pi \geq \rho_j(t)$$

since if we define $\psi_j(t) = \rho_j(t) + f_j(t) - \pi$, then we see that $\psi'(t) \geq 0$ and $\psi(D_j) = 0$. We also note on $\left[D_j - \frac{\pi}{2}, D_j\right]$ that $\frac{\pi}{2} \leq -f_j(t) + \pi \leq \pi$ and $\frac{\pi}{2} \leq \rho_j(t) \leq \pi$. Therefore, we conclude that $\sin^2(f_{j,\epsilon}(t)) = \sin^2(-f_{j,\epsilon}(t) + \pi) \leq \sin^2(\rho_j(t))$ on $\left[D_j - \frac{\pi}{2}, D_j\right]$.

Thus, for all $v \in TM_j$ we have

$$F_{j,\epsilon}^* g_{rd}(v, v) \leq g_j(v, v),$$

which implies

$$\ell_{\mathbb{S}^n}(F_{j,\epsilon} \circ c) \leq \ell_{M_j}(c)$$

where $c : [0, 1] \to (\mathbb{S}^n, g_{rd})$ is a path connecting p and q. This implies that

$$d_{\mathbb{S}^n}(F_{j,\epsilon}(p), F_{j,\epsilon}(q)) \leq d_{M_j}(p, q).$$

Thus, we have that $F_{j,\epsilon}$ is 1-Lipschitz. Moreover, $\deg F_{j,\epsilon} \neq 0$ since $F_{j,\epsilon}$ is a diffeomorphism. □

Lemma 3.1.6. *Let $(\mathbb{S}^3, g_1), (\mathbb{S}^3, g_2)$ be 3-spheres such that there exists a diffeomorphism $F : (\mathbb{S}^3, g_1) \to (\mathbb{S}^3, g_2)$ that is 1-Lipschitz and is isotopic to the identity then*

$$\text{width}(\mathbb{S}^3, g_2) \leq \text{width}(\mathbb{S}^3, g_1).$$

Proof. By the definition of width for any $\delta > 0$ there exists $\{\Sigma_t\}$ such that

$$\sup_t |\Sigma_t|_1 < \text{width}(\mathbb{S}^3, g_1) + \delta;$$

therefore,

$$\text{width}(\mathbb{S}^3, g_2) \leq \sup_t |F(\Sigma_t)|_2 \leq \sup_t |\Sigma_t|_1 \leq \text{width}(\mathbb{S}^3, g_1) + \delta$$

where the first inequality follows since $F(\Sigma_t) \in \Lambda'$ and the second inequality follows since F is 1-Lipschitz. □

Proof of Theorem A. Let M_j be as in Proposition 3.1.1; therefore, $M_j^n \to M_\infty$ where M_∞ is the disjoint union of two spheres. Let $F_j : M_j^n \to \mathbb{S}^n$ be as in Lemma 3.1.5. Then by Arzela-Ascoli Theorem 1.3.4 there is a subsequence F_{j_k} that converges to a 1-Lipschitz map

$$F_\infty : M_\infty \to \mathbb{S}^n.$$

This map is not a Riemannian isometry since $\mathbb{S}^n$ is connected and $N \sqcup N'$ is not. □

Proof of Theorem A′. Let M_j^3 be as in Proposition 3.1.1; therefore, $M_j^3 \to M_\infty$ in $\mathcal{VF}$-sense where M_∞ is the disjoint union of two spheres. Let $F_j : M_j^3 \to \mathbb{S}^n$ be as in Lemma 3.1.5 and define $\tilde{F}_j(r,\theta) = F_j(D_j - r, \theta)$. Consider the diffeomorphism

$$\Phi : [0, D_j] \times \mathbb{S}^2 \to [0, \pi] \times \mathbb{S}^2, \qquad \Phi(r,\theta) = \left(\frac{\pi}{D_j} r, \theta\right)$$

Note that Φ is an isometry between $([0, D_j] \times \mathbb{S}^{n-1}, \Phi^*(dr^2 + \sin^2(r) g_{\mathbb{S}^{n-1}}))$ and $([0,\pi] \times \mathbb{S}^{n-1}, dr^2 + \sin^2(r) g_{\mathbb{S}^{n-1}})$. And now consider

$$(\Phi^{-1} \circ \tilde{F}_j)(r,\theta) = \left(\frac{D_j}{\pi} f_j(D_j - r), \theta\right).$$

This map is a 1-Lipschitz orientation preserving diffeomorphism from M_j^n to the round n-sphere and $\Phi^{-1} \circ F_j$ is isotopic to the identity. Therefore, by Lemma 3.1.6 we have that $\mathrm{width}(M_j^3) \geq 4\pi$. □

3.2 Manifolds with Many Wells

In this section, we will use Proposition 2.1.1 (Wells and Tunnels) to construct sequences of manifolds with many wells. Furthermore, we will prove Theorems B and A′.

Theorem 3.2.1. *Let (M^n, g) be a closed Riemannian manifold of dimension $n \geq 3$ with scalar curvature $R \geq \kappa$. Then there exists a sequence of Riemannian manifolds $M_j^n = (M^n, g_j)$ such that $R^j \geq \kappa - \frac{1}{j}$ and M_j^n converge in the* VADB*-sense and $\mathcal{VF}$-sense to M^n but has no convergent subsequence in the* GH*-topology.*

Proof. Define

$$X_j = \left\{(B(p_i^j, \delta_j), g)\right\}_{i=1}^{j}$$

to be a collection of disjoint geodesic balls in M^n where $0 < \delta_j < \frac{1}{j}$ is chosen small enough so that by Proposition 2.1.1 we replace each $B(p_i^j, \delta_j)$ with the well $W_{i,j} = (B(p_i^j, \delta_j), g_j)$ such that the scalar curvature of each of the wells satisfies $R^j > \kappa - \frac{1}{j}$. Moreover, choose $d = \frac{1}{2}$ in Proposition 2.1.1 so that $\mathrm{diam}(W_{i,j}) \geq \frac{1}{2}$. Call the resulting manifold $M_j^n = (M^n, g_j)$. Now we note that

$$\lim_{j\to\infty} \mathrm{vol}_j\left(M_j^n\right) = \lim_{j\to\infty} \mathrm{vol}_g\left(M^n\right) - \sum_{i=1}^{j} \mathrm{vol}_g\left(B(p_i^j, \delta_j)\right) + \sum_{i=1}^{j} \mathrm{vol}_j\left(W_{i,j}\right).$$

Thus, by Proposition 2.1.1 and Proposition 2.1.10

$$\lim_{j\to\infty} \mathrm{vol}_g\left(M^n\right) - jC\delta_j^n \leq \lim_{j\to\infty} \mathrm{vol}_j\left(M_j^n\right) \leq \lim_{j\to\infty} \mathrm{vol}_g\left(M^n\right) + Cj\left(\delta_j^n + \frac{\delta_j^{n-1}}{2}\right).$$

and so

$$\lim_{j\to\infty} \mathrm{vol}_j\left(M_j^n\right) = \mathrm{vol}_g\left(M^n\right).$$

Also by Proposition 2.1.1 and the triangle inequality, we have that

$$\operatorname{diam}\left(M_j^n\right) \leq \operatorname{diam}\left((M^n, g)\right) + 2\operatorname{diam}\left(W_j\right) \leq \operatorname{diam}\left((M^n, g)\right) + 2\left(C + \frac{1}{2}\right).$$

so the diameters are uniformly bounded.

Consider the identity map $id : (M^n, g_j) \to (M^n, g)$. Denote $id^* g_j = g_j$. Now by construction and Lemma 2.1.12 we have for any $p \in W_{i,j}$ that

$$g(v, v) \leq g_j(v, v) \text{ for all } v \in T_p M$$

because $g_j = dF_j^2 + g$ and if $p \notin W_{i,j}$ then $g(v, v) = g_j(v, v)$ for all $v \in T_p M$.

Therefore, M_j^n converges to (M^n, g) in the VADB-sense and by Theorem 1.3.7 we have that M_j^n converges to (M^n, g) in the $\mathcal{VF}$-sense.

Fix $\epsilon_0 < \frac{1}{4}$. Note that ϵ_0 is less than the diameter of the wells and recall p_i^j are the tip of the wells. Therefore, $B(p_i^j, \epsilon_0) \subset M_j^n$ are disjoint. Therefore,

$$j < \operatorname{Cov}_j(\epsilon_0)$$

and so as $j \to \infty$ we have $\operatorname{Cov}_j(\epsilon_0) \to \infty$ so by Theorem 1.3.1 that M_j does not converge in the GH-sense.

□

Proof of Theorem B. Consider the round n-sphere $(\mathbb{S}^n, g_{rd})$. By Theorem 3.2.1 we see that there exists a sequence $M_j = (\mathbb{S}^n, g_j)$ with scalar curvature $R^j \geq n(n-1) - \frac{1}{j}$ such that $M_j \to (\mathbb{S}^3, g_{rd})$ in the $VADB$ and $\mathcal{VF}$-sense but has no convergent subsequence in the GH-topology. Furthermore, the identity map $id : (\mathbb{S}^n, g_j) \to (\mathbb{S}^n, g_{rd})$ is smooth 1-Lipschitz diffeomorphism. □

Proof of Theorem B′. Consider the round 3-sphere $(\mathbb{S}^3, g_{rd})$. By Theorem 3.2.1 we see that there exists a sequence $(\mathbb{S}^3, g_j)$ with scalar curvature $R^j \geq 6 - \frac{1}{j}$ such that $(\mathbb{S}^3, g_j) \to (\mathbb{S}^3, g_{rd})$ in the $VADB$ and $\mathcal{VF}$-sense but has no convergent subsequence in the GH-topology. Moreover, the identity map $id : (\mathbb{S}^3, g_j) \to (\mathbb{S}^3, g_{rd})$ is 1-Lipschitz and by Lemma 3.1.6 we have that $\operatorname{width}(\mathbb{S}^3, g_j) \geq 4\pi$. □

Proof of Theorem C. Let $\kappa > 0$ and let (M^n, g) be the round sphere of sectional curvature $\frac{2\kappa}{n(n-1)}$. Let $\{p_j\}_{j=1}^{\infty} \subset M^n$ be a sequence of points on a geodesic converging to a point p_∞. Define

$$\{B(p_j, \delta_j)\}_{j=1}^{\infty}$$

to be a collection of disjoint geodesic balls in M^n where $0 < \delta_j < \frac{1}{2^j}$ is chosen small enough so that by Proposition 2.1.1 there exists a well $W_j = (B(p_j, \delta_j), g_j)$ such that the scalar curvature of each of the wells satisfies $R^j > 2\kappa\left(1 - \frac{1}{10j}\right) > \kappa$. Let $\{d_j\}_{j=1}^{\infty} \subset [2, 10]$ be a strictly increasing sequence of positive numbers, and choose $d = d_j$ in Proposition 2.1.1 so that $\operatorname{diam}(W_j) \geq d_j$. Now define M_i^n to be the Riemannian obtained by replacing the first i balls with the corresponding first i wells, i.e.,

$$M_i^n = \left(M^n \setminus \bigcup_{j=1}^{i} B(p_j, \delta_j)\right) \sqcup \bigcup_{j=1}^{i} W_j.$$

We note that M_i^n has scalar curvature strictly larger than κ. We also have by Proposition 2.1.1 that

$$\operatorname{diam}(M_i^n) \leq 25C$$

and

$$\begin{aligned}
\operatorname{vol}(M_i^n) &\leq \operatorname{vol}(M^n) + \sum_{j=1}^{\infty} \operatorname{vol}(W_j) \\
&\leq \operatorname{vol}(M^n) + C\left(\sum_{j=1}^{\infty} \frac{1}{2^{nj}} + 10 \sum_{j=1}^{\infty} \frac{1}{2^{(n-1)j}}\right) \\
&\leq \operatorname{vol}(M^n) + 11C.
\end{aligned}$$

Now we will define M_∞ to be

$$M_\infty = \left(M^n \setminus \bigcup_{j=1}^{\infty} B(p_j, \delta_j)\right) \sqcup \bigcup_{j=1}^{\infty} W_j$$

with its induced length metric and natural current structure T_∞. Therefore, we have that $\operatorname{vol}(M_i^n) \to \operatorname{vol}(M_\infty)$. Note that M_∞ is noncompact since it contains infinitely many disjoint balls of radius 1.

We will show that M_i^n converges to M_∞ in an analogous way to [58, Example A.11]. Let $\epsilon_i = d_{M^n}(p_i, p_\infty)$ and note that if $\tilde{B}_i = B(p_\infty, \epsilon_i - \delta_i)$, then there is an isometry, $\varphi : V_i \to V_i'$ where $U_i = M_i^n \setminus \tilde{B}_i \subset M_i$ and $U_i' \subset M_\infty$. By [58, Lemma A.2], there exists a metric space Z such that

$$\begin{aligned}
d_F^Z(M_i^n, M_\infty) \leq{}& \operatorname{vol}(M_i^n \setminus U_i) + \operatorname{vol}(M_\infty \setminus U_i') \\
&+ \operatorname{vol}(U_i)\left(\sqrt{2\operatorname{diam}_{M_i^n}(\partial U_i)\operatorname{diam}_{M_i^n}(U_i)} + \operatorname{diam}_{M_i^n}(\partial U_i)\right) \\
&+ \operatorname{vol}(U_i')\left(\sqrt{2\operatorname{diam}_{M_i^n}(\partial U_i')\operatorname{diam}_{M_i^n}(U_i')} + \operatorname{diam}_{M_i^n}(\partial U_i')\right).
\end{aligned}$$

We note that

$$\operatorname{vol}(M_i^n \setminus U_i) \leq \pi(\epsilon_i - \delta_i)^n, \qquad \operatorname{vol}(M_\infty \setminus U_i') \leq C\left(\sum_{j=i}^{\infty} \frac{1}{2^{nj}} + 10\sum_{j=i}^{\infty} \frac{1}{2^{(n-1)j}}\right).$$

Also, $\operatorname{diam}(\partial U_i)$ and $\operatorname{diam}(\partial U_i')$ converge to zero. Therefore, the right-hand side of the inequality above goes to zero as $i \to \infty$. We conclude then that M_i^n converges to M_∞ in the $\mathcal{VF}$-sense. □

3.3 Sewing Manifolds

We are able to generalize the sewing examples of Basilio, Dodziuk, and Sormani found in [7] and [9]. There are two methods of sewing developed in [9]. Method I generalizes the curve sewing construction of [7]. Here we will extend the construction using Proposition 2.1.2 (Wells and Tunnels). We start with Method I which says that given a fixed manifold one can tightly sew a compact region to a point.

Proposition 3.3.1. *Let (M^n, g) be a complete Riemannian manifold, and $A_0 \subset M$ a compact subset with an even number of points $p_i \in A_0$, $i = 1, \ldots, n$ with pairwise disjoint balls $B(p_i, 2\delta)$ with scalar curvature greater than κ. For small enough $\delta > 0$, define $A_\delta := T_\delta(A_0)$ and*

$$A'_\delta = A_\delta \setminus \left(\bigcup_{i=1}^{n} B(p_i, \delta) \right) \sqcup \bigcup_{i=1}^{\frac{n}{2}} T_i$$

where T_i are tunnels as in Proposition 2.1.2 (Wells and Tunnels) connecting $\partial B(p_{2j+1}, \delta)$ and $\partial B(p_{2j+2}, \delta)$ for $j = 0, 1, \ldots, \frac{n}{2} - 1$. Then given any ϵ, shrinking δ further, if necessary, we may create a new complete Riemannian manifold, (N^n, h),

$$N^n = (M^n \setminus A_\delta) \sqcup A'_\delta$$

satisfying

$$\operatorname{vol}(A_\delta) - \epsilon \leq \operatorname{vol}(A'_\delta) \leq \operatorname{vol}(A_\delta) + \epsilon$$

and

$$\operatorname{vol}(M) - \epsilon \leq \operatorname{vol}(N) \leq \operatorname{vol}(M) + \epsilon$$

If, in addition, M has scalar curvature, $R^M \geq \kappa$, then N has scalar curvature, $R^N \geq \kappa - \epsilon$. If $\partial M \neq \emptyset$, the balls avoid the boundary and ∂M is isometric to ∂N.

Proof. The proof follows from the proof of [7, Proposition 3.1] while using Proposition 2.1.2 (Wells and Tunnels) and Proposition 2.1.10. □

Proposition 3.3.2. *Let (M^n, g) be a complete Riemannian manifold and $A_0 \subset M$. Let $A_a = T_a(A_0)$ be a tubular neighborhood of A_0. Assume that there is an $a > 0$ such that A_a has scalar curvature greater than κ. Let $r \in (0, a)$. Given $\epsilon > 0$, there exists $\delta = \delta(A_0, \kappa, r, \epsilon) \in (0, r)$ and there exists even $n = \bar{n}(\bar{n} - 1)$ depending on A_0, κ, ϵ, and r and points $p_1, \ldots, p_n \in A_0$ with $B(p_i, \delta)$ pairwise disjoint such that we can "sew the region tightly" to create a new complete Riemannian manifold (N^n, h),*

$$N = (M \setminus A_r) \sqcup A'_r,$$

as in Proposition 3.3.1, with

$$A'_\delta = A_\delta \setminus \left(\bigcup_{i=1}^{2n} B(p_i, \delta) \right) \sqcup \bigcup_{j=0}^{n-1} T_{2j+1}.$$

Moreover,

$$\operatorname{vol}(A'_r) \leq \operatorname{vol}(A_r) + \epsilon$$

and

$$\operatorname{vol}(N) \leq \operatorname{vol}(M) + \epsilon$$

and there is a constant $c > 0$ such that

$$\operatorname{diam}(A'_r) \leq cr.$$

If M has scalar curvature $R^M \geq \kappa$, then N has scalar curvature $R^N \geq \kappa - \epsilon$. If $\partial M \neq \emptyset$, the balls avoid the boundary, and ∂M is isometric to ∂N.

Proof. The proof follows from the proof of [9, Proposition 3.6] while using Propositions 2.1.2, 3.3.1, and Lemma 2.1.3. □

These statements allow us to construct sequences of manifolds with scalar curvature greater than κ which converge to a pulled metric space in a similar manner as in [9]. We recall the following definition from [9].

Definition 3.3.3. Let (M^n, g) be a Riemannian manifold with a compact set $A_0 \subset M$ with tubular neighborhood $A_a = T_a(A_0)$ satisfying the hypotheses of Proposition 3.3.2. We can construct its *sequence of increasingly tightly sewn manifolds*, (N_j^n, g_j), by applying Proposition 3.3.2 taking $\epsilon = \epsilon_j \to 0$, $n = n_j \to \infty$, and $\delta = \delta_j \to 0$ to create each sewn manifold $N^n = N_j^n$ and the edited regions $A'_\delta = A'_{\delta_j}$ which we simply denote A'_j. Since these sequences N_j are created using Proposition 3.3.2, they have scalar curvature greater than $\kappa - \epsilon_j$ when M has scalar curvature greater than κ and $\partial N_j = \partial M$ whenever $\partial M \neq \emptyset$.

Theorem 3.3.4. *The sequence N_j, as in Definition 3.3.3 assuming M^n is compact and A_0 is a compact embedded submanifold of dimension 1 to n, converges in the Gromov–Hausdorff sense and the intrinsic flat sense to N_∞, which is a metric space created by pulling the region A_0 to a point. If, in addition, $\mathcal{H}^{n-1}(A_0) = 0$ then N_j also converges in the metric measure sense to N_∞.*

Proof. The proof follows from the proof of [9, Theorem 3.8] while using Proposition 3.3.2. □

Now we can prove Theorem D.

Proof of Theorem D. Let S be a compact space form of dimension n and constant curvature $\frac{\kappa}{n(n-1)}$ and Σ^m be a constant curvature m-dimensional sphere, $1 \leq m \leq n-1$. We note that there exists an embedding of Σ^m into S. Let (N_j^n, g_j) be a sequence of manifolds constructed from S sewn along an embedded Σ^m with $\delta = \delta_j \to 0$ as in Proposition 3.3.2 and the scalar curvature $R^j \geq \kappa - \frac{1}{j}$. Then by Theorem 3.3.4 we have

$$N_j \xrightarrow{mGH} N_\infty \text{ and } N_j \xrightarrow{\mathcal{F}} N_\infty$$

where N_∞ is the metric space created by taking S and pulling a Σ^m to a point. Moreover, at the pulled point $p_0 \in N_\infty$ we have

$$wR(p_0) = \lim_{r \to 0} 6(n+2) \frac{\mathrm{vol}_{\mathbb{E}^n} B(0,r) - \mathcal{H}^n(B(p_0,r))}{r^2 \cdot \mathrm{vol}_{\mathbb{E}^n} B(0,r)} = -\infty.$$

We can see this because

$$\mathrm{vol}_{N_\infty}(B(p_0,r)) = \mathcal{H}^n_{N_\infty}(B(p_0,r)) = \mathcal{H}^n_{N_\infty}(B(p_0,r) \setminus \{p_0\}) = \mathcal{H}^n_{\mathbb{S}^n_\kappa}(T_r(\mathbb{S}^m)).$$

Moreover, there is a constant $C(n,m,\kappa)$ such that

$$\lim_{r \to 0} \frac{\mathcal{H}^n_{\mathbb{S}^n_\kappa}(T_r(\Sigma^m))}{Cr^{n-m}} = 1.$$

We conclude that

$$wR(p_0) = \lim_{r \to 0} 6(n+2) \frac{\omega_n r^n - Cr^{n-m}}{\omega_n r^{n+2}} = -\infty.$$

□

Moreover, using Proposition 2.1.2 we can extend Method II for sewing manifolds in [9] to the setting where scalar curvature is bounded below. In Method II, given a sequence of Riemannian manifolds whose limit is a Riemannian manifold, then one can create a new sequence where the sewing occurs along the sequence.

Theorem 3.3.5. *Let M_j^n be a sequence of compact Riemannian manifolds each with a compact region $A_{j,0} \subset M_j^3$ with tubular neighborhood, A_j, with scalar curvature greater than κ satisfying the hypotheses of Proposition 3.3.2. We assume M_j^n converge in the biLipschitz sense to M_∞^n and the regions $A_{j,0}$ converge to a compact set $A_{\infty,0} \subset M_\infty^n$ in the sense that there exists biLipschitz maps*

$$\psi_j : M_j^n \to M_\infty^n$$

such that

$$L_j = \log\left(\mathrm{Lip}(\psi_j)\right) + \log\left(\mathrm{Lip}\left(\psi_j^{-1}\right)\right) \to 0$$

and $\psi_j(A_{j,0}) = A_{\infty,0}$. Then there exists $\delta_j \to 0$ and applying Proposition 3.3.2 to $M^n = M_j^n$ to sew the regions $A_0 = A_{j,0}$ with $\delta = \delta_j$, to obtain sewn manifolds $N^n = N_j^n$, we obtain a sequence N_j^n such that

$$N_j^n \xrightarrow{GH} N_\infty \text{ and } N_j^n \xrightarrow{\mathcal{F}} N_{\infty,0},$$

where $\bar{N}_{\infty,0} = N_\infty$ and N_∞ is the metric space created by taking M_∞^n and pulling the region $A_{\infty,0}$ to a point.

Moreover, if the regions $A_{j,0}$ satisfy $\mathcal{H}^n(A_{j,0}) = 0$, then the sequence N_j^n also converges in the metric measure sense

$$N_j^n \xrightarrow{mGH} N_\infty.$$

Proof. The proof follows from the proof of [9, Theorem 5.1] while using Proposition 3.3.2 and Theorem 3.3.4 □

3.4 Intrinsic Flat Limit with No Geodesics

We are able to generalize the result of Basilio, Kazaras, and Sormani from [8] which shows the intrinsic flat limit of Riemannian manifolds need not be geodesically complete. This follows from Proposition 2.1.2 (Wells and Tunnels) and the pipe-filling technique [8, Theorem 3.1]. In particular:

Theorem 3.4.1. *There is a sequence of closed, oriented, Riemannian manifolds (M_j^n, g_j), $n \geq 3$, such that the corresponding integral current spaces converge in the intrinsic flat sense to*

$$M_\infty = \left(N, d_{\mathbb{E}^{n+1}}, \int_N\right),$$

where N is the round n-sphere of section curvature $\frac{2\kappa}{n(n-1)}$ and $d_{\mathbb{E}^{n+1}}$ is the Euclidean distance induced from the standard embedding of N into $\mathbb{E}^{n+1}$. Moreover, M_j may be chosen so that R^j, the scalar curvature of M_j, satisfies $R^j \geq 2\kappa\left(1 - \frac{1}{10j}\right) > \kappa$. Moreover, M_∞ is not a length space and is not locally geodesically complete.

3.5 Novel Min-Oo Counterexamples

In this section, we use quantitative surgery to construct non-perturbative counterexamples to Min-Oo's Conjecture; moreover, we construct counterexamples with more complex topology. But first, we need the following statement of the counterexample of Min-Oo's Conjecture constructed by Brendle, Marques, and Neves.

Theorem 3.5.1 ([13], Corollary 6)**.** *Let $n \geq 3$. There exists a Riemannian metric g on the hemisphere $\mathbb{S}^n_+$ with the following properties:*

i. The scalar curvature, R_g, satisfies $R_g > n(n-1)$.

ii. At each point on $\partial\mathbb{S}^n_+$, $g = g_{rd}$, where g_{rd} is the standard unit round metric on $\mathbb{S}^n_+$.

iii. The boundary $\partial\mathbb{S}^n_+$ is totally geodesic with respect to g.

Now we are ready to prove our main result.

Proof of Theorem F. Let $n \geq 3$. Let (M^n, g) be a Riemannian manifold such that the scalar curvature, R_g, satisfies $R_g > n(n-1)$. Let $(\mathbb{S}^n_+, \bar{g})$ be the Riemannian manifold from Theorem 3.5.1. Choose $d \geq D$ and j large enough in Proposition 2.1.2 so that the constructed metric on $M \# \mathbb{S}^n_+$ satisfies the conditions in Theorem F. □

Proof of Corollary 1. Let $n \geq 3$. In Theorem F, take (M^n, g) to be $\left(\mathbb{S}^n, g^n_{\frac{1}{2}}\right)$ where $g^n_{\frac{1}{2}}$ is the round metric on $\mathbb{S}^n$ of radius $\frac{1}{2}$. □

Proof of Corollary 2. By Proposition 2.1.2, for any $n \in \mathbb{N}$ one can construct a metric g on $M = \#_{i=1}^m \mathbb{S}^n$ that has scalar curvature strictly larger than $n(n-1)$ and whose volume is larger than $\left\lfloor \frac{m}{2} \right\rfloor \omega_n$, where ω_n is the volume of $\mathbb{S}^n$ with its standard unit round metric. Choose m larger enough so that $\mathrm{vol}_g(M) > V$. Now by Theorem F, one can obtain a metric on $\mathbb{S}^n_+$ satisfying the conditions of Corollary 2 □

Proof of Corollary 3. Let $n \geq 3$ and $p+q = n$. In Theorem F, take (M^n, g) to be $(\mathbb{S}^p \times \mathbb{S}^q, h)$ where h is a metric on $\mathbb{S}^p \times \mathbb{S}^q$ of scalar curvature strictly larger than $n(n-1)$. In particular, one can choose $h = \frac{1}{2(n(n-1))}(g^p_1 + g^q_1)$ where g^m_r is the standard round metric on $\mathbb{S}^m$ of radius r. □

Proof of Theorem G. Let $n \geq 3$ and take (M^n, g) in Proposition 2.2.1 to be the manifold from Theorem 3.5.1. □

3.6 Manifold with Volume Constraint and Scalar Curvature greater than $n(n-1)$

In this section, we construct examples that are related to the result of Miao and Tam [46] while keeping the volume arbitrarily close to the volume of the standard unit round $\mathbb{S}^n_+$. Moreover, from the proof of Theorem I one can see outside a set of arbitrarily small volume the metric g is a small perturbation of the standard unit round metric on $\mathbb{S}^n_+$.

Proof of Theorem I. Recall that g_1^n is the standard unit round metric on $\mathbb{S}^n$. By [13, Corollary 15] there exists a smooth function $u : \mathbb{S}^n_+ \to \mathbb{R}$ and a smooth vector field X on $\mathbb{S}^n_+$ such that for all t small enough we have that the metric

$$g(t) = g_1^n + t\mathcal{L}_X g_1^n + \frac{1}{2(n-1)} t^2 u g_1^n$$

has scalar curvature strictly greater than $n(n-1)$, the mean curvature of the boundary is strictly positive, and $g(t) = g_1^{n-1}$ on $\partial\mathbb{S}^n_+$.

Fix $0 < \epsilon < \frac{1}{100}$ and $D > 0$. Now choose t_0 small enough so that

$$|\operatorname{vol}_{g_1^n}(\mathbb{S}^n_+) - \operatorname{vol}_{g(t_0)}(\mathbb{S}^n_+)| < \omega_n \epsilon^n,$$

where ω_n is the volume of the n-sphere and the scalar curvature of $g(t_0)$ is strictly larger than $n(n-1)$. Now using Proposition 2.1.2 we can construct a metric on $\mathbb{S}^n_+$ that satisfies the conditions of Theorem I by constructing a metric g on $\mathbb{S}^n_+ \# \mathbb{S}^n$.

In particular, in the setting of Proposition 2.1.2, we consider $M_1 = (\mathbb{S}^n_+, g(t_0))$ and $M_2 = (\mathbb{S}^n, g^n_{10\epsilon})$. Let $B_1 = B_{g(t_0)}(p, 2\delta) \subset M_1$ and $B_2 = B_{g^n_{10\epsilon}}(p', 2\delta) \subset M_2$. In Proposition 2.1.2, choose $d > D$ and j such that $\min\left(R^{M_1} - \frac{1}{j}, R^{M_2} - \frac{1}{j}\right) > n(n-1)$. Finally, choose $\delta > 0$ such that $200\delta < \epsilon$ and $d\delta^{n-1} < \epsilon^{n-1}$.

Thus, the metric g on $\mathbb{S}^n_+ \# \mathbb{S}^n$ has scalar curvature strictly larger than $n(n-1)$ and $\left|\operatorname{vol}_{g(t_0)}(\mathbb{S}^n_+) - \operatorname{vol}_{g(t_0)}(\mathbb{S}^n_+ \setminus B_1)\right| < \omega_n \epsilon^n$. Note

$$\operatorname{vol}_g(\mathbb{S}^n_+ \# \mathbb{S}^n) = \operatorname{vol}_{g(t_0)}(\mathbb{S}^n_+ \setminus B_1) + \operatorname{vol}(T) + \operatorname{vol}_{g^n_{10\epsilon}}(\mathbb{S}^n \setminus B_2).$$

and

$$\left|\operatorname{vol}_{g_1^n}(\mathbb{S}^n_+) - \operatorname{vol}_{g(t_0)}(\mathbb{S}^n_+ \setminus B_1)\right| < 2\omega_n \epsilon^n.$$

Thus,

$$\begin{aligned}
\operatorname{vol}_{g_1^n}(\mathbb{S}^n_+) &\leq \operatorname{vol}_{g(t_0)}(\mathbb{S}^n_+ \setminus B_1) + 2\omega_n \epsilon^n \\
&\leq \operatorname{vol}_{g(t_0)}(\mathbb{S}^n_+ \setminus B_1) + \operatorname{vol}(T) + (10^n - 1)\omega_n \epsilon^n \\
&\leq \operatorname{vol}_g(\mathbb{S}^n_+ \# \mathbb{S}^n) \\
&= \operatorname{vol}_{g(t_0)}(\mathbb{S}^n_+ \setminus B_1) + \operatorname{vol}(T) + \operatorname{vol}_{g^n_{10\epsilon}}(\mathbb{S}^n \setminus B_2) \\
&\leq \operatorname{vol}_{g_1^n}(\mathbb{S}^n_+) + 2\omega_n \epsilon^n + C(\delta^n + d\delta^{n-1}) + \omega_n 10^n \epsilon^n,
\end{aligned}$$

where in final inequality we use (ii) from Proposition 2.1.2 to estimate $\operatorname{vol}(T)$. Therefore, by our choice of δ and the fact $\epsilon < 1$ we that

$$\frac{1}{2}\omega_n \leq \operatorname{vol}_g(\mathbb{S}^n_+ \# \mathbb{S}^n) \leq \frac{1}{2}\omega_n + (2C + (2 + 10^n)\omega_n)\epsilon^{n-1}.$$

□

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