Understanding the Defining Ideal Through Cones

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Abstract of the Dissertation

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We study the degrees in which the ideal of a smooth projective variety over $\mathbb{C}$ is generated by cones. Our main results focus on the first nontrivial case when the variety is a finite set of points in $\mathbb{P}^2$.

Contents

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For queer and trans mathematicians of color

Chapter 1

Introduction

The purpose of this dissertation is to study when the defining ideal of a projective variety is generated by cones.

Let $X \subseteq \mathbb{P}^r$ be a smooth projective variety of dimension n and degree d. Given an $(r-n-2)$-plane $\Lambda \subseteq \mathbb{P}^r$ disjoint from X, let

$$C_\Lambda(X) \subseteq \mathbb{P}^r$$

denote the cone over X centered at Λ, which is a hypersurface of degree d. This hypersurface is defined by a homogeneous polynomial $c_\Lambda(X) \in S_d$, where $S = \mathbb{C}[x_0, \ldots, x_r]$ is the homogeneous coordinate ring of $\mathbb{P}^r$. We call $c_\Lambda(X)$ the cone polynomial over X centered at Λ. As Λ varies in $\mathbb{P}^r$, one can show that the $c_\Lambda(X)$ cut out X scheme-theoretically. It follows that if $I^{\text{cone}} = I_X^{\text{cone}} \subseteq S$ is the ideal generated by the cone polynomials, then

$$I_k^{\text{cone}} = (I_X)_k \text{ for } k \gg 0. \tag{♪}$$

The motivating question of this dissertation is

Question: In what degrees k does (♪) hold?

In particular, we focus on the first non-trivial case when $X \subseteq \mathbb{P}^2$ is a finite set of points.

One reason that this research question is interesting is that cones give geometrically meaningful polynomials that eventually generate the defining ideal of a projective variety. Understanding the equations defining an abstract variety has been a major area of study in the last century, as described in the survey of Lyubeznik [16]. Recent work in the intersection of algebraic geometry and computational mathematics provides algorithms for finding generators for the defining ideals in specific cases such as points [1], smooth affine varieties [3], and parameterized subgroups in a toric variety [24]. While cone polynomials cannot generate the parts of the defining ideal in degrees less than the degree of the variety, they provide a natural way to see some of the equations of I_X geometrically, so it is interesting to know how close cones come to generating the defining ideal.

Another motivation to study this problem is its relationship to bounds on regularity. Following the results of Castelnuovo [4], Mumford [19] introduced Castelnuovo-Mumford regularity, which gives an integer valued cohomological measure of the algebraic complexity of a smooth variety $Y \subseteq \mathbb{P} = \mathbb{P}^{r+1}$. Recent work [11, 21, 14, 13, 5, 22] in algebraic geometry, commutative algebra, and computational mathematics studies bounds on the regularity of various types of varieties. In practice, knowing for which values of k the cohomology

$$H^1(\mathcal{I}_Y(k)) = 0 \tag{♫}$$

vanishes is key to bounding the Castelnuovo-Mumford regularity of Y. Let

$$X = Y \cap H \subseteq \mathbb{P}^r$$

be a general hyperplane section of Y and consider the exact sequence

$$0 \to \mathcal{I}_Y(k) \to \mathcal{I}_Y(k+1) \to \mathcal{I}_{X/\mathbb{P}^r}(k+1) \to 0.$$

By inducting on dimension, one can usually assume that $H^i(\mathcal{I}_Y(k+1-i)) = 0$ for $i \geq 3$. Then, due to a result of Mumford (see Lemma 3.1.5), $h^1(\mathcal{I}_Y(k))$ is strictly decreasing as a function of k. In this case, the vanishing condition

(♫) is equivalent to knowing the surjectivity of

$$H^0(\mathcal{I}_Y(k+1)) \to H^0(\mathcal{I}_{X/\mathbb{P}^r}(k+1)).$$

The cone polynomials $c_\Lambda(X) \in H^0(\mathcal{I}_{X/\mathbb{P}^r}(d))$ extend to hypersurfaces of degree d in $\mathbb{P}^{r+1}$ vanishing on Y. Thus, an answer to the research question gives an effective bound for the desired surjectivity above.

Now that I have established the underlying motivation, I will discuss the main focus of the dissertation, which is the case when $X \subseteq \mathbb{P}^2$ is a finite set of d points. Prior work done by Fu and Nie [9] examines the ideal generated by cone polynomials when X is a set of generic points. They establish the that cone polynomials fail to generate I_X in degree d when d is odd.

Theorem (c.f. Theorems 3.2.2 and 3.2.3)**.** Let $X \subseteq \mathbb{P}^2$ be a set of d generic points. When $d \geq 3$ is an odd integer with $d \equiv 3 \mod 4$,

$$\operatorname{codim}\left(I_d^{\text{cone}} \subseteq (I_X)_d\right) \geq 1.$$

When $d \geq 5$ and $d \equiv 1 \mod 4$,

$$\operatorname{codim}\left(I_d^{\text{cone}} \subseteq (I_X)_d\right) \geq 2.$$

Their results follow a conjecture (see Conjecture 3.2.1) of Ilic that the inequalities in the theorem above are equalities and when d is even, cones generate all of $(I_X)_d$.

The main theorem of this dissertation is an answer to our research question when $X \subseteq \mathbb{P}^2$ is a collinear set of points.

Theorem (c.f. Theorem 4.1.8)**.** When $X \subseteq \mathbb{P}^2$ is a set of d distinct collinear points, $I_k^{\text{cone}} = (I_X)_k$ if and only if $k \geq 2d-2$.

As the points $X \subseteq \mathbb{P}^2$ move around in the plane, the saturation degree of I^{cone} follows a semicontinuity property (i.e. it may jump up when X lies in special geometric configurations). Then, our theorem for collinear points gives an upper bound on the saturation degree of I^{cone} for arbitrary finite sets in $\mathbb{P}^2$ through a deformation argument.

Theorem (c.f. Theorem 4.2.5)**.** Let $X \subseteq \mathbb{P}^2$ be a set of d distinct points. If $k \geq 2d-2$, then $I_k^{\text{cone}} = (I_X)_k$.

One benefit of working with cone polynomials is that given a concrete $X \subseteq \mathbb{P}^r$, we can generate equations for $c_\Lambda(X)$ using computational software and find the saturation degree of I^{cone}. The final chapter of this dissertation explores some interesting patterns that appear in experimental data generated in `Macaulay2`. Based on these data, I propose conjectures for how cone polynomials behave when X is a finite set in higher projective spaces or a curve in $\mathbb{P}^3$. I also discuss a potential explanation for the phenomenon observed in the work of Fu and Nie regarding the cone polynomials generating in degree d for even degrees only.

The dissertation is organized as follows: In Chapter 2, we discuss relevant background material, including the theories of saturation and Castelnuovo-Mumford regularity. We also dedicate a section to a theorem of Macaulay that allows us to prove our main theorem for collinear points. Chapter 3 introduces cone polynomials and outlines the prior work on cone polynomials done by Fu and Nie [9]. It also establishes that cone polynomials cut out any smooth projective variety scheme-theoretically and a preliminary bound on the saturation degree of the cone ideal coming from sheaf cohomology. Chapter 4 contains the proofs of the main results of the dissertation regarding the saturation degree of the cone ideal for finite sets in $\mathbb{P}^2$. Finally, we examine what experimental data suggests about cone polynomials in higher dimensions in Chapter 5.

Chapter 2

Background

I will survey the relevant background information in this chapter. A central question in this dissertation revolves around the saturation degree of the cone ideal for projective varieties, so Section 2.1 discusses the general theory of saturation. In Section 2.2, I state a classical result of Macaulay that provides a bound on the saturation degree for ideals whose saturation is the irrelevant ideal, which will be applied to our study of cone polynomials. Section 2.3 is an overview of the theory of regularity, which provides motivation for the study of cone polynomials. Section 2.4 reviews some of the basic properties of flat families of sheaves that will be used in our discussion of how the cone degree behaves under degeneration of the base variety.

2.1 Saturation

2.1.1 Basic Definitions and Results

For ideals I, J of a commutative ring R, recall that their ideal quotient $(I : J)$ is the set

$$(I : J) = \{r \in R | rJ \subseteq I\}.$$

Definition 2.1.1. *Let $S = \mathbb{C}[x_0, \ldots, x_r]$ be the homogeneous coordinate ring of $\mathbb{P}^r$ and $\mathfrak{m} = (x_0, \ldots, x_r)$ be the irrelevant ideal. The* ***saturation*** *of an ideal $J \subseteq S$ is*

$$\begin{aligned} J^{\text{sat}} &:= \bigcup_j (I : \mathfrak{m}^j) \\ &= \{f \in S | (x_0, \ldots, x_r)^N \cdot f \subseteq J \text{ for some } N \geq 0\}. \end{aligned}$$

We say that J is ***saturated*** *if $J = J^{\text{sat}}$.*

In other words, J^{sat} is the ideal of polynomials that multiply some power of the irrelevant ideal into J.

Let's show that the defining ideals of projective varieties are saturated.

Proposition 2.1.2. *If $I_X \subseteq S$ is the homogeneous ideal of a (reduced) projective variety $X \subseteq \mathbb{P}^r$, then I_X is a saturated ideal, i.e., $I_X = I_X^{\text{sat}}$.*

Proof. Note that $I_X \subseteq I_X^{\text{sat}}$ because the saturation of an ideal always contains the original ideal. For the other direction, take $f \in I_X^{\text{sat}}$. Then there exists an $N \geq 0$ so that $f \cdot x_0^N, \ldots, f \cdot x_r^N \in I_X$. If $N = 0$, then $f \in I$. If $N > 0$, then we have that $f \cdot x_0^N, \ldots, f \cdot x_r^N$ all vanish on X. Suppose to the contrary that $f \notin I_X$, so there is some $x \in X$ with $f(x) \neq 0$. This means that for all i, $x_i = 0$, but then $x = (0 : \cdots : 0)$, which is not a point in $\mathbb{P}^r$. Thus, $f \in I_X$. □

Furthermore, every saturated ideal in S arises from a subscheme of $\mathbb{P}^r$.

Proposition 2.1.3. *Saturated ideals in $S = \mathbb{C}[x_0, \ldots, x_r]$ correspond precisely to projective subschemes of $\mathbb{P}^r$.*

Proof. Note that two projective subschemes are the same if and only if they agree on all affine charts $x_i = 1$. If $I \subseteq S$ is an ideal, the restriction $I|_{x_i}$ homogenizes to $I_{(i)} := \bigcup_j (I : x_i^j)$. If $J \subseteq S$ is an ideal, then $I|_{x_i=1} = J|_{x_i=1}$ if and only if their rehomogenizations are equal to each other. Thus, I and J define the same subscheme of $\mathbb{P}^r$ precisely when $I_{(i)} = J_{(i)}$ for all i; equivalently, when $\bigcap_i I_{(i)} = \bigcap_i J_{(i)}$. Since $\bigcap_i I_{(i)} = I^{\text{sat}}$, the claim follows. □

Intuitively, an ideal and its saturation will agree in sufficiently high degrees.

Proposition 2.1.4. *For an ideal* $J \subseteq S = \mathbb{C}[x_0, \ldots, x_r]$,

$$(J^{\text{sat}})_k = J_k \text{ for } k \gg 0.$$

Proof. In general, an ideal $J \subseteq S$ is always contained in its saturation, so it is sufficient to show that for k sufficiently large, any degree k element of J^{sat} is in J.

Since S is Noetherian, J^{sat} is finitely generated. Suppose $J^{\text{sat}} = (g_1, \ldots, g_m)$ where $\deg g_i = d_i$. Then by definition of saturation, for each i, there exists $N_i \geq 0$ such that $g_i \cdot \mathfrak{m}^{N_i} \subseteq J$, i.e., for any homogeneous polynomial f_i of degree $\geq N_i$, we have $g_i f_i \in J$. Then, if $M = \max\{N_i\}$ and $\deg f_i \geq M$, $g_i f_i \in J$.

Let $k \geq M + \max\{d_i\}$ and $h \in (J^{\text{sat}})_k$, so $\deg h = k$ and $h = \sum g_i f_i$ where the $f_i \in S$ are degree $k - d_i \geq M$. Then for each i, $g_i f_i \in J$, so since J is closed under addition, $h = \sum g_i f_i \in J$ as well. □

Thus, it makes sense to study the smallest degree in which an ideal agrees with its saturation.

Definition 2.1.5. *The smallest integer k such that $(J^{\text{sat}})_k = J_k$ is called the* ***saturation degree*** $\operatorname{sat.deg}(J)$ *of J.*

Example 2.1.6. Let $S = \mathbb{C}[x_0, x_1, x_2]$ and let $J = (x_0^2, x_0x_1, x_0x_2)$. Then $x_0 \in J^{\text{sat}}$ because x_0 times any degree 1 monomial is in J while $x_1, x_2 \notin J^{\text{sat}}$ because J does not contain any power of those elements. Thus, $J^{\text{sat}} = (x_0)$.

We have

$$(J^{\text{sat}})_2 = \operatorname{span}\{x_0^2, x_0x_1, x_0x_2\} = J_2,$$

so $\operatorname{sat.deg}(J) = 2$.

In the example, we see that geometrically, J “cuts out" the line $L = \{x_0 = 0\}$ even though J does not contain everything in the saturated defining ideal I_L. Let us introduce some language that will allow us to discuss this principle in generality.

Definition 2.1.7. *A subvariety $X \subseteq \mathbb{P}^r$ is* ***scheme-theoretically an intersection of hypersurfaces*** *$H_1, \ldots, H_m$ if as sets,*

$$X = H_1 \cap \cdots \cap H_m,$$

and every $x \in X$ has an affine open neighborhood $U \subseteq \mathbb{P}^r$ such that the ideal $I_{X \cap U} \subseteq \mathbb{C}[U]$ of $X \cap U \subseteq U$ is generated by the affine equations $f_1, \ldots, f_m$ of $H_1, \ldots, H_m$.

If $F_1, \ldots, F_m \in S$ define the hypersurfaces $H_1, \ldots, H_m$, we may also say that the F_i ***scheme-theoretically cut out*** *X.*

Lemma 2.1.8. *Let $X \subseteq \mathbb{P}^r$ be a subvariety with ideal sheaf $\mathcal{I}_X \subseteq \mathcal{O}_{\mathbb{P}}$. Then X is scheme-theoretically an intersection of hypersurfaces H_i if an only if we have a surjective map of sheaves*

$$\oplus_1^m \mathcal{O}_{\mathbb{P}^r}(-d_i) \longrightarrow \mathcal{I}_X \longrightarrow 0$$

given by the homogeneous polynomials F_i defining the hypersurfaces H_i, with $\deg F_i = d_i$.

Proof. Suppose X is scheme-theoretically an intersection of the given hypersurfaces. Then for each $x \in X$, there exists an affine open neighborhood $U \subseteq \mathbb{P}^r$ such that on U the ideal $I(X \cap U)$ is generated by the affine equations f_i of the hypersurfaces defined by F_i. Then the map of sheaves

$$\phi : \oplus_1^m \mathcal{O}_{\mathbb{P}^r}(-d_i) \to \mathcal{I}_X$$

given by the homogeneous F_i is surjective locally on the U, so ϕ is surjective globally.

Conversely, if the surjective map of sheaves is given by the polynomials F_i, then if f_i is the local equation of F_i on U, any $s \in I_{X \cap U}(U)$ is in the image of ϕ_U. In other words, $I_X(U)$ is generated by $f_1, \ldots, f_m$, so the hypersurfaces defined by the f_i scheme-theoretically cut out X. □

Proposition 2.1.9. *The hypersurfaces $F_1, \ldots, F_m$ scheme-theoretically cut out X precisely when $(F_1, \ldots, F_m)^{\text{sat}} = I(X)$.*

Proof. Suppose that the F_i scheme-theoretically cut out X. By Lemma 2.1.8, this is equivalent to the F_i defining a surjective map

$$\oplus_1^m \mathcal{O}_{\mathbb{P}^r}(-d_i) \to \mathcal{I}_X.$$

By applying Serre Vanishing to the kernel of this map, after twisting the sheaves up by $\mathcal{O}_{\mathbb{P}}(k)$, this map will be surjective on H^0 for $k \gg 0$.

Suppose $f \in I_X$ and pick k sufficiently large and greater than $\deg f$. Let $N = k - \deg f$ and $M \in S$ be any monomial of degree N, so that $Mf \in H^0(\mathcal{I}_X(k))$. Then surjectivity of the map implies that $Mf \in (F_1, \ldots, F_m)$, so $f \in J^{\text{sat}}$.

Conversely, if $(F_1, \ldots, F_m)^{\text{sat}} \subsetneq I_X$, we use an analogous argument to show that the F_i cannot scheme-theoretically cut out X. □

2.1.2 Examples of Non-saturated Ideals

In this subsection, we present an extended example of constructing polynomials that cut out a variety scheme-theoretically while the ideal that they generate is non-saturated, which exemplifies that such ideals arise easily in nature.

We will start by establishing that if $X \subseteq \mathbb{P}^r$ is a smooth subvariety and we have polynomials that scheme-theoretically cut out X, we only need $r+1$ general $\mathbb{C}$-linear combinations.

Proposition 2.1.10. *Let L' be a line bundle on a smooth variety P' of dimension n which is generated by its global sections. Then it is generated by $n+1$ general sections.*

Proof. Take a general section $\sigma_0 \in \Gamma(L', P')$ and let $Z_1 = \{\sigma_0 = 0\} \subseteq P'$ be the zero locus, so every component $Z_{1,i}$ of Z_1 has dimension $n-1$. For each i, pick a point $x_1, i \in Z_{1,i}$. Since L' is globally generated, there exists sections $s_{1,i} \in \Gamma(L', P')$ such that $s_{1,i} \neq 0$. Then, there is a section σ_1 which is a $\mathbb{C}-$linear combination of the $s_{1,i}$ such that $\sigma_1(x_{1,i}) \neq 0$ for all i. Thus,

σ does not vanish on any component of Z_1, so the zero locus

$$Z_2 = \{x \in P' | \sigma_0 = \sigma_1 = 0\} \subseteq Z_1 \subseteq P'$$

is pure codimension 2 in P'. Continue inductively until we have chosen sections $\sigma_0, \ldots, \sigma_{n-1}$ such that the common zero locus

$$Z_{n-1} = \{x \in P' | \sigma_0 = \cdots = \sigma_{n-1} = 0\} \subseteq P'$$

is pure codimension n in P'. Since P' is dimension n, that means that Z_{n-1} is a finite set of points in P'. Then, there is a section $\sigma_n \in \Gamma(L', P')$ that is nonzero for all points of Z_{n-1}, so the common zero locus

$$\{x \in P' | \sigma_0 = \cdots = \sigma_n = 0\} = \varnothing.$$

Then, for any point $x \in P'$, there must exist a section σ_j among the $\sigma_0, \ldots, \sigma_n$ such that $\sigma_j(x) \neq 0$, which means that L' is globally generated by the $n+1$ sections $\sigma_0, \ldots, \sigma_n$. □

Lemma 2.1.11. *Let $X \subseteq P$ be a smooth subvariety of a smooth variety P. Let $\mu : P' \to P$ be the blow up of P along X and E be the exceptional divisor of μ. Then $\mu_*(\mathcal{O}_{P'}(-E)) = \mathcal{I}_X$.*

Proof. By Zariski's Main Theorem, $\mu_*(\mathcal{O}_{P'}) \xrightarrow{\sim} \mathcal{O}_P$ and $\mu_*(\mathcal{O}_E) \xrightarrow{\sim} \mathcal{I}_X$ are isomorphisms. The inclusion $E \subseteq P'$ gives rise to the exact sequence

$$0 \longrightarrow \mathcal{O}_{P'}(-E) \longrightarrow \mathcal{O}_{P'} \longrightarrow \mathcal{O}_E \longrightarrow 0 \ .$$

Then, taking the pushforward of this sequence through μ gives a left exact sequence

$$0 \longrightarrow \mu_*(\mathcal{O}_{P'}(-E)) \longrightarrow \mathcal{O}_P \longrightarrow \mathcal{O}_X \longrightarrow 0 \ .$$

Since $X \subseteq P$, the map $\mathcal{O}_P \to \mathcal{O}_X$ is surjective, so the sequence is also right exact. Hence, $\mu_*(\mathcal{O}_{P'}(-E)) = \mathcal{I}_X$. □

Proposition 2.1.12. *If a smooth variety $X \subseteq \mathbb{P}^r$ is scheme-theoretically cut out by hypersurfaces of degree d, then it is cut out by $r+1$ of them.*

Proof. If X is cut out by hypersurfaces of degree d, then then we have sections $s_0, \ldots, s_m \in H^0(\mathcal{O}_{\mathbb{P}^r}(d) \otimes \mathcal{I}_X)$. Let P' be the blow up of $\mathbb{P}^r$ along X and $E \subset P'$ be the exceptional divisor. Then, pulling back the sections through μ gives

$$s'_0, \ldots, s'_m \in H^0(\mu^*(\mathcal{O}_{\mathbb{P}^r}(d)) \otimes \mu^*(\mathcal{I}_X)).$$

By Lemma 2.1.11, $\mu^*(\mathcal{O}_{\mathbb{P}^r}(d)) \otimes \mu^*(\mathcal{I}_X)) = \mu^*(\mathcal{O}_{\mathbb{P}^r}(d)) \otimes \mathcal{O}_{\mathbb{P}^r}(-E)$, which is a line bundle L' on $\mathbb{P}^r$. Since pullback is right exact, the s'_i globally generate L'. By Proposition 2.1.10, L' is generated by $r+1$ general sections. □

This Proposition allows us to use the dimension of the ambient space to bound the number of generators needed to scheme-theoretically cut out a variety. Now we can use this to obtain non-saturated ideals.

Example 2.1.13 (Non-saturated ideals of curves by quadratic polynomials)**.** In this example, we construct examples of non-saturated ideals generated by quadratic polynomials which scheme-theoretically cut out a curve.

Let C be a smooth curve of genus g, degree $d \geq 2g+2$, and L be a line bundle of degree d on C. Then L is very ample by Corollary IV.3.2 in [12] and normally generated (cf. [20]) so it defines an embedding $C \hookrightarrow \mathbb{P}^r$ where $r = d - g$ such that C is projectively normal. Castelnuovo (cf. [2]) shows that C is a scheme-theoretic intersection of quadrics, and by Proposition 2.1.12, $C \subseteq \mathbb{P}^{d-g}$ is scheme-theoretically cut out by $d-g+1$ quadrics. To find out how many quadrics pass through C, consider the exact sequence of sheaves

$$0 \longrightarrow \mathcal{I}_{C/\mathbb{P}^r}(2) \longrightarrow \mathcal{O}_{\mathbb{P}^r}(2) \longrightarrow \mathcal{O}_C(2) \longrightarrow 0.$$

By the Riemann-Roch Theorem,

$$h^0(\mathcal{O}_C(2)) = 2d - g + 1.$$

Since C is projectively normal, the sequence above is exact on global sections, so

$$\begin{aligned} h^0(\mathcal{I}_{C/\mathbb{P}^r(2)}) &= h^0(\mathcal{O}_{\mathbb{P}^r}(2)) - h^0(\mathcal{O}_C(2)) \\ &= \binom{r+2}{2} - (2d-g+1) \\ &= \binom{d-g+2}{2} - (2d-g+1), \end{aligned}$$

which is quadratic in d. Thus, since the number $d-g+1 \leq \frac{d+1}{2}$ of quadratic polynomials needed to scheme theoretically cut out C is linear in d, the ideal that they generate is very non-saturated.

2.2 Macaulay's Theorem

Given a homogeneous ideal $J \subseteq S$, there are some cases in which we can bound the saturation degree of J given the degrees of the generators of J. When the polynomials defining J have no common zeroes, we have the following classical result of Macaulay [17].

Theorem 2.2.1. ***(Macaulay's Theorem)*** *Let $S = \mathbb{C}[x_0, \ldots, x_m]$. Suppose $F_0, \ldots, F_p \in S$ are homogeneous polynomials of degrees $d_0 \geq \cdots \geq d_p$ respectively with no common zeroes in $\mathbb{P}^m$. Let $J = \big(F_0, \ldots, F_p\big)$. Then*

$$J_k = S_k \text{ for } k \geq d_0 + \cdots + d_m - m,$$

and the inequality is sharp when $p = m$.

In terms of saturation, Macaulay's Theorem says that the saturation degree of J is $(\sum d_i) - m$ when the number of polynomials is the dimension of the projective space plus one and otherwise it is an upper bound on the saturation degree.

Example 2.2.2. Here is a case in which the conclusion of Macaulay's theorem is apparent. Consider the monomials $X_0^{d_0}, \ldots, X_m^{d_m} \in S$ and the ideal J that they generate. The highest order monomial that is not contained in J is $X_0^{d_0-1} \cdot \cdots \cdot X_m^{d_m-1}$, which is of degree $(\sum d_i) - m - 1$. Then J contains all monomials of all strictly larger degrees.

The proof of Macaulay's Theorem involves the Koszul complex, which I will briefly review.

Definition 2.2.3. *Given homogeneous polynomials $F_0, \ldots, F_p$ with $\deg F_i = d_i$, let $U = \oplus_{i=0}^{p} \mathcal{O}_{\mathbb{P}}(-d_i)$ so that the F_i define a map of sheaves*

$$U \xrightarrow{\epsilon} \mathcal{O}_{\mathbb{P}}.$$

*The **Koszul complex** $\mathcal{K}(F_0, \ldots, F_p)$ determined by the F_i is*

$$0 \to \mathcal{O}_{\mathbb{P}}(-\sum d_i) \to \Lambda^p U \to \cdots \to \Lambda^2 U \to U \to \mathcal{O}_{\mathbb{P}} \to 0$$

where $\Lambda^i U \to \Lambda^{i-1} U$ is given by the composition

$$\Lambda^i U \hookrightarrow \Lambda^{i-1} U \otimes U \xrightarrow{id \otimes \epsilon} \Lambda^{i-1} U.$$

The Koszul complex detects regular sequences in the local case, as described in the following Proposition.

Proposition 2.2.4 (Theorem A2.49 in [7]). *Let $F_0, \ldots, F_p$ be a sequence of elements in the maximal ideal of the local ring $(S, \mathfrak{m})$. Then $F_0, \ldots, F_p$ forms a regular sequence if and only if $H^p(\mathcal{K}(F_0, \ldots, F_p)) = 0$, in which case the Koszul complex is the minimal free resolution of $\mathcal{I}_X$, where $X = V(F_0, \ldots, F_p)$.*

In particular, if the F_i have no common zeroes, the Koszul complex they determine is globally exact.

When using the Koszul complex, we will often split it into short exact sequences and study the cohomology. The cohomology of line bundles on projective space is useful in studying which twists of an ideal sheaf have vanishing 0th cohomology.

Proposition 2.2.5 (Cohomology of $\mathbb{P}^m$, cf Theorem 5.1 in [12]). *Let k be a non-negative integer. Then*

$$h^0(\mathcal{O}_{\mathbb{P}^m}(k)) = \binom{m+k}{m},$$
$$h^m(\mathcal{O}_{\mathbb{P}^m}(k)) = h^0(\mathcal{O}_{\mathbb{P}^m}(-m-k-1)),$$

and all other H^i vanish.

Now we can prove Macaulay's Theorem.

Proof of Macaulay's Theorem. Since the F_i have no common zeroes, termwise multiplication by the F_i gives a map of sheaves

$$\bigoplus_{i=0}^{p} \mathcal{O}_{\mathbb{P}^m}(k-d_i) \to \mathcal{O}_{\mathbb{P}^m}(k).$$

Note that when $p = m$, the F_i form a regular sequence. In this case, the Koszul complex gives a resolution

$$0 \to \mathcal{O}\left(k - \sum d_i\right) \longrightarrow \dots \longrightarrow \bigoplus_{i=0}^{p} \mathcal{O}(k-d_i) \longrightarrow \mathcal{O}(k) \longrightarrow 0.$$

On H^0, surjectivity of the map

$$\bigoplus_{i=0}^{p} H^0(\mathcal{O}(k-d_i)) \to H^0(\mathcal{O}(k))$$

is equivalent to $J_k = S_k$. By splitting the Koszul complex up into short exact sequences and chasing back to the beginning of the complex, this traces back to whether $H^m(\mathcal{O}_{\mathbb{P}^m}(k-d_0-\cdots-d_m))$ vanishes. By Proposition 2.2.5, this occcurs if and only if $k \geq d_0 + \cdots + d_m - m$, so we get the desired result in the $p = m$ case.

When $p > m$, we will use the fact that $J_k = S_k$ if $(F_{i_0}, \dots, F_{i_m})_k = S_k$ for all possible subcollections of indices $\{i_j\}_{j=0}^m \subseteq \{0, \dots, p\}$. For each such index set $\{i_j\}_{j=0}^m$, it follows that $(F_{i_0}, \dots, F_{i_m})_k = S_k$ when $k \geq \sum_{j=0}^m d_{i_j} - m$. By the ordering $d_0 \geq \cdots \geq d_p$, all of these are achieved when $k \geq d_0 + \cdots + d_m - m$ as desired.

□

Ein, Há, and Lazarsfeld establish that the bound from Macaulay's Theorem holds also in the case when the zero locus of the polynomials is a smooth projective variety.

Theorem 2.2.6. *[6] Suppose that $J = (F_0, \ldots, F_p) \subseteq S$ is generated by forms of degrees $d_0 \geq \cdots \geq d_p$ and that $X = V(J) \subseteq \mathbb{P}^r$ is a smooth variety. Then*

$$\mathrm{sat.deg}(J) \leq d_0 + \cdots + d_r - r,$$

where $d_{p+1} = \cdots = d_r = 0$ if $p < r$.

Note that unlike the situation of Macaulay's Theorem, it is unknown if this bound is sharp in the $p = r$ case.

2.3 Regularity

The vanishing of cohomology groups can be useful when using sheaf cohomology. Recall the well-known theorem of Serre:

Theorem 2.3.1 (Serre Vanishing Theorem, Proposition III.5.3 in [12])**.** *Let X be an irreducible projective variety and $\mathcal{L}$ be an invertible sheaf on X. Then the following conditions are equivalent:*

1. *$\mathcal{L}$ is ample;*
2. *For each coherent sheaf $\mathcal{F}$ on X, there is an integer n_0, depending on $\mathcal{F}$, such that for each $i > 0$ and each $n \geq n_0$, $H^i(\mathcal{F} \otimes \mathcal{L}^n) = 0$.*

If $\mathcal{L}$ is very ample and defines an embedding of X into a projective space $\mathbb{P}$, then in particular, given a coherent sheaf $\mathcal{F}$ on a projective space $\mathbb{P}$ of some dimension, the Serre vanishing theorem says that that the higher cohomology groups of $\mathcal{F}(m)$ vanish for sufficiently large values of m. It is natural to ask whether we can obtain information about what n_0 should be for a given sheaf $\mathcal{F}$. Castelnuovo-Mumford regularity gives a way to approach this.

Definition 2.3.2 (Definition 1.8.1 in [15])**.** *Let $\mathcal{F}$ be a cohererent sheaf on $\mathbb{P}$, and let m be an integer. We call $\mathcal{F}$* ***m-regular*** *if*

$$H^i(\mathbb{P}, \mathcal{F}(m-i)) = 0 \text{ for all } i > 0.$$

Definition 2.3.3. *The* ***Castelnuovo-Mumford regularity*** $\operatorname{reg}(\mathcal{F})$ *of a coherent sheaf $\mathcal{F}$ on $\mathbb{P}$ is the smallest integer m so that $\mathcal{F}$ is m-regular.*

A theorem of Mumford allows us to connect regularity to the questions of when the cohomology of $\mathcal{F}$ becomes simple.

Theorem 2.3.4 (Mumford's Theorem from Lecture 14 in [19])**.** *Let $\mathcal{F}$ be an m-regular sheaf on $\mathbb{P}$. Then for every $k \geq 0$,*

1. *$\mathcal{F}(m+k)$ is generated by its global sections.*
2. *The natural maps*

$$H^0(\mathcal{F}(m)) \otimes H^0(\mathcal{O}_{\mathbb{P}}(k)) \to H^0(\mathcal{F}(m+k))$$

   are surjective.

3. *$\mathcal{F}$ is $(m+k)$-regular.*

One case of particular interest is when $\mathcal{F}$ is the ideal sheaf of a subvariety X of projective space.

Definition 2.3.5. *A subvariety or subscheme $X \subseteq \mathbb{P}$ is* ***m-regular*** *if its ideal sheaf $\mathcal{I}_X$ is. The regularity of X is* $\operatorname{reg}(\mathcal{I}_X)$.

A general goal in the study of regularity is to bound the regularity of projective varieties using their geometric quantities. Towards this, there is the following result due to Gruson, Peskine, and Lazarsfeld.

Theorem 2.3.6 (Regularity for curves, [11])**.** *Let $C \subseteq \mathbb{P}^r$ be an irreducible, nondegenerate, reduced curve of degree d. Then C is $(d+2-r)$-regular.*

Eisenbud and Goto produce a natural extension of this result to the case of higher dimensional varieties.

Conjecture 2.3.7 (Regularity conjecture, [8])**.** *If $X \subseteq \mathbb{P}^r$ is an irreducible subvariety of dimension n and degree d, then X is $(d+n-r+1)$-regular.*

This conjecture has been shown to hold in certain cases, such as arithmetically Cohen-Macaulay varieties [8], subspace arrangements [5], and smooth surfaces [14, 21]. While Theorem 2.3.6 holds for singular curves, [18] provide counterexamples of this conjecture in the singular case overall.

Remark 2.3.8 (Importance of $H^1(\mathcal{I}_X(k))$ in the context of regularity)**.** If $X \subseteq \mathbb{P}^r$ is a subvariety, then we get a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{O}_{\mathbb{P}^r} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Twisting up by some integer k, we can look at a portion of the long exact sequence in cohomology

$$H^{i-1}(\mathcal{O}_{\mathbb{P}^r}(k)) \longrightarrow H^{i-1}(\mathcal{O}_X(k)) \longrightarrow H^i(\mathcal{I}_X(k))) \longrightarrow H^i(\mathcal{O}_{\mathbb{P}^r}(k)).$$

Note that for $1 \leq i \leq r-1$, $H^i(\mathcal{O}_{\mathbb{P}^r}(k))$ vanishes, and for $k \geq -r$, $H^i(\mathcal{O}_{\mathbb{P}^r}(k))$ vanishes for $i = r$. Overall, for $i \geq 2$ and $k \geq -r$, we have an isomorphism

$$H^i(\mathcal{I}_X(k))) \cong H^{i-1}(\mathcal{O}_X(k)).$$

Thus, the cohomology in degrees $i \geq 2$ of $\mathcal{I}_X(k)$ is governed by the cohomology of the line bundle on X defining the embedding of $X \subseteq \mathbb{P}^r$, which is often easy to control. Hence, we are particularly interested in studying $H^1(\mathcal{I}_X(k))$ in order to bound the regularity of X.

2.4 Flatness

If we want to discuss the notion of a continuously varying family of objects in the context of algebraic geometry, the notion of flatness gives a nice way to do this. We start with the idea of a flat module.

Definition 2.4.1. *Let A be a ring and M be an A-module. M is* ***flat*** *over A if the functor $N \mapsto M \otimes_A N$ is an exact functor for every A-module N.*

Proposition 2.4.2 (Proposition 9.1A in [12]). *Let*

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of A-modules. If M' and M'' are both flat, then M is flat; if M and M'' are both flat, then M' is flat.

We can use this to define a flat morphism in the category of $\mathcal{O}_X$ modules.

Definition 2.4.3. *Let $f : X \to Y$ be a morphism of schemes and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is **flat over** Y **at** $x \in X$ if $\mathcal{F}_x$ is a flat $\mathcal{O}_{y,Y}$-module, where $y = f(x)$. We say that $\mathcal{F}$ is **flat over** Y if it is flat at every point of X. We say that f is **flat** if $\mathcal{O}_X$ is flat over $\mathcal{O}_Y$.*

The fibres of a flat morphism satisfy some nice properties.

Proposition 2.4.4 (cf. Proposition 9.5 in [12]). *Let $f : X \to Y$ be a flat morphism of schemes of finite type over a field. Then*

$$\dim_x(X_y) = \dim_x X - \dim_y Y,$$

where X_y is the fibre of f over $y = f(x) \in Y$.

Definition 2.4.5. *The fibres X_y of a flat morphism $f : X \to Y$ are called a **flat family**.*

When the codomain Y of a flat morphism $f : X \to Y$ is one dimensional, we can think of the flat family as an algebraically varying one-parameter family of the fibres X_y over Y. The next result describes the existence of a flat limit over a punctured curve.

Proposition 2.4.6 (Proposition 9.8 in [12]). *Let Y be a curve and $p \in Y$ be a closed point. Let $X \subseteq \mathbb{P}^n|_{Y-p}$ be a closed subscheme which is flat over $Y - \{p\}$. Then there exists a unique closed subscheme $\bar{X} \subseteq \mathbb{P}^n|_Y$ that is flat over Y and restricts to X on $\mathbb{P}^n|_{Y-p}$.*

The cohomology of the members of a flat family follows the following semicontinuity property.

Definition 2.4.7. *Let X be a topological space. A function $f : X \to \mathbb{Z}$ is* ***upper semicontinuous*** *if for each $x \in X$ there is an open neighborhood U of x such that for all $x' \in U$, $f(x') \leq f(x)$.*

Theorem 2.4.8 (Semicontinuity Theorem, Theorem III.12.8 in [12])**.** *Let $f : X \to Y$ be a projective morphism of noetherian schemes, and let $\mathcal{F}$ be a coherent sheaf on X, flat over Y. Then for each $i \geq 0$, the function*

$$h^i(y, \mathcal{F}) = \dim H^i(X_y, \mathcal{F}_y)$$

is an upper semicontinuous function on Y.

In other words, the dimension of the cohomology groups of the flat family may jump up for special values of Y.

Chapter 3

Cone Polynomials

In this chapter, I will introduce cone polynomials which are the key objects of interest in this dissertation. We work over $\mathbb{C}$ unless specified otherwise. Section 3.1 defines the cone polynomial and discusses their connection to Castelnuovo-Mumford regularity. Prior work on cone polynomials is surveyed in Section 3.2. Section 3.3 proves that cone polynomials can be used to scheme-theoretically cut out smooth projective varieties. Finally, in Section 3.4, I find a rough bound on the saturation degree of the ideal generated by cones that follows from sheaf cohomology.

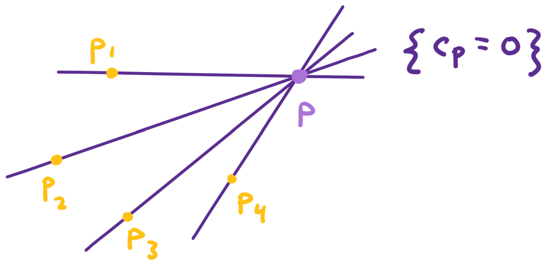
3.1 Definitions and Motivation

Let's start with the definition of a cone polynomial.

Definition 3.1.1. *Let $X \subseteq \mathbb{P}^r$ be a smooth projective variety of degree d and dimension n. For any linear space $\Lambda \subseteq \mathbb{P}^r$ of dimension $r - n - 2$ disjoint from X, let $C_\Lambda(X)$ be the join of Λ and X and call its defining equation $c_\Lambda(X)$. We call $c_\Lambda(X)$ the* ***cone polynomial over X centered at Λ****.*

In the special case when $X \subseteq \mathbb{P}^2$ is a finite set of d distinct points, we will use the notation $c_p(X)$, or simply c_p when the base X is clear, to denote the cone polynomial over X centered at a point p.

Example 3.1.2. Let $X = \{p_1, p_2, p_3, p_4\} \subseteq \mathbb{P}^2$ be a set of 4 distinct points. Then for any point $p \notin X$, c_p is the product of the four linear forms that vanish on p and one of the p_i.



When X is a set of d points in $\mathbb{P}^r$, each $c_\Lambda(X)$ will be a product of d linear forms. When X is an arbitrary degree d variety, $c_\Lambda(X)$ will define a degree d hypersurface in $\mathbb{P}^r$.

Definition 3.1.3. *Let X and $\Lambda \subseteq \mathbb{P}^r$ be as in Definition 3.1.1. Define the* ***cone ideal of*** *X to be the ideal generated by the cone polynomials*

$$I_X^{\text{cone}} = \langle\, c_\Lambda(X) \mid \Lambda \subseteq \mathbb{P}^r \setminus X \,\rangle.$$

When the base variety X is clear, we may leave off the subscript and denote the cone ideal simply as I^{cone}.

Remark 3.1.4. In Section 3.3, we will show that cone polynomials $c_\Lambda(X)$ cut out a smooth projective variety X scheme-theoretically. Thus, by Proposition 2.1.9, $(I^{\text{cone}})^{\text{sat}} = I_X$.

3.1.1 Motivating the Study of Cone Polynomials

As discussed in Remark 2.3.8, understanding $H^1(\mathcal{I}_X(k))$ is important in order to bound the Castelnuovo-Mumford regularity of a projective variety X. In particular, we are interested in finding bounds on k for which this cohomology group vanishes.

One general question of interest is if we can use geometric methods to find such a bound. Towards this, consider a nondegenerate, smooth variety $X \subseteq \mathbb{P}^r$ of degree d and dimension n. A generic hyperplane $H \subseteq \mathbb{P}^r$ gives an exact sequence

$$0 \longrightarrow \mathcal{I}_X(-1) \xrightarrow{\cdot H} \mathcal{I}_X \longrightarrow \mathcal{I}_{X\cap H/H} \longrightarrow 0.$$

The following Lemma, extracted from [19], allows us to study $H^1(\mathcal{I}_X(k))$.

Lemma 3.1.5. *Suppose that $\mathcal{I}_{X\cap H/H}$ is m-regular. Then for $k \geq m-1$,*

$$h^1(\mathcal{I}_X(k))$$

is strictly decreasing as a function of k until it reaches and remains at 0.

Proof. For brevity of notation, let $\mathcal{I} = \mathcal{I}_{\mathcal{X}}$ and $\mathcal{I}_H = \mathcal{I}_{X\cap H/H}$. By definition of m-regularity,

$$H^0(\mathcal{I}(k+1)) \xrightarrow{\rho_{k+1}} H^0(\mathcal{I}_H(k+1)) \to H^1(\mathcal{I}(k)) \to H^1(\mathcal{I}(k+1)) \to 0 \quad (3.1)$$

is exact for $k \geq m-2$ and

$$0 \to H^i(\mathcal{I}(k)) \to H^i(\mathcal{I}(k+1)) \to 0 \quad (3.2)$$

is exact for $i \geq 2$, $k \geq m-i$. Exactness of sequence (3.2) means that $H^i(\mathcal{I}(k)) \cong H^i(\mathcal{I}(k+1))$ for $i \geq 2, k \geq m-i$. Since $H^i(\mathcal{I}(k)) = 0$ for $i \geq 1, k \gg 0$ by Serre Vanishing, we get

$$H^i(\mathcal{I}(k)) = 0 \text{ for } i \geq 2, k \geq m-i,$$

so $\mathcal{I}$ satisfies the m-regular condition on $H^{\geq 2}$.

Consider the map ρ_{k+1} from sequence (3.1). For $k \geq m-2$, either

$$\rho_{k+1} \text{ is surjective } \textbf{ or } \rho_{k+1} \text{ is not surjective, in which case } h^1(\mathcal{I}(k)) > h^1(\mathcal{I}(k+1)). \quad (3.3)$$

Suppose that ρ_k is surjective for some $k \geq m$. Then restriction to H gives a commutative diagram

$$\begin{array}{ccc} H^0(\mathcal{I}_H(k)) \otimes H^0(\mathcal{O}_H(1)) & \twoheadrightarrow & H^0(\mathcal{I}_H(k+1)) \\ \big\uparrow\!\!\uparrow & & \big\uparrow \rho_{k+1} \\ H^0(\mathcal{I}(k)) \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1)) & \longrightarrow & H^0(\mathcal{I}(k+1)), \end{array}$$

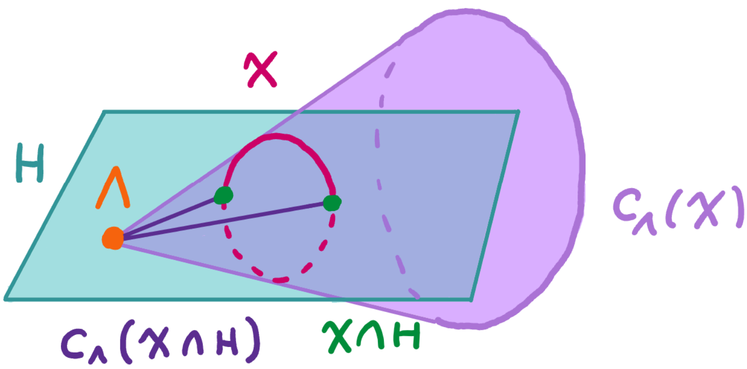
where surjectivity of the upper map comes from the Theorem 2.3.4 and surjectivity of the left map is from our assumption. Then ρ_{k+1} must also be surjective, i.e., once ρ_k is surjective for some $k \geq m$, it must be surjective for all higher k. Considering this along with condition (3.3) yields the Lemma. $\square$

When considering the long exact sequence

$$H^0(\mathcal{I}_X(k)) \to H^0(\mathcal{I}_{X \cap H/H}(k)) \to H^1(\mathcal{I}_X(k-1)) \to H^1(\mathcal{I}_X(k)),$$

the rightmost term is the one we want to be 0. In practice, one can suppose by induction on dimension that we can control $\operatorname{reg}(\mathcal{I}_{X \cap H/H})$, so by Lemma 3.1.5, since $h^1(\mathcal{I}_X(k))$ is strictly decreasing after a certain point, it suffices to show that the leftmost map above is surjective. In other words, can we show that any degree k polynomial in H vanishing on $X \cap H$ comes from (i.e., is a hyperplane section of) a degree k polynomial in $\mathbb{P}^r$ vanishing on X?

This is where the usefulness of cones becomes apparent. When $\Lambda \subseteq H$ is a linear space of dimension $r - n - 2$, the cone polynomial $c_\Lambda(X)$ is a polynomial vanishing on X which extends the polynomial $c_{\Lambda \cap H}(X \cap H)$ that vanishes on $X \cap H$.



This means that when considering the degree d parts of the ideals,

$$(I_X^{\text{cone}})_d \twoheadrightarrow (I_{X\cap H}^{\text{cone}})_d.$$

Since the cone ideals are generated by degree d elements, we also get $(I_X^{\text{cone}})_k \twoheadrightarrow (I_{X\cap H}^{\text{cone}})_k$ for $k \geq d$. Then since $(I_X^{\text{cone}})_k \subseteq (I_X)_k$, if we can find k such that $(I_{X\cap H}^{\text{cone}})_k = (I_{X\cap H})_k$, then for that k, we would obtain the desired surjection $H^0(\mathcal{I}_X(k)) \twoheadrightarrow H^0(\mathcal{I}_{X\cap H}(k))$. In the diagram below, surjectivity of $(*)$ yields surjectivity of $(**)$.

$$\begin{array}{ccc}
H^0(\mathcal{I}_X(k)) & \xrightarrow{(**)} & H^0(\mathcal{I}_{X\cap H}(k)) \\
\| & & \| \\
(I_X)_k & & (I_{X\cap H})_k \\
\uparrow & & \uparrow (*) \\
(I_X^{\text{cone}})_k & \twoheadrightarrow & (I_{X\cap H}^{\text{cone}})_k.
\end{array}$$

This motivates our study of finding the values of k for which $(I_X^{\text{cone}})_k = (I_X)_k$. The smallest such value of k is the saturation degree of the cone ideal.

In this dissertation we attempt to find or bound the saturation degree of the cone ideal and understand how the saturation degree may vary based on geometric properties of the variety.

3.2 Theorems of Fu and Nie

Weibo Fu and Zipei Nie [9] examined the codimension of the span of cone polynomials sitting inside the space of degree d forms. Their work builds on a conjecture of Bo Ilic on whether cone polynomials generate the defining ideal of a set of d generic points in $\mathbb{P}^2$ in the smallest degree possible. I will give an overview of their work here. In this section, projective space is defined over an arbitrary characteristic zero field k.

As previously discussed, cone polynomials can be defined over smooth projective varieties that are at least codimension 2 in some $\mathbb{P}^n$. The first case of this is points in $\mathbb{P}^2$, so let's focus on sets of points in $\mathbb{P}^n$ for $n \geq 2$.

Let X be a set of d points in $\mathbb{P}^n$. Since degree d homogeneous polynomials in $\mathbb{P}^n$ form a vector space of dimension $\binom{d+n}{n}$ and vanishing at a point imposes one linear condition, we have

$$\dim(I_X)_d = \binom{d+n}{n} - d.$$

Then $I_d^{\text{cone}} \subseteq (I_X)_d$ is a vector subspace. Note that for any collection $X \subseteq \mathbb{P}^2$ of d distinct points, $\mathcal{I}_X$ is d-regular.

In the mid 1990s, Bo Ilic ran computational experiments comparing the dimensions of I_d^{cone} and $(I_X)_d$. Based on his observations, he conjectured the following:

Conjecture 3.2.1. *Let $X \subseteq \mathbb{P}^2$ be a set of d generic points. For each integer $d \geq 2$, we have*

$$\dim I_d^{\text{cone}} = \begin{cases} \binom{d+2}{2} - d, & \text{if } d = 0 \mod 2, \\ \binom{d+2}{2} - d - 2, & \text{if } d = 1 \mod 4, \\ \binom{d+2}{2} - d - 1, & \text{if } d = 3 \mod 4. \end{cases}$$

In particular, experimental data suggests that cone polynomials generate all degree d homogeneous polynomials that vanish on X if and only if d is even.

By finding nontrivial linear relations among the cone polynomials, Fu and Nie prove the following two results about an odd number of points in generic position, which are consistent with Conjecture 3.2.1.

Theorem 3.2.2. *For $n \geq 2$, let $X \subseteq \mathbb{P}^n$ be a set of d generic points. When $d \geq 3$ is an odd integer,*

$$\dim I_d^{\text{cone}} \leq \binom{d+2}{2} - d - 1.$$

Theorem 3.2.3. *Let $X \subseteq \mathbb{P}^2$ be a set of d generic points. When $d \geq 5$ and $d \equiv 1 \mod 4$,*

$$\dim I_d^{\text{cone}} \leq \binom{d+2}{2} - d - 2.$$

Note that Theorem 3.2.2 holds in projective spaces in dimensions higher than 2, which may suggest higher-dimensional analogues of Conjecture 3.2.1.

The rest of this section is dedicated to the clever proof of Theorem 3.2.2. We will first fix notation for the proof. Let $X = \{p_1, \ldots, p_d\}$ be a set of d generic points in $\mathbb{P}^n$ and $P_i \in k^{n+1} \setminus \{0\}$ be p_i represented in homogeneous coordinates. Let D_i represent the directional derivative operator along P_i. Let Λ be a $n-2$ plane in $\mathbb{P}^n$. The linear form L_i is the linear form vanishing on Λ and P_i so that the cone polynomial c_Λ is the product $\prod_{i=1}^d L_i$. Let $\{Q_j \in \mathbb{P}^n\}_{j=1}^{n-1}$ be an affine basis of the plane Λ. Define a linear function in $k[x_1, \ldots, x_{n+1}]$ by

$$l_i(v) = \det[v, P_i, Q_1, \ldots, Q_{n-1}].$$

Lemma 3.2.4. *$D_i l_j = -D_j l_i$ for $1 \leq i \neq j \leq d$.*

Proof. We have

$$\begin{aligned} D_i l_j &= \left(\sum_{k=0}^{n} P_{i,k} \frac{\partial}{\partial x_k} \right) \det[v, P_j, Q_1, \ldots, Q_{n-1}] \\ &= \sum_{k=0}^{n} \det[P_j^{@k}, Q_1^{@k}, \ldots, Q_{n-1}^{@k}] \end{aligned}$$

where the $@k$ superscript represents removing the kth position entry. By cofactor expansion,

$$\begin{aligned} D_i l_j &= \det[P_i, P_j, Q_1, \dots, Q_{n-1}] \\ &= -\det[P_j, P_i, Q_1, \dots, Q_{n-1}] \\ &= -D_j l_i. \end{aligned}$$

□

Lemma 3.2.5. *For each odd positive integer $d \geq 3$ and for each cone polynomial c_Λ, we have*

$$\left(\prod_{i=1}^{d} D_i\right) c_\Lambda = 0.$$

Furthermore, this restriction on cone polynomials is nontrivial, i.e., there exists a polynomial $f \in k[x_1, \dots, x_{n+1}]$ of degree d that vanishes on X such that $\left(\prod_{i=1}^{d} D_i\right) f \neq 0$.

Proof. We have

$$\begin{aligned} \left(\prod_{i=1}^{d} D_i\right)\left(\prod_{i=1}^{d} l_i\right) &= \sum_{\sigma \in \mathfrak{S}_d} \prod_{i=1}^{d} D_i l_{\sigma(i)} \\ &= \frac{1}{2} \sum_{\sigma \in \mathfrak{S}_d} \left(\prod_{i=1}^{d} D_i l_{\sigma(i)} + (-1)^d \prod_{i=1}^{d} D_{\sigma(i)} l_i\right). \end{aligned}$$

When d is odd, $(-1)^d = -1$, so the expression continues as

$$\begin{aligned} &= \frac{1}{2}\left(\sum_{\sigma \in \mathfrak{S}_d} \prod_{i=1}^{d} D_i l_{\sigma(i)}\right) - \frac{1}{2}\left(\sum_{\sigma^{-1} \in \mathfrak{S}_d} \prod_{i=1}^{d} D_i l_{\sigma(i)}\right) \\ &= 0. \end{aligned}$$

Since each l_i is a scalar multiple of L_i from the definition of the cone polynomial, the first part of the Lemma holds.

To show that this restriction is nontrivial, consider the polynomial $f \in k[x_1, \ldots, x_{n+1}]$ defined by

$$f(v) = \prod_{i=1}^{d} \det[\pi(v), \pi(P_i), \pi(P_{i+1})],$$

where the indices are taken modulo d and $\pi : k^{n+1} \to k^3$ is the projection $(x_1, \ldots, x_{n+1}) \mapsto (x_1, x_2, x_3)$. It is sufficient to check that $\left(\prod_{i=1}^{d} D_i\right) f$ is nonzero in the case where $P_i = (1, i, i^2, \ldots, i^n)$ for each integer $1 \leq i \leq d$. In this case,

$$\begin{aligned}\left(\prod_{i=1}^{d} D_i\right) f &= \sum_{\sigma \in \mathfrak{S}_d} \prod_{i=1}^{d} D_{\sigma(d)} \det[\pi(v), \pi(P_i), \pi(P_{i+1})] \\ &= \sum_{\sigma \in \mathfrak{S}_d} \prod_{i=1}^{d} \det[P_{\sigma(d), \pi(P_i), \pi(P_{i+1})}] \\ &= \sum_{\sigma \in \mathfrak{S}_d} (d-1)(d-\sigma(d))(\sigma(d)-1) \prod_{i=1}^{d-1} (\sigma(i)-i)(\sigma(i)-i-1),\end{aligned}$$

where the last equality comes from the Vandermonde determinant identity. The final expression must be strictly positive because $(d-1)(d-\sigma(d))(\sigma(d)-1)$ and $(\sigma(i)-i)(\sigma(i)-i-1)$ are always nonnegative, and when $\sigma(i) = i-1 \mod d$, the term is positive. □

As Lemma 3.2.5 provides a nontrivial linear restriction among cone polynomials, the result of Theorem 3.2.2 is implied.

The bound in Theorem 3.2.3 follows a similar idea to obtain an additional linear restriction in the case of $n = 2$ and $d \equiv 1 \mod 4$. If these linear restrictions are the only restrictions among cone polynomials, including the case of d even, Ilic's Conjecture 3.2.1 would hold.

3.3 Cutting out Varieties with Cone Polynomials

Mumford [20] provides the following criterion for when a smooth variety is scheme-theoretically the intersection of hypersurfaces.

Lemma 3.3.1. *If a subvariety $X \subseteq \mathbb{P}^r$ is non-singular, then X is scheme-theoretically the intersection of hypersurfaces $H_1, \ldots, H_n$ if and only if*

1. $X = \bigcap_{i=1}^n H_i$.
2. *for all* $x \in X$, $T_{x,X} = \bigcap_{i=1}^n T_{x,H_i}$.

We can use this to show that cone polynomials can cut out any smooth, projective variety scheme-theoretically.

Proposition 3.3.2. *A smooth, projective variety X is scheme-theoretically an intersection of cone polynomials. In other words, the equations c_Λ where Λ is a linear space of dimension $r-n-2$ disjoint from X generate the ideal sheaf $\mathcal{I}_X$.*

Proof. If $X \subseteq \mathbb{P}^r$ is a projective variety of dimension n (of at least codimension 2) and degree d, and $\Lambda \subseteq \mathbb{P}^r$ a linear space of dimension $r-n-2$ disjoint from X, consider the cone polynomial C_Λ. We have

$$\dim C_\Lambda = \dim X + \dim \Lambda + 1 = n + (r-n-2) + 1 = r-1,$$

so C_Λ is a degree d hypersurface.

To show that X is the intersection of the C_Λ, note that $X \subseteq C_\Lambda$ for each $\Lambda \cap X = \varnothing$, so $X \subseteq \bigcap_{\Lambda \cap X = \varnothing} C_\Lambda$. On the other hand, if $x \in \mathbb{P}^r - X$, consider the projection

$$\pi : \mathbb{P}^r - \{x\} \to \mathbb{P}^{r-1}.$$

Then $\pi(X) \subseteq \mathbb{P}^{r-1}$ is dimension n, so there exists some linear space $M \subseteq \mathbb{P}^{r-1}$ of dimension $(r-1)-n-1$ disjoint from $\pi(X)$. Choose a linear space $\Lambda \subseteq \mathbb{P}^r$ so that $\pi(\Lambda) = M$. Then $x \notin C_\Lambda$, since $x \in C_\Lambda$ means that there exists a line l through X and Λ containing x. Then $\pi(l) = M$ would not be disjoint from $\pi(X)$, which is a contradiction.

Next assume X is smooth. Let's confirm that $T_xX = \bigcap_{\Lambda\cap X=\varnothing} T_x c_\Lambda$ for every $x \in X$. Since every c_Λ contains X, we have $T_xX \subseteq T_x c_\Lambda$ for each $x \in X$, which implies $T_xX \subseteq \bigcap_{\Lambda\cap X=\varnothing} T_x c_\Lambda$. On the other hand, if $v \in T_x\mathbb{P}^r$ is a tangent vector with $v \notin T_xX$, it corresponds to a line $l \subseteq \mathbb{P}^r$ through x. Pick Λ so that $l \subsetneq \Lambda$. For this choice of Λ, $v \notin T_x c_\Lambda$. This shows that the tangent spaces of c_Λ at x cut out T_xX, so by Lemma 3.3.1, cone polynomials cut X out scheme-theoretically. $\square$

Hence, we can apply the theory of saturation to our study of cone polynomials. In particular, bounding the saturation degree of the cone ideal can give quantitative information about how close cone polynomials can get to generating the defining ideal of a smooth variety.

3.4 A Preliminary Bound for the Saturation Degree of the Cone Ideal

In this section, we will examine what sheaf cohomology reveals about the saturation degree of the cone ideal. One goal of this dissertation is to improve upon this bound.

Lemma 3.4.1 (Cohomology Vanishing, Theorem III.2.7 in [12])**.** *If X is a smooth projective variety of dimension n and $\mathcal{F}$ is a coherent sheaf on X, then*

$$H^i(\mathcal{F}) = 0 \text{ for } i > n.$$

Remark 3.4.2. Recall that if $X \subseteq \mathbb{P}^r$ is a finite set of d smooth (i.e., reduced) points, the ideal sheaf is generated by $r+1$ general cone polynomials by Proposition 2.1.12. In other words, we have a surjective map of sheaves

$$\mathcal{O}_{\mathbb{P}^r}(-d)^{r+1} \longrightarrow \mathcal{I}_X \longrightarrow 0.$$

Consider the $r = 2$ case in $\mathbb{P}^2$. Taking the Koszul complex and splitting into short exact sequences, we get the following diagram.

$$\begin{array}{ccccccccccc}
 & & & \nearrow & 0 & & & & & & \\
 & & & & \downarrow & & & & & & \\
0 & \longrightarrow & \mathcal{B}_2 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \mathcal{H}_2 & \longrightarrow & 0 & & \\
 & & \uparrow & & \downarrow & & & & & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(-3d) & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(-2d)^3 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(-d)^3 & \longrightarrow & \mathcal{I}_X & \longrightarrow & 0 \\
 & & & & \downarrow & & \uparrow & & & & \\
 & & 0 & \longrightarrow & \mathcal{B}_1 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \mathcal{H}_1 & \longrightarrow & 0. \\
 & & & & \downarrow & \nearrow & & & & & \\
 & & & & 0 & & & & & &
\end{array}$$

The $r+1 = 3$ cone polynomials do not form a regular sequence, so the Koszul complex is not exact. However, the cohomology of the Koszul complex is annihilated by $\mathcal{I}_X$, so the homology sheaves are supported on X, which is 0-dimensional. This means that the higher cohomology of the $\mathcal{H}_i$ vanish. We can use this to find a bound for the degree in which cone polynomials generate I_X.

Proposition 3.4.3. *(Bound from Sheaf Cohomology.) Let $X \subseteq \mathbb{P}^2$ be a set of d distinct points. Then for $k \geq 3d-2$, we have $(I^{\text{cone}})_k = (I_X)_k$. In other words,* $\text{sat.deg}(I^{\text{cone}}) \leq 3d-2$.

Proof. This result comes from chasing the above diagram twisted up by $\mathcal{O}_{\mathbb{P}^2}(k)$ for an integer k.

We have $(I^{\text{cone}})_k = (I_X)_k$ if and only if

$$H^0(\mathcal{O}_{\mathbb{P}^2}(k-d))^3 \to H^0(\mathcal{I}_X(k))$$

is surjective, which occurs precisely when $H^1(\mathcal{F}_1(k)) = 0$. Since the higher cohomology groups of the $\mathcal{H}_i$ vanish by Lemma 3.4.1, we have $H^1(\mathcal{H}_1(k)) = 0$, so $H^1(\mathcal{B}_1(k)) \twoheadrightarrow H^1(\mathcal{F}_1(k))$. Thus, if we know $H^1(\mathcal{B}_1(k)) = 0$, that would imply the desired $H^1(\mathcal{F}_1(k)) = 0$.

Recall that by Proposition 2.2.5, we have the vanishing of $H^2(\mathcal{O}_{\mathbb{P}^2}(k-2d)^3)$ if and only if $k \geq 2d-2$ and $H^1(\mathcal{O}_{\mathbb{P}^2}(k-2d)^3) = 0$ for any k. Then for $k \geq 2d-2$, exactness of the sequence

$$0 = H^1(\mathcal{O}_{\mathbb{P}^2}(k-2d)^3) \to H^1(\mathcal{B}_1(k)) \to H^2(\mathcal{F}_2(k)) \to 0$$

gives an isomorphism $H^1(\mathcal{B}_1(k)) \cong H^2(\mathcal{F}_2(k))$.

Now since

$$H^2(\mathcal{B}_2(k)) \to H^2(\mathcal{F}_2(k)) \to H^2(\mathcal{H}_2(k)) = 0,$$

we know that the map $H^2(\mathcal{B}_2(k)) \to H^2(\mathcal{F}_2(k))$ is surjective, so vanishing of $H^2(\mathcal{B}_2(k))$ would imply the vanishing of $H^2(\mathcal{F}_2(k))$. We also have that $\mathcal{O}_{\mathbb{P}^2}(k-3d) \cong \mathcal{B}_2(k)$, so $H^2(\mathcal{B}_2(k)) = 0$ if and only if $H^2(\mathcal{O}_{\mathbb{P}^2}(k-3d)) = 0$, which occurs when $k \geq 3d-2$. This yields the stated bound on k. □

In a similar way, applying Theorem [6] of Ein, Há, and Lazarsfeld, we glean the following generalization.

Proposition 3.4.4. *Let $X \subseteq \mathbb{P}^r$ be a smooth variety of codimension at least 2. Then*

$$\text{sat.deg}(I^{\text{cone}}) \leq (r+1)d - r.$$

Chapter 4

Main Results for Finite Sets in the Plane

This chapter contains the main results of this dissertation. In Section 4.1, we prove that the saturation degree of the cone ideal of a collinear set of d points in $\mathbb{P}^2$ is $2d-2$. In Section 4.2, we study the flat degeneration of an arbitrary set of points in $\mathbb{P}^2$ to a collinear one and show that the result for collinear points will yield an upper bound on the saturation degree of the cone ideal for non-collinear sets of points in $\mathbb{P}^2$.

4.1 Collinear Points

We start by considering the most specialized configuration of points in the plane: collinear points. In this section, let $Z = \{p_1, \ldots, p_d\} \subseteq \mathbb{P}^2$ be a set of $d > 1$ distinct points contained in a line L.

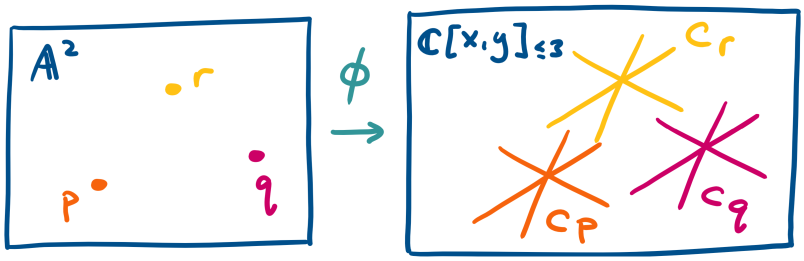
Remark 4.1.1 (Fixing an affine patch)**.** We can choose coordinates x, y, z on $\mathbb{P}^2$ so that the line L containing Z is defined by $z = 0$. Then in the affine patch $\{z \neq 0\} = \mathbb{A}^2 \subseteq \mathbb{P}^2$, the points $p_i \in Z$ all lie in the line at infinity. For any center point $p \in \mathbb{P}^2$ not contained in L, the dehomogenization of the cone polynomial centered at p is $c_p(x : y : 1) \in \mathbb{C}[x, y]_d$. Call $c_p(x : y : 1)$ the **affine cone polynomial** centered at p. The affine curve cut out by $c_p(x : y : 1)$ is the union of d lines meeting at p whose slopes correspond to the d points at infinity.

Lemma 4.1.2. *The set of all affine cone polynomials lies in the ideal spanned by any single affine cone polynomial a and the partials of a of all orders.*

Proof. Consider the map

$$\begin{aligned} \phi : \mathbb{A}^2 &\to \mathbb{C}[x,y]_{\leq d} \\ p &\mapsto c_p(x:y:1) \end{aligned}$$

that sends a point in the affine patch to the affine cone polynomial centered at the point.



Note that the image curves will all be translates of the curve cut out by

$$a(x,y) := \phi(0,0),$$

the affine cone centered at the origin, which is a homogeneous degree d polynomial since it is a product of d linear forms through the origin.

Let s, t be affine coordinates on this patch, so that via Taylor expanding, ϕ can be written as

$$\begin{aligned} \phi : \mathbb{A}^2 &\to \mathbb{C}[x,y]_{\leq d} \\ (s,t) &\mapsto a(x+s, y+t) \\ &= a(x,y) + s\frac{\partial a}{\partial x} + t\frac{\partial a}{\partial y} + s^2\frac{\partial^2 a}{\partial x^2} + 2st\frac{\partial^2 a}{\partial x \partial y} + \cdots + t^d\frac{\partial^d a}{\partial y^d} \\ &= a(x,y) + \sum_{i=1}^{d} \frac{1}{i!}\sum_{j=0}^{i}\binom{i}{j} s^{i-j}t^j \frac{\partial^i a}{\partial x^{i-j} y^j}(x,y). \end{aligned}$$

The image of ϕ contains all of the affine cone polynomials centered at a point $p \notin L$. By the above expresssion, any such cone polynomial can be written as a linear combination combination of partials of $a(x, y)$ with coefficients given by monomials in s, t. □

In order to study the span of the affine cone polynomials, we will use the following Lemma:

Lemma 4.1.3. *Let V be a complex vector space and consider a map ϕ of the following form:*

$$\begin{aligned} \mathbb{C}^2 &\xrightarrow{\phi} V \\ (s,t) &\mapsto v_{00} + sv_{10} + tv_{01} + s^2v_{20} + stv_{11} + t^2v_{02} + \cdots + t^m v_{0m} \\ &= \sum_{i,j=0}^{m} s^i t^j v_{ij}, \end{aligned}$$

where the $v_{ij} \in V$. Then

$$\operatorname{span}(\phi(\mathbb{C}^2)) = \operatorname{span}\{v_{ij}\}.$$

Proof. For each $s, t \in \mathbb{C}$, $\phi(s,t)$ is in the span of the v_{ij}, so $\operatorname{span}(\phi(\mathbb{C}^2)) \subseteq \operatorname{span}\{v_{ij}\}$.

Suppose to the contrary that there is a strict inclusion $\operatorname{span}(\phi(\mathbb{C}^2)) \subsetneq \operatorname{span}\{v_{ij}\}$, which means that there exists some $a, b \in \{1, \ldots, m\}$ so $v_{ab} \notin \operatorname{span}(\phi(\mathbb{C}^2))$. Pick a basis $\{u_1 = v_{ab}, \ldots, u_l\}$ for $\operatorname{span}(v_{ij})$ so that

$$\operatorname{span}(\phi(\mathbb{C}^2)) \subseteq \operatorname{span}\{u_2, \ldots, u_l\} \subsetneq \operatorname{span}\{v_{ij}\}.$$

Using this basis, let v_{ij}^1 be the first coordinate of the vector v_{ij}. Then

$$\phi(s,t) = \begin{bmatrix} v_{00}^1 \\ \vdots \end{bmatrix} + s \begin{bmatrix} v_{10}^1 \\ \vdots \end{bmatrix} + t \begin{bmatrix} v_{01}^1 \\ \vdots \end{bmatrix} + \cdots + t^m \begin{bmatrix} v_{0m}^1 \\ \vdots \end{bmatrix} \in \begin{bmatrix} 0 \\ * \\ \vdots \\ * \end{bmatrix},$$

where the * are arbitrary values. This means that

$$v^1_{00} + v^1_{10}s + v^1_{01}t + v^1_{20}s^2 + \cdots + v^1_{0m}t^m \equiv 0$$

is the zero polynomial. A polynomial is uniformly 0 if and only if all its coefficients are 0, so this means $v^1_{ij} = 0$ for all i, j, and in particular, $v^1_{ab} = 0$. However, by our choice of basis, $v^1_{ab} = 1$, which is a contradiction. □

Considering Lemmas 4.1.2 and 4.1.3 together in the context of the map ϕ, which sends a point to the affine cone polynomial, and viewing the partial derivatives of $a(x, y)$ as vectors in $\mathbb{C}[x, y]_{\leq d}$, we obtain the following result.

Corollary 4.1.4. *With ϕ as described in the proof of Lemma 4.1.2,*

$$\operatorname{span}(\phi(\mathbb{A}^2)) = \operatorname{span}\left\{a, \frac{\partial a}{\partial x}, \frac{\partial a}{\partial y}, \ldots, \frac{\partial^d a}{\partial y^d}\right\}.$$

In other words, affine cone polynomials are spanned by $a(x, y)$ and all of its partial derivatives.

Now, we will rehomogenize the affine cone polynomials so that we can continue to work in projective space. Each $\phi(s, t) = c_p(s : t : 1)$ becomes c_p, so $\operatorname{span}(\phi(\mathbb{A}^2))$, which is the $\mathbb{C}$-linear combinations of affine cone polynomials, rehomogenizes to be the degree d part of I^{cone}. As observed earlier, a is homogeneous and of degree d. An order i partial derivative of a is degree $d - i$, so the rehomogenization will be the partial derivative times z^i. Overall, we get

$$\left(I^{\text{cone}}\right)_d = \left(a, z\frac{\partial a}{\partial x}, z\frac{\partial a}{\partial y}, \ldots, z^d\frac{\partial^d a}{\partial y^d}\right)_d.$$

Since the cone polynomials have degree d, this means that for any $p \notin \{z = 0\}$,

$$c_p \in \left(a, z\frac{\partial a}{\partial x}, z\frac{\partial a}{\partial y}, \ldots, z^d\frac{\partial^d a}{\partial y^d}\right)_d.$$

Hence, $\{a, z\frac{\partial a}{\partial x}, z\frac{\partial a}{\partial y}, \ldots, z^d\frac{\partial^d a}{\partial y^d}\}$ can be taken as a generating set for I^{cone}:

Corollary 4.1.5.

$$I^{\text{cone}} = \left(a, z\frac{\partial a}{\partial x}, z\frac{\partial a}{\partial y}, \ldots, z^d\frac{\partial^d a}{\partial y^d}\right).$$

The last ingredient of our main proof comes from an identity credited to Euler.

Lemma 4.1.6. *[Euler's Homogeneous Function Identity] Let f be a homogeneous degree k function in n variables. Then*

$$k \cdot f(x_1, \ldots, x_n) = \sum_1^n x_i \frac{\partial f}{\partial x_i}(x_1, \ldots, x_n).$$

We will make use of the following generalization:

Lemma 4.1.7. *[Generalized Euler's Identity] Let f be a homogeneous degree k function in n variables. Let $I = \{i_1, \ldots, i_a\}$ be a collection of indices $i_j \in \{1, \ldots, n\}$ and denote the product $x_I := x_{i_1} \cdot \cdots \cdot x_{i_a}$ and the operator $\frac{\partial^a}{\partial x_I} := \frac{\partial^a}{\partial x_{i_1} \ldots \partial x_{i_a}}$. Then for $1 \leq a \leq k$, we have*

$$\frac{k!}{(k-a)!} f(x_1, \ldots, x_n) = \sum_{|I|=a} x_I \frac{\partial^a f}{\partial x_I}(x_1, \ldots, x_n).$$

Proof. We proceed by induction on a. The base case $a = 1$ follows by Euler's Homogeneous Function Identity.

Let $\bar{x} = x_1, \ldots, x_n$. For the inductive step, assume that the identity holds for $a - 1$, i.e.,

$$\frac{k!}{(k-a+1)!} f(\bar{x}) = \sum_{|I|=a-1} x_I \frac{\partial^{a-1} f}{\partial x_I}(\bar{x}). \tag{4.1}$$

Since f is homogeneous, the $(a-1)$-th partials of f are homogeneous of degree $k-a+1$, so

$$(k-a+1)\frac{\partial^{a-1} f}{\partial x_I}(\bar{x}) = \sum_{j=1}^{n} x_j \frac{\partial f}{\partial x_j}\frac{\partial^{a-1} f}{\partial x_I}(\bar{x}).$$

Substituting this into 4.1, we see that

$$\begin{aligned}\frac{k!}{(k-a+1)!}f(\bar{x}) &= \sum_{|I|=a-1} x_I \left(\frac{1}{k-a+1}\sum_{j=1}^{n} x_j \frac{\partial f}{\partial x_j}\frac{\partial^{a-1} f}{\partial x_I}(\bar{x})\right) \\ &= \frac{1}{k-a+1}\sum_{|J|=a} x_J \frac{\partial^a f}{\partial x_J}(\bar{x}).\end{aligned}$$

The statement of the lemma follows. □

Now, we can prove our main result.

Theorem 4.1.8. *If Z is a set of $d > 1$ collinear points in $\mathbb{P}^2$, then*

$$(I^{\text{cone}})_k = I_k$$

if and only if $k \geq 2d-2$.

Proof. The main idea is to filter the elements of I^{cone} based on their degree of vanishing along the line containing Z, which I will assume without loss of generality is $z = 0$.

Take $1 \leq i \leq d$. We start by showing that the order i partial derivatives of the cone polynomial $a(x, y)$ have no common zeroes in $\mathbb{P}^1$.

Suppose $\frac{\partial a}{\partial x}$ and $\frac{\partial a}{\partial y}$ both vanish at $p \in \mathbb{P}^1$. Then by Lemma 4.1.6, $a(x, y)$ must also vanish at p. Since p must be contained in at least one of the affine patches, then without loss of generality, suppose p is in the affine patch defined by $y = 1$. After restricting to this patch, $a(x, y)$ is a univariate polynomial $a(x)$ and

$$a'(x) = \left.\frac{\partial a}{\partial x}\right|_{y=1}.$$

Then a and a' both vanish at p, which means that a has a double root at p. This would contradict our assumption that a has distinct roots in $\mathbb{P}^1$. Hence, the first partials of $a(x, y)$ must have no common zeroes in $\mathbb{P}^1$. Similarly, using Lemma 4.1.7, it follows the order i partials of $a(x, y)$ have no common zeroes in $\mathbb{P}^1$.

Let $S = \mathbb{C}[x, y]$. Then, Macaulay's Theorem 2.2.1 gives

$$\begin{aligned}
\left(\frac{\partial a}{\partial x}, \frac{\partial a}{\partial y}\right)_k &= S_k \qquad \text{for } k \geq 2(d-1) - 1 = 2d - 3, \\
\left(\frac{\partial^2 a}{\partial x^2}, \frac{\partial^2 a}{\partial x \partial y}, \frac{\partial^2 a}{\partial y^2}\right)_k &= S_k \qquad \text{for } k \geq 2(d-2) - 1 = 2d - 5, \\
&\vdots \\
\left(\frac{\partial^{d-1} a}{\partial x^{d-1}}, \ldots, \frac{\partial^{d-1} a}{\partial y^{d-1}}\right) &= S_k \qquad \text{for } k \geq 2(1) - 1 = 1.
\end{aligned}$$

In general, for each order $1 \leq i \leq d - 1$,

$$\left(\frac{\partial^i a}{\partial x^i}, \ldots, \frac{\partial^i a}{\partial y^i}\right)_k = S_k \text{ for } k \geq 2(d-i) - 1. \tag{4.2}$$

Now, look at the ideal

$$I^{\text{cone}} = \left(a, z\frac{\partial a}{\partial x}, z\frac{\partial a}{\partial y}, \ldots, z^d \frac{\partial^d a}{\partial y^d}\right)$$

in degree k for $k \geq 2d - 2$:

$$\begin{aligned}
\left(I^{\text{cone}}\right)_k &= (a)_k + z \cdot \left(\frac{\partial a}{\partial x}, \frac{\partial a}{\partial y}\right)_{k-1} + \cdots + \left(z^d\right)_k \\
&= (a)_k + \sum_{i=1}^{d-1} \left[z^i \cdot \left(\frac{\partial^i a}{\partial x^i}, \ldots, \frac{\partial^i a}{\partial y^i}\right)_{k-i} \right] + \left(z^d\right)_k.
\end{aligned}$$

Note that for $1 \le i \le d-1$, we have $k - i \ge 2d - i - 2 \ge 2(d-i) - 1$, so by equation (4.2), $\left(\frac{\partial^i a}{\partial x^i}, \ldots, \frac{\partial^i a}{\partial y^i}\right)_{k-i} = S_{k-i}$. Thus, continuing the string of equalities above,

$$\begin{aligned} &= (a)_k + \sum_{i=1}^{d-1} \left[z^i \cdot S_{k-i}\right] + \left(z^d\right)_k \\ &= (a)_k + z \cdot \mathbb{C}[x,y,z]_{k-1}. \end{aligned}$$

From this, we get

$$\begin{array}{ccccc} (I^{\text{cone}})_k & \subseteq & I_k & \subseteq & \mathbb{C}[x,y,z]_k \\ \shortparallel & & & & \shortparallel \\ a \cdot S_{k-d} + z \cdot \mathbb{C}[x,y,z]_{k-1} & & \subseteq & & S_k + z \cdot \mathbb{C}[x,y,z]_{k-1}. \end{array}$$

Next, a codimension count:

$$\begin{aligned} \operatorname{codim}\left(I_k^{\text{cone}} \subseteq \mathbb{C}[x,y,z]_k\right) &= \operatorname{codim}\left(a \cdot S_{k-d} \subseteq S_k\right) \\ &= \operatorname{codim}\left(S_{k-d} \subseteq S_k\right) \\ &= \dim S_k - \dim S_{k-d} \\ &= (k+1) - (k-d+1) \\ &= d. \end{aligned}$$

Since I_k is also codimension d in $\mathbb{C}[x,y,z]_k$, there is equality

$$(I^{\text{cone}})_k = I_k$$

for $d > 1, k \ge 2d - 2$.

Now for the converse. The sharpness of Macaulay's Theorem says

$$\left(\frac{\partial a}{\partial x}, \frac{\partial a}{\partial y}\right)_t \subsetneq S_t \text{ for } t < 2(d-1) - 1 = 2d - 3.$$

Now examine I^{cone} in degree $k < 2d-2$:

$$\begin{aligned}(I^{\text{cone}})_k &= (a)_k + z \cdot \left(\frac{\partial a}{\partial x}, \frac{\partial a}{\partial y}\right)_{k-1} + \cdots + z^d \\ &= (a)_k + z \cdot \left(\frac{\partial a}{\partial x}, \frac{\partial a}{\partial y}\right)_{k-1} + \sum_{i=2}^{d} \left[z^i \cdot \left(\frac{\partial^i a}{\partial x^i}, \ldots, \frac{\partial^i a}{\partial y^i}\right)_{k-1} \right]\end{aligned}$$

Earlier, we checked that $\operatorname{codim}\left(a \cdot S_{k-d} + z \cdot \mathbb{C}[x,y,z]_{k-1} \subseteq \mathbb{C}[x,y,z]_k\right) = d$, so as a strict subset,

$$\operatorname{codim}\left[(I^{\text{cone}})_k \subseteq \mathbb{C}[x,y,z]_k\right] \geq d+1,$$

which means

$$(I^{\text{cone}})_k \neq I_k$$

for $d > 1, k < 2d-2$. □

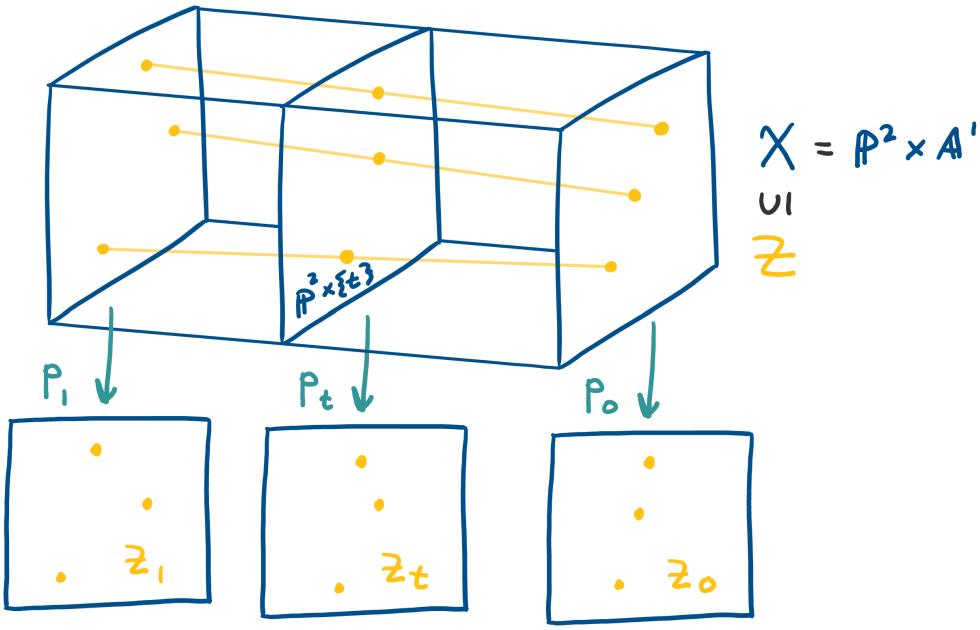
4.2 Implications for Non-colllinear Points

Now that we have established a sharp bound for the saturation degree of the cone ideal for collinear points, we would like to study non-collinear configurations. In this section, we will use the result for collinear points to prove that cones generate the defining ideal in degrees $2d-2$ and higher for any set $Z \subseteq \mathbb{P}^2$ of d reduced points, although the bound is no longer as good as possible in the non-collinear case.

Consider the ambient space $X = \mathbb{P}^2 \times \mathbb{A}^1$, where $\mathbb{A}^1$ can be thought of as a 1-dimensional time parameter. Inside of X, consider two slices $\mathbb{P}^2 \times \{0\}$ and $\mathbb{P}^2 \times \{1\}$. Each of these is a projective plane, in which we can put collections of d distinct points, $Z_0 \in \mathbb{P}^2 \times \{0\}$ and $Z_1 \in \mathbb{P}^2 \times \{1\}$. In particular, we are interested in the case when Z_0 is a set of collinear points and Z_1 is a set of arbitrary reduced points so that we can study the degeneration of Z_1 to Z_0. We will start by describing the construction of a subset

$$\mathcal{Z} \subseteq X$$

where the images of the second projection Z_t form a flat degeneration so that we can use the bound of Theorem 4.1.8 to obtain a rough bound for arbitrary sets of points.



Lemma 4.2.1 (Flat family)**.** *For any arbitrary set Z of d distinct points, there exists a flat family $\{Z_t\}_{t\in\mathbb{A}^1}$ over $\mathbb{A}^1$ where $Z_1 = Z$ and Z_0 is a set of d distinct collinear points. In particular, this can be done so that the ideal sheaves of the Z_t are all isomorphic to one another for $t \neq 0$. Call $\mathcal{Z} \subset X$ the set with fibers Z_t.*

Proof. Let Z_1 be an arbitrary set of d distinct points in $\mathbb{P}^2$. If Y is the union of all lines through each pair of points of Z_1, choose a point P not contained in Y. Choose coordinates on $\mathbb{P}^2$ so that $P = (0 : 0 : 1)$. For each non-zero $t \in \mathbb{C}$, we have an automorphism of $\mathbb{P}^2$ given by

$$\sigma_t : (x : y : z) \mapsto (x : y : tz),$$

which is a projection from the point P onto the line defined by $z = 0$ in the limit as $t \to 0$. By our choice of center P, each point of Z_t approaches a distinct point on the line. For each $t \neq 0$, let $Z_t = \sigma_t(Z_1)$. Note that the all of the ideal sheaves $\mathcal{I}_{Z_t}$ are isomorphic to one another by construction, so in particular, the Z_t form a flat family parameterized by $\mathbb{A}_1 \setminus \{0\}$. Then this extends to a flat family over all of $\mathbb{A}^1$ with the flat limit Z_0 a set of d distinct, collinear points. □

We would like to discuss how cone polynomials over this flat family behave. For simplicity, our next goal is to establish the existence of fixed centers for the cone polynomials, which will generate Z_t as t varies. We will use the following Lemmas:

Lemma 4.2.2. *If a morphism of vector bundles $V \to W$ over $\mathbb{A}^1$ is surjective at $1 \in \mathbb{A}^1$, then it is surjective in a Zariski neighborhood of 1.*

Proof. Suppose $\nu : V \to W$ is a morphism of vector bundles. Then in a neighborhood U around $1 \in \mathbb{A}^1$, $\nu_U : V_U \to W_U$ is given by a matrix of regular functions $[a_{ij}(t)]$. Suppose the map over 1 given by $[a_{ij}(1)]$ is surjective, so the matrix is full rank. Since dropping rank is defined by the vanishing of minors of $[a_{ij}(t)]$, which defines a closed set, the matrix will be full rank in a neighborhood of 1. □

Lemma 4.2.3. *For the flat family $\{Z_t\}$ described in Lemma 4.2.1, there exists a finite collection of fixed points $P_1, \ldots, P_N$ so that the cone polynomials $\{c_{P_i}(Z_t)\}_{i=1}^N$ generate $(I_{Z_t}^{\text{cone}})_d$ for all but finitely many $t \in \mathbb{A}^1$.*

Proof. Choose centers $P_1, \ldots, P_N$ such that $\{c_{P_i}(Z_0)\}_{i=1}^N$ generate $(I_{Z_0}^{\text{cone}})_d$ and $\{c_{P_i}(Z_1)\}_{i=1}^N$ generate $(I_{Z_1}^{\text{cone}})_d$. By the automorphism σ_t of $\mathbb{P}^2$ for $t \neq 0$, all of the subspaces $(I_{Z_t}^{\text{cone}})_d \subseteq S_d$ have the same dimension for $t \neq 0$. Then the $(I_{Z_t}^{\text{cone}})_d$ can be put together to form a vector bundle

$$\mathcal{V} \subseteq S_d \times \mathbb{A}^1$$

over $\mathbb{A}^1$. Consider the map of vector bundles over $\mathbb{A}^1$

$$\mathcal{O}_{\mathbb{A}^1}^N \to \mathcal{V}$$

given by the cone polynomials $\{c_{P_i}(Z_t)\}_{i=1}^N$. By our choice of the P_i, this map is surjective at $t = 1$ and at $t = 0$. Then by Lemma 4.2.2, the map is surjective in a Zariski neighborhood of $1 \in \mathbb{A}^1$, which means that it is surjective for all but finitely many $t \in \mathbb{A}^1$. This is equivalent to the cone polynomials $\{c_{P_i}(Z_t)\}_{i=1}^N$ generating $(I_{Z_t}^{\text{cone}})_d$ for those t. □

Lemma 4.2.4 (Fixed centers). *For the flat family $\{Z_t\}$ described in Lemma 4.2.1, there exists a finite set of fixed points $A = P_1, \ldots, P_M$ for which the cone ideal of Z_t in degree d is generated by the cone polynomials centered at the points $P_1, \ldots, P_M$ for every $t \in \mathbb{A}^1$.*

Proof. Consider the points $P_1, \ldots, P_N$ described in Lemma 4.2.3. Then for the finite number of t_j for which $\{c_{P_i}(Z_t)\}_{i=1}^N$ does not generate $(I_{Z_t}^{\text{cone}})_d$, there are points $P_k^{t_j}$ so that the cones over those points generate $(I_{Z_t}^{\text{cone}})_d$. Let A be the union of the P_i and $P_k^{t_j}$. □

Now, we can prove the main result of this section, which is a bound on the saturation degree of I^{cone} for arbitrary finite sets in $\mathbb{P}^2$.

Theorem 4.2.5. *If Z is a set of d distinct points in $\mathbb{P}^2$ and if $k \geq 2d - 2$, then $(I^{\text{cone}})_k = I_k$.*

Proof. Let $Z_0 = Z$ and consider the flat family Z_t as described previously. By Lemma 4.2.4, there exists some N so that we get a short exact sequence of sheaves over $\mathbb{P}^2 \times \mathbb{A}^1$:

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{O}_{\mathbb{P}^2 \times \mathbb{A}^1}^N(-d) \longrightarrow \mathcal{I}_{\mathcal{Z}} \longrightarrow 0,$$

where $\mathcal{M}$ is the kernel sheaf of the morphism given by the cone polynomials specified in Lemma 4.2.4 and $\mathcal{O}_{\mathbb{P}^2 \times \mathbb{A}^1}^N(-d) := \text{pr}_1^* \mathcal{O}_{\mathbb{P}^2}(-d)$. Since $\mathcal{I}_{\mathcal{Z}}$ is flat over $\mathbb{A}^1$ by Lemma 4.2.1 and $\mathcal{O}_{\mathbb{P}^2}^N(-d)$ is flat over $\mathbb{A}^1$ as it is constant, this implies that $\mathcal{M}$ and its twists are flat over $\mathbb{A}^1$, too, by Proposition 2.4.2.

By the Semicontinuity Theorem, for $k \in \mathbb{Z}$, $h^1(\mathcal{M}_t(k))$ is upper semi-continuous in $t \in \mathbb{A}^1$. Theorem 4.1.8 tells us that cone polynomials generate in degrees at least $2d - 2$, which means that the map

$$H^0(\mathcal{O}_{\mathbb{P}^2}^N(k - d)) \longrightarrow H^0(\mathcal{I}_{Z_0}(k))$$

is surjective for $k \geq 2d-2$. This is equivalent to the vanishing of $H^1(\mathcal{M}_0(k))$ for $k \geq 2d-2$. Upper semicontinuity of $h^1(\mathcal{M}_t(k))$ implies that for $k \geq 2d-2$, the value of h^1 can only jump up. The smallest possible value for the cohomology dimension is 0, which we know is attained at $t = 0$. Thus, there is an open neighborhood around $0 \in \mathbb{A}^1$ on which $h^1(\mathcal{M}_t(k)) = 0$. Then there exists some $t' \neq 0$ such that $H^1(\mathcal{M}_{t'}(k)) = 0$. Since this cohomology group measures the codimension of

$$W = \big(c_{P_1}(Z_{t'}), \ldots, c_{P_N}(Z_{t'})\big)_d \subseteq (I_{Z_{t'}})_k,$$

we know that these cones generate all of $I_{Z_{t'}}$ in degrees $k \geq 2d-2$. Since $\sigma_{t'}$ is an automorphism of $\mathbb{P}^2$, we also have that W' is isomorphic to W, the degree k piece of the ideal generated by the cones

$$c_{\sigma_{t'}^{-1}(P_i)}(\sigma_{t'}^{-1}(Z_{t'})) = c_{\sigma_{t'}^{-1}(P_i)}(Z_1).$$

Then $W = (I_{Z_1})_k$, i.e., I_{Z_1} is generated by cones in degrees $k \geq 2d-2$, which is the desired result. $\square$

This theorem shows that the saturation degree of the cone ideal for arbitrary sets of points in the plane can not exceed $2d-2$. Note that this is not what we expect to be the actual saturation degree in non-collinear cases, which we will discuss in Section 5.1.

Chapter 5

Experimental Data and Further Directions

Interest in the study of cone polynomials began with the experimental observations of Bo Ilic, summarized in Conjecture 3.2.1, that for generic points in $\mathbb{P}^2$, the degree in which cones generate the defining ideal of the points depended on the parity of the number of points. In this chapter, I will describe the results of my computational experiments on cone polynomials. All of the data described was obtained using `Macaulay2` [10]. For trials involving random points in projective space, I made use of the Points package by [23], which includes functions that produce random points in an ideal. My computations are all done over the finite field of order 32749, the largest prime order stored in `Macaulay2`, in the expectation that the experiments would capture what is occurring over $\mathbb{C}$.

Given a projective variety X, we compute the cone ideal I^{cone} by producing sufficiently many cone polynomials over X and looking at the ideal that they generate. We compare the Hilbert function of I^{cone} to the expected size of I_X in a given degree, and search for the first instance in which they agree to find the saturation degree of the cone ideal.

Remark 5.0.1 (Benefits and limitations of computational data)**.** Being able to compute saturation degrees of the cone ideal for a specific variety X is useful as a starting point for finding patterns and forming conjectures. As we will see in this chapter, the data reveal a relationship between the degree d of X and the saturation degree of I^{cone}. For example, the consistent pattern that emerged for collinear points in the plane is what sparked our interest in proving this case, which resulted in Theorem 4.1.8.

A potential issue that could occur with working out concrete examples is that a pattern that occurs for a particular collinear set may not work for every collinear set. To mitigate this, data reported in this chapter were run several times with appropriately randomized attributes and checked for consistency.

Another limitation is the runtime of the programs. The tables in this chapter end at $d = 10$, which was the point at which many of the computations started to slow down. This also impacts the size of the dimension that we can work in, so we stop at curves in $\mathbb{P}^3$.

5.1 Cones over Points in the Plane

Remark 5.1.1 (Cone polynomials in `Macaulay2`). Suppose

$$X = \{P_1, \dots, P_d\} \subseteq \mathbb{P}^2$$

is a set of d points, and we want to find explicit generators of I^{cone} using `Macaualay2`. To work in $\mathbb{P}^2$, set the base ring as $k[x, y, z]$, where $k = \mathbb{Z}/32749\mathbb{Z}$. Using the Points Package, we can generate a matrix whose columns represent the homogeneous coordinates of random center points for the cone polynomials. We choose $\binom{d+2}{2}$ random center points C_j, since cone polynomials live inside $k[x, y, z]_d$ which has dimension $\binom{d+2}{2}$. Then for $1 \leq i \leq d$ and $1 \leq j \leq \binom{d+2}{2}$, the determinant

$$l_{i,j} := \begin{vmatrix} x & | & | \\ y & P_i & C_j \\ z & | & | \end{vmatrix}$$

gives the linear function that vanishes on the points P_i and C_j so that the cone polynomial centered at C_j is

$$c_{C_j}(X) = \prod_{i=1}^{d} l_{i,j}.$$

Then we assume that I^{cone} is the ideal generated by these cones.

Example 5.1.2 (Betti tables and Hilbert functions)**.** As described in the Remark 5.1.1, we can produce generators for the cone ideal of a set of points $X \subseteq \mathbb{P}^2$. When X is a set of 5 randomly generated points, we can print the Betti table of I^{cone} to get the following `Macaulay2` output.

```
        0  1  2 3
total:  1 14 19 6
    0:  1  .  . .
    1:  .  .  . .
    2:  .  .  . .
    3:  .  .  . .
    4:  . 14 19 4
    5:  .  .  . 2
```

In column 1, the 14 in row 4 indicates that there are 14 forms of degree 5 needed to generate I^{cone}. In other words, $\dim(I_5^{\text{cone}}) = 14$. The expected dimension of the ideal of 5 points in degree 5 is $\binom{5+2}{2} - 5 = 16$, so in this case, I^{cone} has codimension 1 in I_5. We can also compute the Hilbert function of I^{cone}, which will give the codimension of I_k^{cone} in $S[x, y, z]_k$. When the value of the Hilbert function in degree k achieves the number of points of X, this means that $I_k^{\text{cone}} = I_k$ for that k. In our case, this first occurs when $k = 6$.

Note that the bottom right entry in the Betti table above indicates that there are two relations of degree 2 among the relations on the generators of I^{cone}. Compare this to the Betti table when X is instead a set of 7 randomly generated points, shown below.

```
       0  1  2  3
total: 1 28 46 19
    0: 1  .  .  .
    1: .  .  .  .
    2: .  .  .  .
    3: .  .  .  .
    4: .  .  .  .
    5: .  .  .  .
    6: . 28 46 18
    7: .  .  .  1
```

In this case, there is one relation of degree 2 among relations instead. In looking at the Betti tables for I^{cone} when X is a set of $d \geq 3$ random points, we observe that the bottom right entry is

$$1 \text{ if } d = 3 \mod 4 \quad \text{and} \quad 2 \text{ if } d = 1 \mod 4$$

for values of d up to 15, which match the number of linear relations among cone polynomials found in Theorems 3.2.2 and 3.2.3 of Fu and Nie.

Similarly, we can work with collinear points by letting X be a set of 5 randomly generated points that lie in the line defined by $z = 0$. Then we get the following Betti table.

```
       0  1  2 3
total: 1 12 17 6
    0: 1  .  . .
    1: .  .  . .
    2: .  .  . .
    3: .  .  . .
    4: . 12 15 4
    5: .  .  1 1
    6: .  .  . .
    7: .  .  1 1
```

Here, I_5^{cone} is codimension 4 in I_5. In the second column, we can read off the syzygies: 15 linear syzygies, 1 degree 2 syzygy, and 1 degree 4 syzygy. By examining the Hilbert function, we get that cones generate the defining ideal of X in degree 8.

Trial 5.1.3 (Generic points)**.** Data on the saturation degree of I^{cone} when X is a set of d randomly generated points in $\mathbb{P}^2$.

FIGURE 5.1: Schematic diagram of X

d	sat. deg
1	1
2	2
3	4
4	4
5	6
6	6
7	8
8	8
9	10
10	10

Based on this data, we can reinterpret Bo Ilic's Conjecture 3.2.1 in terms of saturation degree.

Conjecture 5.1.4. *Let X be a set of $d \geq 2$ generic points in $\mathbb{P}^2$. Then*

$$\text{sat.}\deg(I^{\text{cone}}) = \begin{cases} d & \text{if } d \text{ is even,} \\ d+1 & \text{if } d \text{ is odd.} \end{cases}$$

Trial 5.1.5 (Points split between two lines)**.** For integers $d \geq 2$, let

$$X = \{(0:1:1), \ldots, (0:\lfloor d/2 \rfloor:1), (1:0:1), \ldots, (\lceil d/2 \rceil, 0:1)\},$$

a set of points in $\mathbb{P}^2$ where $\lfloor d/2 \rfloor$ of the points are contained in the line $\{x = 0\}$ and the rest of the points are contained in the line $\{y = 0\}$. Then the saturation degree of X matches the saturation degree of the corresponding number of generic points described in Trial 5.1.3.

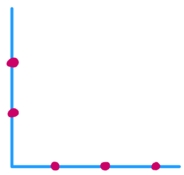


FIGURE 5.2: Schematic diagram of X

These data suggest the following strengthenging of Conjecture 5.1.4:

Conjecture 5.1.6. *Let X be a set of $d \geq 2$ in $\mathbb{P}^2$ such that $\lfloor d/2 \rfloor$ of the points are contained in one line and the rest of the points are contained in a distinct line. Then*

$$\operatorname{sat.deg}(I^{\text{cone}}) = \begin{cases} d & \text{if } d \text{ is even,} \\ d+1 & \text{if } d \text{ is odd.} \end{cases}$$

Trial 5.1.7 (Moving points from one line into two)**.** In this trial, we will look at sets of d points that lie in the two lines in $\mathbb{P}^2$ defined by $x = 0$ and $y = 0$. For $i = 0, 1, 2, 3$, let X_i be a set of d points so that i of the points are randomly generated points in the line $x = 0$ and the rest of them are randomly generated points in the line $y = 0$.

d	sat.deg($I^{\text{cone}}_{X_0}$)	sat.deg($I^{\text{cone}}_{X_1}$)	sat.deg($I^{\text{cone}}_{X_2}$)	sat.deg($I^{\text{cone}}_{X_3}$)
1	1	N/A	N/A	N/A
2	2	N/A	N/A	N/A
3	4	4	N/A	N/A
4	6	5	4	N/A
5	8	6	6	6
6	10	8	7	6
7	12	10	8	8
8	14	12	10	9
9	16	14	12	10
10	18	16	14	12

Recall that Theorem 4.1.8 establishes that for $d \geq 2$, the saturation degree for $I_{X_0}^{\text{cone}}$ is $2d-2$. The data above suggests that for $d \geq 5$, the saturation degree of $I_{X_1}^{\text{cone}}$ is $2d-4$, for $d \geq 7$, the saturation degree of $I_{X_2}^{\text{cone}}$ is $2d-6$, and for $d \geq 9$, the saturation degree of $I_{X_3}^{\text{cone}}$ is $2d-8$. Then, as points are moved one at a time from one line into two, the saturation degree should drop by two until the points are split as evenly as possible between the two lines, which would give a saturation degree of $2d - 2\lfloor d/2 \rfloor = d$ or $d+1$.

Remark 5.1.8 (Semicontinuity and saturation degree)**.** As explained in Section 4.2, the saturation degree of the cone ideal when X is a finite set of d points in $\mathbb{P}^2$ satisfies a semicontinuity property, which means that it may jump up for special configurations of X. To summarize the observations of this Section, we expect the following pattern in the saturation degree of I^{cone}:

$$\text{sat.deg}(I^{\text{cone}}) = \begin{cases} d, d+1 & \text{if } X \text{ is a set of generic points,} \\ d, d+1 & \text{if } X \text{ is split evenly between two lines,} \\ & \vdots \\ 2d-6 & \text{if } X \text{ is collinear except two points,} \\ 2d-4 & \text{if } X \text{ is collinear except one point,} \\ 2d-2 & \text{if } X \text{ is collinear.} \end{cases}$$

5.2 Cones over Points in Higher Projective Spaces

To find cones over points in projective spaces of dimension $n \geq 3$, the centers of the cones are linear spaces Λ of dimension $n-2$.

Trial 5.2.1 (Generic points)**.** Data on the saturation degree for d randomly generated points in $\mathbb{P}^3$ and $\mathbb{P}^4$.

d	sat. deg$(X \subseteq \mathbb{P}^3)$	sat. deg$(X \subseteq \mathbb{P}^4)$
1	1	1
2	2	2
3	4	4
4	4	4
5	6	6
6	6	6
7	8	8
8	8	8
9	10	10
10	10	10

The saturation degree of generic sets of d points in $\mathbb{P}^3$ and $\mathbb{P}^4$ appear to follow the same parity pattern that occurs for $\mathbb{P}^2$: d when d is even and $d+1$ when d is odd.

Note that the saturation degree being greater than d in the odd case will always occur in $\mathbb{P}^{n \geq 2}$ by Theorem 3.2.2 of Fu and Nie.

Trial 5.2.2 (Collinear points in $\mathbb{P}^3$)**.** Data on the saturation degree of d randomly generated points contained in a line in $\mathbb{P}^3$.

d	sat. deg
1	1
2	2
3	4
4	6
5	8
6	10
7	12
8	14
9	16
10	18

We get a saturation degree of $2d-2$, which is the same behavior as collinear points in $\mathbb{P}^2$.

5.3 Cones over Curves

Trial 5.3.1 (Cones over Curves in $\mathbb{P}^3$)**.** I consider two types of curves, although both end up having the same behavior. Let X_d be the union of d randomly generated lines. Let Y_d be the paramerterized curve defined as the image of the map $\mathbb{P}^1 \to \mathbb{P}^3$ which takes

$$(s,t) \mapsto (f_0(s,t), \ldots, f_3(s,t)),$$

where $f_0, \ldots, f_3$ are random linear combinations of the degree d monomials $s^{d-i}t^i$. In both cases, we take $\binom{d+3}{3}$ random center points for the cone polynomials.

d	sat. deg(X_d)	sat. deg(Y_d)
1	1	1
2	2	2
3	3	3
4	4	4
5	5	5
6	6	6
7	7	7
8	8	8
9	9	9
10	10	10

This set of data suggests that the saturation degree for degree d curves in $\mathbb{P}^3$ is d. Notably, the variation depending on the parity of d which was present in the case when X is 0-dimensional does not appear.

5.4 Further Directions

A focus of this dissertation is to study the saturation degrees of ideals generated by cone polynomials. One could naturally ask similar questions for ideals generated by other classes of polynomials.

Example 5.4.1 (Saturation degree of quadrics cutting out a Veronese Curve)**.** Recall that the rational normal curve of degree r is given as the image of the Veronese map

$$\nu : \mathbb{P}^1 \to \mathbb{P}^r$$
$$[s:t] \mapsto [s^r : s^{r-1}t, \dots, st^{r-1}, t^r]$$

and is cut out scheme-theoretically by the 2×2 minors of the $2 \times r$ matrix

$$M = \begin{bmatrix} x_0 & x_1 & \dots & x_{r-1} \\ x_1 & x_2 & \dots & x_r \end{bmatrix}.$$

Let $q_{i,j}$ be the determinant of the matrix consisting of columns i and j of M. There are $\binom{r}{2}$ many of these quadrics $q_{i,j}$. By 2.1.12, taking $r+1$ general linear combinations of these are sufficient to cut out the rational normal curve. For $k = 0, \dots, r$, we set

$$F_k = \sum_{i<j} c_{k,i,j} q_{i,j}$$

where the coefficients $c_{k,i,j}$ are randomly generated values so that the ideal $J := (F_0, \dots, F_r)$ saturates to the defining ideal I of the rational normal curve in $\mathbb{P}^r$. Then, we can compute the saturation degree of I using `Macaulay2`, which yields the following values.

Veronese degree	Saturation degree of J
2	2
3	2
4	3
5	4
6	5
7	6

Since J is generated by elements of degree 2, the smallest possible saturation degree is 2. The data above suggest that for $r \geq 3$, the saturation degree of J is $r - 1$.

Remark 5.4.2 (More questions about cone polynomials)**.** This chapter provides some insight into the behavior of cone polynomials in certain low dimensions, and it would be interesting to study if any patterns analogous to the one confirmed for collinear points in the plane described in Theorem 4.1.8 occur for higher dimensional varieties. As discussed in Section 4.2, the saturation degree of cone ideals follows an upper semicontinuity property. Remark 5.1.8 suggests that the saturation degree of I^{cone} for points in $\mathbb{P}^2$ jumps up based on how close the points are to being collinear.

Here are some open questions related to the work presented in this dissertation.

- If $X \subseteq \mathbb{P}^2$ is a set of d points where all but one of the points are contained in a line, can we prove that the saturation degree is $2d - 4$?

  I expect that if one can figure out a proof of this statement, there is potential for a way to confirm the pattern observed in Remark 5.1.8 by moving points off of the line one by one. This would also prove Conjecture 5.1.4.

- For rational curves of degree d in $\mathbb{P}^3$ (or higher dimensional analogues), do cones generate I_X in degree d?

  This is suggested by the data in Trial 5.3.1.

- For curves $X \subseteq \mathbb{P}^3$ (or higher dimensional analogues), does the saturation degree of I^{cone} jump up when X is contained in a hyperplane? Are there other special geometric conditions on X that result in different saturation degrees?

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