Topics in Quantitative Rectifiability:

Traveling Salesmen, Lipschitz Decompositions, Densities, and Big Pieces

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Abstract of the Dissertation

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We present and prove assorted results in quantitative rectifiability. First, we study the quantitative rectifiability of Jordan arcs in Hilbert spaces, proving a version of the traveling salesman beta number estimate for length minus chord length analogous to an estimate recently attained by Bishop in Euclidean spaces. Second, we prove the existence of Lipschitz decompositions for domains with quantitatively flat boundaries. That is, we show any such domain has an "almost" decomposition into nice Lipschitz domains with control on the total surface area of the decomposition domains in terms of the original domain boundary area. Third, we study the regularity of Hausdorff measure on uniformly rectifiable metric spaces. We show that any such space satisfies the weak constant density condition of David and Semmes. Fourth, we study the iteration of the big pieces operator in Ahlfors regular metric spaces. We prove that iteration stabilizes after two iterations as a result of a more general extension theorem.

Dedication Page

To Mathematics,

An excuse to daydream pretty pictures,

To look beyond myself,

To feel static.

"'What's this? Am I falling? My legs are giving way under me,' he thought, and fell on his back. He opened his eyes, hoping to see how the struggle of the French soldiers with the artilleryman was ending, and eager to know whether the red-haired artilleryman was killed or not, whether the cannons had been taken or saved. But he saw nothing of all that. Above him there was nothing but the sky–the lofty sky, not clear, but still immeasurably lofty, with gray clouds creeping quietly over it." - Lev Tolstoy, War and Peace

"On the last day, she went to see the Watts Towers ... She went around touching things, rubbing her palms over the bright surfaces. She loved the patterns made by jute doormats pressed in cement. She loved the crushed green glass and the brown bottle bottoms that knobbed an archway. And one of the taller towers with its tracery of whirling atoms. And the south wall candied with pebbles and mussel shells ... She felt a static, a depth of spirit, a delectation that took the form of near helplessness. Like laughing helplessly as a girl, collapsing against the shoulder of your best friend. She was weak with sensation, weak with seeing and feeling." - Don DeLillo, Underworld

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Chapter 1

Introduction

Rectifiable sets are central objects of interest in geometric measure theory. If X is a metric space, we say that $E \subseteq X$ is n-rectifiable if there exist countably many subsets $A_i \subseteq \mathbb{R}^n$ and Lipschitz maps $f_i : A_i \to X$ such that $\mathcal{H}^n(E \setminus \bigcup_i f_i(A_i)) = 0$. In other words, all of the $\mathcal{H}^n$ measure of E is captured by Lipschitz image of subsets of $\mathbb{R}^n$. In Euclidean space, rectifiable sets are natural generalizations of smooth n-dimensional submanifolds. Whereas manifolds are locally smoothly parameterized by Euclidean space via charts, rectifiable sets only admit this weak form of measure theoretic covering by Lipschitz maps. Just as differential geometry studies the topological and geometric properties of manifolds, a large part of geometric measure theory seeks to understand what sort of geometric structure remains when we generalize to the class of rectifiable sets.

Some of the most satisfying geometric statements about rectifiable sets concern what happens in the limit as we zoom in on typical points: rectifiable sets have *approximate tangent planes* almost everywhere.

Definition 1.0.1 (Approximate tangent n-planes)**.** For a given n-plane $P \subseteq \mathbb{R}^d, x \in \mathbb{R}^d$, and $s \in (0,1)$, define the cone of aperture s centered at a around P by

$$C(a, P, s) = \{x \in \mathbb{R}^d : \operatorname{dist}(x - a, P) < s|x - a|\}.$$

We say that P is an approximate tangent n-plane for E at a if

$$\limsup_{r \to 0} \frac{\mathcal{H}^n(E \cap B(a, r))}{(2r)^n} > 0, \tag{1.1}$$

and for every $s \in (0,1)$

$$\lim_{r \to 0} \frac{\mathcal{H}^n(E \cap B(a, r) \setminus C(a, P, s))}{(2r)^n} = 0. \tag{1.2}$$

Equation(1.1) says that small balls around a contain at least some mass, while (1.2) says that all of the mass near a concentrates near P in small balls around a. The following theorem characterizes n-rectifiable subsets of $\mathbb{R}^d$ in terms of the existence of approximate tangent n-planes.

Theorem 1.0.1 ([Mat95] Theorem 1.59). *Let E be an $\mathcal{H}^n$ measurable subset of $\mathbb{R}^d$ with $\mathcal{H}^n(E) < \infty$. The following are equivalent:*

(i) E is n-rectifiable,

(ii) For $\mathcal{H}^n$ almost every $x \in E$, there is a unique approximate tangent plane to E at x.

(iii) For $\mathcal{H}^n$ almost every $x \in E$, there is an approximate tangent plane to E at x.

While this characterization is insightful and useful in many settings, we sometimes want to claim that objects we're working with have stronger geometric regularity than that provided by rectifiability alone. This is especially true when studying questions which are *quantitative* in nature. For example, given $E \subseteq \mathbb{R}^d$ we may want to know whether E looks flat in a "typical" ball which requires some way of actually measuring and controlling the "number" of scales and locations on which E is close to an n-plane. The existence of approximate tangent planes only says that there is a good tangent plane (in a measure theoretic sense) on infinitesimally small scales around $\mathcal{H}^n$ almost all points. In order to get this desired stronger control, we need often need to require stronger, more quantitative conditions on E than mere rectifiability.

One such influential quantitative rectifiability condition is the notion of uniform n-rectifiability introduced by David and Semmes. They were motivated by questions related to the boundedness of certain singular integral operators like the following: Given a nice enough class of operators, what geometric conditions must one put on $E \subseteq \mathbb{R}^d$ for these operators to be bounded from $L^2(E)$ to itself? In [DS91] and [DS93], David and Semmes tell a long and beautiful story about how the boundedness of singular integral operators relates to quantitative aspects of rectifiability and geometric measure theory through uniform n-rectifiability.

To define uniformly n-rectifiable sets, we first need the following two conditions.

Definition 1.0.2 (Ahlfors n-regular subsets). We say that a set $E \subseteq \mathbb{R}^d$ is n-Ahlfors regular if there exists a constant $c > 0$ such that for all $x \in E,\ 0 < r < \operatorname{diam}(E)$, we have

$$c^{-1}r^n \leq \mathcal{H}^n(E \cap B(x,r)) \leq cr^n.$$

Definition 1.0.3 (Big pieces of Lipschitz images). We say E has *big pieces of Lipschitz images of* $\mathbb{R}^n$ if there exist constants $L, \theta > 0$ such that for any $x \in E,\ 0 < r < \operatorname{diam}(E)$, there exists an L-Lipschitz map $f : A_{x,r} \subseteq B^n(0,r) \subseteq \mathbb{R}^n \to \mathbb{R}^d$ such that

$$\mathcal{H}^n(E \cap B(x,r) \cap f(A_{x,r})) \geq \theta r^n$$

Finally, we can define uniform rectifiability.

Definition 1.0.4 (Uniformly n-rectifiable sets). We say that $E \subseteq \mathbb{R}^d$ is uniformly n-rectifiable if and only if E is Ahlfors n-regular and E has big pieces of Lipschitz images of $\mathbb{R}^n$.

Ahlfors n-regularity says that the measure of E inside any ball is uniformly comparable to the measure of a ball of equal radius in an n-plane. Assuming Ahlfors n-regularity, the big pieces of Lipschitz images of $\mathbb{R}^n$ condition says that a uniform fraction of the measure of each ball in E is covered by Lipschitz images with uniformly bounded Lipschitz constant of balls of equal size in $\mathbb{R}^n$. Uniformly rectifiable sets have quantitatively large Lipschitz coverings at all scales and locations, while general rectifiable sets may have balls which are arbitrarily badly parameterizable by Lipschitz maps.

David and Semmes proved that, assuming E is Ahlfors n-regular, some nice classes of singular integral operators are bounded from $L^2(E)$ to itself if and only if E is uniformly n-rectifiable. However, they also extensively studied uniformly rectifiable sets as geometric objects of independent interest, proving many equivalent geometric definitions. Many of these give quantitative counterparts of qualitative characterizations of n-rectifiable sets. One of the strongest such characterizations is the bilateral weak geometric lemma (BWGL), which gives a quantitative analog to the existence of approximate tangent n-planes discussed in 1.0.1. Roughly speaking, the BWGL says that inside "most" balls, E is bilaterally close to an n-plane. We measure this bilateral closeness using the Hausdorff distance.

Definition 1.0.5 (Hausdorff distance)**.** For any subsets $A, B \subseteq \mathbb{R}^d$, we define the Hausdorff distance between A and B as

$$d_H(A,B) = \max\left\{ \sup_{b\in B} \operatorname{dist}(b,A),\ \sup_{a\in A} \operatorname{dist}(a,B) \right\}.$$

In words, $d_H(A,B)$ measures the furthest distance any point in one set is from the other set.

Definition 1.0.6 (Bilateral beta numbers)**.** Let $E \subseteq \mathbb{R}^d$, $x \in E,\ r > 0$. We define

$$b\beta_E(x,r) = \inf_{P\ n\text{-plane}} \frac{d_H(E\cap B(x,r), P\cap B(x,r))}{r}.$$

We call $b\beta_E(x,r)$ the bilateral beta number for E inside $B(x,r)$.

In words, $b\beta_E(x,r) < \epsilon$ if and only if there exists a plane P such that every point of $P \cap B(x,r)$ is within distance ϵr of a point of $E \cap B(x,r)$ and every point of $E\cap B(x,r)$ is within distance ϵr of a point of $P \cap B(x,r)$. Before finally stating the bilateral weak geometric lemma, we need a precise way of quantifying the "number" of balls satisfying a certain property. This is given by the notion of a Carleson set

Definition 1.0.7 (Carleson sets)**.** We say that $\mathscr{B} \subseteq E \times \mathbb{R}^+$ is a Carleson set if there exists a constant $C > 0$ such that for every $x \in E$ and $0 < r < \operatorname{diam}(E)$

$$\int_{B(x,r)} \int_0^r \chi_{\mathscr{B}}(x,t) \frac{dxdt}{t} \leq Cr^n.$$

We think of each pair $(x,t) \in E \times \mathbb{R}^+$ as representing the ball $B(x,t) \cap E$. Since the measure $\frac{dt}{t}$ assigns constant mass to each interval $[2^{-k}, 2^{-k+1}]$, each "layer" of balls $\mathscr{L}_k = \{(x,t) \in E \times \mathbb{R}^+ : x \in B(y,r),\ 2^{-(k+1)} \leq t \leq 2^{-k}\}$ inside of a given ball $B(y,r)$ for an Ahlfors n-regular set E has

$$\int\int_{\mathscr{L}_k} \frac{dxdt}{t} \asymp r^n$$

so that

$$\int_{B(y,r)} \int_0^r \chi_{E\times\mathbb{R}^+}(x,t)\frac{dxdt}{t} \asymp \sum_{k=-\log(r)}^{\infty} r^n = \infty.$$

In order for $\mathscr{B}$ to be a Carleson set, $\mathscr{B}$ must be much smaller than $E \times \mathbb{R}^+$. It must become very sparse on small scales.

Finally, we can define what it means for E to satisfy the bilateral weak geometric lemma.

Definition 1.0.8 (Bilateral weak geometric lemma (BWGL))**.** We say that $E \subseteq \mathbb{R}^d$ satisfies the BWGL if and only if for every $\epsilon > 0$, the set

$$\mathscr{B}(\epsilon) = \{(x,t) \in E \times \mathbb{R}^+ : b\beta_E(x,t) > \epsilon\}$$

is a Carleson set.

David and Semmes proved the following equivalent characterization of uniform rectifiability (among many, many more).

Theorem 1.0.2 ([DS93] Theorem I.2.8)**.** *Let $E \subseteq \mathbb{R}^d$ be Ahlfors n-regular. Then E is uniformly n-rectifiable if and only if E satisfies the BWGL.*

Although the topics of the chapters in this thesis vary, they all lie in the general area of quantitative rectifiability, studying ways of quantifying flatness over all scales and all locations. In what follows, we introduce each of the four topics presented. In the final paragraph of each of the following sections, we give a summary of the results we obtain within that topic.

1.1 The analyst's traveling salesman theorem for Jordan arcs

Given an arclength parameterized Jordan arc $\gamma : [0,1] \to \mathbb{R}^d$, we define the length of γ relative to a partition $P = \{x_0, x_1, \ldots, x_N\}$ for $x_i < x_{i+1},\ x_i \in [0,1]$ by

$$\ell(\gamma, P) = \sum_{i=0}^{N-1} |\gamma(x_{i+1}) - \gamma(x_i)|.$$

Then, the length of γ is given by

$$\ell(\gamma) = \sup_P \ell(\gamma, P).$$

This definition of length involves only what happens on infinitesimal scales (partitions with many points are all that matter because refining partitions increases length), reminiscent of how approximate tangent planes only describe the flatness of rectifiable sets on infinitesimal scales. Analyst's traveling salesman theorems for curves attempt to provide a quantification of the length in terms of the local geometry of curves inside balls over all scales and locations inside the curve. The primary tool for this analysis is the Jones beta number introduced by Peter Jones in his proof of the analyst's traveling salesman theorem in $\mathbb{R}^2$. (See the introduction of Chapter 2 for a more detailed exposition of traveling salesman theorems.)

Definition 1.1.1 (Jones beta number)**.** Let $E \subseteq \mathbb{R}^d$ and $Q \in \mathcal{D}(\mathbb{R}^d)$, the set of dyadic cubes in $\mathbb{R}^d$. Define

$$\beta_E(Q) = \inf_{L \text{ line}} \sup_{x \in 3Q} \frac{\operatorname{dist}(x, L)}{\operatorname{diam}(Q)}$$

where $3Q$ is the cube with the same center as Q but with 3 times the side length.

Assume $\operatorname{diam}(\gamma([0,1])) = 1$ and define $P_0 = \{0, 1\}$ where $\gamma(0)$ and $\gamma(1)$ are the endpoints of γ. Consider a sequence $P_1, P_2, \ldots$ of partitions of $[0,1]$ where P_{i+1} is a refinement of P_i, $\gamma(P_i)$ is approximately a 2^{-i}-net for $\Gamma = \gamma([0,1])$, and $\ell(\gamma, P_i) \xrightarrow{i\to\infty} \ell(\gamma)$. Then we can write

$$\ell(\gamma) - \operatorname{crd}(\gamma) = \sum_{i=0}^{\infty} \ell(\gamma, P_{i+1}) - \ell(\gamma, P_i). \tag{1.3}$$

where $\operatorname{crd}(\gamma) = |\gamma(0) - \gamma(1)|$ is defined to be the distance between γ's endpoints. Each term in the sum on the right-hand side of (1.3) is the difference between the length of the polygonal approximations of γ given by P_{i+1} and P_i. Morally speaking, the picture in Figure 2.1 happens everywhere on the scale of the distance between points in P_i, and the Pythagorean theorem as applied there implies we can expect

$$\ell(\gamma) - \operatorname{crd}(\gamma) = \sum_{i=0} \ell(\gamma, P_{i+1}) - \ell(\gamma, P_i) \lesssim \sum_{i=0}^{\infty} \sum_{Q \in \mathcal{D}_i(\mathbb{R}^d)} \beta_\Gamma(Q)^2 \operatorname{diam}(Q) \tag{1.4}$$

where $\mathcal{D}_i(\mathbb{R}^d)$ is the collection of dyadic cubes of side length 2^{-i}. This intuitive argument can be made rigorous with some additional caveats. In fact, the significantly more difficult reverse inequality of (1.4) also holds.

For an idea of the reasons for this difficulty, notice that we might try to prove the reverse inequality by running the construction given previously in reverse. That is, we construct a sequence of polygonal approximations of our curve Γ and attempt to bound the beta number sum in terms of the successive differences in their lengths. This argument can be made to work on certain regions of the curve in which "non-flat arcs" are present, but in general there are balls which have large beta number for which polygonal approximations will always have similar length. One such example is a ball consisting of two line segments forming a "cross" (See the Δ_1 example in Figure 2.3.). In this case, $\beta_\Gamma(Q) \approx 1$, but polygonal approximations on scales smaller than the scale of the ball will just reproduce the line segments. Controlling these beta numbers requires totally different arguments involving geometric martingales and other specially designed tools.

Equation (1.4) and its reverse were first proved by Bishop in $\mathbb{R}^d$.

Theorem 1.1.1 ([Bis22] Theorem 1.2)**.** *Let $\gamma : [0,1] \to \mathbb{R}^d$ be a Jordan arc. Then*

$$\sum_{Q \in \mathcal{D}(\mathbb{R}^d)} \beta_\Gamma(Q)^2 \operatorname{diam}(Q) \asymp_d \ell(\gamma) - \operatorname{crd}(\gamma)$$

In the first part of this thesis, we present the following analog of Bishop's result for curves in Hilbert spaces.

Theorem (See Chapter 2 Theorems A and B). *Let H be a Hilbert space and let $\Gamma \subseteq H$ be a Jordan arc. For any multiresolution family $\mathscr{H}$ associated to Γ with inflation factor $A > 200$, we have*

$$\sum_{Q \in \mathscr{H}} \beta_\Gamma(Q)^2 \operatorname{diam}(Q) \asymp_A \ell(\Gamma) - \operatorname{crd}(\Gamma). \tag{1.5}$$

For the easier direction, we provide a Hilbert space adaptation of Bishop's argument which philosophically follows the above outline. For the harder direction, we refine and adapt Schul's arguments for the general Hilbert space traveling salesman theorem using filtrations and geometric martingales. The most interesting new idea is the introduction of a sort of "reduced length" measure μ which measures the local contribution of subsets of the curve to the overall value of $\ell(\Gamma) - \operatorname{crd}(\Gamma)$.

1.2 Lipschitz decompositions of domains with bilaterally flat boundaries

Let $\Omega \subseteq \mathbb{R}^{d+1}$ be an open set. One would often like to break Ω into smaller pieces satisfying some regularity property and work with the pieces of the domain individually. One such decomposition of Ω is the Whitney decomposition

Definition 1.2.1 (Whitney decomposition). We say that $\mathcal{W}$ is a Whitney decomposition of a domain Ω if $\mathcal{W}$ is a collection of closed cubes $\mathcal{W} = \{Q_j\}_{j \in \mathbb{N}}$ with disjoint interiors such that for all $Q \in \mathcal{W}$,

(i) $\Omega = \bigcup_{Q \in \mathcal{W}} Q$,

(ii) $Q \subseteq \Omega$,

(iii) $\operatorname{dist}(Q, \Omega^c) \asymp_n \operatorname{diam}(Q)$.

The Whitney decomposition always exists, and it is a decomposition into very nice pieces: cubes. However, the Whitney decomposition lacks some desirable properties. For instance, we have no quantitative control over the sum of the surface areas of the boundaries of the constituent cubes. Indeed, no cube in the Whitney decomposition actually intersects $\partial\Omega$, implying there are infinitely many "layers" of cubes extending towards the boundary so that

$$\sum_{Q \in \mathcal{W}} \mathcal{H}^d(\partial Q) \gtrsim \sum_{i=1}^{\infty} 1 = \infty.$$

If we are interested in decompositions into nice pieces with quantitative control on the surface area, then we can do better than the Whitney decomposition in many cases. We will now make our focus more concrete and declare that we are looking for decompositions into *Lipschitz domains*.

Definition 1.2.2 (Lipschitz domains)**.** We say that an open, connected set $\Omega \subseteq \mathbb{C}$ is an *M-Lipschitz domain* if the following holds: After a translation and dilation, we can assume $0 \in \Omega$ and

$$\partial\Omega = \left\{ r(\theta)e^{i\theta} : 0 \leq \theta \leq 2\pi \right\},$$

and for any $\theta_1, \theta_2 \in [0, 2\pi]$,

$$|r(\theta_1) - r(\theta_2)| \leq M|\theta_1 - \theta_2|$$

and for all $\theta \in [0, 2\pi]$,

$$\frac{1}{1+M} \leq r(\theta) \leq 1.$$

If we only require decompositions into Lipschitz domains, then we can immediately find less wasteful decompositions in simple cases. If Ω is a polygon in the plane, then triangulations and dissections provide nice decompositions into finitely many Lipschitz domains. A key difference between these examples and the Whitney decomposition is that elements of the decompositions intersect the boundary in large pieces. This is possible only when the boundary has sufficient geometric regularity. That is, it looks flat enough at most scales and locations. In the planar case, Peter Jones proved a very general decomposition result which was a key step in his proof of the analyst's traveling salesman theorem in the plane.

Theorem 1.2.1 ([Jon90] Theorem 2)**.** *There exists a constant $M > 0$ such that the following holds: For any simply connected domain $\Omega \subseteq \mathbb{C}$ with $\mathcal{H}^1(\partial\Omega) < \infty$, there is a rectifiable curve Γ such that*

$$\Omega \setminus \Gamma = \bigcup_j \Omega_j$$

where Ω_j is an M-Lipschitz domain for each j, and

$$\sum_j \mathcal{H}^1(\partial\Omega_j) \leq M\mathcal{H}^1(\partial\Omega).$$

Jones's theorem provides a Lipschitz decomposition of any simply connected domain with finite boundary length with quantitative control on the sum of the boundary lengths in terms of the length of the original boundary.

In the second part of this thesis, we provide some analogs of Jones's result for higher dimensional domains including the following.

Theorem (See Chapter 3 Theorem D)**.** *Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a domain with $0 \in \partial\Omega$. There exist constants $A(d), L(d), \epsilon(d) > 0$ such that if $\partial\Omega$ is (ϵ, d)-Reifenberg flat, then there exists a collection of L-Lipschitz graph domains $\{\Omega_j\}_{j\in\mathscr{L}}$ such that*

(i) $\Omega_j \subseteq \Omega$,

(ii) $\Omega \cap B(0,1) \subseteq \bigcup_{j=1}^{\infty} \Omega_j$,

(iii) $\exists C(d) > 0$ *such that* $\forall x \in \Omega$, $x \in \Omega_j$ *for at most C values of j,*

(iv) For any $y \in \partial\Omega \cap B(0,1)$ and $0 < r \leq 1$, we have

$$\sum_{j=1}^{\infty} \mathcal{H}^d(\partial\Omega_j \cap B(y,r)) \lesssim_{\epsilon,d,L} \mathcal{H}^d(\partial\Omega \cap B(y,Ar)).$$

Theorem (See Chapter 3 Theorem E)**.** *Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a domain with $0 \in \partial\Omega$. If $\partial\Omega$ is d-uniformly rectifiable, then there exists $L(d), A(d) > 0$ such that there exists a collection of L-Lipschitz graph domains $\{\Omega_j\}_{j \in J_{\mathscr{L}}}$ such that conclusions (i), (ii), (iii), and (iv) (with additional dependence on uniform rectifiability constants) of Theorem D hold.*

We achieve Lipschitz "almost" decompositions with bounded overlap for domains whose boundaries satisfy one of two quantitative rectifiability conditions: uniform d-rectifiability or (ϵ, d)-Reifenberg flatness. In either case, the key idea is to decompose scales and locations in $\partial\Omega$ into a coronization associated to a corona decomposition and use this decomposition to sort a Whitney-type decomposition of scales and locations in the complement into parallel stopping time regions. We then economically decompose these regions further into Lipschitz domains while retaining desired quantitative control over the boundaries in terms of the original domain's boundary.

1.3 Uniformly rectifiable metric spaces satisfy the weak constant density condition

In addition to the existence of approximate tangent n-planes, n-rectifiability in Euclidean spaces is also characterized by regularity of the Hausdorff density.

Theorem 1.3.1. *Let $E \subseteq \mathbb{R}^d$ be $\mathcal{H}^n$ measurable with $\mathcal{H}^n(E) < \infty$. Then E is n-rectifiable if and only if*

$$\lim_{r \to 0} \frac{\mathcal{H}^n(E \cap B(x,r))}{(2r)^n} = 1 \tag{1.6}$$

at $\mathcal{H}^n$-a.e. $x \in E$.

A proof of the forward direction is given in [Mat95] Theorem 16.2. The reverse direction was proven by Marstrand for $n = 2$, $d = 3$ [Mar61] and by Mattila for general n, d [Mat75]. One one hand, the fact that rectifiable sets have approximate tangent n-planes implies (1.6) exactly because $(2r)^n = \mathcal{H}^n(B(x,r) \cap P)$ where P is an n-plane and $x \in P$. The converse requires a more careful geometric argument using the fact that (1.6) implies a form of local n-dimensional symmetry for E.

David, Semmes, and Tolsa were able to show that a more quantitative version of (1.6) called the weak constant density condition (WCD) characterizes uniformly n-rectifiable subsets of Euclidean space.

Definition 1.3.1 (WCD)**.** Let $E \subseteq \mathbb{R}^d$ be Ahlfors n-regular, let $C_0, \epsilon_0 > 0$, and define

$$\mathscr{G}_{\rm cd}(C_0, \epsilon_0) = \left\{ (x,r) \in E \times \mathbb{R}^+ \;\middle|\; \begin{array}{l} \exists \text{Ahlfors } (C_0, n)\text{-regular } \mu,\ \operatorname{spt}(\mu) = E, \\ \forall y \in B(x,r),\ 0 < t \leq r, \\ |\mu(E \cap B(y,t)) - t^n| \leq \epsilon_0 r^n \end{array} \right\}, \tag{1.7}$$

$$\mathscr{B}_{\rm cd}(C_0, \epsilon_0) = E \times \mathbb{R}^+ \setminus \mathscr{G}_{\rm cd}(C_0, \epsilon_0). \tag{1.8}$$

We say that E satisfies the WCD if there exists $C_0 > 0$ such that for all $\epsilon_0 > 0$, $\mathscr{B}_{\rm cd}(C_0, \epsilon_0)$ is a Carleson set.

Roughly speaking, an Ahlfors regular set E satisfies the WCD if in most balls, it supports a measure with nearly constant density in a sufficiently large "neighborhood" of scales and locations.

Theorem 1.3.2. *Let $E \subseteq \mathbb{R}^d$ be Ahlfors n-regular. Then E is uniformly n-rectifiable if and only if E satisfies the WCD.*

David and Semmes proved the forward direction ([DS91] Chapter 6) and the reverse direction in the special cases $n = 1, 2, d-1$ ([DS93] Corollary III.5.4) while Tolsa showed the reverse direction for general n [Tol15].

Recall that the definition of rectifiability was given for subsets of metric spaces, and observe that the definition of uniformly rectifiable subsets given using big pieces of Lipschitz images makes perfect sense for metric spaces as well. So, we can justifiably make the following definition.

Definition 1.3.2 (Uniformly n-recitifable metric spaces)**.** We say that a metric space X is uniformly n-rectifiable if it is Ahlfors n-regular and has big pieces of Lipschitz images of subsets of $\mathbb{R}^n$.

Recently, Bate, Hyde, and Schul proved that uniformly n-rectifiable metric spaces have a number of equivalent geometric definitions analogous to those proven by David and Semmes [BHS23]. These include the existence of metric corona decompositions and the satisfaction of a metric version of the BWGL. Naturally, we would like to know how many equivalent definitions in the Euclidean case have analogs in the metric case.

In the third part of this thesis, we continue the work begun by Bate, Hyde, and Schul by extending a piece of the Euclidean WCD characterization to metric spaces by proving that any uniformly n-rectifiable metric space satisfies the WCD.

Theorem (See Chapter 4 Theorem F)**.** *Uniformly n-rectifiable metric spaces satisfy the WCD.*

The proof uses the fact proven by Bate, Hyde, and Schul that uniformly rectifiable metric spaces have very big pieces of bi-Lipschitz images and an abstract John-Nirenberg Stromberg lemma to reduce to the case of a bi-Lipschitz image of $[0,1]^n$. We then use the area formula to write Hausdorff measure in terms of the metric Jacobian of f and control the variation of the density in terms of a wavelet-like L^2 decomposition of the Jacobian.

1.4 Iterating the big pieces operator

Recall that uniformly n-rectifiable metric spaces are defined as Ahlfors n-regular metric spaces that of big pieces of Lipschitz images of $\mathbb{R}^n$. However, the idea of big pieces is general enough that one can imagine replacing big pieces of Lipschitz images with big pieces of other types of spaces.

Definition 1.4.1 (Big pieces of Ahlfors n-regular metric spaces)**.** Let $\mathscr{F}$ be a class of Ahlfors n-regular subsets of an Ahlfors n-regular metric space X. We say that $X \in \mathrm{BP}(\mathscr{F})$, i.e. X has big pieces of sets in $\mathscr{F}$, if there exists a constant $\theta > 0$ such that for all $x \in X$ and $0 < r < \mathrm{diam}(X)$, there exists $F_{x,r} \in \mathscr{F}$ such that

$$\mathcal{H}^n(B(x,r) \cap F_{x,r}) \geq \theta r^n.$$

In fact, David and Semmes prove the following result.

Theorem 1.4.1. *Let $E \subseteq \mathbb{R}^d$ be Ahlfors n-regular. Then the following are equivalent*

1. *E is uniformly n-rectifiable,*
2. *E has big pieces of bi-Lipschitz images ($E \in \mathrm{BP}(\mathrm{BI})$),*
3. *E has big pieces of sets which have big pieces of Lipschitz graphs ($E \in \mathrm{BP}^2(\mathrm{LG})$),*
4. *$E \in \mathrm{BP}^j(\mathrm{BI})$ for $j \geq 1$,*
5. *$E \in \mathrm{BP}^j(\mathrm{LG})$ for $j \geq 2$.*

The final two items are interesting because, a priori, the conditions $\mathrm{BP}^j(\mathscr{F})$ for $j \geq 1$ get progressively weaker as j increases. David and Semmes show that the big pieces of bi-Lipschitz images condition stabilizes after the first iteration, while the big pieces of Lipschitz graphs condition stabilizes after the second iteration. Bate, Hyde, and Schul also obtain a similar result for $\mathrm{BP}^j(\mathrm{BI})$ in uniformly n-rectifiable metric spaces [BHS23]

In the fourth and final part of this thesis, we show that any Ahlfors n-regular metric space X which has $\mathrm{BP}(\mathrm{BP}(\mathscr{F}))$ admits an Ahlfors n-regular extension $\tilde{X} \supseteq X$ such that $\tilde{X} \in \mathrm{BP}(\mathscr{F})$.

Theorem (See Chapter 5 Theorem G)**.** *Let $\mathscr{F}$ be a class of (closed) Ahlfors-David k-regular sets in a metric space $\mathbb{X}$. Let $E \subseteq \mathbb{X}$ be a Ahlfors-David k-regular set with $E \in \mathrm{BP}(\mathrm{BP}(\mathscr{F}))$. Then there exists a set $F \subset \mathbb{X}$ such that*

(i) *$E \subseteq F$,*

(ii) *F is Ahlfors-David k-regular.*

(iii) *$F \in \mathrm{BP}(\mathscr{F})$.*

The constants in the conclusion are quantitative with dependence on the constants in the assumptions.

The proof constructs the set $F \supset E$ iteratively as the union F_n of a sequence of "big pieces" chosen carefully using Whitney decompositions to retain upper regularity. As a consequence of the theorem, we show that the conditions $\mathrm{BP}^j(\mathscr{F})$ for $j \geq 2$ are all equivalent for any Ahlfors n-regular class of subsets $\mathscr{F}$. This gives a short, direct proof of David and Semmes's above Euclidean result for $\mathrm{BP}^j(\mathrm{LG})$.

Chapter 2

A traveling salesman theorem for Jordan curves in Hilbert space

2.1 Introduction

Given a metric space X and a set $E \subseteq X$, how can one tell if there is a curve γ of finite length containing E? If one does exist, how can one estimate its length in terms of the geometry of E, and how can one construct such a curve with length as short as possible? The problem of answering these questions in X is commonly referred to as the *Analyst's Traveling Salesman Problem* for X. The study of these problems began when Peter Jones introduced and solved the problem in the standard Euclidean plane $\mathbb{R}^2$ [**Jo90**]. Okikiolu later extended the result to curves in $\mathbb{R}^n$ [Oki92], and Schul managed to give an analogue of Okikiolu's and Jones's results in Hilbert space H [Sch07a]. Full solutions have been given for sets in Carnot groups [Li22] and graph inverse limit spaces [DS16], and for Radon measures in $\mathbb{R}^n$ [BS17] and Carnot groups [BLZ23]. Partial results are also available in Banach spaces [BM23a], [BM23b] and general metric spaces [Hah05], [DS21].

Many authors have also studied traveling salesman-type problems for higher-dimensional sets. This includes Hölder curves [BNV19],[BZ20], $C^{1,\alpha}$ surfaces [Ghi20], lower content d-regular sets in $\mathbb{R}^n$ [AS18] and Hilbert space [Hyd22a], analogues of Jordan curves in higher dimensions [Vil20], and even general sets in $\mathbb{R}^n$ [Hyd22b]. Many of these approaches are closely tied to results on parameterization of Reifenberg flat-type sets in $\mathbb{R}^n$ [DT12] and Banach spaces [ENV19].

One of the central matters in traveling salesman problems is finding a specific quantitative relationship between the Hausdorff measure of the set in question and some measure of its local geometry. The traditional traveling salesman theorems in $\mathbb{R}^n$ and ℓ_2 provide a relationship which holds for general subsets of the ambient space. Therefore, it seems plausible that one could find a tighter relationship when one restricts their attention to a more geometrically regular class of subsets. A result in this direction was recently achieved by Bishop [Bis22] as part of his study of Weil-Petersson curves [Bis20]. His result is a sharpening of this quantitative relationship for the class of Jordan arcs in $\mathbb{R}^n$. Bishop posed a natural question: Does a similar relationship exist for Jordan arcs in Hilbert space?

This paper has two primary goals: First, provide a full proof of Schul's necessary condition in the Hilbert space traveling salesman theorem, filling in gaps and correcting errors present in the original presentation in [Sch07a]. Second, we answer Bishop's question in the affirmative by providing an analogous sharpening of the Hilbert space traveling salesman theorem when restricted to Jordan arcs.

The proof of our analogue of Bishop's result differs significantly from Bishop's proof because the latter relies heavily on dimension-dependent estimates. We use dimension-independent pieces of Bishop's argument where possible, but largely focus on implementing an extension of the Hilbert space methods introduced in [Sch07a]. The first goal is motivated by the discovery of several technical errors in Schul's original proof as presented in [Sch07a]. The proof presented here has largely the same outline and general structure as Schul's proof while implementing several new ideas to correct the identified errors. We also mention that the work on the traveling salesman problem in Banach spaces by Badger and McCurdy [BM23a], [BM23b] provides another proof of the Hilbert space necessary condition via methods which diverge more significantly from ours and those of Schul's original proof.

2.1.1 Overview

Jones' solution to the traveling salesman problem in $\mathbb{R}^2$ is based on measuring how close a subset $E \subseteq \mathbb{R}^2$ is to being linear locally. In order to do this, he defined what is now called *Jones' beta number.*

Definition 2.1.1 (*Jones' beta number*)**.** Fix a Hilbert space H and let $E, Q \subseteq H$ where Q has finite diameter. We define the β-number for E in the "window" Q by

$$\beta_E(Q) := \inf_L \sup_{x \in Q \cap E} \frac{\operatorname{dist}(x, L)}{\operatorname{diam}(Q)},$$

where L ranges over all affine lines in H.

One can interpret the number $\beta_E(Q)\operatorname{diam}(Q)$ as the radius of the minimal width cylinder in H which contains the set $E \cap Q$. The factor of $\operatorname{diam}(Q)$ on the right-hand side in the definition ensures that $\beta_E(Q)$ is scale-invariant. We always have $0 \leq \beta_E(Q) \leq 1$ where $\beta_E(Q) = 0$ implies $E \cap Q \subseteq L$ for some line L while $\beta_E(Q) \geq \epsilon$ for some constant $\epsilon > 0$ implies that for every choice of L, there exists a point in $E \cap Q$ of distance $\epsilon \operatorname{diam}(Q)$ from L. Jones and Okikiolu used these numbers to characterize subsets of rectifiable curves in $\mathbb{R}^2$ and $\mathbb{R}^n$ respectively in the following theorem:

Theorem 2.1.1. *(Jones for $n = 2$ [**Jo90**], Okikiolu for $n \geq 2$ [Oki92]) Let $E \subseteq \mathbb{R}^n$. E is contained in a rectifiable curve if and only if*

$$\beta_E^2(\mathbb{R}^n) := \operatorname{diam}(E) + \sum_{Q \in \Delta(\mathbb{R}^n)} \beta_E(3Q)^2 \operatorname{diam}(Q) < \infty$$

where $\Delta(\mathbb{R}^n)$ is the set of all dyadic cubes in $\mathbb{R}^n$ and $3Q$ is the cube with the same center as Q but three times the side length. If Σ is a connected set of shortest length containing E, then

$$\beta_\Sigma^2(\mathbb{R}^n) \lesssim_n \mathcal{H}^1(\Sigma) \tag{2.1}$$

and

$$\beta_E^2(\mathbb{R}^n) \gtrsim_n \mathcal{H}^1(\Sigma). \tag{2.2}$$

It is important to take $3Q$ rather than Q so that the family $\{3Q\}_{Q\in\Delta(\mathbb{R}^n)}$ "covers" $\mathbb{R}^n$ sufficiently well. More precisely, any subset $B \subseteq \mathbb{R}^n$ is contained in a cube $3Q$ of comparable diameter, while there may not exist such a standard dyadic cube Q with this property. The exponent 2 appears in Theorem 2.1.1 because the Pythagorean theorem allows one to estimate the difference in length between a line segment and a slight perturbation of the segment by a small distance d in a perpendicular direction by a factor proportional to d^2 (See Remark 2.1.2). We recommend the reader sees the introduction of [BM23a] for further intuition on the behavior of β-numbers for subsets of rectifiable curves in H (and in Banach spaces).

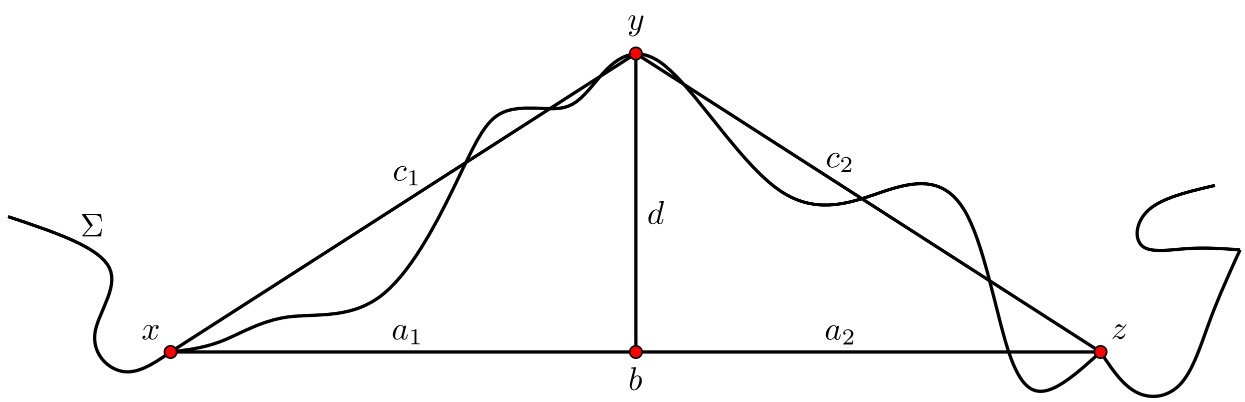

Figure 2.1: Polygonal approximations of Σ

Remark 2.1.2 (The Pythagorean theorem and triangle inequality excess)*.* Let $x, y, z \in \Sigma$. Applying the Pythagorean theorem in Figure 2.1 gives $d^2 = c_i^2 - a_i^2 = (c_i - a_i)(c_i + a_i)$ for $i = 1, 2$. If we assume that $R > 0$ is such that $c_i \simeq a_i \simeq R$, then

$$2d^2 = (c_1 - a_1)(c_1 + a_1) + (c_2 - a_2)(c_2 + a_2) \simeq 2R(c_1 + c_2 - a_1 - a_1).$$

If we further assume that there exists a dyadic cube Q such that $x, y, z \in 3Q, \operatorname{diam}(Q) \simeq R$, and $d \simeq \beta_\Sigma(Q) \operatorname{diam}(Q)$, then

$$\beta_\Sigma(Q)^2 \operatorname{diam}(Q) \simeq c_1 + c_2 - a_1 - a_2 = |x - y| + |y - z| - |x - z|.$$

This final equality demonstrates why beta numbers are often said to measure the *triangle inequality excess* inside a cube $3Q$.

Bishop's results say that if we restrict our attention to *Jordan arcs* then (2.1) and (2.2) can be improved.

Definition 2.1.2 (*Jordan arcs and curves*)**.** For a metric space X, we say that $\Gamma \subseteq X$ is a *rectifiable arc* if $\mathcal{H}^1(\Gamma) < \infty$ and there exists a surjective continuous map $\gamma : I \to \Gamma$ for some closed interval $I := [a, b] \subseteq \mathbb{R}$. We also refer to the map γ as a rectifiable arc. Whenever we refer to such Γ, we implicitly have a particular parameterizing map γ in mind and really mean the pair (Γ, γ). We refer to the points $\gamma(a)$, and $\gamma(b)$ as the *endpoints* of γ (or Γ) and we define

$$\operatorname{crd}(\Gamma) := \operatorname{crd}(\gamma) := \operatorname{dist}(\gamma(a), \gamma(b)).$$

We refer to $\operatorname{crd}(\Gamma)$ as the *chord length* of Γ and refer to the line (segment) which passes through the two endpoints of Γ as its *chord* or its *chord line (segment).* We additionally define

$$\ell(\Gamma) := \ell(\gamma)$$

as the length of Γ (also see 2.14). We call Γ a *Jordan arc* if we can take γ to be bijective. We call Γ a *Jordan curve* if there exists a map $\gamma : [a, b] \to \Gamma$ which is injective on $[a, b)$, but has $\gamma(a) = \gamma(b)$. In general, any rectifiable arc γ with $\gamma(a) = \gamma(b)$ is called *closed.* For Jordan arcs and curves $\ell(\Gamma) = \mathcal{H}^1(\Gamma)$, but this does not hold for general rectifiable arcs (for more on this, see Remark 2.1.9).

The chord length is an integral part of Bishop's following improvement:

Theorem 2.1.3 ([Bis22] Theorem 1.2)**.** *Let $\Gamma \subseteq \mathbb{R}^n$ be a Jordan arc. Then*

$$\sum_{Q \in \Delta(\mathbb{R}^n)} \beta_\Gamma(3Q)^2 \operatorname{diam}(Q) \simeq_n \ell(\Gamma) - \operatorname{crd}(\Gamma). \tag{2.3}$$

This result has the following corollary for Jordan curves:

Corollary 2.1.4 ([Bis22] Corollary 1.3)**.** *If $\Gamma \subseteq \mathbb{R}^n$ is a Jordan curve, then*

$$\sum_{Q \in \Delta(\mathbb{R}^n)} \beta_\Gamma(3Q)^2 \operatorname{diam}(Q) \simeq_n \ell(\Gamma).$$

One can think of deriving this by applying Theorem 2.1.3 while taking $\operatorname{crd}(\Gamma) \to 0$. The significance of the inequalities in (2.3) is easier to see if we compare each directly with its corresponding inequality in Theorem 2.1.1.

Remark 2.1.5 (The $\lesssim$ improvement in (2.3))*.* The inequality (2.1) and the $\lesssim$ direction of (2.3) applied to a Jordan arc Γ are respectively equivalent to

$$\sum_{Q \in \Delta(\mathbb{R}^n)} \beta_\Gamma(3Q)^2 \operatorname{diam}(Q) \leq C_1(n)\ell(\Gamma), \tag{2.4}$$

$$\sum_{Q \in \Delta(\mathbb{R}^n)} \beta_\Gamma(3Q)^2 \operatorname{diam}(Q) \leq C_2(n)(\ell(\Gamma) - \operatorname{crd}(\Gamma)) \tag{2.5}$$

for some constants $C_1(n), C_2(n) > 0$ depending on n where we used the fact that $\operatorname{diam}(\Gamma) \leq \ell(\Gamma)$. This means (2.3) improves the inequality by replacing $\ell(\Gamma)$ with $\ell(\Gamma) - \operatorname{crd}(\Gamma)$ on the right-hand side. One can see how this improvement manifests by considering the case when Γ is a line segment. In this case, $\beta_\Gamma(3Q) = 0$ for all $Q \in \Delta(\mathbb{R}^n)$ while $\operatorname{crd}(\Gamma) = \ell(\Gamma)$. This means (2.4) becomes $0 \leq C_1(n)\ell(\Gamma)$ while (2.5) becomes $0 \leq 0$. Bishop's improvement pulls the slack out of Jones and Okikiolu's inequality by recognizing that $\operatorname{crd}(\Gamma)$ is a global measure that accounts for how much Γ looks like its chord segment: the beta numbers do not see the "component" of Γ along its chord.

Remark 2.1.6 (The $\gtrsim$ improvement in 2.3). The inequality (2.2) and the $\gtrsim$ direction of (2.3) applied to a Jordan arc Γ and $E \subseteq \Gamma$ are respectively equivalent to

$$(1+\delta)\operatorname{diam}(E) + C(\delta, n) \sum_{Q \in \Delta(\mathbb{R}^n)} \beta_E(3Q)^2 \operatorname{diam}(Q) \geq \ell(\Gamma), \tag{2.6}$$

$$\operatorname{crd}(\Gamma) + C_2(n) \sum_{Q \in \Delta(\mathbb{R}^n)} \beta_\Gamma(3Q)^2 \operatorname{diam}(Q) \geq \ell(\Gamma). \tag{2.7}$$

where in (2.6), $\delta > 0$ is arbitrary but $C(\delta, n) \to \infty$ as $\delta \to 0$. This is not exactly as was stated in (2.2), but Jones shows that one can arrange the inequality this way. This means (2.7) changes (2.6) by replacing the $C_1(n)\operatorname{diam}(E)$ term on the left-hand side with $\operatorname{crd}(\Gamma)$ and exchanges the β_E numbers for the (larger) β_Γ numbers.

Bishop [Bis22] shows how this change manifests by considering the example set $E = \{0, 1, i\beta\} \subseteq \mathbb{C}$ with $0 < \beta \ll 1$. One can calculate that $\sum_{Q \in \Delta(\mathbb{R}^2)} \beta_E(3Q)^2 \operatorname{diam}(Q) \leq C\beta^2$ while $\operatorname{diam}(E) = \sqrt{1+\beta^2} \leq 1 + c\beta^2$ and the shortest curve $\Gamma = [i\beta, 0] \cup [0, 1]$ containing E satisfies $\ell(\Gamma) = 1 + \beta$. By taking $\beta \to 0$, this configuration gives a family of examples showing that one cannot take $\delta = 0$ in (2.6). The only cubes Q which contribute to $\sum \beta_E(3Q)^2 \operatorname{diam}(Q)$ are those for which $E \subseteq 3Q$ because E only contains three points total. But, for any cube Q which contains 0, $\beta_\Gamma(3Q) \gtrsim 1$ so that one can show that $\sum \beta_\Gamma(3Q)^2 \operatorname{diam}(Q) \geq 1 + c\beta$. The point is that the connectedness of Γ adds more geometry with more locations and scales to measure curvature at, increasing the contribution of the β-numbers and loosening the requirement on the $\operatorname{diam}(E)$ term.

In each of the previously stated traveling salesman theorems, the implicit constants in the given inequalities increase exponentially as $n \to \infty$. The constants' exponential blowup can be attributed to the exponential increase in the relative number of dyadic cubes on each scale as n increases. To formulate a version of the traveling salesman theorem in Hilbert space H, Schul invented a replacement for the set of dyadic cubes called a *multiresolution family.*

Definition 2.1.3 (*Multiresolution family*)**.** Fix a connected set $\Sigma \subseteq H$. For $\epsilon > 0$, we call a set $E \subseteq \Sigma$ an *ϵ-net* of Σ if both

(i) For all $x, y \in E$, $|x - y| > \epsilon$, and

(ii) $\Sigma \subseteq \bigcup_{x \in E} B(x, \epsilon)$

Any subset satisfying (i) can be extended to an ϵ-net since it can be extended to satisfy (ii) by adding a maximal number of appropriately spaced points. Fixing an integer n_0, let $X_{n_0} \subseteq \Sigma$ be a 2^{-n_0}-net. The extension property implies the existence of a sequence of 2^{-n}-nets $\{X_n\}$ satisfying $X_n \subseteq X_{n+1}$. Fix a constant $A > 1$ and put

$$\mathscr{H} := \{B(x, A2^{-n}) : x \in X_n,\ n \in \mathbb{Z},\ n \geq n_0\} \tag{2.8}$$

where $B(x, A2^{-n})$ is the closed ball of radius $A2^{-n}$ around x. We call $\mathscr{H}$ a *multiresolution family* for Σ and refer to an element $Q \in \mathscr{H}$ as a ball. Given $Q = B(x, A2^{-n})$, define $x_Q := x$ and $\operatorname{rad}(Q) := A2^{-n}$. For $\lambda > 0$, we let $\lambda Q := B(x_Q, \lambda \operatorname{rad}(Q))$. The starting point n_0 is of no consequence; all of the results obtained will be independent of it.

Remark 2.1.7 (Reduction to $H = \ell_2$). If $\Sigma \subseteq H$ is a closed, connected set with $\mathcal{H}^1(\Sigma) < \infty$, then Σ is compact and hence a separable subset of H. This gives the existence of a countable set of vectors $v_1, v_2, \ldots$ such that $\Sigma \subseteq \overline{\text{span}\{v_1, v_2, \ldots\}} =: V$. Hence, Σ is contained in the separable subspace $V \subseteq H$ which is isometric (via a linear transformation) to ℓ_2. Therefore, it suffices to fix $H = \ell_2$ in the following theorems concerning Hilbert space.

The important difference between a multiresolution family and the set of dyadic cubes is that the former is centered on the set, while the latter is a partition of the ambient space. In infinite dimensional space (and general metric spaces), it is necessary to concentrate on the intrinsic properties of the set in question rather than where the set happens to lie relative to pre-defined pieces of the ambient space. We can now state Schul's result:

Theorem 2.1.8 ([Sch07a] Theorem 1.1, Theorem 1.5)**.** *Let $E \subseteq \ell_2$ and let $\mathscr{H}$ be a multiresolution family for E with inflation factor $A > 200$. Then E is contained in a rectifiable curve if and only if*

$$\beta_E^2(\mathscr{H}) := \text{diam}(E) + \sum_{Q \in \mathscr{H}} \beta_E(Q)^2 \,\text{diam}(Q) < \infty.$$

If $\Sigma \subseteq H$ is a connected set of shortest length containing E, then

$$\beta_\Sigma^2(\mathscr{H}) \lesssim_A \mathcal{H}^1(\Sigma) \tag{2.9}$$

and

$$\mathcal{H}^1(\Sigma) \lesssim_A \beta_E^2(\mathscr{H}). \tag{2.10}$$

The exponent 2 can again be attributed to the fact that the Pythagorean theorem holds in Hilbert space. The inflation factor A given in the definition of $\mathscr{H}$ is the analogue of taking $3Q$ rather than Q in the Euclidean space traveling salesman theorems. Schul's proof of (2.10) closely parallels Jones' constructive proof of (2.2), replacing dimension-dependent estimates with dimension-independent estimates needed. We mention here that Badger, Naples, and Vellis provide a refined constructive proof of this result in [BNV19] which produces a nice sequence of parameterizations which they used to prove traveling salesman sufficient conditions for Hölder curves.

On the other hand, Schul's proof of (2.9) differs significantly from Jones's original proof, incorporates some key ideas from Okikiolu's proof in $\mathbb{R}^n$, and introduces several ingenious new constructions to remove dimension-dependent estimates. Unfortunately, several errors have since been discovered in the original presentation in [Sch07a], leaving gaps in the proof. The results of this paper will fill in these gaps, providing a full proof of (2.9) in parallel with the following new results:

Theorem A. *Let $\Gamma \subseteq \ell_2$ be a Jordan arc. For any multiresolution family $\mathscr{H}$ associated to Γ with inflation factor $A > 200$, we have*

$$\sum_{Q \in \mathscr{H}} \beta_\Gamma(Q)^2 \,\text{diam}(Q) \lesssim_A \ell(\Gamma) - \text{crd}(\Gamma). \tag{2.11}$$

The second main result is the other side of the inequality in Theorem 2.1.3 for ℓ_2:

Theorem B. *Let $\Gamma \subseteq \ell_2$ be a Jordan arc. For any multiresolution family $\mathscr{H}$ associated to Γ with inflation factor $A > 30$, we have*

$$\sum_{Q \in \mathscr{H}} \beta_\Gamma(Q)^2 \operatorname{diam}(Q) \gtrsim_A \ell(\Gamma) - \operatorname{crd}(\Gamma). \tag{2.12}$$

These results are to Hilbert space what Bishop's Theorem 2.1.3 is to Euclidean space. One can again look to Remarks 2.1.5 and 2.1.6 to gain intuition about the nature of these improvements over the estimates in Theorem 2.1.8.

Remark 2.1.9 (General rectifiable arcs). Theorems A and B raise a natural question: Do similar results hold for general rectifiable arcs? In this case, we must be careful about the definitions. If $\gamma : [0, \ell(\gamma)] \to \Sigma$ is any constant arc length parameterization of a compact, connected set $\Sigma \subseteq \ell_2$, then we interpret ℓ as the pushforward of Lebesgue measure onto Σ which does not necessarily coincide with $\mathcal{H}^1|_\Sigma$ as it does for a Jordan arc or curve. If Γ' is a rectifiable arc, then $\ell(\Gamma') \geq \mathcal{H}^1(\Gamma')$, so Theorem A is weaker than the more natural inequality

$$\sum_{Q \in \mathscr{H}} \beta_{\Gamma'}(Q)^2 \operatorname{diam}(Q) \lesssim_A \mathcal{H}^1(\Gamma') - \operatorname{crd}(\Gamma'). \tag{2.13}$$

Whether or not (2.13) holds remains open. In Remark 2.4.30 we give some ideas on how one might modify some of our methods in this direction.

2.1.2 Related Results and Questions

2.1.2.1 Weil-Petersson Curves

Theorem 2.1.3 arose as an improvement to the traveling salesman theorem necessary to connect some of the geometric characterizations of *Weil-Petersson* curves discovered in [Bis20] (a few out of 26 total definitions given!). The Weil-Petersson curves are defined to be the closure of smooth curves in $\mathbb{R}^2$ in the Weil-Petersson metric on universal Teichmüller space introduced in [TT06] by Takhtajan and Teo for studying problems related to string theory. This class of curves has also been studied in relation to computer vision [FKL14], [FN17], [SM06], and Schramm-Loewner evolutions [Wan19a], [Wan19b].

The following result gives the aforementioned characterizations when $n = 2$. We say that a curve Γ is *chord-arc* if any two points $x, y \in \Gamma$ are connected by a subarc $\gamma \subseteq \Gamma$ with $\ell(\gamma) \leq C|x - y|$ for some constant C independent of x and y.

Theorem 2.1.10 ([Bis22] Theorem 1.4). *The following are equivalent for a closed Jordan curve $\Gamma \subseteq \mathbb{R}^n$, $n \geq 2$,*

(i) Γ *satisfies*

$$\sum_{Q \in \Delta(\mathbb{R}^n)} \beta_\Gamma(3Q)^2 < \infty.$$

(ii) Γ *is chord-arc, and for any dyadic decomposition of* Γ*, the inscribed polygons* $\{\Gamma_n\}$ *defined by taking the* n*-th generation points as vertices satisfy*

$$\sum_{n=1}^{\infty} 2^n[\ell(\Gamma) - \ell(\Gamma_n)] < \infty$$

with a bound that is independent of the choice of decomposition.

(iii) Γ *has finite Möbius energy. That is,*

$$\text{Möb}(\Gamma) := \int_\Gamma \int_\Gamma \left(\frac{1}{|x-y|^2} - \frac{1}{\ell(x,y)^2} \right) dxdy < \infty$$

where $\ell(x,y)$ *is the length of the shortest arc contained in* Γ *connecting* x *and* y *and the integration is with respect to arc length measure.*

The Möbius energy in (iii) was one of several functionals introduced by O'Hara to study knots [OHa91], [OHa92]. One can interpret (i) as a bound on the total curvature of the curve Γ over all locations and scales. The missing factor of $\text{diam}(Q)$ when compared with the sums that appear in the traveling salesman theorems makes this condition much harder to satisfy in general. For instance, a curve satisfying (i) cannot have a "corner" (conical type singularity) because this would give an infinite collection of cubes Q such that $\beta_\Gamma(3Q) \gtrsim 1$. In (ii), a dyadic decomposition is an ordered collection of points contained in Γ which divide Γ into 2^n intervals of equal length. We let Γ_n be the polygon with these points as vertices. Hence, we interpret (ii) as measuring the rate of convergence of the length of inscribed polygonal approximations to Γ to the length of Γ itself. The term $\ell(\gamma) - \text{crd}(\gamma)$ a subarc γ can be expected to appear because it measures exactly this form of difference in length.

One of the corollaries of Theorem 2.1.3 that Bishop uses to prove Theorem 2.1.10 translates directly to our setting:

Corollary 2.1.11 ([Bis22] Corollary 5.2 in $\mathbb{R}^n$)**.** *If* $\Gamma \subseteq \ell_2$ *is a closed Jordan curve and* $S := \sum_{Q \in \mathscr{H}} \beta_\Gamma(Q)^2 < \infty$*, then* Γ *is chord-arc, i.e.. any pair of points* $z, w \in \Gamma$ *are connected by a subarc* γ *with* $\ell(\gamma) \lesssim |z-w|$*.*

One can check that Bishop's proof of the $\mathbb{R}^n$-version is independent of the dimension n so that this result follows if one replaces usages of Theorem 2.1.3 there with Theorem D. For more on how the traveling salesman theorem applies to Weil-Petersson curves and related subjects, the reader should see Section 4 of [Bis20]. The rest of the paper gives connections between these curves and a plethora of objects such as conformal maps, Schwarzian derivatives, quasiconformal mappings, Sobolev spaces, and minimal surfaces in hyperbolic 3-space.

2.1.2.2 Traveling salesman in Banach spaces

A separate, related branch of research is that of traveling salesman problems in more general metric spaces. Recent success has been achieved by Matthew Badger and Sean McCurdy [BM23a], [BM23b] in attaining traveling salesman-type necessary and sufficient conditions in Banach spaces. Much of their work was inspired by the paper of Edelen, Naber, and Valtorta [ENV19] which implemented the Reifenberg algorithm in Banach spaces.

Roughly speaking, [ENV19] gave a Banach space version of Reifenberg's topological disk theorem [Rei60], which states that any subset $\Sigma \subseteq \mathbb{R}^n$ which is sufficiently bilaterally close to an affine k-dimensional plane at all locations in Σ and all sufficiently small scales is locally homeomorphic to an open subset of $\mathbb{R}^k$, hence is locally topologically a k-dimensional disk. Edelen, Naber, and Valtorta extended this result to infinite-dimensional Banach spaces, and gave a traveling salesman-type application in the form of a structure theorem for measures in Banach spaces ([ENV19] Theorem 2.1). They give a sufficient condition on a Borel measure μ to be well concentrated around a k-dimensional set in terms of the pointwise boundedness of a sum of integral beta numbers β_μ^k which measure how close μ is locally to a k-dimensional affine plane. An important aspect of their result which is particularly relevant to Badger and McCurdy's work is that the exponent on β_μ^k appearing in Edelen, Naber, and Valtorta's sum differs based on the geometric structure of the Banach space. The exponent 2 appears in the Hilbert space case, but one must make other assumptions on the geometry in more general Banach spaces to say something stronger.

Indeed, Badger and McCurdy use the well-studied notions of *modulus of smoothness* and *modulus of convexity* to estimate the triangle inequality excess (recall Remark 2.1.2) from above and below respectively. They apply their results to prove necessary and sufficient conditions in ℓ_p spaces for $1 < p < \infty$. A major difference between ℓ_p, $p \neq 2$ and ℓ_2 is that the sharp necessary and sufficient conditions they prove in ℓ_p using the standard Jones beta number diverge from one another. One reason this result might be expected is that the triangle inequality excess for orthogonally (in the ℓ_2 sense) perturbed vectors differs based on the direction of the perturbed vector.

To illustrate this point, if e_1, e_2 are standard unit basis vectors for ℓ_p and $0 < \delta \ll 1$, then

$$|e_1 + \delta e_2|_p - |e_1|_p = (1 + \delta^p)^{1/p} - 1 \simeq_p \delta^p.$$

On the other hand, suppose we take a "diagonal" vector $v = \frac{1}{2^{1/p}}(e_1 + e_2)$ and perturb it by the orthogonal (in the ℓ_2 sense) vector $w = \frac{1}{2^{1/p}}(e_1 - e_2)$. We have

$$|v + \delta w|_p - |v|_p = \frac{1}{2^{1/p}}\left((1+\delta)^p + (1-\delta)^p\right)^{1/p} - 1 \simeq_p \delta^2.$$

The length gain by small orthogonal perturbation in ℓ_p varies depending on the direction of the perturbed vector in contrast to the ℓ_2 case.

Theorem 2.1.12 ([BM23a] Theorem 1.6)**.** *(sharp sufficient conditions in ℓ_p) Let $1 < p < \infty$. If $E \subseteq \ell_p$ and $S_{E,\min(p,2)}(\mathscr{G}) < \infty$ for some multiresolution family $\mathscr{G}$ for E with inflation factor $A_{\mathscr{G}} \geq 240$, then E is contained in a curve Γ in ℓ_p with*

$$\mathcal{H}^1(\Gamma) \lesssim_{p,A_{\mathscr{G}}} S_{E,\min(p,2)}(\mathscr{G}).$$

The exponent $\min(p,2)$ on beta numbers is sharp.

Theorem 2.1.13 ([BM23a] Theorem 1.7)**.** *(sharp necessary conditions in ℓ_p) Let $1 < p < \infty$. If $\Sigma \subseteq \ell_p$ is a connected set and $\mathscr{H}$ is a multiresolution family for Σ with inflation factor $A_{\mathscr{H}} > 1$, then*

$$S_{\Sigma,\max(2,p)}(\mathscr{H}) \lesssim_{p,A_{\mathscr{H}}} \mathcal{H}^1(\Sigma).$$

The exponent $\max(2,p)$ *on beta numbers is sharp.*

Badger and McCurdy's results give a similar proof of Theorem 2.1.8 by taking the case $p=2$ in their above results.

Remark 2.1.14 (Banach space Jordan arcs). Given the results of this paper, it is natural to ask whether there is any analogue of Theorems A and B in ℓ_p. That is, for a Jordan arc $\Gamma \subseteq \ell_p$ and multiresolution family $\mathscr{H}$ for Γ, could one show that

$$S_{\Gamma,\max(2,p)}(\mathscr{H}) \lesssim_{p,A_{\mathscr{H}}} \ell(\Gamma) - \mathrm{crd}(\Gamma)$$

or

$$S_{\Gamma,\min(p,2)}(\mathscr{H}) \gtrsim_{p,A_{\mathscr{H}}} \ell(\Gamma) - \mathrm{crd}(\Gamma)$$

by combining the methods of [BM23a], [BM23b] and those given here? If these inequalities do not hold, can one find a different geometric function of the endpoints of Γ which could replace $\mathrm{crd}(\Gamma)$? What about for general rectifiable arcs?

2.1.2.3 Traveling salesman in general metric spaces

Some success has also been achieved in the setting of general metric spaces by Hahlomaa [Hah05] and David and Schul [DS21]. Because there is no ambient linear structure in a general metric space which one can use to define the standard Jones beta number, the work in metric spaces uses replacements which directly measure the triangle inequality excess. Hahlomaa originally defined a general *metric beta number* using the notion of Menger curvature, but this definition is equivalent to the following given by David and Schul. Let E be a metric space, $p \in E$ and $r > 0$. Let $Q = B(p,r)$ and define the *metric beta number* by

$$\begin{aligned}\beta_\infty^E(Q)^2 := r^{-1} \sup\{\, &\mathrm{dist}(x,y) + \mathrm{dist}(y,z) - \mathrm{dist}(z,x) :\\ &x,y,z \in E \cap B(p,r) \text{ and } \mathrm{dist}(z,y) \le \mathrm{dist}(y,z) \le \mathrm{dist}(z,x)\}.\end{aligned}$$

If E is ℓ_2, then this is proportional to the normalized length difference between the line segment $[x,z]$ and its perturbed version given by $[x,y] \cup [y,z]$. The exponent 2 is added in the definition as a convention to preserve the form of Theorem 2.1.1. Hahlomaa was the first to give a sufficient condition in general metric spaces:

Theorem 2.1.15 ([Hah05] Theorem 5.3)**.** *Let E be a metric space and let $\mathscr{G}$ be a multiresolution family for E with inflation factor $A \simeq 1$. If*

$$\beta_\infty^E(\mathscr{G}) := \mathrm{diam}(E) + \sum_{Q \in \mathscr{G}} \beta_\infty^E(Q)^2 \,\mathrm{diam}(Q) < \infty,$$

then there exists a set $F \subseteq [0,1]$ and a surjective Lipschitz map $f : F \to E$ with Lipschitz constant $\mathrm{Lip}(f) \lesssim \beta_\infty^E(\mathscr{G})$.

See [Sch07b] Example 3.3.1 for a counterexample to the converse to Hahlomaa's result in $\mathbb{R}^2$ with the ℓ_1 metric. Schul notes however that this counterexample is not fully satisfactory, as Hahlomaa's result can be strengthened, for instance, by defining the metric beta number to be a supremum taken over more restrictive triples. In any case, David and Schul have recently achieved a partial converse to this result. Their result concerns *doubling* metric spaces. We say that a metric space is doubling if there exists a constant N such that every ball of radius $r > 0$ can be covered by at most N balls of radius $\frac{r}{2}$.

Theorem 2.1.16 ([DS21] Theorem A)**.** *Let Σ be a connected doubling metric space with doubling constant N and let $\mathscr{H}$ be a multiresolution family for Σ with inflation factor $A > 1$. For every $\epsilon > 0$,*

$$\operatorname{diam}(Q) + \sum_{Q \in \mathscr{H}} \beta_\infty^\Sigma(Q)^{2+\epsilon} \operatorname{diam}(Q) \lesssim_{\epsilon,A,N} \mathcal{H}^1(\Sigma).$$

The authors conjecture that the doubling hypothesis can be dropped by utilizing the techniques of [Sch07a].

Remark 2.1.17 (Metric space Jordan arcs). It would again be interesting to know whether these results could be strengthened in the special case of a Jordan arc. That is, let Γ be a metric space which is the image of a continuous injective map $\gamma : [0,1] \to \Gamma$. Suppose $\mathscr{G}$ is a multiresolution family for Γ. Do the naïve inequalities

$$\sum_{Q \in \mathscr{G}} \beta_\infty^\Gamma(Q)^2 \operatorname{diam}(Q) \gtrsim_A \ell(\Gamma) - \operatorname{crd}(\Gamma)$$

or, for Γ with doubling constant N,

$$\sum_{Q \in \mathscr{G}} \beta_\infty^\Gamma(Q)^{2+\epsilon} \operatorname{diam}(Q) \lesssim_{\epsilon,A,N} \ell(\Gamma) - \operatorname{crd}(\Gamma)$$

hold? As our methods rely heavily on linear structure, these seem further from proof than the proposed extension to Banach space. But even if these do not hold, can one find a different geometric function of the endpoints of Γ to replace $\operatorname{crd}(\Gamma)$ in the equations above? What about for general rectifiable arcs?

2.1.3 Acknowledgments

The author would first like to thank Raanan Schul for many helpful discussions. This result would not have been possible without both Raanan's edits of several early drafts and the author's numerous conversations with Raanan about [Sch07a] and the new ideas introduced here.

This result would also not exist without the wonderful course "Topics in Real Analysis: The Weil-Petersson class, traveling salesman theorems and minimal surfaces in hyperbolic space" given by Chris Bishop in Fall 2020 at Stony Brook University in which the results from [Bis20] and [Bis22] were presented, and the problem solved here was first presented to the author.

The author would finally like to thank the anonymous referee for a multitude of insightful comments which greatly improved the content and style of the paper. The author also thanks the referee for also pointing out the problems with several aspects of Schul's original proof of [Sch07a] Lemma 3.28 and encouraging the author to write out the details of Schuls' arguments in full.

2.1.4 Preliminaries

2.1.4.1 Parameterizations of finite-length continua and arcs associated to a parameterization in Hilbert space

From this point on, fix a connected, compact set $\Sigma \subseteq \ell_2$ and a rectifiable Jordan arc $\Gamma \subseteq \ell_2$. We are guaranteed that Γ has an injective arc length parameterization $\gamma : [0, \ell(\Gamma)] \to \Gamma$ by definition. It is a vital fact that we also have access to an arc length parameterization of Σ. We deduce the existence of this map as a consequence of the more general results on parameterization of finite-length continua in metric spaces carried out by Alberti and Ottolini [AO17].

Let X be a metric space and $I = [a, b] \subseteq \mathbb{R}$ be a closed interval. Following Alberti and Ottolini, for a continuous map $\gamma : I \to X$ (often referred to as a *path*) and a point $x \in X$, define the *multiplicity* of γ at x as

$$m(\gamma, x) := \#(\gamma^{-1}(x))$$

where for any set A, $\#A$ denotes the cardinality of A. We define the *length* of γ as

$$\ell(\gamma) = \ell(\gamma, I) := \sup \left\{ \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) : n \geq 0,\ t_0 < t_1 < \ldots < t_n,\ t_j \in I \text{ for all } j \right\}. \tag{2.14}$$

Additionally, γ has *constant speed* if there exists a finite constant c such that

$$\ell(\gamma, [t_0, t_1]) = c(t_1 - t_0) \text{ for every } [t_0, t_1] \subseteq I.$$

We will refer to γ as an *arc length* parameterization if γ has constant speed with $c = 1$. We will only consider constant speed parameterizations, and given a fixed parameterization $\gamma : I \to \Sigma$ with constant speed c, we define a finite Borel measure ℓ supported on Σ by

$$d\ell := c\gamma_*(dt)$$

where γ_* denotes the pushforward measure so that $\ell(A) = c \int_{\gamma^{-1}(A)} dt$. Alberti and Ottolini prove the following general parameterization result:

Theorem 2.1.18 ([AO17] Theorem 4.4)**.** *Let X be a connected, compact metric space with $\mathcal{H}^1(X) < \infty$. Then there exists a path $\gamma : [0, 1] \to X$ with the following properties:*

(i) γ is closed, Lipschitz, surjective, and has degree zero;

(ii) $m(\gamma, x) = 2$ for $\mathcal{H}^1$-a.e. $x \in X$, and $\ell(\gamma) = 2\mathcal{H}^1(X)$; and,

(iii) γ has constant speed, equal to $2\mathcal{H}^1(X)$.

See [AO17] Section 4.1 for the definition of degree zero. (Essentially, the path passes through almost every point thee same number of times in one direction as in the opposite direction.) Fix a multiresolution family $\mathscr{H}$ for Σ, and let γ be a constant speed parameterization of Σ, the existence of which is guaranteed by Theorem 2.1.18. We will use γ to properly study the geometry of Σ inside of the balls of $\mathscr{H}$.

Definition 2.1.4 (Arcs)**.** We define an *arc* $\tau := \gamma|_{[a,b]}$ to be the restriction of γ to a subinterval $[a,b] \subseteq I$. Given a ball $Q \subseteq \ell_2$, we define the family of arcs of Σ inside Q as

$$\Lambda(Q) := \{\gamma|_{[a,b]} : [a,b] \subseteq [0,1],\ [a,b] \text{ is a connected component of } \gamma^{-1}(2Q \cap \Sigma)\}.$$

These are arcs inside $2Q$ which intersect Q in the style of [BM23a]. Fix an arc τ as above. Further following [BM23a], we use bold terms to refer to operators acting on arcs and define

$$\mathsf{Domain}(\tau) := [a,b],\ \mathsf{Image}(\tau) := \tau(\mathsf{Domain}(\tau)),\ \mathsf{Diam}(\tau) := \operatorname{diam}(\mathsf{Image}(\tau)),$$

$$\mathsf{Edge}(\tau) := [\tau(a), \tau(b)],\ \mathsf{Line}(\tau) := \{\tau(a) + t(\tau(b) - \tau(a)) : t \in \mathbb{R}\},\ \mathsf{Crd}(\tau) := |\tau(a) - \tau(b)|,$$
$$\mathsf{Start}(\tau) := \tau(a),\ \mathsf{End}(\tau) := \tau(b).$$

where $[\tau(a), \tau(b)] \subseteq \ell_2$ is the line segment connecting the endpoints of τ, and hence $\mathsf{Line}(\tau)$ is the line passing through the endpoints of τ. We will often use the term *arc* to refer to both τ and $\mathsf{Image}(\tau)$, but the referent should be clear from context. If $\xi = \gamma|_{[c,d]}$ for $[c,d] \subseteq [a,b]$, then we will often call ξ a *subarc* of τ. For two general arcs ξ and τ we define shorthand notation by defining (in the sense of logical formulas)

$$(\xi \subseteq \tau) := (\mathsf{Domain}(\xi) \subseteq \mathsf{Domain}(\tau)) \text{ and } (x \in \tau) := (x \in \mathsf{Image}(\tau)),$$

and we define

$$\xi \cap \tau := \gamma|_{\mathsf{Domain}(\tau) \cap \mathsf{Domain}(\xi)} \text{ and } \xi \cup \tau := \gamma|_{\mathsf{Domain}(\tau) \cup \mathsf{Domain}(\xi)}.$$

If $E \subseteq \ell_2$ and μ is a Borel measure on ℓ_2 then we set

$$\tau \cap E := \mathsf{Image}(\tau) \cap E \text{ and } \mu(\tau) := \mu(\mathsf{Image}(\tau)).$$

Definition 2.1.5 (Almost flat and non-flat arcs)**.** In order to measure the flatness of an arc, we define the *arc beta number*

$$\tilde{\beta}(\tau) := \sup_{x \in \mathsf{Image}(\tau)} \frac{\operatorname{dist}(x, \mathsf{Edge}(\tau))}{\mathsf{Diam}(\tau)}.$$

We also set

$$\mathsf{Max}(\tau) := \{y \in \mathsf{Image}(\tau) : \tilde{\beta}(\tau)\,\mathsf{Diam}(\tau) = \operatorname{dist}(y, \mathsf{Edge}(\tau))\} \neq \varnothing,$$
$$\mathsf{Drift}(\tau) := \tilde{\beta}(\tau)\,\mathsf{Diam}(\tau) = \operatorname{dist}(y, \mathsf{Edge}(\tau)) \text{ for any } y \in \mathsf{Max}(\tau).$$

Fix a constant $\epsilon_2 > 0$ whose specific value will be set in Section 2.1.5 Given a ball $Q \in \mathscr{H}$, we define the set of *almost flat arcs* for Q as

$$S(Q) := \{\tau \in \Lambda(Q) : \tilde{\beta}(\tau) \leq \epsilon_2 \beta_\Sigma(Q)\}$$

and refer to any arc $\tau \in S(Q)$ as an *almost flat arc*. We will commonly refer to $\Lambda(Q) \setminus S(Q)$ as the set of *non-flat arcs*. For any collection of arcs $\mathscr{T}$, we define

$$\beta_{\mathscr{T}}(Q) := \beta_{\cup_{\tau \in \mathscr{T}} \mathsf{Image}(\tau)}(Q),$$

and for a single arc τ we set $\beta_\tau(Q) := \beta_{\mathsf{Image}(\tau)}(Q)$.

Considering different configurations of almost flat and non-flat arcs will give us useful ways of classifying balls $Q \in \mathscr{H}$. For $\eta \in S(Q)$, we get that $\mathsf{Image}(\eta)$ lies *very* close to $\mathsf{Edge}(\eta)$ on the scale of $\mathsf{Diam}(\eta) \simeq \mathrm{diam}(Q)$, so that in many cases one can think of η as a line segment. The parameter ϵ_2 will be fixed small enough such that this approximation will work well on all small scales relative to $\mathrm{diam}(Q)$ which are relevant to our almost flat analysis in Section 2.4.

2.1.4.2 The division of $\mathscr{H}$

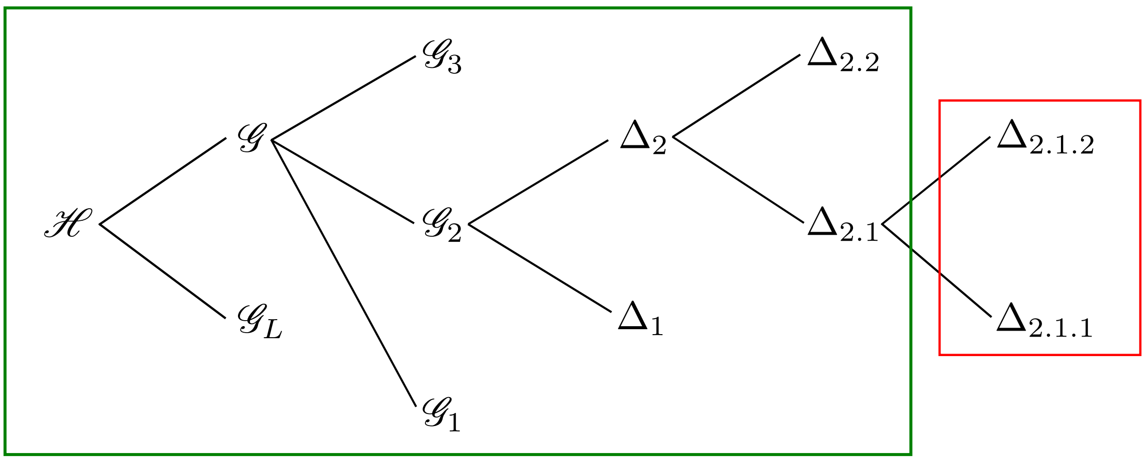

Figure 2.2: A tree denoting the subfamilies of the multiresolution family $\mathscr{H}$. Those in the green rectangle were considered by Schul in his proof of Theorem 2.1.8. The in the red rectangle are new introductions made in the proof of Theorem A

.

We begin classifying balls of $\mathscr{H}$ based on their geometry by splitting off the large balls and balls with $\beta_\Sigma(Q) = 0$: Define

$$\mathscr{G}_L := \{Q \in \mathscr{H} : \Gamma \cap (H \setminus 12Q) = \varnothing \text{ or } \beta_\Sigma(Q) = 0\},$$
$$\mathscr{G} := \mathscr{H} \setminus \mathscr{G}_L.$$

Next, we extend the family $\mathscr{G}$ by considering $\mathscr{G}^\lambda = \{\lambda Q \ : \ Q \in \mathscr{G}\}$ for $\lambda \in \{1, 2, 8, 12\}$ together. For any ball $Q \in \mathscr{G}^1 \cup \mathscr{G}^2 \cup \mathscr{G}^8 \cup \mathscr{G}^{12}$, choose a subarc $\gamma_Q \ni x_Q$ such that we always have $\gamma_Q \subseteq \gamma_{2Q} \subseteq \gamma_{8Q} \subseteq \gamma_{12Q}$. Let $\epsilon_1 > 0$ be small (to be fixed in Section 2.1.5) and partition $\mathscr{G}^\lambda = \mathscr{G}_1^\lambda \cup \mathscr{G}_2^\lambda \cup \mathscr{G}_3^\lambda$ where

$$\mathscr{G}_1^\lambda = \{Q \in \mathscr{G} : \tilde{\beta}(\gamma_{\lambda Q}) > \epsilon_2 \beta(\lambda Q)\},$$
$$\mathscr{G}_2^\lambda = \{Q \in \mathscr{G} : \tilde{\beta}(\gamma_{\lambda Q}) \le \epsilon_2 \beta_\Sigma(\lambda Q); \beta_{S_{\lambda Q}}(\lambda Q) > \epsilon_1 \beta_\Sigma(Q)\},$$
$$\mathscr{G}_3^\lambda = \{Q \in \mathscr{G} : \tilde{\beta}(\gamma_{\lambda Q}) \le \epsilon_2 \beta_\Sigma(\lambda Q); \beta_{S_{\lambda Q}}(\lambda Q) \le \epsilon_1 \beta_\Sigma(Q)\}.$$

See Figure 2.2 for examples. This decomposition is slightly different than Schul's in the style of [BM23a]. Notice that for any $Q \in \mathscr{G}^1 \cup \mathscr{G}^2 \cup \mathscr{G}^8 \cup \mathscr{G}^{12}$, either

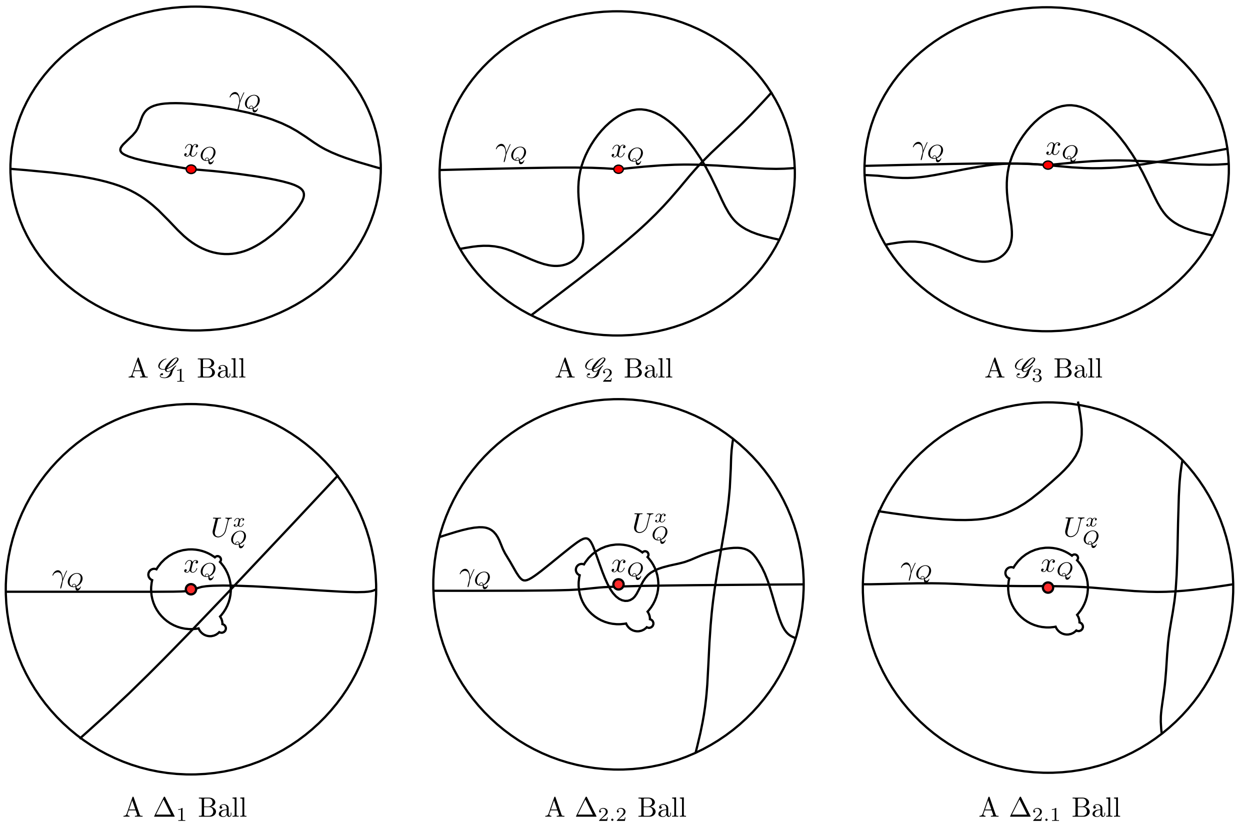

Figure 2.3: Examples of balls in $\mathscr{G}_1, \mathscr{G}_2, \mathscr{G}_3, \Delta_1, \Delta_{2.1}$ and $\Delta_{2.2}$.

(i) $Q \in \bigcup_\lambda \mathscr{G}_1^\lambda \cup \mathscr{G}_2^\lambda$, or

(ii) $Q \in \bigcap_\lambda \mathscr{G}_3^\lambda$.

Hence, it suffices to consider the collections $\mathscr{G}_1 = \cup_\lambda \mathscr{G}_1^\lambda$, $\mathscr{G}_2 = \cup_\lambda \mathscr{G}_2^\lambda$, and $\mathscr{G}_3 = \cap_\lambda \mathscr{G}_3^\lambda$, and our total family of collections of balls is now

$$\{\mathscr{G}_L, \mathscr{G}_1, \mathscr{G}_2, \mathscr{G}_3\}.$$

The collections $\mathscr{G}_L$, $\mathscr{G}_1$, and $\mathscr{G}_3$ will be handled as they are, but $\mathscr{G}_2$ needs further refinement. To describe the refinement of $\mathscr{G}_2$, we will define a ball-like set associated to each ball $Q \in \mathscr{G}$ called its "core".

Definition 2.1.6 (Cores [Sch07a]). Let $Q \in \mathscr{G}$ with $Q = B(x_Q, A2^{-n})$. For any $c \in \mathbb{R},\ 0 < c < \frac{1}{4A}$ and $J \in \mathbb{Z},\ J \geq 10$, define $U_Q^{c,J,0} := cQ$. Let $i \geq 0$ and set

$$U_Q^{c,J,i+1} := U_Q^{c,J,i} \cup \bigcup_{\substack{x_{Q'} \in X_{n+J(i+1)} \\ cQ' \cap U_Q^{c,J,i} \neq \varnothing}} cQ'.$$

We then define the *core* of Q with dilation factor c and scaling factor J to be

$$U_Q^{c,J} := \bigcup_{i \geq 0} U_Q^{c,J,i}.$$

For J as fixed in Section 2.1.5 and $c_0 := \frac{1}{64A}$, we define three successively larger cores for Q as

$$U_Q := U_Q^{c_0,J}, \quad U_Q^x := U_Q^{8c_0,J}, \quad U_Q^{xx} := U_Q^{16c_0,J}.$$

These are the concrete families we will work with.

The cores have nice separation and inclusion properties which will allow us to work with some families of balls more easily. These are given in the following proposition:

Proposition 2.1.19 (properties of core families, cf. [Sch07a] Lemma 3.19). *Let $J \geq 10$ and $c < \frac{1}{4A}$. Fix $1 \leq j \leq J$ and define*

$$\mathscr{Q}_j := \{Q \in \mathscr{G} : Q = B(x_Q, A2^{-n}),\ x \in X_n,\ n \equiv j \mod J\}.$$

Let $Q, Q' \in \mathscr{Q}_j$, with $Q = B(x_Q, A2^{-n}),\ Q' = B(x_{Q'}, A2^{-m})$ and corresponding cores $U_Q^c := U_Q^{c,J},\ U_{Q'}^c := U_{Q'}^{c,J}$. Then

(i) $cQ \subseteq U_Q^c \subseteq (1 + 2^{-J+2})cQ$,

(ii) If $n = m$, then either $Q = Q'$ or $\operatorname{dist}(U_Q^c, U_{Q'}^c) \geq 2^{-n-1}$, and

(iii) If $n > m$ and $U_Q^c \cap U_{Q'}^c \neq \varnothing$, then $U_{Q'}^c \subsetneq U_Q^c$.

Proof. For a proof of (i), one can apply [Sch07b] Lemma 2.16 to show that any point $y \in U_Q$ satisfies

$$\operatorname{dist}(y, U_Q) \leq c \operatorname{rad}(Q) \sum_{k=0}^{\infty} (2 \cdot 2^{-J})^k \leq (1 + 2^{-J+2}) c \operatorname{rad}(Q).$$

For (ii), notice that because $x_Q, x_{Q'} \in X_n$, we know $|x_Q - x_{Q'}| \geq 2^{-n}$. This combined with the second inclusion in (i) completes the proof. Property (iii) follows from the definition. ■

Remark 2.1.20. Property (i) above implies the following containments:

$$\begin{gathered} c_0 Q \subseteq U_Q \subseteq (1 + \epsilon_1) c_0 Q \subseteq 2c_0 Q, \\ 8c_0 Q \subseteq U_Q^x \subseteq (1 + \epsilon_1) 8c_0 Q \subseteq 9c_0 Q, \\ 16c_0 Q \subseteq U_Q^{xx} \subseteq (1 + \epsilon_1) 16c_0 Q \subseteq 17c_0 Q, \end{gathered}$$

where the penultimate containment in each line follows from the fact that $2^{-J+2} < \epsilon_1$ (see Section 2.1.5). It is usually best to think of cores as small perturbations of balls.

Using the cores, we can now refine the family $\mathscr{G}_2$. Let $C_U > 0$ and define

$$\begin{aligned} \Delta_1 &:= \{Q \in \mathscr{G}_2 : C_U \beta_{S(Q)}(U_Q^x) > \beta_{S(Q)}(Q)\}, \\ \Delta_2 &:= \mathscr{G}_2 \setminus \Delta_1. \end{aligned}$$

The constant C_U will be fixed in Section 2.1.5. We further divide Δ_2 by defining

$$\begin{aligned} \Delta_{2.2} &:= \{Q \in \Delta_2 : \exists \tau \in \Lambda(Q) \setminus S(Q),\ \tau \cap U_Q^x \neq \varnothing\}, \\ \Delta_{2.1} &:= \Delta_2 \setminus \Delta_{2.2}. \end{aligned}$$

We note here that it suffices to assume $\mathscr{G}_2 \subseteq \mathscr{G}_2^1$:

Remark 2.1.21 (Reduction of $\mathscr{G}_2$ to $\mathscr{G}_2^1$). Suppose that we have proven the inequality

$$\sum_{Q\in\mathscr{G}_1}\beta_\Sigma(Q)^2\operatorname{diam}(Q)+\sum_{Q\in\mathscr{G}_3}\beta_\Sigma(Q)^2\operatorname{diam}(Q)+\sum_{Q\in\mathscr{G}_2^1}\beta_\Sigma(Q)^2\operatorname{diam}(Q)\lesssim_A\mathcal{H}^1(\Sigma)\tag{2.15}$$

for any multiresolution family with arbitrary inflation factor $A>200$. Recall that $Q\in\mathscr{G}_2$ implies $Q\in\mathscr{G}_2^\lambda$ for some $\lambda\in\{1,2,8,12\}$. We would like to show that

$$\sum_{Q\in\mathscr{G}_2^\lambda}\beta_\Sigma(Q)^2\operatorname{diam}(Q)\lesssim_A\mathcal{H}^1(\Sigma).$$

Let $\lambda\mathscr{H}$ be the multiresolution family with the same net points as $\mathscr{H}$ but inflation factor λA. Define $\hat{Q}:=\lambda Q$. Then $\hat{Q}\in\lambda\mathscr{H}$ and $\tilde{\beta}(\gamma_{\hat{Q}})\le\epsilon_2\beta_\Sigma(\hat{Q})$. If $\beta_{S(\hat{Q})}(\hat{Q})>\epsilon_1\beta_\Sigma(\hat{Q})$, then $\hat{Q}\in\mathscr{G}_2^1(\lambda\mathscr{H})$, the subfamily $\mathscr{G}_2^1$ defined relative to the multiresolution family $\lambda\mathscr{H}$. Otherwise, $\beta_{S(\hat{Q})}(\hat{Q})\le\epsilon_2\beta_\Sigma(\hat{Q})$, implying $\hat{Q}\in\mathscr{G}_3^1(\lambda\mathscr{H})\subseteq\mathscr{G}_1(\lambda\mathscr{H})\cup\mathscr{G}_3(\lambda\mathscr{H})$. Therefore, the desired inequality follows from (2.15) applied to the multiresolution family $\lambda\mathscr{H}$. A similar argument shows that the same reduction holds for Γ where the right side of (2.15) is replaced by $\ell(\Gamma)-\operatorname{crd}(\Gamma)$.

With this remark, we are justified in assuming $\mathscr{G}_2\subseteq\mathscr{G}_2^1$ and need not worry about factors of λ in our analysis of $\mathscr{G}_2$ in Section 2.4. See examples of balls in these families in Figure 2.3. These are all of the subcollections necessary to prove (2.9), the Hilbert space necessary condition. When we restrict to the case of a Jordan arc, we will need to further divide the family $\Delta_{2.1}$. This is carried out in Section 2.4.2 (also see Figure 2.2 for a full diagram of the divisions).

2.1.5 Constants

In this section, we fix the values of constants used in the proof of Theorem A and give general descriptions of their purposes and where the values come from. We fix

$$\begin{aligned}\epsilon_1&:=10^{-10},\\ C_U&:=100A\epsilon_1^{-1},\\ J&:=-\log_2(10^{-3}\epsilon_1c_0),\\ \epsilon_3&:=(100A)^{-1}\epsilon_1^2,\\ \epsilon_2&:=\min((10^5AC_U)^{-1}\epsilon_1^2,100^{-1}c_0\epsilon_3^2).\end{aligned}$$

We first fix ϵ_1, a catch-all, small reference parameter. Next, we fix C_U, the constant used in the definition of Δ_1. It is fixed small enough here to facilitate (2.39) in the proof of Lemma 2.4.6, ensuring that $\beta_{\gamma_Q}(U_Q^x)\lesssim_{\epsilon_1}\beta_{S(Q)}(U_Q^x)$. We now fix the "jump" parameter J. This is fixed large enough so that for any balls $Q,Q'\in\mathscr{Q}_j$, the "thinned" family gotten by skipping J scales in the multiresolution family $\mathscr{H}$, $\operatorname{diam}(Q')<\operatorname{diam}(Q)$ implies

$$\operatorname{diam}(2Q')\le2^{-J+1}\operatorname{diam}(Q)\le100^{-1}\epsilon_1c_0\operatorname{diam}(2Q)<\epsilon_1\operatorname{diam}(U_Q).\tag{2.16}$$

This means any future generation ball Q' is very small even on the scale of U_Q (this is important in Section 2.4.1.3, for instance). We can also conclude from Proposition 2.1.19 that

$$c\operatorname{diam}(Q) \geq \frac{1}{1+2^{-J+2}}\operatorname{diam}(U_Q^c) \geq (1-\epsilon_1)\operatorname{diam}(U_Q^c), \text{ and}$$
$$\operatorname{diam}(U_Q^c) \leq (1+2^{-J+2})c\operatorname{diam}(Q) \leq (1+\epsilon_1)c\operatorname{diam}(Q).$$

for $c < \frac{1}{4A}$. We next fix ϵ_3, a constant introduced in Section 2.4.2 to define the families $\Delta_{2.1.1}$ and $\Delta_{2.1.2}$. This constant is fixed small in terms of ϵ_1 to facilitate the final estimate in the proof of Lemma 2.4.24. It is fixed small in terms of A to ensure that $100\epsilon_3\operatorname{diam}(Q) < \frac{c_0}{4}\operatorname{diam}(Q)$ to facilitate the neighborhood inclusion needed in the proof of Lemma 2.4.25. The constant ϵ_2 is fixed last. It is fixed small in terms of all of the previous parameters to ensure that almost flat arcs stay close to their edge segments on all of the needed scales, i.e., relative to $C_U^{-1}\epsilon_1\operatorname{diam}(U_Q)$ needed in estimates for Δ_1 and relative to $\epsilon_3\operatorname{diam}(U_Q)$ needed in estimates for $\Delta_{2.1.2}$. We have not attempted to optimize these.

2.2 Large-scale balls: $\mathscr{G}_L$

The goal of this section is to prove the following proposition:

Proposition 2.2.1 (cf. [Sch07a] Lemma 3.9, cf. [BM23a] Lemma 3.27)**.** *We have*

$$\sum_{Q\in\mathscr{G}_L}\beta_\Sigma(Q)^2\operatorname{diam}(Q) \lesssim_A \ell(\Sigma) \textit{ and } \sum_{Q\in\mathscr{G}_L}\beta_\Gamma(Q)^2\operatorname{diam}(Q) \lesssim_A \ell(\Gamma) - \operatorname{crd}(\Gamma). \tag{2.17}$$

Proof. We first prove the Σ inequality in (2.17). Since $\Sigma \subseteq 12Q$ for any $Q \in \mathscr{G}_L$, we know $\operatorname{diam}(Q) \geq \frac{\operatorname{diam}(\Sigma)}{12}$. For $k \geq 0$, define

$$\mathscr{B}_k := \left\{Q \in \mathscr{G}_L : \frac{\operatorname{diam}(\Sigma)}{12}2^k \leq \operatorname{diam}(Q) < \frac{\operatorname{diam}(\Sigma)}{12}2^{k+1}\right\}$$

and let $N_k = \#\mathscr{B}_k$. The net spacing for $Q \in \mathscr{B}_k$ must be at least $\frac{\operatorname{diam}(Q)}{2A} \geq \frac{\frac{\operatorname{diam}(\Sigma)}{12}2^k}{2A} \geq \frac{\operatorname{diam}(\Sigma)}{24A}$. Since N_k is maximal when Σ is a line segment with net points separated by distance greater than $\frac{\operatorname{diam}(\Sigma)}{24A}$ along length $\mathcal{H}^1(\Sigma)$, we get

$$N_k \leq 1 + \frac{24A\mathcal{H}^1(\Sigma)}{\operatorname{diam}(\Sigma)} \leq \frac{48A\mathcal{H}^1(\Sigma)}{\operatorname{diam}(\Sigma)}. \tag{2.18}$$

Now, to estimate beta numbers, observe that for any ball $Q \in \mathscr{B}_k$ we have the trivial bound $\beta_\Sigma(Q) \leq \frac{\operatorname{diam}(\Sigma)}{\operatorname{diam}(Q)} \leq 12\cdot 2^{-k}$ so that

$$\beta_\Sigma(Q)^2\operatorname{diam}(Q) \leq 144\cdot 2^{-2k}\frac{\operatorname{diam}(\Sigma)}{12}2^{k+1} \leq 12\operatorname{diam}(\Sigma)2^{-k+1}$$

We now put this all together:

$$\sum_{Q\in\mathscr{G}_L} \beta_\Sigma(Q)^2 \operatorname{diam}(Q) \leq \sum_{k=0}^{\infty}\sum_{Q\in\mathscr{B}_k} \beta_\Sigma(Q)^2 \operatorname{diam}(Q) \leq \sum_{k=0}^{\infty} N_k \cdot (12\operatorname{diam}(\Sigma)2^{-k+1})$$
$$\lesssim_A \mathcal{H}^1(\Sigma)\sum_{k=0}^{\infty} 2^{-k} \lesssim \mathcal{H}^1(\Sigma).$$

This completes the proof of the Σ inequality in (2.17). In order to prove the Γ inequality in (2.17), we first note that it suffices to assume

$$\ell(\Gamma) - \operatorname{crd}(\Gamma) \leq \epsilon_1 \ell(\Gamma). \tag{2.19}$$

Indeed, otherwise (2.17) would imply

$$\sum_{Q\in\mathscr{G}_L} \beta_\Gamma(Q)^2 \operatorname{diam}(Q) \lesssim_A \ell(\Gamma) < \frac{1}{\epsilon_1}(\ell(\Gamma) - \operatorname{crd}(\Gamma))$$

as desired. The only modifications we need to the above proof for this case are improved upper bounds for N_k and $\beta_\Gamma(Q)$. Our assumption (2.19) implies

$$\ell(\Gamma) - \operatorname{diam}(\Gamma) \leq \ell(\Gamma) - \operatorname{crd}(\Gamma) \leq \epsilon_1\ell(\Gamma) \implies \frac{\ell(\Gamma)}{\operatorname{diam}(\Gamma)} \leq \frac{1}{1-\epsilon_1} \leq 2$$

so that (2.18) implies

$$N_k \leq \frac{48A\ell(\Gamma)}{\operatorname{diam}(\Gamma)} \leq 96A.$$

We now give a new estimate for $\beta_\Gamma(Q)$. Assume without loss of generality that the endpoints x and z of Γ satisfy $x := 0$ and $z := \operatorname{crd}(\Gamma)e_1$ so that the chord segment of Γ lies along the e_1 axis. Define $\pi : \ell_2 \to \mathbb{R}$ to be the othogonal projection onto the e_1-axis and let $\pi^\perp : \ell_2 \to \ell_2$ be the projection onto the orthogonal subspace of the e_1-axis. Let $y \in \Gamma$ be a point satisfying

$$|\pi^\perp(y)| = \sup_{u\in\Gamma} |\pi^\perp(u)|.$$

Define $b := \pi(y)e_1$. The two triples of points x, b, y and z, b, y form right triangles with common altitude length $d := |\pi^\perp(y)| = |y - b|$. Let $a_1 := |x - b|$, $a_2 := |b - z|$ be the lengths of the bases of these triangles and let $c_1 := |x - y|$, $c_2 := |y - z|$ be the lengths of their hypotenuses (See Figure 2.1 for a picture). Applying the Pythagorean theorem to each triangle gives

$$d^2 = c_i^2 - a_i^2 = (c_i - a_i)(c_i + a_i) \leq 2\operatorname{diam}(\Gamma)(c_i - a_i)$$

for $i = 1, 2$. Summing these inequalities over i gives

$$d^2 = \operatorname{diam}(\Gamma)(c_1 + c_2 - a_1 - a_2) \leq 2\operatorname{diam}(\Gamma)(\ell(\Gamma) - \operatorname{crd}(\Gamma))$$

where we used the fact that $a_1 + a_2 \geq \operatorname{crd}(\Gamma)$ and $c_1 + c_2 \leq \ell(\Gamma)$ because Γ is a connected set containing x, y, and z. Now, if $Q \in \mathscr{B}_k$, then $\operatorname{diam}(Q) \geq \frac{\operatorname{diam}(\Gamma)}{12} 2^k$ and the definition of $\beta_\Gamma(Q)$ implies $\beta_\Gamma(Q) \leq \frac{d}{\operatorname{diam}(Q)}$ using the e_1 axis as an approximating line. This means

$$\beta_\Gamma(Q)^2 \operatorname{diam}(Q) \leq \frac{d^2}{\operatorname{diam}(Q)} \leq 2 \operatorname{diam}(\Gamma)(\ell(\Gamma) - \operatorname{crd}(\Gamma)) \cdot \frac{12 \cdot 2^{-k}}{\operatorname{diam}(\Gamma)} \leq 24(\ell(\Gamma) - \operatorname{crd}(\Gamma))2^{-k}. \tag{2.20}$$

Therefore,

$$\sum_{Q \in \mathscr{G}_L} \beta_\Gamma(Q)^2 \operatorname{diam}(Q) \leq \sum_{k=0}^{\infty} \sum_{Q \in \mathscr{B}_k} \beta_\Gamma(Q)^2 \operatorname{diam}(Q) \leq \sum_{k=0}^{\infty} N_k \cdot (24(\ell(\Gamma) - \operatorname{crd}(\Gamma))2^{-k})$$
$$\lesssim_A (\ell(\Gamma) - \operatorname{crd}(\Gamma)) \sum_{k=0}^{\infty} 2^{-k} \lesssim \ell(\Gamma) - \operatorname{crd}(\Gamma). \qquad \blacksquare$$

2.3 Non-flat arcs: $\mathscr{G}_1, \mathscr{G}_3, \Delta_{2.2}$

The goal of this section is to prove the following proposition:

Proposition 2.3.1. *Set $\mathscr{N} := \mathscr{G}_1 \cup \mathscr{G}_3 \cup \Delta_{2.2}$. We have*

$$\sum_{Q \in \mathscr{N}} \beta_\Sigma(Q)^2 \operatorname{diam}(Q) \lesssim_A \ell(\gamma) - \operatorname{crd}(\gamma). \tag{2.21}$$

In particular,

$$\sum_{Q \in \mathscr{N}} \beta_\Sigma(Q)^2 \operatorname{diam}(Q) \lesssim_A \mathcal{H}^1(\Sigma) \text{ and } \sum_{Q \in \mathscr{N}} \beta_\Gamma(Q)^2 \operatorname{diam}(Q) \lesssim_A \ell(\Gamma) - \operatorname{crd}(\Gamma). \tag{2.22}$$

Remark 2.3.2. Both inequalities in (2.22) follow from (2.21). For the first, Theorem 2.1.18 gives a parameterization γ of Σ such that $\ell(\gamma) \leq 2\mathcal{H}^1(\Sigma)$. For the second, Γ comes with an injective parameterization γ for which $\ell(\Gamma) = \ell(\gamma)$ and $\operatorname{crd}(\Gamma) = \operatorname{crd}(\gamma)$ by definition.

Recall that $\mathscr{N}$ consists of balls Q which have $\beta_\Sigma(Q) \lesssim_{\epsilon_2} \tilde{\beta}(\tau_Q)$ for some $\tau_Q \in \Lambda(Q)$. That is, their beta number is dominated by the beta-tilde number of some arc they contain. Our strategy to prove (2.22) is to construct an appropriate mapping $Q \mapsto \tau_Q$ and prove that the associated sum $\sum_{Q \in \mathscr{N}} \tilde{\beta}(\tau_Q)^2 \operatorname{diam}(Q)$ is controlled. The first subsection below develops the general method for building an appropriate mapping and proving that the associated sum is controlled, while the second subsection applies the results of the first to proving (2.22).

2.3.1 Filtration construction and properties of $\tilde{\beta}$

It turns out to be most appropriate to derive bounds for sums over $\tilde{\beta}(\tau_Q)$ by only considering certain nice families of arcs called *filtrations*.

Definition 2.3.1 (Filtrations [Oki92], [Sch07a])**.** A *filtration* of γ is a family of subarcs $\mathscr{F} = \bigcup_{n=0}^{\infty} \mathscr{F}_n$ of γ of whose constituent subfamilies $\{\mathscr{F}_n\}_{n \geq 0}$ satisfy the following:

(i) For all $\tau' \in \mathscr{F}_{n+1}$, there exists a unique $\tau \in \mathscr{F}_n$ such that $\mathsf{Domain}(\tau') \subseteq \mathsf{Domain}(\tau)$,

(ii) There exist constants $\underline{A}, A > 0,\ \rho < 1$ such that for all $n \geq 0$ and $\tau \in \mathscr{F}_n$, $\underline{A}\rho^{-n} \leq \mathsf{Diam}(\tau) \leq A\rho^{-n}$,

(iii) For all $\tau, \tau' \in \mathscr{F}_n$, either $\tau = \tau'$ or $\#(\mathsf{Domain}(\tau) \cap \mathsf{Domain}(\tau')) \leq 1$, and

(iv) $\bigcup_{\tau \in \mathscr{F}_0} \mathsf{Domain}(\tau) = \bigcup_{\tau' \in \mathscr{F}_n} \mathsf{Domain}(\tau')$.

We are interested in constructing filtrations with constituent arcs associated to subfamilies of $\mathscr{N}$ because of the following lemma:

Lemma 2.3.3. *([Oki92], cf. [Sch07a] Lemma 3.11) Let $\mathscr{F}$ be a filtration for γ. Then*

$$\sum_{\tau \in \mathscr{F}} \tilde{\beta}(\tau)^2 \,\mathsf{Diam}(\tau) \lesssim_A \ell\left(\bigcup_{\tau \in \mathscr{F}_0} \tau\right) - \sum_{\tau \in \mathscr{F}_0} \mathsf{Crd}(\tau). \tag{2.23}$$

If $\bigcup_{\tau \in \mathscr{F}_0} \tau = \gamma$, then

$$\sum_{\tau \in \mathscr{F}} \tilde{\beta}(\tau)^2 \,\mathsf{Diam}(\tau) \lesssim_A \ell(\gamma) - \mathrm{crd}(\gamma). \tag{2.24}$$

Proof. We refer the reader to the proof of Lemma 3.11 in [Sch07a] for the the proof of (2.23). In order to prove (2.24), we follow Schul's aforementioned proof to the second to last equation of page 349. Summing this equation over n, we replace the first equation on page 350 with

$$\sum_{\tau \in \mathscr{F}} \frac{d_\tau^2}{\mathsf{Diam}(\tau)} \lesssim \sup_n \sum_{\tau \in \mathscr{F}_n} \mathcal{H}^1(I_\tau) - \sum_{\tau \in \mathscr{F}_0} \mathcal{H}^1(I_\tau) \lesssim \ell\left(\bigcup_{\tau \in \mathscr{F}_0} \tau\right) - \sum_{\tau \in \mathscr{F}_0} \mathsf{Crd}(\tau).$$

Finally, replace the following occurrences of $\ell\left(\bigcup_{\tau \in \mathscr{F}_0} \tau\right)$ on page 350 with $\ell\left(\bigcup_{\tau \in \mathscr{F}_0} \tau\right) - \sum_{\tau \in \mathscr{F}_0} \mathsf{Crd}(\tau)$. The result follows from the fact that $\bigcup_{\tau \in \mathscr{F}_0} \tau = \gamma$ and $\mathrm{crd}(\gamma) \leq \sum_{\tau \in \mathscr{F}_0} \mathsf{Crd}(\tau)$ by the triangle inequality. ■

In order to apply this lemma, we must preprocess the collections of dominating arcs $\{\tau_Q\}_{Q \in \mathscr{N}}$ coming from each the families $\mathscr{G}_1, \mathscr{G}_3, \Delta_{2.2} \subseteq \mathscr{N}$ individually into a bounded number of filtrations. [Sch07a] provides Lemma 3.13 for this. However, the statement and proof of the lemma as written contain errors which must be addressed.

First, the statement of the lemma makes the following claim: There exists $c_0 > 0$ such that for any arcs $\tau \subseteq \tau'$ with $\mathsf{Diam}(\tau') \leq 2\,\mathsf{Diam}(\tau)$, we have $\tilde{\beta}(\tau') \geq c_0 \tilde{\beta}(\tau)$. In general, this is false. For example, Figure 2.4 gives two counterexamples for this claim. The problem is not an issue for the results of the paper; although the claim is not true in general, an inequality of this type does hold for the specific arc families we will use. The proof of the lemma also contains a gap which is fixed in a modified, more general version given below. Before we state the lemma, we give a definition:

Definition 2.3.2 (Augmentations)**.** Fix an arc τ. We refer to any arc $\tau' \supseteq \tau$ as a τ-*augmentation* if we can write

$$\tau' = \eta_1 \cup \tau \cup \eta_2 \tag{2.25}$$

where η_1, η_2 are arcs such that

$$\mathsf{Diam}(\eta_i) \leq \frac{1}{1000}\,\mathsf{Diam}(\tau) \text{ and } \mathsf{Domain}(\eta_i) \cap \mathsf{Domain}(\tau) \neq \varnothing. \tag{2.26}$$

This also gives $\mathsf{Diam}(\tau') \leq \left(1 + \frac{1}{100}\right) \mathsf{Diam}(\tau)$.

Lemma 2.3.4 (prefiltration lemma [BM23a])**.** *Let $\mathbb{X}$ be a metric space and let $f : [0,1] \to \overline{\Sigma}$ be a continuous parameterization of a set $\overline{\Sigma} \subset \mathbb{X}$. Assume that $\rho > 1$, $0 < \underline{A} < A < \infty$, and $J \geq 1$ is any integer such that $\rho^J > 6A/\underline{A}$. Then for every family $\mathscr{F}^0 = \bigcup_{n=n_0}^{\infty} \mathscr{F}_n^0$ of arcs in $\overline{\Sigma}$ with $\mathscr{F}_{n_0}^0 \neq \emptyset$ satisfying*

(i) bounded overlap: for every arc $\tau \in \mathscr{F}_n^0$, there exists no more than C arcs $\tau' \in \mathscr{F}_n^0$ such that $\mathsf{Domain}(\tau) \cap \mathsf{Domain}(\tau') \neq \emptyset$ for some constant C independent of τ

(ii) geometric diameters: for every arc $\tau \in \mathscr{F}_n^0$, we have $\underline{A}\rho^{-n} \leq \mathsf{Diam}(\tau) \leq A\rho^{-n}$,

we can construct $5(A/\underline{A})CJ$ or fewer filtrations $\mathscr{F}^1 = \bigcup_{n=n_1}^{\infty} \mathscr{F}_n^1$, $\mathscr{F}^2 = \bigcup_{n=n_2}^{\infty} \mathscr{F}_n^2$, $\ldots$, with starting index $n_j \in \{n_0, n_0+1, \ldots, n_0+J-1\}$ for all j and

$$\frac{1}{1000}\left(\underline{A}\rho^{(J-1)n_j}\right)\rho^{-Jn} \leq \mathsf{Diam}(\tau) < \left(1 + \frac{1}{100}\right)\left(A\rho^{(J-1)n_j}\right)\rho^{-Jn} \tag{2.27}$$

for all j, $\tau \in \mathscr{F}_n^j$, $n \geq n_j$ such that for every index $n \geq n_0$ and arc $\tau \in \mathscr{F}_n^0$, there exists $\mathscr{F}^j$ (in the list of filtrations), an index N with $n - n_j = J(N - n_j)$, and a τ-augmentation $\tau' \in \mathscr{F}_N^j$. The assignment $(n, \tau) \mapsto (\mathscr{F}^j, N, \tau')$ is injective.

Remark 2.3.5 (Changes to the statement of Lemma 2.3.4). In our statement of this lemma, we have first changed (2.27) by replacing a $\frac{1}{4}$ in the diameter lower bound with a $\frac{1}{1000}$ and by replacing a 2 in the corresponding upper bound with $1 + \frac{1}{100}$. The result of this change is that the lemma produces a τ-augmentation τ' such that $\mathsf{Diam}(\tau') \leq \left(1 + \frac{1}{100}\right) \mathsf{Diam}(\tau)$ rather than a general extension τ' such that $\mathsf{Diam}(\tau') \leq 2\,\mathsf{Diam}(\tau)$ as in the statement in [BM23a]. This improvement can be made as long as J is sufficiently large by carefully following the proof in [BM23a]. In the following paragraph, we give a sketch of how one can justify this change.

Indeed, each filtration $\mathscr{F}^j$ produced in the lemma is composed of essentially two types of arcs: τ-type arcs which are extensions of arcs in $\mathscr{F}^0$ originally passed into the lemma and σ-type arcs which are the leftover arcs in-between the τ type arcs. Each arc $\xi \in \mathscr{F}_0$ is extended to a τ-type arc by adding in a chain of arcs of geometrically decreasing diameter, beginning with diameter $\lesssim \rho^{-J} \operatorname{diam}(\xi)$. Hence, each chain can be made to have arbitrarily small diameter compared to ξ as long as we take J small enough. After doing this process to all arcs in one stage of the filtration, the remaining in-between arcs of the curve are broken up appropriately and either added to the filtration themselves or added onto the ends of the recently produced τ-type arcs. By replacing the appropriate factors of $\frac{1}{4}$ in this stage of the proof with $\frac{1}{1000}$, we can ensure each in-between arc is chopped into arcs of no diameter greater than $\left(1 + \frac{1}{100}\right)\left(A\rho^{(J-1)n_j}\right)\rho^{-Jn}$ and no less than $\frac{1}{1000}\left(\underline{A}\rho^{(J-1)n_j}\right)\rho^{-Jn}$. Hence, they satisfy the desired bounds and appending these arcs to the previously produced τ-type arcs gives the form $\tau' = \eta_1 \cup \tau \cup \eta_2$.

If we pass a family of arcs $\mathscr{F}$ into lemma 2.3.4, we receive a finite family of filtrations $\mathscr{F}_j$ such that for any arc $\tau \in \mathscr{F}$, there exists a filtration $\mathscr{F}_i$ and a unique τ-augmentation $\tau' \in \mathscr{F}_i$. In order to effectively apply the filtration estimate in Lemma 2.3.3, we must show that taking the τ' rather than τ does not ruin the arc beta number estimate $\tilde{\beta}(\tau_Q) \gtrsim_{\epsilon_2} \beta_\Sigma(Q)$. That is, we would like to show that $\tilde{\beta}(\tau') \gtrsim \tilde{\beta}(\tau)$ for any τ-augmentation τ'.

Badger and McCurdy do not need this in [BM23a] because they use $\beta_\tau(\mathsf{Image}(\tau))$ instead of $\tilde{\beta}(\tau)$ as their measure of non-flatness of arcs which requires slightly different definitions of the primary arc families. Here, we take the different approach of showing that mapping $\tau' \mapsto \tau$ given in Lemma 2.3.4 also preserves $\tilde{\beta}(\cdot)$ in the sense that there exists a constant $c > 0$ such that $\tilde{\beta}(\tau) \geq c\tilde{\beta}(\tau')$ for any arc τ' in one of the particular families $\mathscr{F}$ which we pass into Lemma 2.3.4.

Fix an arc τ and let τ' be a τ-augmentation. We begin with the simple observation that if τ has large Jones beta number, then $\tilde{\beta}(\tau') \gtrsim \tilde{\beta}(\tau)$ trivially.

Remark 2.3.6. Let $\epsilon > 0$ and suppose $\beta_\tau(\mathsf{Image}(\tau)) \geq \epsilon$. Then by definition,

$$\begin{aligned}\mathsf{Drift}(\tau') \geq \beta_{\tau'}(\mathsf{Image}(\tau'))\,\mathsf{Diam}(\tau') &\geq \frac{1}{2}\beta_{\tau'}(\mathsf{Image}(\tau))\,\mathsf{Diam}(\tau') \\ = \frac{1}{2}\beta_{\tau}(\mathsf{Image}(\tau))\,\mathsf{Diam}(\tau') &\geq \epsilon\frac{\mathsf{Diam}(\tau')}{2}.\end{aligned}$$

where the second inequality follows from the fact that $\mathsf{Image}(\tau') \supseteq \mathsf{Image}(\tau)$ and $\mathsf{Diam}(\tau) \geq \frac{1}{2}\mathsf{Diam}(\tau')$. Hence, $\tilde{\beta}(\tau') \geq \frac{\epsilon}{2} \geq \frac{\epsilon}{2}\tilde{\beta}(\tau)$.

This remark means that we can fix a small constant $\epsilon \gtrsim_A 1$ and achieve $\tilde{\beta}(\tau') \gtrsim_\epsilon \tilde{\beta}(\tau)$ whenever τ satisfies $\beta_\tau(\mathsf{Image}(\tau)) \geq \epsilon$. It turns out that any remaining arc not covered by this case which we will need to pass into Lemma 2.3.4 will be a member of $\Lambda(Q)$ for some Q, meaning its endpoints lie in $\partial(2Q)$ and its image has nonempty intersection with Q. This geometric information is enough to conclude the desired bound.

Lemma 2.3.7. *There exists $c_1 > 0$ such that for all $Q \in \mathscr{G}$ and $\tau \in \Lambda(Q)$, any τ-augmentation τ' satisfies*

$$\tilde{\beta}(\tau') \geq c_1\tilde{\beta}(\tau). \tag{2.28}$$

Our goal for the rest of this section is to prove Lemma 2.3.7. We begin by distinguishing between *tall* and *wide* arcs.

Definition 2.3.3 (Tall and wide arcs)**.** Let $\tau : [a, b] \to \Sigma$ be an arc. One of the following two inequalities holds:

(i) $\mathsf{Crd}(\tau) < 100\,\mathsf{Drift}(\tau)$, or

(ii) $\mathsf{Crd}(\tau) \geq 100\,\mathsf{Drift}(\tau)$

If τ satisfies (i), then we call τ *tall.* If τ instead satisfies (ii), then we call τ *wide.* Tall arcs are allowed to drift very far from the line segment $\mathsf{Edge}(\tau)$ while wide arcs stay relatively close. Figure 2.4 gives an example of each type.

Lemma 2.3.8. *Suppose τ is tall. Then $\tilde{\beta}(\tau') \geq \frac{1}{4}\tilde{\beta}(\tau)$.*

Proof. Let $x, y \in \mathsf{Image}(\tau)$ such that $\mathsf{Diam}(\tau) = \operatorname{dist}(x, y)$. Then

$$\begin{aligned}\mathsf{Diam}(\tau) &\leq \operatorname{dist}(x, \mathsf{Edge}(\tau)) + \operatorname{diam}(\mathsf{Edge}(\tau)) + \operatorname{dist}(y, \mathsf{Edge}(\tau)) \\ &\leq 2\,\mathsf{Drift}(\tau) + \mathsf{Crd}(\tau) \leq 102\,\mathsf{Drift}(\tau).\end{aligned}$$

The augmentation τ' has the form $\tau' = \eta_1 \cup \tau \cup \eta_2$ where $\mathsf{Diam}(\eta_i) \leq \frac{1}{1000}\mathsf{Diam}(\tau) \leq \frac{102}{1000}\mathsf{Drift}(\tau)$. Hence, $\mathsf{Drift}(\tau') \geq \mathsf{Drift}(\tau) - \mathsf{Diam}(\eta_1) \geq \frac{1}{2}\mathsf{Drift}(\tau)$ and $\mathsf{Diam}(\tau') \leq \mathsf{Diam}(\tau) + \mathsf{Diam}(\eta_1) + \mathsf{Diam}(\eta_2) \leq 2\,\mathsf{Diam}(\tau)$. Therefore,

$$\tilde{\beta}(\tau') = \frac{\mathsf{Drift}(\tau')}{\mathsf{Diam}(\tau')} \geq \frac{1}{4}\frac{\mathsf{Drift}(\tau)}{\mathsf{Diam}(\tau)} = \frac{1}{4}\tilde{\beta}(\tau). \qquad \blacksquare$$

Hence, tall arcs extended via Lemma 2.3.4 satisfy (2.28) with $c_1 = \frac{1}{4}$. With Lemma 2.3.8, we now only need a way of proving (2.28) for a wide arc $\tau \in \Lambda(Q)$. The basic idea is as follows. The facts that τ is wide and $\mathsf{Image}(\tau) \cap Q \neq \varnothing$ mean that $\mathsf{Edge}(\tau)$ must have nonempty intersection with $\frac{3}{2}Q$. It suffices to show that there exists $x \in \mathsf{Image}(\tau)$ such that a fixed fraction of the value of $\operatorname{dist}(x, \mathsf{Edge}(\tau))$ comes from the direction perpendicular to $\mathsf{Edge}(\tau)$ rather than the direction parallel. This is proven in the following lemma:

Lemma 2.3.9. *Suppose τ is a wide arc, and there exists $\alpha < 1$ and $x \in \mathsf{Image}(\tau)$ such that $\operatorname{dist}(x, \mathsf{Line}(\tau)) \geq \alpha\,\mathsf{Drift}(\tau)$. Then $\tilde{\beta}(\tau') \geq \frac{\alpha}{2000}\tilde{\beta}(\tau)$.*

Proof. Define $B_a := B\left(\mathsf{Start}(\tau), \frac{\alpha}{1000}\mathsf{Drift}(\tau)\right)$ and $B_b := B\left(\mathsf{End}(\tau), \frac{\alpha}{1000}\mathsf{Drift}(\tau)\right)$. Suppose first that either $\mathsf{Edge}(\tau') \cap B_a = \varnothing$ or $\mathsf{Edge}(\tau') \cap B_b = \varnothing$. Assume without loss of generality that the latter holds. Then $\mathsf{End}(\tau) \in \mathsf{Image}(\tau')$ so that $\mathsf{Drift}(\tau') \geq \operatorname{dist}(\mathsf{End}(\tau), \mathsf{Edge}(\tau')) \geq \frac{\alpha}{1000}\mathsf{Drift}(\tau)$. This implies

$$\tilde{\beta}(\tau') = \frac{\mathsf{Drift}(\tau')}{\mathsf{Diam}(\tau')} \geq \frac{\alpha}{2000}\frac{\mathsf{Drift}(\tau)}{\mathsf{Diam}(\tau)} = \frac{\alpha}{2000}\tilde{\beta}(\tau).$$

Now, suppose that $\mathsf{Edge}(\tau')$ has nonempty intersection with both B_a and B_b. Assume without loss of generality that $\mathsf{Line}(\tau)$ is the e_1-axis. Because τ is wide, $\mathsf{Drift}(\tau) \leq \frac{1}{100}\mathsf{Crd}(\tau)$ so that $B_a \cap B_b = \varnothing$ and $\mathsf{Edge}(\tau)$ hits both ends of the cylinder of length $\mathsf{Crd}(\tau) - 2\frac{\alpha}{1000}\mathsf{Drift}(\tau) \geq \frac{99}{100}\mathsf{Crd}(\tau)$ and radius $\frac{\alpha}{1000}\mathsf{Drift}(\tau)$ whose central axis is collinear with the e_1-axis. Let θ be the angle between $\mathsf{Edge}(\tau')$ and $\mathsf{Edge}(\tau)$. (We measure this by translating $\mathsf{Edge}(\tau')$ to intersect $\mathsf{Edge}(\tau)$, then measuring the angle in the plane containing $\mathsf{Edge}(\tau)$ and $\mathsf{Edge}(\tau')$.) We conclude

$$\tan(\theta) \leq \frac{2\frac{\alpha}{1000}\mathsf{Drift}(\tau)}{\frac{99}{100}\mathsf{Crd}(\tau)} \leq \frac{\alpha}{100}\frac{\mathsf{Drift}(\tau)}{\mathsf{Crd}(\tau)} \leq 10^{-4}\alpha. \tag{2.29}$$

We will derive a lower bound for $\mathsf{Drift}(\tau')$ in terms of $\mathsf{Drift}(\tau)$ by showing that $\mathsf{Edge}(\tau')$ remains much closer to $\mathsf{Line}(\tau)$ than the point x is. We know $\mathsf{Diam}(\tau) \leq 2\,\mathsf{Drift}(\tau) + \mathsf{Crd}(\tau) \leq 2\,\mathsf{Crd}(\tau)$ so that the fact that τ' is a τ-augmentation means that $\mathsf{End}(\tau') \in B(\mathsf{End}(\tau), \frac{1}{1000}\mathsf{Diam}(\tau)) \subseteq B(\mathsf{End}(\tau), \frac{1}{500}\mathsf{Crd}(\tau))$. A similar result holds for $\mathsf{Start}(\tau')$. A very rough estimate gives

$$\sup_{y \in \mathsf{Edge}(\tau')} \operatorname{dist}(y, \mathsf{Line}(\tau)) \leq \frac{\alpha}{1000}\mathsf{Drift}(\tau) + 2\,\mathsf{Crd}(\tau)\tan(\theta) \leq \frac{\alpha}{25}\mathsf{Drift}(\tau).$$

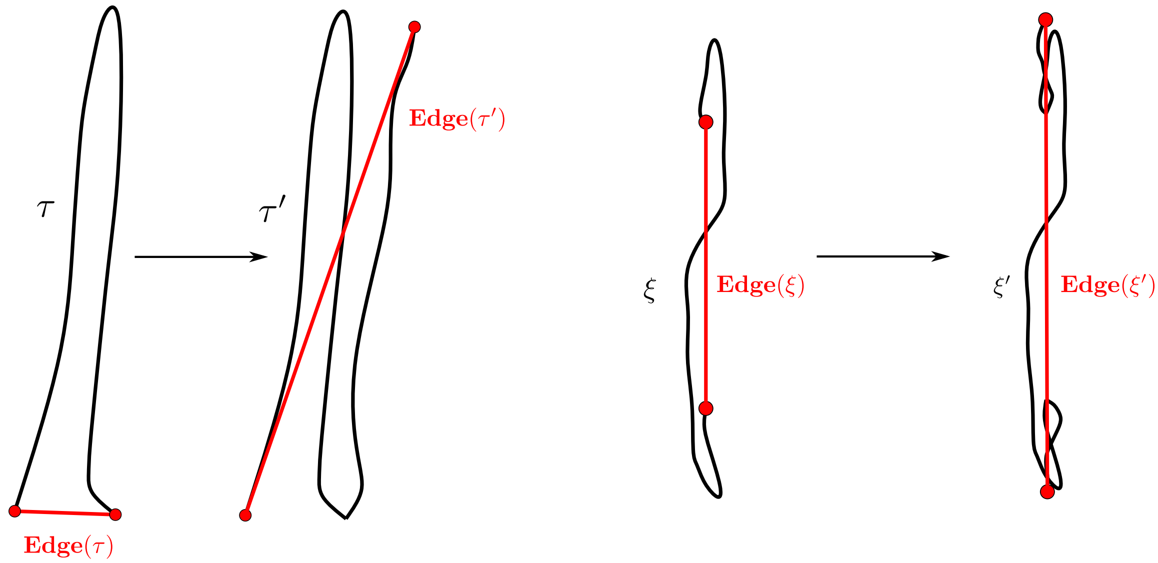

Figure 2.4: The arc τ is an example of a tall arc while ξ is an example of a wide arc (neither are drawn to scale, but we hope the ideas are clear). Both of these arcs admit extensions τ' and ξ' such that $\mathsf{Diam}(\tau') \leq 2\,\mathsf{Diam}(\tau)$ and $\mathsf{Diam}(\xi') \leq 2\,\mathsf{Diam}(\xi)$, but τ' and ξ' have much smaller $\tilde{\beta}$ then τ and ξ. Arcs of the first type are excluded in our analysis by enforcing τ-augmentations to extend only a small distance from the endpoints of τ, while Lemma 2.3.9 gives conditions for excluding arcs of the second type. (Roughly speaking, if $\xi \in \Lambda(Q)$, then it is not allowed to extend outwards in the direction parallel to its chord line outside of the ball $2Q$.)

We conclude

$$\begin{aligned}\mathsf{Drift}(\tau') \geq \mathrm{dist}(x, \mathsf{Edge}(\tau')) &\geq \mathrm{dist}(x, \mathsf{Line}(\tau)) - \sup_{y\in\mathsf{Edge}(\tau')} \mathrm{dist}(y, \mathsf{Line}(\tau)) \\ &\geq \alpha\,\mathsf{Drift}(\tau) - \frac{\alpha}{25}\mathsf{Drift}(\tau) \geq \frac{\alpha}{2}\mathsf{Drift}(\tau).\end{aligned}$$

Therefore, $\tilde{\beta}(\tau') \geq \frac{\alpha}{4}\tilde{\beta}(\tau)$. ■

Remark 2.3.10. One can derive the existence of a point x as in Lemma 2.3.9 by showing that τ lies outside half cones centered at $\mathsf{Start}(\tau)$ and $\mathsf{End}(\tau)$ pointing away from $\mathsf{Edge}(\tau)$ of aperture 2θ such that $\tan(\theta) \geq \alpha$. Indeed, then every point $y \in \mathsf{Image}(\tau)$ satisfies $\mathrm{dist}(y, \mathsf{Line}(\tau)) \geq \alpha\,\mathrm{dist}(y, \mathsf{Edge}(\tau))$ so that any point $x \in \mathsf{Max}(\tau)$ satisfies $\mathrm{dist}(x, \mathsf{Line}(\tau)) \geq \alpha\,\mathsf{Drift}(\tau)$.

With this, we can now give the proof of Lemma 2.3.7.

Proof of Lemma 2.3.7. First, suppose that $\mathsf{Edge}(\tau) \cap \frac{3}{2}Q = \varnothing$. Then since $\tau \in \Lambda(Q)$, $\mathsf{Image}(\tau) \cap Q \neq \varnothing$ and we get $\mathsf{Drift}(\tau) \geq \frac{1}{2}\mathrm{rad}(Q) \geq \frac{1}{8}\mathrm{diam}(2Q) \geq \frac{1}{8}\mathsf{Crd}(\tau)$ so that τ is tall. Lemma 2.3.8 implies $\tilde{\beta}(\tau') \geq \frac{1}{4}\tilde{\beta}(\tau)$ as desired.

Now, suppose that $\mathsf{Edge}(\tau) \cap \frac{3}{2}Q \neq \varnothing$. Our goal is to apply Lemma 2.3.9. By Remark 2.3.10, it suffices to show that there is a $\theta > 0$ independent of τ such that the cone of aperture θ centered at $\mathsf{End}(\tau)$ (and $\mathsf{Start}(\tau)$) pointing away from $\mathsf{Edge}(\tau)$ lies entirely outside the ball

$2Q$. Intuitively, this is true because the fact that $\mathsf{Line}(\tau) \cap \frac{3}{2}Q \neq \varnothing$ implies that every line in the tangent plane to $\partial(2Q)$ at $\mathsf{End}(\tau)$ makes large angle with $\mathsf{Line}(\tau)$. We supply the full details below.

Let P be any two-dimensional affine plane containing $\mathsf{Edge}(\tau)$ and assume without loss of generality that $2Q = B(0,1)$, $\mathsf{Line}(\tau) = \{de_1 + te_2 \ : \ t \in \mathbb{R}\}$ for some $d < \frac{3}{4}$, and $P = \{de_1 + te_2 + sv \ : \ t, s \in \mathbb{R},\ |v| = 1,\ v_2 = 0\}$. First, we show that $\mathsf{Edge}(\tau)$ also intersects a central ball in the disk $P \cap 2Q$.

Claim : $\mathsf{Edge}(\tau) \cap \frac{3}{4}(P \cap 2Q) \neq \varnothing$.

Proof : $P \cap B(0,1)$ is a disk whose boundary has points satisfying the equation

$$(d + sv_1)^2 + t^2 + s^2(1 - v_1^2) = 1 \iff (s + dv_1)^2 + t^2 = 1 - d^2(1 - v_1^2).$$

This is a circle with radius $\sqrt{1 - d^2(1 - v_1^2)}$ and center $(s,t) = (-dv_1, 0)$ which corresponds to the point $de_1 - dv_1 v$. We want to show that $de_1 \in \frac{3}{4}(P \cap B(0,1))$. But $|de_1 - (de_1 - dv_1 v)| = |dv_1| \leq d \leq \frac{3}{4}$, as desired ■

Now, it suffices to assume that $\mathsf{Line}(\tau) = \{de_1 + te_2 \ : \ t \in \mathbb{R}\} \subseteq \mathbb{R}^2$ and to prove that there is θ such that the angle between $\mathsf{Line}(\tau)$ and any line tangent to $\partial B(0,1) \subseteq \mathbb{R}^2$ at $x \in \mathsf{Edge}(\tau) \cap \partial B(0,1) = \{(d, \sqrt{1-d^2}),\ (d, -\sqrt{1-d^2})\}$ makes angle greater than θ. But by implicit differentiation of the equation $x^2 + y^2 = 1$, we see that

$$\left.\frac{dy}{dx}\right|_{(x,y)=(d,-\sqrt{1-d^2})} = -\left.\frac{x}{y}\right|_{(x,y)=(d,-\sqrt{1-d^2})} = \frac{\sqrt{1-d^2}}{d} \geq \frac{\sqrt{7}}{3}.$$

Hence, we can take θ such that $\arctan\theta < \frac{\sqrt{7}}{3} < 2$. We apply Lemma 2.3.9 with $\alpha = \frac{1}{2}$ to get $\tilde{\beta}(\tau') \geq \frac{1}{4000}\tilde{\beta}(\tau)$. This proves we can take $c_1 = \frac{1}{4000}$. ■

2.3.2 Bounds on the $\mathscr{G}_3, \mathscr{G}_1$, and $\Delta_{2.2}$ sums

In this subsection, we use the results from the previous subsection to prove Proposition 2.3.1. The proofs are mostly adaptations of those for the corresponding lemmas in [Sch07a].

The following three proofs share the same structure, each proving the desired bound for a particular family $\mathscr{C} \in \{\mathscr{G}_3, \mathscr{G}_1, \Delta_{2.2}\}$. In each case, we define a mapping from Q to some associated arc τ_Q. We then show that the collection $\{\tau_Q\}_{Q \in \mathscr{C}}$ satisfies the geometric diameters and bounded overlap properties necessary to apply Lemma 2.3.4. This gives a bounded number of filtrations $\mathscr{F}_{\mathscr{C}}^j$ such that each τ_Q has a τ_Q-augmentation τ_Q' as in the conclusion of Lemma 2.3.4. The desired bound then follows from applying Lemma 2.3.3 to each of the filtrations as long as $\beta_\Sigma(Q) \lesssim \tilde{\beta}(\tau_Q')$. This $\tilde{\beta}$ inequality is achieved by either the fact that τ_Q uniformly has $\beta_{\tau_Q}(\mathsf{Image}(\tau_Q)) > \epsilon$ or by showing that τ_Q satisfies the hypotheses of Lemma 2.3.7.

Proposition 2.3.11 (cf. [Sch07a] Lemma 3.16)**.**

$$\sum_{Q \in \mathscr{G}_3} \beta_\Sigma(Q)^2 \operatorname{diam}(Q) \lesssim_{J,A,\epsilon_2} \mathscr{H}^1(\Sigma) \ \textit{and} \ \sum_{Q \in \mathscr{G}_3} \beta_\Gamma(Q)^2 \operatorname{diam}(Q) \lesssim_{J,A,\epsilon_2} \ell(\Gamma) - \operatorname{crd}(\Gamma).$$

Proof. We begin by defining a new family $\mathscr{E} \subseteq \mathscr{G}_3$ and proving the claim for $\mathscr{E} \cap \mathscr{G}_3$ in place of $\mathscr{G}_3$. Define

$$\mathscr{E} := \Big\{ Q \in \mathscr{G}_3 : \exists \eta \subseteq \gamma_{12Q},\ \mathsf{Diam}(\eta) \geq \operatorname{diam}(Q) \text{ and } \tilde{\beta}(\eta') \geq 10^{-6}\epsilon_2 \beta_\Sigma(Q) \\ \text{for all } \eta' \supseteq \eta \text{ with } \mathsf{Diam}(\eta) \leq \mathsf{Diam}(\eta') \leq \left(1 + \frac{1}{100}\right) \mathsf{Diam}(\eta) \Big\}.$$

We will build an appropriate mapping $Q \mapsto \tau_Q \subseteq \gamma_{12Q}$ to pass into Lemma 2.3.4. We will then apply Lemma 2.3.3 to conclude the result.

Now, for any $Q \in \mathscr{E}$ we have the existence of an arc $\eta_Q \subseteq \gamma_{12Q}$ with $\operatorname{diam}(Q) \leq \mathsf{Diam}(\eta_Q) \leq 24 \operatorname{diam}(Q)$ and $\tilde{\beta}(\eta_Q) \geq 10^{-6}\epsilon_2\beta_\Sigma(Q)$. We define $\tau_Q := \eta_Q$.

In order to apply Lemma 2.3.4, we must verify that the family $\{\tau_Q\}_{Q \in \mathscr{E}}$ has geometric diameters and bounded overlap. The diameter requirement is satisfied by definition, so we only need to prove that $\mathscr{E}_Q := \{R \in \mathscr{E} : \operatorname{diam}(R) = \operatorname{diam}(Q),\ \tau_R \cap \tau_Q \neq \varnothing\}$ satisfies $\#(\mathscr{E}_Q) \leq C$ for some C independent of Q. Using the parameterization γ, we can put a total order on balls with diameter equal to $\operatorname{diam}(Q)$ by setting $R < Q$ if and only if $\gamma_R^{-1}(x_R) < \gamma_Q^{-1}(x_Q)$. Because X_n is finite, there exist balls $R_1, R_2 \in \mathscr{E}_Q$ which are respectively maximal and minimal in $\mathscr{E}_Q$ with respect to this ordering. By definition, any $R \in \mathscr{E}_Q$ must satisfy $x_R \in \gamma_{12R_1} \cup \gamma_{12Q} \cup \gamma_{12R_2}$. But, since $\tilde{\beta}(\gamma_{12Q'}) \leq \epsilon_2$ for all $Q' \in \mathscr{E}$, the set $\mathsf{Image}(\gamma_{12Q'})$ is contained in a cylinder of width at most $\epsilon_2(24 \operatorname{diam}(Q'))$ and length at most $24 \operatorname{diam}(Q')$. Since net points on the scale of Q' must be separated by distance at least $\frac{\operatorname{diam}(Q')}{2A}$, the net points must be separated by at least distance $\frac{\operatorname{diam}(Q')}{4A}$ along the axis of the cylinder because $\epsilon_2 \ll A^{-1}$. This means the number of net points on each of $\gamma_{12Q'}, \gamma_{12R_1}, \gamma_{12R_2}$ is less than $24 \operatorname{diam}(Q') \cdot \frac{\operatorname{diam}(Q')}{4A} + 1 \leq 100A$ so that $\#(\mathscr{E}_Q) \leq 300A$ as desired.

This verifies the geometric diameter and bounded overlap conditions for $\{\tau_Q\}_{Q\in\mathscr{E}}$. We apply Lemma 2.3.4 to receive a bounded number of filtrations $\mathscr{F}^j_{\mathscr{E}},\ j \in J_{\mathscr{E}}$ such that for any τ_Q, there exists a τ_Q-augmentation $\tau'_Q \in \mathscr{F}^j_{\mathscr{E}}$ for some j. Therefore, the definition of $\mathscr{E}$ also implies that $\tilde{\beta}(\tau'_Q) \geq 10^{-6}\epsilon_2\beta_\Sigma(Q)$. Therefore, we have

$$\sum_{Q \in \mathscr{G}_3 \cap \mathscr{E}} \beta_\Sigma(Q)^2 \operatorname{diam}(Q) \lesssim_{\epsilon_2} \sum_{j \in J} \sum_{\tau \in \mathscr{F}^j_{\mathscr{E}}} \tilde{\beta}(\tau)^2 \, \mathsf{Diam}(\tau) \lesssim_{J,A} \ell(\gamma) - \operatorname{crd}(\gamma)$$

using Lemma 2.3.3. This proves the desired inequalities for $\mathscr{G}_3 \cap \mathscr{E}$. We will now prove this for $\mathscr{G}_3 \setminus \mathscr{E}$.

Indeed, fix $Q \in \mathscr{G}_3 \setminus \mathscr{E}$. We look to build an appropriate mapping $Q \mapsto \tau_Q$ to pass into Lemma 2.3.4. Let $x \in \Sigma \cap Q$ be such that $\operatorname{dist}(x, \bigcup_{\tau \in S(Q)} \mathsf{Image}(\tau))$ is maximal and let $\xi_Q \in \Lambda(Q) \setminus S(Q)$ be such that $x \in \mathsf{Image}(\xi_Q)$. By the definition of $\mathscr{G}_3$, $\beta_{S(Q)}(Q) \leq \epsilon_1 \beta_\Sigma(Q)$ so that

$$\beta_{S(Q) \cup \xi_Q}(Q) \geq \frac{1}{3}\beta_\Sigma(Q).$$

Indeed, otherwise $\Sigma \cap Q$ is contained in a cylinder of width

$$\frac{2}{3}\beta_\Sigma(Q) \operatorname{diam}(Q) + c\epsilon_1 \beta_\Sigma(Q) \operatorname{diam}(Q)$$

with $\epsilon_1 \ll c$ contradicting the definition of $\beta_\Sigma(Q)$. We set $\tau_Q := \xi_Q$. We now verify that the family $\{\tau_Q\}_{Q\in\mathscr{G}_3\setminus\mathscr{E}}$ has geometric diameters and bounded overlap in order to apply the pre-filtration lemma.

Since $\tau_Q \in \Lambda(Q)$, we know $\tau_Q \cap Q \neq \varnothing$ and $\tau_Q \cap H \setminus 2Q \neq \varnothing$ so that

$$2\,\mathrm{diam}(Q) \geq \mathsf{Diam}(\tau_Q) \geq \mathrm{rad}(2Q) - \mathrm{rad}(Q) = \frac{1}{2}\,\mathrm{diam}(Q).$$

In order to verify bounded overlap, set $\mathscr{G}_Q := \{R \in \mathscr{G}_3\setminus\mathscr{E} : \mathrm{diam}\, R = \mathrm{diam}\, Q,\ \tau_R \cap \tau_Q \neq \varnothing\}$ so that we want to show $\#(\mathscr{G}_Q) \leq C$ for C independent of Q. Assume first that $\beta_\Sigma(R) \leq \beta_\Sigma(Q)$. Because $\tau_R \cap \tau_Q \neq \varnothing$, we have $2Q \cap 2R \neq \varnothing$ so that $x_R \in 8Q \subseteq 12R$. Let $\gamma_{12R}|_{8Q}$ be a largest diameter subarc of γ_{12R} which is in $\Lambda(8Q)$. We want to show that $\gamma_{12R}|_{8Q} \in S_{8Q}$. This is the reason for the addition of $\mathscr{E}$. Because $R \notin \mathscr{E}$ and $\mathsf{Diam}(\gamma_{12R}|_{8Q}) \geq \mathrm{diam}\, R$, there exists some extension $\eta' \supseteq \gamma_{12R}|_{8Q}$ such that $\tilde{\beta}(\eta') < 10^{-6}\epsilon_2\beta_\Sigma(R)$. But, $x_R \in 4Q = \frac{1}{2}8Q$ implies we can apply Lemma 2.3.7 to conclude

$$\tilde{\beta}(\gamma_{12R}|_{8Q}) \leq 4000\tilde{\beta}(\eta') \leq \frac{4000}{10^6}\epsilon_2\beta_\Sigma(R) < \epsilon_2\beta_\Sigma(8Q).$$

This proves that $\gamma_{12R}|_{8Q} \in S_{8Q}$. In particular, the fact that $\beta_{S_{8Q}}(8Q) \leq \epsilon_1\beta_\Sigma(Q)$ implies that x_R is contained in a small tube around γ_{8Q}. We assumed that $\beta_\Sigma(R) \leq \beta_\Sigma(Q)$ for this, but if $\beta_\Sigma(R) > \beta_\Sigma(Q)$, then running the argument for $\gamma_{12Q}|_{8R}$ in place of $\gamma_{12R}|_{8Q}$ proves the same claim with Q and R reversed. In either case, all $R \in \mathscr{G}_Q$ are contained in a small neighborhood of the almost flat arc γ_{12Q}, proving $\#(\mathscr{G}_Q) \leq 100A$.

This verifies the geometric diameters and bounded overlap condition, so we apply Lemma 2.3.4 to get a bounded family of filtrations $\mathscr{F}^j_{\mathscr{G}_3}$, $j \in J_{\mathscr{G}_3}$ such that for each $Q \in \mathscr{G}_3 \setminus \mathscr{E}$, there exists $\tau'_Q \in \mathscr{F}^j_{\mathscr{G}_3}$ for some j which is a τ_Q-augmentation. Because $\tau_Q \in \Lambda(Q) \setminus S_Q$, we apply Lemma 2.3.7 to conclude

$$\tilde{\beta}(\tau'_Q) \geq \frac{1}{4000}\tilde{\beta}(\tau_Q) \geq \frac{\epsilon_2}{4000}\beta_\Sigma(2Q) \geq \frac{\epsilon_2}{8000}\beta_\Sigma(Q).$$

Therefore, Lemma 2.3.3 implies

$$\sum_{Q\in\mathscr{G}_3\setminus\mathscr{E}} \beta_\Sigma(Q)^2\,\mathrm{diam}(Q) \lesssim_{\epsilon_2} \sum_{j\in J_{\mathscr{G}_3}} \sum_{\tau\in\mathscr{F}^j_{\mathscr{G}_3}} \tilde{\beta}(\tau)^2\,\mathsf{Diam}(\tau) \lesssim_{J,A} \ell(\gamma) - \mathrm{crd}(\gamma). \qquad \blacksquare$$

Proposition 2.3.12 (cf. [Sch07a] Lemma 3.14)**.**

$$\sum_{Q\in\mathscr{G}_1} \beta_\Sigma(Q)^2\,\mathrm{diam}(Q) \lesssim_{J,A} \mathscr{H}^1(\Sigma) \ \textit{and}\ \sum_{Q\in\mathscr{G}_1} \beta_\Gamma(Q)^2\,\mathrm{diam}(Q) \lesssim_{J,A} \ell(\Gamma) - \mathrm{crd}(\Gamma).$$

Proof. Let us build an appropriate mapping $Q \mapsto \tau_Q$. Put $Q = B(x_Q, A\lambda 2^{-n})$. We define τ_Q in one of two ways:

(i) If $\#(\gamma_Q \cap X_n) \leq 3\lambda A$, then set $\tau_Q := \gamma_Q$.

(ii) Otherwise, let τ_Q be a subarc of γ_Q containing x_Q such that $\#(\tau_Q \cap X_n) = 3\lambda A$.

In order to apply Lemma 2.3.4, we must check the geometric diameters and bounded overlap conditions. The geometric diameters condition follows in case (i) because $\gamma_Q \in \Lambda(Q)$. It follows in case (ii) because the net point condition implies that $2^{-n} \leq \mathsf{Diam}(\tau_Q) \leq 2\,\mathrm{diam}(Q) = 4\lambda A 2^{-n}$. In either case, bounded overlap follows from a similar argument to that in the proof of Proposition 2.3.11. Because each arc is centered on a unique net point in X_n and each arc contains at most $3\lambda A$ net points, ordering the net points via the parameterization γ shows that there can be at most $6\lambda A$ net points (inclusive) between intersecting arcs τ_Q and τ_R for either arcs of type (i) or (ii) above. This proves $\#(\{R \in \mathscr{G}_1 : \mathrm{diam}(R) = \mathrm{diam}(Q),\ \tau_R \cap \tau_Q \neq \varnothing\}) \leq 12\lambda A$.

Applying Lemma 2.3.4 to the collections of type (i) and (ii) arcs above gives a family of filtrations $\mathscr{F}^j_{\mathscr{G}_1},\ j \in J_{\mathscr{G}_1}$ such that for any $Q \in \mathscr{G}_3^\lambda$, there exists a τ_Q-augmentation $\tau'_Q \in \mathscr{F}^j_{\mathscr{G}_1}$ for some j. In order to finish, we only have to check that $\tilde{\beta}(\tau'_Q) \gtrsim_A \beta_\Sigma(Q)$. For arcs of type (i), $\tau_Q = \gamma_Q \in \Lambda(Q)$ so that Lemma 2.3.7 gives the result. For arcs of type (ii), observe that $\#(\tau_Q \cap X_n) = 3\lambda A$ implies that $\beta_{\tau'_Q}(\tau'_Q) \gtrsim \beta_{\tau_Q}(\tau_Q) \gtrsim_A 1 \gtrsim_A \beta_\Sigma(Q)$. Therefore, applying Lemma 2.3.3 to this collection of filtrations gives

$$\sum_{Q \in \mathscr{G}_1} \beta_\Sigma(Q)^2 \,\mathrm{diam}(Q) \lesssim_A \sum_{j \in J_{\mathscr{G}_1}} \sum_{\tau \in \mathscr{F}^j_{\mathscr{G}_1}} \tilde{\beta}(\tau)^2 \,\mathrm{diam}(\tau) \lesssim_{J,A} \ell(\gamma) - \mathrm{crd}(\gamma). \qquad \blacksquare$$

Proposition 2.3.13 (cf. [Sch07a] Lemma 3.24)**.**

$$\sum_{Q \in \Delta_{2.2}} \beta_\Sigma(Q)^2 \,\mathrm{diam}(Q) \lesssim_{J,A} \mathcal{H}^1(\Sigma) \ \textit{and} \ \sum_{Q \in \Delta_{2.2}} \beta_\Gamma(Q)^2 \,\mathrm{diam}(Q) \lesssim_{J,A} \ell(\Gamma) - \mathrm{crd}(\Gamma).$$

Proof. Let us build an appropriate mapping $Q \mapsto \tau_Q$ as in the previous two propositions. Again, let $Q = B(x_Q, \lambda A 2^{-n})$ By definition, there exists $\xi_Q \in \Lambda(Q) \setminus S_Q$ such that $\xi_Q \cap U_Q^x \neq \varnothing$. We define τ_Q in one of two ways

(i) If $\#(\{R \in \Delta_{2.2} : \mathrm{diam}(R) = \mathrm{diam}(Q),\ \xi_Q \cap U_R^x \neq \varnothing\}) \leq 9\lambda A$, then set $\tau_Q = \xi_Q$.

(ii) Otherwise, let τ_Q be a subarc of ξ_Q such that $\tau_Q \cap U_Q^x \neq \varnothing$ and $\#(\{R \in \Delta_{2.2} : \mathrm{diam}(R) = \mathrm{diam}(Q),\ \xi_Q \cap U_R^x \neq \varnothing\}) = 9\lambda A$.

Type (i) arcs have geometric diameters since $\xi_Q \in \Lambda(Q)$. Type (ii) arcs have nonempty intersection with two distinct, disjoint cores U_Q^x and U_R^x so that $\mathsf{Diam}(\xi_Q) \geq \mathrm{dist}(\partial U_Q^x, \partial U_Q^{xx}) \gtrsim \mathrm{diam}(Q)$. To check bounded overlap, we argue almost exactly as in the corresponding part of the proof of Proposition 2.3.12. Indeed, $\tau_Q \cap U_Q^x \neq \varnothing$ so that we can order the arcs τ_Q via the parameterization γ by the ordering of $x_Q \in U_Q^x$. There can be at most $18\lambda A$ net points separating x_Q and x_R for admissible R so that $\#(\{R \in \Delta_{2.2} : \mathrm{diam}(R) = \mathrm{diam}(Q),\ \tau_R \cap \tau_Q \neq \varnothing\}) \leq 36\lambda A$.

Applying Lemma 2.3.4 gives a bounded number of filtrations $\mathscr{F}^j_\Delta$ such that each $Q \in \Delta_{2.2}$ has an associated τ_Q-augmentation τ'_Q. We only need to show that $\tilde{\beta}(\tau'_Q) \gtrsim \beta_\Sigma(Q)$. This follows for type (i) arcs by Lemma 2.3.7 and for type (ii) arcs by the fact that $\#(\{R \in \Delta_{2.2} : \mathrm{diam}(R) = \mathrm{diam}(Q),\ \xi_Q \cap U_R^x \neq \varnothing\}) = 9\lambda A$ implies $\tilde{\beta}(\tau'_Q) \gtrsim \beta_{\tau_Q}(\tau_Q) \gtrsim_A 1 \gtrsim_A \beta_\Sigma(Q)$ as in the proof of Proposition 2.3.12. The result follows by applying Lemma 2.3.3 to each filtration to get

$$\sum_{Q \in \Delta_{2.2}} \beta_\Sigma(Q)^2 \,\mathrm{diam}(Q) \lesssim_A \sum_{j \in J_{\Delta_{2.2}}} \sum_{\tau \in \mathscr{F}^j_{\Delta_{2.2}}} \tilde{\beta}(\tau)^2 \,\mathrm{diam}(\tau) \lesssim_{J,A} \ell(\gamma) - \mathrm{crd}(\gamma). \qquad \blacksquare$$

2.4 Almost flat arcs: $\Delta_1, \Delta_{2.1}$

Our goal now is to prove the following proposition:

Proposition 2.4.1. *Set $\mathscr{A} := \Delta_1 \cup \Delta_{2.1}$. We have*

$$\sum_{Q\in\mathscr{A}} \beta_\Sigma(Q)^2 \operatorname{diam}(Q) \lesssim_A \ell(\Sigma) \text{ and } \sum_{Q\in\mathscr{A}} \beta_\Gamma(Q)^2 \operatorname{diam}(Q) \lesssim_A \ell(\Gamma) - \operatorname{crd}(\Gamma). \tag{2.30}$$

Recall that $\mathscr{A} \subseteq \mathscr{G}_2$ so that for any $Q \in \mathscr{A}$, $\beta_\Sigma(Q) \leq \epsilon_1^{-1}\beta_{S(Q)}(Q)$. That is, the beta number of the union of images of almost flat arcs controls the total beta number for Q. For the purposes of estimating the beta-squared sum, we can essentially think of Σ (or Γ) inside of Q as consisting of a union of line segments (we have taken the parameter ϵ_2 sufficiently small so that this heuristic holds at all scales we will perform estimates at). In Section 2.4.1 we prove the first inequality in (2.30), finishing the proof of the Hilbert space necessary condition. In Section 2.4.2 we prove the second, finishing the proof of Theorem A.

We begin by giving some comments on the structure of almost flat arcs. Recall that an almost flat arc $\tau \in S(Q)$ satisfies the inequality

$$\tilde{\beta}(\tau) \leq \epsilon_2\beta_\Sigma(Q). \tag{2.31}$$

We interpret this as saying that τ is a *very* small perturbation of $\mathsf{Edge}(\tau)$ relative to the overall flatness of Σ inside Q. This means that τ is *bilaterally* close to $\mathsf{Edge}(\tau)$, forcing τ to be "diametrical" and giving it the crossing property we prove in Lemma 2.4.2 below. The condition (2.31) is importantly stronger than the similar inequality

$$\beta_\tau(\mathsf{Image}(\tau)) \leq \epsilon_2\beta_\Sigma(Q). \tag{2.32}$$

This condition only forces the image of τ to be *unilaterally* close to some line L relative to the overall flatness of Σ inside Q. This allows almost flat arcs which are "radial" rather than "diametrical". This is an important point at which the results here diverge from results of [BM23b] in which analogous results are proven for this weaker notion of almost flat arcs in Banach spaces.

We now record two lemmas needed in the following sections.

Definition 2.4.1 (Cylinders)**.** Let $a, b \in \ell_2$, let $s := [a, b]$ be a line segment, and let $r > 0$. We define the cylinder C of radius r around s as

$$C(s, r) := \{z \in \pi_s^{-1}(s) : \pi_s^\perp(z) \leq r\},$$

where π_s is the orthogonal projection onto the line collinear with the line segment s and $\pi_s^\perp : \ell_2 \to \ell_2$ is the projection onto the corresponding affine orthogonal hyperplane. We also allow s to be an affine line. For a segment s as above, we define the faces

$$F_a(s, r) := \{z \in C(s, r) : \pi_s(z) = a\} \text{ and } F_b(s, r) := \{z \in C(s, r) : \pi_s(z) = b\}.$$

For any $\tau \in \Lambda(Q)$, $\tau \subseteq C(\mathsf{Line}(\tau), \tilde{\beta}(\tau)\,\mathsf{Diam}(\tau)) \subseteq C(\mathsf{Line}(\tau), \tilde{\beta}(\tau) \operatorname{diam}(2Q))$..

Lemma 2.4.2 (Crossing Property)**.** *Let $Q \in \mathscr{G}$, $\tau \in \Lambda(Q)$, and $\tau' := [a_{\tau'}, b_{\tau'}] \subseteq \mathsf{Edge}(\tau) := [a_\tau, b_\tau]$. Let $\epsilon > 0$ such that $\tilde{\beta}(\tau) \leq \frac{\epsilon}{2}$. There exists an arc τ_0 such that*

(i) $\mathsf{Domain}(\tau_0) \subseteq \mathsf{Domain}(\tau) \cap \gamma^{-1}(C(\tau', \epsilon \operatorname{diam}(Q)))$, *and*

(ii) $\mathsf{Diam}(\tau_0) \geq \mathsf{Diam}(\tau')$.

Proof. Let $C := C(\mathsf{Line}(\tau'), \epsilon \operatorname{diam}(Q))$. Because $\tilde{\beta}(\tau) \leq \frac{\epsilon}{2}$ and $\mathsf{Diam}(\tau) \leq 2 \operatorname{diam}(Q)$, we know $\tau \subseteq C$. Because $\mathsf{Image}(\tau)$ is connected, $\pi_{\tau'}$ is continuous, and $a_\tau, b_\tau \in \mathsf{Edge}(\tau) \cap \tau$, there must exist $u \in \tau \cap F_{a_{\tau'}}$ and $v \in \tau \cap F_{b_{\tau'}}$. This implies that $\{t \in \mathsf{Domain}(\tau) : \tau(t) \in F_{b_{\tau'}}\} \neq \varnothing$ and $\{t \in \mathsf{Domain}(\tau) : \tau(t) \in F_{a_{\tau'}}\} \neq \varnothing$ so that we can further suppose without loss of generality that

$$0 \leq \inf\{t \in \mathsf{Domain}(\tau) : \tau(t) \in F_{a_{\tau'}}\} < \inf\{t \in \mathsf{Domain}(\tau) : \tau(t) \in F_{b_{\tau'}}\}. \tag{2.33}$$

That is, τ enters $F_{a_{\tau'}}$ before $F_{b_{\tau'}}$. We define

$$\begin{aligned} t_2 &:= \inf\{t \in \mathsf{Domain}(\tau) : \tau(t) \in F_{b_{\tau'}}\}, \\ t_1 &:= \sup\{t \in \mathsf{Domain}(\tau) : t \leq t_2,\ \tau(t) \in F_{a_{\tau'}}\}, \\ \tau_0 &:= \tau|_{[t_1, t_2]}. \end{aligned}$$

Suppose without loss of generality that $\pi_{\tau'}(a_{\tau'}) \leq \pi_{\tau'}(b_{\tau'})$. We know $\tau(t_2) \in F_{b_{\tau'}}$ by the continuity of τ. By the continuity of $\pi_{\tau'}$ and the definition of t_2, we also know that $\pi_{\tau'}(\tau(t)) \leq \pi_{\tau'}(b_{\tau'})$ for all $t \leq t_2$. On the other hand, the definition of t_1 implies that $\pi_{\tau'}(\tau(t)) \geq \pi_{\tau'}(a_{\tau'})$ for all $t_1 \leq t \leq t_2$ so that $\tau|_{[t_1,t_2]} \subseteq C(\tau', \epsilon \operatorname{diam}(Q))$. Item (i) follows. In fact, we can conclude $\tau(t_1) \in F_{a_{\tau'}}$ because the supremum in the definition of t_1 is over a non-empty set by (2.33). Item (ii) follows because $\mathsf{Diam}(\tau_0) \geq |b_{\tau'} - a_{\tau'}| = \mathsf{Diam}(\tau')$. ∎

For convenience, we also record an estimate for lower-bounding the diameter of chord segments of arcs which touch central balls inside of Q:

Lemma 2.4.3. *Let $Q = B(x_Q, R)$ be a ball and let $0 < \alpha < 1$ be such that $\alpha^2 < 1/2$. Let $\varphi' := [a_{\varphi'}, b_{\varphi'}]$ be a line segment such that $a_{\varphi'}, b_{\varphi'} \in \partial Q$ and $\varphi' \cap \alpha Q \neq \varnothing$. Then,*

$$\mathcal{H}^1([a_{\varphi'}, b_{\varphi'}]) \geq \operatorname{diam}(Q)\left(1 - 2\alpha^2\right).$$

Proof. We will first give a lower bound for the function $\sqrt{1-x}$, then apply this to a Pythagorean theorem estimate. Let $0 < x < \frac{1}{2}$ and observe that, by the generalized binomial theorem,

$$\sqrt{1-x} = \sum_{n=0}^{\infty} (-1)^n \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-n+1)}{n!} x^n \geq 1 - \sum_{n=1}^{\infty} x^n \geq 1 - x - \frac{x}{1-x} \geq 1 - 2x$$

using our assumption that $x \leq \frac{1}{2}$ in the last line. Now, let $y_{\varphi'}$ be the point in φ' closest to x_Q. Then, using the Pythagorean theorem, we get $|y_{\varphi'} - a_{\varphi'}|^2 = |a_{\varphi'} - x_Q|^2 - |x_Q - y_{\varphi'}|^2$ from which we can estimate

$$|y_{\varphi'} - a_{\varphi'}| \geq \sqrt{R^2 - (\alpha R)^2} \geq R\sqrt{1-\alpha^2} \geq R\left(1 - 2\alpha^2\right).$$

Applying the same argument to $|y_{\varphi'} - b_{\varphi'}|$, we get $\mathcal{H}^1([a_{\varphi'}, b_{\varphi'}]) = |y_{\varphi'} - b_{\varphi'}| + |y_{\varphi'} - a_{\varphi'}| \geq \operatorname{diam}(Q)(1 - 2\alpha^2)$. ∎

2.4.1 Almost flat arcs for Σ

In this section, we complete our proof of (2.9), the necessary condition in the Hilbert space traveling salesman theorem. We begin in Section 2.4.1.1 by giving a general presentation of Schul's martingale construction. In Section 2.4.1.2, we give the first application of the martingale construction by repeating Schul's proof of the beta-squared sum bound for the family Δ_1. Finally, in Section 2.4.1.3 we give a new proof of the beta-squared sum bound for the family $\Delta_{2.1}$ using Schul's martingales again, filling in the final gap in proof of the Hilbert space necessary condition in [Sch07a].

2.4.1.1 Schul's Martingale Lemma

The martingale argument relies heavily on the structure of the cores for balls constructed in Proposition 2.1.19, so we begin by giving some definitions and notation related to the families of cores. For the rest of this section, fix $0 < c < \frac{1}{4A}$ and $J \geq 10$.

Definition 2.4.2 (The tree structure of cores)**.** Fix a collection $\mathscr{L} \subseteq \mathscr{G}$. Proposition 2.1.19 gives a partition of $\mathscr{G}$ into J families $\{\mathscr{Q}_j\}_{j=1}^J$ such that cores for balls inside $\mathscr{Q}_j$ satisfy the inclusion and separation properties (i), (ii), and (iii) in Proposition 2.1.19. Defining $\mathscr{L}_j := \mathscr{L} \cap \mathscr{Q}_j$, we see that for any $Q \in \mathscr{L}_j$, either

(a) For all $Q' \in \mathscr{L}_j$ such that $U^c_{Q'} \cap U_Q \neq \varnothing$, $U^c_{Q'} \subseteq U^c_Q$, or

(b) There exists $Q' \in \mathscr{L}_j$ such that $U^c_Q \subsetneq U^c_{Q'}$.

These set inclusion properties induce a partial order on $\mathscr{L}$, giving it the structure of a forest in which the balls satisfying the first condition above are the roots of trees in the forest while the balls satisfying the second condition are descendants of some root. We denote the forest of trees (whose partial order depends on the constants c and J) by $\mathcal{T}^{c,J}_{\mathscr{L}} = \mathcal{T}^c_{\mathscr{L}} = \mathcal{T}_{\mathscr{L}}$ where we often suppress the constants when understood (in practice, we suppress J more often than c in the construction because J will be fixed once and for all while c will vary). We refer to the *root* of $T \in \mathcal{T}^{c,J}_{\mathscr{L}}$ as $Q(T)$. For each $Q \in T,\ Q \neq Q(T)$, there exists a unique minimal ball $P(Q)$ respect to the ordering of T such that $U^c_Q \subsetneq U^c_{P(Q)}$. We call $P(Q)$ the *parent* of Q. Similarly, for any $Q \in T$ we define the collection of *children* of Q in T by

$$C(Q) := \{Q' \in T : Q' \text{ is maximal in } T \text{ such that } U^c_{Q'} \subsetneq U^c_Q\}.$$

We also think of $C^1(Q) := C(Q)$ as the first generation descendants of Q. Given the set $C^n(Q)$ for some $n \geq 1$, we define the $n+1$-th generation descendants of Q as

$$C^{n+1}(Q) := \{Q'' \in C(Q') : Q' \in C^n(Q)\}.$$

Because each ball is either a root or its core is contained in the core of some root ball, we have

$$U^c_{\mathscr{L}} := \bigcup_{Q \in \mathcal{T}^c_{\mathscr{L}}} U^c_{Q(T)} = \bigcup_{Q \in \mathscr{L}} U^c_Q.$$

If $\mathscr{L} \subseteq \mathscr{Q}_j$ for some $j,\ 1 \leq j \leq J$, then the union above over trees is disjoint. Otherwise it is a union in which each point is contained in at most J constituent sets.

We now give the definition of a martingale and relevant notions from probability theory.

Definition 2.4.3 (Martingales)**.** Given a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, we define a *filtration* of $\mathcal{A}$ to be an increasing sequence $(\mathcal{F}_n)_{n\geq 0}$ of sub-σ-algebras of $\mathcal{A}$. We say that a collection of real-valued random variables $(X_n)_{n\geq 0}$ on Ω is a *martingale* with respect to $(\mathcal{F}_n)_{n\geq 0}$ if for all $n \geq 0$,

(i) X_n is $\mathcal{F}_n$-measurable,

(ii) $\mathbb{E}(X_n) < \infty$,

(iii) $\mathbb{E}(X_{n+1}|\mathcal{F}_n) = X_n$.

where $\mathbb{E}(X_{n+1}|\mathcal{F}_n)$ denotes the conditional expectation of X_{n+1} with respect to $\mathcal{F}_n$. Importantly, it is well-known that positive martingales converge pointwise almost surely. That is, if $X_n \geq 0$ for all $n \geq 0$, then there exists a positive random variable X such that $X(\omega) = \lim_{n\to\infty} X_n(\omega)$ for $\mathbb{P}$ almost all $\omega \in \Omega$. We will only consider positive martingales.

Remark 2.4.4 (Schul's martingales). Let $\mathscr{L} \subseteq \mathscr{G}$ and form the forest $\mathcal{T}^c_{\mathscr{L}}$ which gives $\mathscr{L}$ a partial order, hence a child-parent structure as defined above. For each $Q \in \mathscr{L}$, Schul constructs a martingale $(w_Q^n)_{n\geq 0}$ supported inside $U_Q^c \cap \Sigma$. We define the remainder

$$R_Q := U_Q \cap \Sigma \setminus \left(\bigcup_{Q' \in C(Q)} U^c_{Q'} \cap \Sigma \right)$$

so that

$$U^c_Q \cap \Sigma = \left(\bigcup_{Q' \in C(Q)} U^c_{Q'} \cap \Sigma \right) \cup R_Q$$

where the collection $\{U^c_{Q'} \cap \Sigma\}_{Q' \in C(Q)} \cup \{R_Q\}$ is pairwise disjoint. Applying this partitioning scheme iteratively to each $U^c_{Q'} \cap \Sigma$ in the union above, we see that for any $n \geq 0$, we can write $U^c_Q \cap \Sigma$ as the partition.

$$U^c_Q \cap \Sigma = R_Q \cup \left(\bigcup_{Q' \in C^1(Q)} R_{Q'} \right) \cup \ldots \cup \left(\bigcup_{Q' \in C^n(Q)} R_{Q'} \right) \cup \left(\bigcup_{Q' \in C^{n+1}(Q)} U^c_{Q'} \cap \Sigma \right) \tag{2.34}$$

This gives a decomposition of $U^c_Q \cap \Sigma$ into "atoms" at the $(n+1)$-th level, from which we will define a filtration by setting $\mathcal{F}_n$ to be the sigma algebra generated by $\cup_{k\leq n}\{U^c_{Q'} \cap \Sigma\}_{Q' \in C^k(Q)}$. We will form the martingale $(w^n_Q)_{n\geq 0}$ by setting w^0_Q to be constant on $U^c_Q \cap \Sigma$ and defining w^{n+1}_Q by distributing the mass that w^n_Q assigns to $U^c_{Q'} \cap \Sigma$ for any $Q' \in C^n(Q)$ onto its constituent pieces $R_{Q'} \cup \bigcup_{Q'' \in C(Q')} U^c_{Q''} \cap \Sigma = U^c_{Q'} \cap \Sigma$ with weighting factors depending on the size and number of children in $C(Q')$ and the length of the remainder $R_{Q'}$.

Lemma 2.4.5 (Martingale construction)**.** *Fix a constant $D > 0$, $c < \frac{1}{4A}$ and let $\mathscr{L} \subseteq \mathscr{G} \cap \mathscr{Q}_j$. Suppose that there exists a constant $q < 1$ such that for any $Q \in \mathscr{L}$*

$$\frac{\operatorname{diam}(U^c_Q)}{\sum_{Q' \in C(Q)} \operatorname{diam}(U^c_{Q'}) + D\ell(R_Q)} \leq q. \tag{2.35}$$

Then, there exists a collection of positive real-valued functions $\{w_Q\}_{Q\in\mathscr{L}}$ *satisfying*

(i) $\int_\Sigma w_Q d\ell = \mathrm{diam}(U_Q^c)$,

(ii) $\sum_{Q\in\mathscr{L}} w_Q(x) \le \frac{D}{1-q}\chi_{U^c_{\mathscr{L}}}(x)$ *for almost every* $x\in\Sigma$,

(iii) *supp*$(w_Q) \subseteq U_Q^c \cap \Sigma$.

Proof. We will suppress the superscript of the cores and write $U_Q = U_Q^c$. Fix $Q \in \mathscr{L}$. For any set E and function $w : \Sigma \to \mathbb{R}$, we let $w(E) = \int_E w d\ell$. Let $\mathcal{F}_n$ be the σ-algebra generated by $\cup_{k\le n}\{U_{Q'}\cap\Sigma\}_{Q'\in C^k(Q)}$. We construct the function w_Q as the pointwise limit of a martingale $(w_Q^n)_{n\ge0}$ adapted to the filtration $(\mathcal{F}_n)_{n\ge0}$ with underlying finite measure $\ell|_{U_Q\cap\Sigma}$. We begin by defining the function w_Q^0:

$$w_Q^0(x) := \frac{\mathrm{diam}(U_Q)}{\ell(U_Q\cap\Sigma)} \text{ for any } x\in U_Q\cap\Sigma.$$

The martingale sequence will have fixed total mass $w_Q^0(U_Q) = \mathrm{diam}(U_Q)$. We next define

$$s_Q := \sum_{Q'\in C(Q)} \mathrm{diam}(U_{Q'}) + D\ell(R_Q) < \infty.$$

Given the function w_Q^n, we define w_Q^{n+1} by readjusting the distribution of mass inside the cores $\{U_{Q'}\}_{Q'\in C^{n+1}(Q)}$ and leaving the remainders $R_{Q''}$ constant for any ancestor balls $Q''\in C^j(Q)$ for $j<n$. Let $Q'\in C^n(Q)$, $Q''\in C(Q')\subseteq C^{n+1}(Q)$, and $Q_0\in C^j(Q)$ for some $j<n$. We define $w_Q^{n+1}(x)$ by declaring w_Q^{n+1} to be constant on each of $R_{Q_0}, R_{Q'}$, and $U_{Q''}\cap\Sigma$ and imposing

$$w_Q^{n+1}(R_{Q_0}) = w_Q^n(R_{Q_0}), \tag{2.36}$$

$$w_Q^{n+1}(R_{Q'}) = w_Q^n(U_{Q'})\cdot\frac{D\ell(R_{Q'})}{s_{Q'}}, \tag{2.37}$$

$$w_Q^{n+1}(U_{Q''}) = w_Q^n(U_{Q'})\cdot\frac{\mathrm{diam}(U_{Q''})}{s_{Q'}}. \tag{2.38}$$

We could find the pointwise value for w^{n+1} on each set by dividing the three above equations by $\ell(R_{Q_0}), \ell(R_{Q'})$, and $\ell(U_{Q''}\cap\Sigma)$ respectively. It follows from the definition that $w_Q^n(U_Q)$ is $\mathcal{F}_n$ measurable. In order to show that (w_Q^n) is a martingale adapted to the filtration $(\mathcal{F}_n)$, we must prove

$$\mathbb{E}(w_Q^{n+1}|\mathcal{F}_n) = w_Q^n.$$

It suffices to show that $w_Q^{n+1}(U_{Q'}) = w_Q^n(U_{Q'})$ for any $Q'\in\cup_{k\le n}C^k(Q')$. First, suppose $Q'\in C^n(Q)$. Then, using (2.38) and (2.37),

$$\begin{aligned} w_Q^{n+1}(U_{Q'}) &= w_Q^{n+1}(R_{Q'}) + \sum_{Q''\in C(Q')} w_Q^{n+1}(U_{Q'}) \\ &= w_Q^n(U_{Q'})\cdot\frac{D\ell(R_{Q'})}{s_{Q'}} + \sum_{Q''\in C(Q')} w_Q^n(U_{Q'})\cdot\frac{\mathrm{diam}(U_{Q''})}{s_{Q'}} = w_Q^n(U_{Q'}). \end{aligned}$$

On the other hand, if $Q' \in C^k(Q)$ for some $k < n$, we can apply (2.34) to Q' to write $U_{Q'}$ in terms of the remainders down to level $n-1$ and the cores at level n inside of $U_{Q'}$:

$$\begin{aligned}
w_Q^{n+1}(U_{Q'}) &= \sum_{j=0}^{n-k-1} \sum_{Q'' \in C^j(Q')} w_Q^{n+1}(R_{Q'}) + \sum_{Q'' \in C^{n-k}(Q')} w_Q^{n+1}(U_{Q'}) \\
&= \sum_{j=0}^{n-k-1} \sum_{Q'' \in C^k(Q')} w_Q^{n}(R_{Q'}) + \sum_{Q'' \in C^{n-k}(Q')} w_Q^{n}(U_{Q'}) = w_Q^n(U_{Q'})
\end{aligned}$$

using the previous two cases. This also shows that $w_Q^{n+1}(U_Q) = w_Q^n(U_Q) = \ldots = \operatorname{diam}(U_Q)$, verifying the finite expectation condition. Hence, $(w_Q^n)_{n\geq 0}$ is a positive martingale so that it converges pointwise ℓ almost everywhere to a function

$$w_Q(x) = \lim_{n\to\infty} w_Q^n(x).$$

By definition, $\operatorname{supp}(w_Q) \subseteq \operatorname{supp}(w_Q^0) = U_Q \cap \Sigma$ and $\int_Q w_Q d\ell = w_Q(U_Q) = w_Q^0(U_Q) = \operatorname{diam}(U_Q)$ verifying properties (i) and (iii) above. We now prove (ii). Fix $x \in U_Q \cap \Sigma$ for which $\lim_{n\to\infty} w_Q^n(x)$ exists and suppose that $x \in R_{Q_N} \cap U_{Q_N} \subset U_{Q_{N-1}} \subset \cdots \subset U_{Q_0} = U_Q$. Then (2.38) and (2.35) imply

$$\frac{w_Q(U_{Q_N})}{\operatorname{diam}(U_{Q_N})} = \frac{w_Q(U_{Q_{N-1}})}{s_{Q_{N-1}}} = \frac{w_Q(U_{Q_{N-1}})}{\operatorname{diam}(U_{Q_{N-1}})} \frac{\operatorname{diam}(U_{Q_{N-1}})}{s_{Q_{N-1}}} < q \frac{w_Q(U_{Q_{N-1}})}{\operatorname{diam}(U_{Q_{N-1}})}.$$

Applying this N times, we get

$$\frac{w_Q(U_{Q_N})}{\operatorname{diam}(U_{Q_N})} < q^N \frac{w_Q(U_Q)}{\operatorname{diam}(U_Q)} = q^N.$$

Therefore, using (2.37), we conclude

$$w_Q(x) = \frac{w_Q(R_{Q_N})}{\ell(R_{Q_N})} \leq \frac{w_Q(U_{Q_N})}{\ell(R_{Q_N})} \frac{D\ell(R_{Q_N})}{s_{Q_N}} \leq D \frac{w_Q(U_{Q_N})}{s_{Q_N}} < Dq^N.$$

In particular, if x is contained in an infinite sequence of nested cores, then $w_Q(x) = 0$ for all Q. Applying the above calculation for each $Q_k,\ 0 \leq k \leq N$, we see that $w_{Q_k}(x) \leq Dq^{N-k}$. . Because $\operatorname{supp}(w_Q) = U_Q \cap \Sigma$, we also know that $\bigcup_{Q\in\mathscr{L}} \operatorname{supp}(w_Q) \subseteq U_{\mathscr{L}}$ and we can compute

$$\begin{aligned}
\sum_{Q\in\mathscr{L}} w_Q(x) = \sum_{Q\in\mathscr{L}} w_Q(x)\chi_{U_Q}(x) &\leq \left(\sum_{Q\in\mathscr{L}} w_Q(x)\right) \chi_{U_{\mathscr{L}}}(x) \\
&\leq \left(\sum_{n=0}^{\infty} Dq^n\right) \chi_{U_{\mathscr{L}}}(x) \leq \frac{D}{1-q} \chi_{U_{\mathscr{L}}}(x).
\end{aligned}$$

This concludes the proof of (ii). ■

2.4.1.2 Bound on the Δ_1 sum for Σ

The ideas of this section are all present in [Sch07a]. We present them here in greater detail out of a desire for completeness. For $M \in \mathbb{N}$, define

$$\Delta(M) = \{Q \in \Delta_1 : 2^{-M} \leq \beta_{S(Q)}(U_Q^x) < 2^{-M+1}\}.$$

Fix $K \in \mathbb{N}$ such that $1 \leq K \leq MJ$ and define

$$\Delta' := \Delta'(M, K) := \{Q \in \Delta(M) : \mathrm{rad}(Q) = A2^{K+MJn},\ n \in \mathbb{Z}\}.$$

Intuitively, Δ' is obtained by starting at an offset K and skipping all elements in $\Delta(M)$ on the nearest MJ scales so that the difference in scale between adjacent levels within Δ' is large. We want to apply the martingale lemma, Lemma 2.4.5, with $\mathscr{L} = \Delta'$, so we need to prove the following lemma:

Lemma 2.4.6 (cf. [Sch07a] Lemma 3.25)**.** *For any $Q \in \Delta'$,*

$$\frac{\mathrm{diam}(U_Q^{xx})}{\sum_{Q' \in C(Q)} \mathrm{diam}(U_{Q'}^{xx}) + \ell(R_Q)} \leq \frac{1}{1 + \frac{1}{10}}.$$

Proof. (See Figure 2.5 for a picture of this proof). Recall that the definition of Δ_1 implies $\beta_{S(Q)}(U_Q^x) \geq C_U^{-1}\beta_{S(Q)}(Q) \geq C_U^{-1}\epsilon_1\beta_\Sigma(Q)$. For any $\eta \in S(Q)$, we get the bound

$$\tilde{\beta}(\eta) \leq \epsilon_2\beta_\Sigma(Q) < \epsilon_1(10^5 AC_U)^{-1}\epsilon_1\beta_\Sigma(Q) \leq 10^{-5}A^{-1}\epsilon_1\beta_{S(Q)}(U_Q^x) < 10^{-5}A^{-1}\epsilon_1 2^{-M+1}. \tag{2.39}$$

Therefore, because $\gamma_Q \in S(Q)$ we conclude

$$\beta_{\gamma_Q}(U_Q^x) \leq 16A\beta_{\gamma_Q}(Q) \leq 32A\tilde{\beta}(\gamma_Q) \leq 10^{-3}\epsilon_1\beta_{S(Q)}(U_Q^x).$$

This implies that there exists $\xi_Q \in S(Q)$, $\xi_Q \neq \gamma_Q$ such that we have $y \in \xi_Q \cap U_Q^x$ with $\mathrm{dist}(y, \mathsf{Edge}(\gamma_Q)) \geq \beta_{S(Q)}(U_Q^x)\,\mathrm{diam}(U_Q^x) \geq 2^{-M}\,\mathrm{diam}(U_Q^x)$ by using the line collinear with $\mathsf{Edge}(\gamma_Q)$ as an approximating line for $\beta_{S(Q)}(U_Q^x)$. Define

$$\gamma' := \mathsf{Edge}(\gamma_Q), \quad B_{\gamma'} := B(\gamma', \epsilon_1 2^{-M}\,\mathrm{diam}(U_Q^x)),$$
$$\xi' := \mathsf{Edge}(\xi_Q) \cap 15c_0Q, \quad B_{\xi'} := B(\xi', \epsilon_1 2^{-M}\,\mathrm{diam}(U_Q^x)).$$

By Lemma 2.4.2, there exists $y_{\xi'} \in \xi'$ such that

$$|y - y_{\xi'}| \leq \tilde{\beta}(\xi_Q)\,\mathrm{diam}(2Q) \leq \epsilon_1 2^{-M-1}\,\mathrm{diam}(U_Q^x)$$

so that $\mathrm{dist}(y_{\xi'}, \gamma') \geq 2^{-M-1}\,\mathrm{diam}(U_Q^x)$, and hence $y_{\xi'} \in 9c_0Q$. Write ξ' as the union of two subsegments $\xi' = [a_{\xi'}, y_{\xi'}] \cup [y_{\xi'}, b_{\xi'}]$ where $a_{\xi'}$ and $b_{\xi'}$ are the endpoints of ξ'. Because the line segments $[a_{\xi'}, y_{\xi'}]$ and $[b_{\xi'}, y_{\xi'}]$ extend in opposite directions away from $y_{\xi'}$, one of them, suppose it is $[a_{\xi'}, y_{\xi'}]$, satisfies $\mathrm{dist}([a_{\xi'}, y_{\xi'}], \gamma') \geq \mathrm{dist}(y_{\xi'}, \gamma') \geq 2^{-M-5}\,\mathrm{diam}(U_Q^{xx})$ also using the fact that γ' is a line segment. In addition, $[a_{\xi'}, y_{\xi'}]$ has nonempty intersection with both $9c_0Q$ and $15c_0Q \subseteq U_Q^{xx}$ so that we can assume both of the following hold:

$$\mathrm{dist}([a_{\xi'}, y_{\xi'}], \gamma') \geq 2^{-M-5}\,\mathrm{diam}(U_Q^{xx}), \tag{2.40}$$
$$\mathrm{diam}([a_{\xi'}, y_{\xi'}]) \geq 6c_0\,\mathrm{rad}(Q) = 3c_0\,\mathrm{diam}(Q). \tag{2.41}$$

Therefore, we can apply Lemma 2.4.2 to the segment $[a_{\xi'}, y_{\xi'}]$ to get an arc $\xi_0 \subseteq \xi_Q$ such that $\xi_0 \subseteq B_{\xi'} \subseteq 16c_0Q \subseteq U_Q^{xx}$ and $\mathsf{Diam}(\xi_0) \geq 3c_0 \operatorname{diam}(Q)$. Now, $Q \in \Delta'$ implies, for all $Q' \in C(Q)$,

$$\operatorname{diam}(U_{Q'}^{xx}) \leq \operatorname{diam}(Q') \leq 2^{-MJ} \operatorname{diam}(Q) \leq 2^{-M} \cdot 2^{-J+2} \operatorname{diam}(Q) \leq 2^{-M} \epsilon_1 \operatorname{diam}(U_Q^{xx}) \quad (2.42)$$

Using (2.40) and the fact that $\gamma_Q \subseteq B_{\gamma'}$, this implies that ξ_0 satisfies the following:

$$U_{Q'}^{xx} \cap \gamma_Q = \varnothing \text{ for all } Q' \in C(Q) \text{ such that } U_{Q'}^{xx} \cap \xi_0 \neq \varnothing.$$

Hence, we can estimate

$$\begin{aligned} \sum_{Q' \in C(Q)} \operatorname{diam}(U_{Q'}^{xx}) + \ell(R_Q) &\geq \sum_{\substack{Q' \in C(Q) \\ U_{Q'}^{xx} \cap \gamma_Q \neq \varnothing}} \operatorname{diam}(U_{Q'}^{xx}) + \ell(R_Q \cap \gamma_Q) + \sum_{\substack{Q' \in C(Q) \\ U_{Q'}^{xx} \cap \xi_0 \neq \varnothing}} \operatorname{diam}(U_{Q'}^{xx}) + \ell(R_Q \cap \xi_0) \\ &\geq \operatorname{diam}(\gamma_Q \cap U_Q^{xx}) + \mathsf{Diam}(\xi_0) \geq 15c_0 \operatorname{diam}(Q) + 3c_0 \operatorname{diam}(Q) \\ &\geq \left(1 + \frac{1}{10}\right) \operatorname{diam}(U_Q^{xx}). \end{aligned} \quad (2.43)$$

■

Proposition 2.4.7 (cf. [Sch07a] Lemma 3.25)**.**

$$\sum_{Q \in \Delta_1} \beta_\Sigma(Q)^2 \operatorname{diam}(Q) \lesssim_{A,J} \mathcal{H}^1(\Sigma).$$

Proof. Fix $\Delta'(M, K) \subseteq \Delta_1$ as defined above and order it via the forest $\mathcal{T}_{\Delta'}^{16c_0}$. By Lemma 2.4.6, we can apply Lemma 2.4.5 with $\mathscr{L} = \Delta'$, $D = 1$, $q = \frac{1}{1+\frac{1}{10}}$ to get a collection of positive real-valued functions $\{w_Q\}_{Q \in \Delta'}$ such that

(i) $\int_Q w_Q \, d\ell = \operatorname{diam}(U_Q^{xx})$, and

(ii) $\sum_{Q \in \Delta'} w_Q(x) \lesssim \chi_{U_{\Delta'}^{16c_0}}(x)$ for almost every $x \in \Sigma$,

Therefore, we have

$$\begin{aligned} \sum_{Q \in \Delta'} \beta_\Sigma(Q) \operatorname{diam}(Q) &\lesssim_A 2^{-M} \sum_{Q \in \Delta'} \int_Q w_Q d\ell \leq 2^{-M} \int_\Gamma \sum_{Q \in \Delta'} w_Q d\ell \\ &\lesssim 2^{-M} \int_{U_{\Delta'}^{16c_0} \cap \Sigma} d\ell \leq 2^{-M} \sum_{T \in \mathcal{T}_{\Delta'}^{16c_0}} \ell(U_{Q(T)}^{xx} \cap \Sigma) \lesssim 2^{-M} \ell(\Sigma) \end{aligned}$$

where the final inequality follows because the collection $\{U_{Q(T)}^{xx}\}_{T \in \mathcal{T}_{\Delta'}^{16c_0}}$ is pairwise disjoint. Summing this over $M \geq 0$ and $1 \leq K \leq MJ$, we get

$$\sum_{Q \in \Delta_1} \beta_\Sigma(Q)^2 \operatorname{diam}(Q) \leq \sum_{M=0}^{\infty} \sum_{K=1}^{MJ} \sum_{Q \in \Delta'(M,K)} \beta_\Sigma(Q) \operatorname{diam}(Q) \lesssim_{A,J} \sum_{M=0}^{\infty} M 2^{-M} \ell(\Sigma) \lesssim \mathcal{H}^1(\Sigma).$$

■

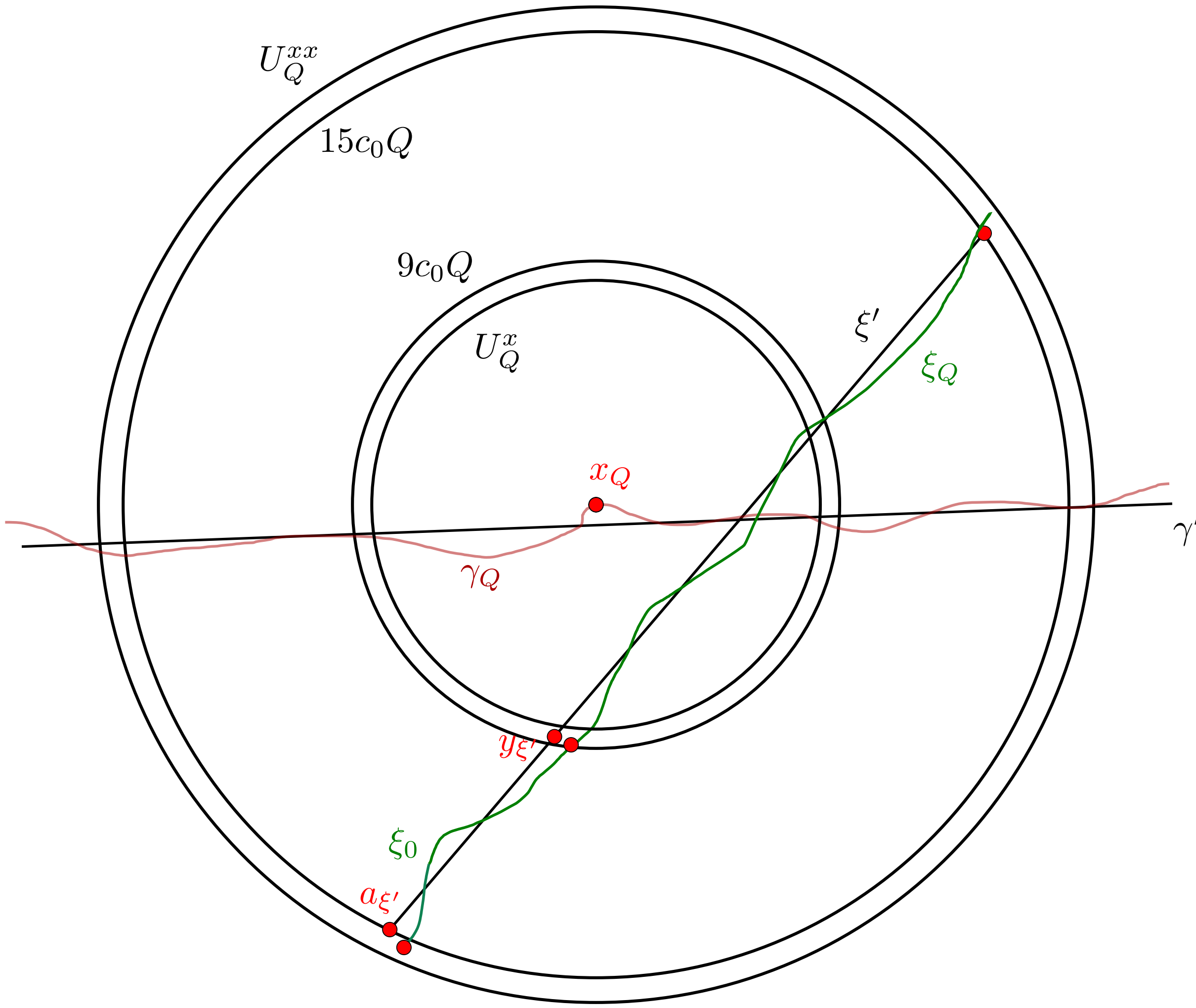

Figure 2.5: A picture of the proof of Lemma 2.4.6.

2.4.1.3 Bound on the $\Delta_{2.1}$ sum for Σ

Our proof of the beta sum bound for $\Delta_{2.1}$ relies heavily on the construction of weights analogous to those of Proposition 2.4.7 adapted to $\Delta_{2.1}$ balls rather than Δ_1 balls. Unfortunately, the proof of the existence of these weights in [Sch07a] Lemma 3.28 contains technical errors which leave gaps in the proof (see [BM23b] Appendix C. for further explanation of the issues). In this section, we provide a new proof.

We begin with a general lemma which gives a nice approximating line segment for almost flat γ_Q inside the core of a general ball $Q \in \mathscr{G}$ for a range of core sizes. We will use this line segment γ' as an accounting tool for proving the analogue of Lemma 2.4.6 for $\Delta_{2.1}$ balls.

Lemma 2.4.8. *Let $Q \in \mathscr{G}$ such that $\gamma_Q \in S(Q)$ and fix $\frac{2\epsilon_2}{\epsilon_1} < c < \frac{1}{4A}$. There exists a line segment $\gamma' = [a_{\gamma'}, b_{\gamma'}] \subseteq \mathsf{Edge}(\gamma_Q)$ such that*

(i) $\mathcal{H}^1(\gamma') \geq (1 - 30\epsilon_1)\,\mathrm{diam}(U_Q^c)$,

(ii) $B(\gamma', 5\epsilon_1\,\mathrm{diam}(U_Q^c)) \subseteq U_Q^c$,

(iii) $\gamma' \subseteq \pi_{\gamma'}(\gamma_Q \cap cQ)$, *and*

(iv) If $\beta_\Sigma(U_Q^c) < \epsilon_1$, *then for any* $Q' \in C(Q)$, $\pi_{\gamma'}(U_{Q'}^c) \cap \gamma' \neq \varnothing \implies 2Q' \subseteq U_Q^c$.

Proof. Let $\gamma'' := [a_{\gamma''}, b_{\gamma''}] := \mathsf{Edge}(\gamma_Q) \cap cQ$. Define γ' to be the line segment gotten by chopping off the segments of length $10\epsilon_1 \operatorname{diam}(U_Q^c)$ from either end of γ'':

$$\gamma' := \left[a_{\gamma''} - 10\epsilon_1 \operatorname{diam}(U_Q^c)\frac{b_{\gamma''} - a_{\gamma''}}{|b_{\gamma''} - a_{\gamma''}|}, b_{\gamma''} - 10\epsilon_1 \operatorname{diam}(U_Q^c)\frac{a_{\gamma''} - b_{\gamma''}}{|b_{\gamma''} - a_{\gamma''}|}\right] =: [a_{\gamma'}, b_{\gamma'}].$$

Because $x_Q \in \gamma_Q \in S(Q)$ implies $\operatorname{dist}(x_Q, \gamma'') < 2\epsilon_2 \operatorname{diam}(Q) \leq \epsilon_1 c \operatorname{rad}(Q)$, and $a_{\gamma''}, b_{\gamma''} \in \partial(cQ)$, Lemma 2.4.3 implies

$$\mathcal{H}^1(\gamma'') = |a_{\gamma''} - b_{\gamma''}| \geq c \operatorname{diam}(Q)\left(1 - 2\epsilon_1^2\right) \geq (1 - 2\epsilon_1)\operatorname{diam}(U_Q^c)$$

where we used the fact that $c \operatorname{diam}(Q) \geq \frac{1}{1+2^{-J+2}} \operatorname{diam}(U_Q^c) \geq (1-\epsilon_1)\operatorname{diam}(U_Q^c)$. This gives

$$\mathcal{H}^1(\gamma') \geq \mathcal{H}^1(\gamma'') - 20\epsilon_1 \operatorname{diam}(U_Q^c) \geq (1 - 30\epsilon_1)\operatorname{diam}(U_Q^c).$$

This proves (i). In a similar vein, we can use the Pythagorean theorem to estimate

$$\begin{aligned} |a_{\gamma'} - x_Q| &\leq \sqrt{(1-10\epsilon_1)^2 \operatorname{diam}(U_Q^c)^2 + \epsilon_1^2 \operatorname{diam}(U_Q^c)^2} \leq \operatorname{diam}(U_Q^c)\sqrt{1 - 20\epsilon_1 + 101\epsilon_1^2} \\ &\leq \operatorname{diam}(U_Q^c)\sqrt{1-19\epsilon_1} \leq (1-9\epsilon_1)\operatorname{diam}(U_Q^c) \end{aligned}$$

using the Taylor expansion estimate $\sqrt{1-x} = 1 - \frac{x}{2} - \frac{x^2}{8} - \ldots \leq 1 - \frac{x}{2}$. A similar inequality holds for $b_{\gamma'}$, implying

$$B(\gamma', 5\epsilon_1 \operatorname{diam}(U_Q^c)) \subseteq cQ \subseteq U_Q^c \tag{2.44}$$

by the triangle inequality, the convexity of balls in ℓ_2, and the fact that γ' is a line segment. This proves (ii). To prove (iii), we observe that $\tilde{\beta}(\gamma_Q)\operatorname{diam}(2Q) \leq 2\epsilon_2 \operatorname{diam}(Q) < c\epsilon_1 \operatorname{diam}(Q) \leq \epsilon_1 \operatorname{diam}(U_Q^c)$ and apply Lemma 2.4.2 to the segment γ' to get a subarc $\gamma_0 \subseteq \gamma_Q$ such that $\mathsf{Domain}(\gamma_0) \subseteq \mathsf{Domain}(\gamma_Q) \cap \gamma^{-1}(C(\gamma', \epsilon_1 \operatorname{diam}(U_Q^c)))$ such that $\pi_{\gamma'}(\gamma_0) = \gamma'$. We now prove (iv). We compute

$$\operatorname{dist}(\pi_{\gamma'}(x_{Q'}), \gamma') \leq \operatorname{diam}(2Q') \leq 2^{-J+1}\operatorname{diam}(Q) < 2(10A)^{-2}\epsilon_1 \operatorname{diam}(Q) \leq \epsilon_1 \operatorname{diam}(U_Q^c). \tag{2.45}$$

On the other hand, $\beta_\Sigma(U_Q^c) < \epsilon_1$ implies $|\pi_{\gamma'}^{\perp}(x_{Q'})| \leq 2\epsilon_1 \operatorname{diam}(U_Q^c)$ which combined with (2.45) gives $2Q' \subseteq B(\gamma', 5\epsilon_1 \operatorname{diam}(U_Q^c)) \subseteq U_Q^c$ by (ii). ■

For the rest of this section, we consider $\Delta_{2.1}$ with the ordering given by the forest $\mathcal{T}_{\Delta_{2.1}}^{c_0}$. We now identify a good family of balls from which we will extract excess length in order to prove the existence of $q < 1$ such that $\operatorname{diam}(U_Q) \leq q s_Q$ for $Q \in \Delta_{2.1}$.

Definition 2.4.4 (Dominant balls)**.** Fix $Q \in \Delta_{2.1}$ and let γ' be as in Lemma 2.4.8. Define the "interior" and "exterior" children as

$$\begin{aligned} C_I(Q) &:= \{Q' \in C(Q) : \pi_{\gamma'}(U_{Q'}) \cap \gamma' \neq \varnothing\}, \\ C_E(Q) &:= \{Q' \in C(Q) : \pi_{\gamma'}(U_{Q'}) \cap \gamma' = \varnothing\}, \\ &= C(Q) \setminus C_I(Q). \end{aligned}$$

We have $\bigcup_{Q' \in C_I(Q)} 2Q' \subseteq U_Q$ by Lemma 2.4.8 (iv). For any $Q' \in C_I(Q)$, one of the following two properties holds:

(i) For all $Q'' \in C(Q)$ such that $U_{Q''} \cap 2Q' \neq \varnothing$, $\operatorname{diam}(Q'') \leq \operatorname{diam}(Q')$, or

(ii) There exists $Q'' \in C(Q)$ such that $U_{Q''} \cap 2Q' \neq \varnothing$ and $\operatorname{diam}(Q'') > \operatorname{diam}(Q')$

Define the "dominant" and "minor" balls as

$$\begin{aligned} C_D(Q) &:= \{Q' \in C_I(Q) : Q' \text{ satisfies (i)}\}, \\ C_M(Q) &:= \{Q' \in C_I(Q) : Q' \text{ satisfies (ii)}\} \\ &= C_I(Q) \setminus C_D(Q). \end{aligned}$$

The balls in $C_D(Q)$ have dominant projections on γ' in the sense of the following lemma:

Lemma 2.4.9. *Let $Q \in \Delta_{2.1}$ be such that $\ell(R_Q) < \epsilon_1 \operatorname{diam}(U_Q)$. Then*

$$\mathcal{H}^1\left(\bigcup_{Q' \in C_D(Q)} \pi_{\gamma'}(U_{Q'})\right) \geq (1 - 50\epsilon_1)\operatorname{diam}(U_Q).$$

Proof. The collection $\{\pi_{\gamma'}(U_{Q'})\}_{Q' \in C_I(Q)} \cup \{\pi_{\gamma'}(R_Q \cap \gamma_Q)\}$ is a covering of γ' by Lemma 2.4.8 (iii) and (iv). Hence,

$$\begin{aligned} \mathcal{H}^1\left(\bigcup_{Q' \in C_I(Q)} \pi_{\gamma'}(U_{Q'})\right) + \ell(R_Q \cap \gamma_Q) &\geq \mathcal{H}^1\left(\bigcup_{Q' \in C_I(Q)} \pi_{\gamma'}(U_{Q'})\right) + \mathcal{H}^1(\pi_{\gamma'}(R_Q \cap \gamma_Q)) \\ &\geq \mathcal{H}^1(\gamma') \geq (1 - 30\epsilon_1)\operatorname{diam}(U_Q) \end{aligned}$$

by Lemma 2.4.8 (i). Using the fact that $\ell(R_Q) < \epsilon_1 \operatorname{diam}(U_Q)$, we get

$$\mathcal{H}^1\left(\bigcup_{Q' \in C_I(Q)} \pi_{\gamma'}(U_{Q'})\right) \geq (1 - 31\epsilon_1)\operatorname{diam}(U_Q).$$

The rest of the proof amounts to showing that the projections of cores of balls in the subfamily $C_D(Q) \subseteq C_I(Q)$ cover most of the projected cores of balls in $C_I(Q)$.

We begin by defining a many-to-one mapping $\psi : C_M(Q) \to C(Q)$. Fix $Q_0 \in C_M(Q)$. By definition, there exists some $Q_1 \in C(Q)$ such that $U_{Q_1} \cap 2Q_0 \neq \varnothing$ and $\operatorname{diam}(Q_1) > \operatorname{diam}(Q_0)$. If $Q_1 \in C_D(Q) \cup C_E(Q)$, then define $\psi(Q_0) = Q_1$. Otherwise, $Q_1 \in C_M(Q)$ and, applying the same logic to Q_1 as we did to Q_0, we get the existence of $Q_2 \in C(Q)$ satisfying condition (ii) for Q_1. Repeating this argument recursively, we get a finite chain of balls $Q_0, Q_1, Q_2, \ldots, Q_N$ with strictly increasing diameter such that $Q_N \in C_D(Q) \cup C_E(Q)$ and $Q_0, \ldots, Q_{N-1} \in C_M(Q)$ with $2Q_i \cap U_{Q_{i+1}} \neq \varnothing$ (the chain must be finite because there is an absolute upper bound on the diameter for balls in $C(Q)$). Set $\psi(Q_0) = \psi(Q_1) = \cdots = \psi(Q_{N-1}) = Q_N$.

Now, let $x \in Q_0 \in C_M(Q)$ with the above described chain $Q_0, \ldots, Q_{N-1}, \psi(Q_0)$. By the triangle inequality, we get

$$\begin{aligned} \operatorname{dist}(x, U_{\psi(Q_0)}) &\leq \sum_{i=0}^{N-1} \operatorname{diam}(2Q_i) \leq 2^{-J+1} \sum_{i=0}^{N-1} (2^{-J})^i \operatorname{diam}(\psi(Q_0)) \leq \frac{2^{-J+1}}{1 - 2^{-J}} \operatorname{diam}(\psi(Q_0)) \\ &< \epsilon_1 \operatorname{diam}(U_{\psi(Q_0)}). \end{aligned}$$

This implies that for any $Q' \in C_D(Q) \cup C_E(Q)$, the set of balls $Q'' \in C_M(Q)$ such that $\psi(Q'') = Q'$ are contained in a neighborhood of radius $\epsilon_1 \operatorname{diam}(U_{Q'})$ around $U_{Q'}$. Therefore, the projection $\pi_{\gamma'}(U_{Q'})$ is an interval of length at least $c_0 \operatorname{diam}(Q') \geq (1-\epsilon_1)\operatorname{diam}(U_{Q'})$ while the set $\left(\bigcup_{\psi(Q'')=Q'} \pi_{\gamma'}(U_{Q''})\right) \setminus \pi_{\gamma'}(U_{Q'})$ is contained in the union of two intervals of length $\epsilon_1 \operatorname{diam}(U_{Q'})$ adjoined to either end of $\pi_{\gamma'}(U_{Q'})$. This means that if $Q' \in C_M(Q)$ is such that $\psi(Q') \in C_E(Q)$, then $\pi_{\gamma'}(U_{Q''}) \cap \gamma'$ is contained in an interval of width less than $\epsilon_1 \operatorname{diam}(U_{Q'}) \leq \epsilon_1 \operatorname{diam}(U_Q)$ containing one of the two endpoints of γ'. Hence,

$$\mathcal{H}^1\left(\bigcup_{\substack{Q'\in C_M(Q)\\ \psi(Q')\in C_E(Q)}} \pi_{\gamma'}(U_{Q'})\right) \leq 2\epsilon_1 \operatorname{diam}(U_Q).$$

This implies

$$\begin{aligned}
&\mathcal{H}^1\left(\bigcup_{Q'\in C_M(Q)} \pi_{\gamma'}(U_{Q'}) \setminus \bigcup_{Q'\in C_D(Q)} \pi_{\gamma'}(U_{Q'})\right)\\
&\qquad \leq \mathcal{H}^1\left(\bigcup_{\substack{Q'\in C_M(Q)\\ \psi(Q')\in C_D(Q)}} \pi_{\gamma'}(U_{Q'}) \setminus \bigcup_{Q'\in C_D(Q)} \pi_{\gamma'}(U_{Q'})\right) + \mathcal{H}^1\left(\bigcup_{\substack{Q'\in C_M(Q)\\ \psi(Q')\in C_E(Q)}} \pi_{\gamma'}(U_{Q''})\right)\\
&\qquad \leq 4\epsilon_1 \mathcal{H}^1\left(\bigcup_{Q'\in C_D(Q)} \pi_{\gamma'}(U_{Q'})\right) + 2\epsilon_1 \operatorname{diam}(U_Q).
\end{aligned}$$

Therefore, we conclude

$$\begin{aligned}
(1-31\epsilon_1)\operatorname{diam}(U_Q) &\leq \mathcal{H}^1\left(\bigcup_{Q'\in C_I(Q)} \pi_{\gamma'}(U_{Q'})\right)\\
&\leq \mathcal{H}^1\left(\bigcup_{Q'\in C_D(Q)} \pi_{\gamma'}(U_{Q'})\right) + \mathcal{H}^1\left(\bigcup_{Q'\in C_M(Q)} \pi_{\gamma'}(U_{Q'}) \setminus \bigcup_{Q'\in C_D(Q)} \pi_{\gamma'}(U_{Q'})\right)\\
&\leq (1+4\epsilon_1)\mathcal{H}^1\left(\bigcup_{Q'\in C_D(Q)} \pi_{\gamma'}(U_{Q'})\right) + 2\epsilon_1 \operatorname{diam}(U_Q)
\end{aligned}$$

from which we get the result. ■

We now want to show that each ball $Q' \in C(Q)$ has double $2Q'$ which contains a significant amount of excess length which contributes to the value of s_Q. We begin by isolating an almost flat arc $\tau_{Q'}$ of large diameter which does not overlap with γ_Q too badly.

Remark 2.4.10 (Existence of $\tau_{Q'}$). Fix $Q' \in \Delta_{2.1}$. Because $\beta_{S_{Q'}}(U^x_{Q'}) < C_U^{-1}\beta_{S_{Q'}}(Q')$, there must exist an arc $\tau_{Q'} \in S_{Q'}$ such that both

(i) $\tau_{Q'} \cap U^x_{Q'} = \varnothing$, and

(ii) $\beta_{\gamma_{Q'} \cup \tau_{Q'}}(2Q') \gtrsim_A 1$.

Intuitively, we think of $\tau_{Q'}$ as an "additional" arc alongside $\gamma_{Q'}$ which makes a significant contribution to $\sum_{Q'' \in C(Q)} \operatorname{diam}(U_{Q''}) + \ell(R_Q)$ inside $2Q'$ because it carries a large number of child cores disjoint from those on γ_Q.

In order to estimate the core diameter sum, we will use line segment approximations to $\tau_{Q'}$ and $\gamma_{Q'}$ with the idea of first isolating appropriate subsegments which are far apart, then applying Lemma 2.4.2 to get associated arcs which are far apart. We define $\tau' := \mathsf{Edge}(\tau_{Q'}) \cap (1-3c_0)2Q'$. Then τ' is a line segment with endpoints in the boundary of $(1-3c_0)2Q'$ such that $\tau' \cap (1+c_0)Q' \neq \varnothing$ because $\tilde{\beta}(\tau_{Q'}) \leq \epsilon_2$ and $\tau \cap Q \neq \varnothing$ because $\tau \in \Lambda(Q)$. Similarly, we define $\eta' := \mathsf{Edge}(\gamma_{Q'}) \cap (1-3c_0)2Q'$. Because $x_{Q'} \in \gamma_{Q'}$ and $\gamma_{Q'} \in S_{Q'}$, we have $\operatorname{dist}(\eta', x_{Q'}) \leq \tilde{\beta}(\gamma_Q) \operatorname{diam}(2Q') \leq \epsilon_2 \operatorname{diam}(2Q') = \frac{\epsilon_2}{(1-3c_0)} \cdot (1-3c_0)\operatorname{diam}(2Q')$ so that we can apply Lemma 2.4.3 and receive

$$\begin{aligned}\operatorname{diam}(\eta') &\geq \left(1 - 2\left(\frac{\epsilon_2}{1-3c_0}\right)^2\right)(1-3c_0)\operatorname{diam}(2Q') \\ &\geq (1-8\epsilon_2^2)(1-3c_0)\operatorname{diam}(2Q') \geq \frac{999}{1000}\operatorname{diam}(2Q')\end{aligned}$$

because $(1-8\epsilon_2^2)(1-3c_0) \geq (1-4c_0) \geq (1-4\cdot 10^{-4}) \geq \frac{999}{1000}$. We will use η' and τ' in the following lemma:

Lemma 2.4.11. *Let $Q \in \Delta_{2.1}$. For any $Q' \in C_D(Q)$,*

$$\sum_{\substack{Q'' \in C(Q) \\ U_{Q''} \subseteq 2Q' \\ U_{Q''} \cap (\gamma_{Q'} \cup \tau_{Q'}) \neq \varnothing}} \operatorname{diam}(U_{Q''}) + \ell(R_Q \cap 2Q') \geq \left(1 + \frac{1}{10}\right)\operatorname{diam}(2Q').$$

Proof. Our plan is to apply Lemma 2.4.2 to η' and a large diameter subsegment $\tau'' \subseteq \tau'$ which is far from η' to get arcs $\gamma_0 \subseteq \gamma_{Q'}$ and $\tau_0 \subseteq \tau_{Q'}$ such that no child core of Q touches both (See Figure 2.6 for a picture of the proof). Because $\tilde{\beta}(\gamma_{Q'})\operatorname{diam}(2Q') \leq \epsilon_2 \operatorname{diam}(2Q') \leq \epsilon_1 \operatorname{diam}(U_{Q'})$, we can apply Lemma 2.4.2 to the segment η' to get an arc $\gamma_0 \subseteq \gamma_{Q'}$ such that $\mathsf{Domain}(\gamma_0) \subseteq \mathsf{Domain}(\gamma_{Q'}) \cap \gamma^{-1}(C(\eta', \epsilon_1 \operatorname{diam}(U_{Q'})))$ with $\mathsf{Diam}(\gamma_0) \geq \operatorname{diam}(\eta') \geq \frac{999}{1000}\operatorname{diam}(Q)$. We claim that for any $Q'' \in C(Q)$,

$$U_{Q''} \cap \gamma_0 \neq \varnothing \implies U_{Q''} \subseteq 2Q'. \tag{2.46}$$

For proof, first note that because $Q' \in C_D(Q)$, $U_{Q''} \cap 2Q' \neq \varnothing$ implies $\operatorname{diam}(Q'') \leq \operatorname{diam}(Q')$ so that

$$\operatorname{diam}(U_{Q''}) \leq (1+2^{-J+2})c_0 \operatorname{diam}(Q') \leq (1+\epsilon_1)c_0 \operatorname{diam}(Q') = (1+\epsilon_1)c_0 \operatorname{rad}(2Q'). \tag{2.47}$$

Hence, if $Q'' \in C(Q)$ such that $U_{Q''} \cap \gamma_0 \neq \varnothing$, then $U_{Q''} \cap (1 - \frac{3}{2}c_0)2Q' \neq \varnothing$ so that any $x \in U_{Q''}$ satisfies

$$\begin{aligned} \operatorname{dist}(x, x_Q) &\leq \left(1 - \frac{3}{2}c_0\right) \operatorname{rad}(2Q') + \operatorname{diam}(U''_Q) \\ &\leq \left(1 - \frac{3}{2}c_0\right) \operatorname{rad}(2Q') + (1+\epsilon_1)c_0 \operatorname{rad}(2Q') \leq \operatorname{rad}(2Q') \end{aligned}$$

so that $U_{Q''} \subseteq 2Q'$. Because $Q' \in C_D(Q)$, we also know that $\gamma_0 \subseteq \gamma_{Q'} \subseteq 2Q' \subseteq U_Q$ so that the family $\{U_{Q''} : Q'' \in C(Q),\ U_{Q''} \cap \gamma_0 \neq \varnothing\} \cup \{R_Q \cap \gamma_0\}$ is a covering of γ_0. We can then estimate

$$\begin{aligned} \sum_{\substack{Q'' \in C(Q) \\ U_{Q''} \cap \gamma_{Q'} \neq \varnothing}} \operatorname{diam}(U_{Q''}) + \ell(R_Q \cap \gamma_{Q'}) &\geq \sum_{\substack{Q'' \in C(Q) \\ U_{Q''} \subseteq 2Q' \\ U_{Q''} \cap \gamma_0 \neq \varnothing}} \operatorname{diam}(U_{Q''}) + \ell(R_Q \cap \gamma_0) \\ &\geq \mathsf{Diam}(\gamma_0) \geq \frac{999}{1000} \operatorname{diam}(2Q'). \end{aligned}$$

We want to apply a similar argument to τ', this time finding an arc τ_0 which lies close to a subsegment τ'' of τ' which is far from η', hence from γ_Q. Indeed, suppose first that there exists a point $y_{\tau'} \in \tau' \cap (1 + c_0)Q$ such that $\operatorname{dist}(y_{\tau'}, \eta') \geq 7c_0 \operatorname{rad}(Q')$. Write τ' as the union of two subsegments $\tau' = [a_{\tau'}, y_{\tau'}] \cup [y_{\tau'}, b_{\tau'}]$ where $a_{\tau'}, b_{\tau'}$ are the endpoints of τ'. Because the line segments $[a_{\tau'}, y_{\tau'}]$ and $[y_{\tau'}, b_{\tau'}]$ extend in opposite directions away from $y_{\tau'}$, we know that one of them, suppose it is $[a_{\tau'}, y_{\tau'}]$, satisfies $\operatorname{dist}([a_{\tau'}, y_{\tau'}], \eta') \geq \operatorname{dist}(y_{\tau'}, \eta') \geq 7c_0 \operatorname{rad}(Q)$ using the fact that η' is a line segment. Set $\tau'' := [a_{\tau'}, y_{\tau'}]$. This completes the definition of τ'' in the first case.

If instead there is no such point $y_{\tau'} \in (1+c_0)Q' \cap \tau'$, then $(1+c_0)Q' \cap \tau' \subseteq B(\eta', 7c_0 \operatorname{rad}(Q'))$. We claim that τ' is nearly perpendicular to η'. Indeed, consider $E := \partial((1 + c_0)Q') \cap \tau'$ and let C_1, C_2 be the two connected components of the set $B(\eta', 7c_0 \operatorname{rad}(Q')) \cap \partial((1 + c_0)Q')$. First, we claim there cannot exist distinct points $e_1, e_2 \in E$ such that $e_1 \in C_1$ and $e_2 \in C_2$. If there did exist such points, then because τ' is a line segment and $B(\eta', 7c_0 \operatorname{rad}(Q'))$ is convex, there would exist $e' \in \tau' \cap B(\eta', 7c_0 \operatorname{rad}(Q'))$ with $\pi_{\eta'}(e) = \pi_{\eta'}(x_{Q'})$ so that $e' \in \frac{15}{2}c_0 Q'$. Hence, we would have $\tau_{Q'} \cap U^x_{Q'} \neq \varnothing$, contradicting the definition of $\tau_{Q'}$. Therefore, without loss of generality we can assume that $E \subseteq C_1$.

Let $P \subseteq H$ be the affine plane containing the line segments η' and τ' (this is at most 3-dimensional). By translating and rotating, we can assume without loss of generality that $x_{Q'} = 0$ and η' is collinear with the x_1-axis so that

$$E \subseteq S := \left\{ x \in P : x_1 > 0,\ |x| = (1 + c_0) \operatorname{rad}(Q'),\ |x^\perp| \leq \frac{1}{1000} \operatorname{rad}(Q') \right\}$$

where $|x^\perp|^2 = |x|^2 - |x_1|^2$, and we have used the fact that $8c_0 < \frac{1}{1000}$. The set S is a small spherical cap of the (at most 2-dimensional) sphere $\{x \in P : |x| = (1 + c_0) \operatorname{rad}(Q')\}$ around the point $((1 + c_0) \operatorname{rad}(Q'), 0, 0, \ldots)$. Fix $e_{\tau'} \in E$. We can write $[a_{\tau'}, e_{\tau'}] = \{e_{\tau'} + tv : 0 \leq t \leq |a_{\tau'} - e_{\tau'}|\}$ where $|v| = 1$ and we claim v is parallel to a tangent vector to S. Indeed, if $\#(E) = 1$, then τ' is tangent to S while if $\#(E) = 2$, then $\tau' \cap (1 + c_0)Q'$ is a line segment with two endpoints in S and the claim follows from considering S as a graph over the plane $\{x_1 = 0\}$ and applying the mean value theorem (geometrically, one can imagine translating the line segment to be tangent to S).

One can compute by implicit differentiation in $S \subseteq P \subseteq \mathbb{R}^3$ that $\frac{|v^\perp|}{|v|} \geq \frac{1}{2}$. We can also assume $\operatorname{dist}(e_{\tau'} + tv, \eta')$ is increasing in t by exchanging $[a_{\tau'}, e_{\tau'}]$ with $[e_{\tau'}, b_{\tau'}]$ or v with $-v$ if necessary. Therefore, $\operatorname{dist}(e_{\tau'} + (20c_0 \operatorname{rad}(Q'))v, \eta') \geq 7c_0 \operatorname{rad}(Q')$. Define $\tau'' := [a_{\tau'}, e_{\tau'} + (20c_0 \operatorname{rad}(Q'))v]$.

With τ'' defined as in either of the two cases above, we get the following two lower bounds:

$$\operatorname{dist}(\tau'', \eta') \geq 7c_0 \operatorname{rad}(Q'), \tag{2.48}$$

$$\begin{aligned}\operatorname{diam}(\tau'') &\geq 2(1-3c_0)\operatorname{rad}(Q') - (1+c_0)\operatorname{rad}(Q') - 20c_0 \operatorname{rad}(Q') \\ &\geq \frac{1}{4}(1-27c_0)\operatorname{diam}(2Q') \geq \frac{1}{5}\operatorname{diam}(2Q').\end{aligned}$$

Applying Lemma 2.4.2 to the segment τ'', we get an arc $\tau_0 \subseteq C(\tau'', \epsilon_1 c_0 \operatorname{rad}(Q'))$ with $\mathsf{Diam}(\tau_0) \geq \frac{1}{5}\operatorname{diam}(2Q')$. Therefore, we conclude from (2.48) and (2.47) that $U_{Q''} \cap \tau_0 \neq \varnothing$ implies $U_{Q''} \cap \gamma_0 = \varnothing$ and $U_{Q''} \subseteq 2Q'$ as in (2.46) so that we can estimate

$$\begin{aligned}\sum_{\substack{Q'' \in C(Q) \\ U_{Q''} \cap \gamma_{Q'} = \varnothing \\ U_{Q''} \cap \tau_{Q'} \neq \varnothing}} \operatorname{diam}(U_{Q''}) + \ell(R_Q \cap \tau_{Q'}) &\geq \sum_{\substack{Q'' \in C(Q) \\ U_{Q''} \subseteq 2Q' \\ U_{Q''} \cap \tau_0 \neq \varnothing}} \operatorname{diam}(U_{Q''}) + \ell(R_Q \cap \tau_0) \\ &\geq \mathsf{Diam}(\tau_0) \geq \frac{1}{5}\operatorname{diam}(2Q').\end{aligned}$$

By summing the estimates for γ_0 and τ_0, we conclude

$$\sum_{\substack{Q'' \in C(Q) \\ U_{Q''} \subseteq 2Q' \\ U_{Q''} \cap (\gamma_{Q'} \cup \tau_{Q'}) \neq \varnothing}} \operatorname{diam}(U_{Q''}) + \ell(R_Q \cap 2Q') \geq \left(\frac{999}{1000} + \frac{1}{5}\right)\operatorname{diam}(2Q') \geq \left(1 + \frac{1}{10}\right)\operatorname{diam}(2Q').$$

■

We can now combine this lemma with Lemma 2.4.9 on dominant projections to prove that the martingale construction can be applied to $\Delta_{2.1}$.

Lemma 2.4.12 (cf. [Sch07a] Lemma 3.28)**.** *For any* $Q \in \Delta_{2.1}$,

$$\frac{\operatorname{diam}(U_Q)}{\sum_{Q' \in C(Q)} \operatorname{diam}(U_{Q'}) + 2\epsilon_1^{-1}\ell(R_Q)} \leq \frac{1}{1 + \frac{1}{50}}.$$

Proof. First, observe that if $\ell(R_Q) > \epsilon_1 \operatorname{diam}(U_Q)$, then

$$\frac{\operatorname{diam}(U_Q)}{s_Q} = \frac{\operatorname{diam}(U_Q)}{\sum_{Q' \in C(Q)} \operatorname{diam}(U_{Q'}) + 2\epsilon_1^{-1}\ell(R_Q)} < \frac{\operatorname{diam}(U_Q)}{2\operatorname{diam}(U_Q)} = \frac{1}{2} < 1.$$

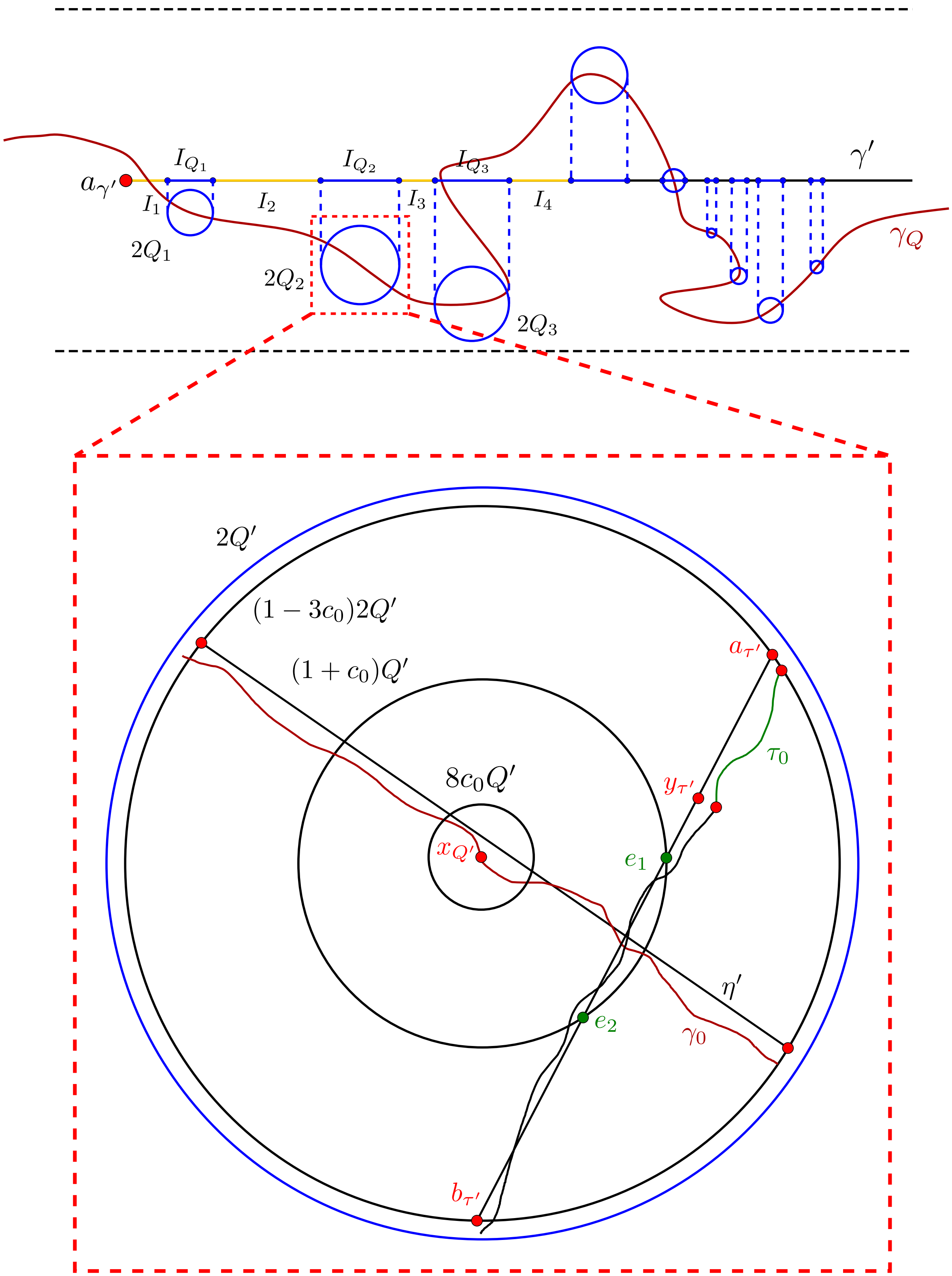

Figure 2.6: A picture of the proofs of Lemmas 2.4.12 and 2.4.11. At the top is a picture of $\gamma_Q \cap U_Q$ in Lemma 2.4.12 on which lies members of the large family of disjoint dominant balls. In the image below, we have zoomed-in on one of these balls and have labeled pieces present in the proof of Lemma 2.4.11.

Therefore, we can assume without loss of generality that $\ell(R_Q) \leq \epsilon_1 \operatorname{diam}(U_Q)$. For $Q' \in C_D(Q)$, define $I_{Q'} := \pi_{\gamma'}(2Q')$. Identifying γ' with $\mathbb{R}$, we can apply a covering lemma for the real line (see [Ald91] Lemma 2.1, for example) to the collection $\{I_{Q'}\}_{Q' \in C_D(Q)}$ to get a collection $\mathscr{Q} \subseteq C_D(Q)$ of balls with pairwise disjoint doubles so that

$$\mathcal{H}^1\left(\bigcup_{Q' \in \mathscr{Q}} I_{Q'}\right) \geq \frac{1}{3}\mathcal{H}^1\left(\bigcup_{Q' \in C_D(Q)} I_{Q'}\right) \geq \frac{1}{3}(1 - 50\epsilon_1)\operatorname{diam}(U_Q) \geq \frac{1}{4}\operatorname{diam}(U_Q) \tag{2.49}$$

where we used Lemma 2.4.9 in the penultimate inequality. We can then enumerate the components of $\gamma' \setminus \bigcup_{Q' \in \mathscr{Q}} I_{Q'}$ as

$$\gamma' \setminus \bigcup_{Q' \in \mathscr{Q}} I_{Q'} =: \bigcup_{j \in J_{\mathscr{Q}}} I_j.$$

Define $\mathscr{Q}_c := \{Q' \in C(Q) : U_{Q'} \setminus \bigcup_{Q'' \in \mathscr{Q}} 2Q'' \neq \varnothing\}$. We have

$$\bigcup_{j \in J_{\mathscr{Q}}} I_j \subseteq \pi_{\gamma'}\left(R_Q \setminus \bigcup_{Q' \in \mathscr{Q}} 2Q'\right) \cup \bigcup_{Q' \in \mathscr{Q}_c} \pi_{\gamma'}(U_{Q'})$$

using Lemma 2.4.8 (iii). Therefore, combining this fact with Lemma 2.4.11,

$$\begin{aligned}
&\sum_{Q' \in C(Q)} \operatorname{diam}(U_{Q'}) + 2\epsilon_1^{-1}\ell(R_Q) \\
&\qquad \geq \sum_{Q' \in \mathscr{Q}} \sum_{\substack{Q'' \in C(Q) \\ U_{Q''} \subseteq 2Q'}} \operatorname{diam}(U_{Q''}) + \ell(R_Q \cap 2Q') + \sum_{Q' \in \mathscr{Q}_c} \operatorname{diam}(U_{Q'}) + \ell\left(R_Q \setminus \bigcup_{Q' \in \mathscr{Q}} 2Q'\right) \\
&\qquad \geq \sum_{Q' \in \mathscr{Q}} \left(1 + \frac{1}{10}\right)\operatorname{diam}(2Q') + \sum_{Q' \in \mathscr{Q}_c} \mathcal{H}^1(\pi_{\gamma'}(U_{Q'})) + \mathcal{H}^1\left(\pi_{\gamma'}\left(R_Q \setminus \bigcup_{Q' \in \mathscr{Q}} 2Q'\right)\right) \\
&\qquad \geq \sum_{Q' \in \mathscr{Q}} \left(1 + \frac{1}{10}\right)\mathcal{H}^1(I_{Q'}) + \sum_{j \in J_{\mathscr{Q}}} \mathcal{H}^1(I_j) \\
&\qquad \geq \frac{1}{10}\sum_{Q' \in \mathscr{Q}} \mathcal{H}^1(I_{Q'}) + \mathcal{H}^1(\gamma') \\
&\qquad \overset{(2.49)}{\geq} \frac{1}{40}\operatorname{diam}(U_Q) + (1 - 30\epsilon_1)\operatorname{diam}(U_Q) > \left(1 + \frac{1}{50}\right)\operatorname{diam}(U_Q).
\end{aligned}$$

■

Proposition 2.4.13 (cf. [Sch07a] Lemma 3.28)**.**

$$\sum_{Q \in \Delta_{2.1}} \beta_\Sigma(Q)^2 \operatorname{diam}(Q) \lesssim_A \sum_{Q \in \Delta_{2.1}} \operatorname{diam} U_Q \lesssim_J \ell(\Sigma)$$

Proof. Order $\Delta_{2.1}$ via the forest $\mathcal{T}_{\Delta_{2.1}}^{c_0}$. Using Lemma 2.4.12, we apply Lemma 2.4.5 with $\mathscr{L} = \Delta_{2.1}$, $D = 2\epsilon_1^{-1}$, $q = \frac{1}{1+\frac{1}{50}}$ to get the existence of a collection of positive real-valued functions $\{w_Q\}_{Q \in \Delta_{2.1}}$ satisfying

(i) $\int_Q w_Q d\ell = \operatorname{diam}(U_Q)$, and

(ii) $\sum_{Q\in\Delta'} w_Q(x) \lesssim \epsilon_1^{-1}\chi_{U^{c_0}_{\Delta_{2.1}}}(x)$ for almost every $x \in \Sigma$.

Using these properties, we can finish the proof of the lemma as follows:

$$\sum_{Q\in\Delta_{2.1}} \beta(Q)^2 \operatorname{diam}(Q) \lesssim_A \sum_{Q\in\Delta_{2.1}} \operatorname{diam}(U_Q) = \sum_{Q\in\Delta_{2.1}} \int_Q w_Q d\ell = \int_\Gamma \sum_{Q\in\Delta_{2.1}} w_Q d\ell$$
$$\lesssim_{\epsilon_1} \int_{U^{c_0}_{\Delta_{2.1}}} d\ell = \sum_{T\in\mathcal{T}^{c_0}_{\Delta_{2.1}}} \ell(U_{Q(T)}) \lesssim_J \mathcal{H}^1(\Sigma). \qquad \blacksquare$$

2.4.2 Almost flat arcs for Γ

The goal of this section is to finish the proof of Proposition 2.4.1 by proving the second inequality in (2.30). In Section 2.4.2.1, we give preliminary definitions and lemmas needed to refine the results of the previous section. In Section 2.4.2.2 we use these tools to strengthen the previously given martingale arguments for the family Δ_1 and the newly defined family $\Delta_{2.1.1} \subseteq \Delta_2$. Finally, in Section 2.4.2.3 we analyze the leftover family $\Delta_{2.1.2}$ and finish the proof of Proposition 2.4.1, and hence the proof of Theorem A.

2.4.2.1 New Definitions and Tools

Recall that $\Gamma \subseteq \ell_2$ is a Jordan arc with an injective arc length parameterization $\gamma : I \to \Gamma$ where we fix $I := [0, \ell(\Gamma)]$. We assume without loss of generality that the chord line of Γ is the e_1-axis. Let $\pi : \ell_2 \to \mathbb{R}$ is the orthogonal projection onto the e_1-axis and let $\pi^\perp : \ell_2 \to \ell_2$ be the orthogonal projection onto the orthogonal hyperplane to the e_1-axis. For every $i \in \mathbb{N}$ the function $\gamma_i(t) := \langle \gamma(t), e_i\rangle$ is 1-Lipschitz, hence differentiable almost everywhere. We let $\gamma_i'(t)$ denote the derivative and write

$$\gamma(t) = \sum_{i=1}^{\infty} \gamma_i(t) e_i \text{ and } \gamma'(t) := \sum_{i=1}^{\infty} \gamma_i'(t) e_i.$$

The fact that γ is an arc length parameterization means that $|\gamma'(t)| = 1$ almost everywhere. In particular, $\gamma'(t)$ gives an almost everywhere well-defined notion of tangent vector to Γ at $\gamma(t)$. For $x \in \Gamma$, we let $t(x) \in I$ be the unique number such that $\gamma(t(x)) = x$.

We begin by defining a new measure $\mu \ll \ell$ which quantifies how much subsets of Γ contribute to the value of $\ell(\Gamma) - \operatorname{crd}(\Gamma)$.

Definition 2.4.5 (μ measure)**.** Let $\rho : I \to [0, 2]$ be given by

$$\rho(t) := \begin{cases} 1 - \gamma_1'(t), & \gamma_1'(t) \text{ exists} \\ 1, & \text{otherwise.} \end{cases}$$

Define the finite Borel measure μ supported on Γ as

$$d\mu := \gamma_*(\rho\ dt).$$

where $\mu(A) = \gamma_*(\rho\ dt)(A) := \int_{\gamma^{-1}(A)} \rho(t)dt$ is the pushforward of $\rho(t)dt$ by γ.

The definition of μ is motivated by the fundamental theorem of calculus in the following way:

Lemma 2.4.14. *Let $a, b \in I$ with $a \leq b$. Then*

$$\mu(\gamma|_{[a,b]}) = \ell(\gamma|_{[a,b]}) - (\pi(\gamma(b)) - \pi(\gamma(a))).$$

In particular,

$$\mu(\Gamma) = \ell(\Gamma) - \text{crd}(\Gamma).$$

Proof. We compute

$$\mu(\gamma|_{[a,b]}) := \mu(\mathsf{Image}(\gamma_{[a,b]})) = \int_{\gamma([a,b])} \gamma_*(\rho \; dt) = \int_a^b \rho(t) \; dt = \int_a^b 1 - \gamma_1'(t) \; dt$$
$$= (b-a) - (\gamma_1(b)) - \gamma_1(a)) = \ell(\gamma|_{[a,b]}) - (\pi(\gamma(b)) - \pi(\gamma(a))).$$

Setting $a = 0$ and $b = \ell(\Gamma)$ gives $\mu(\Gamma) = \ell(\Gamma) - \text{crd}(\Gamma)$. ■

Remark 2.4.15 (Null sets and examples). Fix $x, y \in \Gamma$ with $t(x) < t(y)$ and suppose ξ is a subarc of Γ such that $\mathsf{Start}(\xi) = x$ and $\mathsf{End}(\xi) = y$. If $\mu(\xi) = 0$, then

$$\ell(\xi) = \pi(y) - \pi(x) = y_1 - x_1.$$

This forces $y_1 > x_1$ and forces ξ to be a parameterization of the line segment $[x, y] = [x, x + (y_1 - x_1)e_1$ which is parallel to the chord line of Γ. Now, suppose $\tau(t) := x + t\frac{y-x}{|y-x|}$ for $t(x) \leq t \leq t(x) + |y - x|$. That is, τ parameterizes $[x, y] \subseteq \Gamma$. In general, we have the formula

$$\mu(\tau) = \ell(\tau) - (\pi(y) - \pi(x)) = |y - x| - (y_1 - x_1). \tag{2.50}$$

If $y_1 < x_1$, then μ is larger than ℓ on τ; this measure assigns "bonus" length to arcs which "backtrack" along the direction of the chord line of Γ. If $y_1 > x_1$, the right side of (2.50) bears resemblance to triangle inequality excess estimates. Indeed, let $x, y, z \in \Gamma$ and suppose there exists a subarc η such that

$$\eta(t) := \begin{cases} x + t\frac{y-x}{|y-x|}, & t(x) \leq t \leq t(x) + |y - x| \\ y + t\frac{z-y}{|z-y|}, & t(x) + |y - x|t \leq t(x) + |y - x| + |z - y|. \end{cases}$$

The arc η injectively parameterizes the line segments $[x, y]$ and $[y, z]$. We compute

$$\mu(\eta) = \ell(\eta) - (\pi(x) - \pi(z)) = |x - y| + |y - z| - (z_1 - x_1).$$

When $x_1 < y_1 < z_1$, this is something like a triangle inequality excess estimate (see Remark 2.1.2) where instead of subtracting the length of the triangle base $[x, z]$, we subtract the length of the projection of $[x, z]$ along the chord line of Γ .

Our goal for proving Theorem A is to bound the beta sums above by $\mu(\Gamma)$ rather than $\ell(\Gamma)$. Intuitively, this is plausible since μ assigns small measure only to those regions of Γ which are nearly parallel to the chord and are directed via the parameterization γ towards the terminal endpoint of Γ, i.e., have $\gamma_1' > 1 - \delta$ for $\delta > 0$ small. One would expect $\beta_\Gamma(Q)$ to be small on average for Q centered in such a region. There is a problem with this definition of μ, however. We would like to have a bound of the form

$$\mu(U_Q) \gtrsim \ell(U_Q) \tag{2.51}$$

for individual cores in some family because this would allow a translation of the preceding martingale arguments to this setting. However, such a result cannot hold, as $\mu(U_Q) = 0$ may hold even for U_Q with $\beta_\Gamma(U_Q) \approx 1$ (see Figure 2.7). However, in order for a situation like Figure 2.7 to occur, there must be some "backtracking" arc (given in the figure by the bottom-most horizontal piece of Γ outside of U_Q). On this arc, $\gamma_1' < 0$ so that $d\mu \geq d\ell$. To recover inequalities like (2.51), we will construct a new, larger measure $\tilde{\mu}$ that fills in the μ measure gaps in Figure 2.7 by "borrowing" mass from backtracking arcs. We begin by isolating these regions of change as maximal disjoint arcs where Γ "bends" back on itself along the e_1 axis in the sense that the projection map π is non-injective. This is made more precise with the following definition.

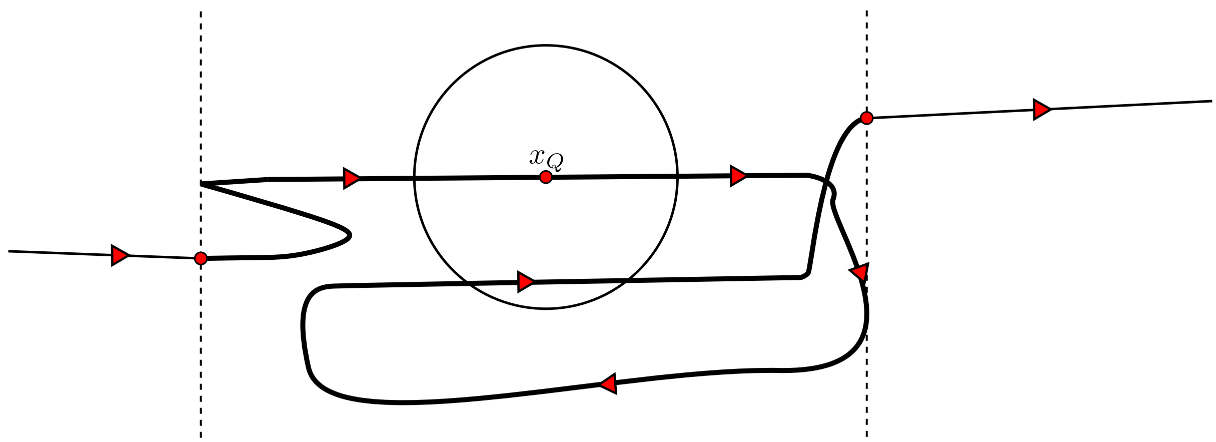

Figure 2.7: A core with $\beta_\Gamma(U_Q) \approx 1$ but $\mu(U_Q) = 0$. The red arrows indicate the direction of the parameterization such that $\rho \equiv 0$ on the two horizontal lines passing through U_Q. The thickened piece of Γ in between the vertical dotted lines is a bend (assume that Γ's chord line is horizontal).

Definition 2.4.6 (Multiplicity)**.** For $t \in I$, define

$$\text{perp}(t) := \gamma^{-1}(\pi^{-1}(\pi(\gamma(t))) \cap \Gamma).$$

This is the set of points in I which map to points in Γ that have the same first coordinate as $\gamma(t)$. Define $M : I \to \mathbb{N} \cup \{\infty\}$, the multiplicity function, by

$$M(t) := \#\,\text{perp}(t).$$

Additionally, we let $E := \{t \in I : \pi(\gamma(t)) = \min(\pi(\Gamma)) \text{ or } \pi(\gamma(t)) = \max(\pi(\Gamma))\}$ and set

$$S_\pi := \{t \in I : M(t) = 1\},$$
$$M_\pi := \{t \in I : M(t) \geq 2\} \cup E.$$

S_π is the set where $\text{perp}(t) = \{t\}$ is a singleton while M_π is the set where either $\text{perp}(t)$ has multiple elements or $\gamma(t)$ is an extremal point of Γ along the e_1-axis. If one of these latter points is also a member of S_π, then it is isolated by connectedness.

We will add mass to μ by raising the value of ρ in a carefully chosen neighborhood of M_π. In order to define this neighborhood, we first provide a decomposition of M_π into the maximal arcs promised above. The following structure lemma for M_π will aid us:

Lemma 2.4.16. *Suppose $a, b \in I$ with $\pi(\gamma(a)) = \pi(\gamma(b))$ and $a \le b$. Then $[a, b] \subseteq M_\pi$.*

Proof. Let $r \in (a, b)$. If $\pi(\gamma(r)) = \pi(\gamma(a))$, then $r \in M_\pi$. Otherwise, $\pi(\gamma(r)) \neq \pi(\gamma(a))$ and

$$\inf_{t\in[a,b]} \pi(\gamma(t)) \le \pi(\gamma(r)) \le \sup_{t\in[a,b]} \pi(\gamma(t)).$$

Since π is continuous, there exist points $s, u \in [a, b]$ on which $\pi \circ \gamma$ achieves the infimum and supremum above respectively. Suppose first that $\pi(\gamma(r)) = \pi(\gamma(s))$. If there exists $s' \in I$, $s' \neq r$ such that $\pi(\gamma(s')) = \pi(\gamma(r))$, then $r \in M_\pi$ by definition. Otherwise, $\pi(x) > \pi(\gamma(r))$ for all $x \in \Gamma \setminus \gamma(r)$ so that $\gamma(r) = \min(\pi(\Gamma)) \in E \subseteq M_\pi$. Therefore, it suffices to consider the case when $\pi(\gamma(r)) > \pi(\gamma(s))$. By a similar argument, we can also assume $\pi(\gamma(r)) < \pi(\gamma(u))$. Hence, the function $f : [a, b] \to \mathbb{R}$ given by $f(t) = \pi(\gamma(t)) - \pi(\gamma(a))$ is continuous and satisfies

(i) $f(r) \neq 0$,

(ii) $f(s) < f(r) < f(u)$, and

(iii) $f(a) = f(b) = 0$.

We claim that the intermediate value theorem implies the existence of a point $r' \in [a, b]$, $r \neq r'$ with $f(r') = f(r)$ so that $\gamma(r) \in M_\pi$. Indeed, suppose without loss of generality that $f(r) > 0$. If $u < r$, then $f(a) < f(r) < f(u)$ so that there exists such $r' \in [a, u]$. Otherwise, $u > r$ and $f(b) < f(r) < f(u)$ so that there exists such $r' \in [u, b]$. ∎

Definition 2.4.7 (Bends)**.** It follows from Lemma 2.4.16 that for any $t \in M_\pi$, the family

$$\mathcal{C}_t := \{[a, b] \subseteq I : t \in [a, b] \subseteq M_\pi\}$$

contains a non-degenerate interval so that the union $I_t := \bigcup \mathcal{C}_t$ is also a non-degenerate interval. The union $\bigcup_{t\in I} I_t = M_\pi$ has countably many disjoint connected components, each of which is a non-degenerate subinterval of I (which is possibly open). Thus, the closure $\overline{M}_\pi$ has connected components that are closed, non-degenerate intervals which we enumerate as $\overline{M}_\pi = \bigcup_{k\in K}[s_k, u_k]$. We define

$$\Phi := \{\gamma|_{[s_k,u_k]} : k \in K\}.$$

We refer the elements of Φ as *bends.*

An important fact is that these regions contribute a proportionally large amount of measure to μ.

Lemma 2.4.17. *Let $\phi \in \Phi$. Then*

$$\mu(\phi) \geq \frac{1}{2}\ell(\phi).$$

Proof. Let $\phi \in \Phi$. Lemma 2.4.14 implies

$$\mu(\phi) = \ell(\phi) - [\pi(\mathsf{End}(\phi)) - \pi(\mathsf{Start}(\phi))].$$

But π is at least two-to-one almost everywhere on $\mathsf{Image}(\phi) \subseteq \overline{M}_\pi$, implying $\ell(\phi) \geq 2|\pi(\mathsf{End}(\phi)) - \pi(\mathsf{Start}(\phi))|$. ■

Lemma 2.4.17 says that the bends are arcs on which μ measure is globally comparable to length measure. This allows us to promote μ to a bigger measure $\tilde{\mu}$ which is pointwise comparable to length inside bends at the cost of increasing the total measure by a bounded factor independent of Γ. In fact, at the cost of further increasing μ's mass by a bounded factor, we can take our proposed larger measure $\tilde{\mu}$ to be comparable to length on regions of Γ which extend a distance comparable to $\ell(\phi)$ out from ϕ in the e_1 direction.

Definition 2.4.8 ($\tilde{\mu}$ measure)**.** For any $\phi \in \Phi$, define

$$N_\phi := \{t \in I : \mathrm{dist}(\pi(\gamma(t)), \pi(\mathsf{Image}(\phi))) \leq 100\ell(\phi)\},$$

and set

$$N(\Phi) := \bigcup_{\phi \in \Phi} N_\phi.$$

We define a new weight $\tilde{\rho} : I \to [0, 2]$ as

$$\tilde{\rho}(t) := \begin{cases} 2, & t \in \mathsf{Domain}(\phi) \text{ for some } \phi \in \Phi \\ 1, & t \in N(\Phi) \setminus \bigcup_{\phi \in \Phi} \mathsf{Domain}(\phi) \\ \rho(t), & t \in I \setminus N(\Phi). \end{cases}$$

We use the value 2 in the first case so that $\tilde{\rho}(t) \geq \rho(t)$ for all $t \in I$. Put

$$d\tilde{\mu} := \gamma_*(\tilde{\rho}\ dt).$$

In words, $\tilde{\mu}$ is equal to twice length measure on bends, equal to length measure just outside of bends, and equal to μ measure far away from bends.

Looking back at Figure 2.7, we can see that $\tilde{\mu}(U_Q) = 2\ell(U_Q)$ because $\Gamma \cap U_Q$ is contained in a single bend ϕ. We will use $\tilde{\mu}$ as our primary accounting tool for bounding the beta-squared sum from above. We begin by verifying that the total mass of $\tilde{\mu}$ is controlled by the total mass of μ.

Lemma 2.4.18.

$$\tilde{\mu}(\Gamma) \lesssim \mu(\Gamma).$$

Proof. Fix $\phi \in \Phi$ and consider the region

$$U_\phi := \gamma\left(N_\phi \setminus \bigcup_{\substack{\phi' \in \Phi \\ \phi' \neq \phi}} \mathsf{Image}(\phi')\right)$$

Observe that we can bound the mass added in U_ϕ as follows:

$$\begin{aligned}
\tilde{\mu}(U_\phi) - \mu(U_\phi) &= \tilde{\mu}(U_\phi \setminus \mathsf{Image}(\phi)) + \tilde{\mu}(\phi) - [\mu(U_\phi \setminus \mathsf{Image}(\phi)) + \mu(\phi)] \\
&= \ell(U_\phi \setminus \mathsf{Image}(\phi)) - \mu(U_\phi \setminus \mathsf{Image}(\phi)) + 2\ell(\phi) - \mu(\phi) \\
&\leq 2\ell(\phi) + \int_{U_\phi \setminus \mathsf{Image}(\phi)} \gamma_1'(t)\ dt \\
&\leq \operatorname{diam}(\pi(U_\phi \setminus \mathsf{Image}(\phi))) + 2\ell(\phi) \leq 200\ell(\phi) + 2\ell(\phi) \leq 404\mu(\phi)
\end{aligned}$$

where the final inequality follows from Lemma 2.4.17. The fact that $\tilde{\rho}(t) \geq \rho(t)$ for all $t \in I$ implies $\tilde{\mu} - \mu$ is a positive measure so that, because $\tilde{\rho}(t) = \rho(t)$ for all $t \in I \setminus N(\Phi)$ and $N(\Phi) = \bigcup_{\phi \in \Phi} \gamma^{-1}(U_\phi)$,

$$\begin{aligned}
\tilde{\mu}(\Gamma) - \mu(\Gamma) &\leq \tilde{\mu}\left(\bigcup_{\phi \in \Phi} U_\phi\right) - \mu\left(\bigcup_{\phi \in \Phi} U_\phi\right) = (\tilde{\mu} - \mu)\left(\bigcup_{\phi \in \Phi} U_\phi\right) \\
&\leq \sum_{\phi \in \Phi} (\tilde{\mu} - \mu)(U_\phi) = \sum_{\phi \in \Phi} \tilde{\mu}(U_\phi) - \mu(U_\phi) \leq 404 \sum_{\phi \in \Phi} \mu(\phi) \leq 404\mu(\Gamma).
\end{aligned}$$

■

The final lemma we will prove in this section gives sufficient conditions for an inequality like $\tilde{\mu}(B) \gtrsim \ell(B)$ to hold for any Borel set B. The fact that $\tilde{\mu}$ is comparable to ℓ on $\gamma(M_\pi)$ means that in order for $\tilde{\mu}(B) \ll \ell(B)$ to hold, most of $\Gamma \cap B$ must be contained in $\gamma(S_\pi)$. Even then, it must be true that $\tilde{\rho} \ll 1$ on most of $\Gamma \cap B$ so that $\gamma_1' \approx 1$ inside B, constraining the total amount of length that B is allowed to contain. Lemma 2.4.20 proves a sort of contrapositive of this observation, showing that a lower bound on $\ell(B)$ translates into a lower bound on $\tilde{\mu}(B)$ in terms of $\ell(B)$. First, we will need a version of the area formula:

Lemma 2.4.19. *(Area formula) Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz map and let $g : \mathbb{R} \to \mathbb{R}$ be an integrable function. Then the map*

$$z \mapsto \sum_{x \in f^{-1}(\{z\})} g(x),$$

is measurable, and

$$\int_{\mathbb{R}} g(y)|f'(y)|dy = \int_{\mathbb{R}} \sum_{x \in f^{-1}(\{z\})} g(x) dz.$$

For proof of this result, see [Fed69] Theorem 3.2.5.

Lemma 2.4.20. *Fix $\delta < \frac{1}{10}$ and let $B \subseteq \ell_2$ be Borel. If either* *(i)* $\ell(B) \geq (1+\delta)\operatorname{diam}(B)$, *or*

(ii) $\tilde{\mu}(B) \geq \delta \operatorname{diam}(B)$,

then,

$$\tilde{\mu}(B) \geq \frac{\delta^3}{2}\ell(B).$$

Proof. We first prove that (i) implies the conclusion. Consider the set $E_{\delta^2} = \{t \in I : \tilde{\rho}(t) < \delta^2\} \cap \gamma^{-1}(B)$. Observe that $\tilde{\rho}(t) < \delta^2$ implies $\gamma_1'(t) > 1 - \delta^2$ so that $E_{\delta^2} \subseteq S_\pi$. The former inequality directly implies $t \in S_\pi$ so that $E_{\delta^2} \subseteq S_\pi$. Applying the area formula (Lemma 2.4.19) with the Lipschitz function $\gamma_1 : I \to \mathbb{R}$ and the integrable function $\frac{1}{\gamma_1'}\chi_{E_{\delta^2}}$ gives

$$\int_{E_{\delta^2}} \frac{1}{\gamma_1'}|\gamma_1'|\ dt = \int_{\mathbb{R}} \sum_{s \in \gamma_1^{-1}(u)} \frac{1}{\gamma_1'(s)}\chi_{E_{\delta^2}}(s)\ du = \int_{\mathbb{R}} \sum_{s \in \gamma^{-1}(\pi^{-1}(u)) \cap E_{\delta^2}} \frac{1}{\gamma_1'(s)}\ du.$$

Because $E_{\delta^2} \subseteq S_\pi$ and γ is injective, the set $\gamma^{-1}(\pi^{-1}(u)) \cap E_{\delta^2}$ contains at most one element, and is nonempty only if $u \in \pi(\gamma(E_{\delta^2})) \subseteq B$. Therefore, we get

$$\sum_{s \in \gamma^{-1}(\pi^{-1}(u)) \cap E_{\delta^2}} \frac{1}{\gamma_1'(s)} = \sum_{s \in \gamma^{-1}(\pi^{-1}(u)) \cap E_{\delta^2}} \frac{1}{\gamma_1'(s)}\chi_{\gamma_1(E_{\delta^2})}(u) \leq \frac{1}{1-\delta^2}\chi_{\gamma_1(E_{\delta^2})}(u).$$

Using these statements, the area formula simplifies to

$$\ell(E_{\delta^2}) = \int_{\mathbb{R}} \sum_{s \in \gamma^{-1}(\pi^{-1}(u)) \cap E_{\delta^2}} \frac{1}{\gamma_1'(s)}\ du \leq \int_{\gamma_1(E_{\delta^2})} \frac{1}{1-\delta^2}\ du \leq \frac{\operatorname{diam}(B)}{1-\delta^2}.$$

Now, define $C_{\delta^2} := B \cap \Gamma \setminus \gamma(E_{\delta^2})$. We have

$$\tilde{\mu}(B) \geq \tilde{\mu}(C_{\delta^2}) \geq \delta^2 \ell(C_{\delta^2}) = \delta^2(\ell(B) - \ell(E_{\delta^2})).$$

Adding in the lemma's hypothesis, we have both

$$\ell(B) \geq (1+\delta)\operatorname{diam}(B), \text{ and}$$
$$\ell(E_{\delta^2}) \leq \frac{\operatorname{diam}(B)}{1-\delta^2}.$$

This means

$$\frac{\ell(E_{\delta^2})}{\ell(B)} \leq \frac{\operatorname{diam}(B)}{1-\delta^2} \cdot \frac{1}{(1+\delta)\operatorname{diam}(B)} = 1 + \left(\frac{1}{(1+\delta)(1-\delta^2)} - 1\right)$$
$$= 1 - \frac{\delta - \delta^2 - \delta^3}{(1+\delta)(1-\delta^2)} \leq 1 - \frac{\delta - \frac{\delta}{10} - \frac{\delta}{100}}{1 + \frac{1}{10}} \leq 1 - \frac{\delta}{2}.$$

using the fact that $\delta < \frac{1}{10}$. Rearranging this inequality gives $\ell(B) - \ell(E_{\delta^2}) \geq \frac{\delta}{2}\ell(B)$. Therefore

$$\tilde{\mu}(B) \geq \delta^2(\ell(B) - \ell(E_{\delta^2})) \geq \frac{\delta^3}{2}\ell(B).$$

This concludes the proof that (i) implies the conclusion. We now show that (ii) implies the conclusion. From (i), it suffices to assume $\ell(B) < (1+\delta)\operatorname{diam}(B)$. Then,

$$\tilde{\mu}(B) \geq \delta \operatorname{diam}(B) \geq \delta\frac{1+\delta}{2}\operatorname{diam}(B) \geq \frac{\delta}{2}\ell(B).$$ ■

2.4.2.2 Martingale refinement: Bounds on the Δ_1 and $\Delta_{2.1.1}$ sums for Γ

In this section, we provide the refinements of Proposition 2.4.7 and (part of) Proposition 2.4.7 for a rectifiable Jordan arc Γ.

Lemma 2.4.21. *For any $Q \in \Delta'(M,K)$*

$$\tilde{\mu}(U_Q^{xx}) \gtrsim \ell(U_Q^{xx}).$$

Proof. In the proof of Lemma 2.4.6, we gave the existence of an arc $\xi_0 \subseteq U_Q^{xx}$ such that

$$\ell(U_Q^{xx}) \geq \ell(\gamma_Q \cap U_Q^{xx}) + \ell(\xi_0) \geq \left(1 + \frac{1}{10}\right) \operatorname{diam}(U_Q^{xx})$$

as in (2.43). Applying Lemma 2.4.20 gives the result. ∎

Proposition 2.4.22.

$$\sum_{Q \in \Delta_1} \beta_\Gamma(Q)^2 \operatorname{diam}(Q) \lesssim_A \sum_{Q \in \Delta_1} \operatorname{diam}(U_Q) \lesssim_{A,J} \tilde{\mu}(\Gamma).$$

Proof. Fix $\Delta' = \Delta'(M,K)$ and follow the proof of Proposition 2.4.7 to get

$$\sum_{Q \in \Delta'} \beta_\Gamma(Q) \operatorname{diam}(Q) \lesssim_A 2^{-M} \sum_{T \in \mathcal{T}_{\Delta'}^{16c_0}} \ell(U_{Q(T)}^{xx}) \lesssim 2^{-M} \tilde{\mu}(U_{Q(T)}^{xx}) \leq 2^{-M} \tilde{\mu}(\Gamma).$$

The result follows by summing over M and K. ∎

We wish to argue for similarly for $\Delta_{2.1}$, but an inequality like that of Lemma 2.4.21 does not hold for $\Delta_{2.1}$ balls. We proceed by splitting $\Delta_{2.1}$ into a subfamily where $\tilde{\mu}(U_Q) \gtrsim \ell(U_Q)$ on which we can run the martingale argument and a leftover subfamily on which $\tilde{\mu}(U_Q) \ll \ell(U_Q)$. We define

$$\begin{aligned}
\Delta_{2.1.1} &= \{Q \in \Delta_{2.1} : \tilde{\mu}(U_Q) \geq \epsilon_3^2 \ell(U_Q)\}, \\
\Delta_{2.1.2} &= \{Q \in \Delta_{2.1} : \tilde{\mu}(U_Q) < \epsilon_3^2 \ell(U_Q)\} \\
&= \Delta_{2.1} \setminus \Delta_{2.1.1}.
\end{aligned}$$

The collection $\Delta_{2.1.1}$ can be handled with the addition of one inequality to the proof of Proposition 2.4.13.

Proposition 2.4.23.

$$\sum_{Q \in \Delta_{2.1.1}} \beta_\Gamma(Q)^2 \operatorname{diam}(Q) \lesssim_A \sum_{Q \in \Delta_{2.1.1}} \operatorname{diam}(U_Q) \lesssim_{J,\epsilon_1,\epsilon_3} \mu(\Gamma).$$

Proof. We apply Lemma 2.4.5 with $D = 2\epsilon_1^{-1}, q = \frac{1}{1+\frac{1}{50}}$ to the family $\mathscr{L} = \Delta_{2.1.1}$ with ordered by the forest structure $\mathcal{T}_{\Delta_{2.1.1}}^{c_0}$ and use the produced collection $\{w_Q\}_{Q \in \Delta_{2.1.1}}$ to calculate, as in the proof of Proposition 2.4.13,

$$\sum_{Q \in \Delta_{2.1.1}} \operatorname{diam}(U_Q) \lesssim_{\epsilon_1} \sum_{T \in \mathcal{T}_{\Delta_{2.1.1}}^{c_0}} \ell(U_{Q(T)}) \lesssim_{\epsilon_3} \sum_{T \in \mathcal{T}_{\Delta_{2.1.1}}^{c_0}} \tilde{\mu}(U_{Q(T)}) \lesssim_J \tilde{\mu}(\Gamma).$$ ∎

2.4.2.3 Bound on the $\Delta_{2.1.2}$ sum

We now handle the family $\Delta_{2.1.2}$, beginning with a general summary of the argument. We show that $Q \in \Delta_{2.1.2}$ implies $U_Q \cap \Gamma$ essentially consists of a small perturbation (in length) of a line segment through the center of Q which is parallel to the chord line of Γ (see Lemma 2.4.24). Because of the definition of the bends and $\tilde{\mu}$, the nearly-segment pieces inside disjoint cores in this family project to line segments on the chord line of Γ which have controlled overlap (see Lemma 2.4.25). By decomposing $\Delta_{2.1.2}$ into a sequence of "levels", each of which consists of a disjoint subfamily of $\Delta_{2.1}$, we can exploit this packing lemma by controlling the number of balls which have overlapping τ_Q arcs (see Remark 2.4.10), controlling the core diameter sum on each level in terms of a disjoint collection of subarcs of τ_Q's (see Lemma 2.4.27 and 2.4.28). This all works for the subcollection of cores which lie on the "inner" region of parent cores. The proof is completed by showing that the "outer" family of cores is controlled by the inner family (see Lemma 2.4.26).

Let us begin the proof. Define

$$\eta_Q := \{x_Q + te_1 : t \in \mathbb{R}\}.$$

This is the line parallel to the chord of Γ which passes through the center of Q. The next lemma states that any $Q \in \Delta_{2.1.2}$ has γ_Q close to η_Q with constant dependent on ϵ_3.

Lemma 2.4.24. *Let $Q \in \Delta_{2.1.2}$ and $\xi \subseteq \Gamma$ such that $\xi \cap \gamma_Q = \varnothing$. Then,*

(i) $\gamma_Q \subseteq B(\eta_Q, 100\epsilon_3 \operatorname{diam}(Q))$, and

(ii) $\xi \cap \pi^{-1}((1-10\epsilon_1)c_0 Q) = \varnothing$.

Proof. We begin by proving (i). We will assume that $\gamma_Q \not\subseteq B(\eta_Q, 100\epsilon_3 \operatorname{diam}(Q))$ and show that $\tilde{\mu}(U_Q) > \epsilon_3^2 \ell(U_Q)$. Let $\gamma'' := \mathsf{Edge}(\gamma_Q)$, $\gamma' := \gamma'' \cap c_0 Q =: [x, y]$, and let φ be a connected component of $\gamma_Q \cap c_0 Q$ of largest diameter. We will show that γ' makes angle of order ϵ_3 with η_Q, derive a lower bound for the "excess" length of γ', and then use that to bound $\frac{\tilde{\mu}(\varphi)}{\ell(\varphi)}$ from below. First, observe that $\tilde{\beta}(\gamma_Q) \operatorname{diam}(2Q) \leq 2\epsilon_2 \operatorname{diam}(Q)$ implies, by Lemma 2.4.2,

$$\gamma_Q \subseteq B(\gamma'', 2\epsilon_2 \operatorname{diam}(Q)) \text{ and } \gamma'' \subseteq B(\gamma_Q, 2\epsilon_2 \operatorname{diam}(Q)). \tag{2.52}$$

Let $z \in \gamma_Q$ be such that $\operatorname{dist}(z, \eta_Q) \geq 100\epsilon_3 \operatorname{diam}(Q)$. Then, there exists $z'' \in \gamma''$ such that $|z - z''| \leq 2\epsilon_2 \operatorname{diam}(Q) \leq \epsilon_3 \operatorname{diam}(Q)$ so that $\operatorname{dist}(z'', \eta_Q) \geq \operatorname{dist}(z, \eta_Q) - |z - z''| \geq 99\epsilon_3 \operatorname{diam}(Q)$. We can define the angle $\theta := \angle(\gamma', \eta_Q) = \angle(\gamma'', \eta_Q)$ by translating the segment γ'' so that one of its endpoints lies in η_Q and measuring the angle in the (at most 2-dimensional) plane containing η_Q and this translated segment. The previous estimates then imply $\tan(\theta) \geq \frac{99\epsilon_3 \operatorname{diam}(Q)}{\operatorname{diam}(2Q)} \geq 45\epsilon_3$. Using the Pythagorean theorem, we get $|x-y|^2 = |\pi(x) - \pi(y)|^2 + |\pi^\perp(x) - \pi^\perp(y)|^2$, and the lower bound on $\tan(\theta)$ implies $|\pi^\perp(x) - \pi^\perp(y)| \geq 45\epsilon_3 |\pi(x) - \pi(y)|$. Using the difference of squares formula with the Pythagorean theorem estimate, we compute

$$\begin{aligned}\frac{|x-y| - |\pi(x) - \pi(y)|}{|x-y|} &= \frac{|\pi^\perp(x) - \pi^\perp(y)|^2}{|x-y|(|x-y| + |\pi(x) - \pi(y)|)} \geq \frac{45^2\epsilon_3^2 |\pi(x) - \pi(y)|^2}{|x-y|(|x-y| + |\pi(x) - \pi(y)|)} \\ &\geq 45^2\epsilon_3^2 \frac{\frac{1}{4}\operatorname{diam}(U_Q)^2}{\operatorname{diam}(U_Q) \cdot 2\operatorname{diam}(U_Q)} \geq 100\epsilon_3^2 \end{aligned} \tag{2.53}$$

Now, let x', y' be the endpoints of φ on $\partial(c_0 Q)$. By (2.52) and the fact that $x_Q \in \gamma_Q$, we have

$$|x - x'| \leq 8\epsilon_2 \operatorname{diam}(Q) \text{ and } |y - y'| \leq 8\epsilon_2 \operatorname{diam}(Q).$$

We estimate

$$\begin{aligned}\frac{\tilde{\mu}(\varphi)}{\ell(\varphi)} &\geq \frac{\mu(\varphi)}{\ell(\varphi)} \geq 1 - \frac{|\pi(x') - \pi(y')|}{|x' - y'|} = \frac{|x' - y'| - |\pi(x') - \pi(y')|}{|x' - y'|} \\ &\geq \frac{|x - y| - |\pi(x) - \pi(y)| - 32\epsilon_2 \operatorname{diam}(Q)}{|x - y| + 16\epsilon_2 \operatorname{diam}(Q)} \\ &\geq \frac{1}{2}\frac{|x - y| - |\pi(x) - \pi(y)|}{|x - y|} - \frac{32\epsilon_2 \operatorname{diam}(Q)}{|x - y| + 16\epsilon_2 \operatorname{diam}(Q)} \geq 50\epsilon_3^2 - 64c_0^{-1}\epsilon_2 \geq 40\epsilon_3^2\end{aligned} \tag{2.54}$$

where we used (2.53) and $|x-y| \geq \frac{1}{2}c_0 \operatorname{diam}(Q)$ in the penultimate inequality. We would like to show that $\tilde{\mu}(\gamma_Q \cap U_Q) \geq \epsilon_3^2 \ell(\gamma_Q \cap U_Q)$. Given the preceding inequality, the only possible obstruction is the existence of components of $\gamma_Q \cap U_Q$ with long length and small $\tilde{\mu}$ measure. It suffices to consider the case where $\tilde{\mu}(\gamma_Q \cap U_Q \setminus \varphi) \leq 40\epsilon_3^2 \ell(\gamma_Q \cap U_Q \setminus \varphi)$. Unpacking this inequality, we see

$$\begin{aligned}40\epsilon_3^2 \ell(\gamma_Q \cap U_Q \setminus \varphi) &\geq \tilde{\mu}(\gamma_Q \cap U_Q \setminus \varphi) \geq \mu(\gamma_Q \cap U_Q \setminus \varphi) \\ &\geq \ell(\gamma_Q \cap U_Q \setminus \varphi) - \operatorname{diam}(\pi(\gamma_Q \cap U_Q \setminus \varphi)).\end{aligned}$$

Rearranging gives

$$\ell(\gamma_Q \cap U_Q \setminus \varphi) \leq \frac{\operatorname{diam}(\pi(\gamma_Q \cap U_Q \setminus \varphi))}{1 - \epsilon_3^2} \leq \frac{\operatorname{diam}(U_Q)}{1 - 40\epsilon_3^2} \leq 2\ell(\varphi) \tag{2.55}$$

where the final inequality follows since $x_Q \in \gamma_Q \in S(Q)$. Using (2.54) and (2.55),

$$\tilde{\mu}(\gamma_Q \cap U_Q) \geq \tilde{\mu}(\varphi) \geq 40\epsilon_3^2 \ell(\varphi) \geq \frac{1}{2}(40\epsilon_3^2 \ell(\varphi) + 10\epsilon_3^2 \ell(\gamma_Q \cap U_Q \setminus \varphi)) \geq 5\epsilon_3^2 \ell(\gamma_Q \cap U_Q).$$

With this intermediate inequality, we can now prove the lemma. Arguing as in the proof of $\tilde{\mu}(\gamma_Q \cap U_Q) \geq 5\epsilon_3^2 \ell(\gamma_Q \cap U_Q)$ above, it suffices to assume that $\tilde{\mu}(U_Q \setminus \gamma_Q) \leq 5\epsilon_3^2 \ell(U_Q \setminus \gamma_Q)$. We get

$$\ell(U_Q \setminus \gamma_Q) \leq \frac{\operatorname{diam}(\pi(U_Q \setminus \gamma_Q))}{1 - 5\epsilon_3^2} \leq 2\ell(\gamma_Q \cap U_Q).$$

Multiplying this inequality on both sides by ϵ_3^2, we use this to estimate

$$\tilde{\mu}(U_Q) \geq \tilde{\mu}(U_Q \cap \gamma_Q) \geq 5\epsilon_3^2 \ell(U_Q \cap \gamma_Q) \geq 3\epsilon_3^2 \ell(U_Q \cap \gamma_Q) + \epsilon_3^2 \ell(U_Q \setminus \gamma_Q) > \epsilon_3^2 \ell(U_Q).$$

This concludes the proof of (i). We now prove (ii). Suppose $Q \in \Delta_{2.1.2}$ is such that $\xi \cap \pi^{-1}(1-10\epsilon_1)c_0 Q \neq \varnothing$ and let $x \in \xi \cap \pi^{-1}((1-10\epsilon_1)c_0 Q)$. We will show that $\tilde{\mu}(U_Q) > \epsilon_3^2 \ell(U_Q)$. By Lemma 2.4.24, we have either $\gamma_Q \cap U_Q \cap \{z : \pi(z) \geq \pi(x)\} \subseteq \gamma(\overline{M}_\pi)$ or $\gamma_Q \cap U_Q \cap \{z : \pi(z) \leq \pi(x)\} \subseteq \gamma(\overline{M}_\pi)$ because the extension of γ_Q to an arc containing ξ must cross from the boundary of $2Q$ to x with an arc disjoint from γ_Q. Therefore, we conclude that $\gamma_Q \cap U_Q$ contains an arc $\zeta \subseteq \gamma_Q \cap c_0 Q \cap (1-10\epsilon_1)c_0 Q$ with $\tilde{\mu}(\zeta) = 2\ell(\zeta) \geq 2(10\epsilon_1 c_0)\operatorname{rad}(Q) \geq 10\epsilon_1 \operatorname{diam}(U_Q)$. This means $\tilde{\mu}(U_Q) \geq 2\epsilon_1 \operatorname{diam}(U_Q)$ so that, by Lemma 2.4.20,

$$\tilde{\mu}(U_Q) \geq \frac{(2\epsilon_1)^3}{2}\ell(U_Q) \geq \epsilon_1^3 \ell(U_Q) > \epsilon_3^2 \ell(U_Q). \qquad \blacksquare$$

This lemma places strong restrictions of the geometry of $\Gamma \cap 2Q$. The fact that γ_Q is restricted to be nearly parallel to η_Q on the scale of $\operatorname{diam}(Q)$ allows us to derive packing estimates for disjoint families of balls in $\Delta_{2.1.2}$ along the direction of the chord of Γ as in the following lemma.

Lemma 2.4.25. *For any $Q_1, Q_2 \in \Delta_{2.1.2}$ such that $U_{Q_1} \cap U_{Q_2} = \varnothing$,*

$$\pi\left(\frac{1}{4}c_0 Q_1\right) \cap \pi\left(\frac{1}{4}c_0 Q_2\right) = \varnothing.$$

Proof. Suppose by way of contradiction that $\pi\left(\frac{1}{4}c_0 Q_1\right) \cap \pi\left(\frac{1}{4}c_0 Q_2\right) \neq \varnothing$ and assume that $\operatorname{diam}(Q_2) \leq \operatorname{diam}(Q_1)$. The fact that $c_0 Q_1 \cap c_0 Q_2 = \varnothing$ and $\pi\left(\frac{1}{4}c_0 Q_1\right) \cap \pi\left(\frac{1}{4}c_0 Q_2\right) \neq \varnothing$ imply, respectively,

$$\begin{aligned} |x_{Q_1} - x_{Q_2}| &\geq c_0(\operatorname{rad}(Q_1) + \operatorname{rad}(Q_2)), \\ |\pi(x_{Q_1}) - \pi(x_{Q_2})| &\leq \frac{c_0}{4}(\operatorname{rad}(Q_1) + \operatorname{rad}(Q_2)). \end{aligned}$$

From these, we estimate

$$\begin{aligned} |\pi^\perp(x_{Q_1}) - \pi^\perp(x_{Q_2})| &\geq |x_{Q_1} - x_{Q_2}| - |\pi(x_{Q_1}) - \pi(x_{Q_2})| \\ &\geq \frac{3c_0}{4}(\operatorname{rad}(Q_1) + \operatorname{rad}(Q_2)) \geq \frac{c_0}{2}\operatorname{rad}(Q_1) + c_0 \operatorname{rad}(Q_2). \end{aligned}$$

Therefore, $B(\eta_{Q_1}, 100\epsilon_3 \operatorname{diam}(Q_1)) \cap U_{Q_2} = \varnothing$ because $100\epsilon_3 \operatorname{diam}(Q_1) < \frac{c_0}{4}\operatorname{rad}(Q_1)$ so that Lemma 2.4.24 implies $\gamma_{Q_1} \cap U_{Q_2} = \varnothing$ and $\pi(U_{Q_2}) \subseteq \pi(\gamma_{Q_1})$. Because $\pi(U_{Q_2}) \subseteq \pi(\gamma_{Q_2})$, we get $U_{Q_2} \subseteq M_\pi$, implying $\tilde{\mu}(U_{Q_2}) = 2\ell(U_{Q_2})$, contradicting the fact that $\tilde{\mu}(U_{Q_2}) < \epsilon_3^2 \ell(U_{Q_2})$ because $Q_2 \in \Delta_{2.1.2}$. $\blacksquare$

Definition 2.4.9 (Levels and inner/outer cores)**.** Because we would prefer to work with pairwise disjoint subfamilies of $\Delta_{2.1.2}$ in view of Lemma 2.4.25, we will divide $\Delta_{2.1.2}$ into pairwise disjoint "levels" using its tree structure. Indeed, fix j, $1 \leq j \leq J$ and recall $\mathscr{Q}_j$ is one of the J families of balls ordered by inclusion of cores constructed in Proposition 2.1.19. Consider the family $\Delta^j_{2.1.2} := \Delta_{2.1.2} \cap \mathscr{Q}_j$. We define the k-th level of $\Delta^j_{2.1.2}$ for $k \geq 0$ as

$$\mathscr{L}_k := \{Q \in \Delta^j_{2.1.2} : Q \in C^k(Q(T)),\ T \in \mathcal{T}^{c_0}_{\Delta^j_{2.1.2}}\}$$

where we set $C^0(Q(T)) = \{Q(T)\}$. The family $\mathscr{L}_k$ is pairwise disjoint for any $k \geq 0$ and $\Delta^j_{2.1.2} = \bigcup_{k\geq 0} \mathscr{L}_k$. We additionally want to single out balls which live away from from the boundary of the core of their parent in the tree structure. We define the inner and outer balls:

$$\begin{aligned} &\Delta_I := \mathscr{L}_0 \cup \{Q \in \Delta^j_{2.1.2} \setminus \mathscr{L}_0 : 2Q \subseteq (1-5\epsilon_1)c_0 P(Q)\}, \\ &\Delta_O := \Delta^j_{2.1.2} \setminus \Delta_I. \end{aligned}$$

We can show that the diameters of the outer cores are controlled by the diameters of the inner cores using Lemma 2.4.25 and some algebra.

Lemma 2.4.26.
$$\sum_{Q\in\Delta_O} \operatorname{diam}(U_Q) \lesssim \sum_{Q\in\Delta_I} \operatorname{diam}(U_Q).$$

Proof. Fix $Q \in \Delta^j_{2.1.2}$ and let $Q' \in C(Q)$. Recall that $\operatorname{diam}(2Q') \le \epsilon_1 \operatorname{diam}(U_Q)$ as in (2.45). Hence, if $Q' \in \Delta_O$, then Lemma 2.4.24 implies

$$(\Gamma \setminus \gamma_Q) \cap 2Q' \cap \pi^{-1}\left((1-10\epsilon_1)c_0 Q\right) = \varnothing. \tag{2.56}$$

Because the cores of balls in $C(Q)$ are pairwise disjoint, the projection lemma 2.4.25 implies

$$\sum_{Q'\in C(Q)\cap\Delta_O} \operatorname{diam}(U_{Q'}) \le 5 \sum_{Q'\in C(Q)\cap\Delta_O} \mathcal{H}^1\left(\pi\left(\frac{c_0}{4}Q'\right)\right) \le 200\epsilon_1 c_0 \operatorname{diam}(Q) \le 200\epsilon_1 \operatorname{diam}(U_Q).$$

Summing this inequality over $Q \in \mathscr{L}_k$, we get

$$\begin{aligned}\sum_{Q'\in\mathscr{L}_{k+1}\cap\Delta_O} \operatorname{diam}(U_{Q'}) &= \sum_{Q\in\mathscr{L}_k}\sum_{Q'\in C(Q)\cap\Delta_O} \operatorname{diam}(U_{Q'}) \le 200\epsilon_1 \sum_{Q\in\mathscr{L}_k} \operatorname{diam}(U_Q)\\ &\le 200\epsilon_1 \sum_{Q'\in\mathscr{L}_k\cap\Delta_O} \operatorname{diam}(U_{Q'}) + 200\epsilon_1 \sum_{Q'\in\mathscr{L}_k\cap\Delta_I} \operatorname{diam}(U_{Q'}).\end{aligned} \tag{2.57}$$

In order to simplify the notation, define

$$\begin{aligned}S_k^O &:= \sum_{Q'\in\mathscr{L}_k\cap\Delta_O} \operatorname{diam}(U_{Q'}),\\ S_k^I &:= \sum_{Q'\in\mathscr{L}_k\cap\Delta_I} \operatorname{diam}(U_{Q'}).\end{aligned}$$

The lemma will follow from some algebraic manipulations of (2.57). We can restate (2.57) in this notation as

$$S_{k+1}^O \le 200\epsilon_1 S_k^O + 200\epsilon_1 S_k^I.$$

Iterating this inequality over k, we get

$$S_k^O \le \sum_{n=0}^{k} (200\epsilon_1)^n S_{k-n}^I$$

which gives

$$\begin{aligned}\sum_{Q\in\Delta_O} \operatorname{diam}(U_Q) &= \sum_{k=0}^{\infty} S_k^O \le \sum_{k=0}^{\infty}\sum_{n=0}^{k} (200\epsilon_1)^n S_{k-n}^I = \sum_{\substack{(k,n)\in\mathbb{N}\times\mathbb{N}\\ n\le k}} (200\epsilon_1)^n S_{k-n}^I\\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (200\epsilon_1)^n S_m^I \lesssim \sum_{m=0}^{\infty} S_m^I = \sum_{Q\in\Delta_I} \operatorname{diam}(U_Q).\end{aligned}$$ ■

With this lemma, we now concentrate on proving $\sum_{Q\in\Delta_I}\operatorname{diam}(U_Q) \lesssim \tilde{\mu}(\Gamma)$. Because $\Delta_I \subseteq \Delta_{2.1}$, any $Q \in \Delta_I$ has the existence of an arc τ_Q as described in the section following Lemma 2.4.9. It follows from the neighborhood containment of γ_Q in Lemma 2.4.24 that there exists a subarc $\zeta_Q \subseteq \tau_Q \cap (1-c_0)2Q \subseteq \gamma(\overline{M}_\pi)$ such that

$$\mathsf{Diam}(\zeta_Q) \geq \frac{1}{10}\operatorname{diam}(2Q).$$

These properties imply

$$\tilde{\mu}(\zeta_Q) \geq \frac{1}{5}\operatorname{diam}(2Q). \tag{2.58}$$

Fix a level $\mathscr{L}_k^I := \mathscr{L}_k \cap \Delta_I$ and define an equivalence relation on $\mathscr{L}_k^I$ by putting $Q \sim Q'$ if and only if there exists a collection $\{Q_n\}_{n\geq 1} \subseteq \mathscr{L}_k^I$ such that $\zeta_{Q_i} \cap \zeta_{Q_{i+1}} \neq \varnothing$ while both $\zeta_Q \cap \bigcup_{n\geq 1}\zeta_{Q_n} \neq \varnothing$ and $\zeta_{Q'} \cap \bigcup_{n\geq 1}\zeta_{Q_n} \neq \varnothing$. That is, $Q \sim Q'$ if and only if ζ_Q and $\zeta_{Q'}$ can be connected by a connected path of ζ arcs from $\mathscr{L}_k^I$. This partitions $\mathscr{L}_k^I$ into equivalence classes $\mathscr{L}_k^I = \bigcup_{i\in I_k}\mathscr{C}_{k,i}$. In each equivalence class $\mathscr{C}_{k,i}$, there exists a ball $Q_{k,i}^M$ of maximal diameter. The arc $\zeta_{Q_{k,i}^M}$ dominates the sum of diameters of balls in this equivalence class in the sense of the following lemma:

Lemma 2.4.27.

$$\sum_{Q\in\mathscr{C}_{k,i}}\operatorname{diam}(U_Q) \lesssim \tilde{\mu}(\zeta_{Q_{k,i}^M}).$$

Proof. Define $\zeta_{k,i} := \bigcup_{Q\in\mathscr{C}_{k,i}}\zeta_Q$. We claim that for any $Q \in \mathscr{C}_{k,i}$,

$$0 < \operatorname{dist}(\pi(x_Q), \pi(\mathsf{Image}(\zeta_{k,i}))) \leq \operatorname{diam}(2Q) \leq \operatorname{diam}(2Q_{k,i}^M).$$

Indeed, to prove the left inequality, suppose that $\operatorname{dist}(\pi(x_Q), \pi(\mathsf{Image}(\zeta_{k,i}))) = 0$. Because $\zeta_Q \subseteq \gamma(\overline{M}_\pi)$ for any $Q \in \mathscr{L}_k$, we know that $\zeta_{k,i} \subseteq \gamma(\overline{M}_\pi)$. Therefore, there exists $\phi \in \Phi$ such that $\zeta_{k,i} \subseteq \phi$ and $\ell(\phi) \geq \ell(\zeta_{Q_{k,i}^M}) \geq \frac{1}{10}\operatorname{diam}(2Q_{k,i}^M) \geq \operatorname{diam}(U_Q)$. Recalling the definition of $N(\Phi)$ (see Definition 2.4.8), it follows from the fact that $\sup_{x\in U_Q}\operatorname{dist}(\pi(x), \pi(\mathsf{Image}(\phi))) \leq \operatorname{diam}(U_Q)$ that $\gamma^{-1}(U_Q) \subseteq N(\Phi)$, implying $\tilde{\mu}(U_Q) \geq \ell(U_Q)$ which contradicts the fact that $Q \in \Delta_{2.1.2}$. For the right inequality above, notice that $\zeta_Q \subseteq 2Q$ so that $\zeta_{k,i} \cap 2Q \neq \varnothing$. Therefore, by Lemma 2.4.25, $\{\pi\left(\frac{c_0}{4}Q\right)\}_{Q\in\mathscr{C}_{k,i}}$ is a collection of pairwise disjoint intervals of total length less than $10\operatorname{diam}(Q_{k,i}^M)$. This means

$$\sum_{Q\in\mathscr{C}_{k,i}}\operatorname{diam}(U_Q) \leq 5\sum_{Q\in\mathscr{C}_{k,i}}\mathcal{H}^1\left(\pi\left(\frac{c_0}{4}Q\right)\right) \leq 50\operatorname{diam}(Q_{k,i}^M) \leq 250\tilde{\mu}(\zeta_{Q_{k,i}^M})$$

using (2.58) in the final inequality. ■

The following lemma gives the reason for restricting this argument to the inner cores.

Lemma 2.4.28. *The arcs in the collection $\{\zeta_{Q_{k,i}^M}\}_{k\geq 0, i\in I_k}$ have pairwise disjoint images. As a result,*

$$\sum_{Q\in\Delta_I}\operatorname{diam}(U_Q) \lesssim \tilde{\mu}(\Gamma).$$

Proof. Fix $k, k' \geq 0$ and $i \in I_k,\ i' \in I_{k'}$. Because the cores of balls in $\mathscr{L}_k$ are pairwise disjoint for any $k \geq 0$, we can assume $k > k' \geq 0$. For ease of notation, let $Q_1 := Q^M_{k,i},\ Q_2 := Q^M_{k',i'}$, and $Q' = P(Q_1)$ which exists because $k > 0$. Suppose by way of contradiction that $\mathsf{Image}(\zeta_{Q_1}) \cap \mathsf{Image}(\zeta_{Q_2}) \neq \varnothing$. We first claim that $U_{Q'} \cap U_{Q_2} = \varnothing$. Indeed, further suppose by way of contradiction that $U_{Q_2} \cap U_{Q'} \neq \varnothing$. Then $k < k'$ implies $Q' \in C^m(Q_2)$ for some $m \geq 0$. Because $Q_1 \in \Delta_I$ and because $\operatorname{diam}(Q_2) \geq \operatorname{diam}(Q') > \operatorname{diam}(Q_1)$, we have

$$2Q_1 \subseteq (1 - 5\epsilon_1)c_0 Q', \text{ and}$$

$$\operatorname{diam}(2Q_1) \leq \epsilon_1 \operatorname{diam}(U_{Q'}).$$

We conclude $2Q_1 \subseteq U_{Q'} \subseteq U_{Q_2}$, which contradicts $\mathsf{Image}(\zeta_{Q_1}) \cap \mathsf{Image}(\zeta_{Q_2}) \neq \varnothing$ because $\zeta_{Q_2} \cap U_{Q_2} = \varnothing$ by definition as a subarc of τ_{Q_2}.

From this claim, we see that $\operatorname{dist}(2Q_1, x_{Q_2}) \geq 4\epsilon_1 \operatorname{diam}(U_{Q'})$ so that we can again conclude $\operatorname{diam}(Q_2) \geq \operatorname{diam}(Q')$, for otherwise we would have $2Q_1 \cap 2Q_2 = \varnothing$ which is in contradiction to our starting assumption that $\mathsf{Image}(\zeta_{Q_1}) \cap \mathsf{Image}(\zeta_{Q_2}) \neq \varnothing$. Now, because $\zeta_{Q_2}, \zeta_{Q_1} \subseteq \gamma(M_\pi)$ and $\mathsf{Image}(\zeta_{Q_2}) \cap \mathsf{Image}(\zeta_{Q_1}) \neq \varnothing$, we can conclude that there exists a bend $\phi \in \Phi$ such that $\mathsf{Image}(\zeta_{Q_1}) \cup \mathsf{Image}(\zeta_{Q_2}) \subseteq \mathsf{Image}(\phi)$, hence $\ell(\phi) \geq \ell(\zeta_{Q_2}) \geq \frac{1}{10} \operatorname{diam}(2Q_2)$. Because $2Q_1 \cap 2Q_2 \neq \varnothing$, this implies $\gamma^{-1}(Q_1) \subseteq N(\Phi)$ so that $\tilde{\mu}(U_{Q_1}) \geq \ell(U_{Q_1})$, contradicting the fact that $Q \in \Delta_{2.1.2}$ and implying our assumption that $\mathsf{Image}(\zeta_{Q_1}) \cap \mathsf{Image}(\zeta_{Q_2}) \neq \varnothing$ must be false. This proves the first claim of the lemma. Using Lemma 2.4.27, we get

$$\sum_{Q \in \Delta_I} \operatorname{diam}(U_Q) = \sum_{k \geq 0} \sum_{i \in \mathscr{C}_{k,i}} \sum_{Q \in \mathscr{C}_{k,i}} \operatorname{diam}(U_Q) \lesssim \sum_{k \geq 0} \sum_{i \in \mathscr{C}_{k,i}} \tilde{\mu}(\zeta_{Q^M_{k,i}}) \leq \tilde{\mu}(\Gamma). \qquad \blacksquare$$

Now that we have controlled the inner cores, we can finish the proof of the bound for $\Delta_{2.1.2}$ and of the proof of Theorem A.

Proposition 2.4.29.

$$\sum_{Q \in \Delta_{2.1.2}} \beta_\Gamma(Q)^2 \operatorname{diam}(Q) \lesssim_A \sum_{Q \in \Delta_{2.1.2}} \operatorname{diam}(U_Q) \lesssim_J \tilde{\mu}(\Gamma).$$

Proof. Using Lemmas 2.4.26 and 2.4.28, we get

$$\sum_{Q \in \Delta_{2.1.2}} \operatorname{diam}(U_Q) = \sum_{j=1}^{J} \sum_{Q \in \Delta^j_{2.1.2}} \operatorname{diam}(U_Q) \lesssim_J \sum_{Q \in \Delta_I} \operatorname{diam}(U_Q) + \sum_{Q \in \Delta_O} \operatorname{diam}(U_Q) \lesssim \tilde{\mu}(\Gamma). \quad \blacksquare$$

This completes the proof of Theorem A.

Remark 2.4.30 (General rectifiable arcs). Given a rectifiable arc Γ_r with arc length parameterization γ_r, one can proceed as for Jordan arcs and define the measure μ by $d\mu := (\gamma_r)_*(\rho\, d\ell)$. Lemma 2.4.16 likely holds, and one can define bends and likely carry out a similar program to that of Section 2.4.2 to show that $\sum_{Q \in \mathscr{A}} \beta_{\Gamma_r}(Q)^2 \operatorname{diam}(Q) \lesssim_A \ell(\Gamma_r) - \operatorname{crd}(\Gamma_r)$. This is importantly weaker than the more desirable inequality

$$\sum_{Q \in \mathscr{A}} \beta_{\Gamma_r}(Q)^2 \operatorname{diam}(Q) \lesssim_A \mathcal{H}^1(\Gamma_r) - \operatorname{crd}(\Gamma_r). \tag{2.59}$$

If one would like to achieve 2.59 via methods similar to those used here, one likely needs a stronger definition of μ. For any $t \in I$, set $m(t) := \inf\{\gamma_1'(s) : \gamma(s) = \gamma(t)\}$. A more prudent choice of μ might be something like

$$d\mu_r := \sigma_r d\mathcal{H}^1$$

where

$$\sigma_r(x) := \begin{cases} 1 - \gamma_1'(t), & \gamma_1'(t) \text{ exists and } \gamma_1'(t) = \inf\{\gamma_1'(s) : s \in \gamma^{-1}(x)\} \\ 0 & \text{otherwise.} \end{cases}$$

That is, we assign to x the maximal ρ value achieved on $\gamma^{-1}(x)$. We do not investigate this approach further here.

2.5 Theorem B

In this section, we show how making minor modifications to the proof of the $\gtrsim$ direction of Theorem 1.3 in [Bis20] gives a proof of Theorem B. First, we need a slightly weaker version of Theorem B.

Theorem 2.5.1 ([Bis20] Theorem 1.1 in $\mathbb{R}^n$)**.** *Let $\Gamma \subseteq \ell_2$ be a rectifiable Jordan arc. For any multiresolution family $\mathscr{H}$ associated to Γ with inflation factor $A > 30$, we have*

$$\ell(\Gamma) - \operatorname{diam}(\Gamma) \lesssim \sum_{Q \in \mathscr{H}} \beta_\Gamma(Q)^2 \operatorname{diam}(Q).$$

Proof. The proof is similar in form to Bishop's proof in $\mathbb{R}^n$, but we construct coverings of the curve by pieces of Γ inside Voronoi cells centered at net points rather than dividing convex hulls of pieces of the curve along diameter segments. Assume $\operatorname{diam}(\Gamma) = 1$. Define

$$\mathcal{V}_n := \{V_n(x) : x \in X_n\}$$

where, given any $x \in X_n$,

$$V_n(x) := \{y \in \Gamma : \forall z \in X_n,\ |x - y| \leq |z - y|\}.$$

Since X_n is a 2^{-n}-net centered on Γ, it is clear that

$$2^{-n-1} \leq \operatorname{diam}(V_n(x)) < 2^{-n+1}. \tag{2.60}$$

By definition, for any $n \geq 0$ we also have $\Gamma = \bigcup_{x \in X_n} V_n(x)$ so that

$$\ell(\Gamma) = \mathcal{H}^1(\Gamma) \leq \limsup_{n \to \infty} \sum_{x \in X_n} \operatorname{diam}(V_n(x)).$$

This means is suffices to prove

$$\sum_{x \in X_n} \operatorname{diam}(V_n(x)) \leq \operatorname{diam}(\Gamma) + C \sum_{Q \in \mathscr{H}} \beta_\Gamma(Q)^2 \operatorname{diam}(Q) \tag{2.61}$$

for any $n \geq 0$ and some $C > 0$. We will show

$$\sum_{x \in X_n} \operatorname{diam}(V_n(x)) \leq \sum_{y \in X_{n-1}} \operatorname{diam}(V_{n-1}(y)) + C' \sum_{Q \in \mathscr{G}_n} \beta_\Gamma(Q)^2 \operatorname{diam}(Q) \tag{2.62}$$

where each ball $Q \in \mathscr{H}$ will only appear in $\mathscr{G}_n$ for a bounded number of values of n. We can prove Theorem 2.5.1 by repeatedly applying inequality (2.62). Indeed,

$$\begin{aligned}
\sum_{x \in X_n} \operatorname{diam}(V_n(x)) &\leq \sum_{y \in X_{n-1}} \operatorname{diam}(V_{n-1}(y)) + C' \sum_{Q \in \mathscr{G}_n} \beta_\Gamma(Q)^2 \operatorname{diam}(Q) \\
&\leq \sum_{y \in X_{n-2}} \operatorname{diam}(V_{n-2}(y)) + C' \sum_{Q \in \mathscr{G}_{n-1} \cup \mathscr{G}_n} \beta_\Gamma(Q)^2 \operatorname{diam}(Q) \\
&\vdots \\
&\leq \operatorname{diam}(V_0(x_0)) + C' \sum_{k=1}^{n} \sum_{Q \in \mathscr{G}_k} \beta_\Gamma(Q)^2 \operatorname{diam}(Q) \\
&\leq \operatorname{diam}(\Gamma) + C \sum_{Q \in \mathscr{H}} \beta_\Gamma(Q)^2 \operatorname{diam}(Q).
\end{aligned}$$

We now turn to proving (2.62). Fix $\epsilon < 2^{-10}$. For any net point $x \in X_n$, call x *flat* if $\beta_\Gamma(B(x, 10 \cdot 2^{-n})) < \epsilon$ and call x *non-flat* if $\beta_\Gamma(B(x, 10 \cdot 2^{-n})) \geq \epsilon$. We will construct a function $P : \cup_n X_n \to \cup_n X_n$ which assigns each $y \in X_{n+1}$ to a parent $P(y) \in X_n$. Since $\operatorname{diam}(\Gamma) = 1$, $\mathcal{V}_0 = \{V_0(x_0)\}$ so for any $y \in X_1$, define $P(y) = x_0$. Fix $n > 0$ and a point $y \in X_{n+1}$. If there exists $x' \in X_n$ such that x' is non-flat and $y \in V_n(x')$, then define $P(y) = x'$. Otherwise, every $x \in X_n$ such that $y \in V_n(x)$ is flat. Choose one such x and let $x', y' \in V_n(x)$ such that $\operatorname{diam}(V_n(x)) = |x' - y'|$. We define $d_x := [x', y']$ and call d_x a diameter segment of $V_n(x)$. Let $\pi_{d_x} : \ell_2 \to \mathbb{R}$ be the orthogonal projection onto the line containing d_x. Since ϵ is small, we can write

$$\begin{aligned}
X_n \cap B(x, 10 \cdot 2^{-n}) &= \{v_1, \ldots, v_N\} \\
X_{n+1} \cap V_n(x) &= \{u_1, \ldots, u_M\}
\end{aligned}$$

where

$$\begin{aligned}
&\pi_{d_x}(v_1) < \ldots < \pi_{d_x}(v_N), \\
&\pi_{d_x}(u_1) < \ldots < \pi_{d_x}(u_M)
\end{aligned}$$

We define $E_x := \{u_1, u_M\}$. If $y \notin E_x$, then define $P(y) = x$. If $y \in E_x$ we define $P(y)$ dependent on the behavior of points adjacent to x in X_n. Assuming $x = v_j$ with $0 < j < N$, suppose first that $y = u_1$. Then if v_{j-1} is non-flat, define $P(y) = v_{j-1}$. Similarly, if $y = u_M$ and v_{j+1} is non-flat, then put $P(y) = v_{j+1}$. Otherwise, put $P(y) = x$.

With the function P defined, we write

$$\sum_{y \in X_{n+1}} \operatorname{diam}(V_{n+1}(y)) = \sum_{\substack{y \in X_{n+1} \\ P(y) \text{ flat}}} \operatorname{diam}(V_{n+1}(y)) + \sum_{\substack{z \in X_{n+1} \\ P(z) \text{ non-flat}}} \operatorname{diam}(V_{n+1}(z)).$$

If $P(z)$ is non-flat, then $|z-P(z)|<4\cdot 2^{-n}$ so that $B(P(z),10\cdot 2^{-n})\subseteq B(z,A2^{-n-1})$ because $A>30$. Hence, $\beta_\Gamma(B(z,A2^{-n-1}))\gtrsim_A \epsilon$. Using (2.60), this means

$$\begin{aligned}\sum_{\substack{z\in X_{n+1}\\ P(z)\text{ non-flat}}}\operatorname{diam}(V_{n+1}(z)) &\lesssim_{\epsilon,A} \sum_{\substack{z\in X_{n+1}\\ P(z)\text{ non-flat}}}\beta_\Gamma(B(z,A2^{-n-1}))^2\operatorname{diam}(B(z,A2^{-n-1}))\\ &\leq \sum_{\substack{x\in X_n\\ x\text{ non-flat}}}\left[\operatorname{diam}(V_n(x))+C'\sum_{P(z)=x}\beta_\Gamma(B(z,A2^{-n-1}))^2\operatorname{diam}(B(z,A2^{-n-1}))\right]\\ &\leq \sum_{\substack{x\in X_n\\ x\text{ non-flat}}}\operatorname{diam}(V_n(x))+C'\sum_{\operatorname{rad}(Q)=A2^{-n-1}}\beta_\Gamma(Q)^2\operatorname{diam}(Q). \qquad (2.63)\end{aligned}$$

Now, let $x\in X_n$ be flat. We will construct a set $\operatorname{seg}(x)$ of subsets of diameter segments of the sets $\{V_{n+1}(y)\}$ for y with $P(y)$ flat. For $y\in X_{n+1}$ with $P(y)$ flat, let d_y be a diameter segment. If $y\in X_{n+1}\cap V_n(x)$ but $y\notin E_x$, then put d_y into $\operatorname{seg}(x)$. If $y=u_1\in E_x$, then we have the decomposition

$$V_{n+1}(y)=(V_{n+1}(y)\cap V_n(x))\cup(V_{n+1}(y)\cap V_n(v_{j-1}))$$

because ϵ is small. Put the segment $d_y\cap V_n(v_{j-1})$ in $\operatorname{seg}(v_{j-1})$ and the segment $d_y\cap V_n(x)$ in $\operatorname{seg}(x)$. We similarly handle the case when $y=u_N$. In this case, put the segment $d_y\cap V_n(x)$ in $\operatorname{seg}(x)$ and the segment $d_y\cap V_n(v_{j+1})$ in $\operatorname{seg}(v_{j+1})$. With these sets constructed, we can now write

$$\sum_{\substack{y\in X_{n+1}\\ P(y)\text{ flat}}}\operatorname{diam}(V_{n+1}(y))=\sum_{\substack{y\in X_{n+1}\\ P(y)\text{ flat}}}d_y=\sum_{\substack{x\in X_n\\ x\text{ flat}}}\sum_{s\in\operatorname{seg}(x)}\mathcal{H}^1(s). \qquad (2.64)$$

With (2.64) in place, we only need to give an appropriate bound for $\sum_{s\in\operatorname{seg}(x)}\mathcal{H}^1(s)$. Define $Q_x=B(x,A2^{-n})$. We claim

$$\sum_{s\in\operatorname{seg}(x)}\mathcal{H}^1(s)\leq\operatorname{diam}(V_n(x))+C\beta_\Gamma(Q_x)^2\operatorname{diam}(Q_x) \qquad (2.65)$$

for some large $C>0$. In order to prove this statement, we first state a lemma given in [BS17]:

Lemma 2.5.2. *([BS17] Lemma 8.3) Suppose that $V\subseteq\mathbb{R}^n$ is a 1-separated set with $\#V\geq 2$ and there exist lines ℓ_1 and ℓ_2 and a number $0\leq\alpha\leq 1/16$ such that*

$$\operatorname{dist}(v,\ell_i)\leq\alpha\quad\textit{for all } v\in V \textit{ and } i=1,2.$$

Let π_i denote the orthogonal projection onto ℓ_i. There exist compatible identifications of ℓ_1 and ℓ_2 with $\mathbb{R}$ such that $\pi_1(v')\leq\pi_1(v'')$ if and only if $\pi_2(v')\leq\pi_2(v'')$ for all $v',v''\in V$. If v_1 and v_2 are consecutive points in V relative to the ordering of $\pi_1(V)$, then

$$\mathcal{H}^1([u_1,u_2])<(1+3\alpha^2)\cdot\mathcal{H}^1([\pi_1(u_1),\pi_1(u_2)])\quad\textit{for all } [u_1,u_2]\subseteq[v_1,v_2]. \qquad (2.66)$$

Moreover,

$$\mathcal{H}^1([y_1,y_2])<(1+12\alpha^2)\cdot\mathcal{H}^1([\pi_1(y_1),\pi_1(y_2)])\quad\textit{for all } [y_1,y_2]\subseteq\ell_2. \qquad (2.67)$$

Applying this lemma with $\ell_1 = d_x$ and $\ell_2 = d_y$, for any segment $[s_1, s_2] \subseteq d_y$, we have

$$\mathcal{H}^1([s_1, s_2]) < (1 + C\beta_\Gamma(Q_x)^2 \operatorname{diam}(Q_x))\mathcal{H}^1([\pi_{d_x}(s_1), \pi_{d_x}(s_2)]).$$

Enumerate $\operatorname{seg}(x) = \{s_1, \ldots, s_N\}$. With this, we can write

$$\begin{aligned}\sum_{s\in\operatorname{seg}(x)} \mathcal{H}^1(s) &< (1 + C\beta_\Gamma(Q_x)^2 \operatorname{diam}(Q_x)) \sum_{s\in\operatorname{seg}(x)} \mathcal{H}^1(\pi_{d_x}(s)) \\ &\leq (1 + C\beta_\Gamma(Q_x)^2 \operatorname{diam}(Q_x)) \left(\mathcal{H}^1(d_x) + 2\sum_{i=1}^{N-1} \mathcal{H}^1(\pi_{d_x}(s_i) \cap \pi_{d_x}(s_{i+1}))\right) \\ &= (1 + C\beta_\Gamma(Q_x)^2 \operatorname{diam}(Q_x)) \left(\operatorname{diam}(V_n(x)) + 2\sum_{i=1}^{N-1} \mathcal{H}^1(\pi_{d_x}(s_i) \cap \pi_{d_x}(s_{i+1}))\right).\end{aligned} \tag{2.68}$$

Because x is flat, $\#\operatorname{seg}(x)$ is bounded above by a universal constant, so we only need to show that

$$\mathcal{H}^1(\pi_L(s_i) \cap \pi_L(s_{i+1})) \leq C'\beta_\Gamma(Q_x)^2 \operatorname{diam}(Q_x). \tag{2.69}$$

for some $C' > 0$. It suffices to bound the length of the overlap between the projections of consecutive Voronoi cells $V_{n+1}(u_i), V_{n+1}(u_{i+1})$ within the tube of radius $2\beta_\Gamma(Q_x)\operatorname{diam}(Q_x)$ around d_x. The boundary between the cells is the intersection of this tube with the hyperplane of points of equal distance from both u_i and u_{i+1}. A simple geometric estimate of the type carried out in [BS17] pages 41 and 42 gives (2.69) (also see Figure 2.8). Combining (2.68) and (2.69) gives (2.65). Applying (2.65) to (2.64), we can finally write

$$\begin{aligned}\sum_{y\in X_{n+1}} \operatorname{diam}(V_{n+1}(y)) &= \sum_{\substack{y\in X_{n+1}\\ P(y)\ \text{flat}}} \operatorname{diam}(V_{n+1}(y)) + \sum_{\substack{z\in X_{n+1}\\ P(z)\ \text{non-flat}}} \operatorname{diam}(V_{n+1}(z)) \\ &\leq \sum_{\substack{x\in X_n\\ x\ \text{non-flat}}} \operatorname{diam}(V_n(x)) + C' \sum_{\operatorname{rad}(Q)=A2^{-n-1}} \beta_\Gamma(Q)^2 \operatorname{diam}(Q) \\ &\quad + \sum_{\substack{x\in X_n\\ x\ \text{flat}}} \sum_{s\in\operatorname{seg}(x)} \mathcal{H}^1(s) \\ &\leq \sum_{\substack{x\in X_n\\ x\ \text{non-flat}}} \operatorname{diam}(V_n(x)) + C' \sum_{\operatorname{rad}(Q)=A2^{-n-1}} \beta_\Gamma(Q)^2 \operatorname{diam}(Q) \\ &\quad + \sum_{\substack{x\in X_n\\ x\ \text{flat}}} \operatorname{diam}(V_n(x)) + C'' \sum_{\operatorname{rad}(Q)=A2^{-n}} \beta_\Gamma(Q)^2 \operatorname{diam}(Q) \\ &\leq \sum_{x\in X_n} \operatorname{diam}(V_n(x)) + C \sum_{\operatorname{rad}(Q)\in\{A2^{-n-1}, A2^{-n}\}} \beta_\Gamma(Q)^2 \operatorname{diam}(Q).\end{aligned}$$

This proves inequality (2.61) and finishes the proof of Theorem 2.5.1. ■

The method for replacing $\operatorname{diam}(\Gamma)$ with $\operatorname{crd}(\Gamma)$ in Theorem 2.5.1 is nearly identical to that given in [Bis20]. Since most of the argument is dimension independent, we only need

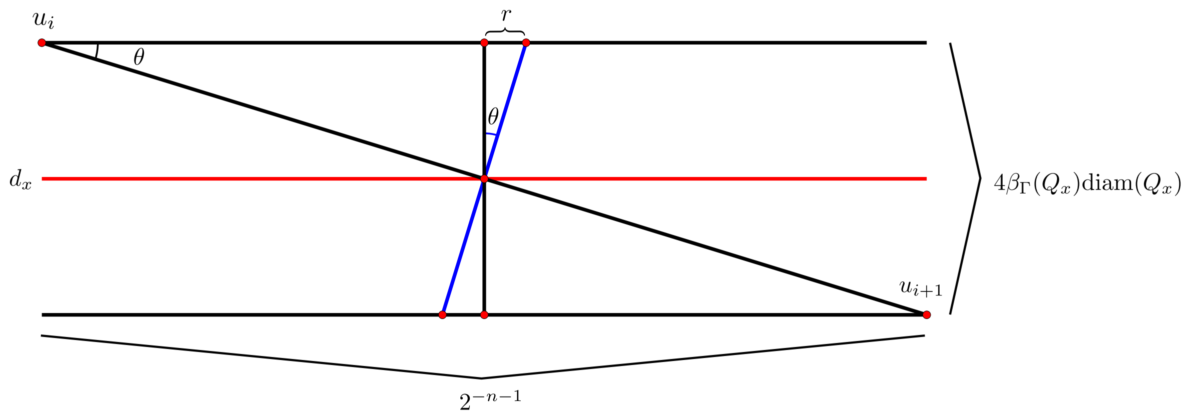

Figure 2.8: This figure depicts a worst case estimate for the overlap of $V_{n+1}(u_i)$ and $V_{n+1}(u_{i+1})$ along the diameter d_x for $u_i, u_{i+1} \in X_{n+1} \cap V_n(x)$. The blue line depicts the hyperplane which forms the boundary between $V_{n+1}(u_i)$ and $V_{n+1}(u_{i+1})$ while the overlap is bounded above by the quantity r. We have $\tan\theta = \frac{2\beta_\Gamma(Q_x)\,\mathrm{diam}(Q_x)}{2^{-n-1-1}} = \frac{r}{2\beta_\Gamma(Q_x)\,\mathrm{diam}(Q_x)}$. Rearranging, we find $r = 8\beta_\Gamma(Q_x)^2\,\mathrm{diam}(Q_x)^2 \cdot 2^{n+1} \lesssim_A \beta_\Gamma(Q_x)^2\,\mathrm{diam}(Q_x)$ as desired.

to replace certain collections of dyadic cubes with appropriate collections of balls in a multiresolution family, and switch out applications of the version of Theorem 2.5.1 proven there with Theorem 2.5.1 itself. We now give a summary of the needed modifications to Bishop's proof.

Proof. (Theorem B) We will make some amendments to Section 4 of [Bis20] beginning on page 15, but the vast majority of the proof is identical because Bishop's arguments are mostly dimension-independent from this point on. We reproduce most of the argument here for convenience.

Assume the $\sum_{Q\in\mathscr{H}} \beta_\Gamma(Q)^2\,\mathrm{diam}(Q) < \infty$ and that $\mathrm{diam}(\Gamma) = 1$. Let $Q_0 \in \mathscr{H}$ with $2 < \mathrm{diam}(Q_0) \leq 4$ so that $\Gamma \subseteq Q$. Suppose β_0 is a small positive number to be chosen below. If $\beta_\Gamma(Q_0) > \beta_0$, then we have

$$\mathrm{crd}(\Gamma) \leq \mathrm{diam}(\Gamma) = 1 \leq \frac{\beta_\Gamma(Q_0)^2}{\beta_0^2}\,\mathrm{diam}(Q_0) \lesssim \beta_\Gamma(Q_0)^2\,\mathrm{diam}(Q_0).$$

Therefore, by Theorem 2.5.1,

$$\ell(\Gamma) - \mathrm{crd}(\Gamma) \leq \mathrm{diam}(\Gamma) - \mathrm{crd}(\Gamma) + C\sum_{Q\in\mathscr{H}} \beta_\Gamma(Q)^2\,\mathrm{diam}(Q) \lesssim \sum_{Q\in\mathscr{H}} \beta_\Gamma(Q)^2\,\mathrm{diam}(Q).$$

Hence, we may assume that $\beta_\Gamma(Q_0) < \beta_0$. Let $x, y \in \Gamma$ be such that $|x - y| = \mathrm{diam}(\Gamma)$ and re-orient Γ so that $x = 0$ and $y = (1, 0, 0, \ldots)$. Assuming $t(x) < t(y)$, let $\gamma_1 = \gamma(0, t(x))$ and $\gamma_2 = \gamma(t(y), \ell(\Gamma))$. That is, γ_1 is the subarc of Γ from the beginning of Γ to x and γ_2 is the subarc from y to the end of Γ. Observe that

$$\mathrm{crd}(\Gamma) \geq \mathrm{diam}(\Gamma) - \ell(\gamma_1) - \ell(\gamma_2)$$

so that applying Theorem 2.5.1 three times (using the fact that γ_1, γ_2 are Jordan arcs) gives

$$\begin{aligned}\ell(\Gamma) - \operatorname{crd}(\Gamma) &\le \ell(\Gamma) - \operatorname{diam}(\Gamma) + \ell(\gamma_1) + \ell(\gamma_2) \\ &\le C(A) \sum_{Q \in \mathscr{H}} \beta_\Gamma(Q)^2 \operatorname{diam}(Q) + \operatorname{diam}(\gamma_1) + \operatorname{diam}(\gamma_2). \end{aligned} \tag{2.70}$$

This means we only need to show that $\operatorname{diam}(\gamma_1) + \operatorname{diam}(\gamma_2) \lesssim_A \sum_{Q\in\mathscr{H}} \beta_\Gamma(Q)^2 \operatorname{diam}(Q)$. Because the arguments for both arcs are the same, we only consider γ_1.

Let $\epsilon = \operatorname{diam}(\gamma_1) > 0$. Let $Q_1, \ldots, Q_k$ be balls with diameter going from $\operatorname{diam}(Q_0)$ to ϵ such that among all balls of their given radius, their center is closest to x. We have that $k \simeq \log(\operatorname{diam}(Q_0)/\epsilon)$. If any of these balls satisfies $\beta_\Gamma(Q_j) > \beta_0$, then

$$\operatorname{diam}(\gamma_1) \le \frac{\beta_\Gamma(Q_j)^2}{\beta_0^2} \operatorname{diam}(\gamma_1) \lesssim \beta_\Gamma(Q_j)^2 \operatorname{diam}(Q_j)$$

so that $\operatorname{diam}(\gamma_1)$ satisfies the desired bound. Hence, we assume that $\beta_\Gamma(Q_j) \le \beta_0$ for all $1 \le j \le k$. Let L_j be an infimizing line in the definition of $\beta_\Gamma(Q_j)$. We measure the angle that L_j makes with the x_1-axis by translating it to intersect 0, then measuring the angle between these lines in the (at most 2-dimensional) plane containing them.

Case 1: Assume that L_i makes an angle larger than $10\beta_0$ with the x_1 axis for some i. Since the angle between L_0 and L_i is bounded by $C\sum_{j=1}^{i} \beta_\Gamma(Q_j)$ and the best line for Q_0 is within an angle β_0 of the x_1-axis, we have $\sum_{j=1}^{k} \beta_\Gamma(Q_j) \gtrsim \beta_0 \gtrsim 1$. The Cauchy-Schwarz inequality implies

$$1 \lesssim \left(\sum_{j=1}^{k} \beta_\Gamma(Q_j) \right)^2 \lesssim \left(\sum_{j=1}^{k} \beta_\Gamma(Q_j)^2 2^{-j} \right) \cdot \left(\sum_{j=1}^{k} 2^j \right) \simeq 2^k \sum_{i=j}^{k} \beta_\Gamma(Q_j)^2 2^{-j}.$$

Therefore, $\sum_{j=1}^{k} \beta(Q_j) 2^{-j} \gtrsim 2^{-k} \gtrsim \epsilon$ so that

$$\epsilon = \operatorname{diam}(\gamma_1) \lesssim \sum_{j=1}^{k} \beta_\Gamma(Q_j)^2 \operatorname{diam}(Q_j)$$

as desired.

Case 2: Now, assume that all of the lines L_j, $1 \le j \le k$ make angle less than $10\beta_0$ with the x_1-axis. Consider a subarc $\gamma_1' \subseteq \gamma_1$ that is contained in and connects the boundary components of the annulus

$$B\left(x, \frac{1}{5}\operatorname{diam}(\gamma_1)\right) \Big\backslash B\left(x, \frac{1}{10}\operatorname{diam}(\gamma_1)\right).$$

Because γ_1' and γ_1 have comparable diameters, it suffices to bound $\operatorname{diam}(\gamma_1')$.

Given any $p \in \gamma_1'$, one of the following two statements holds:

(i) Every ball $Q = B(x, A2^{-n})$ with $p \in B(x, 2^{-n})$ and $\operatorname{diam}(Q) \le \frac{1}{10}\operatorname{diam}(\gamma_1)$ satisfies $\beta_\Gamma(Q) \le \beta_0$.

(ii) There exists a ball Q_p of the above form such that $\beta_\Gamma(Q_p) > \beta_0$.

We let $E \subseteq \gamma_1'$ be the set of points p where a ball Q_p as in (ii) exists. Since γ_1' is rectifiable, it has tangents almost everywhere. Bishop provides the following two lemmas

Lemma 2.5.3. *([Bis20] Lemma 4.1 for $\mathscr{H}$) If $p \in \gamma_1' \backslash E$ and p is a tangent point of Γ, then p has a "crossing property": If $Q \in \mathscr{H}$ has $p \in \frac{1}{2}Q$ with $\operatorname{diam}(Q) \leq \frac{\operatorname{diam}(\gamma_1)}{10}$ then γ_1 must "cross" Q in the sense that γ_1 must connect the two components of $\partial Q \cap W$ where W is a cylinder of radius $\frac{\operatorname{diam}(Q)}{10}$ containing p.*

Lemma 2.5.4. *([Bis20] Lemma 4.2) $\ell(E) = \ell(\gamma_1')$.*

We have changed the statement of Lemma 2.5.3 only by replacing the setting from $\mathbb{R}^n$ to ℓ_2 and replacing dyadic cubes with balls in a multiresolution family. Bishop proves this lemma by constructing a "dividing" hypersurface which γ_1 can only cross once because γ_1 is a Jordan arc. It is straightforward to modify Bishop's construction by replacing $n-1$-dimensional planes in $\mathbb{R}^n$ with corresponding hyperplanes in ℓ_2. Given Lemma 2.5.3, Lemma 2.5.4 follows directly from Bishop's original argument. For the proofs of these results, we direct the reader to [Bis20] (especially see Figure 6 there for a good picture of Lemma 2.5.3).

Given these lemmas, we can complete the proof of Theorem B by noting that Lemma 2.5.4 implies that γ_1' is nearly covered by the balls $\{Q_p\}_{p \in E}$ so that there is subcollection of distinct balls $\{Q_{p_j}\}_j$ with

$$\operatorname{diam}(\gamma_1') \leq 5 \sum_j \operatorname{diam}(Q_{p_j}) \lesssim_{\beta_0} \sum_j \beta_\Gamma(Q_{p_j})^2 \operatorname{diam}(Q_{p_j}) \leq \sum_{Q \in \mathscr{H}} \beta_\Gamma(Q)^2 \operatorname{diam}(Q).$$

Combining this result with (2.70) completes the proof of Theorem B. ■

Chapter 3

Lipschitz decompositions of domains with bilaterally flat boundaries

3.1 Introduction

3.1.1 Overview

In many areas of analysis, general domains which are somehow "close" or well-approximated by a *Lipschitz domain* tend to have many desirable properties.

Definition 3.1.1 (Lipschitz domains)**.** We say that $\Omega \subseteq \mathbb{R}^{d+1}$ is a *Lipschitz domain* if for each $p \in \partial\Omega$, there exists $r > 0$ such that $B(p,r) \cap \partial\Omega$ is a Lipschitz graph.

For instance, the idea of finding good Lipschitz domains inside of more general domains has an important place in the study of harmonic measure in the plane and beyond [DJ90], [Dah77], [Bad10], [Azz18]. Lipschitz domains have similarly been used to give characterizations of rectifiability and uniform rectifiability [Akm+19], [Akm+17], [BH17], [GMT18]. The slightly stronger notion of a *Lipschitz graph domain* has also played an important role in quantitative geometric measure theory.

Definition 3.1.2 (Lipschitz graph domains)**.** We say that an open, connected set $\Omega \subseteq \mathbb{R}^{d+1}$ is an *M-Lipschitz graph domain* if the following holds: There exists a composition of a translation, dilation, and rotation A with image domain $\widetilde{\Omega} = A(\Omega)$ such that there exists a function $r_{\widetilde{\Omega}} : \mathbb{S}^d \to \mathbb{R}^+$ with

$$\partial\widetilde{\Omega} = \left\{ r_{\widetilde{\Omega}}(\theta)\theta : \theta \in \mathbb{S}^d \right\}$$

and, for any $\theta, \psi \in \mathbb{S}^d$

$$|r_{\widetilde{\Omega}}(\theta) - r_{\widetilde{\Omega}}(\psi)| \leq M|\theta - \psi|,$$

$$\frac{1}{1+M} \leq r_{\widetilde{\Omega}}(\theta) \leq 1.$$

Intuitively, a Lipschitz graph domain is a "Lipschitz graph over a sphere". These domains appear in the following striking result due to Peter Jones which is the primary inspiration for this paper:

Theorem 3.1.1 ([Jon90] Theorem 2)**.** *There exists a constant $M > 0$ such that the following holds: For any simply connected domain $\Omega \subseteq \mathbb{C}$ with $\mathcal{H}^1(\partial\Omega) < \infty$, there is a rectifiable curve Γ such that*

$$\Omega \setminus \Gamma = \bigcup_j \Omega_j$$

where Ω_j is an M-Lipschitz graph domain for each j, and

$$\sum_j \mathcal{H}^1(\partial\Omega_j) \leq M\mathcal{H}^1(\partial\Omega).$$

We informally say that Theorem 3.1.1 gives a *Lipschitz decomposition* of a domain Ω in the sense that Ω is written as a union of closures of disjoint Lipschitz graph domains with boundary lengths controlled by the boundary length of Ω. Also see [GJM92] for a similar result for minimal surfaces in $\mathbb{R}^n$. Despite being geometrically interesting in and of itself, Theorem 3.1.1 has an important place in the history of quantitative geometric measure theory because it is central to Jones's original proof of the Analyst's Traveling Salesman Theorem in $\mathbb{R}^2$. This central result gives a characterization of subsets of rectifiable curves and an estimate on their lengths in terms of a quantity called the *Jones beta number* which measures how close a subset $E \subseteq \mathbb{R}^2$ is to being linear locally.

Definition 3.1.3 (Jones beta number)**.** Let $E, Q \subseteq \mathbb{R}^2$ where Q has finite diameter. We define the β-number for E in the "window" Q by

$$\beta_E(Q) = \inf_L \sup_{x \in Q \cap E} \frac{\operatorname{dist}(x, L)}{\operatorname{diam}(Q)},$$

where L ranges over all affine lines in $\mathbb{R}^2$.

Theorem 3.1.2 (cf. [Jon90] Theorem 1, [Oki92] in $\mathbb{R}^n$, $n > 2$)**.** *Let $E \subseteq \mathbb{R}^2$. E is contained in a rectifiable curve if and only if*

$$\beta_E^2(\mathbb{R}^2) = \operatorname{diam}(E) + \sum_{Q \in \Delta(\mathbb{R}^2)} \beta_E(3Q)^2 \operatorname{diam}(Q) < \infty$$

where $\Delta(\mathbb{R}^2)$ is the set of all dyadic cubes in $\mathbb{R}^2$ and $3Q$ is the cube with the same center as Q but three times the side length. If Σ is a connected set of shortest length containing E, then

$$\beta_\Sigma^2(\mathbb{R}^2) \lesssim \mathcal{H}^1(\Sigma) \tag{3.1}$$

and

$$\beta_E^2(\mathbb{R}^2) \gtrsim \mathcal{H}^1(\Sigma). \tag{3.2}$$

There are now many results referred to as "Traveling Salesman Theorems" which share the general structure and philosophy of Theorem 2.1.1 but take place in different spaces such as Hilbert space [Sch07a], Banach spaces [BM23a], [BM23b], Carnot groups [Li22], graph inverse limit spaces [DS16], and general metric spaces [DS21], [Hah05]. Many also apply to different geometric objects such as Jordan arcs [Bis22], Hölder curves [BNV19], higher-dimensional sets [AS18], [Hyd22a], [Hyd22b], [**Ghi17**], or measures [BS17], [BLZ23].

Jones proves Theorem 2.1.1 essentially as a corollary of Theorem 3.1.1. Roughly speaking, given a rectifiable curve $\Gamma \subseteq \mathbb{D}$, one can apply the Lipschitz decomposition result to each component of $\mathbb{D}\backslash\Gamma$ and use the boundaries of the produced Lipschitz graph domains to control the beta numbers of Γ. In fact, this shows that rectifiable curves in $\mathbb{R}^2$ admit extensions of controlled length which are quasiconvex: if one considers the union of the boundaries as a new curve $\widetilde{\Gamma} = \cup_j \partial\Omega_j \cup \Gamma$, then $\mathcal{H}^1(\widetilde{\Gamma}) \lesssim \mathcal{H}^1(\Gamma)$ and $\widetilde{\Gamma}$ is quasiconvex (see [AS12b] for a generalization of this corollary to higher dimensions).

Jones's result is powerful, but it is confined to two dimensions. In this paper, we consider the following question:

Question 3.1.3. *For $\Omega \subseteq \mathbb{R}^{d+1}$, $d > 1$, what geometric conditions on $\partial\Omega$ are sufficient for Ω to admit a Lipschitz decomposition?*

One of the attractive features of Theorem 3.1.1 is the minimality of its assumptions on Ω; Jones only assumes simple connectivity and finite boundary length. These assumptions suffice essentially because they give access to a nicely behaved parameterization in the form of a conformal map $\varphi : \mathbb{D} \to \Omega$. The lack of similar conformal maps in higher dimensions precludes one from directly translating Jones's original argument from $\mathbb{R}^2$ to higher dimensions, but, by assuming stronger control of the geometry of $\partial\Omega$, one does get access to nicely behaved parameterizations which are sufficient replacements. The vital geometric condition on $\partial\Omega$ is called *Reifenberg flatness*, which states that $\partial\Omega$ is bilaterally close to a d-plane at all scales and all locations. This bilateral closeness is measured by the *bilateral beta number*.

Definition 3.1.4 (bilateral beta number)**.** For $E \subseteq \mathbb{R}^n$, P a $d-$plane, and B a ball, the *d-bilateral beta number relative to P* for E inside B is

$$b\beta_E^d(B, P) = \frac{2}{\text{diam}(B)} d_H(B \cap E, B \cap P).$$

The full *bilateral beta number* for E inside B is then

$$b\beta_E^d(B) = \inf_{P \text{ } d\text{-plane}} b\beta_E^d(B, P).$$

Definition 3.1.5 ((ϵ, d)-Reifenberg flatness)**.** For fixed $\epsilon > 0$ and $d, n \in \mathbb{N}$ with $0 < d < n$, we say a set $E \subseteq \mathbb{R}^n$ is *(ϵ, d)-Reifenberg flat* if, for all $x \in E$ and $r > 0$,

$$b\beta_E^d(B(x, r)) \leq \epsilon.$$

Sets that are (ϵ, d)-Reifenberg flat for small enough $\epsilon \leq \epsilon_0(d, n)$ admit bi-Hölder parameterizations which we informally call *Reifenberg parameterizations*. This was first shown by Reifenberg in [Rei60], but was later generalized by David and Toro [DT12] to produce parameterizations of Reifenberg flat sets "with holes" along with giving a condition under which the parameterization can be upgraded from bi-Hölder to bi-Lipschitz.

Theorem 3.1.4 (cf. [DT12] Theorem 1.10)**.** *For any $d, n \in \mathbb{N}$ with $0 < d < n$ and $0 < \tau < \frac{1}{10}$, there exists a constant $\epsilon_0(d, n)$ such that if $\epsilon \leq \epsilon_0$ and $0 \in E \subseteq \mathbb{R}^n$ is (ϵ, d)-Reifenberg* *flat, then there exists a bijection* $g : \mathbb{R}^n \to \mathbb{R}^n$ *satisfying the following conditions: For any* $z, x, y \in \mathbb{R}^n$ *with* z *arbitrary,* $|x - y| \leq 1$,

$$|g(z) - z| \leq \tau,$$

$$\frac{1}{4}|x-y|^{1+\tau} \leq |g(x) - g(y)| \leq 3|x-y|^{1-\tau},$$

and, for some d*-plane* P *such that* $b\beta_E(B(0,10), P) \leq \epsilon$,

$$E \cap B(0,1) = g(P) \cap B(0,1).$$

Given a domain $\Omega \subseteq \mathbb{R}^{d+1}$ such that $\partial\Omega$ is (ϵ, d)-Reifenberg flat, we use the Reifenberg parameterization g produced by Theorem 3.1.4 as a replacement for the conformal map in Jones's original argument to first prove the following new result

Theorem C. *Let* $\Omega \subseteq \mathbb{R}^{d+1}$ *be a domain with* $0 \in \partial\Omega$*. There exists* $\epsilon_0(d) > 0$ *such that for any* $L > 0$*, if* $\epsilon \leq \epsilon_0$ *and*

(i) $\partial\Omega$ *is* (ϵ, d)*-Reifenberg flat,*

(ii) $\sum_{k=1}^{\infty} \beta_{\partial\Omega}^{d,1}(B(x, 2^{-k}))^2 \leq L$ *for all* $x \in \partial\Omega$,

then there exists a Ahlfors d*-regular,* d*-rectifiable set* Σ *such that*

$$\Omega \cap B(0,1) \setminus \Sigma = \bigcup_{j=1}^{\infty} \Omega_j$$

and there exists $L_1(\epsilon, L, d) > 0$ *such that* $\mathscr{L} = \{\Omega_j\}_{j \in J_{\mathscr{L}}}$ *is a collection of disjoint* L_1*-Lipschitz graph domains. In addition, for any* $y \in \partial\Omega \cap B(0,1)$ *and* $0 < r < 1$*, we have*

$$\sum_{j=1}^{\infty} \mathcal{H}^d(\partial\Omega_j \cap B(y,r)) \lesssim_{\epsilon,L,d} r^d.$$

See Definition 3.2.10 for the definition of $\beta_{\partial\Omega}^{d,1}(B(x, 2^{-k}))$. Hypothesis (ii) is used to ensure that David and Toro's bi-Lipschitz condition for the Reifenberg parameterization is satisfied. If this hypothesis is not satisfied, then one can still run the proof of Theorem C to produce a collection of Lipschitz graph domains whose total boundary measure and Lipschitz constants blow up near where the sum in (ii) diverges. However, we conjecture that a result similar to Theorem C holds without assumption (ii).

If one is willing to weaken the conclusion of $\{\Omega_j\}$ being disjoint to having bounded overlap, then one can show that similar Lipschitz decompositions exist for domains with weaker assumptions on the boundary. We prove the following result of this type:

Theorem D. *Let* $\Omega \subseteq \mathbb{R}^{d+1}$ *be a domain with* $0 \in \partial\Omega$*. There exist constants* $A(d), L(d), \epsilon(d) > 0$ *such that if* $0 \in \partial\Omega$ *is* (ϵ, d)*-Reifenberg flat, then there exists a collection of* L*-Lipschitz graph domains* $\{\Omega_j\}_{j \in \mathscr{L}}$ *such that*

(i) $\Omega_j \subseteq \Omega$,

(ii) $\Omega \cap B(0,1) \subseteq \bigcup_{j=1}^{\infty} \Omega_j$,

(iii) $\exists C(d) > 0$ *such that* $\forall x \in \Omega$, $x \in \Omega_j$ *for at most* C *values of* j,

(iv) For any $y \in \partial\Omega \cap B(0,1)$ *and* $0 < r \leq 1$, *we have*

$$\sum_{j=1}^{\infty} \mathcal{H}^d(\partial\Omega_j \cap B(y,r)) \lesssim_{\epsilon,d,L} \mathcal{H}^d(\partial\Omega \cap B(y,Ar)).$$

To prove this result, we use a collection of $(1 + C\delta)$-bi-Lipschitz Reifenberg parameterizations to produce a large collection of disjoint Lipschitz graph domains with controlled boundaries and expand these domains with Whitney-type "buffer zones" to form a true covering of $\Omega \cap B(0,1)$. This method carries over to the well-known *d-uniformly rectifiable* sets of David and Semmes who give many different equivalent definitions of d-uniform rectifiability [DS93]. One such definition involves the *bilateral weak geometric lemma* (BWGL), which roughly says that E looks Reifenberg flat on most scales and locations.

Definition 3.1.6 (bilateral weak geometric lemma)**.** Given a family of Christ-David cubes $\mathscr{D}$ for E (see Theorem 3.2.5) and constants $M, \epsilon > 0$, define

$$\text{BWGL}(M,\epsilon) = \{Q \in \mathscr{D} : b\beta_E^d(MB_Q) > \epsilon\}.$$

For $Q \in \mathscr{D}$, define

$$\text{BWGL}(Q,M,\epsilon) = \sum_{\substack{R \subseteq Q \\ R \in \text{BWGL}(M,\epsilon)}} \ell(R)^d.$$

We say that E satisfies the *bilateral weak geometric lemma* if for any $M, \epsilon > 0$, there exists a constant $C_0(M,\epsilon)$ such that for all $Q \in \mathscr{D}$,

$$\text{BWGL}(Q,M,\epsilon) \leq C_0 \ell(Q)^d. \tag{3.3}$$

If E is (ϵ,d)-Reifenberg flat, then $\text{BWGL}(Q,M,\epsilon) = 0$ for all Q and M. Equation (3.3) is often referred to as a *Carleson packing condition.* One can define a d-uniformly rectifiable set as a Ahlfors d-regular set which satisfies the BWGL.

Definition 3.1.7 (Ahlfors d-regularity)**.** We say that a set $E \subseteq \mathbb{R}^n$ is *Ahlfors d-regular* if E is closed and there exists a constant $C_0 > 0$ such that for any $x \in E$ and $0 < r < \text{diam}(E)$, we have

$$C_0^{-1} r^d \leq \mathscr{H}^d(E \cap B(x,r)) \leq C_0 r^d.$$

Definition 3.1.8 (d-uniform rectifiability)**.** We say a Ahlfors d-regular set $E \subseteq \mathbb{R}^n$ is *d-uniformly rectifiable* if E satisfies the BWGL.

Using similar methods to those of the proof of Theorem D, we prove an analogue of Theorem D for d-uniformly rectifiable sets.

Theorem E. *Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a domain with $0 \in \partial\Omega$. If $\partial\Omega$ is d-uniformly rectifiable, then there exists $L(d), A(d) > 0$ such that there exists a collection of L-Lipschitz graph domains $\{\Omega_j\}_{j \in J_{\mathscr{L}}}$ such that conclusions (i), (ii), (iii), and (iv) (with additional dependence on uniform rectifiability constants) of Theorem D hold.*

Uniform rectifiability was studied in detail by David and Semmes in [DS93] where connections between the BWGL and numerous other equivalent definitions involving boundedness of singular integral operators, approximation by Lipschitz graphs (the existence of corona decompositions), "big piece" parameterizations by Lipschitz maps, and more. Uniform rectifiability has recently become of interest in the study of harmonic measure and the solvability of the homogeneous Dirichlet problem in rough domains. In [Azz+20], the authors give a geometric characterization of open sets $\Omega \subseteq \mathbb{R}^{d+1}$ such that there exists $p < \infty$ such that the $L^p(\partial\Omega)$-Dirichlet problem is solvable given the background hypotheses that $\partial\Omega$ is Ahllfors-David d-regular and Ω satisfies the interior corkscrew condition. They prove that solvability is equivalent to $\partial\Omega$ being d-uniformly rectifiable and Ω satisfying a quantitative connectivity condition called the weak local John condition. A related line of research studies L^p solvability of inhomogeneous problems on rough domains. In the course of preparing this work, the author was notified of [MPT22] in which the authors study equivalences of solutions to boundary value problems in rough domains and show that the regularity problem for so-called DKP operators is L^p-solvable on certain geometrically nice domains. In the course of their study, the authors derive a very similar result to Theorem E with the added assumption that Ω satisfies the interior corkscrew condition and the added conclusion that the nice approximating domains are adapted to a DKP operator (see Section 4.3 of [MPT22] and see also [MT22] for an earlier version of their construction).

3.1.2 Outlines of the paper and proofs of the theorems

In Section 3.2, we introduce the necessary notation and basic facts about Reifenberg parameterizations, Whitney decompositions, Christ-David cubes, coronizations, Reifenberg flat sets, and uniformly rectifiable sets.

In Sections 3.3 and 3.4 we respectively prove Theorems C and Theorems D and E while taking for granted the results on Lipschitz graph domains proved in Section 3.5 and results on controlling the derivative of Reifenberg parameterizations proved in Section 3.6.

Roughly speaking, the proof of Theorem C in Section 3.3 proceeds as follows. The fact that $\partial\Omega$ is Reifenberg flat means that we can produce a Reifenberg parameterization $g : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ such that $g(\mathbb{R}^d \times \{0\}) \cap B(0,1) = \partial\Omega \cap B(0,1)$. The uniform bound on the beta-squared sum in condition (ii) of Theorem C ensures that g is $L'(d,L)$-bi-Lipschitz so that $\partial\Omega$ is in fact a bi-Lipschitz image, hence uniformly rectifiable. This means that there exists a Christ-David lattice $\mathscr{D}$ for $\partial\Omega$ with a graph coronization whose stopping time regions $\mathscr{F} = \{S\}$ consist of cubes well-approximated by Lipschitz graphs with Lipschitz constant small in terms of d and L' (this is a coronization that produces a corona decomposition). Proposition 3.6.7 implies that Dg is nearly constant on parts of its domain which are mapped into regions of Ω sitting "above" a stopping time region S on the scale of the cubes inside S. By the results of Section 3.5, g maps forward Lipschitz graph domains to Lipschitz graph domains when the change in Dg is small compared to the Lipschitz constants of the mapped domains. Therefore, to produce a Lipschitz decomposition of $\Omega \cap B(0,1)$, it suffices to produce a Lipschitz decomposition $\mathscr{L}_0$ (see Definition 3.3.3) of the domain of g into domains over which Dg is nearly constant so that the collection of images $\mathscr{L} = \{g(\mathcal{D}) : \mathcal{D} \in \mathscr{L}_0\}$ is a Lipschitz decomposition.

In order to form this decomposition, we produce a "coronization" of a lattice of Whitney boxes which parallels the coronization for $\mathscr{D}$ on $\partial\Omega$ (see 3.3.1). That is, we separate Whitney boxes into bad boxes which g maps near bad cubes in $\mathscr{D} \cap \mathscr{B}$ or cubes on the "edges" in scale and location of stopping time regions in $\mathscr{D}$. This decomposition then maps forward to a collection of domains whose total surface measure is bounded by the surface measure of $\partial\Omega$ plus the Carleson packing sums for the bad and "edge" cubes of $\mathscr{D}$.

The proofs of Theorems D and E both follow a single similar argument to that of Theorem C. In the Reifenberg flat case, the difference is that any single global Reifenberg parameterization g produced for the set is not in general bi-Lipschitz, so we have no uniform estimates on how g distorts any given cube. In the uniformly rectifiable case, we have no global Reifenberg parameterization because there can be many scales and locations at which Reifenberg flatness fails. In either case, we sidestep these by producing a collection of local $(1+\delta)$-bi-Lipschitz parameterizations by parameterizing pieces of the domain above stopping time regions in a graph coronization (see Definition 3.2.9) using single stopping time domains composed of Whitney cubes. By similar arguments, the surface measure of these domains is controlled by the surface masure of $\partial\Omega \cap B(0,1)$ plus the Carleson packing sum of the same bad set of cubes in $\mathscr{D} \cap \mathscr{B}$ and near "edges" of stopping time domains. We then fill parts of $\Omega \cap B(0,1)$ that are missed by these domains with "buffer zones" of cubes on the exterior of these domains as well as families of cubes which sit above surface cubes in the bad set. By similar reasoning, the surface measure of these domains is bounded by the same Carleson packing sums as above.

3.1.3 Acknowledgements

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3.2 Preliminaries

3.2.1 Conventions and basic definitions

Whenever we write $A \lesssim B$, we mean that there exists some constant C independent of A and B such that $A \leq CB$. If we write $A \lesssim_{a,b,c} B$ for some constants a, b, c, then we mean that the implicit constant C mentioned above is allowed to depend on a, b, c. We will sometimes write $A \simeq B$ to mean that both $A \lesssim B$ and $B \lesssim A$ hold.

In many computations, we use a constant C to denote a catch-all, general constant which is allowed to vary significantly from one line to the next.

Definition 3.2.1 (Hausdorff measure, Hausdorff distance, Nets)**.** For $F, E \subseteq \mathbb{R}^n$, $a \in \mathbb{R}^n$, we let

$$\begin{aligned} \operatorname{dist}(E,F) &= \inf\{|x-y| : x \in F,\ y \in E\}, \\ \operatorname{dist}(a,E) &= \operatorname{dist}(\{a\},E) \end{aligned}$$

and define

$$\operatorname{diam}(F) = \sup\{|x-y| : x,y \in F\}.$$

For any $r > 0$, we let

$$B(E,r) = \{x \in \mathbb{R}^{d+1} : \operatorname{dist}(x,E) < r\}.$$

For any subset $F \subseteq \mathbb{R}^{d+1}$, an integer $m \geq 0$, and constant $0 < \delta \leq \infty$, we define

$$\mathscr{H}^m(F) = \inf\left\{\sum \operatorname{diam}(E_i)^d : F \subseteq \bigcup E_i, \operatorname{diam}(E_i) < \delta\right\}.$$

The Hausdorff m-measure of F is defined as

$$\mathscr{H}^m(F) = \lim_{\delta \to 0} \mathscr{H}^m_\delta(F),$$

We will only use this in the case $m = d$, and we often use the notation $|F| = \mathscr{H}^d(F)$. We refer to the function $\mathscr{H}^m_\infty$ as the m-dimensional Hausdorff content. Given two closed sets $E, F \subseteq \mathbb{R}^{d+1}$, and a third set $B \subseteq \mathbb{R}^{d+1}$ we define the Hausdorff distance between E and F inside B as

$$d_B(E,F) = \frac{2}{\operatorname{diam} B} \max\left\{\sup_{y \in E \cap B} \operatorname{dist}(y,F),\ \sup_{y \in F \cap B} \operatorname{dist}(y,E)\right\}.$$

Given a subset $E \subseteq \mathbb{R}^{d+1}$ and $r > 0$, we let $\operatorname{Net}(E,r)$ denote the set of r-nets of E. That is, $X \in \operatorname{Net}(E,r)$ if $X \subseteq E$ such that both

(i) For any $x \neq y \in X$, $|x-y| \geq r$,

(ii) $E \subseteq \bigcup_{x \in X} B(x,r)$.

3.2.2 Reifenberg parameterizations

In this section, we record the basic facts about Reifenberg parameterizations needed from [DT12].

3.2.2.1 Coherent Collections of Balls and Planes (CCBP)

Set $r_k = 10^{-k}$ and let $x_{j,k} \in \mathbb{R}^{d+1}$, $j \in J_k$ satisfy

$$|x_{j,k} - x_{i,k}| \geq r_k. \tag{3.4}$$

Put $B_{j,k} = B(x_{j,k}, r_k)$ and for $\lambda > 0$ define $V_k^\lambda = \bigcup_{j \in J_k} \lambda B_{j,k} = \bigcup_{j \in J_k} B(x_{j,k}, \lambda r_k)$ where λB is always the ball with the same center as B and radius dilated by a factor of λ. We also assume

$$x_{j,k} \in V_{k-1}^2. \tag{3.5}$$

We will always use a d-plane as the initial surface Σ_0. We require

$$\mathrm{dist}(x_{j,0}, \Sigma_0) \le \epsilon \text{ for } j \in J_0. \tag{3.6}$$

Finally, the coherent collection of planes is a collection of planes (in general of any dimension $m < d+1$, although here we only take d-planes) $P_{j,k}$ associated to $x_{j,k}$ such that the compatibility conditions

$$d_{x_{j,k},100r_k}(P_{i,k}, P_{j,k}) \le \epsilon \ \text{ for } k \ge 0 \text{ and } i,j \in J_k \text{ such that } |x_{i,k} - x_{j,k}| \le 100r_k \tag{3.7}$$

$$d_{x_{i,0},100}(P_{i,0}, P_x) \le \varepsilon \ \text{ for } i \in J_0 \text{ and } x \in \Sigma_0 \text{ such that } |x_{i,0} - x| \le 2, \tag{3.8}$$

$$d_{x_{i,k},20r_k}(P_{i,k}, P_{j,k+1}) \le \varepsilon \ \text{ for } i \in J_k \text{ and } j \in J_{k+1} \text{ such that } |x_{i,k} - x_{j,k+1}| \le 2r_k. \tag{3.9}$$

With these conditions, we can define a CCBP

Definition 3.2.2. A CCBP is a triple $(\Sigma_0, \{B_{j,k}\}, \{P_{j,k}\})$ such that conditions (3.4), (3.5), (3.6), (3.7), (3.8), (3.9) are satisfied with ϵ sufficiently small in terms of d.

We first state a small modification of a lemma in [AS18] which gives criteria for a triple $(\Sigma_0, \{B_{j,k}\}, \{P_{j,k}\})$ to be a CCBP.

Lemma 3.2.1 (cf. [AS18] Theorem 2.5)**.** *For any $k \in \mathbb{N} \cup \{0\}$, let $r_k = 10^{-k}$. Let $\{x_{j,k}\}_{j \in J_k}$ be a collection of points such that for some d-plane P_0 we have*

$$\mathrm{dist}(x_{j,0}, P_0) < \epsilon,$$

$$|x_{j,k} - x_{i,k}| \ge r_k,$$

and, with $B_{j,k} = B(x_{j,k}, r_k)$,

$$x_{i,k} \in V_{k-1}^2$$

where

$$V_k^\lambda = \bigcup_{j \in J_k} \lambda B_{j,k}.$$

Let $P_{j,k}$ be a d-plane such that $x_{j,k} \in P_{j,k}$. There is $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$, if

$$\epsilon_k'(x_{j,k}) \lesssim \epsilon \text{ for all } k \ge 0 \text{ and } j \in J_k$$

then $(P_0, \{B_{j,k}\}, \{P_{j,k}\})$ is a CCBP. See (3.74) for the definition of the ϵ_k' numbers.

CCBPs allow the construction of Reifenberg parameterizations which we will denote by the letter g. David and Toro give the following Theorem

Theorem 3.2.2 ([DT12] Theorems 2.15, 2.23)**.** *Let $(\Sigma_0, \{B_{j,k}\}, \{P_{j,k}\})$ be a CCBP with ϵ sufficiently small. Then there exists a bijection $g : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$g(z) = z \ \text{ when } \mathrm{dist}(z, \Sigma_0) \ge 2, \tag{3.10}$$

$$|g(z) - z| \le C\varepsilon \ \text{ for } z \in \mathbb{R}^n, \tag{3.11}$$

$$\frac{1}{4}|z' - z|^{1+C\varepsilon} \leq |g(z') - g(z)| \leq 3|z' - z|^{1-C\varepsilon} \tag{3.12}$$

for $z, z' \in \mathbb{R}^n$ such that $|z' - z| \leq 1$, and $\Sigma = g(\Sigma_0)$ is a $C\varepsilon$-Reifenberg flat set that contains the accumulation set

$$E_\infty = \Big\{x \in \mathbb{R}^n\,;\ x \text{ can be written as } x = \lim_{m\to+\infty} x_{j(m),k(m)},\ \text{with } k(m) \in \mathbb{N} \\ \text{and } j(m) \in J_{k(m)} \text{ for } m \geq 0,\ \text{and } \lim_{m\to+\infty} k(m) = +\infty\Big\}.$$

If in addition there exists $M > 0$ such that

$$\sum_{k\geq 0} \epsilon'_k(f_k(z))^2 \leq L \quad \text{for all } z \in \Sigma_0,$$

then g is bi-Lipschitz: there is a constant $C(n,d,L) \geq 1$ such that

$$C(n,d,L)^{-1}|z - z'| \leq |g(z) - g(z')| \leq C(n,d,L)|z - z'|.$$

3.2.2.2 The definition of g

Following Chapter 3 of [DT12], we take ψ_k to be a smooth function vanishing outside V_k^8 and $\theta_{j,k}$ to be a collection of smooth compactly supported functions in $10B_{j,k}$ such that $|\nabla^m \theta_{j,k}(y)| \leq C_m r_k^{-m}$ and $\psi_k(y) + \sum_{j\in J_k} \theta_{j,k}(y) = 1$. We then define a sequence of maps f_k by

$$f_0(y) = y,\ f_{k+1} = \sigma_k \circ f_k$$

where

$$\sigma_k(y) = y + \sum_{j\in J_k} \theta_{j,k}(y)\,[\pi_{j,k}(y) - y] = \psi_k(y)y + \sum_{j\in J_k} \theta_{j,k}(y)\,\pi_{j,k}(y),$$

where $\pi_{j,k}$ is orthogonal projection onto $P_{j,k}$. In our application, we only care about points inside V_k^8, so $\psi_k(y) = 0$ and the formula simplifies to

$$\sigma_k(y) = \sum_{j\in J_k} \theta_{j,k}(y)\,\pi_{j,k}(y).$$

The map σ_k also satisfies

$$|\sigma_k(y) - y| \leq C\epsilon r_k \tag{3.13}$$

for $k \geq 0$ and $y \in \Sigma_k$.

The map g is constructed by, roughly speaking, interpolating between adjacent maps in the sequence f_k at distance r_k from the surface $\Sigma_k = f_k(\Sigma_0)$. In order to construct this, David and Toro define a collection of linear isometries R_k on $\mathbb{R}^n$. The following proposition summarizes the properties of R_k that we need

Proposition 3.2.3 ([DT12] Proposition 9.29). *Let $\mathcal{R}$ denote the set of linear isometries of $\mathbb{R}^n$. Also set*

$$T_k(x) = T\Sigma_k(f_k(x)) \quad \text{for } x \in \Sigma_0 \text{ and } k \geq 0.$$

There exist C^1 mappings $R_k : \Sigma_0 \to \mathcal{R}$, with the following properties:

$$R_0(x) = I \text{ for } x \in \Sigma_0,$$

$$R_k(x)(T_0(x)) = T_k(x) \quad \text{for } x \in \Sigma_0 \text{ and } k \geq 0,$$
$$|R_{k+1}(x) - R_k(x)| \leq C\varepsilon \quad \text{for } x \in \Sigma_0 \text{ and } k \geq 0, \tag{3.14}$$

In addition, we record the bounds the distance between generations of tangent planes and between planes at different locations

Lemma 3.2.4 ([DT12] Lemma 9.2)**.** *We have that for $k \geq 0$ and $x, x' \in \Sigma_0$ such that $|x' - x| \leq 10$,*

$$D(T\Sigma_{k+1}(f_{k+1}(x)), T\Sigma_k(f_k(x))) \leq C_1\varepsilon$$
$$D(T\Sigma_k(f_k(x')), T\Sigma_k(f_k(x))) \leq C_2\varepsilon\, r_k^{-1}|f_k(x') - f_k(x)|.$$

Now, following Chapter 10 in [DT12], we define a collection ρ_k of positive, smooth, radial functions such that $\sum_{k\geq 0} \rho_k(y) = 1$ for $y \in \mathbb{R}^n \setminus \{0\}$ and $\rho_k(y) = 0$ unless $r_k < |y| < 20r_k$. Because $[r_k, 20r_k] \cap [r_{k-2}, 20r_{k-2}] = [r_k, 20r_k] \cap [100r_k, 2000r_k] = \varnothing$, we always have at most two values of k such that $\rho_k(y) \neq 0$ for any fixed y. In order to single out specific values of k, we define functions $l, n : \mathbb{R}^+ \to \mathbb{N}$ by

$$l(y) = \min\{k \in \mathbb{N} : \rho_k(y) > 0\}, \tag{3.15}$$
$$n(y) = \max\{k \in \mathbb{N} : \rho_k(y) > 0\} = l(y) + 1. \tag{3.16}$$

More concretely, we have

$$n(y) = n \iff 20r_{n+1} = 2r_n < y \leq 20r_n \tag{3.17}$$

because then $\rho_{n+1}(y) = 0$ while $\rho_n(y) > 0$. Roughly speaking, $n(y)$ gives the index of the maps $f_{n(y)}$ which is most relevant for the behavior of g on points roughly distance $|y|$ from Σ_0. We will now assume $\Sigma_0 = \mathbb{R}^d$ and write

$$g(z) = \sum_{k\geq 0} \rho_k(y)\,\{f_k(x) + R_k(x) \cdot y\} \quad \text{for } z = (x, y)$$

We will commonly use the notation $z = (x, y)$ as understood above when discussing points in the domain of g.

3.2.3 Whitney cubes, Whitney boxes, and Christ-David cubes

We will make significant use of the standard Whitney decomposition of the upper half-space with respect to $\mathbb{R}^d \times \{0\} \subseteq \mathbb{R}^{d+1}$.

Definition 3.2.3 (Whitney cubes)**.** Define

$$\mathscr{W} = \left\{[k_1 2^{-n}, (k_1+1)2^{-n}] \times \cdots \times [k_d 2^{-n}, (k_d+1)2^{-n}] \times [2^{-n}, 2^{-n+1}] : k_1, \ldots, k_d, n \in \mathbb{Z}\right\}.$$

$\mathscr{W}$ consists of exactly the dyadic cubes in $\mathbb{R}^d \times [0,\infty)$ which satisfy $\ell(W) = h(W) = \operatorname{dist}(W,\mathbb{R}^d)$ where $\ell(W) = h(W)$ denote the *side length* of W and the *height* of W. Cubes $W, R \in \mathscr{W}$ have a natural partial order induced by distance to $\mathbb{R}^d \times \{0\}$. We define the projection $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^d \times \{0\}$ by $\pi(x,y) = x$ where $(x,y) \in \mathbb{R}^d \times \mathbb{R}$ and write

$$W \leq R$$

if and only if $\pi(W) \subset \pi(R)$. If $h(W) = \frac{1}{2}h(R)$, we call W a *child* of R and R a *parent* of W. This gives a partial order on $\mathscr{W}$ which we use to define the *descendants* of W

$$D(W) = \{R \in \mathscr{W} : R \leq W\}.$$

This partial order imposes a natural tree structure on $\mathscr{W}$ which we will use in stopping time constructions. It will additionally be useful to refine the family of Whitney cubes into rectangular Whitney boxes in which the side length of the boxes in the first d-coordinate directions is allowed to vary.

Definition 3.2.4 (Whitney boxes)**.** We define the set of p-th order *Whitney boxes* by

$$\mathscr{R}_p = \left\{[k_1 2^{-p-n}, (k_1+1)2^{-p-n}] \times \cdots \times [k_d 2^{-p-n}, (k_d+1)2^{-p-n}] \times [2^{-n}, 2^{-n+1}] : k_i, n \in \mathbb{Z}\right\}.$$

These are like Whitney cubes, but they have lengths along the first d coordinate directions contracted by a factor of 2^p. Given $R \in \mathscr{R}_p$, we call $\ell(R) = 2^{-p-n}$ the *side length* and $h(R) = 2^{-n} = \operatorname{dist}(R,\mathbb{R}^d)$ the *height* so that

$$\ell(R) = 2^{-p}h(R).$$

Any collection of Whitney boxes has a tree structure induced by the same partial order as in Definition 3.2.3. We set $\mathscr{R} = \cup_p \mathscr{R}_p$.

We will later construct stopping time regions composed of Whitney boxes in the upper half space. We will also need the following notion of "closeness".

Definition 3.2.5 (A-close subsets)**.** We call two subsets $W, R \subseteq \mathbb{R}^{d+1}$ *A-close* (as in [DS93] pg. 59) if the following hold:

$$\begin{aligned}&\frac{1}{A}\operatorname{diam} W \leq \operatorname{diam} R \leq A \operatorname{diam} W,\\ &\operatorname{dist}(W,R) \leq A(\operatorname{diam} W + \operatorname{diam} R).\end{aligned}$$

We will also use the notation

$$W \simeq_A R$$

when W is A-close to R.

We will also need families of partitions of $\partial\Omega \subseteq \mathbb{R}^{d+1}$ which function as dyadic cubes do for $\mathbb{R}^{d+1}$. These were originally devised by Christ in [Chr90], but the formulation given here is due to Hytonen and Martikainen from [HM12].

Theorem 3.2.5 (Christ-David cubes)**.** *Let X be a doubling metric space. Let X_k be a nested sequence of maximal ρ^k-nets for X where $\rho < 1/1000$ and let $c_0 = 1/500$. For each $k \in \mathbb{Z}$ there is a collection $\mathscr{D}_k$ of "cubes," which are Borel subsets of X such that the following hold.*

(i) $X = \bigcup_{Q\in\mathscr{D}_k} Q$.

(ii) *If* $Q, Q' \in \mathscr{D} = \bigcup \mathscr{D}_k$ *and* $Q \cap Q' \neq \emptyset$, *then* $Q \subseteq Q'$ *or* $Q' \subseteq Q$.

(iii) *For* $Q \in \mathscr{D}$, *let* $k(Q)$ *be the unique integer so that* $Q \in \mathscr{D}_k$ *and set* $\ell(Q) = 5\rho^{k(Q)}$. *Then there is* $\zeta_Q \in X_k$ *so that*

$$B(x_Q, c_0\ell(Q)) \subseteq Q \subseteq B(x_Q, \ell(Q))$$

and

$$X_k = \{x_Q : Q \in \mathscr{D}_k\}.$$

If in addition we assume $X \subseteq \mathbb{R}^{d+1}$ and X is Ahlfors-David d-regular, then these cubes also satisfy

(iv) $|Q| \simeq_d (\operatorname{diam} Q)^d \simeq_d \ell(Q)^d$.

For any $Q \in \mathscr{D}$, we will use the notation $Q^{(1)}$ to refer to the parent of Q

We will refer to any family of Christ-David Cubes for $\partial\Omega$ by $\mathscr{D}$ and define

$$B_Q = B(x_Q, \ell(Q)).$$

3.2.4 Coronizations for Reifenberg flat and uniformly rectifiable sets

The boundary measure bounds for our Lipschitz decompositions come from Carleson packing conditions for well-chosen *coronizations* of a Christ-David lattice for $\partial\Omega$. Coronizations essentially consist of a partition of $\mathscr{D}$ into "good" cubes $\mathscr{G}$ and "bad" cubes $\mathscr{B}$ and a further partition of $\mathscr{G}$ into disjoint *stopping time regions* $\mathscr{F} = \{S_i\}_i$.

Definition 3.2.6 (stopping time regions)**.** We call $S \subseteq \mathscr{G} \subseteq \mathscr{D}$ a *stopping time region* if it is *coherent*, i.e.

(i) There exists a "top cube" $Q(S) \in S$ such that $R \subseteq Q(S)$ for all $R \in S$,

(ii) If $Q \in S$ and $R \in \mathscr{D}$ satisfies $Q \subseteq R \subseteq Q(S)$, then $R \in S$,

(iii) If $Q \in S$, then either every child of Q is in S, or none of them are.

Remark 3.2.6. We note that Definition 3.2.6 makes sense with any well-ordered collection of subsets of $\mathbb{R}^{d+1}$ in place of $\mathscr{D}$. For instance, we will use the term stopping time region to refer to such collections of Whitney boxes with the partial order defined in Definition 3.2.3.

Definition 3.2.7 (Coronizations (cf. [DS93] Definition 3.13))**.** We say that a triple $(\mathscr{G}, \mathscr{B}, \mathscr{F})$ is a *coronization* of $\mathscr{D}$ if

(i) $\mathscr{F}$ is a collection of disjoint stopping time regions as in Definition 3.2.6 with $\mathscr{G} = \bigcup_{S\in\mathscr{F}} S$,

(ii) $\mathscr{G} \cup \mathscr{B} = \mathscr{D}$ and $\mathscr{G} \cap \mathscr{B} = \varnothing$,

(iii) $\mathscr{B}$ and $\{Q(S)\}_{S\in\mathscr{F}}$ satisfy Carleson packing conditions. That is, there exist constants $C_1, C_2 > 0$ such that for any $Q \in \mathscr{D}$

$$\sum_{\substack{R\in\mathscr{B}\\ R\subseteq Q}} \ell(R)^d \leq C_1\mathcal{H}^d(Q), \quad \text{and} \quad \sum_{\substack{S\in\mathscr{F}\\ Q(S)\subseteq Q}} \ell(Q(S))^d \leq C_2\mathcal{H}^d(Q).$$

The stopping time regions in coronizations collect scales and locations into good, "connected" packages on which $\partial\Omega$ behaves well. David and Semmes used the concept of a coronization to produce a definition of uniform rectifiability involving *corona decompositions*

Definition 3.2.8 (Corona decomposition (cf. [DS93] Definition 3.19))**.** We say that a set $E \subseteq \mathbb{R}^n$ admits a d-dimensional *corona decomposition* if for any constants $\eta, \theta > 0$, there exists a coronization $(\mathscr{G}, \mathscr{B}, \mathscr{F})$ of a d-dimensional lattice $\mathscr{D}$ for E such that for each $S \in \mathscr{F}$, there exists a d-dimensional Lipschitz graph $\Gamma(S)$ with Lipschitz constant less than η such that for each $x \in 2Q$ and $Q \in S$

$$\operatorname{dist}(x, \Gamma(S)) \leq \theta \operatorname{diam}(Q). \tag{3.18}$$

If one has an appropriate coronization, then one can use Reifenberg parameterizations to produce the approximating Lipschitz graphs in the definition of the corona decomposition directly. We call these specific good coronizations *graph coronizations*

Definition 3.2.9 (graph coronizations)**.** For constants $M, \epsilon, \delta > 0$, we say that $(\mathscr{G}, \mathscr{B}, \mathscr{F})$ is a d-dimensional $(M, \epsilon, \delta)-$*graph coronization* if it is a coronization such that $\mathscr{B} \supseteq \mathrm{BWGL}(M, \epsilon)$ and for each $S \in \mathscr{F}$ and $Q \in S$, there exists a d-plane $P_Q \ni x_Q$ such that

(i) $b\beta_E(MB_Q, P_Q) \leq 2b\beta_E(MB_Q) \leq 2\epsilon$,

(ii) $\sum_{x\in Q\in S} \beta_E^{d,1}(MB_Q)^2 \leq \epsilon^2$ for any $x \in Q(S)$.

(iii) $\mathrm{Angle}(P_Q, P_{Q(S)}) \leq \delta$,

Condition (ii) above uses the *content beta number* introduced by Azzam and Schul in [AS18]. This is closely related to the more standard L^p *beta numbers* used by David and Semmes in characterizing uniform rectifiability via the *strong geometric lemma.*

Definition 3.2.10 (L^p beta numbers and content beta numbers)**.** Let $B = B(x, r) \subseteq \mathbb{R}^{d+1}$ and let P be a d-plane. We define

$$\beta^d_{E,p}(B, P) = \left(\frac{1}{r^d}\int_{B\cap E}\left(\frac{\operatorname{dist}(y, P)}{r}\right)^p \, d\mathscr{H}^d(y)\right)^{1/p},$$

and we define the L^p *beta number* as

$$\beta^d_{E,p}(B) = \inf\{\beta^d_{E,p}(B, P) : P \text{ is a } d\text{-dimensional plane in } \mathbb{R}^{d+1}\}.$$

Similarly, we define

$$\beta_E^{d,p}(B,P) = \left(\frac{1}{r_B^d} \int_0^\infty \mathscr{H}_\infty^d \{x \in B \cap E : \mathrm{dist}(x,P) > tr_B\} t^{p-1} dt \right)^{1/p},$$

and we define the L^p *content beta number* as

$$\beta_E^{d,p}(B) = \inf\{\beta_E^{d,p}(B,P) : P \text{ is a } d\text{-dimensional plane in } \mathbb{R}^{d+1}\}.$$

If E is Ahlfors d-regular, then these two beta numbers are comparable with constants depending on d and the regularity constant.

Proposition 3.2.7 (cf. [DS93] Part I, Theorem 1.57 and Theorem 2.4; Part IV Proposition 2.1)**.** *Let $E \subseteq \mathbb{R}^{d+1}$ be Ahlfors d-regular for $d \geq 1$. The following are equivalent:*

(i) E is d-uniformly rectifiable,

(ii) E satisfies the strong geometric lemma: For any $Q \in \mathscr{D}$, $M > 1$, and $1 \leq p < \frac{2d}{d-2}$,

$$\sum_{R \subseteq Q} \beta_{E,p}^d(MB_R)^2 \ell(R)^d \lesssim_{M,d} \ell(Q)^d,$$

(iii) E admits a corona decomposition.

(iv) E admits an (M,ϵ,δ)-graph coronization for any $M,\epsilon,\delta > 0$.

The main tool we will use to create Lipschitz decompositions is the graph coronization. In Appendix 3.7, we review the d-dimensional traveling salesman results of [AS18] and [Hyd22a] which give a similar analysis for general Reifenberg flat sets. By collecting these results, we prove the following proposition:

Proposition 3.2.8. *For any $d,n \in \mathbb{N}$ with $0 < d < n$, there exists $\epsilon_0(d,n), \delta(d,n) > 0$ such that if $\epsilon \leq \epsilon_0 \ll \delta^4$ and $E \subseteq \mathbb{R}^n$ is (ϵ,d)-Reifenberg flat, then E admits an (M,ϵ,δ)-graph coronization for any $M > 0$.*

We will use the existence of graph coronizations as in the previous two propositions to prove Theorems D and E.

We also record some important facts about using beta numbers to control the Hausdorff distance of planes. Given a set $E \subseteq \mathbb{R}^{d+1}$ and a Christ-David lattice $\mathscr{D}$ for E, we define epsilon numbers adapted to the lattice $\mathscr{D}$ and a collection of planes $\{P_Q\}_{Q\in\mathscr{D}}$. Fix $K = \frac{10^4}{\rho}$. We define

$$\epsilon(Q) = \sup\left\{ d_{KB_R}(P_U,P_R) : k(R) \in \{k(Q), k(Q)-1\},\ k(U) = k(Q),\ x_Q \in \frac{K}{10}B_Q \cap \frac{K}{10}B_R \right\}.$$

This is essentially a version of David and Toro's ϵ_k' numbers which is adapted to a cube structure rather than a general collection of nets. Now, let $M \geq \frac{10K}{\rho^2}$. We will use these to control ϵ_k' in terms of $\beta^{d,1}(MB_Q)$ in the second lemma below. First, we give a recall a general result that allows one to bound the Hausdorff distance between planes by beta numbers:

Lemma 3.2.9 ([AS18] Lemma 2.16)**.** *Suppose $E \subseteq \mathbb{R}^n$ and B is a ball centered on E such that for all balls $B' \subseteq B$, $\mathscr{H}^d_\infty(B') \geq c r^d_{B'}$. Let P and P' be two d-planes. Then*

$$d_{B'}(P, P') \lesssim_{d,c} \left(\frac{r_B}{r_{B'}}\right)^{d+1} \beta^{d,1}_E(B, P) + \beta^{d,1}_E(B', P').$$

The next lemma applies this to bound $\epsilon(Q)$ by $\beta^{d,1}_E(MB_Q)$:

Lemma 3.2.10. *Let $\mathscr{D}$ be a Christ-David lattice for a lower content d-regular set E and $K, M > 0$ be constants such that $\frac{10^4}{\rho} \leq K \leq 10^{-1}\rho^2 M$. If $\{P_Q\}_{Q\in\mathscr{D}}$ is a family of d planes satisfying $\beta^{d,1}_E(2\rho^{-1}KB_Q, P_Q) \lesssim \beta^{d,1}_E(2\rho^{-1}KB_Q)$, then*

$$\epsilon(Q) \lesssim_{\rho,M,d} \beta^{d,1}_E(MB_Q).$$

Proof. Let $U, R \in \mathscr{D}$ be cubes which achieve the supremum in the definition of $\epsilon(Q)$. Then

$$\epsilon(Q) = d_{KB_R}(P_U, P_R).$$

We want to apply Lemma 3.2.9 with $B = B' = KB_R$. First, we prove some ball inclusions. We claim

$$KB_R \subseteq 2\rho^{-1}KB_U. \tag{3.19}$$

Indeed, we let $y \in KB_R$ and we compute

$$\begin{aligned}|y - x_U| &\leq |y - x_R| + |x_R - x_Q| + |x_Q - x_U| \\ &\leq K\ell(R) + \frac{K}{10}\ell(R) + \frac{K}{10}\ell(U) \leq 2K\ell(R) \leq 2\rho^{-1}K\ell(U).\end{aligned}$$

Second, we claim

$$2\rho^{-1}KB_U \subseteq MB_Q \text{ and } 2\rho^{-1}KB_R \subseteq MB_Q. \tag{3.20}$$

Because $\ell(R) \geq \ell(U)$, it suffices to prove $2\rho^{-1}KB_R \subseteq MB_Q$. We let $y \in 2\rho^{-1}KB_R$ and compute

$$|y - x_Q| \leq |y - x_R| + |x_R - x_Q| \leq 4\rho^{-1}K\ell(R) + \frac{K}{10}\ell(R) \leq 10K\rho^{-2}\ell(Q) < M\ell(Q).$$

Now, we apply Lemma 3.2.9 with $B = B' = KB_R$, then

$$\begin{aligned}d_{KB_R}(P_U, P_R) &\lesssim \beta^{d,1}_E(KB_R, P_R) + \beta^{d,1}_E(KB_R, P_U) \\ &\lesssim_\rho \beta^{d,1}_E(2\rho^{-1}KB_R, P_R) + \beta^{d,1}_E(2\rho^{-1}KB_U, P_U) \\ &\lesssim \beta^{d,1}_E(2\rho^{-1}KB_R) + \beta^{d,1}_E(2\rho^{-1}KB_U) \\ &\lesssim_M \beta^{d,1}_E(MB_Q).\end{aligned}$$

where the second line follows from (3.19), the third line follows from the hypothesis on P_Q, and the final line from (3.20). ■

3.3 The proof of Theorem C

Fix constants $\rho = \frac{1}{1000}$, $K = \frac{10^4}{\rho}$, $M = \frac{10K}{\rho^2}$, $A_0 = \frac{1000\sqrt{d}}{c_0\rho}$. Throughout this section, assume that $\Omega \subseteq \mathbb{R}^{d+1}$ satisfies the hypotheses of Theorem C. We begin by constructing a Reifenberg parameterization for $\partial\Omega \cap B(0,1)$

3.3.1 The CCBP adapted to $\mathscr{D}$

We want to form a CCBP adapted to the Christ-David lattice $\mathscr{D}$ for $\partial\Omega$ with the aim of applying David and Toro's bi-Lipschitz Reifenberg parameterization result Theorem 3.2.2. For any $k \in \mathbb{Z}$, let $s(k)$ be an integer such that

$$50\rho^{s(k)} \le r_k < 50\rho^{s(k)-1} \tag{3.21}$$

We note that if $Q \in \mathscr{D}_{s(k)}$, then this means

$$10\ell(Q) \le r_k < 10\rho^{-1}\ell(Q) \tag{3.22}$$

and

$$\frac{\rho}{5000} r_k \le \frac{c_0}{10\rho} r_k \le c_0\ell(Q) \le \operatorname{diam} Q \le \ell(Q) \le \frac{r_k}{10}.$$

For any $k \ge 0$, define

$$Y_k = \{x_Q : Q \in \mathscr{D}_{s(k)},\ Q \cap B(0, A_0) \neq \varnothing\}, \tag{3.23}$$
$$X_k \in \operatorname{Net}(Y_k, r_k). \tag{3.24}$$

We enumerate $X_k = \{x_{j,k}\}_{j \in J_k}$ and often use the notation $x_{j,k} = x_Q = x_{Q_{j,k}}$. Let P_0 achieve the infimum in the definition of $b\beta_{\partial\Omega}(B(0, 10A_0))$ and define

$$B_{j,k} = B(x_{j,k}, r_k),$$
$$P_{j,k} = P_{Q_{j,k}},$$

where $P_{Q_{j,k}} \ni x_{Q_{j,k}}$ are such that $b\beta^{d,1}_{\partial\Omega}(2\rho^{-1}KB_{Q_{j,k}}) \lesssim b\beta^{d,1}_{\partial\Omega}(2\rho^{-1}KB_{Q_{j,k}})$ as in the hypotheses of Lemma 3.2.10. We first show that $\epsilon'_k(x_{j,k})$ is controlled by $\epsilon(Q_{j,k})$.

Lemma 3.3.1. *Fix $k \ge 0$ and $Q \in \mathscr{D}_{s(k)}$. For any $z \in \mathbb{R}^{d+1}$ such that $|z - x_Q| < 200\rho^{-1}\ell(Q)$,*

$$\epsilon'_k(z) \le K\epsilon(Q).$$

Proof. We first show that the supremum in the definition of $\epsilon(Q)$ is over a larger collection of pairs of planes than that in the definition of $\epsilon'_k(z)$. Let $i \in J_k$ be such that $z \in 10B_{i,k}$. Then by (3.22),

$$|x_Q - x_{i,k}| < |x_Q - z| + |z - x_{i,k}| < 200\rho^{-1}\ell(Q) + 10r_k < 300\rho^{-1}\ell(Q_{i,k}) < \frac{K}{10}\ell(Q)$$

because $K \ge 10^4\rho^{-1}$ and $\ell(Q) = \ell(Q_{i,k})$. Therefore, $x_Q \in \frac{K}{10}B_{Q_{i,k}}$. If instead $z \in 11B_{i,k-1}$ for some $i \in J_{k-1}$, then

$$|x_Q - x_{i,k-1}| < |x_Q - z| + |z - x_{i,k-1}| < 200\rho^{-1}\ell(Q) + 11r_{k-1} < 310\rho^{-1}\ell(Q_{i,k-1}) < \frac{K}{10}\ell\left(Q_{i,k-1}\right).$$

Therefore, $x_Q \in \frac{K}{10} B_{Q_{i,k-1}}$. In addition, for any admissible $x_{i,l}$ in the definition of $\epsilon'_k(z)$ we can write $100 r_l \leq 1000\rho^{-1}\ell(Q_{i,l}) < K\ell(Q_{i,l})$ so that $100 B_{i,l} \subseteq K B_{Q_{i,l}}$. Let $P_{i,k}$ and $P_{m,l}$ be planes which achieve the supremum in the definition of $\epsilon'_k(z)$. Then

$$d_{x_{m,l},100B_{m,l}}(P_{i,k}, P_{m,l}) \leq \frac{K\ell(B_{Q_{m,l}})}{100 r_l} d_{KB_{Q_{m,l}}}(P_{Q_{i,k}}, P_{Q_{m,l}}) \leq K d_{KB_{Q_{m,l}}}(P_{Q_{i,k}}, P_{Q_{m,l}})$$

using the fact that $\ell(Q_{m,l}) < r_l$. ■

Applying this result for $z = x_{j,k}$ shows that $\epsilon'_k(x_{j,k}) \lesssim \epsilon(Q_{j,k})$ which we can use to prove that the triple $\mathscr{X} = (P_0, \{B_{j,k}\}, \{P_{j,k}\})$ is a CCBP.

Lemma 3.3.2. *$\mathscr{X}$ is a CCBP.*

Proof. We will use Lemma 3.2.1. First, we will show that for any $j \in J_0$, $\text{dist}(x_{j,0}, P_0) \lesssim \epsilon$. Indeed, $x_{j,0} = x_Q$ for some $Q \in \mathscr{D}_{s(0)}$ with $Q \cap B(0, A_0) \neq \varnothing$. Hence, $x_Q \in B(0, 2A_0) \cap \partial\Omega$ so that $b\beta_{\partial\Omega}(B(0, 10A_0), P_0) \leq \epsilon$ implies

$$\text{dist}(x_Q, P_0) \lesssim b\beta(B(0, 10A_0)) \cdot 10A_0 \lesssim_d \epsilon.$$

Now, we fix $k > 0$ and $j \in J_k$ and prove the following claim:

Claim T:here exists $i \in J_{k-1}$ such that $x_{j,k} \in B_{i,k-1}$

Proof I:ndeed, let $x_{j,k} = x_{Q_{j,k}}$. If $s(k) = s(k-1)$, then $Y_{k-1} = Y_k$ so that $x_{Q_{j,k}} \in Y_{k-1}$. The claim follows since X_{k-1} is an r_{k-1}-net for Y_{k-1}. If instead $s(k) > s(k-1)$, then $x_{Q^{(1)}_{j,k}} \in Y_{k-1}$ so that there exists $i \in J_{k-1}$ such that $x_{Q^{(1)}_{j,k}} \in B_{i,k-1}$. We have

$$\ell\left(Q^{(1)}_{j,k}\right) = 5\rho^{s(k-1)} \leq 5\rho^{s(k)-1} \leq r_{k-1}$$

so that

$$\text{dist}(x_{Q_{j,k}}, x_{i,k-1}) \leq \text{dist}(x_{Q_{j,k}}, x_{Q^{(1)}_{j,k}}) + \text{dist}(x_{Q^{(1)}_{j,k}}, x_{i,k-1}) \leq \ell\left(Q^{(1)}_{j,k}\right) + r_{k-1} \leq 2r_{k-1}$$

which proves $x_{Q_{j,k}} \in 2B_{i,k-1}$. ■

By Lemma 3.3.1, it suffices to show that $\epsilon(Q_{j,k}) \lesssim \epsilon$. But by the definition of $P_{Q_{j,k}}$ and Lemma 3.2.10, we have $\epsilon(Q_{j,k}) \lesssim_{M,d} \beta^{d,1}_{\partial\Omega}(MB_{Q_{j,k}}) \lesssim \epsilon$. ■

Since we've shown that $\mathscr{X}$ is a CCBP, Theorem 3.2.2 gives a Reifenberg parameterization $g : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ such that

$$g(P_0) \cap B(0,1) = \partial\Omega \cap B(0,1)$$

Without loss of generality, we can assume $P_0 = \mathbb{R}^d \times \{0\}$ and translate the Whitney decomposition $\mathscr{W}$ as in Definition 3.2.3 to a new decomposition $\mathscr{W}'$ such that $W_0 = [-2,2]^d \times [4,8] \in \mathscr{W}'$. We have that

$$\Omega \cap B(0,1) \subseteq g([-2,2]^d \times [0,8])$$

because $\partial\Omega$ is contained in the closure of $\cup_k X_k$, so in practice we only need to consider the set of descendants of W_0 to cover $\Omega \cap B(0,1)$:

$$\mathscr{W}_0 = \{W \in \mathscr{W}' : W \in D(W_0)\}. \tag{3.25}$$

We can now derive some useful properties of g.

Lemma 3.3.3 (Properties of g)**.**

(i) For any $x \in [-2,2]^d \times \{0\}$ and $n \in \mathbb{N}$,

$$f_n(x) \in V_n^8,$$

(ii) For any $z = (x,y) \in [-2,2]^d \times [0,8]$

$$(1 - C(d)\epsilon)|y| \leq \operatorname{dist}(g(z), \partial\Omega) \leq (1 + C(d)\epsilon)|y|. \tag{3.26}$$

(iii) For any $x \in [-2,2]^d \times \{0\}$ and $p, n \in \mathbb{N}$ with $p < n$, there exists a collection of cubes $Q_n \subseteq Q_{n-1} \subseteq \cdots \subseteq Q_p$ such that for any k with $p \leq k \leq n$, $\operatorname{dist}(g(x, r_k), Q_{s(k)}) \lesssim r_k$ and

$$\sum_{k=p}^{n} \epsilon'(f_k(x))^2 \lesssim_{M,\rho,d} \sum_{k=p}^{n} \beta_{\partial\Omega}^{d,1}(MB_{Q_k})^2.$$

In particular, g is $L'(L, \rho, M, d)$-bi-Lipschitz.

Proof. Let $z = (x,y)$ be as in (ii) with $n = n(y)$. We first prove (ii) with the added hypothesis that $f_n(x) \in V_n^8$. We will then prove (i) which will complete the proof of (ii).

Observe that

$$g(z) - f_n(x) = \sum_k \rho_k(y) \left\{ f_k(x) - f_n(x) + R_k(x) \cdot y \right\}$$

where $|f_k(x) - f_n(x)| \lesssim \epsilon r_n$ and $R_k(x) \cdot y$ is a vector of norm $|y|$ which is orthogonal to the tangent plane $T_k(x)$ to Σ_k at $f_k(x)$. The fact that $f_n(x) \in V_n^8$ implies the existence of $Q \in \mathscr{D}_{s(n)}$ such that $|f_n(x) - x_Q| \leq 8r_n$. The fact that $b\beta_{\partial\Omega}(MB_Q, P_Q) \lesssim \epsilon$ combined with Lemma 3.6.1 and (3.77) implies

$$d_{f_n(x),19r_n}(\Sigma_n, \partial\Omega) \leq d_{f_n(x),19r_n}(\Sigma_n, P_Q) + d_{f_n(x),19r_n}(P_Q, \partial\Omega) \lesssim \epsilon. \tag{3.27}$$

We conclude $d_{f_n(x),19r_n}(T_n(x) + f_n(x), \partial\Omega) < C\epsilon$, which implies

$$(1 - C\epsilon)|y| \leq \operatorname{dist}(g(z), \partial\Omega) \leq (1 + C\epsilon)|y|$$

as desired.

We now prove (i) by induction on n. For the base case $n = 0$, notice that $f_0(x) = x \in B(0, 5\sqrt{d}) \cap P_0$ so that $b\beta_{\partial\Omega}(B(0, 10A_0), P_0) \leq \epsilon$ implies the existence of $y \in \partial\Omega$ with $\operatorname{dist}(y, x) \lesssim_{A_0} \epsilon$. There exists $Q_0 \in \mathscr{D}_{s(0)}$ such that $y \in Q_0$ and $\operatorname{dist}(Q, 0) \leq 10\sqrt{d}$ so that x_{Q_0} is a member of the set Y_0 (see (3.23)) from which the maximal net X_0 forming the CCBP is taken. Notice that

$$|f_0(x) - x_{Q_0}| \leq |x - y| + |y - x_{Q_0}| \leq C\epsilon + \ell(Q_0) \leq 2r_0.$$

Hence, we are done if $x_{Q_0} \in X_0$. Otherwise, there exists $x_{Q_0'} \in X_0$ such that $|x_{Q_0} - x_{Q_0'}| \leq r_0$ so that $|f_0(x) - x_{Q_0'}| \leq 3r_0$ implying $f_0(x) \in V_0^3$. This proves the base case for (i).

For the inductive step, assume that $f_n(x) \in V_n^8$ for some $n \in \mathbb{N}$. Using (3.27), we find $y \in \partial\Omega$ such that

$$|f_{n+1}(x) - y| \leq |f_{n+1}(x) - f_n(x)| + |f_n(x) - y| \lesssim \epsilon r_{n+1}$$

and hence there exists $Q_{n+1} \in \mathscr{D}_{s(n+1)}$ with $\operatorname{dist}(Q_{n+1}, 0) \leq 10\sqrt{d}$ such that $|f_{n+1}(x) - x_{Q_{n+1}}| \leq 2r_{n+1}$. By a similar argument to the base case, this finishes the proof of (i).

To prove (iii), notice that $f(x) \in \partial\Omega$ so that there exists an infinite chain of (possibly repeating) cubes $Q_0 \supseteq Q_1 \supseteq \cdots \ni f(x)$ where $Q_k \in \mathscr{D}_{s(k)}$. We claim that

$$\epsilon'(f_k(x)) \lesssim \epsilon(Q_k) \lesssim \beta_{\partial\Omega}^{d,1}(MB_{Q_k}).$$

Indeed, by Lemmas 3.3.1 and 3.2.10, we only need to show that $|f_k(x) - x_{Q_k}| < 200\rho^{-1}\ell(Q_k)$ to verify the first inequality. But we have

$$\begin{aligned} |f_k(x) - x_{Q_k}| &\leq |f_k(x) - f(x)| + |f(x) - x_{Q_k}| \\ &\leq C\epsilon r_k + 10r_k + \ell(Q_k) \leq (C\epsilon + 100)\rho^{-1}\ell(Q_k) + \ell(Q_k) \leq 102\rho^{-1}\ell(Q_k) \end{aligned}$$

as desired. Because the set $\{n : s(k) = s(n)\}$ has a uniformly bounded number of elements in terms of ρ, it follows that

$$\sum_{k=p}^{n} \epsilon_k'(f_k(x))^2 \lesssim_{M,\rho} \sum_{k=p}^{n} \epsilon(Q_k)^2 \lesssim_{M,\rho,d} \sum_{k=p}^{n} \beta_{\partial\Omega}^{d,1}(MB_{Q_k})^2.$$

The claim that $\operatorname{dist}(g(x, r_k), Q_k) \lesssim r_k$ follows from (ii). By the hypotheses on Ω, we have $\sum_{k=1}^{\infty} \epsilon'(f_k(x))^2 \lesssim \sum_{f(x)\in Q} \beta_{\partial\Omega}^{d,1}(MB_Q)^2 \leq L$ so that g is $L'(L, \rho, M, d)$-bi-Lipschitz by Theorem 3.2.2. ■

Now that we know that g is L'-bi-Lipschitz, we define $p(L') \in \mathbb{Z}$ such that

$$2^{p-1} \leq L' < 2^p \tag{3.28}$$

and we replace $\mathscr{W}_0$ with

$$\mathscr{R}_w = \{R \in \mathscr{R}_p : \exists W \in \mathscr{W}_0,\ R \subseteq W\}.$$

That is, $\mathscr{R}_w$ is the set of Whitney boxes R with $\ell(R) = 2^{-p}h(R)$ which are contained in some member of $\mathscr{W}_0$. This ensures that

$$L'\ell(R) = L'2^{-p}h(R) \leq h(R) \tag{3.29}$$

so that g does not stretch R across too far of a region on the scale of $h(R)$.

We say more about the shape of image boxes in the following lemma:

Lemma 3.3.4 (Image boxes)**.** *For any $W \in \mathscr{R}_w$, we have*

$$(1 - C\epsilon)h(W) \le \operatorname{dist}(g(W), \partial\Omega) \le (1 + C\epsilon)h(W), \tag{3.30}$$

$$(1 - C\epsilon)h(W) \le \operatorname{diam} g(W) \le 5\sqrt{d}h(W). \tag{3.31}$$

There exists constants $C_0(L'), C_1(d)$ such that

$$B(g(c_W), C_0^{-1}h(W)) \subseteq g(W) \subseteq B(g(c_W), C_1 h(W)) \tag{3.32}$$

where c_W is the center of W.

Proof. We first note that (3.30) follows from (3.26) and the fact that $\operatorname{dist}(W, \mathbb{R}^d) = h(W)$ by definition. To prove (3.31), let $z, z' \in R$ with $z = (x, y), z' = (x', y')$. We have

$$\begin{aligned} |g(x,y) - g(x',y')| &\le |g(x,y) - g(x',y)| + |g(x',y) - g(x',y')| \\ &\le L'|x - x'| + 2|y - y'| \\ &\le L'\sqrt{d}\ell(R) + 2h(R) \\ &\le 5\sqrt{d}h(R) \end{aligned}$$

The lower bound follows from (3.26) by considering the distance between images of points in the lower and upper faces of W. To prove (3.32), we first observe that each box $W \in \mathscr{R}$ contains a small ball $B(c_W, c(L')h(W))$ around its center. Since g is L'-bi-Lipschitz, we get a larger constant $C_0(L')$ such that the lower containment in (3.32) holds. The existence of $C_1(d)$ as in the upper containment follows from the upper inequality in (3.31). We also note that because g is injective and distinct boxes $R, W \in \mathscr{R}_w$ have disjoint interiors, we have

$$B(g(c_W), C_0^{-1}h(W)) \cap B(g(c_R), C_0^{-1}h(R)) = \varnothing. \tag{3.33}$$

■

3.3.2 Whitney coronizations and the Lipschitz decomposition

In Item (iii) of Lemma 3.3.3, we showed that the mapping g was bi-Lipschitz so that $\partial\Omega$ is locally a bi-Lipschitz image. Hence, $\partial\Omega$ is d-uniformly rectifiable and therefore has an (M, ϵ, δ)-graph coronization for arbitrarily small values of ϵ and δ by Proposition 3.2.7. Take $\epsilon'(d, L), \delta'(d, L) > 0$ fixed later sufficiently small and let $\mathscr{C} = (\mathscr{G}, \mathscr{B}, \mathscr{F})$ be an (M, ϵ', δ')-graph coronization for $\partial\Omega$.

The plan for the proof of Theorem C is to construct a "coronization" of $\mathscr{R}_w$ which "follows" the coronization $\mathscr{C}$ of $\partial\Omega$. That is, we will construct a triple

$$\mathscr{C}_w = (\mathscr{G}_w, \mathscr{B}_w, \mathscr{T})$$

of good boxes, bad boxes, and stopping time regions $\mathscr{T} = \{T\}_{T \in \mathscr{T}}$ (see Remark 3.2.6) partitioning $\mathscr{G}_w$ such that for each $T \in \mathscr{T}$, there exists some $S \in \mathscr{F}$ such that the images of all boxes in T under g are "surrounded" in scale and location by cubes in S.

Definition 3.3.1 (g-Whitney coronizations)**.** Let $g, \mathscr{R}_w$ be as above. We now give a partition of $\mathscr{R}_w$ into a bad set $\mathscr{B}_w$ and good set $\mathscr{G}_w$ which picks out all Whitney boxes whose images under g are "A_0-surrounded" by surface cubes within a single stopping time region $S \in \mathscr{F}$:

$$\mathscr{G}_w = \{W \in \mathscr{W} : \exists S \in \mathscr{F},\ \forall Q \in \mathscr{D} \text{ such that } Q \simeq_{A_0} g(W) \text{ we have } Q \in S\}, \tag{3.34}$$
$$\mathscr{B}_w = \mathscr{W} \setminus \mathscr{G}_w. \tag{3.35}$$

(See Definition 3.2.5.) Given a root box $W \in \mathscr{G}_w$, we define the stopping time region T_W with top cube W to be the maximal sub tree of $D(W) \cap \mathscr{G}_w$ such that for any $R \in T_W$, either all of its children are in T_W, or none are. Any such stopping time region has associated minimal cubes and stopped cubes

$$m(T_W) = \{R \in T_W : R \text{ has a child not in } T_W\},$$
$$\text{Stop}(T_W) = \{R \in \mathscr{W} : R \text{ has a parent in } m(T_W)\}.$$

We initialize our construction with the lattice $\mathscr{R}_w$ and triple $(\mathscr{G}_w, \mathscr{B}_w, \mathscr{T}_0 = \varnothing)$. Given the k-th stage stopping time collection $\mathscr{T}_k$, we choose a root box $W \in \mathscr{G}_w \setminus \cup_{T \in \mathscr{T}_k} T$ and form the stopping time region T_W. We set $\mathscr{T}_{k+1} = \mathscr{T}_k \cup \{T_W\}$. Repeating this process inductively, we obtain a partition $\mathscr{T} = \bigcup_{k=1}^{\infty} \mathscr{T}_k$ of $\mathscr{G}_w$ into coherent stopping time regions. This gives the triple $\mathscr{C}_w = (\mathscr{G}_w, \mathscr{B}_w, \mathscr{T})$. We call $\mathscr{C}_w$ the *g-Whitney coronization* of $\mathscr{R}_w$ with respect to $\mathscr{C} = (\mathscr{G}, \mathscr{B}, \mathscr{F})$.

Remark 3.3.5 (improving the stopping time). In this construction, we used Whitney boxes with side length $\ell(R) = 2^{-p} h(R)$ to ensure that for any $R \in \mathscr{R}_w$, $\operatorname{diam} g(R) \lesssim_d h(R)$. Without this condition or some other method of controlling the size of image boxes, we could have $z = (x, y),\ z = (x', y')$ such that $h(R) \ll |g(z) - g(z')|$ which would cause us to lose control of the change in Dg across R we desire in Lemma 3.3.8 below.

What we really want are image pieces of some kind which satisfy the conclusions of Lemma 3.3.4 along with small parameterization derivative change across the pieces as in Lemma 3.3.8 below. If one could form reasonable stopping time domains out of similar pieces whose images satisfy the conclusions of 3.3.4 with constant C_0 dependent only on d, this would essentially prove a version of Theorem C without hypothesis (ii). If g were $K(d)$-quasiconformal, then this could likely be accomplished by adding modifications to the stopping time by dynamically either combining or cutting apart children boxes for a given top box $W(T)$ along coordinate directions according to the size and shape of Dg inside. In general though, Dg can distort boxes so badly that coordinate boxes cannot be mapped forward appropriately in general, so one would need to devise a better way of partitioning the domain into pieces which are mapped forward well under a more wild parameterization.

We will use $\mathscr{C}_w$ to break up $\mathscr{R}_w$ into regions which will map forward under g to the Lipschitz graph domains we desire as in the conclusion of Theorem C

Definition 3.3.2 (Stopping time domains)**.** Let $\mathscr{C}_w = (\mathscr{G}_w, \mathscr{B}_w, \mathscr{T})$ be a g-Whitney coronization as above. For each $T \in \mathscr{T}$, we define a *stopping time domain*

$$\mathcal{D}_T = \bigcup_{W \in T} W.$$

For each $W \in \mathscr{B}_w$, we note that $\ell(W) = 2^{-p}h(W)$ where p is as in (3.28) and define a collection of associated trivial domains by chopping W into 2^p cubes of common side length $\ell(W)$. That is, assuming $W = [0, \ell(W)]^d \times [h(W), 2h(W)]$, we set

$$\mathscr{L}_W = \{[0, \ell(W)]^d \times [h(W) + k\ell(W), h(W) + (k+1)\ell(W)] : 0 \leq k \leq 2^p - 1\}.$$

The collection $\mathscr{L}' = \{\mathcal{D}_T\}_{T \in \mathscr{T}} \cup \bigcup_{W \in \mathscr{B}_w} \mathscr{L}_W$ is a partition of $\bigcup_{W \in \mathscr{R}} W = [-2, 2]^d \times [0, 8]$ up to finite overlaps on boundaries. Each cube domain $R_W \in \mathscr{L}_W$ is $C(d)$-Lipschitz graphical, but the domain $\mathcal{D}_T$ is not Lipschitz graphical in general. However, T consists of a coherent collection of boxes of a given ratio of side length to height $\ell(R) = 2^{-p}h(R)$. Therefore, applying a dilation A_p by a factor of 2^p in the first d coordinates gives a domain $A_p(\mathcal{D}_T)$ consisting of cubes. Proposition 3.5.1 then gives the existence of a d-rectifiable, d-Ahlfors upper regular set Σ_T such that

$$A_p(\mathcal{D}_T) \setminus \Sigma_T = \bigcup_{j \in J_T} A_p(\mathcal{D}_T^j)$$

where $\{A_p(\mathcal{D}_T^j)\}_{j \in J_T}$ is a collection of $C(d)$-Lipschitz graph domains with disjoint interiors. By Lemma 3.5.4, we then get the existence of a constant $C'(L, d)$ such that $\mathcal{D}_T^j$ is a $C'(L, d)$-Lipschitz graph domain. We then finally define

$$\mathscr{L}_0 = \left\{\mathcal{D}_T^j\right\}_{T \in \mathscr{T}, j \in J_T} \cup \bigcup_{W \in \mathscr{B}_w} \mathscr{L}_W.$$

We can now define the collection of Lipschitz graph domains $\mathscr{L}$ as desired in Theorem C:

Definition 3.3.3 (Lipschitz decomposition)**.** Let $\mathscr{L}_0$ be as in Definition 3.3.2. We define the *Lipschitz decomposition* of $\Omega \cap B(0, 1)$ as

$$\mathscr{L} = \{g(\mathcal{D}) : \mathcal{D} \in \mathscr{L}_0\}. \tag{3.36}$$

In order to prove Theorem C, it suffices to prove Propositions 3.3.6 and 3.3.7 below.

Proposition 3.3.6. *Let Ω be as in Theorem C and $\mathscr{L} = \{\Omega_j\}_{j \in J_{\mathscr{L}}}$ be as in* (3.36)*. There exists $L_1(L, d, \epsilon) > 0$ such that for any $j \in J_{\mathscr{L}}$, Ω_j is an L_1-Lipschitz graph domain.*

To prove Proposition 3.3.6, we use the fact that the graph coronization $\mathscr{C}$ of $\partial\Omega$ and the Whitney coronization $\mathscr{C}_w$ of Definition 3.3.1 adapted to $\mathscr{C}$ were chosen so that Dg is very close to being constant on any given domain $\mathcal{D} \in \mathscr{L}_0$. This uses the explicit calculations for Dg given in Proposition 3.6.7. This means g distorts $\mathcal{D}$ only slightly such that $\mathcal{D}$ remains a Lipschitz graph domain (see Proposition 3.5.6). The refinement of Whitney cubes to smaller Whitney boxes ensures that $\operatorname{diam}(g(W)) \simeq_d h(W)$ holds for any box W so that $g(W)$ does not stretch across too long of a region of $\partial\Omega$ compared to its distance from $\partial\Omega$. If $W \in \mathscr{B}_W$, then this ensures that $Dg|_W$ varies at a rate determined at worst by the Reifenberg flatness constant ϵ. Because in this case, W is divided into the set $\mathscr{L}_W$ of cubes, which are $C(d)$-Lipschitz graph domains (note C is independent of L), g maps them forward to Lipschitz graph domains given that ϵ is fixed small enough with respect to d.

The construction of stopping time regions $\mathscr{T}$ proceeds in such a way that any $T \in \mathscr{T}$ is a coherent collection of (potentially long and thin) Whitney boxes such that the change in Dg on $\mathcal{D}_T$ is controlled by the geometry of $\partial\Omega$ inside some surface stopping time region $S \in \mathscr{F}$. These regions are defined such that $\partial\Omega$ looks like a Lipschitz graph with small constant $\epsilon'(L,d)$ and angle variation $\delta'(L,d)$ on the scale of cubes in S from which we derive that $Dg|_{\mathcal{D}_T}$ varies at a rate determined by $\delta'(L,d)$ (See Lemma 3.3.8), giving Lipschitz graphicality for domains in $\{g(\mathcal{D}_T^j)\}_{j\in J_T, T\in\mathscr{T}}$ by Proposition 3.5.6 again as long as ϵ', δ' are fixed small enough with respect to L and d.

Proposition 3.3.7. *Let Ω be as in Theorem C and $\mathscr{L} = \{\Omega_j\}_{j\in J_{\mathscr{L}}}$ be as in* (3.36)*. For any $y \in \partial\Omega \cap B(0,1)$ and $0 < r < 1$, we have*

$$\sum_{j\in j_{\mathscr{L}}} \mathcal{H}^d(\partial\Omega_j \cap B(y,r)) \lesssim_{\epsilon,L,d} r^d. \tag{3.37}$$

To prove 3.3.7 we use the fact that the Whitney coronization is chosen in such a way that the images of boxes in the bad set $\mathscr{B}_w$ have surface measure controlled by the measure of the A_0-close bad cubes $\mathscr{B}$ or cubes in $\mathscr{D}$ on the "edges" of stopping time regions which we collect in the set $\mathscr{B}_e$ in (3.45) below. These cubes form a Carleson set (see Lemma 3.3.12) which gives Carleson packing type estimates for the surface measure of the image cubes $\{g(R_W)\}_{W\in\mathscr{B}_w,\ R_W\in\mathscr{L}_W}$. Because the only time we stop in the construction of $T \in \mathscr{T}$ is when we hit some $W \in \mathscr{B}_w$ the surface measure of domains in $\{g(\mathcal{D}_T)\}_{T\in\mathscr{T}}$ is controlled by the measure of nearby cubes in $\mathscr{B}_e$. The fact that g is bi-Lipschitz and preserves distances to the boundary means that the family $\{g(W)\}_{W\in\mathscr{R}}$ behaves in many ways like a Whitney decomposition itself (see Lemma 3.3.10) so that we can bound the number of image boxes which are A_0-close to any fixed bad cube $Q \in \mathscr{B}_e$, giving the desired Carleson type estimates.

3.3.3 Lipschitz bounds for Theorem C

The goal of this section is to prove Proposition 3.3.6. The following lemma allows us to control the change in Dg on any stopping time domain T.

Lemma 3.3.8 (Variation of Dg)**.** *For any $T \in \mathscr{T}$ and $z, w \in \mathcal{D}_T$, we have*

$$|Dg(z) \cdot Dg(w)^{-1} - I| \leq C\delta' \tag{3.38}$$

Proof. First, fix some $T \in \mathscr{T}$. We want to apply Proposition 3.6.7 with $M_0 \lesssim_d 1$ and $z = c(W(T)) = (x,y)$ by showing that $\mathcal{D}_T \subseteq G_z^{M_0}$. So, let $z' = (x',y') \in R \in T$ and let $n = n(y'),\ p = l(y)$. We need to prove the following three statements:

(i) $|f_p(x) - f_p(x')| \lesssim_d r_p$,

(ii) $\sum_{k=p}^n \epsilon_k'(f_k(x'))^2 \lesssim \epsilon'$,

(iii) $\text{Angle}(T_k(x'), T_p(x')) \lesssim \delta'$.

We begin by observing that (i) follows from the fact that f_p is L'-bi-Lipschitz so that

$$|f_p(x) - f_p(x')| \leq L'|x - x'| \leq 2L'\sqrt{d}\ell(W(T)) \lesssim_d h(W(T)) \lesssim r_p$$

using (3.29). To prove (ii), let $Q_p \supseteq Q_{p+1} \supseteq \cdots \supseteq Q_n$ be the cubes given by Lemma 3.3.3. For any k with $p \leq k \leq n$ the fact that $\operatorname{dist}(g(x', r_k), Q_k) \lesssim r_k$ means that $(x', r_k) \in R \leq W(T)$ with

$$\operatorname{diam} Q_k \geq c_0\ell(Q_{k+1}) \geq \frac{c_0\rho}{10} r_k \geq \frac{c_0\rho}{200} h(R) \geq \frac{c_0\rho}{1000\sqrt{d}} \operatorname{diam}(R) = A_0 \operatorname{diam} R.$$

so that $Q_k \simeq_{A_0} R$. This means there exists $S \in \mathscr{F}$ such that $Q_k \in S$ for any k by the definition of the stopping time region T. We conclude that

$$\sum_{k=p}^{n} \epsilon_k'(f_k(x))^2 \lesssim \sum_{k=p}^{n} \beta_{\partial\Omega}^{d,1}(MB_{Q_k})^2 \lesssim \epsilon'.$$

To prove (iii), observe that

$$\begin{aligned}\operatorname{Angle}(T_p(x'), T_n(x')) &\leq \operatorname{Angle}(T_p(x'), P_{Q_p}) + \operatorname{Angle}(P_{Q_p}, P_{Q_n}) + \operatorname{Angle}(P_{Q_n}, T_n(x')) \\ &\lesssim \epsilon' + \delta' + \epsilon' \lesssim \delta'.\end{aligned}$$

where we used Lemma 3.6.1 and the fact that $Q_p, Q_n \in S$. ■

Using the results of Section 3.5, we can now prove Proposition 3.3.6.

Proof of Proposition 3.3.6. Every domain in $\mathscr{L}$ is either of the form $g(\mathcal{D}_T^j)$ for some $T \in \mathscr{T}, j \in J_T$ or $g(R_W)$ for some $W \in \mathscr{B}_w$, $R \in \mathscr{L}_W$. We first consider domains of the first form.

Let $T \in \mathscr{T}$ and let $A_p : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^{d+1}$ be given by $A_p(x, y) = (2^p x, y)$. By definition, the image stopping time region $\mathcal{D}_T' = A_p(\mathcal{D}_T)$ is composed of cubes and Proposition 3.5.1 implies there exists a constant $L_0(d)$ such that $\mathcal{D}_T'$ has a decomposition into L_0-Lipschitz graph domains which passes to a decomposition of $\mathcal{D}_T$ into $L_0'(d, L')$-Lipschitz graph domains $\{\mathcal{D}_T^j\}_{j\in J_T}$ by applying A_p^{-1}. Now, using Lemma 3.3.8, we see (3.38) holds on $\mathcal{D}_T$ so that by taking $\epsilon'(L', d), \delta'(L', d)$ sufficiently small, Proposition 3.5.6 implies $g(\mathcal{D}_T^j)$ is an $L_1(L', d)$-Lipschitz graph domain.

Now, let $W \in \mathscr{B}_w$ and $R_W \in \mathscr{L}_W$. The proof of Lemma 3.3.8 shows that $|Dg(z) \cdot Dg(w)^{-1} - I| \leq C\epsilon$ using only the fact that $\partial\Omega$ is ϵ Reifenberg flat. Since R_W is a cube, it is a $C(d)$-Lipschitz graph domain so that Proposition 3.5.6 implies $g(R_W)$ is a $C'(d)$-Lipschitz graph domain as long as ϵ is sufficiently small with respect to d. ■

3.3.4 Surface area bounds for Theorem C

We now focus on proving Proposition 3.3.7. We will justify the name coronization by proving Carleson estimates for the g-Whitney coronization which will imply the desired estimates for our domains.

Definition 3.3.4 (C_0-Whitney family)**.** Let $\Omega_0 \subseteq \mathbb{R}^{d+1}$ be a domain and let $C_0 \geq 1$. We say that a collection $\mathscr{V}$ of subsets of Ω_0 is a C_0-*Whitney family* if for every $V \in \mathscr{V}$, we have

$$C_0^{-1} \operatorname{diam} V \leq \operatorname{dist}(V, \Omega_0^c) \leq C_0 \operatorname{diam} V, \tag{3.39}$$

there exists $c_V \in V$ such that

$$B(c_V, C_0^{-1} \operatorname{diam} V) \subseteq V, \tag{3.40}$$

and, if $V \neq V'$, then

$$B(c_V, C_0^{-1} \operatorname{diam} V) \cap B(c_{V'}, C_0^{-1} \operatorname{diam} V') = \varnothing. \tag{3.41}$$

Lemma 3.3.9. *Let $\Omega_0, \mathscr{V}$ be as in Definition 3.3.4. Let $A \geq 1$, $U \subseteq \mathbb{R}^{d+1}$ and set*

$$\mathscr{V}_{A,U} = \{V \in \mathscr{V} : V \simeq_A U\}.$$

Then,

$$\#(\mathscr{V}_{A,U}) \lesssim_{A,C_0,d} 1. \tag{3.42}$$

If $\mathscr{U}$ is a collection of subsets such that for any $V \in \mathscr{V}$, there exists $U \in \mathscr{U}$ such that $V \simeq_A U$, then

$$\sum_{V \in \mathscr{V}} (\operatorname{diam} V)^d \lesssim_{A,C_0,d} \sum_{U \in \mathscr{U}} (\operatorname{diam} U)^d. \tag{3.43}$$

Proof. For any $V \in \mathscr{V}$, we have

$$\operatorname{dist}(U, V) \leq A \operatorname{diam} U,$$

$$A^{-1} \operatorname{diam} U \leq \operatorname{diam} V \leq A \operatorname{diam} U.$$

Let $B_V = B(c_V, C_0^{-1} \operatorname{diam} V)$ and fix $u \in U$. It follows that $B_V \subseteq V \subseteq B(u, 3A \operatorname{diam} U)$ and $C_0^{-1} \operatorname{diam} V \geq (C_0 A)^{-1} \operatorname{diam} U$ so that $\{B_V\}_{V \in \mathscr{V}_{A,U}}$ is a collection of disjoint balls with radius $r(B_V) \geq (C_0 A)^{-1} \operatorname{diam} U$ contained in the ball $B(u, 3A \operatorname{diam} U)$ and hence has cardinality bounded in terms of C_0, A, and d. This proves (3.42).

To prove (3.43), notice that

$$\sum_{V \in \mathscr{V}} (\operatorname{diam} V)^d \leq \sum_{U \in \mathscr{U}} \sum_{V \in \mathscr{V}_{A,U}} (\operatorname{diam} V)^d \lesssim_A \sum_{U \in \mathscr{U}} \#(\mathscr{V}_{A,U}) (\operatorname{diam} U)^d \lesssim_{A,C_0,d} \sum_{U \in \mathscr{U}} (\operatorname{diam} U)^d \quad \blacksquare$$

We define

$$\mathcal{G}_0 = \{g(W) : W \in \mathscr{R}_w\} \tag{3.44}$$

and observe that $\mathcal{G}_0$ is a $\Lambda_0(L', d)$-Whitney family by equations (3.30) - (3.33):

Lemma 3.3.10. *There exists a constant $\Lambda_0(L', d) > 0$ such that $\mathcal{G}_0$ is a $\Lambda_0(L', d)$-Whitney family.*

Combining this fact with Lemma 3.3.9 will allow us to bound the surface measure of images of stopped boxes in terms of the side-length of A_0-close bad and stopped cubes in $\mathscr{D}$. The following two lemmas will give a Carleson packing condition on this bad subset $\mathscr{B}_e \subseteq \mathscr{D}$ defined in (3.45) below from which we will be able to conclude the desired surface measure bound (3.37). We begin with the following lemma due to David and Semmes.

Lemma 3.3.11 (cf. [DS93] Part I Lemma 3.27, (3.28)). *Let $A \geq 1$, let $\mathscr{D}$ be a Christ-David lattice with coronization $(\mathscr{G}, \mathscr{B}, \mathscr{F})$. Then,*

(a) The set

$$\mathscr{A} = \{Q \in \mathscr{G} : \exists Q' \in S' \neq S \ni Q \text{ such that } Q \simeq_A Q'\}$$

satisfies a Carleson packing condition.

(b) Suppose $\mathscr{H} \subseteq \mathscr{D}$ satisfies a Carleson packing condition. The set

$$\mathscr{H}_A = \{Q \in \mathscr{D} : \exists Q' \in \mathscr{H} \text{ such that } Q \simeq_{A_0} Q'\}$$

satisfies a Carleson packing condition.

This lemma will directly give us a Carleson packing condition on the set

$$\mathscr{B}_e = \mathscr{B} \cup \{Q \in \mathscr{G} : \exists Q' \in S' \neq S \ni Q \text{ such that } Q \simeq_{2A_0^2} Q'\}. \tag{3.45}$$

Lemma 3.3.12 ($\mathscr{B}_e$ Carleson packing condition). *The family $\mathscr{B}_e$ satisfies a Carleson packing condition. For any $W \in \mathscr{B}_w$, there exists $Q_W \in \mathscr{B}_e$ such that $g(W) \simeq_{A_0} Q_W$.*

Proof. The fact that $\mathscr{B}_e$ satisfies a Carleson packing condition follows from Lemma 3.3.11. For the second statement, let $W \in \mathscr{B}_w$. By definition, $W \notin \mathscr{G}_w$ so that either

(i) $\exists Q \in \mathscr{B}$ such that $g(W) \simeq_{A_0} Q$,

(ii) $\exists S_1, S_2 \in \mathscr{F}$ such that $Q_1 \in S_1 \neq S_2 \ni Q_2$ with $g(W) \simeq_{A_0} Q_1$ and $g(W) \simeq_{A_0} Q_2$.

The first case gives the desired cube Q_W immediately. In the second case, a calculation using the definition of A_0-closeness shows that $Q_1 \simeq_{2A^2} Q_2$ so that $Q_1, Q_2 \in \mathscr{B}_e$ and we can set $Q_W = Q_1$. ∎

We now fix $y \in \partial\Omega \cap B(0,1)$ and $0 < r \leq 1$. In order to pick out the pieces of the domains which actually intersect $B(y,r)$, for any $T \in \mathscr{T}$ we define

$$\mathscr{T}'_{y,r} = \{T \in \mathscr{T} : \Omega_T \cap B(y,r) \neq \varnothing\}.$$

We break up $\mathscr{T}'_{y,r}$ into regions with large and small top cubes:

$$\begin{aligned} \mathscr{T}_{L,r} &= \{T \in \mathscr{T}_{y,r'} : h(W(T)) > 10r\}, \\ \mathscr{T}_{y,r} &= \mathscr{T}'_{y,r} \setminus \mathscr{T}_L. \end{aligned}$$

It is also convenient to collect all of the boundaries associated with a given stopping time domain $T \in \mathscr{T}$ into one set:

$$\mathcal{B}_T = \bigcup_{j \in J_T} \partial\Omega_T^j.$$

We note that $\mathcal{B}_T$ is d-upper Ahlfors regular by Proposition 3.5.1. Proposition 3.3.7 will follow from the following three lemmas below. The first gives a bound for the domains in $\mathscr{T}_{L,r}$ while the second gives a bound for those in $\mathscr{T}_{y,r}$.

Lemma 3.3.13.

$$\sum_{T\in\mathscr{T}_{L,r}} \mathcal{H}^d(\mathcal{B}_T\cap B(y,r)) \lesssim_{L',d} r^d \le \mathcal{H}^d(\partial\Omega\cap B(y,r)).$$

Proof. We will show that $\#(\mathscr{T}_{L,r})$ is bounded independent of y and r. For any $T\in\mathscr{T}_{L,r}$ we claim that there exists some $W_T\in T$ such that $h(W_T)\simeq r$ and $\operatorname{dist}(g(W_T),y)\simeq r$. Indeed, by definition there exists $R_T\in T$ such that $g(R_T)\cap B(y,r)\neq\varnothing$. There then exists a box $W_T\in T$ with $W_T\ge R_T$ with the desired properties because of (3.26) and the inequality $h(W(T))>10r$. But, since the collection $\{g(W_T)\}_{T\in\mathscr{T}_{L,r}}$ is a Whitney family, it follows that $N=\#(\mathscr{T}_{L,r})=\#(\{g(W_T)\}_{T\in\mathscr{T}_{L,r}})\lesssim_{L',d} 1$. Therefore, since $\mathcal{B}_T$ is d-upper Ahlfors regular,

$$\sum_{T\in\mathscr{T}_{L,r}} \mathcal{H}^d(\mathcal{B}_T\cap B(y,r)) \lesssim_d \#(\mathscr{T}_{L,r})r^d \lesssim_{L',d} r^d. \qquad \blacksquare$$

We now handle the regions with small top boxes:

Lemma 3.3.14.

$$\sum_{T\in\mathscr{T}_{y,r}} \mathcal{H}^d(\mathcal{B}_T\cap B(y,r)) \lesssim_{L',d,\epsilon'} \mathcal{H}^d(\partial\Omega\cap B(y,A_0^2r)) \lesssim_{L',d} r^d. \tag{3.46}$$

Proof. We first note that since $\mathcal{H}^d(\mathcal{B}_T)\lesssim_d \mathcal{H}^d(\partial\Omega_T)$, we have

$$\sum_{T\in\mathscr{T}_{y,r}} \mathcal{H}^d(\mathcal{B}_T\cap B(y,r)) \le \sum_{T\in\mathscr{T}_{y,r}} \mathcal{H}^d(\mathcal{B}_T) \lesssim_d \sum_{T\in\mathscr{T}_{y,r}} \mathcal{H}^d(\partial\Omega_T).$$

Therefore, it suffices to prove $\sum_{T\in\mathscr{T}_{y,r}} \mathcal{H}^d(\partial\Omega_T)\lesssim_{L',d,\epsilon} \mathcal{H}^d(\partial\Omega\cap B(y,A_0^2r))$.

For any $T\in\mathscr{T}_{y,r}$, (3.29) gives $\operatorname{diam} g(\operatorname{Bot}(W)))\lesssim_d h(W)$ so that Lemma 3.3.8 and the fact that g is L'-bi-Lipschitz give an analogue of (3.67):

$$\begin{aligned}\mathcal{H}^d(\partial\Omega_T) &\lesssim_{d,L'} \mathcal{H}^d(\partial\Omega_T\cap\partial\Omega) + \sum_{W\in m(T)} \mathcal{H}^d(g(\operatorname{Bot}(W)))\\ &\lesssim_d \mathcal{H}^d(\partial\Omega_T\cap\partial\Omega) + \sum_{W\in m(T)} h(W)^d. \end{aligned}\tag{3.47}$$

Now, $W\in m(T)$ implies that there exists a child $W'\in\operatorname{Stop}(T)\cap\mathscr{B}_w$ for which we have $Q\in\mathscr{B}_e$ with $g(W')\simeq_{A_0} Q$ by Lemma 3.3.12. For any $x\in Q$, we compute

$$\begin{aligned}|x-y| &\le \operatorname{diam} Q + \operatorname{dist}(Q,g(W')) + \operatorname{diam} g(W') + \operatorname{dist}(y,g(W'))\\ &\le 2A_0\operatorname{diam} g(W') + 2A_0\operatorname{diam} g(W') + \operatorname{diam} g(W') + 10r\\ &\le 10\sqrt{d}A_0h(W') + 10r \le 100\sqrt{d}A_0r \le A_0^2r \end{aligned}\tag{3.48}$$

This shows that $Q\subseteq B(y,A_0^2r)$. Hence, applying Lemma 3.3.9 with $\mathscr{V}=\{g(W):W\in m(T)\}$ and $\mathscr{U}=\{Q\in\mathscr{B}_e : Q\subseteq B(y,A_0^2r)\}$, we get

$$\sum_{W\in m(T)} h(W)^d \lesssim_{A_0,d,L'} \sum_{\substack{Q\in\mathscr{B}_e\\ Q\subseteq B(y,A_0^2r)}} \ell(Q)^d \lesssim_{d,\epsilon'} \mathcal{H}^d(\partial\Omega\cap B(y,A_0^2r)) \tag{3.49}$$

where the last inequality follows from the Carleson packing condition for $\mathscr{B}_e$. By observing that $\partial\Omega_T \cap \partial\Omega \subseteq B(y, 50\sqrt{d}r)$ for any $T \in \mathscr{T}_{y,r}$ and $\mathcal{H}^d(\partial\Omega_T \cap \partial\Omega_{T'} \cap \partial\Omega) = 0$ for $T \neq T'$, (3.47) implies

$$\sum_{T\in\mathscr{T}_{y,r}} \mathcal{H}^d(\partial\Omega_T) \lesssim_{A_0,L',d,\epsilon'} \mathcal{H}^d(\partial\Omega \cap B(y, A_0^2 r)) \lesssim_{d,L'} r^d$$

using the fact that g is bi-Lipschitz and parameterizes $\partial\Omega$ in the last inequality. ■

Finally, we handle the boundaries of "trivial" cube domains associated to the bad boxes in $\mathscr{B}_w$. To do so, we collect the boundaries associated to fixed $W \in \mathscr{B}_w$ into the set

$$\mathcal{B}_W = \bigcup_{R\in\mathscr{L}_W} \partial R.$$

Lemma 3.3.15.

$$\sum_{W\in\mathscr{B}_w} \mathcal{H}^d(\mathcal{B}_W \cap B(y,r)) \lesssim_{L',d,\epsilon} \mathcal{H}^d(\partial\Omega \cap B(y, A_0^2 r)) \lesssim_{A_0,d,L'} r^d.$$

Proof. We first note that

$$\sum_{W\in\mathscr{B}_w} \mathcal{H}^d(\mathcal{B}_W \cap B(y,r)) \leq \sum_{\substack{W\in\mathscr{B}_w \\ g(W)\cap B(y,r)\neq\varnothing}} \mathcal{H}^d(\mathcal{B}_W) \lesssim_{L'} \mathcal{H}^d(\partial g(W)) \lesssim h(W)^d$$

using $\mathcal{H}^d(g(\mathrm{Bot}(W))) \lesssim_d h(W)^d$ as in (3.47) above. In addition, there exists some cube $Q \in \mathscr{B}_e$ such that $g(W) \simeq_{A_0} Q$ and, as in (3.48), $Q \subseteq B(y, A_0^2 r)$. Hence, we have

$$\begin{aligned}\sum_{W\in\mathscr{B}_w} \mathcal{H}^d(\mathcal{B}_W \cap B(y,r)) &\lesssim_d \sum_{\substack{W\in\mathscr{B}_w \\ g(W)\cap B(y,r)\neq\varnothing}} h(W)^d \lesssim_{A_0,d,L'} \sum_{\substack{Q\in\mathscr{B}_e \\ Q\subseteq B(y,A_0^2 r)}} \ell(Q)^d \\ &\lesssim_{d,\epsilon'} \mathcal{H}^d(\partial\Omega \cap B(y, A_0^2 r)) \lesssim_{d,L'} r^d.\end{aligned}$$ ■

Proof of Proposition 3.3.7. First, consider $\Omega_j \in \mathscr{L}$ such that either there exists j_0, T_0 such that $\Omega_j = \Omega_{T_0}^{j_0}$ or there exists $W \in \mathscr{B}_w$ and $R \in \mathscr{L}_W$ such that $\Omega_j = g(R)$. Therefore, we have

$$\begin{aligned}&\sum_{j\in J_{\mathscr{L}}} \mathcal{H}^d(\partial\Omega_j \cap B(y,r)) \\ &\quad\leq \sum_{T\in\mathscr{T}_{L,r}} \sum_{j\in J_T} \mathcal{H}^d(\partial\Omega_T^j \cap B(y,r)) + \sum_{T\in\mathscr{T}_{y,r}} \sum_{j\in J_T} \mathcal{H}^d(\partial\Omega_T^j \cap B(y,r)) \\ &\qquad + \sum_{W\in\mathscr{B}_w} \sum_{R\in\mathscr{L}_W} \mathcal{H}^d(\partial R \cap B(y,r)) \\ &\quad\lesssim \sum_{T\in\mathscr{T}_{L,r}} \mathcal{H}^d(\mathcal{B}_T \cap B(y,r)) + \sum_{T\in\mathscr{T}_{y,r}} \mathcal{H}^d(\mathcal{B}_T \cap B(y,r)) + \sum_{W\in\mathscr{B}_w} \mathcal{H}^d(\mathcal{B}_W \cap B(y,r)) \\ &\quad\lesssim_{L',d,\epsilon'} r^d\end{aligned}$$

by Lemmas 3.3.13, 3.3.14, and 3.3.15. ■

This completes the proof of Theorem C.

3.4 The proofs of Theorems D and E

We now turn to proving Theorems D and E. Both of these theorems will follow from the following result

Theorem 3.4.1. *Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a domain. There exists constants $A(d), L(d), \epsilon_0(d) > 0$ such that if $\partial\Omega$ admits a d-dimensional graph coronization with $\epsilon \leq \epsilon_0$, then there exists a collection $\mathscr{L} = \{\Omega_j\}_{j \in J_{\mathscr{L}}}$ of L-Lipschitz graph domains such that*

(i) $\Omega_j \subseteq \Omega$,

(ii) $\Omega \cap B(0,1) \subseteq \bigcup_{j=1}^{\infty} \Omega_j$,

(iii) $\exists C(d) > 0$ *such that* $\forall x \in \mathbb{R}^{d+1}$, $x \in \Omega_j$ *for at most* C *values of* j,

(iv) For any $y \in \partial\Omega \cap B(0,1)$ *and* $0 < r \leq 1$, *we have*

$$\sum_{j=1}^{\infty} \mathcal{H}^d(\partial\Omega_j \cap B(y,r)) \lesssim_{\epsilon,d} \mathcal{H}^d(\partial\Omega \cap B(y,Ar)).$$

The proof will be via relatively minor modifications of the argument for Theorem C. The idea is to construct a collection of CCBPs with associated maps $\{g_i\}_{i \in I}$ where $g_i : \mathcal{D}_i \to \overline{\Omega}$ which individually parameterize only a little piece of $\overline{\Omega}$ at a time. These maps will be $(1 + C\delta)$-bi-Lipschitz at the cost of introducing an outer "buffer zone" of domains in the image of these mappings having bounded overlap.

We now fix constants ρ, A_0, K as in Section 3.3 and set

$$A_1 = \max\left\{20A_0^2,\ \frac{2000\sqrt{d}A_0}{c_0\rho}\right\},$$
$$M = \max\left\{\frac{10K}{\rho^2},\ A_1^2\right\}.$$

3.4.1 Local CCBPs adapted to $\mathscr{D}$

We will construct Reifenberg parameterizations as in subsection 3.3.1 centered around the points of a Whitney-like net $\mathcal{C}_0$ of $\Omega \cap B(0,1)$ rather than having a single global map.

For every $n \geq 0$, define

$$s_n = 3 \cdot 2^{-n+1},$$
$$D_n = \{z \in B(0,1) : \operatorname{dist}(z, \partial\Omega) = s_n\},$$
$$C_n = \operatorname{Net}(D_n, s_n) = \{p_{i,n}\}_{i \in I_n}.$$

Set $\mathcal{C}_0 = \bigcup_n C_n$.

Definition 3.4.1 (flat and non-flat points)**.** Let $p \in \Omega \cap B(0,1)$. Define

$$\mathscr{Q}_p = \left\{ Q \in \mathscr{D} : Q \simeq_{10\sqrt{d}A_1} B\left(p, \frac{1}{2}\operatorname{dist}(p, \partial\Omega)\right)\right\}$$

We say that p is *flat* if there exists $S \in \mathscr{F}$ such that $\mathscr{Q}_p \subseteq S$. Otherwise, we say that p is *non-flat*. Given the set $\mathcal{C}_0$ above, we define the flat and non-flat points of $\mathcal{C}_0$ by

$$\begin{aligned} \mathcal{F}_0 &= \{p \in \mathcal{C}_0 : \exists S \in \mathscr{F},\ \mathscr{Q}_p \subseteq S\}, \\ \mathcal{N}_0 &= \mathcal{C}_0 \setminus \mathcal{F}_0. \end{aligned}$$

Fix $p \in \mathcal{F}_0$ and let $S_p \in \mathscr{F}$ be such that $\mathscr{Q}_p \subseteq S_p$. Without loss of generality, assume that $\operatorname{dist}(p,0) = \operatorname{dist}(p,\partial\Omega) = 6 = s_0$. The fact that $p \in \mathcal{F}_0$ implies there exists $Q_p \in \mathscr{D}_{s(0)}$ with $\operatorname{dist}(p, Q_p) \leq 6$ and $c_0\rho \leq \operatorname{diam}(Q_p) \leq 1 = r_0$ so that $Q_p \in \mathscr{Q}_p$ because $A_1 \geq 10(c_0\rho)^{-1}$. Hence, $b\beta_{\partial\Omega}(MB_{Q_p}) \leq \epsilon$. Without loss of generality, suppose $P_{Q_p} = \mathbb{R}^d$ achieves the infimum in the definition of $b\beta_{\partial\Omega}(MB_{Q_p})$.

For any $k \geq 0$, let

$$Y_k^p = \{x_Q : Q \in S_p \cap \mathscr{D}_{s(k)}\}, \tag{3.50}$$

$$X_k^p \in \operatorname{Net}(Y_k^p, r_k). \tag{3.51}$$

We enumerate $X_k^p = \{x_{j,k}\}_{j \in J_k}$ and define

$$\begin{aligned} B_{j,k} &= B(x_{j,k}, r_k), \\ P_{j,k} &= P_{Q_{j,k}}, \\ \mathscr{L}_p &= (P_{Q_p}, \{B_{j,k}\}, \{P_{j,k}\}). \end{aligned}$$

where $P_{Q_{j,k}} \ni x_{Q_{j,k}}$ satisfy $\beta_{\partial\Omega}^{d,1}(2\rho^{-1}KB_{Q_{j,k}}, P_{Q_{j,k}}) \lesssim \beta_{\partial\Omega}^{d,1}(2\rho^{-1}KB_{Q_{j,k}}B_{Q_{j,k}})$ as in the hypotheses of Lemma 3.2.10. Using the fact that $Q \in S_p \subseteq \mathscr{G}$ so that $b\beta(MB_Q) \leq \epsilon$, a nearly identical argument to that of Lemma 3.3.2 gives that $\mathscr{L}_p$ is a CCBP:

Lemma 3.4.2. *For any $p \in \mathcal{F}_0$, $\mathscr{L}_p$ is a CCBP.*

We will now prove the following analogue of Lemma 3.3.3

Lemma 3.4.3 (properties of g_p)**.** *There exists a choice of constant $A_1 \lesssim_d A_0$ such that for any $z = (x,y) \in \widehat{\mathcal{D}}_p$, the following hold:*

(i) $f_{n(y)}(x) \in V_{n(y)}^8$,

(ii) $(1 - C\epsilon)|y| \leq \operatorname{dist}(g_p(z), \partial\Omega) \leq (1 + C\epsilon)|y|$.

(iii) For any $m \in \mathbb{N}$ with $m < n(y)$, there exists a collection of cubes $Q_{n(y)} \subseteq Q_{n(y)-1} \subseteq \cdots \subseteq Q_m$ such that for any k with $m \leq k \leq n$, $Q_k \in S_p$ and $\operatorname{dist}(g(x,r_k), Q_k) \lesssim r_k$ and

$$\sum_{k=m}^{n} \epsilon'(f_k(x))^2 \lesssim_{M,\rho,d} \sum_{k=m}^{n} \beta_{\partial\Omega}^{d,1}(MB_{Q_k})^2 \lesssim \epsilon.$$

Proof. The proof is similar to that of 3.3.3 with the only complication being that we need the map g_p to also behave nicely on the buffer region of A_0 close cubes to those in $\mathscr{W}_0$. We will prove this for fixed $z = (x, y)$ by first assuming that (i) holds and showing that items (ii) and (iii) hold. We will then prove item (i) by induction, considering the points $(x, r_k) \in \widehat{\mathcal{D}}_p$ for $0 \le k < n(y)$ (assume without loss of generality that $h(W(T)) = 4$).

So, first assume that item (i) holds. Given this, item (ii) follows exactly as in Lemma 3.3.3 item (ii). Similarly, item (iii) follows as in Lemma 3.3.3 item (iii) by replacing the infinite chain of cubes with a chain terminating in $Q_{n(y)} \in \mathscr{D}_{s(n(y))} \cap S_p$.

We now prove item (i) by the induction discussed above. For the base case, recall that $f_0(x) = x$ so that $(x, y) \in \widehat{\mathcal{D}}_p$ means $\operatorname{dist}(x, W(T_p)) \le 2A_0 \operatorname{diam} W(T)$. Since we've chosen M large enough, $x \in MB_{Q_p} \cap P_{Q_p}$ so that $\operatorname{dist}(x, \partial\Omega) \lesssim_M \epsilon$. This means $p \in \mathcal{F}$ implies that there exists $Q_0 \in \mathscr{D}_{s(0)} \cap S_p$ such that $|x - x_{Q_0}| \le 2r_0$ from which the claim follows. We will finish the proof by proving the following claim:

Claim F:or any $k < n(y)$, $f_k(x) \in V_k^8$ implies that $f_{k+1}(x) \in V_{k+1}^8$.

Proof T:he fact that $(x, r_k) \in \widehat{\mathcal{D}}_p$ means that $(x, r_k) \in R_k \in \mathscr{W}_p'$ and there exists $W \in T_p$ such that $R_k \simeq_{A_0} W$. This gives $\operatorname{dist}(R_k, W) \lesssim_{A_0} h(W)$ and $r_k \simeq h(R_k) \simeq_d \operatorname{diam} R_k \simeq_{A_0} h(W)$. If now $f_k(x) \in V_k^8$, then there exists $Q \in S_p \cap \mathscr{D}_{s(k)}$ such that $|f_k(x) - x_Q| \le 8r_k$, so that $b\beta_{\partial\Omega}(MB_Q) \le \epsilon$ implies there is $Q_{k+1} \in \mathscr{D}_{s(k+1)}$ with $|f_{k+1}(x) - x_{Q_{k+1}}| \le 2r_{k+1}$. Applying item (ii) and (3.31) gives

$$\begin{aligned}\operatorname{dist}(Q_{k+1}, g(W)) &\le \operatorname{dist}(Q_{k+1}, g(R_k)) + \operatorname{diam} g(R_k) + \operatorname{dist}(g(R_k), g(W)) \\ &\le 2\sqrt{d}h(R_k) + 2\sqrt{d}h(R_k) + A_0(\operatorname{diam} g(R_k) + \operatorname{diam} g(W)) \\ &\le 5\sqrt{d}A_0 h(R_k) + 5\sqrt{d}A_0(h(R_k) + \operatorname{diam} g(W) \le A_0^2 \operatorname{diam} g(W)\end{aligned}$$

and

$$\operatorname{diam} Q_{k+1} \le r_{k+1} \le h(R) \le A_0 h(W) \le 2A_0 \operatorname{diam} g(W).$$

$$\operatorname{diam} Q_{k+1} \ge c_0 \ell(Q) \ge \frac{c_0\rho}{10} r_{k+1} \ge \frac{c_0\rho}{200} h(R_k) \ge \frac{c_0\rho}{200A_0} h(W) \ge \frac{c_0\rho}{1000\sqrt{d}A_0} \operatorname{diam} g(W).$$

Therefore, $Q_{k+1} \simeq_{A_1} g(W)$. By the definition of T_p, we then have $Q_{k+1} \in S_p$ so that $x_{Q_{k+1}} \in Y_{k+1}^p$ and $f_{k+1}(x) \in V_{k+1}^8$ as in Lemma 3.3.3 (i). ■ ■

3.4.2 The Lipschitz decomposition with bounded overlaps

Hence, by Theorem 3.2.2, we get a Reifenberg parameterization $g_p : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ and we let $\mathscr{W}_p'$ be the Whitney decomposition of $\mathbb{H}^{d+1}$ such that $W_0 = [-2, 2]^d \times [4, 8] \in \mathscr{W}_p'$ so that $p = c(W_0) = (0, 6)$ and let $\mathscr{W}_0 = \{W \in \mathscr{W}_p' : W \in D(W_0)$ as in (3.25). We now give a one-step version of the stopping time construction in Definition 3.2.6 to produce a single domain $\mathcal{D}_p$ and an extended version $\widehat{\mathcal{D}}_p$ which contains additional "buffer" cubes which g_p maps forward to approximating Lipschitz graph domains

Definition 3.4.2 (Stopping time regions around flat p)**.** Fix a constant $A_1 > 1$ and $p \in \mathcal{F}$ and form the map g_p and Whitney lattices $\mathscr{W}_p'$ and $\mathscr{W}_0$ as above. As in (3.34), we define

$$\mathscr{G}_p = \{W \in \mathscr{W}_0 : \forall Q \in \mathscr{D} \text{ such that } Q \simeq_{A_1} g_p(W) \text{ we have } Q \in S_p\}$$

By definition, $p \in \mathcal{F}_0$ implies $W_0 \in \mathscr{G}_p$. Define a single stopping time region $T_p \subseteq \mathscr{W}_0$ by setting T_p to be the maximal subtree of $D(W_0) \cap \mathscr{G}_p$ such that for any $R \in T_p$, either all of its children are in T_p or none are.

Definition 3.4.3 (Stopping time domains around flat p)**.** For any $p \in \mathcal{F}_0$, we define a *stopping time domain*

$$\mathcal{D}_p = \bigcup_{W \in T_p} W.$$

Additionally, we extend $\mathcal{D}_p$ by a "buffer" region of A_0-close cubes on the boundary of $\mathcal{D}_p$ by defining the *extended stopping time region* and *extended stopping time domain* by

$$\begin{aligned} \widehat{T}_p &= \{W \in \mathscr{W}_p' : \exists R \in T_p,\ W \simeq_{A_0} R\}, \\ \widehat{\mathcal{D}}_p &= \bigcup_{W \in \widehat{T}_p} W. \end{aligned}$$

We will carve up the image domains

$$\begin{aligned} \Omega_p &= g_p(\mathcal{D}_p), \\ \widehat{\Omega}_p &= g_p(\widehat{\mathcal{D}}_p) \end{aligned}$$

to construct one family of our desired Lipschitz graph domains in the conclusion of Theorem 3.4.1.

We will also need to construct Lipschitz graph domains around non-flat $q \in \mathcal{N}_0$. Because $\partial\Omega$ admits a graph coronization, there are a controlled number of such q so that we can cover the regions around them by "trivial" domains without adding too much total boundary.

Definition 3.4.4 (Trivial domains around non-flat q)**.** Fix once and for all an auxiliary Whitney decomposition $\widetilde{\mathscr{W}}$ of Ω. For any $q \in \mathcal{N}_0$, there exists a Whitney cube $W_q \in \widetilde{\mathscr{W}}$ such that $q \in W_q$ and

$$\operatorname{diam} W_q \leq \operatorname{dist}(q, \partial\Omega) \leq 8 \operatorname{diam} W_q.$$

We directly define

$$\begin{aligned} \mathcal{D}_q &= \Omega_q = W_q, \\ \widehat{\mathcal{D}}_q &= \{W \in \widetilde{\mathscr{W}} : W \simeq_{A_0} W_q\}, \\ \widehat{\Omega}_q &= \bigcup_{W \in \widehat{\mathcal{D}}_q} W. \end{aligned}$$

We will get our final collection of domains by choosing a well-spaced subsets $\mathcal{F} \subseteq \mathcal{F}_0$ and $\mathcal{N} \subseteq \mathcal{N}_0$ and carving up the domains in $\{\widehat{\Omega}_p\}_{p \in \mathcal{F}} \cup \{\widehat{\Omega}_q\}_{q \in \mathcal{N}}$.

To choose our collections $\mathcal{F}$ and $\mathcal{N}$, we put an ordering on the points of $\mathcal{C}_0$ by choosing some ordering on each finite set C_n and then imposing $p_n < p_m$ for any $p_n \in C_n,\ p_m \in C_m$ with $n < m$. $\mathcal{C}_0$ has a least element which we call c_0 and we define an auxiliary collection

$\mathcal{P}_0 = \{p_0\}$. Given the definitions of $\mathcal{C}_0$ and $\mathcal{P}_0$, we define $\mathcal{C}_{n+1}$ and $\mathcal{P}_{n+1}$ inductively for any $n \geq 0$ by

$$\mathcal{C}_{n+1} = \mathcal{C}_n \setminus \left\{ p \in \mathcal{C}_n \ : \ \operatorname{dist}\left(p, \bigcup_{p' \in \mathcal{P}_n} \Omega_{p'}\right) < \frac{A_0}{30} \operatorname{dist}(p, \partial\Omega). \right\}, \tag{3.52}$$

$$\mathcal{P}_{n+1} = \mathcal{P}_n \cup \{p_{n+1}\}, \tag{3.53}$$

where p_{n+1} is the least element of $\mathcal{C}_{n+1}$ with respect to the ordering inherited from $\mathcal{C}_0$. Finally, put

$$\mathcal{C} = \bigcup_{n=0}^{\infty} \mathcal{P}_n,$$
$$\mathcal{F} = \mathcal{F}_0 \cap \mathcal{C},$$
$$\mathcal{N} = \mathcal{N}_0 \cap \mathcal{C}.$$

We can now give the definition of our desired Lipschitz decomposition with bounded overlaps

Definition 3.4.5 (Lipschitz decomposition with bounded overlap)**.** For any $p \in \mathcal{F}$, Proposition 3.5.1 implies there exists an Ahlfors regular d-rectifiable set Σ_{T_p} such that

$$\mathcal{D}_p \setminus \Sigma_{T_p} = \bigcup_{j \in J_p} \mathcal{D}_p^j$$

where $\mathcal{D}_p^j$ is an $L_0(d)$-Lipschitz graph domain. We set $\Omega_p^j = g_p(\mathcal{D}_p^j)$ and define our *Lipschitz decomposition with bounded overlap*

$$\mathscr{L} = \{g_p(W)\}_{p \in \mathcal{F},\ W \in \widehat{\mathcal{D}}_p \setminus \mathcal{D}_p} \cup \{\Omega_p^j\}_{p \in \mathcal{F},\ j \in J_p} \cup \{R\}_{q \in \mathcal{N},\ R \in \widehat{\mathcal{D}}_q}. \tag{3.54}$$

In analogy to Propositions 3.3.6 and 3.3.7, we will finish the proof of Theorem 3.4.1 if we can prove the following propositions:

Proposition 3.4.4. *Let Ω be as in Theorem 3.4.1 and $\mathscr{L} = \{\Omega_j\}_{j \in J_{\mathscr{L}}}$ be as in* (3.54)*. There exists $L_1(d) > 0$ such that for any $j \in J_{\mathscr{L}}$, Ω_j is an L_1-Lipschitz graph domain. In addition, we have*

(i) $\Omega_j \subseteq \Omega$,

(ii) $\Omega \subseteq \bigcup_{j \in J_{\mathscr{L}}} \overline{\Omega_j}$,

(iii) $\exists C(d) > 0$ *such that* $\forall x \in \Omega$, $x \in \Omega_j$ *for at most C values of j.*

Proposition 3.4.5. *Let Ω be as in Theorem 3.4.1 and $\mathscr{L} = \{\Omega_j\}_{j \in J_{\mathscr{L}}}$ be as in* (3.54)*. For any $y \in \partial\Omega \cap B(0,1)$ and $0 < r < 1$, we have*

$$\sum_{j \in j_{\mathscr{L}}} \mathcal{H}^d(\partial\Omega_j \cap B(y,r)) \lesssim_{\epsilon,d} \mathcal{H}^d(\partial\Omega \cap B(y, A_1 r)). \tag{3.55}$$

3.4.3 Lipschitz bounds and covering / overlap properties for Theorems D and E

In order to prove Propositions 3.4.4 and 3.4.5, we must show that the mapping g_p behaves on $\widehat{\mathcal{D}_p}$ as our single Reifenberg parameterization g did on each $\mathcal{D}_T$ in the setting of Theorem C.

The following analogue of Lemma 3.3.8 allows us to control the change in Dg_p on any extended stopping time domain $\widehat{\mathcal{D}}_p$.

Lemma 3.4.6 (Variation of Dg_p). *For any $p \in \mathcal{F}$ and $z \in \widehat{\mathcal{D}}_p$, we have*

$$|Dg_p(z) - I| \leq C\delta \tag{3.56}$$

In particular, $g_p|_{\widehat{\mathcal{D}}_p}$ is $(1 + C\delta)$-bi-Lipschitz.

This result follows directly from the proof of Lemma 3.3.8. Equation (3.56) follows from the added observation that $p \in \mathcal{D}_p$ and $\text{dist}(p, \partial\Omega) \geq 2$ (after normalizing) implies $Dg_p(p) = I$ so that the claim follows from (3.38) by taking $w = p$.

We now have enough to show that each domain in $\mathscr{L}$ as in (3.54) is Lipschitz graphical

Lemma 3.4.7. *There exists a constant $L_1(d) > 0$ such that Ω_j is an L_1-Lipschitz graph domain for all $j \in J_{\mathscr{L}}$.*

Proof. Each domain in the set $\{R\}_{q\in\mathcal{N},\ R\in\widehat{\mathcal{D}}_q}$ is a cube, which is an $L_0(d)$-Lipschitz graph domain trivially. Each domain Ω_j in the set $\{g_p(W)\}_{p\in\mathcal{F},\ W\in\widehat{\mathcal{D}}_p\setminus\mathcal{D}_p} \cup \{\Omega_p^j\}_{p\in\mathcal{F},\ j\in J_p}$ is the image under g_p of an L_0-Lipschitz graph domain. Therefore, by Lemma 3.4.6 and 3.5.6 there exists $L_1(d) > L_0$ such that each such Ω_j is an L_1-Lipschitz graph domain. ■

In order to prove the remaining statements of Proposition 3.4.4, we first show that the buffer region $\widehat{\Omega}_p \setminus \Omega_p$ contains a cone around Ω_T with respect to the distance to $\partial\Omega$ for any $p \in \mathcal{C}$:

Lemma 3.4.8. *For any $p \in \mathcal{C}$, $\widehat{\Omega}_p$ contains a $\frac{A_0}{10}$-cone around Ω_p with respect to distance from $\partial\Omega$. That is,*

$$F = \left\{ w \in \Omega : \text{dist}(w, \Omega_p) < \frac{A_0}{10} \min\left\{\text{dist}(w, \partial\Omega), \text{dist}(g_p(W(T_p)), \partial\Omega)\right\} \right\} \subseteq \widehat{\Omega}_p \tag{3.57}$$

Proof. First, suppose that $p \in \mathcal{F}$ and let $z \in F$. Since $\widehat{\Omega}_p = g_p(\widehat{\mathcal{D}}_p)$ where g_p is $(1 + C\delta)$-bi-Lipschitz by Lemma 3.4.6 and translates distance in the domain to $\mathbb{R}^d$ to distance to $\partial\Omega$ in the image by Lemma 3.4.3 (ii), it suffices to show

$$\left\{ z \in \Omega : \text{dist}(z, \mathcal{D}_p) < \frac{A_0}{4} \min\left\{\text{dist}(z, \mathbb{R}^d), \text{dist}(W(T_p), \mathbb{R}^d)\right\} \right\} \subseteq \widehat{\mathcal{D}}_p \tag{3.58}$$

because the desired containment then follows by mapping (3.58) forward. Now, there exists $W \in T_p$ such that $\text{dist}(z, W) = \text{dist}(z, \mathcal{D}_p)$ and there exists a cube $W_z \in \mathscr{W}_p'$ such that

$z \in W_z$. By the definition of $\widehat{\mathcal{D}}_p$, it suffices to show that $W \simeq_{A_0} W_z$. We estimate

$$\begin{aligned} \operatorname{dist}(W, W_z) &\leq \operatorname{dist}(z, \mathcal{D}_p) < \frac{A_0}{4} \min\{\operatorname{dist}(z, \mathbb{R}^d), \operatorname{dist}(W(T_p), \mathbb{R}^d)\} \\ &\leq \frac{A_0}{2} \min\{h(W_z), h(W(T_p))\} = \frac{A_0}{2} \min\{\ell(W_z), \ell(W(T_p))\}. \end{aligned} \tag{3.59}$$

Using this we get

$$\ell(W) = h(W) \leq \operatorname{dist}(W, W_z) + \operatorname{diam} W_z + h(W_z) \leq \left(\frac{A_0}{2} + \sqrt{d+1} + 1\right) \ell(W_z) \leq A_0 \ell(W_z)$$

given that $A_0 \geq 4\sqrt{d}$. A similar calculation shows that $\ell(W_z) \leq A_0 \ell(W)$ which completes the proof in the case when $p \in \mathcal{F}$. If $q \in \mathcal{N}$, then $\Omega_q = W_q$. Let $w \in F$ and let $W_w \in \widetilde{\mathscr{W}}$ with $w \in W_w$. By a similar computation to the above, one can show that $W_q \simeq_{A_0} W_w$ from which the result follows. ∎

With the help of Lemma 3.4.8, we can prove the bounded overlap and covering properties of $\mathscr{L}$.

Lemma 3.4.9. *Let $p, p' \in \mathcal{C}$, $p \neq p'$. The following hold:*

(i) $\Omega_p \cap \Omega_{p'} = \varnothing$,

(ii) $\Omega \cap B(0,1) \subseteq \bigcup_{p \in \mathcal{C}} \overline{\widehat{\Omega}}_p$,

(iii) $\exists C(d) > 0$ *such that* $\forall x \in \Omega$, $x \in \widehat{\Omega}_p$ *for at most C values of j,*

(iv) $\widehat{\Omega}_p \subseteq \Omega$.

Proof. We begin with proving (i). Using the partial order on $\mathcal{C}$, assume without loss of generality that $p' < p$. By the definition of $\mathscr{C}$, we have $\operatorname{dist}(p, \Omega_{p'}) \geq \frac{A_0}{30} \operatorname{dist}(p, \partial\Omega)$. We claim that

$$\Omega_p \subseteq B(p, 3\sqrt{d} \operatorname{dist}(p, \partial\Omega)) \subseteq B\left(p, \frac{A_0}{30} \operatorname{dist}(p, \partial\Omega)\right)$$

where the final inclusion follows because $A_0 \geq 120\sqrt{d}$. Indeed, if $p \in \mathcal{N}$, then $\Omega_p = W_p \ni p$ with $\operatorname{diam} W_p \leq \operatorname{dist}(p, \partial\Omega)$. If instead $p \in \mathcal{F}$, then $\Omega_p = g_p(\mathcal{D}_p)$ where $\mathcal{D}_p$ is composed of a union of cubes in the descendants $D(W(T_p))$ where $\operatorname{dist}(p, \partial\Omega) \geq \ell(W(T_p))$ so that the fact that g_p is $(1 + C\delta)$-bi-Lipschitz means $\sqrt{d+1} \operatorname{dist}(p, \Omega) \geq \operatorname{diam}(g_p(W))$ and $\operatorname{dist}(p, \Omega) \geq \operatorname{dist}(g_p(W), p)$ for any $W \in T_p$. The claim follows.

We now prove (ii). Let $z \in \Omega \cap B(0,1)$ and let $k \geq 0$ be such that $s_{k+1} \leq \operatorname{dist}(z, \partial\Omega) \leq s_k$. By the definition of C_k, there exists $p_k \in C_k$ such that

$$|z - p_k| \leq 3 s_k = 6 s_{k+1} \leq 6 \operatorname{dist}(z, \partial\Omega).$$

Now, if $p_k \in \mathcal{C}$, then by Lemma 3.4.8, $z \in \widehat{\Omega}_{p_k}$. Otherwise, $p_k \notin \mathcal{C}$ so that by (3.52) there exists $p \in \mathcal{C}$ such that $p < p_k$ and $\operatorname{dist}(p_k, \Omega_p) < \frac{A_0}{30} \operatorname{dist}(p_k, \partial\Omega) = \frac{A_0}{30} s_k = \frac{A_0}{15} s_{k+1}$. But then

$$\operatorname{dist}(z, \Omega_p) \leq |z - p_k| + \operatorname{dist}(p_k, \Omega_p) \leq 6 s_{k+1} + \frac{A_0}{15} s_{k+1} \leq \frac{A_0}{10} \operatorname{dist}(z, \partial\Omega)$$

so that $z \in \widehat{\Omega}_p$ by Lemma 3.4.8 as long as $\operatorname{dist}(z, \Omega_p) \leq \frac{A_0}{10} \operatorname{dist}(g(W_p), \partial\Omega)$ which follows from the fact that $p < p_k$ (p is not a net point of smaller scale).

We now prove (iii). Let $z \in \Omega \cap B(0,1)$ and define $\mathcal{C}_z = \{p \in \mathcal{C} : z \in \widehat{\Omega}_p\}$. It suffices to prove

$$\#(\mathcal{C}_z) \lesssim_d 1$$

First, suppose $p \in \mathcal{C}_z \cap \mathcal{F}$. Then there exists $W \in \widehat{T}_p$ such that $z \in g_p(W)$ and the definition of $\widehat{\Omega}_p$ then implies that there exists $R_p \in T_p$ such that $R_p \simeq_{A_0} W$. Let $r_z = \operatorname{dist}(z, \partial\Omega)$. Lemmas 3.4.3 and 3.4.6 imply that $\operatorname{diam} g_p(R_p) \simeq_d \operatorname{dist}(g_p(R_p), \partial\Omega) \simeq_{A_0} r_z$ and there exists $C_0(d, A_0) > 0$ such that

$$B(g_p(c_{R_p}), C_0^{-1} r_z) \subseteq g_p(R_p) \subseteq B(z, C_0 r_z) \tag{3.60}$$

Since $\Omega_p \cap \Omega_{p'} = \varnothing$ for $p \neq p'$, we have

$$g_p(R_p) \cap g_{p'}(R_{p'}) = \varnothing. \tag{3.61}$$

it follows from (3.60) and (3.61) that $\#(\mathcal{C}_z \cap \mathcal{F}) \lesssim_{A_0,d} 1$. A similar argument shows that $\#(\mathcal{C}_z \cap \mathcal{N}) \lesssim_{d,A_0} 1$ from which the claim follows.

Item (iv) follows from Lemma 3.4.3 (ii). ■

Remark 3.4.10 (Whitney family). In fact, (3.60) and (3.61) in combination with Lemma 3.4.3 show that there exists a constant $\Lambda_1(d)$ such that the family

$$\mathcal{G}_1 = \bigcup_{\substack{p \in \mathcal{F} \\ W \in T_p}} g_p(W) \cup \bigcup_{q \in \mathcal{N}} W_q. \tag{3.62}$$

is a Λ_1-Whitney family in the sense of Definition 3.3.4 (compare with Lemma 3.3.10).

We can now finish the proof of Proposition 3.4.4.

Proof of Proposition 3.4.4. We showed the existence of L_1 such that Ω_j is L_1-Lipschitz graphical forany $j \in J_{\mathscr{L}}$ in Lemma 3.4.7. The fact that $\Omega_j \subseteq \Omega$ follows from Lemma 3.4.9 (iv) while $\Omega \subseteq \bigcup_{j \in J_{\mathscr{L}}} \Omega_j$ follows from Lemma 3.4.9 (ii). Finally, item (iii) of Proposition 3.4.4 follows from Lemma 3.4.9 (iii) because for each $p \in \mathcal{C}$, there is by definition at most one index j_p such that $x \in \Omega_{j_p} \subseteq \widehat{\Omega}_p$. ■

3.4.4 Surface area bounds for Theorems D and E

In this section, we prove Proposition 3.4.5. The proof is similar to that of Proposition 3.3.7 given Remark 3.4.10. Fix $y \in \partial\Omega \cap B(0,1)$ and $0 < r \leq 1$ and let $A_2 = 100\sqrt{d}A_0^2$, $A_3 = 50\sqrt{d}A_1A_2$. If $p \in \mathcal{F}$ is such that $\widehat{\Omega}_p \cap B(y,r) \neq \varnothing$, then there exists a cube R with $\ell(R) \leq 2r$ such that $g_p(R) \cap B(y,r) \neq \varnothing$ and $g_p(R) \simeq_{A_0} W$ with $W \in T_p$. Then

$$\begin{aligned} \operatorname{dist}(g_p(W), y) &\leq \operatorname{dist}(g_p(W), g_p(R)) + \operatorname{diam} g_p(R) \leq A_0(1 + A_0) \operatorname{diam} g_p(R) \\ &\leq 3\sqrt{d}A_0^2 \ell(R) < 10\sqrt{d}A_0^2 r. \end{aligned}$$

Therefore, since $A_2 > 50\sqrt{d}A_0^2$, we get that $\Omega_p \cap B(y, A_2 r) \neq \varnothing$. We set

$$\mathscr{T}'_{y,A_2r} = \{T_p : p \in \mathcal{F},\ \Omega_p \cap B(y, A_2 r) \neq \varnothing\}.$$

The above discussion gives that $\widehat{\Omega}_p \cap B(y,r) \neq \varnothing \implies \Omega_p \cap B(y, A_2 r) \neq \varnothing$, so it suffices to consider stopping time domains in the family $\mathscr{T}'_{y,A_2r}$. Break up $\mathscr{T}'_{y,A_2r}$ into regions with large and small top cubes:

$$\begin{aligned}\mathscr{T}_{L,A_2r} &= \{T_p \in \mathscr{T}'_{y,A_2r} : h(W(T_p)) > 10A_2 r\},\\ \mathscr{T}_{y,A_2r} &= \mathscr{T}'_{y,A_2r} \setminus \mathscr{T}_{L,A_2r}.\end{aligned}$$

We also collect all of the boundaries of domains in our decomposition $\mathscr{L}$ associated with a given flat point $p \in \mathcal{F}$ into the set

$$\mathcal{B}_p = \bigcup_{j \in J_{T_p}} \partial\Omega_p^j \cup \bigcup_{W \in \widehat{\mathcal{D}}_p \setminus \mathcal{D}_p} g(\partial W). \tag{3.63}$$

We note that $\mathcal{B}_p$ is Ahlfors d-regular with constant depending on d and A_0 by Proposition 3.5.1 and the fact that each cube $W \subseteq \widehat{\mathcal{D}}_p \setminus \mathcal{D}_p$ is A_0-close to a cube $W' \in T_p$ with at least one face inside $\partial\mathcal{D}_p$.

We can then use the arguments of the previous section to get the following analogues of Lemmas 3.3.13 and 3.3.14.

Lemma 3.4.11.

$$\sum_{\substack{p \in \mathcal{F} \\ T_p \in \mathscr{T}_{L,A_2r}}} \mathcal{H}^d(\mathcal{B}_p \cap B(y,r)) \lesssim_d r^d \leq \mathcal{H}^d(\partial\Omega \cap B(y,r)).$$

Proof. It follows from the proof of Lemma 3.3.13 and the fact that $\mathcal{G}_1$ is a Whitney family (see Remark 3.4.10) that $\#(\mathscr{T}_{L,A_2r}) \lesssim_{A_2,d} 1$. Since $\mathcal{B}_p$ is Ahlfors d-regular, we have

$$\sum_{\substack{p \in \mathcal{F} \\ T_p \in \mathscr{T}_{L,A_2r}}} \mathcal{H}^d(\mathcal{B}_p \cap B(y,r)) \lesssim_{A_0,d} \#(\mathscr{T}_{L,A_2r}) r^d \lesssim_{A_2,d} r^d. \qquad \blacksquare$$

We now handle the regions with small top boxes:

Lemma 3.4.12.

$$\sum_{\substack{p \in \mathcal{F} \\ T_p \in \mathscr{T}_{y,A_2r}}} \mathcal{H}^d(\mathcal{B}_p \cap B(y,r)) \lesssim_{d,\epsilon} \mathcal{H}^d(\partial\Omega \cap B(y, A_3 r)). \tag{3.64}$$

Proof. We modify the proof of Lemma 3.3.14. We first observe that since $\mathcal{H}^d(\mathcal{B}_p) \lesssim_{A_0,d} \mathcal{H}^d(\partial\Omega_p)$, we have

$$\sum_{\substack{p \in \mathcal{F} \\ T_p \in \mathscr{T}_{y,A_2r}}} \mathcal{H}^d(\mathcal{B}_p \cap B(y,r)) \leq \sum_{\substack{p \in \mathcal{F} \\ T_p \in \mathscr{T}_{y,A_2r}}} \mathcal{H}^d(\mathcal{B}_p) \lesssim_{A_0,d} \sum_{\substack{p \in \mathcal{F} \\ T_p \in \mathscr{T}_{y,A_2r}}} \mathcal{H}^d(\partial\Omega_p).$$

Therefore, it suffices to prove (3.64) with $\mathcal{B}_p \cap B(y,r)$ replaced by $\partial\Omega_p$. For any $T_p \in \mathscr{T}_{L,A_2r}$, we get

$$\mathcal{H}^d(\partial\Omega_p) \lesssim_d \mathcal{H}^d(\partial\Omega_p \cap \partial\Omega) + \sum_{W \in m(T_p)} h(W)^d \tag{3.65}$$

Now, $W \in m(T_p)$ implies that there exists a child $W' \in \mathrm{Stop}(T_p)$ for which we have $Q \in \mathscr{B}_e$ of (3.45) with $g(W') \simeq_{A_1} Q$ by Lemma 3.3.12. By replacing A_0 with A_1 and r with A_2r in (3.48), we get $Q \subseteq B(y, 50\sqrt{d}A_1A_2r) \subseteq B(y, A_3r)$. Hence, applying Lemma 3.3.9 with $\mathscr{V} = \{g(W) : W \in m(T_p),\ T_p \in \mathscr{T}_{y,A_2r}\}$ and $\mathscr{U} = \{Q \in \mathscr{B}_e : Q \subseteq B(y, A_3r)\}$, we get

$$\sum_{\substack{p \in \mathcal{F} \\ T \in \mathscr{T}_{y,A_2r}}} \sum_{W \in m(T_p)} h(W)^d \lesssim_d \sum_{\substack{Q \in \mathscr{B}_e \\ Q \subseteq B(y,A_3r)}} \ell(Q)^d \lesssim_{d,\epsilon} \mathcal{H}^d(\partial\Omega \cap B(y, A_3r)) \tag{3.66}$$

where the last inequality follows from the Carleson packing condition for $\mathscr{B}_e$. By observing that $\partial\Omega_p \cap \partial\Omega \subseteq B(y, 50\sqrt{d}A_2r)$ and $\mathcal{H}^d(\partial\Omega_p \cap \partial\Omega_{p'} \cap \partial\Omega) = 0$ for any $p \neq p'$, (3.65) implies

$$\sum_{T_p \in \mathscr{T}_{y,A_2r}} \mathcal{H}^d(\partial\Omega_p) \lesssim_{d,\epsilon} \mathcal{H}^d(\partial\Omega \cap B(y, A_3r)). \qquad \blacksquare$$

We also need to bound the surface measure associated to trivial domains around non-flat $q \in \mathcal{N}$. For any $q \in \mathcal{N}$, we define the set of boundaries

$$\mathcal{B}_q = \partial W_q \cup \bigcup_{\substack{W \in \widetilde{\mathscr{W}} \\ W \subseteq \widehat{\mathcal{D}}_q \setminus \mathcal{D}_q}} \partial W.$$

We note that $\mathcal{H}^d(\mathcal{B}_q) \lesssim_{d,A_0} \ell(W_q)^d$.

Lemma 3.4.13.

$$\sum_{q \in \mathcal{N}} \mathcal{H}^d(\mathcal{B}_q \cap B(y,r)) \lesssim_{d,\epsilon} \mathcal{H}^d(\partial\Omega \cap B(y, A_3r))$$

Proof. Observe that $\mathcal{B}_q \cap B(y,r) \neq \varnothing$ implies there exists $Q \in \mathscr{B}_e$ such that $W_q \simeq_{10A_1} Q$ and $Q \subseteq B(y, A_3r)$ so that we have

$$\sum_{q \in \mathcal{N}} \mathcal{H}^d(\mathcal{B}_q \cap B(y,r)) \leq \sum_{\substack{q \in \mathcal{N} \\ \mathcal{B}_q \cap B(y,r) \neq \varnothing}} \ell(W_q)^d \lesssim_{A_1,d} \sum_{\substack{Q \in \mathscr{B}_e \\ Q \subseteq B(y,A_3r)}} \ell(Q)^d \lesssim_{d,\epsilon} \mathcal{H}^d(\partial\Omega \cap B(y, A_3r)). \qquad \blacksquare$$

Proof of Proposition 3.4.5. $\Omega_j \in \mathscr{L}$ implies that there either there exists j_0, T_0 such that $\Omega_j = \Omega_{T_0}^{j_0}$ or $q \in \mathcal{N}$ such that $\Omega_j = R \in \widetilde{W}$ where $R \simeq_{A_0} W_q$. This means that

$$\begin{aligned}
&\sum_{j \in j_{\mathscr{L}}} \mathcal{H}^d(\partial\Omega_j \cap B(y,r)) \\
&\quad \leq \sum_{T \in \mathscr{T}_{L,A_2r}} \sum_{j \in J_T} \mathcal{H}^d(\partial\Omega_T^j \cap B(y,r)) + \sum_{T \in \mathscr{T}_{y,A_2r}} \sum_{j \in J_T} \mathcal{H}^d(\partial\Omega_T^j \cap B(y,r)) + \sum_{q \in \mathcal{N}} \mathcal{H}^d(\mathcal{B}_q \cap B(y,r)) \\
&\quad \lesssim \sum_{T \in \mathscr{T}_{L,A_2r}} \mathcal{H}^d(\mathcal{B}_T \cap B(y,r)) + \sum_{T \in \mathscr{T}_{y,A_2r}} \mathcal{H}^d(\mathcal{B}_T \cap B(y,r)) + \sum_{q \in \mathcal{N}} \mathcal{H}^d(\mathcal{B}_q \cap B(y,r)) \\
&\quad \lesssim_{L',d,\epsilon} \mathcal{H}^d(\partial\Omega \cap B(y, A_3r))
\end{aligned}$$

by Lemmas 3.4.11, 3.4.12, and 3.4.13. $\blacksquare$

3.5 Lipschitz graph domains

Because each stopping time domain is not necessarily a Lipcshitz graph domain, we will construct a Ahlfors d-regular, d-rectifiable set Σ_T which carves $\mathcal{D}_T$ into a collection of $c(d)$-Lipschitz graph domains. The images of these nicer domains under a Reifenberg parameterization whose derivative is nearly constant on the domain will then map them forward to Lipcshitz graph domains as desired in the conclusions of Theorems C, D, and E.

3.5.1 Carving up stopping time domains

We want to prove the following proposition:

Proposition 3.5.1. *There exists a constant $L_0(d) > 0$ such that for any stopping time region $T \subseteq \mathscr{W}$, there exists a d-Ahlfors upper regular set Σ_T which is a union of subsets of d-planes such that*

$$\mathcal{D}_T \setminus \Sigma_T = \bigcup_{j \in J_T} \mathcal{D}_T^j$$

where

$$\sum_{j \in J_T} \mathcal{H}^d(\partial \mathcal{D}_T^j) \lesssim_d \mathcal{H}^d(\partial \mathcal{D}_T) \simeq_d \mathcal{H}^d(\mathcal{D}_T \cap \mathbb{R}^d) + \sum_{W \in m(T)} \ell(W)^d \tag{3.67}$$

and $\mathcal{D}_T^j$ is an L_0-Lipschitz graph domain.

Remark 3.5.2. In Proposition 3.5.1, we only care that T is a coherent collection of cubes in the sense of Definition 3.2.6, not that they are produced by the specific g-Whitney coronization construction in Definition 3.3.1.

Σ_T will be defined as a union of more local sets Σ_W for $W \in m(T)$. The basic idea is to use a "cover" emanating from the bottom face of every minimal cube W downwards at a $\frac{\pi}{4}$ angle with the vertical in order to turn the jagged right angles made by stopped cubes into smoother $\frac{\pi}{4}$ angles which look Lipschitz to a point sitting above them higher up in the domain. This is essentially a modification of Peter Jones's algorithm for turning chord arc domains composed of Whitney boxes in the disk into Lipschitz graph domains in his proof of the Analyst's Traveling Salesman Theorem in the complex plane (see pg. 8 of [Jon90]). We now construct Σ_W.

Fix T and $W \in m(T)$. By translating and dilating, we can without loss of generality assume $W = [-1, 1]^d \times [2, 4]$. For any function $f : \mathbb{R}^d \to \mathbb{R}$, we let Graph$(f)$ denote the graph of f in $\mathbb{R}$ over $\mathbb{R}^d \times \{0\}$. We begin by defining, for $1 \leq j \leq d$,

$$H_0(x) = 2,$$
$$H_{2j-1}(x) = 3 + x_j,$$
$$H_{2j}(x) = 3 - x_j.$$

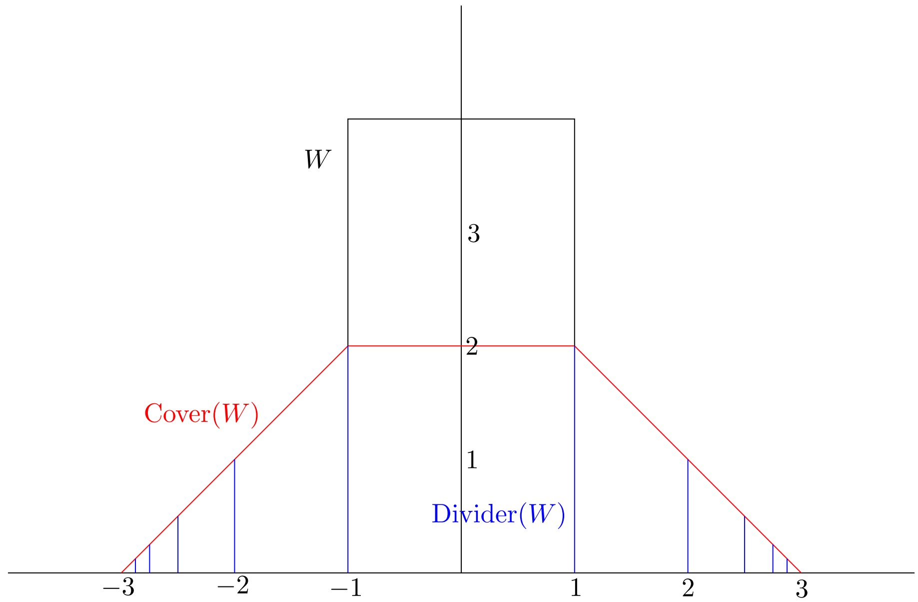

Figure 3.1: A representation of W, $\text{Cover}(W)$, and $\text{Divider}(W)$ in $\mathbb{R}^2$.

The graphs of these functions (except H_0) over $\mathbb{R}^d$ are planes which make an angle of $\frac{\pi}{4}$ with $\mathbb{R}^d$ and contain the edges of $\text{Bot}(W)$ with $x_j = -1$ and $x_j = 1$ respectively. We define

$$H_W(x) = \min_{0\le i\le 2d} H_i(x),$$
$$\text{Cover}(W) = \text{Graph}(H_W) \cap \mathbb{H}^{d+1}.$$

$\text{Cover}(W)$ is the lower envelope of the collection of planes given by the graphs of the H_i. In $\mathbb{R}^3$, $\text{Cover}(W)$ forms the sides of a square pyramid minus its tip with base $[-3,3]^2 \times \{0\}$. In general, $\text{Cover}(W)$ divides $\mathbb{H}^{d+1}$ into two components: a bounded component C_W with boundary $\text{Cover}(W) \cup [-3,3]^d \times \{0\}$ and the unbounded complimentary component. It also follows that

$$\mathcal{H}^d(\text{Cover}(W)) \lesssim_d \mathcal{H}^d(\text{Bot}(W)) = \ell(W)^d. \tag{3.68}$$

$\text{Cover}(W)$ is one of two parts of Σ_W. The second part will be called $\text{Divider}(W)$ because its purpose will be to ensure that all future domains beneath $\text{Cover}(W)$ look similar to the top domain by separating future domains from one another with vertical plane extensions of the sides of cubes sliced by $\text{Cover}(W)$.

We begin by defining $t_n = 1 + \sum_{j=0}^{n-1} 2^{-j}$ and

$$\mathscr{Q}_n = \Big\{ Q \in \Delta^d([-3,3]^d \times \{0\}) : \ell(Q) = t_{n+1} - t_n = 2^{-n},$$
$$\exists j,\ 1 \le j \le d,\ a_j = \pm t_n,\ Q = \prod_{j=1}^{d} [a_j, b_j] \Big\}$$

where $\Delta^d([-3,3]^d \times \{0\})$ is the set of d-dimensional dyadic cubes contained in $[-3,3]^d \times \{0\}$. Intuitively, we think of t_n as the radii of growing balls in the ℓ_∞ metric centered at 0,

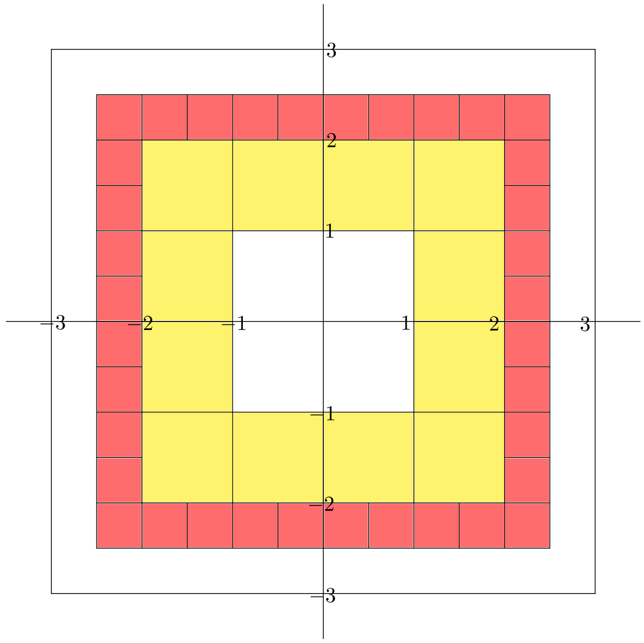

Figure 3.2: A representation of $[-3,3]^2 \times \{0\}$ split into $\mathscr{Q}_1$ in yellow, $\mathscr{Q}_2$ in red, and $\cup_{n=3}^{\infty} \mathscr{Q}_n$ left uncolored at the edge of $\mathscr{Q}_2$ (The white square in the middle sits below the cube $W \in m(T)$, hence nothing above it lies in $\mathcal{D}_T$). The set Divider(W) shoots out of the page as a union of extensions of the sides of the squares up to the points at which they hit the slanting top of Cover(W).

and the cubes inside $\mathscr{Q}_n$ as the natural collection of dyadic cubes tiling the set difference between successive balls with side length exactly equal to the gap between the two square rings forming the boundaries of the ℓ_∞ balls (See Figure 3.2). Set $\mathscr{Q} = \bigcup_{n=1}^{\infty} \mathscr{Q}_n$ and define

$$\text{Divider}(W) = C_W \cap \bigcup \{F_j \times [0, 2\ell(Q)] : F_j \in \text{Faces}(Q),\ Q \in \mathscr{Q}\}.$$

Because $\sum_{j=1}^{2d} \mathcal{H}^d(F_j \times [0, 2\ell(Q)]) \lesssim_d \mathcal{H}^d(Q)$ and $[-3,3]^d \times \{0\} = \bigcup_{Q \in \mathscr{Q}} Q$ is a disjoint union, it follows immediately that

$$\mathcal{H}^d(\text{Divider}(W)) \lesssim_d \mathcal{H}^d(\text{Bot}(W)) = \ell(W)^d. \tag{3.69}$$

Now, we define

$$\Sigma_W = \text{Cover}(W) \cup \text{Divider}(W),$$
$$\Sigma_T = \bigcup_{W \in m(T)} \Sigma_W \cap \mathcal{D}_T.$$

We first prove the upper regularity claim of Proposition 3.5.1.

Lemma 3.5.3. *Σ_T is upper d-Ahlfors upper regular with constant $C \lesssim_d 1$.*

Proof. Fix $R > 0$ and $x \in \Sigma_W \subseteq \Sigma_T$ for some $W \in m(T)$. We write

$$\mathcal{H}^d(\Sigma_T \cap B(x,R)) = \sum_{\substack{W \in m(T) \\ h(W) < 10R}} \mathcal{H}^d(\Sigma_W \cap B(x,R)) + \sum_{\substack{W \in m(T) \\ h(W) \geq 10R}} \mathcal{H}^d(\Sigma_W \cap B(x,R)).$$

We note that $\pi(W)$ and $\pi(W')$ have disjoint interiors for any $W, W' \in m(T)$ with $W \neq W'$, so that

$$\sum_{\substack{W \in m(T) \\ h(W) < 10R}} \mathcal{H}^d(\Sigma_W \cap B(x,R)) \lesssim_d \sum_{\substack{W \in m(T) \\ h(W) < 10R}} \mathcal{H}^d(\mathrm{Bot}(W)) \leq (20R)^d.$$

On the other hand, there are a uniformly bounded number of minimal cubes $N(d)$ with $h(W) \geq 10R$ such that $B(x,R) \cap \Sigma_W \neq \varnothing$ so that

$$\sum_{\substack{W \in m(T) \\ h(W) \geq 10R}} \mathcal{H}^d(\Sigma_W \cap B(x,R)) \leq N(d) \cdot c(d) R^d \lesssim_d R^d$$

because $\mathcal{H}^d(\Sigma_W \cap B(x,R)) \leq c(d)R^d$ for any particular W by construction. Therefore, Σ_T is upper regular. ■

We now finish the proof of Proposition 3.5.1.

Proof of Proposition 3.5.1. It follows from (3.68) and (3.69) that

$$\mathcal{H}^d(\Sigma_T) \leq \sum_{W \in m(T)} \mathcal{H}^d(\Sigma_W) \lesssim_d \sum_{W \in m(T)} \mathcal{H}^d(\mathrm{Bot}(W)) \leq \mathcal{H}^d(\mathrm{Bot}(W(T))) \lesssim_d \mathcal{H}^d(\partial \mathcal{D}_T)$$

which proves (3.67). We now need to show that the resulting domains $\mathcal{D}_T^j$ are Lipschitz-graphical. If $\mathcal{D}_T^j$ is the domain containing $W(T)$, then the claim follows with the choice of central point $c_{W(T)}$. Indeed, the cube $W(T)$ is clearly Lipschitz-graphical with respect to $c_{W(T)}$, and any boundary point of $\mathcal{D}_T^j$ not in $\partial W(T)$ is either in a vertical plane containing one of the vertical faces of $W(T)$, or is part of the Lipschitz graph consisting of the horizontally planar faces $\mathrm{Bot}(W)$ for $W \in m(T)$ and the planes of $\mathrm{Cover}(W)$ making $\frac{\pi}{4}$ angles with the bottom faces.

Now, suppose $\mathcal{D}_T^j \cap W(T) = \varnothing$. We have set up the construction such that this will not differ too much from the top cube case. Let $W \in m(T)$ be a cube of minimal height such that $\mathcal{D}_T^j \subseteq C_W$ and $\mathcal{H}^d(\partial \mathcal{D}_T^j \cap \mathrm{Cover}(W)) > 0$. Such W exists because its minimality implies that for any $W' \in m(T)$ of smaller side length than W, $\mathrm{Cover}(W')$ can only be part of the "lower" boundary of $\mathcal{D}_T^j$ while the only non-vertical planar pieces in Σ_T are bottoms and covers of minimal cubes. Then the cube R of maximal height such that $R \cap \mathcal{D}_T^j \neq \varnothing$ is exactly the cube of length $\ell(Q)$ sitting above $Q \subseteq \mathbb{R}^d \times \{0\}, Q \in \mathscr{Q}$ used in the definition of $\mathrm{Divider}(W)$.

Therefore, $R \cap \mathcal{D}_T^j$ is a cube sliced by finitely many d-planes passing through its sides and corners at $\frac{\pi}{4}$ angles. By the geometry described above, $\mathcal{D}_T^j$ contains the convex hull of c_R and $\mathrm{Bot}(R)$, so we have that $\mathcal{D}_T^j$ is Lipschitz-graphical with respect to $\frac{1}{2}(c_R + c_{\mathrm{Bot}(R)})$. Indeed, Lipschitz-graphicality follows for points in $R \cap \mathcal{D}_T^j$ immediately, and follows for the rest of $\mathcal{D}_T^j$ by the same argument as for the region containing $W(T)$ because the definition of $\mathrm{Divider}(W)$ ensures that all cubes which make up $\mathcal{D}_T^j$ are children of R. Indeed, the boundary outside of ∂R consists of vertical planes containing one of the vertical faces of R or is part of a Lipschitz graph consisting of horizontally planar faces $\mathrm{Bot}(W')$ for $W' \in m(T)$ with $W' \leq R$ and the planes of $\mathrm{Cover}(W')$ making $\frac{\pi}{4}$ angles with the bottom faces. ■

3.5.2 Images of Lipschitz graph domains

We now show that the Lipschitz graph domain property is preserved under images of maps whose derivatives are nearly constant. We begin by observing that linear transformations preserve Lipschitz graph domains

Lemma 3.5.4. *Let $\Omega \subseteq \mathbb{R}^{d+1}$ be an L_0-Lipschitz graph domain and let $A : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ be an L'-bi-Lipschitz affine map. Then there exists a constant $L_1(L_0, L')$ such that $A(\Omega)$ is an L_1-Lipschitz graph domain.*

Proof. Without loss of generality, assume $A(0) = 0$ and set $\Omega' = A(\Omega)$. Then since $\Omega = \{t\theta : 0 \leq t \leq r(\theta),\ \theta \in \mathbb{S}^d\}$, we know that $\Omega' = \{tA(\theta) : 0 \leq t \leq A(\theta),\ \theta \in \mathbb{S}^d\}$ so that Ω' is star-shaped and $r_{\Omega'}$ is well-defined. We have

$$\partial\Omega' = A(\partial\Omega) = \{A(r(\theta)\theta) = r(\theta)A(\theta) : \theta \in \mathbb{S}^d\}.$$

Therefore, given $\psi \in \mathbb{S}^d$, we see that

$$r_{\Omega'}(\psi) = r_\Omega\left(\frac{A^{-1}(\psi)}{|A^{-1}(\psi)|}\right)\frac{1}{|A^{-1}(\psi)|}.$$

Because A^{-1} is L'-bi-Lipschitz and r_Ω is L_0-Lipschitz on $\mathbb{S}^d$, $r_{\Omega'}$ is composed of products and compositions of bounded Lipschitz functions and it follows that there exists $L_1(L_0, L')$ such that $r_{\Omega'}$ satisfies the requirements of Definition 3.1.2 after scaling. ∎

We now move from affine maps to maps whose derivative is sufficiently close to the identity. In preparation, define ℓ_z for any $z \in \mathbb{R}^{d+1}$ to be the line passing through 0 and z and let $P_z = \ell_z^\perp + z$. Define the radial cone at x of aperture α and radius R as

$$C_x(\alpha, R) = \left\{y \in B(x, R) : \frac{\operatorname{dist}(y, \ell_x)}{\operatorname{dist}(y, P_x)} < \tan(\alpha)\right\} \setminus \{x\}.$$

Lemma 3.5.5. *Let $\Omega \subseteq \mathbb{R}^{d+1}$ be an L_0-Lipschitz graph domain. There exists a constant $\delta_0(L_0, d) > 0$ such that if $\delta < \delta_0$ and $\varphi : \overline{\Omega} \to \varphi(\overline{\Omega})$ is a $(1+\delta)$-bi-Lipschitz C^1 map satisfying*

$$|D\varphi(z) - I| \leq \delta \tag{3.70}$$

for all $z \in \overline{\Omega}$, then there exists $L_1 \lesssim_{L_0,d} 1$ such that $\varphi(\Omega)$ is a L_1-Lipschitz graph domain.

Proof. Assume without loss of generality that Ω is Lipschitz graphical with respect to 0 and $\varphi(0) = 0$. We first verify that $r_\Omega : \mathbb{S}^d \to \mathbb{R}^+$ is well-defined, i.e., the domain is star-shaped with respect to 0. Let $\varphi(x) \in \partial\Omega$ and let $\gamma(t) = t\varphi(x)$. We want to show $\gamma \cap \partial\varphi(\Omega) = \{\varphi(x)\}$. Set $\tilde{\gamma}(t) = \varphi^{-1}(\gamma(t))$. We would like to prove

$$|\tilde{\gamma}'(t) - x| \leq 5\delta|x| \tag{3.71}$$

for all $t \in [0, 1]$. First note that

$$|D\varphi(z)^{-1} - I| = |D\varphi(z)^{-1} \cdot \left[I - D\varphi(z)^{-1}\right]| \leq 2\delta||D\varphi(z)^{-1}| \leq 3\delta$$

using the bound $|D\varphi(z)^{-1}| \le \frac{1}{\sigma_{\min}(D\varphi(z))} \le (1+2\delta)$ where $\sigma_{\min}(D\varphi(z))$ is the smallest singular value of $D\varphi(z)$. This means

$$\begin{aligned}|\tilde{\gamma}'(t) - x| &= |D\varphi^{-1}(\gamma(t)) \cdot \gamma'(t) - x| = |\big[D\varphi(\tilde{\gamma}(t))^{-1} - I\big] \cdot \gamma'(t) + \gamma'(t) - x| \\ &\le 3\delta|\varphi(x)| + |\varphi(x) - x| \le 5\delta|x|\end{aligned}$$

where the final line follows from the fact that $\varphi(x) = \int_0^1 D\varphi(tx)\cdot x\,dt = x + \int_0^1 (D\varphi(tx) - I)\cdot x\,dt$ so that $|\varphi(x) - x| \le \delta|x|$. It follows from the mean value theorem that $\tilde{\gamma} \subseteq C_x(10\delta, |x|)$. Since $C_x(10\delta, |x|) \cap \partial\Omega = \varnothing$ for δ sufficiently small in terms of L_0, it follows that choosing δ_0 small enough gives $\tilde{\gamma} \cap \partial\Omega = \{x\}$ so that $\gamma \cap \partial\varphi(\Omega) = \{\varphi(x)\}$ as desired.

Set $\Omega' = \varphi(\Omega)$. Now, $r_{\Omega'}$ is well-defined and (3.70) implies

$$\frac{1}{2(L_0+1)} \le r_{\Omega'}(\theta) \le 2$$

so that we only need to show that $r_{\Omega'}$ satisfies the Lipschitz bound in Definition 3.1.2 for some constant $L_1(L_0, d)$. Let $a, b \in \partial\Omega'$ with $a = |a|\psi_1$ and $b = |b|\psi_2$. Let $\psi = |\psi_1 - \psi_2|$. If $\psi \ge \frac{\pi}{4}$, then the result follows directly from the fact that φ is $(1+\delta)$-bi-Lipschitz. If instead $\psi < \frac{\pi}{4}$, then there exist unique $x, y \in \partial\Omega$ such that $a = \varphi(x)$ and $b = \varphi(y)$ and we assume without loss of generality that $|x| \ge |y|$. Let $x = r_\Omega(\theta_1)\theta_1 = |x|\theta_1,\ y = r_\Omega(\theta_2)\theta_2 = |y|\theta_2$ and set $\theta = |\theta_1 - \theta_2|$.

We first claim that it suffices to show

$$|\theta_1 - \theta_2| \lesssim_{L_0,d} |\psi_1 - \psi_2|. \tag{3.72}$$

Indeed, if (3.72) holds, then

$$\begin{aligned}|r_{\Omega'}(\psi_1) - r_{\Omega'}(\psi_2)| &= ||a| - |b|| \le |a - b| = |\varphi(x) - \varphi(y)| \le (1+\delta)|x - y| \\ &\le (1+\delta)(|x - \theta_1|y|| + |\theta_1|y| - y|) \\ &= (1+\delta)(r_\Omega(\theta_1) - r_\Omega(\theta_2) + |y||\theta_1 - \theta_2|) \\ &\le (1+\delta)(L_0+1)|\theta_1 - \theta_2| \lesssim_{L_0,d} |\psi_1 - \psi_2|.\end{aligned}$$

Now, we concentrate on proving 3.72.

Put $z = (1 - |x-y|)x$ and $c = (1 - |a-b|)a$ and define

$$\alpha = \angle zxy,\ \alpha' = \angle\varphi(z)ab,\ \beta = \angle cab.$$

By the law of cosines,

$$\begin{aligned}\cos\alpha &= \frac{|z-x|^2 + |x-y|^2 - |z-y|^2}{2|z-x||x-y|} = 1 - \frac{|z-y|^2}{2|z-x|^2}, \\ \cos\alpha' &= \frac{|\varphi(z) - \varphi(x)|^2 + |\varphi(x) - \varphi(y)|^2 - |\varphi(z) - \varphi(y)|^2}{2|\varphi(x) - \varphi(z)||\varphi(x) - \varphi(y)|} \\ &\le \frac{2(1+\delta)^2|z-x|^2 - (1-\delta)^2|z-y|^2}{2(1-\delta)^2|z-x|^2} \le 1 - \frac{|z-y|^2}{2|z-x|^2} + 5\delta = \cos\alpha + 5\delta.\end{aligned}$$

Because Ω is L_0-Lipschitz-graphical, $\alpha \gtrsim_{L_0} 1$ so that if δ is sufficiently small, then $\alpha' \geq \frac{\alpha}{2}$. In addition, (3.71) implies that $\varphi([x,z]) \subseteq C_{\varphi(x)}(10\delta, 2(|\varphi(x)| - |c|)$ so that $|\beta - \alpha'| \leq 20\delta$, meaning $\beta \geq \frac{\alpha}{4}$ as long as δ is small enough. To complete the proof, observe that $|\psi_1 - \psi_2| \simeq \angle a0b$, $|\theta_1 - \theta_2| \simeq \angle x0y$, $\beta = \angle 0ab$, and $\alpha = \angle 0xy$ so that $\beta \geq \frac{\alpha}{4}$ implies (3.72) using the fact that φ is $(1+\delta)$-bi-Lipschitz. ■

Finally, by chaining Lemmas 3.5.4 and 3.5.5, we can prove the following desired proposition:

Proposition 3.5.6. *Let $\Omega \subseteq \mathbb{R}^{d+1}$ be an L_0-Lipschitz graph domain and suppose $g : \Omega \to g(\Omega) \subseteq \mathbb{R}^{d+1}$ is C^1 and L-bi-Lipschitz. There exist constants $L_1, \delta_0(L_0, L) > 0$ such that if $\delta < \delta_0$ and*

$$|Dg(z) \cdot Dg(w)^{-1} - I| \leq \delta \tag{3.73}$$

for all $z, w \in \Omega$, then $g(\Omega)$ is an L_1-Lipschitz graph domain.

Proof. Suppose Ω is Lipschitz graphical around 0 and set

$$L(z) = Dg(0) \cdot z.$$

By Lemma 3.5.4, $L(\Omega)$ is $L_0'(L, L_0)$-Lipschitz graphical. The map $\varphi : L(\Omega) \to g(\Omega)$ given by

$$\varphi = g \circ L^{-1}$$

satisfies

$$D\varphi(z)(L(w)) = Dg(L^{-1}(L(w))) \cdot DL^{-1}(L(w)) = Dg(w) \cdot Dg(0)^{-1} \cdot w$$

so that

$$|D\varphi - I| \leq \delta.$$

By taking δ_0 sufficiently small in terms of L_0', Lemma 3.5.5 implies that there exists $L_1(L_0')$ such that $g(\Omega)$ is L_1-Lipschitz graphical. ■

3.6 Controlling the change in the derivative of Reifenberg parameterizations

The goal of this appendix is to give conditions under which we can say that the change in the derivative of a Reifenberg parameterization $g : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ is small. This is specified exactly in Proposition 3.6.7 below.

3.6.1 Preliminary derivative estimates and regularity

In this section, we review some properties of the maps used in the construction of a Reifenberg parameterization g that we need to make specific estimates on the change in Dg. First, the surface Σ_k has a nice local Lipschitz representation:

Lemma 3.6.1 ([DT12] Lemma 6.12)**.** *For $k \geq 0$ and $y \in \Sigma_k$, there is an affine d-plane P through y and a $C\varepsilon$-Lipschitz and C^2 function $A : P \to P^\perp$ such that $|A(x)| \leq C\epsilon r_k$ for all $x \in B(y, 19r_k)$ and*

$$\Sigma_k \cap B(y, 19r_k) = \Gamma \cap B(y, 19r_k).$$

where Γ denotes the graph of A over P.

Now, we record distortion estimates for $D\sigma_k$ as in [DT12] chapter 7. Importantly, $D\sigma_k$ is very close to the identity in the following sense:

Lemma 3.6.2 ([DT12] Lemma 7.1)**.** *For $k \geq 0$, σ_k is a C^2-diffeomorphism from Σ_k to Σ_{k+1} and, for $y \in \Sigma_k$,*

$$D\sigma_k(y) : T\Sigma_k(y) \to T\Sigma_{k+1}(\sigma_k(y)) \text{ is bijective and } (1 + C\varepsilon)\text{-bi-Lipschitz.}$$

In addition,

$$|D\sigma_k(y) \cdot v - v| \leq C\varepsilon|v| \quad \text{for } y \in \Sigma_k \text{ and } v \in T\Sigma_k(y)$$
$$|\sigma_k(y) - \sigma_k(y') - y + y'| \leq C\varepsilon|y - y'| \quad \text{for } y, y' \in \Sigma_k.$$

More precise estimates can be obtained when restricting $D\sigma_k$ to its action on vectors tangent to Σ_k. The best way to capture this is to define quantities which take into account exactly how close the nearby planes of appropriate scale in the CCBP are. These are the ϵ_k' numbers, defined by

$$\epsilon_k'(y) = \sup\big\{d_{x_{i,l},100r_l}(P_{j,k}, P_{i,l})\,;\, j \in J_k,\, l \in \{k-1, k\},\\ i \in J_l, \text{ and } y \in 10B_{j,k} \cap 11B_{i,l}\big\} \tag{3.74}$$

The following lemma gives estimates in terms of these numbers

Lemma 3.6.3 ([DT12] Lemma 7.32)**.** *For $k \geq 1$ and $y \in \Sigma_k \cap V_k^8$, choose $i \in J_k$ such that $|y - x_{i,k}| \leq 10r_k$. Then*

$$|D\pi_{i,k} \circ D\sigma_k(y) \circ D\pi_{i,k} - D\pi_{i,k}| \leq C\varepsilon_k'(y)^2, \tag{3.75}$$

and

$$\big||D\sigma_k(y) \cdot v| - 1\big| \leq C\varepsilon_k'(y)^2 \quad \text{for every unit vector } v \in T\Sigma_k(y). \tag{3.76}$$

Similarly, these numbers also control the distance between tangent planes to the surface and nearby $P_{j,k}$. For any $k \geq 0$ and $y \in \Sigma_k \cap V_k^8$ and $i \in J_k$ such that $|y - x_{i,k}| \leq 10r_k$, we have ([DT12] (7.22))

$$\text{Angle}(T\Sigma_k(y), P_{i,k}) \leq C\epsilon_k'(y). \tag{3.77}$$

Finally, we also use an estimate on $D^2\sigma_k$ obtained in by Ghinassi in [**Ghi17**] in work on constructing $C^{1,\alpha}$ parametrizations.

Lemma 3.6.4 ([**Ghi17**] Lemma 3.16)**.** *For $k \geq 0$, $y \in \Sigma_k \cap V_k^8$,*

$$|D^2\sigma_k(y)| \leq C\frac{\epsilon_k(y)}{r_k} \leq C\frac{\epsilon}{r_k}$$

where we interpret the norm on the tensor $D^2\sigma_k$ as the Euclidean norm on $\mathbb{R}^{n^3}$. We also provide the following lemma and proof adapted from a proof of [DT12] to fit our needs.

Lemma 3.6.5 (cf. [DT12] (11.22))**.** *Suppose Σ_0 is such that for any $x, x' \in \Sigma_0$, there exists a curve γ_0 connecting x and x' with $\ell(\gamma_0) \leq (1 + C\epsilon)|x - x'|$. Let $1 \leq M^3\epsilon < c(d) < 1$ with $c(d)$ sufficiently small and $k \geq 0$ be such that $|f_k(x) - f_k(x')| < Mr_k$. Then there is a curve $\gamma : I \to \Sigma_k$ such that $\ell(\gamma) \leq 2|f_k(x) - f_k(x')|$.*

Proof. We first prove the following claim:
Claim F:or any $0 \leq p \leq k$,

$$|f_{k-p}(x) - f_{k-p}(x')| < \frac{Mr_{k-p}}{5^p}. \tag{3.78}$$

Proof W:e prove this by induction. Indeed, observe that

$$|f_{k-p-1}(x) - f_{k-p-1}(x')| = |\sigma^{-1}_{k-p-1}(f_{k-p}(x)) - \sigma^{-1}_{k-p-1}(f_{k-p}(x'))| \leq (1+C\epsilon)|f_{k-p}(x) - f_{k-p}(x')|.$$

by (3.13). Applying this for $p = 1$ gives

$$|f_{k-1}(x) - f_{k-1}(x')| < (1 + C\epsilon)(Mr_k) < \frac{Mr_{k-1}}{5}$$

This proves the base case. Assuming the claim holds for some p, we get

$$|f_{k-p-1}(x) - f_{k-p-1}(x')| \leq (1 + C\epsilon)\frac{Mr_{k-p}}{5^p} < \frac{Mr_{k-p-1}}{5^{p+1}}.$$

■

To continue the proof of the lemma, we modify the proof of [DT12] (11.22). If $|f_k(x) - f_k(x')| < 18r_k$, then the claim follows immediately from the local Lipschitz graph description of Σ_k in Lemma 3.6.1. So, assume $|f_k(x) - f_k(x')| > 18r_k$ and suppose first that there exists an integer $0 \leq m \leq k$ such that $|f_m(x) - f_m(x')| < 5r_m$. We calculate

$$\frac{Mr_m}{5^{k-m}} < 5r_m \iff \log_5 M - 1 < k - m$$

so that by the above claim we can assume $k - m < \log_5 M < \log M$. Applying the Lipschitz graph lemma for $B(f_m(x), 19r_m)$, we see that there exists a path $\gamma_m \subseteq \Sigma_m$ such that

$$\begin{aligned}\ell(\gamma_m) \leq (1 + C\epsilon)|f_m(x) - f_m(x')| &\leq (1 + C\epsilon)(C\epsilon r_m + |f_k(x) - f_k(x')|) \\ &\leq (1 + C\epsilon)|f_k(x) - f_k(x')| + C\epsilon r_k \log M.\end{aligned}$$

On the other hand, since $|f_m(x) - f_m(x')| < 5r_m$, we get $\ell(\gamma_m) \leq 10r_m$ and so we can choose a chain of $N \leq \frac{10r_m}{10r_k} = 10^{k-m} \leq M^2$ points contained in γ_m with consecutive points separated by a distance of at least $11r_k$ beginning at $f_m(x)$ and ending at $f_m(x')$. Call this collection of points $\{f_m(x_l)\}_{l=1}^N$ for $x_l \in \Sigma_0$. This implies the total length of the string of points $\{f_k(x_l)\}$ is

$$\begin{aligned}L' = \sum_{l=1}^N |f_k(x_l) - f_k(x_{l+1})| \leq \sum_{l=1}^N [C\epsilon r_m + |f_m(x_l) - f_m(x_{l+1})|] &\leq C\epsilon r_k M^2 \log M + \ell(\gamma_m) \\ &\leq C\epsilon r_k M^2 \log M + (1 + C\epsilon)|f_k(x) - f_k(x')|.\end{aligned}$$

In addition, for any admissible l we can calculate

$$|f_k(x_l) - f_k(x_{l+1})| \leq C\epsilon r_m + |f_m(x_l) - f_m(x_{l+1})| \leq C\epsilon r_k \log M + 11 r_k < 12 r_k. \tag{3.79}$$

Using (3.79) and Lemma 3.6.1 once again, we connect each pair $(f_k(x_l), f_k(x_{l+1}))$ by a curve γ_l of length $\ell(\gamma_l) \leq (1 + C\epsilon)|f_k(x_l) - f_k(x_{l+1})|$ to get a curve γ with

$$\ell(\gamma) \leq (1 + C\epsilon)L' \leq (1 + C\epsilon)|f_k(x) - f_k(x)| + C\epsilon r_k M^2 \log M \leq 2|f_k(x) - f_k(x')|$$

using the fact that $|f_k(x) - f_k(x')| > 18 r_k$ and $M \ll \epsilon^{-1}$, i.e. ϵ is sufficiently small compared to M. This completes the proof if there exists such an m where $f_k(x)$ and $f_k(x')$ pull back to a Lipschitz neighborhood in Σ_m. If there does not exist such an m (i.e., k is too small), then we instead use the assumed curve $\gamma_0 \subseteq \Sigma_0$ in place of γ_m and argue as in the previous case. ■

We also recall a reverse triangle inequality:

Lemma 3.6.6. *(Reverse Triangle Inequality) Let $u, v \in \mathbb{R}^{d+1}$ with $\langle u, v\rangle \geq -\frac{1}{2}|u||v|$. Then*

$$|u| + |v| \leq 2|u + v|. \tag{3.80}$$

3.6.2 Controlling the change in Dg

Proposition 3.6.7 follows from a series of computations involving the derivative of the map g produced by Theorem 3.2.2 for a given CCBP. Proposition 3.6.7 says that given a "central" point $z \in \mathbb{R}^{d+1}$ and an inflation factor $1 \leq M_0$ such that $M_0\epsilon$ is sufficiently small, we can get a set $G_z^{M_0}$ such that $w \in G_z^{M_0}$ means $Dg(w)$ is very close to $Dg(z)$ in the sense of (3.81).

Proposition 3.6.7 follows from Lemmas 3.6.8 and 3.6.10 which give separate horizontal and vertical estimates respectively. Define the sets of horizontal and vertical vectors by

$$H = \mathbb{R}^d \times \{0\}, \ V = \{0\}^d \times \mathbb{R}.$$

These lemmas show how to appropriately bound the individual pieces of the difference $Dg(x, y) - Dg(x', y)$ and $Dg(x', y) - Dg(x', y')$ between points $z = (x, y), z' = (x', y')$ respectively when acting on $v \in H \cup V$. Corollaries 3.6.9 and 3.6.11 put these pieces together to get the requisite Dg estimates, from which we prove Proposition 3.6.7.

Proposition 3.6.7. *Let $0 < \epsilon < \delta < 1$ and $M_0 > 0$ such that $1 \leq M_0^3\epsilon < c(d)$ with $c(d)$ sufficiently small. Fix a CCBP $(\Sigma_0, \{B_{j,k}\}, \{P_{j,k}\})$. Let $z, z' \in \mathbb{R}^d \times \mathbb{R}$ with $z = (x, y),\ z' = (x', y')$ where $|y'| \leq |y|$ and assume $f_{n(y)}(x) \in V^8_{n(y)}$ and $f_{n(y')}(x') \in V^8_{n(y')}$ (see (3.16)). Define*

$$G_z^{M_0} = \Big\{z' = (x', y') \in \mathbb{R}^{d+1} : |f_{n(y)}(x) - f_{n(y)}(x')| < M_0 r_{n(y)}, \sum_{k=n(y)}^{n(y')} \epsilon_k'(f_k(x'))^2 < \epsilon, \\ \text{Angle}(T_k(x'), T_{n(y)}(x') \leq \delta\Big\}.$$

Then there exists $C(d) > 0$ such that for any $w \in G_z^{M_0}$, we have

$$|Dg(w) \cdot Dg(z)^{-1} - I| \leq C(d)\delta. \tag{3.81}$$

Lemma 3.6.8 (Horizontal Estimates)**.** *Let z, z', M_0, ϵ be as in Proposition 3.6.7 and let $v \in H \cup V$. Let k be such that $\rho_k(y) > 0$. If $|f_k(x) - f_k(x')| < M_0 r_k$, then*

$$|Df_k(x) \cdot v - Df_k(x') \cdot v| \le C\epsilon |Df_k(x) \cdot v|, \tag{3.82}$$

$$|(D_x(R_k(x) \cdot y)) \cdot v - (D_x(R_k(x') \cdot y)) \cdot v| \le C\epsilon |Df_k(x) \cdot v|. \tag{3.83}$$

In any case,

$$\left|\frac{\partial g}{\partial y}(x, y) - \frac{\partial g}{\partial y}(x', y)\right| \le C\epsilon \left|\frac{\partial g}{\partial y}(x, y)\right| \tag{3.84}$$

where the constant C depends on M.

Proof. We begin with proving (3.82). We have

$$\begin{aligned} |Df_k(x) \cdot v - Df_k(x') \cdot v| = \Big| &\big[D\sigma_{k-1}(f_{k-1}(x)) - D\sigma_{k-1}(f_{k-1}(x'))\big] Df_{k-1}(x) \cdot v \\ &+ D\sigma_{k-1}(f_{k-1}(x'))\big[Df_{k-1}(x) - Df_{k-1}(x')\big] \cdot v\Big| \\ \le\ &|Df_{k-1}(x) \cdot v||D\sigma_{k-1}(f_{k-1}(x)) - D\sigma_{k-1}(f_{k-1}(x'))| \\ &+ |D\sigma_{k-1}(f_{k-1}(x'))||Df_{k-1}(x) \cdot v - Df_{k-1}(x') \cdot v| \end{aligned}$$

Recursively applying this inequality for decreasing values of k gives

$$\begin{aligned} &|Df_k(x) \cdot v - Df_k(x') \cdot v| \\ &\le |Df_{k-1}(x) \cdot v||D\sigma_{k-1}(f_{k-1}(x)) - D\sigma_{k-1}(f_{k-1}(x'))| \\ &\qquad + |D\sigma_{k-1}(f_{k-1}(x'))|\big(|Df_{k-2}(x) \cdot v||D\sigma_{k-2}(f_{k-2}(x)) - D\sigma_{k-2}(f_{k-2}(x'))| \\ &\qquad + |D\sigma_{k-2}(f_{k-2}(x'))||Df_{k-2}(x) \cdot v - Df_{k-2}(x') \cdot v|\big) \\ &\le |Df_{k-1}(x) \cdot v||D\sigma_{k-1}(f_{k-1}(x)) - D\sigma_{k-1}(f_{k-1}(x'))| \\ &+ \sum_{p=1}^{k} \left(\prod_{m=1}^{p} |D\sigma_{k-m}(f_{k-m}(x'))|\right) |Df_{k-p-1}(x) \cdot v| \cdot |D\sigma_{k-p-1}(f_{k-p-1}(x)) - D\sigma_{k-p-1}(f_{k-p-1}(x')|. \end{aligned} \tag{3.85}$$

Now, Lemma 3.6.2 implies

$$\prod_{m=1}^{p} |D\sigma_{k-m}(f_{k-m}(x'))| \le (1 + C\epsilon)^p, \tag{3.86}$$

and

$$|Df_{k-p-1}(x) \cdot v| = \left|\prod_{m=1}^{p+1} D\sigma_{k-m}^{-1}(f_{k-m+1}(x)) Df_k(x) \cdot v\right| \le (1 + C\epsilon)^{p+1} |Df_k(x) \cdot v|. \tag{3.87}$$

Using Lemma 3.6.5, we see that (3.78) implies $|f_{k-p-1}(x) - f_{k-p-1}(x')| < \frac{M_0 r_{k-p-1}}{5^{p+1}} \le M_0 r_{k-p-1}$ so that we get a rectifiable curve γ_{k-p-1} connecting $f_{k-p-1}(x)$ and $f_{k-p-1}(x')$ such that $\ell(\gamma_{k-p-1}) \leq 2|f_{k-p-1}(x) - f_{k-p-1}(x')|$. Lemma 3.6.4 gives

$$\begin{aligned}
|D\sigma_{k-p-1}(f_{k-p-1}(x)) - D\sigma_{k-p-1}(f_{k-p-1}(x'))| &= \left|\int_I D^2\sigma_{k-p-1}(\gamma_{k-p-1}(t)) \cdot \gamma'_{k-p-1}(t)\ dt\right| \\
&\leq \int_I |D^2\sigma_{k-p-1}(\gamma_{k-p-1}(t))||\gamma'_{k-p-1}(t)|dt \\
&\leq C\frac{\epsilon}{r_{k-p-1}} \cdot 2|f_{k-p-1}(x) - f_{k-p-1}(x')| \\
&\leq C\frac{\epsilon}{r_{k-p-1}}\frac{M_0 r_{k-p-1}}{5^{p+1}} \\
&\leq CM_0\frac{\epsilon}{5^{p+1}}. \qquad (3.88)
\end{aligned}$$

Applying (3.86), (3.87), and (3.88) to (3.85) gives

$$\begin{aligned}
|Df_k(x) \cdot v - Df_k(x') \cdot v| &\leq (1 + C\epsilon)C\frac{\epsilon}{5}|Df_k(x) \cdot v| + \sum_{p=1}^{k}(1 + C\epsilon)^p(1 + C\epsilon)^{p+1}|Df_k(x) \cdot v|CM_0\frac{\epsilon}{5^{p+1}} \\
&\leq C\epsilon|Df_k(x) \cdot v| + C\epsilon M_0|Df_k(x) \cdot v|\sum_{p=1}^{k}\frac{(1 + C\epsilon)^{2p}}{5^{p+1}} \\
&\leq C\epsilon|Df_k(x) \cdot v|.
\end{aligned}$$

We now prove (3.83). For any $t > 0$, Proposition 3.2.3 implies that the quantity $R_k(x + tv)\cdot e_{d+1} - R_k(x)\cdot e_{d+1}$ is the difference between the unit normal vectors to the linear subspaces $T\Sigma_k(f_k(x + tv))$ and $T\Sigma_k(f_k(x))$. But by Lemma 3.2.4, we have

$$|R_k(x+tv) \cdot e_{d+1} - R_k(x) \cdot e_{d+1}| \leq D(T\Sigma_k(f_k(x+tv)), T\Sigma_k(f_k(x))) \leq C\frac{\epsilon}{r_k}|f_k(x+tv) - f_k(x)|. \qquad (3.89)$$

Hence, we can write

$$\begin{aligned}
|(D_x(R_k(x) \cdot y)) \cdot v| &\leq |y| \lim_{t\to 0}\frac{|R_k(x + tv) \cdot e_{d+1} - R_k(x) \cdot e_{d+1}|}{|t|} \leq C\epsilon\frac{|y|}{r_k}\lim_{t\to 0}\frac{|f_k(x + tv) - f_k(x)|}{|t|} \\
&\leq C\epsilon|Df_k(x) \cdot v| \qquad (3.90)
\end{aligned}$$

where $|y| \lesssim r_k$ since $\rho_k(y) > 0$. We then have

$$|(D_x(R_k(x) \cdot y)) \cdot v - (D_x(R_k(x') \cdot y)) \cdot v| \leq C\epsilon(|Df_k(x) \cdot v| + |Df_k(x') \cdot v|) \leq C\epsilon|Df_k(x) \cdot v|$$

using (3.82).

Finally, we prove (3.84). First, we compute

$$\begin{aligned}
\frac{\partial g}{\partial y}(x, y) - \frac{\partial g}{\partial y}(x', y) &= \sum_{k\geq 0}\frac{\partial\rho_k}{\partial y}(y)\left\{f_k(x) - f_k(x') + R_k(x) \cdot y - R_k(x') \cdot y\right\} \\
&\quad + \sum_{k\geq 0}\rho_k(y)(R_k(x) \cdot e_{d+1} - R_k(x') \cdot e_{d+1}) \\
&=: I + II
\end{aligned}$$

Let $p, p+1$ be the values of k such that $\rho_k(y) > 0$. Since $\rho_p(y) + \rho_{p+1}(y) = 1$, we have $\frac{\partial \rho_p}{\partial y}(y) + \frac{\partial \rho_{p+1}}{\partial y}(y) = 0$. This implies

$$
\begin{aligned}
I = & \frac{\partial \rho_p}{\partial y}(y)\left(f_p(x) - f_{p+1}(x) + R_p(x)\cdot y - R_{p+1}(x)\cdot y\right) \\
& + \frac{\partial \rho_p}{\partial y}(y)\left(f_p(x') - f_{p+1}(x') + R_p(x')\cdot y - R_{p+1}(x')\cdot y\right)
\end{aligned}
$$

But using (3.13) and (3.14), we have

$$|I| \leq \frac{C}{r_p}(C\epsilon r_p + C\epsilon|y|) \leq C\epsilon.$$

By (3.89) we have

$$
\begin{aligned}
|II| &\leq |\rho_p(y)|\frac{C\epsilon}{r_p}|f_p(x) - f_p(x')| + |\rho_{p+1}(y)|\frac{C\epsilon}{r_{p+1}}|f_{p+1}(x) - f_{p+1}(x')| \\
&\leq CM_0\epsilon(|\rho_p(y)| + |\rho_{p+1}(y)|) \leq C\epsilon.
\end{aligned}
$$

We've proven that $\left|\frac{\partial g}{\partial y}(x,y) - \frac{\partial g}{\partial y}(x',y)\right| \leq C\epsilon$. We will complete the proof of (3.84) by showing that $\left|\frac{\partial g}{\partial y}(x,y)\right| \gtrsim 1$. Indeed,

$$
\begin{aligned}
\frac{\partial g}{\partial y}(x,y) = \Bigg| & \left[\frac{\partial \rho_p}{\partial y}(y)\left(f_p(x) - f_{p+1}(x) + R_p(x)\cdot y - R_{p+1}(x)\cdot y\right)\right] \\
& + \left[\rho_p(y)R_p(x)\cdot e_{d+1} + \rho_{p+1}(y)R_{p+1}(x)\cdot e_{d+1}\right]\Bigg|.
\end{aligned} \tag{3.91}
$$

But the previous computation shows that the first expression has norm $\leq C\epsilon$, while the second expression is a convex combination of two nearly parallel unit vectors because $R_p(x)$ and $R_{p+1}(x)$ are orthogonal matrices which are $C\epsilon$ close. Hence, we get

$$\left|\frac{\partial g}{\partial y}\right| \gtrsim 1. \tag{3.92}$$

■

Corollary 3.6.9. *Let z, z' be as in Lemma 3.6.8 and set $p = n(y'), m = n(y)$. Then for any vector $v \in H \cup V$, we have*

$$|Dg(x,y)\cdot v - Dg(x',y)\cdot v| \leq C\epsilon|Dg(x,y)\cdot v|$$

Proof. First, suppose $v = v_x \in H$. Since $v_x \cdot e_{d+1} = 0$, we have

$$Dg(x,y)\cdot v_x = \sum_{k\geq 0}\rho_k(y)\left\{Df_k(x)\cdot v_x + D(R_k(x)\cdot y)\cdot v_x\right\}. \tag{3.93}$$

Therefore, we get

$$
\begin{aligned}
&|Dg(x,y)\cdot v_x - Dg(x',y)\cdot v_x| \\
&\quad \leq \sum_{k\geq 0}\rho_k(y)\big\{|Df_k(x)\cdot v_x - Df_k(x')\cdot v_x| + |D(R_k(x)\cdot y)\cdot v_x - D(R_k(x'))\cdot v_x|\big\} \\
&\quad \leq CM_0\epsilon\sum_{k\geq 0}\rho_k(y)\left\{|Df_k(x)\cdot v_x|\right\}.
\end{aligned} \tag{3.94}
$$

using (3.82) and (3.83). We now want to bound $\sum_{k\geq 0}\rho_k(y)\left\{|Df_k(x)\cdot v_x|\right\}$ by $|Dg(x,y)\cdot v_x|$. In order to do so, we first simplify notation by setting

$$s=\rho_p(y),\ t=\rho_{p+1}(y),$$
$$v_1=Df_p(x)\cdot v_x,\ u_1=Df_{p+1}(x)\cdot v_x,$$
$$v_2=D(R_p(x)\cdot y)\cdot v_x,\ u_2=D(R_{p+1}(x)\cdot y)\cdot v_x.$$

Putting $v=v_1+v_2,\ u=u_1+u_2$, we have $Dg(x,y)\cdot v_x=sv+tu$. In this notation,

$$|v_1-u_1|\leq C\epsilon|v_1|, \tag{3.95}$$
$$|v_2|,|u_2|\leq C\epsilon|v_1|, \tag{3.96}$$

by Lemma 3.6.2 and (3.90). We then want to prove the following claim:

Claim : $s|v_1|+t|u_1|\lesssim|sv+tu|$.

Proof U:sing (3.96), we get $s|v_1|\leq s|v_1|+s|v_2|\leq s|v|$ and similarly $t|u_1|\leq t|u|$. We now just need to show that $|sv|+|tu|\lesssim|sv+tu|$. By Lemma 3.6.6, this follows if we can show $\langle sv,tu\rangle\geq-\frac{1}{2}|sv||tu|$. Indeed, we have

$$\begin{aligned}\langle sv,tu\rangle&=st\left(\langle v_1,u_1\rangle+\langle v_1,u_2\rangle+\langle v_2,u_1\rangle+\langle v_2,u_2\rangle\right),\\&\geq st(|v_1|^2-\langle v_1,u_1-v_1\rangle-C\epsilon|v_1|^2)\geq st(1-C\epsilon)|v_1|^2\geq 0.\end{aligned}$$

■

This completes the proof for $v=v_x$. If instead $v=v_y\in V$, then $Dg(z)\cdot v_y=v_y\cdot\frac{\partial g}{\partial y}(z)$ and the result follows from (3.84) in Lemma 3.6.8. ■

Lemma 3.6.10 (Vertical Estimates)**.** *Let z,z',v,p,m be as in Corollary 3.6.9. If $\sum_{k=p}^{m}\epsilon_k'(f_k(x'))^2\leq C\epsilon$ and* $\operatorname{Angle}(T\Sigma_p(f_p(x')),T\Sigma_m(f_m(x')))\leq C\delta$*, we have*

$$\left|\sum_{k\geq 0}(\rho_k(y)-\rho_k(y'))Df_k(x')\cdot v\right|\leq C\delta|Df_p(x')\cdot v|, \tag{3.97}$$

$$\left|\sum_{k\geq 0}\rho_k(y)D(R_k(x')\cdot y)\cdot v-\rho_k(y')D(R_k(x')\cdot y')\cdot v\right|\leq C\delta|Df_p(x')\cdot v|, \tag{3.98}$$

$$\left|\frac{\partial g}{\partial y}(x',y)-\frac{\partial g}{\partial y}(x',y')\right|\leq C\delta\left|\frac{\partial g}{\partial y}(x',y)\right|. \tag{3.99}$$

Proof. We being by proving (3.97). First, since $D\sigma_k$ is $(1+C\epsilon)$-bi-Lipschitz for any k, we have

$$|Df_p(x')\cdot v-Df_{p+1}(x')\cdot v|\leq C\epsilon|Df_p(x')\cdot v|.$$

This implies

$$\begin{aligned}\left|\sum_{k\geq 0}\rho_k(y)Df_k(x')\cdot v-Df_p(x')\cdot v\right|&\leq\sum_{k\geq 0}\rho_k(y)|Df_k(x')\cdot v-Df_p(x')\cdot v| \\&\leq C\epsilon|Df_p(x')|\end{aligned} \tag{3.100}$$

because $\rho_k(y) \neq 0$ only for $k = p, p+1$. An identical argument gives (3.100) with y replaced by y' and p replaced by m. We now want a similar bound for $|Df_m(x') \cdot v - Df_p(x') \cdot v|$. For ease of notation, define $u = Df_p(x') \cdot v$ and $w = \prod_{k=p}^{m-1} D\sigma_k(f_k(x')) \cdot u$. We can then write

$$|Df_m(x') \cdot v - Df_p(x') \cdot v| = \left| \left[\prod_{k=p}^{m-1} D\sigma_k(f_k(x')) \right] \cdot u - u \right| = |w - u|.$$

The fact that $\sum_{k=p}^{m} \epsilon_k'(f_k(x'))^2 \leq C\epsilon^2$ means

$$\left| \prod_{k=p}^{m-1} D\sigma_k(f_k(x')) \right| \leq \prod_{k=p}^{m-1} 1 + CM_0^2 \epsilon_k'(f_k(x'))^2 \leq 1 + CM_0^2 \epsilon^2. \tag{3.101}$$

Hence, $||w| - |u|| \leq CM_0^2 \epsilon^2 |u|$. Since $w \in T\Sigma_m(f_m(x'))$, $u \in T\Sigma_p(f_p(x'))$, and we've assumed that $\text{Angle}(T\Sigma_p(f_p(x')), T\Sigma_m(f_m(x'))) \leq C\delta$, we have $\text{Angle}(w, u) \leq C\delta$ and it follows that

$$|w - u| \lesssim_{M_0} \delta |u| \tag{3.102}$$

as long as δ and ϵ are sufficiently small. Finally, using (3.100) and (3.102), we see

$$\begin{aligned}
&\left| \sum_{k \geq 0} (\rho_k(y) - \rho_k(y')) Df_k(x') \cdot v \right| \\
&\qquad \leq \left| \sum_{k \geq 0} \rho_k(y) Df_k(x') \cdot v - Df_p(x') \cdot v \right| + \left| \sum_{k \geq 0} \rho_k(y') Df_k(x') \cdot v - Df_m(x') \cdot v \right| \\
&\qquad\quad + |Df_p(x') \cdot v - Df_m(x') \cdot v| \\
&\qquad \leq C\epsilon |Df_p(x') \cdot v| + C\epsilon |Df_m(x') \cdot v| + C\delta |Df_p(x') \cdot v| \\
&\qquad \leq C\delta |Df_p(x') \cdot v|.
\end{aligned}$$

The proof of (3.98) follows from (3.90) and (3.97). Indeed,

$$\begin{aligned}
&\left| \sum_{k \geq 0} \rho_k(y) D(R_k(x') \cdot y) \cdot v - \rho_k(y') D(R_k(x') \cdot y') \cdot v \right| \\
&\qquad\qquad \leq C\epsilon |Df_p(x') \cdot v| + C\delta |Df_p(x') \cdot v| + C\epsilon |Df_m(x') \cdot v| \\
&\qquad\qquad \leq C\delta |Df_p(x') \cdot v|.
\end{aligned}$$

Finally, we prove (3.99). We have

$$\begin{aligned}
\left| \frac{\partial g}{\partial y}(x', y) - \frac{\partial g}{\partial y}(x', y') \right| &\leq \sum_{k \geq 0} \left| \frac{\partial \rho_k}{\partial y}(y) \left\{ f_k(x') + R_k(x') \cdot y \right\} \right| + \left| \frac{\partial \rho_k}{\partial y}(y') \left\{ f_k(x') + R_k(x') \cdot y' \right\} \right| \\
&\quad + |(\rho_k(y) - \rho_k(y')) R_k(x') \cdot e_{d+1}| \\
&=: \delta_1 + \delta_2 + \delta_3.
\end{aligned}$$

We first handle δ_1 and δ_2. We have

$$\delta_1 \leq \left| \frac{\partial \rho_p}{\partial y}(y) \right| (|f_p(x') - f_{p+1}(x')| + |R_p(x') - R_{p+1}(x')||y|) \leq \frac{C}{r_p}(C\epsilon r_p + C\epsilon r_p) \leq C\epsilon$$

by (3.13) and (3.14). A nearly identical calculation gives the same bound for δ_2. We now handle δ_3. First, notice that

$$\left|\sum_{k\geq 0}\rho_k(y)R_k(x')\cdot e_{d+1}-R_p(x')\cdot e_{d+1}\right|\leq\sum_{k\geq 0}|\rho_k(y)||R_k(x')\cdot e_{d+1}-R_p(x')\cdot e_{d+1}|\leq C\epsilon \tag{3.103}$$

by (3.14). Because $R_k(x')$ is an isometry such that $R_k(x')(T\Sigma_0(x'))=T\Sigma_k(f_k(x))$, $R_k(x')\cdot e_{d+1}$ is the unit normal to $T\Sigma_k(f_k(x'))$ so that

$$|R_p(x')\cdot e_{d+1}-R_m(x')\cdot e_{d+1}|\leq C\,\mathrm{Angle}(T\Sigma_p(f_p(x')),T\Sigma_m(f_m(x')))\leq C\delta. \tag{3.104}$$

Finally, (3.103) and (3.104) imply

$$\begin{aligned}\delta_3\leq&\left|\sum_{k\geq 0}\rho_k(y)R_k(x')\cdot e_{d+1}-R_p(x')\cdot e_{d+1}\right|+\left|\sum_{k\geq 0}\rho_k(y')R_k(x')\cdot e_{d+1}-R_m(x')\cdot e_{d+1}\right|\\&+|R_p(x')\cdot e_{d+1}-R_m(x')\cdot e_{d+1}|\\\leq&\, C\epsilon+C\epsilon+C\delta\leq C\delta\left|\frac{\partial g}{\partial y}(y')\right|.\end{aligned}$$

where the final inequality uses (3.92). ■

Corollary 3.6.11. *Let z,z' be as in Lemma 3.6.10. Then for any vector $v\in H\cup V$, we have*

$$|Dg(x',y)\cdot v-Dg(x',y')\cdot v|\leq C\delta|Dg(x',y)\cdot v|$$

Proof. Suppose first that $v=v_x\in H$. Then using (3.93), we compute

$$|Dg(x',y)\cdot v_x-Dg(x',y')\cdot v_x| \tag{3.105}$$

$$=\left|\sum_{k\geq 0}(\rho_k(y)-\rho_k(y'))Df_k(x')\cdot v_x+\rho_k(y)D(R_k(x')\cdot y)\cdot v_x-\rho_k(y')D(R_k(x')\cdot y')\cdot v_x\right| \tag{3.106}$$

$$\begin{aligned}&\leq C\delta|Df_p(x')\cdot v_x|\leq C\delta|Dg(x',y)\cdot v_x|\\&\leq C\delta(1+C\delta)|Dg(x,y)\cdot v_x|\leq C\delta|Dg(x,y)\cdot v_x|\end{aligned} \tag{3.107}$$

using (3.97) and (3.98) in the first inequality, (3.90) in the second, and (3.94) in the third. If instead $v=v_y\in V$, then $Dg(x',y)=v_y\cdot\frac{\partial g}{\partial y}(x',y)$ and the result follows from (3.99) and (3.84). ■

Using Corollaries 3.6.9 and 3.6.11, we can prove Proposition 3.6.7.

Proof of Proposition 3.6.7. Let $z'=(x',y')\in G_z^{M_0}$. We will show that for any vector $v\in H\cup V$,

$$|Dg(x,y)\cdot v-Dg(x',y)\cdot v|\leq C\delta|Dg(x,y)\cdot v|. \tag{3.108}$$

The set $G_z^{M_0}$ is designed exactly so that $z' \in G_z^{M_0}$ implies that the hypotheses of Lemmas 3.6.8 and 3.6.10 are satisfied. Hence, we can apply Corollaries 3.6.9 and 3.6.11 so that

$$\begin{aligned}|Dg(x,y)\cdot v - &Dg(x',y')\cdot v| \\ &\leq |Dg(x,y)\cdot v - Dg(x',y)\cdot v| + |Dg(x',y)\cdot v - Dg(x',y')\cdot v| \\ &\leq C\delta|Dg(x,y)\cdot v| + C\delta|Dg(x',y)\cdot v| \\ &\leq C\delta|Dg(x,y)\cdot v|.\end{aligned}$$

By decomposing an arbitrary $v' \in \mathbb{R}^{d+1}$ as $v' = v_x + v_y$ where $v_x \in H$ and $v_y \in V$, we write

$$\begin{aligned}|Dg(x,y)\cdot v' - &Dg(x',y')\cdot v'| \\ &\leq |Dg(x,y)\cdot v_x - Dg(x',y')\cdot v_x| + |Dg(x,y)\cdot v_y - Dg(x',y')\cdot v_y| \\ &\leq C\delta(|Dg(x,y)\cdot v_x| + |Dg(x,y)\cdot v_y|) \\ &\leq C\delta|Dg(x,y)\cdot v'|, \end{aligned} \tag{3.109}$$

Where the final inequality follows from an application of the reverse triangle inequality in Lemma 3.6.6. We justify the application of the lemma by looking at the equations (3.93) and (3.91). These imply that the vector $Dg(x,y)\cdot v_x$ is nearly parallel to $T\Sigma_k(x)$ while the vector $Dg(x,y)\cdot v_y$ is nearly perpendicular to $T\Sigma_k(x)$ for some value of k where the deviations described are on the order of ϵ. This implies $|\langle Dg(x,y)\cdot v_x, Dg(x,y)\cdot v_y\rangle| \leq \frac{1}{2}|Dg(x,y)\cdot v_x|\cdot|Dg(x,y)\cdot v_y|$ so that the lemma applies. With this, we now compute,

$$\begin{aligned}|Dg(z')\cdot Dg(z)^{-1}\cdot v' - v'| = |[Dg(z') - Dg(z)]\cdot Dg(z)^{-1}\cdot v'| &\leq C\delta|Dg(z)\cdot Dg(z)^{-1}\cdot v'| \\ &= C\delta|v'|. \end{aligned}$$ ∎

This concludes the computations we need to bound the change in Dg. By integrating Dg over paths in a quasiconvex domain Ω, we get a companion result to Proposition 3.6.7 which roughly states that the map $g|_\Omega$ is a $(1 + C\delta)$-bi-Lipschitz perturbation of $Dg(z_0)$ for any $z_0 \in \Omega$. More precisely, for any $z \in \mathbb{R}^{d+1}$ define

$$L_{z_0}(z) = z_0 + Dg(z_0)(z - z_0). \tag{3.110}$$

This is the affine transformation which approximates g near z_0. Define

$$\varphi_{z_0} = g \circ L_{z_0}^{-1} \tag{3.111}$$

Proposition 3.6.12. *Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a quasiconvex domain with constant M_0 such that $\Omega \subseteq G_{z_0}^{M_0}$ for some $z_0 \in \Omega$ and M_0, ϵ be as in Proposition 3.6.7. Then the map $\varphi_{z_0} : L_{z_0}(\Omega) \to g(\Omega)$ is $(1 + C\delta)$-bi-Lipschitz and*

$$|D\varphi_{z_0}(w) - I| \leq C\delta \tag{3.112}$$

for all $w \in L_{z_0}(\Omega)$.

Proof. Because $w \in L_{z_0}(G_{z_0}^{M_0})$ by assumption, we get

$$D\varphi_{z_0}(w) = Dg(L_{z_0}^{-1}(w))\cdot DL_{z_0}^{-1}(w) = Dg(z)\cdot Dg(z_0)^{-1}$$

for $z' = L_{z_0}^{-1}(w) \in G_{z_0}^{M_0}$. Equation (3.112) follows from (3.6.7).

To prove that φ_{z_0} is $(1+C\delta)$-bi-Lipschitz, let $\gamma : [0,1] \to \mathbb{R}^{d+1}$ be a path with $\gamma(0) = z_0$, $\gamma(1) = z$, and $\ell(\gamma) \lesssim_{M_0} |z_0 - z|$. Put $\tilde{\gamma}(t) = L_{z_0}(\gamma(t))$ and $w_0 = L_{z_0}(z) = z$. Observe that

$$L_{z_0}^{-1}(w) = z_0 + Dg(z_0)^{-1}(w - z_0).$$

We estimate

$$\begin{aligned}
|\varphi_{z_0}(w) - \varphi_{z_0}(w_0)| &= \left| \int_0^1 D(g \circ L_{z_0}^{-1})(\tilde{\gamma}(t)) \cdot \tilde{\gamma}'(t) dt \right| \\
&= \left| \int_0^1 Dg(\gamma(t)) \cdot DL_{z_0}^{-1}(\tilde{\gamma}(t)) \cdot \tilde{\gamma}'(t) dt \right| \\
&= \left| \int_0^1 Dg(\gamma(t)) \cdot Dg(z_0)^{-1} \cdot \tilde{\gamma}'(t) dt \right| \\
&= \left| w - w_0 + \int_0^1 \left[Dg(\gamma(t)) \cdot Dg(z_0)^{-1} - I \right] \cdot \tilde{\gamma}'(t) dt \right|.
\end{aligned}$$

Using the fact that $\gamma(t) \in G_{z_0}^{M_0}$ for all t, Proposition 3.6.7 implies, on one hand

$$\begin{aligned}
|\varphi_{z_0}(w) - \varphi_{z_0}(w_0)| &\le |w - w_0| + \int_0^1 \left| Dg(\gamma(t)) \cdot Dg(z_0)^{-1} - I \right| \cdot |\tilde{\gamma}'(t)| dt \\
&\le |w - w_0| + C\delta |Dg(z_0)| \cdot \ell(\gamma) \\
&\le (1 + C\delta)|w - w_0|.
\end{aligned}$$

On the other,

$$\begin{aligned}
|\varphi_{z_0}(w) - \varphi_{z_0}(w_0)| &\ge |w - w_0| - \int_0^1 \left| Dg(\gamma(t)) \cdot Dg(z_0)^{-1} - I \right| \cdot |\tilde{\gamma}'(t)| dt \\
&\ge |w - w_0| - C\delta |Dg(z_0)| \cdot \ell(\gamma) \\
&\ge (1 - C\delta)|w - w_0|
\end{aligned}$$

where the final inequality on both hands comes from the fact that $|w' - w| = |Dg(z) \cdot (z' - z)| \le |Dg(z)| \cdot |z - z'|$ and our assumption that $\ell(\gamma) \lesssim_{M_0} |z - z'|$. ■

3.7 Graph coronizations for Reifenberg flat sets

The goal of this section is to provide a proof of Proposition 3.2.8 which states that there exist (sufficiently small in terms of d) constants $\epsilon, \delta > 0$ such that Reifenberg flat sets admit (M, ϵ, δ)-graph coronizations.

Reifenberg flat sets are a subset of a more general class of sets called *lower content d-regular sets* studied by Azzam and Schul [AS18] and later Hyde [Hyd22a] as a class of objects for d-dimensional traveling salesman results.

Definition 3.7.1 (lower content d-regularity)**.** A set $E \subseteq \mathbb{R}^{d+1}$ is said to be *lower content d-regular* in a ball $B(x, r)$ if there exists a constant $c > 0$ and $r_B > 0$ such that

$$\mathscr{H}_\infty^d(E \cap B(x, r)) \ge cr^d \text{ for all } x \in E \cap B \text{ and } r \in (0, r_B).$$

A set E is *lower content d-regular* if there exists a constant c such that E is lower content regular with constant c in every ball centered on E.

Since a Reifenberg flat set Σ satisfies $b\beta_{\Sigma}(B) \leq \epsilon$ in every ball by definition, the only remaining requirements for the existence of a graph coronization are control over $\beta_{\Sigma}^{d,1}$-squared sums and control over the frequency of angle turning of well-approximating planes. The necessary control over β-sums is contained in the following traveling salesman theorems formulated for general lower content regular sets:

Theorem 3.7.1 ([Hyd22a] Theorem 1.6)**.** *Let H be a Hilbert space and $1 \leq d < \dim(H)$, $1 \leq p < p(d)$, $C_0 > 1$, and $A > 10^5$. Let $E \subseteq H$ be a lower content d-regular set with regularity constant c and Christ-David cubes $\mathscr{D}$. There exists $\epsilon > 0$ small enough so that the following holds. Let $Q_0 \in \mathscr{D}$ and*

$$\beta_{E,C_0,d,p}(Q_0) = \ell(Q_0)^d + \sum_{Q \subseteq Q_0} \beta_E^{d,p}(C_0 B_Q)^2 \ell(Q)^d.$$

Then

$$\beta_{E,C_0,d,p}(Q_0) \lesssim_{A,d,c,p,C_0,\epsilon} \mathcal{H}^d(Q_0) + \mathrm{BWGL}(Q_0, A, \epsilon). \tag{3.113}$$

Theorem 3.7.2 ([Hyd22a] Theorem 1.7)**.** *Let H be a Hilbert space, $1 \leq d < \dim(H)$, $1 \leq p < \infty$, $A > 1$, $\epsilon > 0$, and $C_0 > 2\rho^{-1}$ where ρ is as in the construction of the Christ-David lattice $\mathscr{D}$. Let $E \subseteq H$ be lower content d-regular with regularity constant c and Christ-David cubes $\mathscr{D}$. For $Q_0 \in \mathscr{D}$, we have*

$$\mathcal{H}^d(Q_0) + \mathrm{BWGL}(Q_0, A, \epsilon) \lesssim_{A,d,c,C_0,\epsilon} \beta_{E,C_0,d,p}(Q_0).$$

If E is (ϵ, d)-Reifenberg flat, then the BWGL terms above vanish and (3.113) gives a Carleson packing condition for the content beta number sum reminiscent of the strong geometric lemma for uniformly rectifiable sets from which we will conclude the desired β^2 sum control.

We will require small technical tweaks of the stopping time machinery of Azzam and Schul on Reifenberg flat sets. We review the necessary definitions here, but refer to [AS18] sections 5-8 for a full treatment of the construction.

Definition 3.7.2 (d-dimensional traveling salesman stopping time)**.** We fix constants $0 < \epsilon \ll \alpha^4$ with $\alpha(d), \epsilon(d)$ to be chosen sufficiently small in terms of δ as required in [AS18] . For any cube $Q \in \mathscr{D}$, we define a stopping time region S_Q^{α} by adding cubes $R \subseteq Q$ to S_Q if

(i) $R^{(1)} \in S_Q^{\alpha}$,

(ii) $\mathrm{Angle}(P_U, P_Q) < \alpha$ for any sibling U of R (including R itself).

For any collection of cubes $\mathscr{Q}$, define a distance function

$$d_{\mathscr{Q}}(x) = \inf\{\ell(Q) + \mathrm{dist}(x, Q) : Q \in \mathscr{Q}\}.$$

For any $Q \in \mathscr{D}$, define

$$d_{\mathscr{Q}}(Q) = \inf_{x \in Q} d_{\mathscr{Q}}(x) = \inf\{\ell(R) + \mathrm{dist}(Q, R) : R \in \mathscr{Q}\}.$$

We let $m(S)$ be the set of minimal cubes of S, those which have no children contained in S and define

$$z(S) = Q(S) \setminus \bigcup_{Q \in m(S)} Q.$$

Let

$$\mathrm{Stop}(-1) = \mathscr{D}_0$$

and fix a small constant $\tau \in (0,1)$. Suppose we have defined $\mathrm{Stop}(N-1)$ for some integer $N \geq 0$ and define

$$\mathrm{Layer}(N) = \bigcup \{S_Q^\alpha : Q \in \mathrm{Stop}(N-1)\}.$$

We then set $\mathrm{Up}(-1) = \varnothing$ and put

$$\begin{aligned}\mathrm{Stop}(N) &= \{Q \in \mathscr{D} : Q \text{ maximal such that } Q \text{ has a sibling } Q' \text{ with } \ell(Q') < \tau d_{\mathrm{Layer}(N)}(Q')\},\\ \mathrm{Up}(N) &= \mathrm{Up}(N-1) \cup \{Q \in \mathscr{D} : Q \supset R \text{ for some } R \in \mathrm{Stop}(N) \cup \mathrm{Layer}(N)\}\end{aligned}$$

[AS18] Lemma 5.5 says that, in fact

$$\mathrm{Up}(N) = \{Q \in \mathscr{D} : Q \not\subset R \text{ for any } R \in \mathrm{Stop}(N)\}.$$

Essentially, $\mathrm{Layer}(N)$ is a layer of stopping time regions S_Q^α beginning at the stopped cubes of the previous generation and continuing until reaching a cube R with a child R' such that $\mathrm{Angle}(P_Q, P_{R'}) > \alpha$. $\mathrm{Stop}(N)$ is formed by taking a "smoothing" of $\mathrm{Layer}(N)$ that ensures that nearby minimal cubes in $\mathrm{Stop}(N)$ are of similar size. One forms a CCBP from the centers and $b\beta$-minimizing planes of cubes in $\mathrm{Up}(N)$ which gives a surface Σ_N for any $N \geq 0$ which converges to Σ as $N \to \infty$. Azzam and Schul give tools for proving bounds on the degree of stopping in this construction in the following lemma

Lemma 3.7.3 ([Hyd22a] Lemma 4.4 (5))**.** *Let Σ be (ϵ, d)-Reifenberg flat and $\mathscr{D}$ a Christ-David lattice for Σ. Let $N \geq 0$. For any $Q_0 \in \mathscr{D}$,*

$$\sum_{N \geq 0} \sum_{\substack{Q \in \mathrm{Stop}(N) \\ Q \subseteq Q_0}} \ell(Q)^d \lesssim_{d,\alpha,\epsilon} \mathcal{H}^d(Q_0)$$

Proof of Proposition 3.2.8. Fix $M \geq 1$, and $\epsilon, \alpha > 0$ sufficiently small in terms of M, d, n determined by Lemma 3.7.3 and Theorem 3.7.1 and let $\delta = 100\alpha$. Let $\mathscr{D}$ be a Christ-David lattice for Σ and let $\{P_Q\}_{Q \in \mathscr{D}}$ be a family of d-planes such that $x_Q \in P_Q$ and $\beta_\Sigma^{d,1}(MB_Q, P_Q) \leq 2\beta_\Sigma^{d,1}(MB_Q)$. Fix $Q_0 \in \mathscr{D}$ and form a collection of stopping time regions $\mathscr{F} = \{S_Q\}$ contained within Q_0 satisfying the stopping conditions Items (ii) and (iii) of Definition 3.2.9. We set $\mathscr{G} = \mathscr{D}$, $\mathscr{B} = \varnothing$. To prove that $\mathscr{C} = (\mathscr{G}, \mathscr{B}, \mathscr{F})$ is an (M, ϵ, δ)-graph coronization, we only need to show that $\mathscr{C}$ is a coronization, i.e., there exists a constant $C(M, \epsilon, \delta, d)$ such that

$$\sum_{S \in \mathscr{F}} \ell(Q(S))^d \leq C\mathcal{H}^d(Q_0).$$

Define

$$S_\delta = \{Q \in \mathscr{D} : \exists S \in \mathscr{F},\ Q \in \mathrm{Stop}(S),\ \mathrm{Angle}(P_Q, P_{Q(S)}) > \delta\},$$
$$S_\beta = \left\{ Q \in \mathscr{D} : \exists S \in \mathscr{F},\ Q \in \mathrm{Stop}(S),\ \sum_{Q \subseteq R \subseteq Q(S)} \beta_\Sigma^{d,1}(MB_R)^2 > \eta \right\}.$$

It suffices to show that $\sum_{Q \in S_\delta \cup S_\beta} \ell(Q)^d \leq C\mathcal{H}^d(Q_0)$. We define

$$\mathrm{Stop}(-1,\delta) = \{Q_0\},$$

and, given $\mathrm{Stop}(N-1,\delta)$ for some integer $N \geq 0$, we define

$$\mathrm{Stop}(N,\delta) = \{R \in S_\delta : R \text{ maximal such that } R \subseteq Q \in \mathrm{Stop}(N-1,\delta)\}.$$

With this, we have

$$S_\delta = \bigcup_{N \geq 0} \mathrm{Stop}(N,\delta).$$

We will use this to show that $\sum_{Q \in S_\delta} \ell(Q)^d \leq C\mathscr{H}^d(Q_0)$.

Fix $Q \in \mathrm{Stop}(N,\delta)$ and let $x \in Q \setminus z(S)$. Then there exists $R \in S_\delta,\ R \subset Q$ such that $x \in R$ and, since $\delta \geq 100\alpha$, there exists a cube $R' \in \mathrm{Stop}(K)$ for some $K \geq 0$ such that $R \subseteq R' \subseteq Q$. Set

$$\mathrm{Stop}(Q) = \left\{ R \in \mathscr{D} : R \text{ maximal such that } R \in \bigcup_{N \geq 0} \mathrm{Stop}(N) \text{ and } R \subseteq Q \right\}.$$

The above argument has shown that $Q \setminus z(S_Q) \subseteq \cup_{R \in \mathrm{Stop}(Q)} R$. We see

$$\ell(Q)^d \lesssim_d \sum_{R \in \mathrm{Stop}(Q)} \ell(R)^d + \mathscr{H}^d(z(S_Q)).$$

This means

$$\begin{aligned}
\sum_{Q \in S_\delta} \ell(Q)^d &= \sum_{N \geq 0} \sum_{Q \in \mathrm{Stop}(N,\delta)} \ell(Q)^d \\
&\lesssim_d \sum_{N \geq 0} \sum_{Q \in \mathrm{Stop}(N,\delta)} \left(\sum_{R \in \mathrm{Stop}(Q)} \ell(R)^d + \mathcal{H}^d(z(S_Q)) \right) \\
&\lesssim \mathcal{H}^d(Q_0) + \sum_{K \geq 0} \sum_{R \in \mathrm{Stop}(K)} \ell(R)^d \\
&\lesssim_{d,\delta,\epsilon} \mathcal{H}^d(Q_0)
\end{aligned}$$

where the penultimate line follows from the fact that $\mathrm{Stop}(Q) \cap \mathrm{Stop}(Q') = \varnothing$ for $Q, Q' \in S_\delta,\ Q \neq Q'$, and the final line follows from Lemma 3.7.3.

We now show that $\sum_{Q\in S_\beta}\ell(Q)^d \leq C\mathcal{H}^d(Q_0)$. We have

$$\sum_{Q\in S_\beta}\ell(Q)^d \leq \sum_{Q\in S_\beta}\ell(Q)^d\left[\epsilon^{-2}\sum_{\substack{Q\subseteq R\subseteq Q(S)\\ Q\in S\in\mathscr{F}}}\beta_\Sigma^{d,1}(MB_R)^2\right] = \epsilon^{-2}\sum_{R\in\mathscr{D}}\beta_\Sigma^{d,1}(MB_R)^2\sum_{\substack{Q\text{ maximal }\subseteq R\\ Q\in S_\beta}}\ell(Q)^d,$$
$$\lesssim \epsilon^{-2}\sum_{R\in\mathscr{D}}\beta_\Sigma^{d,1}(MB_R)^2\ell(R)^d \lesssim_{d,\eta} \mathcal{H}^d(Q_0) \qquad \blacksquare$$

using Theorem 3.7.1 in the last line.

Chapter 4

Uniformly rectifiable metric spaces satisfy the weak constant density condition

4.1 Introduction

The relation between the rectifiability properties and density properties of sets and measures has long been a topic of interest in geometric measure theory. For a metric space X, we say that $E \subseteq X$ with $\mathscr{H}^n(E) < \infty$ is n-rectifiable if E can be covered $\mathscr{H}^n$ almost everywhere by a countable union of Lipschitz images of subsets of $\mathbb{R}^n$. In Euclidean space, rectifiable sets give a natural generalization of C^1 manifolds. If $E \subseteq X$ is n-rectifiable, then for $\mathscr{H}^n$ almost every $x \in E$,

$$\lim_{r \to 0} \frac{\mathscr{H}^n(B(x,r) \cap E)}{(2r)^n} = 1 \tag{4.1}$$

as holds for an n-dimensional submanifold of Euclidean space (See [Kir94] Theorem 9 for a proof of (4.1)). In fact, if $E \subseteq \mathbb{R}^d$, then the above condition also implies n-rectifiability and hence gives a characterization of rectifiability equivalent to coverings by Lipschitz images. This was first proven by Marstrand for $n = 2,\ d = 3$ [Mar61] and Mattila for general n, d [Mat75]. Later, Preiss showed that any measure in $\mathbb{R}^d$ whose n-dimensional density merely exists and is finite $\mathscr{H}^n$-a.e. is rectifiable, generalizing this result [Pre87].

The weak constant density condition (WCD) is one of several conditions meant to provide an analog of (4.1) in the world of *uniform rectifiability* pioneered by David and Semmes in their foundational works [DS91] and [DS93].

Definition 4.1.1 (uniform n-rectifiability)**.** We say that a set $E \subseteq \mathbb{R}^d$ is *uniformly n-rectifiable* if there exists a constant $C_0 > 0$ such that E is *Ahlfors (C_0, n)-regular*, i.e., for all $x \in E$ and $0 < r < \mathrm{diam}(E)$,

$$C_0^{-1} r^n \leq \mathscr{H}^n(B(x,r) \cap E) \leq C_0 r^n, \tag{4.2}$$

and E has *Big Pieces of Lipschitz images of* $\mathbb{R}^n$ (BPLI), i.e., there exist constants $L, \theta > 0$ such that for all $x \in E$ and $0 < r < \mathrm{diam}(E)$, there exists an L-Lipschitz map $f : B(0,r) \subseteq$

$\mathbb{R}^n \to \mathbb{R}^d$ such that

$$\mathscr{H}^n(B(x,r) \cap E \cap f(B(0,r))) \geq \theta r^n. \tag{4.3}$$

One can think of this as a stronger form of n-rectifiability in which one requires a uniform percentage of the measure of each ball to be covered by a Lipschitz image. David and Semmes introduced the WCD as a way of quantifying (4.1) by requiring that in almost every *ball*, there exists a measure supported on the set with *nearly constant* density nearby.

Definition 4.1.2 (Weak constant density condition, Carleson sets and measures)**.** Let $E \subseteq \mathbb{R}^d$ be Ahlfors n-regular, let $C_0, \epsilon_0 > 0$, and define

$$\mathscr{G}_{\text{cd}}(C_0,\epsilon_0) = \left\{ (x,r) \in E \times \mathbb{R}^+ \;\middle|\; \begin{array}{l} \exists \text{ Ahlfors } (C_0,n)\text{-regular } \mu,\ \operatorname{spt}(\mu) = E, \\ \forall y \in B(x,r),\ 0 < t \leq r, \\ |\mu(E \cap B(y,t)) - t^n| \leq \epsilon_0 r^n \end{array} \right\}, \tag{4.4}$$

$$\mathscr{B}_{\text{cd}}(C_0,\epsilon_0) = E \times \mathbb{R}^+ \setminus \mathscr{G}_{\text{cd}}(C_0,\epsilon_0). \tag{4.5}$$

We say that E satisfies the weak constant density condition if there exists $C_0 > 0$ such that for all $\epsilon_0 > 0$, $\mathscr{B}_{\text{cd}}(C_0,\epsilon_0)$ is a Carleson set. That is, there exists a constant $C_1 > 0$ such that for all $z \in E$ and $0 < r < \operatorname{diam}(E)$,

$$\int_{B(z,r)} \int_0^r \chi_{\mathscr{B}_{\text{cd}}(C_0,\epsilon_0)}(x,t)\, d\mathscr{H}^n(x) \frac{dt}{t} \leq C_1 r^n.$$

If this holds, we say that $\chi_{\mathscr{B}_{\text{cd}}(C_0,\epsilon_0)} d\mathscr{H}^n(x)\frac{dt}{t}$ is a *Carleson measure* and say that $\mathscr{B}_{\text{cd}}(C_0,\epsilon_0)$ is C_1-*Carleson.*

For related quantitative conditions involving densities, see [Cho+16], [AH22], and [TT15]. The work of David, Semmes, and Tolsa combine to prove the following Theorem:

Theorem 4.1.1. *Let $E \subseteq \mathbb{R}^d$ be Ahlfors n-regular. Then E is uniformly n-rectifiable if and only if E satisfies the WCD.*

David and Semmes proved the forward implication in Chapter 6 of [DS91] using a characterization of uniform rectifiability (condition C2 of [DS91]) more closely related to the boundedness of singular integral operators. We will say more about this when we discuss our result.

They proved the reverse implication only in the case $n = 1, 2$, and $d-1$. Their proof uses the fact that if a measure is very close to having constant density in a large neighborhood of scales and locations, then its support is well-approximated by the support of an *n-uniform measure*, a measure μ for which there exists $c > 0$ such that $\mu(B(x,r)) = cr^n$ for all $x \in \operatorname{spt}(\mu)$ and $r > 0$. Because uniform measures in Euclidean space are completely classfied for $n = 1, 2$ (they are all multiples of Hausdorff measure on a plane) and for $n = d-1$ (they are Hausdorff measure on products of planes and light cones [KP87]), David and Semmes are able to show that a WCD set is very close to flat on most balls which are good for the WCD. The absence of a classification for uniform measures in intermediate dimensions prevented a direct adaptation of their arguments. However, Tolsa completed the proof of the reverse direction in Theorem 4.1.1 in [Tol15] by replacing elements of David and Semmes's argument specific to their examples of uniform measures with general flatness properties of uniform measures derived by Preiss [Pre87] in addition to new arguments using the Riesz transform.

In general, classifying uniform measures is a difficult open problem, but see [Nim22] for an interesting family of examples. For further studies of uniform measures in Euclidean spaces see [Pre87] (and [De 08] for a more gentle presentation of Preiss), [KP02], [Nim17], and [Nim19]. For research into uniform measures in the Heisenberg group see [CMT20] and [Mer22] and for a related result in ℓ^3_∞, see [Lor03].

Given the fact that (4.1) is valid even in rectifiable metric spaces, it is natural to ask whether there is some extension of the above theory to *uniformly rectifiable metric spaces*, i.e., metric spaces which are Ahlfors n-regular and have big pieces of Lipschitz images of subsets of $\mathbb{R}^n$. In this paper, we extend a piece of this theory by proving the following theorem.

Theorem F. *Uniformly n-rectifiable metric spaces satisfy the WCD.*

Our proof is made possible by the recent work of Bate, Hyde, and Schul [BHS23] which adapted a substantial portion of the theory of uniformly rectifiable subsets of Euclidean spaces to metric spaces. While our argument uses this theory, it does not follow David and Semmes's original proof very closely. Roughly speaking, David and Semmes proved the Euclidean version of Theorem F by showing that for most balls centered on an $n-$uniformly rectifiable set E, there exists a d-plane P such that the pushforward of Hausdorff measure for E onto P must be very close to symmetric. They do this by showing that the non-symmetric balls contribute substantially to the value of a Carleson measure defined using integrals over E of a family of smooth odd functions designed to detect asymmetry.

We prove Theorem F by first proving that bi-Lipschitz images satisfy the WCD. Then, using the fact that uniformly rectifiable spaces have very big pieces of bi-Lipschitz images in Banach spaces proven by Bate, Hyde, and Schul [BHS23], we adapt the bi-Lipschitz image arguments to the general case. To handle the bi-Lipschitz image case, we first prove Lemma 4.3.3, a form of quantitative Lebesgue differentiation theorem for L^2 functions very similar to theorems considered by David and Semmes (see [DS93] Lemma IV.2.2.14, Corollary IV.2.2.19, etc.), although our proof proceeds by contradiction and uses a compactness argument, a method which differs significantly from their proofs. We apply this lemma to the Jacobian of the metric derivative of our bi-Lipschitz function $f : \mathbb{R}^n \to \Sigma$ to control the variation of its averages over neighborhoods of scales and locations and receive control over the variation of the Hausdorff measure of Σ using the area formula. To the knowledge of the author, this gives a new proof of the WCD even in the Euclidean case.

We note here that the naive converse of Theorem 4.1.1 is false: There exist Ahlfors regular metric spaces which satisfy the WCD, yet are not uniformly rectifiable. Indeed, the metric space $(X, d) = (\mathbb{R}, |\cdot|^{1/2}_{\text{Euc}})$ is in fact 2-uniform: $\mathscr{H}^2(B(x,r)) = cr^2$ for all $x \in \mathbb{R}$ and $r \geq 0$, hence X satisfies the weak constant density condition, yet X is purely 2-unrectifiable. Some different examples of this failure are given by Bate [Bat23]. He proves that every 1-uniform metric measure space is either $\mathbb{R}$, a particular union of disjoint circles of radius d, or a purely unrectifiable "limit" of the circle spaces. These last two spaces are examples of 1-uniform spaces which are not uniformly rectifiable.

Analyzing connectedness plays a special role in the proof because any 1-uniform connected component must be locally isometric to $\mathbb{R}$, implying any connected 1-uniform space is itself $\mathbb{R}$.

From these examples, it seems reasonable to think that some connectedness and topological conditions are necessary hypotheses for any type of converse to hold. It also follows from work of Schul [Sch07a], [Sch09], and Fassler and Violo [FV23] (see also [Hah05]) that any Ahlfors 1-regular connected subset of a metric space is uniformly 1-rectifiable, although perhaps adding some form of weaker hypothesis could provide an interesting converse to our result in the one-dimensional case using Bate's classification.

4.2 Preliminaries

Whenever we write $A \lesssim B$, we mean that there exists some constant C independent of A and B such that $A \leq CB$. If we write $A \lesssim_{a,b,c} B$ for some constants a, b, c, then we mean that the implicit constant C mentioned above is allowed to depend on a, b, c. We will sometimes write $A \asymp_{a,b,c} B$ to mean that both $A \lesssim_{a,b,c} B$ and $B \lesssim_{a,b,c} A$ hold. We use the notation $f : E \twoheadrightarrow F$ to mean f is a surjective map from E to F.

Let (X, d) be a metric space. For any subset $F \subseteq X$, integer $n \geq 0$, and constant $0 < \delta \leq \infty$, we define

$$\mathscr{H}^n_\delta(F) = \inf\left\{ \sum \operatorname{diam}(E_i)^d : F \subseteq \bigcup E_i,\ \operatorname{diam}(E_i) < \delta \right\}$$

where $\operatorname{diam}(E) = \sup_{x,y\in E} d(x, y)$. The Hausdorff n-measure of F is defined as

$$\mathscr{H}^n(F) = \lim_{\delta\to 0} \mathscr{H}^m_\delta(F).$$

Occasionally, we will specify a subset $\Sigma \subseteq X$ and write $\mathscr{H}^n_\Sigma = \mathscr{H}^n|_\Sigma$. For any $\mathscr{H}^n$ measurable $A \subseteq X$ and measurable $f : A \subseteq X \to \mathbb{R}$, we define

$$⨍_A f = \frac{1}{\mathscr{H}^n(A)} \int_A f(x) d\mathscr{H}^n(x).$$

We let $\mathcal{D}(\mathbb{R}^n)$ denote the family of dyadic cubes in $\mathbb{R}^n$. For $Q \in \mathcal{D}(\mathbb{R}^n)$, we let $\ell(Q)$ denote the side length of Q. If $R \in \mathcal{D}(\mathbb{R}^n)$ and $k \in \mathbb{Z}$, we define

$$\begin{aligned} \mathcal{D}(R) &= \{ Q \in \mathcal{D}(\mathbb{R}^n) \mid Q \subseteq R \}, \\ \mathcal{D}_k(R) &= \left\{ Q \in \mathcal{D}(R) \;\middle|\; \ell(Q) = 2^{-k}\ell(R) \right\}. \end{aligned}$$

We will also need a version of "cubes" associated to a metric space. David [Dav88] introduced this idea first, and it was later generalized by [Chr90] and [HM12]. The following formulation draws most from the latter two.

Theorem 4.2.1 (Christ-David cubes)**.** *Let X be a doubling metric space. Let X_k be a nested sequence of maximal ρ^k-nets for X where $\rho < 1/1000$ and let $c_0 = 1/500$. For each $k \in \mathbb{Z}$ there is a collection $\mathscr{D}_k$ of "cubes," which are Borel subsets of X such that the following hold.*

(i) $X = \bigcup_{Q\in\mathscr{D}_k} Q$.

(ii) *If $Q, Q' \in \mathscr{D} = \bigcup \mathscr{D}_k$ and $Q \cap Q' \neq \emptyset$, then $Q \subseteq Q'$ or $Q' \subseteq Q$.*

(iii) *For $Q \in \mathscr{D}$, let $k(Q)$ be the unique integer so that $Q \in \mathscr{D}_k$ and set $\ell(Q) = 5\rho^{k(Q)}$. Then there is $x_Q \in X_k$ so that*

$$B(x_Q, c_0\ell(Q)) \subseteq Q \subseteq B(x_Q, \ell(Q))$$

and

$$X_k = \{x_Q : Q \in \mathscr{D}_k\}.$$

(iv) *If X is Ahlfors n-regular, then there exists $C \geq 1$ such that*

$$\mathscr{H}^n(\left\{ x \in Q \;\middle|\; d(x, X \setminus Q) \leq \eta\rho^k \right\}) \lesssim \eta^{1/C}\ell(Q)^n$$

for all $Q \in \mathscr{D}$ and $\eta > 0$.

In addition, we define

$$B_Q = B(x_Q, \ell(Q)).$$

In analogy to the dyadic cube notation, for any $R \in \mathscr{D}$ and $k \in \mathbb{Z}$ we also write

$$\mathscr{D}(R) = \{ Q \in \mathscr{D} \mid Q \subseteq R \},$$
$$\mathscr{D}_k(R) = \left\{ Q \in \mathscr{D}(R) \;\middle|\; \ell(Q) = \rho^{-k}\ell(R) \right\}.$$

We will actually prove a form of the WCD adapted to Christ-David cubes. The following two lemmas will allow us to show that the cube WCD in Definition 4.4.1 implies the WCD from Definition 4.1.2. Recall from Theorem 4.2.1 that $c_0 = \frac{1}{500}$.

Lemma 4.2.2. *Let X be a doubling metric space with doubling constant C_d. There exists $N(C_d) \in \mathbb{N}$ such that the following holds: There exist N Christ-David systems of cubes $\{\mathscr{D}_i\}_{i=1}^N$ for X such that for any $x \in X$, $0 < t < \operatorname{diam}(X)$, there exists $i \in \{1, \ldots, N\}$ and $Q \in \mathscr{D}_i$ with $\ell(Q) \leq \frac{5}{\rho c_0}t$ such that $x \in \frac{c_0}{4}B_Q$ and $t < \frac{c_0}{4}\ell(Q)$.*

Proof. Fix $\rho < \frac{1}{1000}$. For each k, let $\tilde{X}_k$ be a maximal $c_0\rho^k$-net for X. We now iteratively construct maximal ρ^k-nets $X_k^1, X_k^2, \ldots, X_k^N, \ldots$ in the following way. Let X_k^1 be a completion of a maximal ρ^k-separated subset of $\tilde{X}_k$ to a maximal ρ^k-net for X. Given X_k^i for any $i > 0$, construct X_k^{i+1} by completing a maximal ρ^k-separated subset of $Y_k^i := \tilde{X}_k \backslash (X_k^1 \cup X_k^2 \cup \ldots \cup X_k^i)$ to a maximal ρ^k-net for X. We claim that this process terminates in $N(C_d)$ steps, giving for each $k \in \mathbb{Z}$ a collection of maximal ρ^k-nets $X_k^1, \ldots, X_k^N$. Indeed, let B be a ball of radius $2\rho^k$. By doubling, there exists $N(C_d) < \infty$ such that $\#(B \cap \tilde{X}_k) \leq N(C_d)$. Suppose that $\frac{1}{2}B \cap Y_k^j \neq \varnothing$ for some $j > 0$. Then, because X_k^{j+1} is maximal, there exists some $x \in B \cap Y_k^j$ such that $x \in X_k^{j+1}$. Therefore, $\#(B \cap Y_k^{j+1}) < \#(B \cap Y_k^j)$ whenever $\frac{1}{2}B \cap Y_k^j \neq \varnothing$. This means $\frac{1}{2}B \cap Y_k^{N+1} = \varnothing$ for any such B, implying $Y_k^{N+1} = \varnothing$ and $\tilde{X}_k \subseteq \cup_{i=1}^N X_k^i$ as desired.

We now show that the lemma follows from this. Recall that Theorem 4.2.1 takes as input a collection $\{X_k\}_{k\in\mathbb{Z}}$ of maximal ρ^k-nets for X and outputs a system of cubes $\mathscr{D}$ such that every $x_k^\alpha \in X_k$ is the "center" of a cube $Q_k^\alpha \in \mathscr{D}$ with $B_X(x_k^\alpha, c_0 5\rho^k) = c_0 B_{Q_k^\alpha} \subseteq Q_k^\alpha$. We apply Theorem 4.2.1 to the collection $\{X_k^i\}_{k\in\mathbb{Z}}$ for every $1 \leq i \leq N$ and receive a Christ-David system $\mathscr{D}_i$ such that each point $\tilde{x}_k \in \tilde{X}_k$ is the center of some $Q \in \mathscr{D}_i$ for some i. So, let $x \in X$, $0 < t < \operatorname{diam}(X)$, and let $k \in \mathbb{Z}$ such that $c_0\rho^{k-1} \leq t < c_0\rho^k$. Because $\tilde{X}_k$ is a maximal $c_0\rho^k$-net for X, there exists $\tilde{x}_k \in \tilde{X}_k$ such that $d(x, \tilde{x}_k) < c_0\rho^k$. Because $\tilde{X}_k \subseteq \cup_{i=1}^N X_k^i$, there then exists $1 \leq i \leq N$ and $Q \in \mathscr{D}_i$ such that $\tilde{x}_k = x_Q$ so that $x \in B(x_Q, c_0\rho^k) = \frac{1}{5}B(x_Q, c_0\ell(Q)) = \frac{c_0}{5}B_Q$. Similarly, $\frac{\rho c_0}{5}\ell(Q) \leq t < \frac{c_0}{5}\ell(Q)$. ■

Lemma 4.2.3. *Let X be Ahlfors (C_0, n)-regular. If X satisfies the cube WCD of Definition 4.4.1, then X satisfies the WCD.*

Proof. We first note that if an $a_Q > 0$ as in (4.14) exists, then $(2C_0)^{-1} \leq a_Q \leq 2C_0$ by Ahlfors regularity. Therefore, whenever $|\mathscr{H}^n(X \cap B(y,r)) - a_Q r^n| \leq \epsilon_0 \ell(Q)$, we have

$$|(a_Q)^{-1}\mathscr{H}^n(X \cap B(y,r)) - r^n| = (a_Q)^{-1}|\mathscr{H}^n(X \cap B(y,r)) - a_Q r^n| \leq (2C_0)^{-1}\epsilon_0\ell(Q).$$

This means that one can replace $\mathscr{H}^n$ with a multiple of $\mathscr{H}^n$ and $a_Q r^n$ with r^n in the definition of the cube WCD at the cost of increasing ϵ_0. Therefore, it suffices to show that the complement of

$$\mathscr{G}_0(C_0, \epsilon_0) = \left\{ (x,t) \in X \times \mathbb{R}^+ \;\middle|\; \begin{array}{l} \exists a_{(x,t)} > 0, \text{ such that } \forall y \in B(x,t),\ 0 < r \leq t, \\ |\mathscr{H}^n(X \cap B(y,r)) - a_{(x,t)} r^n| \leq \epsilon_0 t^n \end{array} \right\}$$

is a Carleson set. In order to show this, we apply Lemma 4.2.2 to X and receive a finite number of Christ-David systems $\{\mathscr{D}_i\}_{i=1}^N$ with N depending only on n and C_0 such that for any $x \in X,\ 0 < t < \mathrm{diam}(X)$, there exists $i \in \{1, \ldots, N\}$ and $Q \in \mathscr{D}_i$ with $\ell(Q) \lesssim t$ such that $x \in \frac{c_0}{4}B_Q$ and $t < \frac{c_0}{4}\ell(Q)$. It follows that if if $Q \in \mathscr{G}_{\mathrm{cd}}(C_0, \epsilon_0)$, then $(x,t) \in \mathscr{G}_0(C_0, C(n)\epsilon_0)$ for any $x \in \frac{c_0}{4}B_Q$ and $\frac{\rho^2 c_0}{4}\ell(Q) \leq t < \frac{c_0}{4}\ell(Q)$ by choosing $a_{(x,t)} = a_Q$. Therefore, if X satisfies the cube WCD then $\mathscr{D}_i$ has a Carleson packing condition for its bad set, implying a Carleson condition for the bad balls of the WCD with a larger choice of ϵ_0 and with larger Carleson constant. ■

4.3 Oscillation of means of L^2 functions

In this section, we review necessary facts about wavelets and prove Lemma 4.3.3, one of our main tools for the proof of Theorem F.

Definition 4.3.1. We follow the presentation of [Tol12]. Given $h : \mathbb{R}^n \to \mathbb{R}$ and $Q \in \mathcal{D}(\mathbb{R}^n)$, define

$$\Delta_Q h(x) = \begin{cases} \fint_P h(z)dz - \fint_Q h(z)dz & \text{if } x \in P, \text{ where } P \text{ is a child of } Q, \\ 0 & \text{otherwise} \end{cases}$$

If $h \in L^2(\mathbb{R}^n)$, then

$$h = \sum_{Q \in \mathcal{D}(\mathbb{R}^n)} \Delta_Q h \quad \text{and} \quad h\chi_Q = \fint_Q h + \sum_{R \subseteq Q} \Delta_R h$$

where the sums converge in L^2 and $\langle \Delta_Q h, \Delta_{Q'} h\rangle_{L^2} = 0$ when $Q \neq Q'$ so that $\|h\|_2 = \sum_{Q\in\mathcal{D}} \|\Delta_Q h\|_2^2$. One can view $\Delta_Q h$ as a projection of h onto the subspace of L^2 formed by the Haar wavelets $h_Q^\epsilon,\ \epsilon \in \{0,1\}^n \setminus \{(0,0,\ldots,0)\}$ associated to Q. For any $k \in \mathbb{N}$, define

$$\Delta_k^h(Q)^2 = \sum_{j=0}^{k} \sum_{R \in \mathcal{D}_j(Q)} \|\Delta_R h\|_2^2. \tag{4.6}$$

Remark 4.3.1 (Properties of Δ_k^h). Notice that if $h \in L^\infty(\mathbb{R}^n)$, then Δ_k^h has a form of geometric lemma since

$$\sum_{Q\subseteq Q_0} \Delta_k^h(Q)^2 = \sum_{Q\subseteq Q_0} \sum_{j=0}^{k} \sum_{R\in\mathcal{D}_j(Q)} \|\Delta_R h\|_2^2 \lesssim_{k,n} \sum_{R\subseteq Q_0} \|\Delta_R h\|_2^2 \lesssim_{\|h\|_\infty} \ell(Q_0)^n.$$

This gives the Carleson condition

$$\sum_{\substack{Q\subseteq Q_0 \\ \Delta_k^h(Q)>\delta\ell(Q)^{n/2}}} \ell(Q)^n \lesssim_\delta \sum_{Q\subseteq Q_0} \Delta_k^h(Q)^2 \lesssim_{k,n,\|h\|_\infty} \ell(Q_0)^n. \tag{4.7}$$

Δ_k^h also scales appropriately in the following manner: Let $Q, \tilde{Q} \in \mathcal{D}(\mathbb{R}^n)$ and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the affine map sending $\tilde{Q}$ onto Q by

$$T(x) = x_Q + \left(\frac{x - x_{\tilde{Q}}}{\ell(\tilde{Q})}\right) \ell(Q) \tag{4.8}$$

where $x_{\tilde{Q}}$ is the center of $\tilde{Q}$. Let $\tilde{h} \in L^2(\tilde{Q})$ and set $h = \left(\frac{\ell(\tilde{Q})}{\ell(Q)}\right)^{n/2} \tilde{h} \circ T^{-1}$, a scaled and translated version of $\tilde{h}$ satisfying $\|h\|_2 = \|\tilde{h}\|_2$ because, by change of variables,

$$\int h^2 = \frac{\ell(\tilde{Q})^n}{\ell(Q)^n} \int (\tilde{h} \circ T^{-1})^2 = \frac{\ell(\tilde{Q})^n}{\ell(Q)^n} \int \tilde{h}^2 \frac{\ell(Q)^n}{\ell(\tilde{Q})^n} = \int \tilde{h}^2.$$

Similarly, notice that if $V \subseteq Q$ and $\tilde{V} \subseteq \tilde{Q}$ with $T(\tilde{V}) = V$, then

$$\begin{aligned} \|\Delta_V h\|_2^2 &= \int_V (\Delta_V h(x))^2 dx = \int_{T(\tilde{V})} (\Delta_{\tilde{V}} \tilde{h}(T^{-1}(x)))^2 dx \\ &= \left(\frac{\ell(Q)}{\ell(\tilde{Q})}\right)^n \int_{\tilde{V}} (\Delta_{\tilde{V}} \tilde{h})^2 dx = \left(\frac{\ell(Q)}{\ell(\tilde{Q})}\right)^n \|\Delta_{\tilde{V}} \tilde{h}\|_2^2 \end{aligned}$$

which gives $\Delta_k^h(Q)^2 = \left(\frac{\ell(Q)}{\ell(\tilde{Q})}\right)^n \Delta_k^{\tilde{h}}(\tilde{Q})^2$.

Definition 4.3.2 (normed balls)**.** Given $L > 0$, we define the set of norms on $\mathbb{R}^n$ which are L-bi-Lipschitz to the Euclidean norm by

$$\mathcal{N}_L = \{\|\cdot\| \;:\; L^{-1}\|x\| \le |x| \le L\|x\|\}.$$

Given a dyadic cube $Q \in \mathcal{D}(\mathbb{R}^n)$ and $L > 0$, we define a collection of L-bi-Lipschitz normed balls inside Q by

$$\mathcal{B}_L(Q) = \{B_{\|\cdot\|}(x, r) \subseteq Q \;:\; \|\cdot\| \in \mathcal{N}_L,\; r \ge L^{-1}\ell(Q)\}.$$

The following lemma gives a form of compactness result for $\mathcal{B}_L(Q)$.

Lemma 4.3.2. *Fix $Q \in \mathcal{D}(\mathbb{R}^n)$ and $L > 0$. Let $B_j = B_{\|\cdot\|_j} \in \mathcal{B}_L(Q)$ for $j \in \mathbb{N}$. There exists a subsequence of $B_j = B_{\|\cdot\|_j}(x_j, r_j)$ and a normed ball $B = B_{\|\cdot\|}(x, r) \in \mathcal{B}_L(Q)$ for which the following holds: For every $\eta > 0$, there exists $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$,*

$$B_{\|\cdot\|}(x, (1-5\eta)r) \subseteq B_j \subseteq B_{\|\cdot\|}(x, (1+5\eta)r). \tag{4.9}$$

Proof. Because $\|\cdot\|_j \in \mathcal{N}_L$ for all j, the functions $f_j : B(0,1) \to \mathbb{R}$ given by $f_j(x) = \|x\|_j$ are an equicontinuous, uniformly bounded family of continuous functions. Therefore, they subconverge uniformly to some limit function $f : B(0,1) \to \mathbb{R}$. It is straightforward to show this function gives a norm $\|\cdot\|$ when extended homogeneously to $\mathbb{R}^n$. By passing to further subsequences, we can assume that $x_j \to x \in Q$ and $r_j \to r$ with $L^{-1}\ell(Q) \leq r$. We set $B = B_{\|\cdot\|}(x, r)$ and fix $\eta > 0$ as in the statement of the lemma. By the convergences assumed, we can take j_0 large enough such that for $j \geq j_0$, we have $\|x_j - x\| \leq \eta \min\{r, r_j\}$, $|r_j - r| \leq \eta \min\{r, r_j\}$, and

$$(1-\eta)\|x\| \leq \|x\|_j \leq (1+\eta)\|x\|, \text{ for all } x \in \mathbb{R}^n,$$

We now aim to prove (4.9). Let $y \in B_j$. Then

$$\begin{aligned}\|y - x\| &\leq \|y - x_j\| + \|x_j - x\| \leq (1+2\eta)\|y - x_j\|_j + \eta r \\ &\leq (1+2\eta)(1+\eta)r + \eta r \leq (1+5\eta)r.\end{aligned}$$

so that $y \in B_{\|\cdot\|}(x, (1+5\eta)r)$. On the other hand, if $z \in B_{\|\cdot\|}(x, (1-5\eta)r)$, then

$$\begin{aligned}\|z - x_j\|_j &\leq \|z - x\|_j + \|x - x_j\|_j \leq (1+2\eta)\|z - x\| + (1+2\eta)\|x - x_j\| \\ &\leq (1+2\eta)(1-5\eta)r_j + (1+2\eta)\eta r_j \leq r_j\end{aligned}$$

showing $z \in B_j$. ■

Lemma 4.3.3. *For all $\epsilon, M, L > 0$, $n \in \mathbb{N}$, there exist $k(\epsilon, M, L, n) \in \mathbb{N}$ and $\delta(\epsilon, M, L, n) > 0$ such that the following holds: Let $h \in L^2(\mathbb{R}^n)$ with $h \geq 0$ and $\|h\|_2 \leq M$. Let $Q \in \mathcal{D}(\mathbb{R}^n)$. If $\Delta_k^h(Q) \leq \delta \ell(Q)^{n/2}$, then for any normed ball $B \in \mathcal{B}_L(Q)$, we have*

$$\left| \fint_B h - \fint_Q h \right| \leq \epsilon \left| \fint_Q h \right|. \tag{4.10}$$

Proof. Suppose the conclusion of the lemma is false. Then there exist ϵ, M, L, n and a sequence of maps $\tilde{h}_j \in L^2(\mathbb{R}^n)$ with $\tilde{h}_j \geq 0$ and $\|\tilde{h}_j\|_2 \leq M$, cubes $\tilde{Q}_j \in \mathcal{D}(\mathbb{R}^n)$, and normed balls $\tilde{B}_j \in \mathcal{B}_L(\tilde{Q}_j)$ so that (4.10) does not hold for $\tilde{h}_j, \tilde{Q}_j, \tilde{B}_j$, yet $\Delta_j^{\tilde{h}_j}(\tilde{Q}_j) \leq \frac{1}{j}\ell(\tilde{Q}_j)^{n/2}$. For any $j \in \mathbb{N}$, let T_j be the affine transformation sending $\tilde{Q}_j$ onto $Q = [0,1]^n$ as in (4.8) and define a scaled and translated copy of $\tilde{h}_j$ called $h_j : Q \to \mathbb{R}$ as in Remark 4.3.1 by

$$h_j = \chi_Q \left(\frac{\ell(\tilde{Q}_j)}{\ell(Q)} \right)^{n/2} \tilde{h}_j \circ T_j^{-1}$$

Let $c_j = \fint_Q h_j$ and observe that $\Delta_j^{h_j}(Q) = \left(\frac{\ell(Q)}{\ell(\tilde{Q}_j)}\right)^{n/2} \Delta_j^{\tilde{h}_j}(Q_j) \leq \frac{1}{j}\ell(Q)^{n/2} = \frac{1}{j}$ by Remark 4.3.1. In addition, we define $B_j = T_j(\tilde{B}_j)$ to be the appropriately translated and scaled copy of $\tilde{B}_j$. By the weak compactness of bounded closed balls in L^2, there exists some $h \in L^2(Q)$ such that $h_j \rightharpoonup h$ in L^2 for some subsequence of h_j. By further refining subsequences and using Lemma 4.3.2, we can further assume the subsequence is chosen so that a limiting normed ball $B = B_{\|\cdot\|}(x, r) \in \mathcal{B}_L(Q)$ as in the lemma's conclusion exists. Let $c = \fint_Q h$.

We will first show that $h = c$ by showing that $\Delta_V h = 0$ for all $V \subseteq Q$. By weak convergence we have

$$c_j = \fint_Q h_j \to \fint_Q h = c. \tag{4.11}$$

Write $h_j = c_j + \sum_{R \subseteq Q} \Delta_R h_j$ and $h = c + \sum_{R \subseteq Q} \Delta_R h$. Fix $V \subseteq Q$ and observe

$$\begin{aligned}\int_Q h_j \Delta_V h &= \int_Q \left(c_j + \sum_{R \subseteq Q} \Delta_R h_j \right) \Delta_V h = c_j \int_Q \Delta_V h + \sum_{R \subseteq Q} \langle \Delta_R h_j, \Delta_V h \rangle \\ &= \langle \Delta_V h_j, \Delta_V h \rangle.\end{aligned}$$

where the final equality follows since $\langle \Delta_R f_1, \Delta_V f_2 \rangle = 0$ whenever $f_1, f_2 \in L^2$ and $R \neq V$. Similarly, we have

$$\int_Q h \Delta_V h = \langle h, \Delta_V h \rangle = \langle \Delta_V h, \Delta_V h \rangle = \|\Delta_V h\|_2^2$$

Using weak convergence again, we get

$$\langle \Delta_V h_j, \Delta_V h \rangle = \int_Q h_j \Delta_V h \to \int_Q h \Delta_V h = \|\Delta_V h\|_2^2.$$

Using Cauchy-Schwarz, we can now conclude that $\|\Delta_V h\|_2 \leq \lim_j \|\Delta_V h_j\|_2$. We claim that $\|\Delta_V h\|_2 = 0$. Indeed, if j is sufficiently large, then both $V \in \mathcal{D}_j(Q)$ and $\|\Delta_V h\|_2 \leq 2\|\Delta_V h_j\|_2$. This means

$$\|\Delta_V h\|_2^2 \leq 4\|\Delta_V h_j\|^2 \leq 4 \sum_{k=0}^{j} \sum_{R \in \mathcal{D}_k(Q)} \|\Delta_R h_j\|_2^2 = 4\Delta_j^{h_j}(Q) \leq \frac{4}{j}.$$

for all large j. This shows that $\Delta_V h = 0$ for all $V \subseteq Q$, hence $h = c$ as desired.

We will now show how this leads to a contradiction. Let $\eta > 0$ and choose j large enough so that

$$B_1 := B_{\|\cdot\|}(x, (1 - 5\eta)r) \subseteq B_j \subseteq B_{\|\cdot\|}(x, (1 + 5\eta)r) =: B_2.$$

Using the fact that $h_j \geq 0$ (this is the first use of this hypothesis),

$$\int_{B_1} h_j \leq \int_{B_j} h_j \leq \int_{B_2} h_j \tag{4.12}$$

so that

$$\frac{\mathscr{L}^n(B_1)}{\mathscr{L}^n(B_j)} \fint_{B_1} h_j - \fint_Q h_j \leq \fint_{B_j} h_j - \fint_Q h_j \leq \frac{\mathscr{L}^n(B_2)}{\mathscr{L}^n(B_j)} \fint_{B_2} h_j - \fint_Q h_j.$$

Because $\mathscr{L}^n(B_1) \leq \mathscr{L}^n(B_j) \leq \mathscr{L}^n(B_2)$ and $\frac{\mathscr{L}^n(B_2)}{\mathscr{L}^n(B_1)} \leq (1 + c'(n)\eta)$, we can assume without loss of generality that

$$\limsup_j \left| \fint_{B_j} h_j - \fint_Q h_j \right| \leq \limsup_j \left| \frac{\mathscr{L}^n(B_2)}{\mathscr{L}^n(B_1)} \fint_{B_2} h_j - \fint_Q h_j \right| \leq |(1 + c'\eta)c - c| \lesssim_n \eta c.$$

Since this holds for all $\eta > 0$, we get $\limsup_j \left| \fint_{B_j} h_j - \fint_Q h_j \right| = 0$. On the other hand, by hypothesis

$$\left| \fint_{B_j} h_j - \fint_Q h_j \right| > \epsilon \left| \fint_Q h_j \right|$$

for all j so that $\limsup_j \left| \fint_{B_j} h_j - \fint_Q h_j \right| > \epsilon c \geq 0$, giving a contradiction. ■

Remark 4.3.4. Suppose that we only want to conclude (4.10) with normed balls $B \in \mathcal{B}_L(Q)$ replaced by $Q' \in \mathcal{D}_j(Q)$ for $j \leq k \in \mathbb{N}$. The following stronger condition holds even without the positivity assumption for h: Let $\alpha : \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n)$ where $\alpha(Q) \in \mathcal{D}_j(Q)$. There exists $C_0(k, n) > 0$ such that

$$\sum_{Q \subseteq R} \left| \fint_{\alpha(Q)} h - \fint_Q h \right|^2 \ell(Q)^n \leq C_0 \|h\|_2^2. \tag{4.13}$$

The proof is straightforward: because $\alpha(Q) \in \mathcal{D}_j(Q)$, there is a chain of at most $k+1$ cubes $\alpha(Q) = Q_j \subseteq Q_{j-1} \subseteq \ldots \subseteq Q_0 = Q$ such that Q_{j+1} is a child of Q_j. Therefore, we can use the triangle inequality to write

$$\left| \fint_{\alpha(Q)} h - \fint_Q h \right|^2 \ell(Q)^n \lesssim_{k,n} \sum_{i=1}^{j} \left| \fint_{Q_i} h - \fint_{Q_{i-1}} h \right|^2 \ell(Q_{i-1})^n \leq \sum_{i=1}^{j} \|\Delta_{Q_i} h\|_2^2.$$

Because each cube $Q' \subseteq R$ can appear in at most $N(n, k) < \infty$ chains of the above type, this gives

$$\left| \fint_{\alpha(Q)} h - \fint_Q h \right|^2 \ell(Q)^n \lesssim_{n,k} \sum_{Q \subseteq R} \sum_{i=1}^{j(Q)} \|\Delta_{Q_i} h\|_2^2 \lesssim_{n,k} \sum_{Q \subseteq R} \|\Delta_R h\|_2^2 = \|h\|_2^2.$$

The reader should also see [DS93] Lemma IV.2.2.14 for a version of this statement where $\alpha(Q)$ is only required to be N-close to Q rather than contained in Q.

Corollary 4.3.5 (cf. [DS93] Corollary IV.2.2.19)**.** *Let $L, \epsilon, M > 0$ and let $h \in L^\infty(\mathbb{R}^n)$ with $\|h\|_\infty \leq M$. Let*

$$\mathscr{G}_h(L, \epsilon) = \left\{ Q \in \mathcal{D}(\mathbb{R}^n) \;\middle|\; \left| \fint_B h - \fint_Q h \right| \leq \epsilon \text{ for all } B \in \mathcal{B}_L(Q) \right\}.$$

Then $\mathscr{B}_h(L, \epsilon) = \mathcal{D}(\mathbb{R}^n) \setminus \mathscr{G}_h(L, \epsilon)$ is $C(M, L, n, \epsilon)$-Carleson.

Proof. Let $\tilde{h} = h + \|h\|_\infty + 1$, let $\epsilon, M, L > 0$, and choose $k, \delta > 0$ such that the conclusion of Lemma 4.3.3 holds. Let

$$\mathscr{B}'_{\tilde{h}} = \left\{ Q \in \mathcal{D}(\mathbb{R}^n) \;\middle|\; \exists B \in \mathcal{B}_L(Q),\ \left| \fint_B \tilde{h} - \fint_Q \tilde{h} \right| > \epsilon \left| \fint_Q \tilde{h} \right| \right\}$$

and fix $R \in \mathcal{D}$. By Lemma 4.3.3, $Q \in \mathscr{B}'_{\tilde{h}}$ implies $\Delta_k^{\tilde{h}}(Q) > \delta \ell(Q)^{n/2}$ so that by (4.7).

$$\sum_{\substack{Q \subseteq R \\ Q \in \mathscr{B}_{\tilde{h}}}} \ell(Q)^n \leq \sum_{\substack{Q \subseteq R \\ \Delta_k^{\tilde{h}}(Q) > \delta \ell(Q)^{n/2}}} \ell(Q)^n \lesssim_{\delta,k,n,M} \ell(R)^n.$$

The result follows since $\fint_B \tilde{h} - \fint_Q \tilde{h} = \fint_B h - \fint_Q h$ and $|\fint_Q \tilde{h}| \leq 2M+1$ so that $\mathscr{B}_h(L, (2M+1)\epsilon) \subseteq \mathscr{B}'_{\tilde{h}}$. ■

4.4 Bi-Lipschitz images satisfy the weak constant density condition

In this section, we use the tools from Section 4.3 to prove that metric spaces which are bi-Lipschitz images of Euclidean spaces satisfy the WCD. In this section and the next, we will use the following version of the WCD adapted to Christ-David cubes using only $\mathscr{H}^n$.

Definition 4.4.1 (Cube weak constant density condition)**.** Let (X, d) be an Ahlfors n-regular metric space, $\mathscr{D}$ be a system of Christ-David cubes for X, and let $C_0, \epsilon_0 > 0$. Define

$$\mathscr{G}_{\mathrm{cd}}(C_0, \epsilon_0) = \left\{ Q \in \mathscr{D} \;\middle|\; \begin{array}{l} \exists a_Q > 0, \text{ such that } \forall y \in \frac{c_0}{2} B_Q,\ 0 < r \leq \frac{c_0}{2}\ell(Q), \\ |\mathscr{H}^n(B(y,r)) - a_Q r^n| \leq \epsilon_0 \ell(Q)^n \end{array} \right\}, \tag{4.14}$$

$$\mathscr{B}_{\mathrm{cd}}(C_0, \epsilon_0) = \mathscr{D} \setminus \mathscr{G}_{\mathrm{cd}}(C_0, \epsilon_0). \tag{4.15}$$

We say that X satisfies the cube WCD if there exists $C_0 > 0$ such that for all choices of system $\mathscr{D}$ and $\epsilon_0 > 0$, $\mathscr{B}_{\mathrm{cd}}(C_0, \epsilon_0)$ is Carleson.

See Lemma 4.2.3 for a proof that this version of the WCD implies the version given in Definition 4.1.2. We will also need to review some of the theory of rectifiability in metric spaces.

Definition 4.4.2 (metric derivatives, jacobians)**.** Let $f : \mathbb{R}^n \to \Sigma$ be L-Lipschitz. We say a seminorm on $\mathbb{R}^n$ $|Df|(x)$ is a *metric derivative* of f at x if

$$\lim_{y,z \to x} \frac{d(f(y), f(z)) - |Df|(x)(y-z)|}{|y-x| + |z-x|} = 0.$$

Given a seminorm s on $\mathbb{R}^n$, define $\mathscr{J}(s)$, the *jacobian of* s, by

$$\mathscr{J}(s) = \alpha(n) n \left(\int_{\mathbb{S}^{n-1}} (s(x))^{-n} d\mathscr{H}^{n-1}(x) \right)^{-1}.$$

Kircheim used these ideas to prove the following metric analogs of Rademacher's theorem and the area formula for Lipschitz maps from $\mathbb{R}^n$ into metric spaces.

Theorem 4.4.1 (cf. [Kir94] Theorem 2, Corollary 8)**.** *Let $f : \mathbb{R}^n \to \Sigma$ be L-Lipschitz and let $\mathscr{J}_f(x) = \mathscr{J}(|Df|(x))$. A metric derivative for f exists at $\mathscr{L}^n$ almost every $x \in \mathbb{R}^n$. In addition, for any Lebesgue integrable function $g : \mathbb{R}^n \to \mathbb{R}$,*

$$\int_{\mathbb{R}^n} g(x)\mathscr{J}_f(x)d\mathscr{L}^n(x) = \int_\Sigma \left(\sum_{x \in f^{-1}(y)} g(x) \right) d\mathscr{H}^n(y).$$

In their work on Lipschitz analogs of Sard's Theorem, Azzam and Schul developed the following quantitative measure of how far a function f is from being given by a seminorm.

Definition 4.4.3. Let $f : \mathbb{R}^n \to X$ and $Q \in \mathcal{D}(\mathbb{R}^n)$. Define

$$\mathrm{md}_f(Q) = \frac{1}{\ell(Q)} \inf_{\|\cdot\|} \sup_{x,y \in Q} \Big| d(f(x), f(y)) - \|x - y\| \Big|$$

They used this to prove the following metric quantitative differentiation theorem.

Theorem 4.4.2 ([AS14] Theorem 1.1)**.** *Let $f : \mathbb{R}^n \to X$ be an L-Lipschitz function. Let $\delta > 0$. Then for each $R \in \mathcal{D}(\mathbb{R}^n)$,*

$$\sum_{\substack{Q \in \mathcal{D}(R) \\ \mathrm{md}_f(3Q) > \delta L}} \ell(Q)^n \le C(\delta, n)\ell(R)^n.$$

Finally, we will need to extend the standard system of dyadic cubes.

Definition 4.4.4 (one-third trick lattices)**.** The following family of dyadic systems were introduced by Okikiolu [Oki92]. For any $e \in \{0,1\}^n$ and cube $Q_0 \in \mathcal{D}(\mathbb{R}^n)$, define the shifted dyadic lattice

$$\mathcal{D}^e_j(Q_0) = \left\{ Q + \frac{\ell(Q)}{3}e \;\middle|\; Q \in \mathcal{D}_j(Q_0) \right\},$$
$$\mathcal{D}^e(Q_0) = \bigcup_{j \ge 0} \mathcal{D}^e_j(\mathbb{R}^n)$$

and set

$$\tilde{\mathcal{D}}(Q_0) = \bigcup_{e \in \{0,1\}^n} \mathcal{D}^e(Q_0).$$

$\tilde{\mathcal{D}}(Q_0)$ has the following property: For any $x \in Q_0$ and $j \ge 0$, there exists $Q \in \tilde{\mathcal{D}}(Q_0)$ such that $x \in \frac{2}{3}Q$ (See [Ler03] Proposition 3.2).

We now begin setting up the proof of the WCD for bi-Lipschitz images. We use the following good family of dyadic cubes from our collection of dyadic trees $\tilde{\mathcal{D}}$ to do analysis in the domain of our bi-Lipschitz maps.

Definition 4.4.5 (L-good I_Q)**.** Let $f : [0,1]^n \twoheadrightarrow \Sigma$ be L-bi-Lipschitz. Fix $Q \in \mathscr{D}(\Sigma)$. We call a dyadic cube $I_Q \in \tilde{\mathcal{D}}$ *L-good for Q* if the following hold:

(i) $\ell(I_Q) \asymp_L \ell(Q)$,

(ii) $3B_Q \subseteq f(I_Q)$,

where the implicit constant in 4.4.5(i) is independent of Q and I_Q.

Using the special property of the one-third trick lattices and the definition of bi-Lipschitz maps, the following lemma is standard.

Lemma 4.4.3. *Let $f : [0,1]^n \twoheadrightarrow \Sigma$ be L-bi-Lipschitz. For each $Q \in \mathscr{D}(\Sigma)$ with $\ell(Q) \lesssim_L 1$, there exists an L-good $I_Q \in \tilde{\mathcal{D}}$.*

For $k \in \mathbb{N},\ \delta > 0$, define

$$\mathscr{G}_\Sigma(k,\delta) = \left\{ Q \in \mathscr{D} \;\middle|\; \begin{array}{l} \exists\ L\text{-good } I_Q \in \tilde{\mathcal{D}} \text{ with} \\ \Delta_k^{\mathscr{J}_f}(I_Q) \le \delta \ell(I_Q)^{n/2},\ \mathrm{md}_f(I_Q) \le \delta \end{array} \right\}, \tag{4.16}$$

$$\mathscr{B}_\Sigma(k,\delta) = \mathscr{D} \setminus \mathscr{G}_\Sigma(k,\delta). \tag{4.17}$$

Lemma 4.4.4. *Let $f : [0,1]^n \twoheadrightarrow \Sigma$ be L-bi-Lipschitz and let $\epsilon > 0$. There exist $k(\epsilon, L, n),\ \delta(\epsilon, L, n) > 0$ such that the following holds: For any $Q \in \mathscr{G}_\Sigma(k,\delta)$ there exists a constant $c_Q \asymp_{L,n} 1$ such that for any normed ball $B \in \mathcal{B}_L(I_Q)$,*

$$|\mathscr{H}^n(f(B)) - c_Q \mathscr{L}^n(B)| \le \epsilon c_Q \mathscr{L}^n(B)$$

Proof. Let $I_Q \in \tilde{\mathcal{D}}$ for Q be as in (4.16), let $\epsilon > 0$, and assume k, δ are small enough to satisfy the hypotheses of Lemma 4.3.3 with respect to $0 \le \mathscr{J}_f \in L^\infty$ and ϵ. Then $\Delta_k^{\mathscr{J}_f}(I_Q) \le \delta$ implies that

$$\left| \fint_B \mathscr{J}_f - \fint_Q \mathscr{J}_f \right| \le \epsilon \left| \fint_Q \mathscr{J}_f \right|.$$

for any normed ball $B \in \mathcal{B}_L(I_Q)$. By setting $c_Q = \fint_Q \mathscr{J}_f$ and using the area formula, we get the desired inequality. ■

Lemma 4.4.5. *Let $f : [0,1]^n \twoheadrightarrow \Sigma$ be L-bi-Lipschitz, let C_0 be a regularity constant for Σ, and let $\epsilon_0 > 0$. There exist $k(\epsilon_0, L, n),\ \delta(\epsilon_0, L, n) > 0$ such that $\mathscr{G}_\Sigma(k,\delta) \subseteq \mathscr{G}_{cd}(2C_0, \epsilon_0)$. In fact, for any $Q \in \mathscr{G}_\Sigma(k,\delta)$, there exists a constant $(2C_0)^{-1} \le a_Q \le 2C_0$ such that for any $y \in B_Q,\ 0 < r \le \ell(Q)$, we have*

$$|\mathscr{H}^n(B(y,r)) - a_Q r^n| \le \epsilon_0 \ell(Q)^n. \tag{4.18}$$

That is, the condition on cubes in $\mathscr{G}_{cd}(2C_0, \epsilon_0)$ is attained with a multiple of $\mathscr{H}^n$.

Proof. First, we note that if a constant a_Q such as in (4.18) exists, it must satisfy $(2C_0)^{-1} \le a_Q \le 2C_0$ for small enough ϵ_0 because Σ is (C_0, n)-regular. Let I_Q be as in (4.16) and let

$\epsilon > 0$. By Lemma 4.4.4, we can choose k large enough and $\delta > 0$ small enough so there exists $c_Q \asymp_{L,n} 1$ such that for any $B \in \mathcal{B}_{2L}(I_Q)$

$$|\mathscr{H}^n(f(B)) - c_Q \mathscr{L}^n(B))| \leq \epsilon c_Q \mathscr{L}^n(B). \tag{4.19}$$

In addition, the fact that $\mathrm{md}_f(I_Q) \leq \delta$ implies that there exists a norm $\|\cdot\|_Q$ such that

$$\sup_{x,y \in I_Q} \big| d(f(x), f(y)) - \|x-y\|_Q \big| \leq \delta \ell(I_Q). \tag{4.20}$$

Let $c_{\|\cdot\|_Q} \asymp_{L,n} 1$ be such that $\mathscr{L}^n(B_{\|\cdot\|_Q}(0,r)) = c_{\|\cdot\|_Q} r^n$. We set

$$a_Q = c_Q c_{\|\cdot\|_Q}$$

and begin the proof of (4.18).

Let $y_0 = f^{-1}(y)$. We claim that there exists a constant $c_1(n,L) > 0$ such that

$$B_1 := B_{\|\cdot\|_Q}(y_0, (1-c_1\delta)r) \subseteq f^{-1}(B(y,r)) \subseteq B_{\|\cdot\|_Q}(y_0, (1+c_1\delta)r) =: B_2. \tag{4.21}$$

For the first inclusion, let $x_0 \in B_1$. By (4.20),

$$d(f(x_0), f(y_0)) \leq \|x_0 - y_0\|_Q + 3\delta \ell(I_Q) \leq (1-c_1\delta)r + C(L,n)\delta r < r$$

where the final inequality holds if c_1 is large enough. Similarly, let $z_0 \in f^{-1}(B(y,r)) \subseteq I_Q$. Then

$$\|z_0 - y_0\|_Q \leq d(f(z_0), f(y_0)) + \delta \ell(3I_Q) \leq r + C(L,n)\delta r \leq (1+c_1\delta)r$$

with the same restriction on δ as above This finishes the proof of (4.21). Because $3B_Q \subseteq f(I_Q)$, we immediately have that $B_1, B_2 \subseteq I_Q$ for small enough δ so that $B_1, B_2 \in \mathcal{B}_{2L}(I_Q)$. Using (4.19), this implies the existence of a constant $c_2(n,L)$ so that

$$\mathscr{H}^n(B(y,r)) \leq \mathscr{H}^n(f(B_2)) \leq c_Q(1+\epsilon)c_{\|\cdot\|_Q}(1+c_1\delta)^n r^n \leq a_Q r^n + c_2(\epsilon+\delta)\ell(Q)^n.$$

A similar computation using $\mathscr{H}^n(f(B_1))$ gives a similar lower bound for $\mathscr{H}^n(B(y,r))$. This shows that

$$|\mathscr{H}^n_\Sigma(B(y,r)) - a_Q r^n| \leq c_2(\epsilon+\delta)\ell(Q)^n$$

By choosing ϵ small enough, then k large enough and δ small enough, we get the conclusion of the lemma. ■

Lemma 4.4.6. *Let $f : [0,1]^n \twoheadrightarrow \Sigma$ be L-bi-Lipschitz. For any $k, \delta > 0$, $\mathscr{B}_\Sigma(k,\delta)$ is $C(k,\delta,n,L)$-Carleson.*

Proof. Let $R \in \mathscr{D}$. By Lemma 4.4.3, Remark 4.3.1, and Theorem 4.4.2 we have

$$\begin{aligned}
\sum_{\substack{Q \in \mathscr{B}(k,\delta) \\ Q \subseteq R}} \ell(Q)^n &\lesssim \sum_{\substack{Q \subseteq R \\ \Delta_k^{\mathscr{J}f}(I_Q) > \delta \ell(I_Q)^{n/2}}} \ell(Q)^n + \sum_{\substack{Q \subseteq R \\ \mathrm{md}_f(I_Q) > \delta}} \ell(Q)^n + \sum_{\substack{Q \subseteq R \\ \ell(Q) > C(L)}} \ell(Q)^n \\
&\lesssim_{L,n} \sum_{\substack{I_Q \subseteq I_R \\ \Delta_k^{\mathscr{J}f}(I_Q) > \delta \ell(I_Q)^{n/2}}} \ell(I_Q)^n + \sum_{\substack{I_Q \subseteq I_R \\ \mathrm{md}_f(I_Q) > \delta}} \ell(I_Q)^n + C(L,n)\ell(R)^n \\
&\lesssim_{k,\delta,L,n} \ell(I_R)^n + \ell(R)^n \lesssim_{L,n} \ell(R)^n.
\end{aligned}$$

■

Theorem 4.4.7. *The WCD holds for any bi-Lipschitz image of* $[0,1]^n$.

Proof. Let f be L-bi-Lipschitz $f : [0,1]^n \twoheadrightarrow \Sigma$. Choose $C_0(L,n)$ such that Σ is Ahlfors (C_0,n)-regular and let $\epsilon_0 > 0$. Choose k large enough and δ small enough with respect to ϵ_0, L, n so that $\mathscr{G}_\Sigma(k,\delta) \subseteq \mathscr{G}_{\mathrm{cd}}(2C_0,\epsilon_0)$. That is, the conclusion of Lemma 4.4.5 holds. Then $\mathscr{B}_{\mathrm{cd}}(\epsilon_0, 2C_0) \subseteq \mathscr{B}_\Sigma(k,\delta)$. Lemma 4.4.6 implies that $\mathscr{B}_\Sigma(k,\delta)$ is $C(\epsilon_0,L,n)$-Carleson, implying $\mathscr{B}_{\mathrm{cd}}(2C_0,\epsilon_0)$ is also $C(\epsilon_0,L,n)$-Carleson which says exactly that Σ satisfies the WCD. ■

4.5 Stability of the weak constant density condition under big pieces

The goal of this section is to prove Theorem F: uniformly n-rectifiable metric spaces satisfy the WCD. We will prove this via a stability argument. That is, we will use the fact that uniformly rectifiable metric spaces have big pieces of bi-Lipschitz images (in fact, very big pieces of bi-Lipschitz images with worsening constants) to transfer our bi-Lipschitz image estimates to the uniformly rectifiable case. The primary tool for this argument is the following abstract analog of the John-Nirenberg-Stromberg theorem.

Lemma 4.5.1 ([BHS23] Lemma 4.2.8, [DS93] Lemma IV.1.12)**.** *Let X be an Ahlfors n-regular metric space and $\mathscr{D}$ a system of Christ-David cubes for X. Let $\alpha : \mathscr{D} \to [0,\infty)$ be given and suppose there are $N, \eta > 0$ such that*

$$\mathscr{H}^n\left(\left\{ x \in R \;\middle|\; \sum_{\substack{Q\subseteq R\\ x\in Q}} \alpha(Q) \le N \right\}\right) \ge \eta \ell(R)^n \tag{4.22}$$

for all $R \in \mathscr{D}$. Then,

$$\sum_{Q\subseteq R} \alpha(Q)\ell(Q)^n \lesssim_{N,\eta} \ell(R)^n$$

for all $R \in \mathscr{D}$.

For our application, we will take $\alpha(Q) = \chi_{\mathscr{B}_{\mathrm{cd}}(C_0,\epsilon_0)}(Q)$ where $\mathscr{B}_{\mathrm{cd}}(C_0,\epsilon_0)$ is as in (4.15). If we can show that (4.22) holds for this choice of α, then we will conclude

$$\sum_{Q\subseteq R} \chi_{\mathscr{B}_{\mathrm{cd}}(C_0,\epsilon_0)}(Q)\ell(Q)^n = \sum_{\substack{Q\subseteq R\\ Q\in\mathscr{B}_{\mathrm{cd}}(C_0,\epsilon_0)}} \ell(Q)^n \lesssim_{N,\eta} \ell(R)^n \tag{4.23}$$

which is exactly the desired Carleson packing condition for $\mathscr{B}_{\mathrm{cd}}(C_0,\epsilon_0)$. We will need the following result from Bate, Hyde, and Schul's paper which states that uniformly rectifiable metric spaces have very big pieces of bi-Lipschitz images.

Theorem 4.5.2 (cf. [BHS23] Theorem B, Proposition 9.0.2)**.** *Let $\epsilon > 0$ and let X be uniformly n-rectifiable. There is an $L \ge 1$ depending only on ϵ, n, the Ahlfors regularity constant for X, and the BPLI constants for X such that for each $x \in X$ and $r > 0$ there exists $F \subseteq B(x,r)$, satisfying $\mathscr{H}^n_X(B(x,r)\setminus F) \le \epsilon r^d$ and an L-bi-Lipschitz map $g : F \to \mathbb{R}^n$.*

Remark 4.5.3. If we embed X isometrically into ℓ_∞, then we can take the map $g^{-1} : g(F) \to F \subseteq \ell_\infty$ above and extend it to an $L'(L,n)$-bi-Lipschitz map $f : \mathbb{R}^n \to \ell_\infty$ satisfying the same conclusions with respect to the isometric embedding of X. (See [BHS23] Lemma 4.3.2 for a proof.)

We now begin setting up the proof of Theorem F. Fix a uniformly n-rectifiable metric space X with regularity constant C_0 and a system of Christ-David cubes $\mathscr{D}$ for X. Let $\epsilon_0 > 0$ and $R \in \mathscr{D}(X)$. By applying Theorem 4.5.2 to the ball $3B_R$, we get an L-bi-Lipschitz map $f : \mathbb{R}^n \twoheadrightarrow \Sigma \subseteq \ell_\infty$ such that

$$\mathscr{H}^n_X(3B_R \setminus \Sigma) \leq \frac{\epsilon_0}{4}\ell(R)^n. \tag{4.24}$$

We will only need to use f near where it parameterizes $3B_R$, so it suffices to consider $f|_{I_R}$ where I_R is L-good for R (See Definition 4.4.5). We can assume without loss of generality that $I_R = [0,1]^n$ so that the results of the previous section for bi-Lipschitz images of $[0,1]^n$ apply to f.

Because Σ has such large intersection with $3B_R$, we can use the following lemma to find a substantial subset $\tilde{R} \subseteq R$ such that for every $x \in \tilde{R}$, every cube $Q \subseteq R$ with $x \in Q$ has very large intersection with Σ.

Lemma 4.5.4. *Let X be a doubling metric space with a system of Christ-David cubes $\mathscr{D}$. Let $\epsilon > 0$, $F \subseteq X$ measurable, and let $R \in \mathscr{D}$ be such that $\mathscr{H}^n(R \setminus F) \leq \epsilon \mathscr{H}^n(R)$. Define*

$$\tilde{R} = \left\{ x \in R \;\middle|\; \begin{array}{l} \text{For all } Q \in \mathscr{D} \text{ such that } x \in Q \subseteq R, \\ \mathscr{H}^n(Q \cap F) \geq (1-2\epsilon)\mathscr{H}^n(Q) \end{array} \right\}. \tag{4.25}$$

We have $\mathscr{H}^n(\tilde{R}) \geq \epsilon \mathscr{H}^n(R)$.

Proof. This proof is essentially contained in the proof of Lemma IV.2.2.38 in [DS93], but we need to be more precise than them about the constant ϵ. If $x \in R \setminus \tilde{R}$, then x is contained in some cube Q such that $\mathscr{H}^n(Q \cap F) \leq (1-2\epsilon)\mathscr{H}^n(Q)$. Let $\{Q_i\}_i$ be a maximal disjoint family of such cubes so that $R \setminus \tilde{R} = \bigcup_i Q_i$. Then

$$\begin{aligned} \mathscr{H}^n((R \setminus \tilde{R}) \cap F) &= \sum_i \mathscr{H}^n(Q_i \cap F) \leq (1-2\epsilon)\sum_i \mathscr{H}^n(Q_i) \\ &\leq (1-2\epsilon)\mathscr{H}^n(R \setminus \tilde{R}) \leq (1-2\epsilon)\mathscr{H}^n(R). \end{aligned} \tag{4.26}$$

On the other hand,

$$\begin{aligned} \mathscr{H}^n((R \setminus \tilde{R}) \cap F) &= \mathscr{H}^n((R \cap F) \setminus \tilde{R}) \geq \mathscr{H}^n(R \cap F) - \mathscr{H}^n(\tilde{R}) \\ &\geq (1-\epsilon)\mathscr{H}^n(R) - \mathscr{H}^n(\tilde{R}). \end{aligned} \tag{4.27}$$

Combining (4.26) and (4.27) and rearranging gives

$$\mathscr{H}^n(\tilde{R}) \geq (1-\epsilon)\mathscr{H}^n(R) - (1-2\epsilon)\mathscr{H}^n(R) = \epsilon \mathscr{H}^n(R). \qquad \blacksquare$$

While this lemma allows us to control the measure of the part of X outside of Σ, we will also use separate control of the maximal distance of points in $Q \in \mathscr{D}(R)$ from Σ as measured by the following quantity.

Definition 4.5.1. Let (Z, d) be a metric space and suppose $X, Y \subseteq Z$. For $x \in X$ and $0 < r < \operatorname{diam}(X)$, define

$$I_{X,Y}(x,r) = \frac{1}{r} \sup_{\substack{y \in X \cap B(x,r) \\ \operatorname{dist}(y,Y) \leq r}} \operatorname{dist}(y, Y)$$

The following lemma gives Carleson control over cubes where $I_{X,Y}$ is large.

Lemma 4.5.5 ([BHS23] Lemma 4.2.6)**.** *Let (Z, d) be a metric space with $X, Y \subseteq Z$ Ahlfors (C_0, n)-regular subsets and $\mathscr{D}$ a system of Christ-David cubes for X. For any $\delta > 0$, the set $\{ Q \in \mathscr{D} \mid I_{X,Y}(3B_Q) > \delta \}$ is $C(C_0, n, \delta)$-Carleson.*

We can now define the good family of descendants of R we want to consider. Let $E = f^{-1}(X)$. For any $k \in \mathbb{N}$ and $\delta > 0$, consider the following three conditions applicable to $Q \in \mathscr{D}(R) \subseteq \mathscr{D}(X)$:

(i) $\mathscr{H}^n(Q \setminus \Sigma) \leq \frac{\epsilon_0}{2} \mathscr{H}^n(Q)$,

(ii) $I_{X,\Sigma}(3B_Q) \leq \delta$,

(iii) $\exists L$-good $I_Q \in \tilde{\mathcal{D}}(\mathbb{R}^n)$ for which the following hold:

  a) $\Delta_k^{\mathscr{J}_f \cdot \chi_E}(I_Q) \leq \delta \ell(I_Q)^{n/2}$,

  b) $\operatorname{md}_f(I_Q) \leq \delta$.

Define

$$\mathscr{G}_{R,f}(k,\delta) = \{ Q \in \mathscr{D}(R) \mid \ Q \text{ satisfies (i), (ii), and (iii)} \},$$
$$\mathscr{B}_{R,f}(k,\delta) = \mathscr{D} \setminus \mathscr{G}_{R,f}(k,\delta).$$

We first show that $\mathscr{G}_{R,f}(k,\delta)$ cubes are good for the WCD for X. The reader should compare the following lemma with Lemma 4.4.5.

Lemma 4.5.6. *Let X be uniformly n-rectifiable with regularity constant C_0, let $\epsilon_0 > 0$, and let $R \in \mathscr{D}(X)$. Let $f : \mathbb{R}^n \twoheadrightarrow \Sigma$ be L-bi-Lipschitz and satisfy (4.24). There exist $k, \delta > 0$ dependent on C_0, ϵ_0, n, L such that $\mathscr{G}_{R,f}(k,\delta) \subseteq \mathscr{G}_{cd}(2C_0, \epsilon_0)$.*

Proof. Let $Q \in \mathscr{G}_{R,f}(k,\delta)$, $y \in \frac{c_0}{2} B_Q$, $0 < r \leq \frac{c_0}{2}\ell(Q)$ and let I_Q be the cube guaranteed from condition (iii). Notice that $B(y,r) \subseteq c_0 B_Q$. By condition (ii), there exists $y_0 \in \Sigma$ satisfying $|y - y_0| \leq 3\delta\ell(Q)$. Let $r_\pm = r \pm 3\delta\ell(Q)$ so that

$$B_\Sigma(y_0, r_-) \cap X \subseteq B_X(y,r) \cap \Sigma \subseteq B_\Sigma(y_0, r_+) \cap X.$$

Since $\operatorname{md}_f(I_Q) \leq \delta$, the proof of Lemma 4.4.5, specifically of (4.9), shows that there exists a norm $\| \cdot \|_Q$ and $c_1(n, L) > 0$ such that the balls

$$B_1 = B_{\|\cdot\|_Q}(f^{-1}(y_0), (1 - c_1\delta)r_-),$$
$$B_2 = B_{\|\cdot\|_Q}(f^{-1}(y_0), (1 + c_1\delta)r_+)$$

satisfy $B_1, B_2 \in \mathcal{B}_{2L}(I_Q)$ and

$$B_1 \cap E \subseteq f^{-1}(B_X(y,r) \cap \Sigma) \subseteq B_2 \cap E. \tag{4.28}$$

Let $\epsilon > 0$. By taking δ small enough and k large enough so that the hypotheses of Lemma 4.3.3 are satisfied for $\mathscr{J}_f \chi_E$, $\Delta_k^{\mathscr{J}_f \chi_E}(I_Q) \leq \delta \ell(I_Q)^{n/2}$ gives

$$\left| \fint_B \mathscr{J}_f \chi_E - \fint_Q \mathscr{J}_f \chi_E \right| \leq \epsilon \left| \fint_Q \mathscr{J}_f \chi_E \right|$$

for any normed ball $B \in \mathcal{B}_{2L}(I_Q)$. Set $c_Q = \left| \fint_Q \mathscr{J}_f \chi_E \right|$. After rearranging and applying the area formula, this becomes

$$|\mathscr{H}^n(f(B \cap E)) - c_Q \mathscr{L}^n(B)| \leq \epsilon c_Q \mathscr{L}^n(B). \tag{4.29}$$

Let $\mathscr{L}^n(B_{\|\cdot\|_Q}(0,r)) = c_{\|\cdot\|_Q} r^n$ and set $a_Q = c_Q c_{\|\cdot\|_Q}$. Combining (4.28) and (4.29) gives

$$\begin{aligned}\mathscr{H}^n(B_X(y,r) \cap \Sigma) &\leq \mathscr{H}^n(f(B_2 \cap E)) \leq c_Q(1+\epsilon)\mathscr{L}^n(B_2) = a_Q(1+\epsilon)(1-c_1\delta)^n r_+^n \\ &= a_Q(1+\epsilon)(1-c_1\delta)^n(r + 3\delta\ell(Q))^n \\ &\leq a_Q r^n + C(n,L)(\epsilon+\delta)\ell(Q)^n.\end{aligned}$$

A similar argument using B_1 gives a similar lower bound so that

$$|\mathscr{H}^n(B_X(y,r) \cap \Sigma) - a_Q r^n| \leq C(n,L)(\epsilon+\delta)\ell(Q)^n.$$

Finally, using (i), we have

$$\begin{aligned}|\mathscr{H}^n(B_X(y,r)) - a_Q r^n| &\leq \mathscr{H}^n(B_X(y,r) \setminus \Sigma) + |\mathscr{H}^n(B_X(y,r)) \cap \Sigma - a_Q r^n| \\ &\leq \mathscr{H}^n(c_0 B_Q \cap X \setminus \Sigma) + C(n,L)(\epsilon+\delta)\ell(Q)^n \\ &\leq \frac{\epsilon_0}{2}\ell(Q)^n + C(n,L)(\epsilon+\delta)\ell(Q)^n \leq \epsilon_0 \ell(Q)^n\end{aligned}$$

where the final inequality follows by first fixing ϵ sufficiently small in terms of ϵ_0, L, n then δ small and k large in terms of $\epsilon, \epsilon_0, n, L$. ■

We now show that $\mathscr{B}_{R,f}(k,\delta)$ is not too big. The reader should compare this lemma with Lemma 4.4.6.

Lemma 4.5.7. *Let X be uniformly n-rectifiable with regularity constant C_0, let $\epsilon_0, \delta, k > 0$, and let $R \in \mathscr{D}(X)$. Let $f : \mathbb{R}^n \twoheadrightarrow \Sigma$ be L-bi-Lipschitz and satisfy (4.24). There exist constants $N, \eta > 0$ dependent on $k, \delta, \epsilon_0, n, L$ such that*

$$\mathscr{H}^n\left(\left\{ x \in R \,\middle|\, \sum_{\substack{Q \subseteq R \\ x \in Q}} \chi_{\mathscr{B}_{R,f}(k,\delta)} \leq N \right\}\right) \geq \eta \ell(R)^n. \tag{4.30}$$

Proof. Define

$$\mathscr{B}_1 = \{\, Q \subseteq R \mid I_{X,\Sigma}(3B_Q) > \delta \,\},$$
$$\mathscr{B}_2 = \{\, Q \subseteq R \mid \text{there is no } I_Q \text{ satisfying (iii)} \,\}.$$

Lemma 4.5.5 shows that $\mathscr{B}_1$ is $C(\delta, C_0, n)$-Carleson and Lemma 4.4.6 shows that $\mathscr{B}_2$ is $C(k, \delta, n, L)$-Carleson. Let $\tilde{R}$ be as in (4.25) and write $\tilde{R} = \bigcup_i Q_i$ for a maximal disjoint set of cubes. By Chebyshev's inequality,

$$\begin{aligned}
\mathscr{H}^n\left(\left\{ x \in \tilde{R} \;\middle|\; \sum_{\substack{Q\subseteq R\\ x\in Q}} \chi_{\mathscr{B}_{R,f}(k,\delta)}(Q) \geq N \right\}\right) &\leq \frac{1}{N}\int_{\tilde{R}} \sum_{\substack{Q\subseteq R\\ x\in Q}} \chi_{\mathscr{B}_{R,f}(k,\delta)}(Q) \\
&\lesssim \frac{1}{N}\int_{\tilde{R}} \sum_{\substack{Q\subseteq Q_i \subseteq \tilde{R}\\ x\in Q}} \chi_{\mathscr{B}_1}(Q) + \chi_{\mathscr{B}_2}(Q) \\
&\lesssim \frac{1}{N}\sum_i \left[\sum_{\substack{Q\subseteq Q_i\\ Q\in \mathscr{B}_a}} \ell(Q)^n + \sum_{\substack{Q\subseteq Q_i\\ Q\in \mathscr{B}_b}} \ell(Q)^n \right] \\
&\lesssim_{k,\delta,C_0,\epsilon_0,n,L} \frac{1}{N}\sum_i \ell(Q_i)^n \leq \frac{1}{N}\ell(R)^n.
\end{aligned}$$

The result follows by taking N sufficiently large and using Lemma 4.5.4. ■

We finally observe that these pieces combine to prove Theorem F:

Proof of Theorem F. Choose $R \in \mathscr{D}(X)$ and apply Theorem 4.5.2 to get an L-bi-Lipschitz map $f : \mathbb{R}^n \twoheadrightarrow \Sigma \subseteq \ell_\infty$ satisfying (4.24). Fix k large enough and δ small enough in terms of C_0, ϵ_0, n, L so that $\mathscr{G}_{R,f}(k,\delta) \subseteq \mathscr{G}_{\mathrm{cd}}(2C_0, \epsilon_0)$. That is, the conclusion of Lemma 4.5.6 holds. Then we have $\mathscr{B}_{\mathrm{cd}}(2C_0, \epsilon_0) \cap \mathscr{D}(R) \subseteq \mathscr{B}_{R,f}(k,\delta)$ so that $\chi_{\mathscr{B}_{\mathrm{cd}}(2C_0,\epsilon_0)}(Q) \leq \chi_{\mathscr{B}_{R,f}(k,\delta)}(Q)$ for all $Q \subseteq R$. Lemma 4.5.7 gives the existence of N, η independent of R so that

$$\mathscr{H}^n\left(\left\{ x \in R \;\middle|\; \sum_{\substack{Q\subseteq R\\ x\in Q}} \chi_{\mathscr{B}_{\mathrm{cd}}(2C_0,\epsilon_0)} \leq N \right\}\right) \geq \eta \ell(R)^n.$$

By Lemma 4.5.1, this implies $\mathscr{B}_{\mathrm{cd}}(2C_0, \epsilon_0)$ is Carleson, implying X satisfies the WCD. ■

Chapter 5

Iterating the big pieces operator and larger sets

5.1 Introduction

A closed set E (with more than one point) in a metric space $\mathbb{X}$ is said to be Ahlfors-David k-regular if there is a constant $C > 1$ such that for all $r \in (0, \operatorname{diam}(E))$ and $x \in E$ we have $C^{-1}r^k < \mathcal{H}^k(E \cap B(x,r)) < Cr^k$. For some given class $\mathscr{F}$ of Ahlfors-David k-regular subsets (of a metric space $\mathbb{X}$), we define $\mathrm{BP}(\mathscr{F})$ as follows: $F \in \mathrm{BP}(\mathscr{F})$ if F is a Ahlfors-David k-regular set for which there exists a constant $\theta > 0$ such that for any $x \in F$ and $R > 0$, there is a set $G_{x,R} \in \mathscr{F}$ such that

$$\mathcal{H}^k(B(x,R) \cap F \cap G_{x,R}) \geq \theta \mathcal{H}^k(B(x,R) \cap F).$$

Conditions involving $\mathrm{BP}(\mathscr{F})$ for various classes of sets $\mathscr{F}$ play an important role in the theory of uniformly rectifiable sets in $\mathbb{R}^n$ developed By David and Semmes (see e.g. [Dav91], [DS91]). While the original motivation was the study of singular integral operators, the study of such conditions has taken on a life of its own.

In the context of singular integrals, the condition $\mathrm{BP}(\mathscr{F})$ is important because it allows the uniform boundedness of a family of SIOs given by convolution with 'nice' kernels over sets in $\mathscr{F}$ to be transported to sets in $\mathrm{BP}(\mathscr{F})$. In particular, one can define successively weaker conditions $\mathrm{BP}^j(\mathscr{F})$ for all $j > 0$ which all imply boundedness given that the SIOs are bounded on $\mathscr{F}$; the initial case David and Semmes considered [DS91] used Lipschitz graphs as the base class, i.e., $E \in \mathrm{BP}^j(\mathrm{LG})$. This raised a natural question: how do the collections $\mathrm{BP}^j(\mathrm{LG})$ behave as j grows? It turned out that for $j \geq 2$ the collections $\mathrm{BP}^j(\mathrm{LG})$ are all the same and their elements are called *Uniformly rectifiable sets*. We refer the reader to [DS93], [DS91], specifically to Proposition 2.2 on page 97 of [DS91], and Theorem 2.29 on page 336 of [DS91]. For $n \in [k, 2d+1)$ one also needs [AS12a] to show that BPBI implies $\mathrm{BP}(\mathrm{BP}(\mathrm{LG}))$, but this is not where most of the work goes – the proofs by David and Semmes of that stability (for $j \geq 2$) are quite sophisticated and rely on a Euclidean ambient space.

There has recently been interest in other families $\mathscr{F}$, in particular for the purpose of studying *Parabolic Uniform Rectifiability.* See e.g. the work in [Bor+22], where the questions about Uniform Rectifiability in the metric setting are discussed for this purpose. In fact, we refer to [Bor+22] for a great introduction on contemporary applications of the idea of Big Pieces.

An immediate corollary of the main result contained in this essay (Theorem G) is that stabilization of the operator BP^j occurs in the setting of metric spaces for $j \geq 2$ as well. Our proof is both simple and direct.

5.2 A Theorem

Theorem G. *Let $\mathscr{F}$ be a class of (closed) Ahlfors-David k-regular sets in a metric space $\mathbb{X}$. Let $E \subseteq \mathbb{X}$ be a Ahlfors-David k-regular set with $E \in \mathrm{BP}(\mathrm{BP}(\mathscr{F}))$. Then there exists a set $F \subset \mathbb{X}$ such that*

(i) $E \subseteq F$,

(ii) F *is Ahlfors-David k-regular.*

(iii) $F \in \mathrm{BP}(\mathscr{F})$.

The constants in the conclusion are quantitative with dependance on the constants in the assumptions.

Corollary 5.2.1. *Let $\mathscr{F}$ be a class of closed Ahlfors-David $k-$regular sets in a metric space $\mathbb{X}$. For any $j > 2$, and any constants $\theta_1, ..., \theta_j > 0$ defining $\mathrm{BP}^j(\mathscr{F})$, there are $\theta_1', \theta_2' > 0$ such that the family $\mathrm{BP}(\mathrm{BP}(\mathscr{F}))$ defined using θ_1', θ_2' is equal to $\mathrm{BP}^j(\mathscr{F})$ defined using $\theta_1, ..., \theta_j$*

Proof of Corollary 5.2.1. Let $E \in \mathrm{BP}^3(\mathscr{F})$. Then for any $x \in E$ and $R < \operatorname{diam}(E)$ we have a set $E'_{x,R} \in \mathrm{BP}^2(\mathscr{F})$ such that $\mathcal{H}^k(B(x,R) \cap E) \lesssim \mathcal{H}^k(B(x,R) \cap E \cap E'_{x,R})$. By Theorem G, there is a set $F_{x,R} \in \mathrm{BP}(\mathscr{F})$ so that $F_{x,R} \supset E'_{x,R}$. Clearly $\mathcal{H}^k(B(x,R) \cap E) \lesssim \mathcal{H}^k(B(x,R) \cap E \cap F_{x,R})$. We have shown $E \in \mathrm{BP}^2(\mathscr{F})$. This gives for any $j \geq 3$ that $\mathrm{BP}^j(\mathscr{F}) = \mathrm{BP}^{j-1}(\mathscr{F})$, and so we are done by induction. ■

Proof of Theorem G for the case $\operatorname{diam} E < \infty$. We suppose that $\operatorname{diam} E < \infty$. In order to construct the set F, we first fix a dyadic cube decomposition of E denoted by $\Delta = \Delta(E)$ with root cube $\operatorname{root}(\Delta) = Q_0 = E$. By construction, for each cube $Q \in \Delta$ there exists a point $c(Q) \in Q$ which we call the center of Q satisfying

$$\operatorname{dist}(B(c(Q), c_1 \operatorname{diam} Q),\ E \setminus Q) \geq c_2 \operatorname{diam} Q. \tag{5.1}$$

for some constants $c_1, c_2 > 0$ (see e.g. [Chr90] and [HM12]). From now on, define $B_{c(Q)} = B(c(Q), c_1 \operatorname{diam}(Q))$. We construct the set F desired in the theorem inductively. At stage 0, use the fact that $E \in \mathrm{BP}(\mathrm{BP}(\mathscr{F}))$ to find a closed set $F_{Q_0} \in \mathrm{BP}(\mathscr{F})$ such that $F_{Q_0} \subseteq B_{c(Q_0)}$ and

$$\mathcal{H}^k(B_{c(Q_0)} \cap E \cap F_{Q_0}) \gtrsim_{\theta_1} \mathcal{H}^k(B_{c(Q_0)} \cap E) \gtrsim_{c_1,c_2} \operatorname{diam}(Q_0)^k.$$

We define

$$F_0 = F_{Q_0}.$$

We continue the construction by defining a dyadic decomposition $\mathscr{Q}_1$ of the set $E \setminus F_{Q_0}$. Indeed, since F_{Q_0} is closed, $E \setminus F_{Q_0}$ is relatively open in E and for any $x \in E \setminus F_{Q_0}$, there exists some dyadic cube $Q \ni x$ of maximal diameter such that $\operatorname{dist}(Q, F_{Q_0}) > \operatorname{diam} Q$. We call the disjoint family of all such maximal cubes $\mathscr{Q}_1$, so that we have

$$E \setminus F_{Q_0} = \bigcup_{Q \in \mathscr{Q}_1} Q.$$

We now give stage 1 of the construction of F. For each $Q \in \mathscr{Q}_1$, Again find closed a set $F_Q \in \mathrm{BP}(\mathscr{F})$ such that

$$\mathcal{H}^k(B_{c(Q)} \cap E \cap F_Q) \gtrsim_{\theta_1, c_1, c_2} \mathcal{H}^k(Q). \tag{5.2}$$

We define

$$F_1 = F_{Q_0} \cup \bigcup_{Q \in \mathscr{Q}_1} F_Q.$$

Continue the construction inductively. Given the construction completed up to stage m, we define the set $\mathscr{Q}_{m+1}$ to be the collection of dyadic cubes with maximal diameter contained in $E \setminus F_m$ such that $Q \in \mathscr{Q}_{m+1}$ satisfies

$$\operatorname{dist}(Q, F_m) > \operatorname{diam}(Q). \tag{5.3}$$

$\mathscr{Q}_{m+1}$ is a disjoint decomposition of $E \setminus F_m$ so that

$$E \setminus F_m = \bigcup_{Q \in \mathscr{Q}_{m+1}} Q. \tag{5.4}$$

Given such a Q, let $F_Q \in \mathrm{BP}(\mathscr{F})$ with $F_Q \subseteq B_{c(Q)}$ be such that (5.2) holds and define

$$F_{m+1} = F_m \cup \bigcup_{Q \in \mathscr{Q}_{m+1}} F_Q = F_{Q_0} \cup \bigcup_{Q \in \mathscr{Q}_1} F_Q \cup \cdots \cup \bigcup_{Q \in \mathscr{Q}_{m+1}} F_Q. \tag{5.5}$$

Finally, set

$$F = \overline{\bigcup_{m=0}^{\infty} F_m} \tag{5.6}$$

and define $\mathscr{Q} = \cup_m \mathscr{Q}_m$. Now that we have constructed the set F, we note two of its simple properties. First, given any $Q \neq Q' \in \mathscr{Q}$, equality (5.3) implies

$$\operatorname{dist}(F_Q, F_{Q'}) > \min\{\operatorname{diam}(Q), \operatorname{diam}(Q')\}. \tag{5.7}$$

Second,

$$\lim F \subseteq E \cup \bigcup_{m=0}^{\infty} F_m$$

where $\lim F$ denotes the set of limit points of F. Indeed, suppose $x \in \lim F$ with $x_j \to x$, $x_j \in F_{Q_j}$. If the set $\{Q_j\}_j$ is finite, then (5.7) implies the sequence F_{Q_j} is eventually constant, say $F_{Q_j} \to F_{Q_i}$ meaning $x \in F_{Q_i}$ since F_{Q_i} is closed. If instead $\{Q_j\}_j$ is infinite, then consider a subsequence $x_{k_j} \to x$ such that $Q_{k_j} \neq Q_{k_i}$ for any i, j. The fact that x_{k_j} converges combined with (5.7) then implies $\operatorname{diam} Q_j \to 0$. Since $\operatorname{dist}(F_{Q_j}, E) \leq \operatorname{diam} Q_j$, we have $\operatorname{dist}(x, E) = 0$ which implies $x \in E$. (In particular, we will soon see that this implies $\mathcal{H}^k(\lim F \setminus \cup_m F_m) = 0$.)

Proof of (i): Notice that for any $N \in \mathbb{N}$,

$$\mathcal{H}^k(E \setminus F) \leq \mathcal{H}^k\left(E \setminus \bigcup_{m=0}^{\infty} F_m\right) \leq \mathcal{H}^k(E \setminus F_N)$$

because the sets F_m are increasing. Letting $0 < c_0 < 1$ be the constant implicit in inequality (5.2), we can write

$$\begin{aligned}\mathcal{H}^k(E \setminus F_N) &\overset{(5.5)}{=} \mathcal{H}^k\left(E \setminus F_{N-1} \setminus \bigcup_{Q \in \mathscr{Q}_N} F_Q\right) \overset{(5.4)}{=} \mathcal{H}^k\left(\bigcup_{Q \in \mathscr{Q}_N} Q \setminus \bigcup_{Q \in \mathscr{Q}_N} F_Q\right) \\ &= \sum_{Q \in \mathscr{Q}_N} \mathcal{H}^k\left(Q \setminus F_Q\right) \leq (1 - c_0) \sum_{Q \in \mathscr{Q}_N} \mathcal{H}^k(Q) = (1 - c_0)\mathcal{H}^k(E \setminus F_{N-1})\end{aligned}$$

where we used the fact that $F_Q \cap F_{Q'} = \varnothing$ for $Q, Q' \in \mathscr{Q}_N$. Since this holds for any N, we can iterate this inequality to get

$$\mathcal{H}^k(E \setminus F_N) \leq (1 - c_0)^N \mathcal{H}^k(E)$$

from which we conclude $\mathcal{H}^k(E \setminus F) = 0$. To finish the proof of (i), let $x \in E$ be arbitrary. Since E is Ahlfors-David regular, for any $R > 0$, $\mathcal{H}^k(B(x, R)) > 0$ so that $F \cap B(x, r) \neq \varnothing$. This means x is a limit point of F, implying $x \in F$ because F is closed.

Proof of (ii): Fix any point $x \in F$ and some $R < \operatorname{diam} F$. If $x \in F \setminus \cup_m F_m$, then we can find a particular F_Q with $\operatorname{dist}(x, F_Q) < \frac{R}{100}$ and $\operatorname{dist}(x, F_Q) = \operatorname{dist}(x, z)$ for $z \in F_Q$. Then, we have $B(z, R/2) \subseteq B(x, R) \subseteq B(z, 2R)$, and substitute the first ball or final ball for $B(x, R)$ in the proofs of lower and upper regularity respectively. Hence, we can assume $x \in \cup_m F_m$. By definition, there exists $Q_m \in \mathscr{Q}_m$ such that $x \in F_{Q_m}$ for some $m \in \mathbb{N}$. Write

$$\mathcal{H}^k(B(x, R) \cap F) = \sum_{\substack{F_Q \cap B(x,R) \neq \varnothing \\ \operatorname{diam} Q > 10R}} \mathcal{H}^k(B(x, R) \cap F_Q) + \sum_{\substack{F_Q \cap B(x,R) \neq \varnothing \\ \operatorname{diam} Q \leq 10R}} \mathcal{H}^k(B(x, R) \cap F_Q). \quad (5.8)$$

We will first show that F is upper regular. Let $\mathscr{Q}_I$ be the collection of cubes summed over in the first term of (5.8). By (5.7), we have that for any $Q, Q' \in \mathscr{Q}_I$, $\operatorname{dist}(F_Q, F_{Q'}) > 10R$. This means $\mathscr{Q}_I$ has at most one element. Given such a Q, choose $y \in B(x, R) \cap F_Q$ and write

$$\mathcal{H}^k(B(x, R) \cap F_Q) \leq \mathcal{H}^k(F_Q \cap B(y, 2R)) \lesssim R^k$$

using the fact that F_Q is itself Ahlfors-David k-regular. This proves the first sum in (5.8) has the appropriate upper bound. Let $\mathscr{Q}_{II}$ be the collection of cubes summed over in the second term of (5.8). Since $\operatorname{diam}(Q) < 10R$, any $Q \in \mathscr{Q}_{II}$ satisfies $Q \subseteq B(x, 20R)$. We first prove a lemma

Lemma 5.2.2. *Let $Q \in \mathscr{Q}$, and let $D(Q)$ be the descendants of Q in $\mathscr{Q}$. Then*

$$\mathcal{H}^k\left(\bigcup_{Q' \in D(Q)} F_{Q'}\right) = \sum_{Q' \in D(Q)} \mathcal{H}^k(F_{Q'}) \lesssim_{\theta_1, c_1, c_2} \mathcal{H}^k(Q).$$

Proof of Lemma 5.2.2. Suppose for simple notation that $Q = Q_0$. Using the regularity of each F_Q, we have

$$\mathcal{H}^k\left(\bigcup_{Q \in D(Q_0)} F_Q\right) = \sum_{m=0}^{\infty} \sum_{Q \in D(Q_0) \cap \mathscr{Q}_m} \mathcal{H}^k(F_Q) \le C \sum_{m=0}^{\infty} \sum_{Q \in D(Q_0) \cap \mathscr{Q}_m} \mathcal{H}^k(Q).$$

In analogy to (5.4), $Q_0 \setminus F_{m-1} = \bigcup_{Q \in D(Q_0) \cap \mathscr{Q}_m} Q$ holds so that

$$\begin{aligned}
\sum_{Q \in D(Q_0) \cap \mathscr{Q}_m} \mathcal{H}^k(Q) = \mathcal{H}^k(Q_0 \setminus F_{m-1}) &= \mathcal{H}^k\left(Q_0 \setminus F_{m-2} \setminus \bigcup_{Q \in D(Q_0) \cap \mathscr{Q}_{m-1}} F_Q\right) \\
&= \mathcal{H}^k\left(\bigcup_{Q \in D(Q_0) \cap \mathscr{Q}_{m-1}} Q \setminus \bigcup_{Q \in D(Q_0) \cap \mathscr{Q}_{m-1}} F_Q\right) \\
&\le \sum_{Q \in D(Q_0) \cap \mathscr{Q}_{m-1}} \mathcal{H}^k(Q \setminus F_Q) \\
&\le (1 - c_0) \sum_{Q \in D(Q_0) \cap \mathscr{Q}_{m-1}} \mathcal{H}^k(Q)
\end{aligned}$$

where c_0 was defined as the implicit constant in (5.2). Iterating this inequality, we find

$$\mathcal{H}^k\left(\bigcup_{Q \in D(Q_0)} F_Q\right) \le C \sum_{m=0}^{\infty} (1 - c_0)^m \mathcal{H}^k(Q_0) \lesssim_{c_0} \mathcal{H}^k(Q_0).$$

■

Using this lemma, we can write

$$\begin{aligned}
\sum_{\substack{F_Q \cap B(x,R) \neq \varnothing \\ \operatorname{diam} Q \le 10R}} \mathcal{H}^k(B(x,R) \cap F_Q) &\le \sum_{\substack{Q \text{ maximal} \\ Q \in \mathscr{Q}_{II}}} \sum_{Q' \in D(Q)} \mathcal{H}^k(F_{Q'}) \lesssim \sum_{\substack{Q \text{ maximal} \\ Q \in \mathscr{Q}_{II}}} \mathcal{H}^k(Q) \\
&\le \mathcal{H}^k(E \cap B(x, 20R)) \lesssim R^k.
\end{aligned}$$

This proves the desired bound for the second sum in (5.8), proving the upper regularity of F. Now we show that F is lower regular. If $R < 100 \operatorname{diam} Q_m$, then the claim follows immediately from the lower regularity of F_Q. If $100 \operatorname{diam} Q_m \le R < \operatorname{diam} F$, then since $F_{Q_m} \cap Q_m \neq \varnothing$, there exists $z \in Q$ (and thus, $z \in E$) with $B(x, R) \supseteq B(z, R/2)$ and

$$\mathcal{H}^k(B(x,R) \cap F) \ge \mathcal{H}^k(B(z, R/2) \cap E) \gtrsim R^k$$

using the fact that $E \subseteq F$. This completes the proof of lower regularity, hence of (ii) as well.

Proof of (iii): Fix $x \in F_{Q_m}$ and $R > 0$ as in the proof of (ii). Fix a constant $\alpha > 10$ to be chosen later. If $R < \alpha \operatorname{diam} Q_m$, then since $F_{Q_m} \in \mathrm{BP}(\mathscr{F})$, there exists $G_{x,R} \in \mathscr{F}$ such that

$$\mathcal{H}^k(B(x,R) \cap F_{Q_m} \cap G_{x,R}) \geq \theta_2 \mathcal{H}^k(B(x,R) \cap F_{Q_m}) \gtrsim_{C',\alpha} R^k \gtrsim_{C''} \mathcal{H}^k(B(x,R) \cap F) \quad (5.9)$$

where C' is the regularity constant for F_{Q_m} and C'' is the regularity constant for F. Now, suppose that $\alpha \operatorname{diam} Q_m \leq R < \operatorname{diam} F$. Since $x \in F_{Q_m}$, there exists a chain of cubes $Q_i \in \mathscr{Q}_i,\ 0 \leq i \leq m$ such that

$$Q_m \subseteq Q_{m-1} \subseteq \ldots \subseteq Q_1 \subseteq Q_0.$$

Next, notice that for any choice of $\alpha > 10$, there exists a smallest cube Q_j in the above chain such that $R < \alpha \operatorname{diam} Q_j$ since for all admissible R, $R < 10 \operatorname{diam} Q_0$. Choose the constant α such that for any $y \in E \setminus F_i$, the cube $Q_{i+1} \ni y$ satisfies

$$\operatorname{dist}(Q_{i+1}, F_{Q_i}) < \frac{\alpha}{10} \operatorname{diam} Q_{i+1}. \quad (5.10)$$

In general, α will depend on the constants used in the construction of Δ, as it may be the case that all of the children of the cube Q_i are small relative to Q_i with bounds given in terms of these constants. With such an α chosen, let Q_j be the smallest cube in the above chain for x such that $R < \alpha \operatorname{diam} Q_j$. This means that $R \geq \alpha \operatorname{diam} Q_{j+1}$ so that (5.10) implies that there exists $y \in F_{Q_j}$ such that $B(y, R/2) \subseteq B(x,R)$. We can now repeat the argument of (5.9) with Q_j in place of Q_m to finish the proof. This completes the proof of Theorem G for the case $\operatorname{diam} E < \infty$. ■

Before we turn to the case $\operatorname{diam} E = \infty$, we need the following lemma. It says, roughly, that finite diameter subsets of E can be made regular by extending them slightly. This extension also preserves the $\mathrm{BP}(\mathcal{F})$ property.

Lemma 5.2.3. *Let $E \subseteq \mathbb{X}$ be a Ahlfors-David k-regular set and suppose that $G \subseteq E$ satisfies* $\operatorname{diam} G = D < \infty$. *For any $A \geq 1$, there exists a set $\tilde{G} \subseteq E$ such that*

(i) $G \subseteq \tilde{G} \subseteq B(G, \frac{3D}{A}) \cap E = \{x \in E :\ d(x,G) < \frac{3D}{A}\}$,

(ii) $\tilde{G}$ *is Ahlfors-David k-regular with constant $C(k, C_E, A)$.*

Furthermore, if $E \in \mathrm{BP}(\mathcal{F})$ with constant θ_E for some class of Ahlfors-David k-regular sets, then $\tilde{G} \in \mathrm{BP}(\mathcal{F})$ with constant $\theta(k, \theta_E, A)$.

Proof of Lemma 5.2.3. We define an "interior" of the set $G \subseteq E$ by

$$I_A(G) = \left\{ x \in G : d(x, E \setminus G) \geq \frac{D}{A} \right\}.$$

The corresponding "boundary" is then

$$G \setminus I_A(G) = \left\{ x \in G : d(x, E \setminus G) < \frac{D}{A} \right\}$$

We will construct the set $\tilde{G}$ inductively. In the first stage, we will take a maximal net of appropriate size inside $G \setminus I_A(G)$ and add in balls around each net point to G. In the second step, we consider a smaller "boundary" of this new set and repeat the above process with a finer net and smaller balls. If we continue this process indefinitely while adding balls of exponentially decreasing radii, we get the desired set by taking a closure. We now give this construction explicitly.

Let $G_0 = G$ and let X_1 be a maximal $\frac{D}{A}$-net for the set $G \setminus I_A(G) \subseteq E$. Define

$$G_1 = G \cup \bigcup_{x \in X_1} B\left(x, \frac{2D}{A}\right) \cap E.$$

Given the set G_n, we define X_{n+1} to be a maximal $4^{-n}\frac{D}{A}$-net for $G_n \setminus I_{4^n \cdot A}(G_n)$ and we let

$$G_{n+1} = G_n \cup \bigcup_{x \in X_{n+1}} B\left(x, 4^{-n}\frac{2D}{A}\right) \cap E.$$

Finally, define

$$\tilde{G} = \overline{\bigcup_{n=0}^{\infty} G_n}.$$

We will now show that $\tilde{G}$ satisfies the desired properties in the statement of the lemma.

Proof of (i): The maximal distance of a point $x \in \tilde{G}$ from G is just given by the sum of the radii of the balls added in each step:

$$d(x, G) \le \frac{2D}{A} \sum_{n=0}^{\infty} 4^{-n} = \frac{8}{3}\frac{D}{A} < \frac{3D}{A}.$$

Proof of (ii): First, we observe that since $\tilde{G} \subseteq E$, we immediately have, for all $x \in \tilde{G}$, $R > 0$,

$$\mathcal{H}^k(B(x,R) \cap \tilde{G}) \le \mathcal{H}^k(B(x,R) \cap E) \le C_E R^k.$$

Hence, $\tilde{G}$ is upper Ahlfors-David k-regular with constant C_E. We will now show that $\tilde{G}$ is lower regular. In order to do so, we will first prove that there exists a constant $0 < c < 1$ dependent only on A such that

$$\forall x \in \tilde{G},\ \forall R,\ 0 < R < \operatorname{diam} \tilde{G},\ \exists y \in E \text{ such that } B(y, cR) \cap E \subseteq B(x,R) \cap \tilde{G}. \qquad (5.11)$$

We note that (ii) will follow from this since for any relevant pair (x, R), we get the existence of $y \in E$ such that

$$\mathcal{H}^k(B(x,R) \cap \tilde{G}) \ge \mathcal{H}^k(B(y, cR) \cap E) \ge \frac{c^k R^k}{C_E}$$

by the lower regularity of E. We now prove (5.11). We begin by using the constant $c' = \frac{1}{10 \cdot 4^4 \cdot A}$ (we will only need to decrease it by a factor of $\frac{1}{2}$ at the end of the proof). Let $x \in \tilde{G}$ and assume $x \in G_m$ for some m. There exists some minimal n such that $x \in I_{4^n A}(G_n)$ because

$x \in G_m \setminus I_{4^m A}(G_m)$ implies $x \in I_{4^{m+1}A}(G_{m+1})$ by the triangle inequality. Indeed, let $t \in X_{m+1}$ be a nearest net point to x and let $z \in E \setminus G_{m+1}$. We can calculate

$$d(x,z) \geq d(t,z) - d(t,x) \geq 4^{-m}\frac{2D}{A} - 4^{-m}\frac{D}{A} = 4^{-m}\frac{D}{A} > 4^{-m-1}\frac{D}{A}.$$

Therefore, $d(x, E \setminus G_{m+1}) > 4^{-m-1}\frac{D}{A}$ so that $x \in I_{4^{m+1}A}(G_{m+1})$. Suppose first that $n \leq 4$. In this case, we will take $y = x$, and we must show the inclusion of the balls given in (5.11) for any admissible value of R. For $0 < R \leq 4^{-4}\frac{D}{A}$, note that $x \in I_{4^4 A}(G_4)$ implies

$$d(x, E \setminus \tilde{G}) \geq d(x, E \setminus G_4) > \frac{D}{4^4 A} \tag{5.12}$$

so that $B(x,R) \cap \tilde{G} = B(x,R) \cap E$. If instead $4^{-4}\frac{D}{A} < R < \operatorname{diam} \tilde{G} < D + \frac{6D}{A} < 10D$,

$$c'R = \frac{R}{10 \cdot 4^4 \cdot A} < \frac{10D}{10 \cdot 4^4 \cdot A} = 4^{-4}\frac{D}{A}.$$

Which shows that

$$B(x, c'R) \cap E \subseteq B\left(x, 4^{-4}\frac{D}{A}\right) \cap E = B\left(x, 4^{-4}\frac{D}{A}\right) \cap \tilde{G}$$

by (5.12). Now, suppose $n > 4$. This means $x \in I_{4^n A}(G_n) \setminus I_{4^{n-1}A}(G_{n-1})$. Hence, if $R < 4^{-n}\frac{D}{A}$, then we can take $y = x$ and note that $B(x,R) \cap \tilde{G} = B(x,R) \cap E$ in analogy to (5.12). Now, suppose $4^{-m}\frac{D}{A} \leq R < 4^{-m+1}\frac{D}{A}$ for $0 \leq m \leq n-3$. There exist net points $x_p \in X_p$ for $m+3 \leq p \leq n$ such that

$$d(x, x_n) \leq 4^{-n}\frac{2D}{A},$$
$$d(x_{p+1}, x_p) \leq 4^{-p}\frac{2D}{A}.$$

Hence, the triangle inequality implies

$$d(x, x_{m+3}) \leq \frac{2D}{A}\sum_{p=m+2}^{n} 4^{-p} \leq \frac{2D}{A}\left(4^{-m-2} \cdot 2\right) = 4^{-m-1}\frac{D}{A}. \tag{5.13}$$

In this case, we choose $y = x_{m+3}$. We calculate

$$B(y, c'R) = B\left(x_{m+3}, \frac{R}{10 \cdot 4^4 \cdot A}\right) \subseteq B\left(x_{m+3}, 4^{-(m+3)}\frac{D}{10A^2}\right) \subseteq B\left(x, 4^{-m}\frac{D}{A}\right) \subseteq B(x,R)$$

using (5.13) and the fact that $4^{-m}\frac{D}{A} \leq R < 4^{-m+1}\frac{D}{A}$. In the case when $\frac{D}{A} < R < 10D$, choose $y = x_3$, the nearest net point in X_3 and observe that

$$B(y, c'R) = B\left(x_3, \frac{R}{10 \cdot 4^4 \cdot A}\right) \subseteq B\left(x_{m+3}, 4^{-4}\frac{D}{A}\right) \subseteq B\left(x, \frac{D}{A}\right) \subseteq B(x,R)$$

again using (5.13). This proves (5.11) for all $x \in G_n$ for some n. If $x \notin G_n$ for all n, then given any admissible $R > 0$, there is a net point $t \in X_N$ for arbitrarily large N such that $d(x,t) < \frac{R}{4}$ so that $B(t,\frac{R}{2}) \subseteq B(x,R)$ and, applying (5.11) to $B(t,\frac{R}{2})$, we get a point $y \in B(t,\frac{R}{2})$ such that $B(y,c'\frac{R}{2}) \subseteq B(t,\frac{R}{2}) \subseteq B(x,R)$. Take $c = \frac{c'}{2}$ and $B(y,cR) \subseteq B(x,R)$ so that (5.11) holds with $c = \frac{1}{20\cdot 4^4\cdot A}$.

Proof that $\tilde{G} \in \mathrm{BP}(\mathcal{F})$. This follows from (5.11). Indeed, for any admissible pair (x,R), choose y as given by (5.11). Applying the $\mathrm{BP}(\mathcal{F})$ condition for E in the ball $B(y,cR)$ gives a set $H_{y,cR} \in \mathcal{F}$ such that

$$\mathcal{H}^k(B(x,R) \cap \tilde{G} \cap H_{y,cR}) \geq \mathcal{H}^k(B(y,cR) \cap E \cap H_{y,cR}) \gtrsim_{A,\theta_E,k} R^k \gtrsim_C \mathcal{H}^k(B(x,R) \cap \tilde{G}).$$

This concludes the proof of the lemma. ■

Proof of Theorem G for the case $\operatorname{diam} E = \infty$. Fix $x_0 \in E$. Let $A > 1$ and, for $n \geq 0$, set

$$B_n = B(x_0, A^n)$$

where the constant A is sufficiently large in terms of C_E, k, and θ_E, the BP constant. Let E_n be the Ahlfors-David regular extension of the set $E \cap B_n$ with constant A in Lemma 5.2.3 replaced with 100 so that $E_n \subseteq B(E \cap B_n, \frac{A^n}{4})$. E_n satisfies the hypotheses of the finite diameter case of the theorem, so apply the theorem to get a regular set $F_n \in \mathrm{BP}(\mathcal{F})$ satisfying

$$E_n \subseteq F_n \subseteq B\left(x_0, \frac{5A^n}{4}\right).$$

In order to ensure bounded overlap, we then define $\tilde{F}_0 = F_0$ and $\tilde{F}_n$ for $n \geq 1$ to be the regular extension of $F_n \setminus \frac{1}{2}B_{n-1}$ given by the lemma with constant A there replaced by $100A$ here. By construction, $\tilde{F}_n \subseteq B(F_n, \frac{A^{n-1}}{10})$ so that $\tilde{F}_n \cap \frac{1}{4}B_{n-1} = \varnothing$ and $\tilde{F}_n \subseteq B(x_0, 2A^n)$. We also have $\tilde{F}_n \in \mathrm{BP}(\mathcal{F})$ with constant $\tilde{\theta}_F$ independent of n. We now define

$$F = \bigcup_{n=0}^{\infty} \tilde{F}_n$$

and claim that F satisfies conditions (i)-(iii).

Proof of (i): By definition, $E \cap (B_n \setminus \frac{1}{2}B_{n-1}) \subseteq \tilde{F}_n$ so $E = \bigcup_{n=0}^{\infty} E \cap (B_n \setminus \frac{1}{2}B_{n-1}) \subseteq F$.

Proof of (ii): For any n, $\tilde{F}_n$ is regular with some constant $\tilde{C}_F(A, C_E, k)$ independent of n. Lower regularity of F with constant $\tilde{C}_F$ follows immediately, so we only need to show that F is upper regular. Let $x \in \tilde{F}_n$ for some n. Observe that, for $j \geq 2$

$$d(x, \tilde{F}_{n+j}) \geq d\left(\tilde{F}_n, \frac{1}{4}B_{n+j-1}\right) \geq \frac{1}{4}A^{n+j-1} - 2A^n > A^{n+j-2}$$

provided we choose A sufficiently large. Hence, if $R \leq A^{n-2}$, then $B(x,R) \cap \tilde{F}_j = \varnothing$ for $|n-j| \geq 2$. In this case,

$$\mathcal{H}^k(B(x,R) \cap F) = \sum_{j=-1}^{1} \mathcal{H}^k(B(x,R) \cap \tilde{F}_{n+j}) \lesssim_{\tilde{C}_F} R^k$$

independent of n because $\tilde{F}_{n+j}$ is regular with constant independent of n. Now, suppose $A^j < R \leq A^{j+1}$ for $j \geq n-2$. We can write

$$\begin{aligned}\mathcal{H}^k(B(x,R)\cap F) &= \sum_{i=0}^{j+2}\mathcal{H}^k(B(x,R)\cap\tilde{F}_i) \leq \sum_{i=0}^{j+2}\mathcal{H}^k(\tilde{F}_i) \leq \tilde{C}_F\sum_{i=0}^{j+2}\operatorname{diam}(\tilde{F}_i)^k \\ &\leq \tilde{C}_F\sum_{i=0}^{j+2}(4A)^{ik} \leq 2\tilde{C}_F(4A)^{(j+2)k} \leq (4A)^{2k+1}\tilde{C}_F(4R)^k.\end{aligned}$$

This proves upper regularity and finishes the proof of (ii). From now on, let $C_F = C_F(C_E, A, k)$ be the regularity constant for F.

Proof of (iii): Let $x \in \tilde{F}_n$ and $R > 0$. Suppose first that $0 < R \leq A^{n+2}$. Because $\tilde{F}_n \in \mathrm{BP}(\mathcal{F})$ by the lemma with constant $\tilde{\theta}_F(\theta_E, A, k)$ independent of n, we get the existence of a set $G_{x,R} \in \mathcal{F}$ such that

$$\begin{aligned}\mathcal{H}^k(B(x,R)\cap F\cap G_{x,R}) &\geq \mathcal{H}^k(B(x,R)\cap\tilde{F}_n\cap G_{x,R}) \gtrsim_{\tilde{\theta}_F,A} \mathcal{H}^k(B(x,R)\cap\tilde{F}_n) \\ &\gtrsim_{C_F} R^k \gtrsim_{C_F} \mathcal{H}^k(B(x,R)\cap F).\end{aligned}$$

using the fact that $\tilde{F}_n$ is regular. Now, suppose $A^j < R \leq A^{j+1}$ for $j \geq n+2$. Because $x \in \tilde{F}_n$, $\frac{1}{4}A^{n-1} \leq d(x,x_0) \leq 2A^n$ so that

$$\tilde{F}_{j-2} \subseteq B\left(x_0, 2A^{j-2}\right) \subseteq B\left(x, 2A^{j-2}+2A^n\right) \subseteq B(x, A^{j-1}) \subseteq B(x,R).$$

Using the above containment and the fact that $\tilde{F}_{j-2} \in \mathrm{BP}(\mathcal{F})$, there exists a set $G_{x,R} \in \mathcal{F}$ with both $G_{x,R} \subseteq B(x,R)$ and

$$\mathcal{H}^k(G_{x,R}\cap\tilde{F}_{j-2}) \gtrsim_{\tilde{\theta}_F} \operatorname{diam}(\tilde{F}_{j-2})^k \gtrsim A^{(j-2)k} \gtrsim_{A,k} R^k.$$

Hence, we have $\mathcal{H}^k(B(x,R)\cap G_{x,R}\cap F) \gtrsim_{\tilde{\theta}_F,A,k} R^k \gtrsim_{C_F} \mathcal{H}^k(B(x,R)\cap F)$ as desired. ■

Bibliography

[AH22] Jonas Azzam and Matthew Hyde. "The weak lower density condition and uniform rectifiability". *Ann. Fenn. Math.* 47.2 (2022), pp. 791–819.

[Akm+17] Murat Akman et al. "Rectifiability and elliptic measures on 1-sided NTA domains with Ahlfors-David regular boundaries". *Trans. Amer. Math. Soc.* 369.8 (2017), pp. 5711–5745.

[Akm+19] Murat Akman et al. "Rectifiability, interior approximation and harmonic measure". *Ark. Mat.* 57.1 (2019), pp. 1–22.

[Ald91] Jesus M. Aldaz. "A general covering lemma for the real line". *Real. Anal. Exchange* 17.1 (1991), pp. 394–398.

[AO17] Giovanni Alberti and Martino Ottolini. "On the structure of continua with finite length and Golab's semicontinuity theorem". *Nonlinear Anal.* 153 (2017), pp. 35–55.

[AS12a] J. Azzam and R Schul. "Hard Sard: quantitative implicit function and extension theorems for Lipschitz maps". *Geom. Funct. Anal.* 22.5 (2012), pp. 1062–1123.

[AS12b] Jonas. Azzam and Raanan. Schul. "How to take shortcuts in Euclidean space: making a given set into a short quasi-convex set". *Proc. Lond. Math. Soc.* 105.2 (2012), pp. 367–392.

[AS14] Jonas Azzam and Raanan Schul. "A quantitative metric differentiation theorem". *Proc. Amer. Math. Soc.* 142.4 (2014), pp. 1351–1357.

[AS18] Jonas Azzam and Raanan Schul. "An analyst's traveling salesman theorem for sets of dimension larger than one". *Math. Ann.* 370.3-4 (2018), pp. 1389–1476.

[Azz+20] Jonas Azzam et al. "Harmonic measure and quantitative connectivity: geometric characterization of the L^p-solvability of the Dirichlet problem". *Invent. Math.* 222 (3 2020), pp. 881–993.

[Azz18] Jonas Azzam. "Tangents, rectifiability, and corkscrew domains". *Publ. Mat.* 62.1 (2018), pp. 161–176.

[Bad10] Matthew Badger. "Null sets of harmonic measure on NTA domains: Lipschitz approximation revisited". *Math. Z.* 270 (2010), pp. 241–262.

[Bat23] David Bate. *On 1-regular and 1-uniform metric measure spaces.* Youtube. 2023.

[BH17] Simon Bortz and Steve Hofmann. "Harmonic measure and approximation of uniformly rectifiable sets". *Rev. Mat. Iberoam.* 33.1 (2017), pp. 351–373.

[BHS23] David Bate, Matthew Hyde, and Raanan Schul. "Uniformly rectifiable metric spaces: Lipschitz images, Bi-Lateral Weak Geometric Lemma and Corona Decompositions". arXiv preprint, arXiv:2306.12933. 2023.

[Bis20] Christopher J. Bishop. "Weil-Petersson curves, conformal energies, β-numbers, and minimal surfaces". Preprint. 2020.

[Bis22] Christopher J. Bishop. "The traveling salesman theorem for Jordan curves". *Adv. Math.* 404 (2022). Paper No. 109443.

[BLZ23] Matthew Badger, Sean Li, and Scott Zimmerman. *Analysis and Geometry in Metric Spaces* 11.1 (2023), p. 20230102.

[BM23a] Matthew Badger and Sean McCurdy. "Subsets of rectifiable curves in Banach spaces I: Sharp exponents in traveling salesman theorems". *Illinois Journal of Mathematics* 67.2 (2023), pp. 203–274.

[BM23b] Matthew Badger and Sean McCurdy. "Subsets of rectifiable curves in Banach spaces II: Universal estimates for almost flat arcs". *Illinois Journal of Mathematics* 67.2 (2023), pp. 275–331.

[BNV19] Matthew Badger, Lisa Naples, and Vyron Vellis. "Hölder curves and parameterizations in the Analyst's Traveling Salesman theorem". *Adv. Math.* 349 (2019), pp. 564–647.

[Bor+22] Simon Bortz et al. "Coronizations and big pieces in metric spaces". *Ann. Inst. Fourier (Grenoble)* 72.5 (2022), pp. 2037–2078.

[BS17] Matthew Badger and Raanan Schul. "Multiscale Analysis of 1-rectifiable Measures II: characterizations". *Anal. Geom. Metr. Spaces* 5 (2017), pp. 1–39.

[BZ20] Zóltan M. Balogh and Roger Züst. "Box-counting by Hölder traveling salesman". *Arch. Math.* 114.5 (2020), pp. 561–572.

[Cho+16] Vasileios Chousionis et al. “Square functions and uniform rectifiability”. *Trans. Amer. Math. Soc.* 368.9 (2016), pp. 6063–6102.

[Chr90] Michael Christ. “A T(b) theorem with remarks on analytic capacity and the Cauchy integral”. *Colloq. Math.* 60/61.2 (1990), pp. 601–628.

[CMT20] Vasilis Chousionis, Valentino Magnani, and Jeremy T. Tyson. “On uniform measures in the Heisenberg group”. *Adv. Math.* 363 (2020), pp. 106980, 42.

[Dah77] B. Dahlberg. “Estimates of Harmonic Measure”. *Arch. Rational Mech. Anal.* 65.3 (1977), pp. 275–288.

[Dav88] Guy David. “Morceaux de graphes lipschitziens et intégrales singulières sur une surface”. *Rev. Mat. Iberoamericana* 4.1 (1988), pp. 73–114.

[Dav91] Guy David. *Wavelets and singular integrals on curves and surfaces.* Vol. 1465. Lecture Notes in Mathematics. Berlin: Springer-Verlag, 1991, pp. x+107.

[De 08] Camillo De Lellis. *Rectifiable sets, densities and tangent measures.* Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008, pp. vi+127.

[DJ90] Guy David and David Jerison. “Lipschitz approximation to hypersurfaces, harmonic measure, and singular integrals”. *Indiana Univ. Math. J.* 39.3 (1990), pp. 831–845.

[DS16] Guy C. David and Raanan Schul. “The analyst’s traveling salesman theorem in graph inverse limits”. *Ann. Acad. Sci. Fenn. Math.* 42.2 (2016), pp. 649–692.

[DS21] Guy C. David and Raanan Schul. “A sharp necessary condition for rectifiable curves in metric spaces”. *Rev. Mat. Iberoam.* 37.3 (2021), pp. 1007–1044.

[DS91] Guy David and Stephen Semmes. “Singular integrals and rectifiable sets in $\mathbf{R}^n$: Beyond Lipschitz graphs”. *Astérisque* 193 (1991), p. 152.

[DS93] Guy David and Stephen Semmes. *Analysis of and on uniformly rectifiable sets.* Vol. 38. Mathematical Surveys and Monographs. Providence, RI: American Mathematical Society, 1993.

[DT12] Guy David and Tatiana Toro. “Reifenberg parameterizations for sets with holes”. *Mem. Amer. Math. Soc.* 215.1012 (2012), pp. vi+102.

[ENV19] Nick Edelen, Aaron Naber, and Daniele Valtorta. “Effective Reifenberg theorems in Hilbert and Banach spaces”. *Math. Ann.* 374.3-4 (2019), pp. 1139–1218.

[Fed69] Herbert Federer. *Geometric Measure Theory.* Classics in Mathematics. Springer-Verlag, Berlin Heidelberg, 1969.

[FKL14] Matt Feiszli, Sergey Kushnarev, and Kathryn Leonard. "Metric spaces of shapes and applications: compression, curve matching and low-dimensional representation". *Geom. Imaging Comput.* 1.2 (2014), pp. 173–221.

[FN17] Matt Feiszli and Akil Narayan. "Numerical computation of Weil-Petersson geodesics in the universal Teichmüller space". *SIAM J. Imaging Sci.* 10.3 (2017), pp. 1322–1345.

[FV23] Katrin Fassler and Ivan Yuri Violo. "On various Carleson-type geometric lemmas and uniform rectifiability in metric spaces". arXiv preprint, arXiv:2310.10519. 2023.

[Ghi20] Silvia Ghinassi. "Sufficient conditions for $C^{1,\alpha}$ paramerization and rectifiability". *Ann. Acad. Sci. Fenn. Math.* 45.2 (2020), pp. 1065–1094.

[GJM92] John Garnett, Peter Jones, and Donald Marshall. "A Lipschitz decomposition of minimal surfaces". *Journal of Differential Geometry* 35.3 (1992), pp. 659–673.

[GMT18] John Garnett, Mihalis Mourgoglou, and Xavier Tolsa. "Uniform rectifiability from Carleson measure estimates and ε-approximability of bounded harmonic functions". *Duke Math. J.* 167.8 (2018), pp. 1473–1524.

[Hah05] Immo Hahlomaa. "Menger curvature and Lipschitz parametrizations in metric spaces". *Fund. Math.* 185.2 (2005), pp. 143–169.

[HM12] T. Hytönen and H. Martikainen. "Non-homogeneous Tb theorem and random dyadic cubes on metric measure spaces". *J. Geom. Anal.* 22.4 (2012), pp. 1071–1107.

[Hyd22a] Matthew Hyde. "A d-dimensional Analyst's Traveling Salesman Theorem for subsets of Hilbert space". *Math. Ann.* (2022).

[Hyd22b] Matthew Hyde. "The restricted content and the d-dimensional Analyst's Travelling Salesman Theorem for general sets". *Adv. Math.* 397 (2022). Paper No. 108189.

[Jon90] Peter Jones. "Rectifiable sets and the traveling salesman problem". *Invent. Math.* 102.1 (1990), pp. 1–15.

[Kir94] Bernd Kirchheim. "Rectifiable metric spaces: local structure and regularity of the Hausdorff measure". *Proc. Amer. Math. Soc.* 121.1 (1994), pp. 113–123.

[KP02] Bernd Kirchheim and David Preiss. "Uniformly distributed measures in Euclidean spaces". *Math. Scand.* 90.1 (2002), pp. 152–160.

[KP87] Oldřich Kowalski and David Preiss. "Besicovitch-type properties of measures and submanifolds". *J. Reine Angew. Math.* 379 (1987), pp. 115–151.

[Ler03] Gilad Lerman. "Quantifying curvelike structures of measures by using L_2 Jones quantities". *Comm. Pure Appl. Math.* 56.9 (2003), pp. 1294–1365.

[Li22] Sean Li. "Stratified β-numbers and traveling salesman in Carnot groups". *J. London Math. Soc.* (2022).

[Lor03] Andrew Lorent. "Rectifiability of measures with locally uniform cube density". *Proc. London Math. Soc. (3)* 86.1 (2003), pp. 153–249.

[Mar61] J. M. Marstrand. "Hausdorff two-dimensional measure in 3-space". *Proc. London Math. Soc. (3)* 11 (1961), pp. 91–108.

[Mat75] Pertti Mattila. "Hausdorff m regular and rectifiable sets in n-space". *Trans. Amer. Math. Soc.* 205 (1975), pp. 263–274.

[Mat95] Pertti Mattila. *Geometry of sets and measures in Euclidean spaces*. Vol. 44. Cambridge Studies in Advanced Mathematics. Fractals and rectifiability. Cambridge University Press, Cambridge, 1995, pp. xii+343.

[Mer22] Andrea Merlo. "Geometry of 1-codimensional measures in Heisenberg groups". *Invent. Math.* 227.1 (2022), pp. 27–148.

[MPT22] Mihlais Mourgoglou, Bruno Poggi, and Xavier Tolsa. "L^p-solvability of the Poisson-Dirichlet problem and its applications to the regularity problem". Preprint. 2022.

[MT22] Mihalis Mourgoglou and Xavier Tolsa. "The regularity problem for the Laplace equation in rough domains". arxiv preprint, arxiv. 2022.

[Nim17] A. Dali Nimer. "A sharp bound on the Hausdorff dimension of the singular set of a uniform measure". *Calc. Var. Partial Differential Equations* 56.4 (2017), Paper No. 111, 31.

[Nim19] A. Dali Nimer. "Uniformly distributed measures have big pieces of Lipschitz graphs locally". *Ann. Acad. Sci. Fenn. Math.* 44.1 (2019), pp. 389–405.

[Nim22] A. Dali Nimer. "Conical 3-uniform measures: a family of new examples and characterizations". *J. Differential Geom.* 121.1 (2022), pp. 57–99.

[OHa91] Jun O'Hara. "Energy of a Knot". *Topology* 30.2 (1991), pp. 241–247.

[OHa92] Jun O'Hara. "Family of energy functionals of knots". *Topology Appl.* 48.2 (1992), pp. 147–161.

[Oki92] Kate Okikiolu. "Characterization of subsets of rectifiable curves in $\mathbf{R}^n$". *Journal of the London Mathematical Society* s2-46.2 (1992), pp. 336–348.

[Pre87] David Preiss. "Geometry of measures in $\mathbf{R}^n$: distribution, rectifiability, and densities". *Ann. of Math. (2)* 125.3 (1987), pp. 537–643.

[Rei60] Ernst Reifenberg. "Solution of the Plateau problem for m-dimensional surfaces of varying topological type". *Acta Mathematica* 104 (Oct. 1960), pp. 1–92.

[Sch07a] Raanan Schul. "Ahlfors-regular curves in metric spaces". *Ann. Acad. Sci. Fenn. Math.* 32.2 (2007), pp. 437–460.

[Sch07b] Raanan Schul. "Ahlfors-regular curves in metric spaces". *Ann. Acad. Sci. Fenn. Math.* 32.2 (2007), pp. 431–460.

[Sch09] Raanan Schul. "Bi-Lipschitz decomposition of Lipschitz functions into a metric space". *Rev. Mat. Iberoam.* 25.2 (2009), pp. 521–531.

[SM06] Eitan. Sharon and David Mumford. "2D-Shape analysis using conformal mapping". *Int. J. Comput. Vis.* 70 (2006), pp. 55–75.

[Tol12] Xavier Tolsa. "Mass transport and uniform rectifiability". *Geom. Funct. Anal.* 22.2 (2012), pp. 478–527.

[Tol15] Xavier Tolsa. "Uniform measures and uniform rectifiability". *J. Lond. Math. Soc. (2)* 92.1 (2015), pp. 1–18.

[TT06] Leon A. Takhtajan and Lee-Peng Teo. "Weil-Petersson metric on the universal Teichmüller space". *Mem. Amer. Math. Soc.* 183.861 (2006), pp. viii+119.

[TT15] Xavier Tolsa and Tatiana Toro. "Rectifiability via a square function and Preiss' theorem". *Int. Math. Res. Not. IMRN* 13 (2015), pp. 4638–4662.

[Vil20] Michele Villa. "Higher dimensional Jordan curves". Preprint. 2020.

[Wan19a] Yilin Wang. "Equivalent descriptions of the Loewner energy". *Invent. Math.* 21.7 (2019), pp. 573–621.

[Wan19b] Yilin Wang. “The energy of a deterministic Loewner chain: reversibility and interpretation via SLE_{0+}”. *J. Eur. Math. Soc. (JEMS)* 21.7 (2019), pp. 1915–1941.