One parameter families of Schwarz reflection maps arising from Shabat-Belyi maps

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Abstract of the Dissertation

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We study the iteration of Schwarz reflection maps, arising from polynomials which are univalent on a closed disk.These form an anti-holomorphic counterpart to the correspondences studied by S. Bullett, C. Penrose, L. Lomonaco, and others, which give matings of rational maps and Kleinian groups. Adapting the straightening theorem of Douady and Hubbard we show that such Schwarz reflection maps with connected non-escaping set are in a dynamical bijection with a certain class of parabolic rational maps.

We then turn our attention to special one-parameter families of such Schwarz reflections, which arise when the uniformizing polynomials are Shabat polynomials, and are indexed by rooted planar trees. We show that the parameter spaces and escape loci are connected and simply connected, and hence the connectedness loci are themselves connected. We give a partial combinatorial description of the connectedness loci. We show, as in the Mandelbrot set, that there are many renormalizable parameters, giving rise to little copies of Multibrot and Multicorns contained in the connectedness loci. We also use the recent results of Clark-Drach-Kozlovsky-van Strien (generalizing earlier results of Yoccoz and Avila-Kahn-Lyubich-Shen), to show that any parameters which are not renormalizable are combinatorially rigid.

Dedication

To Herb and Gladys. To Ben. To Esther.

Table of Contents

Acknowledgements

Who is wise? The one who
learns from all people.

Avot 4:1

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Chapter 1

Introduction

Complex dynamics is the study of holomorphic (and anti-holomorphic) actions on complex manifolds. The two oldest and most well studied cases within complex dynamics are the actions of discrete groups of Möbius transformations on the Riemann sphere $\widehat{\mathbb{C}}$, and the iteration of rational maps. Pierre Fatou, and later Dennis Sullivan, noted similarities between these two situations.

In the 90s, Shaun Bullett and Chris Penrose demonstrated that one could produce *holomorphic correspondences* which acted as matings of quadratic polynomials with the modular group. Bullett expanded on this with a number of co-authors, and with Luna Lomonaco showed that *any* parabolic quadratic rational map could be mated with the modular group, and that furthermore this mating respected the structure of the parameter spaces.

More recently, there has been interest in the study of *Schwarz reflection maps*, a class of *anti-holomorphic* dynamics. There have been a number of works by Seung-Yeop Lee, Mikhail Lyubich, Nikolai Makarov, and Sabyasachi Mukherjee, the author, and a number of other contributors. We refer the reader to the survey article [LM23] for an overview of some recent results.

This thesis details work done by the author which gives a coplementary, anti-holomorphic perspective to the Bullett-Lomonaco-Penrose correspondences. Results from the first half of this thesis appear in [LMM23], and are primarily *dynamical* results in nature, and the second half focuses on *one-parameter families* of Schwarz reflections.

The Schwarz reflections in question always have parabolic behavior at the cusp point. To develop a suitable semi-local theory for the dynamics of these Schwarz reflections, we introduce notions of *pinched* polynomial-like maps, and give a straightening theorem for them under suitable hypotheses (see definition 3.3.2).

Theorem A. *A simple pinched anti-polynomial-like map is hybrid conjugate to a Schwarz reflection, and to a parabolic anti-rational map. Furthermore, if the non-escaping set is connected, then the straightened maps are unique up to a holomorphic automorphism.*

The proof of this theorem is an adaptation of the classical Douady-Hubbard straightening theorem [DH85] and relies crucially on an analytic estimate for uniformizing strips due to Warschawski [War42].

This results naturally lends itself to the following.

Theorem B. *There is a dynamical bijection between a family* $\S_{\mathcal{R}_\Gamma}$ *of Schwarz reflection maps and a corresponding family* $\mathcal{F}_d$ *of parabolic anti-rational maps. Furthermore, this bijection is continuous at hyperbolic and quasiconformally rigid parameters.*

Straightneing results do not, in general, have any ability to describe what happens in parameter spaces when they are of complex dimension greater than one. We are able to obtain a dynamical bijection in higher dimension in this case as we have exactly two external classes we are concerned with, and the surgery replaces one with the other.

We then turn our attention to *parameter* questions regarding these maps. In particular, we consider Schwarz reflection maps for which there is a single unrestricted critical value. Such maps arise when the uniformizing map for their domains have at most two critical values, polynomials known as *Shabat polynomials*. These Shabat polynomials are classified by combinatorial bi-colored planar embedded trees, giving us many families of Schwarz reflections. We denote a family of such reflections by $\mathcal{S}_{\mathcal{T}}$.

Theorem C. *The family $\mathcal{S}_{\mathcal{T}}$ is the interior of a quadrilateral with real-analytic boundary arcs, together with one of the boundary arcs.*

One point of interest about this theorem that it is essentially an *analytic* result. It is a statement about disks on which a given polynomial is univalent. We prove it, however, using **dynamical** methods to analyze the degeneration of quadrature domains. It seems likely that such methods could be fruitful in the study of singularities for quadrature domains more broadly.

Following this we focus more closely on the connectedness locus for this space.

Theorem D. *The connectedness locus $\mathcal{C}(\mathcal{S}_{\mathcal{T}})$ is a hull, i.e. it is compact, connected, and has connected complement.*

We present a partial combinatorial model of the connectedness loci for these families, and prove that non-renormalizable maps within this family are combinatorially rigid.

1.1 Organization of chapters

Let us detail the organization of the rest of this thesis.

In Chapter 2 we detail background results in complex dynamics. First, for context, we recall the theory of polynomial-like maps, straightening and renormalization for unicritical polynomials introduced in [DH85] as well as Yoccoz's rigidity theorem for non-renormalizable quadratic polynomials. We then define *generalized polynomial-like mappings* (also known as *complex box mappings*) and recall more recent results on their rigidity given in [Avi+09; Cla+22].

We conclude the chapter with a theorem due to Alfredo Poirier which appears in [Poi13], based on the realization result of W. Thurston appears in [DH93], which gives a condition

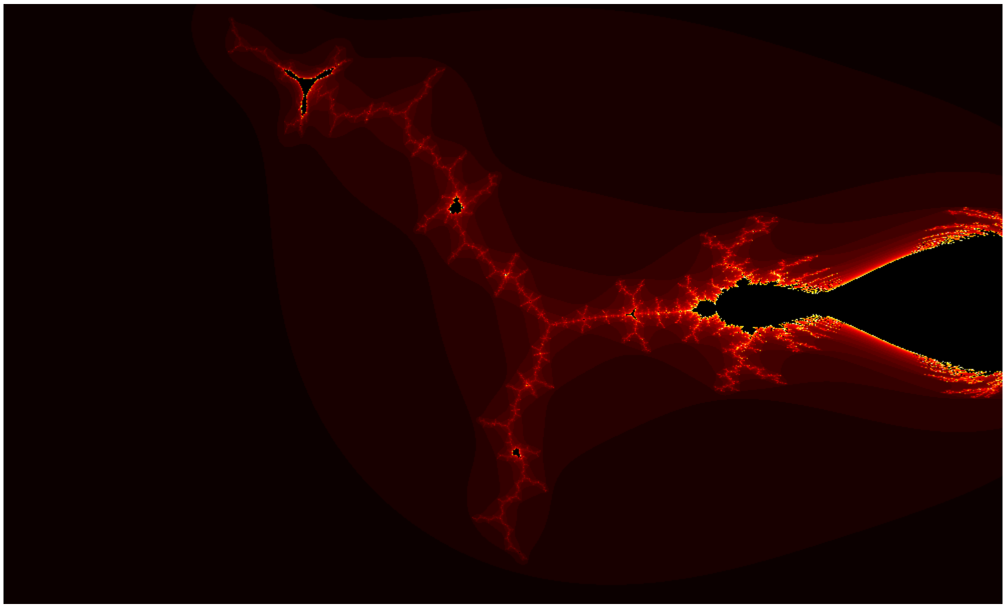

Figure 1.1: Pictured is the parameter space for a family of Shabat-Belyi polynomials. It gives a combinatorially equivalent picture to a corresponding family of Schwarz reflections.

for when combinatorial models called Hubbard trees may be realized by postcritically finite polynomials.

Chapter 3 concerns the dynamics of Schwarz reflections, anti-holomorphic maps on a domain which extend identity on the boundary. Such domains are called quadrature domains. We restrict our attention to those quadrature domains which are Jordan domains with one boundary singularity, and such that the reflection has a single maximal degree critical value lying outside the domain of definition. We describe an external class for such maps, playing an analogous role to the Böttcher coordinates for polynomials.

We then introduce the notion of pinched polynomial-like-maps. These are a natural class of semi-local restrictions of Schwarz reflections and parabolic anti-rational maps alike. We show that when one has a single pinching for these maps and that the behavior at such a pinching is of simple parabolic type, that one can straighten this map to a Schwarz reflection or a parabolic anti-rational map, giving theorem A.

This is not enough to prove the entirety of theorem B, as both Schwarz reflections and parabolic anti-rational maps contain higher order parabolic parameters. To prove a full bijection we introduce instead a different quasi-conformal surgery. The other surgery however does not yield as much information about how the straightening map varies under parameter change, and so both techniques are necessary.

In Chapter 4 we impose critical orbit relations on our Schwarz reflections so that there are two critical values, one of them necessarily being the singular point on the boundary of the quadrature domain. This implies that the domain for the reflection map is uniformized by a Shabat polynomial. We consider the combinatorics of such maps and give a realization theorem for post-critically finite ones, using a result of Poirier [Poi13]. We analyze the ways in which the corresponding domains can degenerate to prove theorem C.

Following this, we show that a hyperbolic component in our family can be described by the multiplier of the attracting periodic point together with some finite covering data, and that the boundaries of hyperbolic components consists of maps with neutral periodic points.

We then give a tessellation for the escape locus, and use this to prove theorem D.

After that we turn our attention to finer detail of the structure of the connectedness loci, $\mathcal{C}(\mathcal{S}_{\mathcal{T}})$. In particular, we introduce a puzzle structure for maps, show that renormalizable maps have associated with them full Multibrot or Multicorn combinatorial families in $\mathcal{C}(\mathcal{S}_{\mathcal{T}})$, and non-renormalizable parameters are combinatorially rigid.

We conclude by showing that the critical dessin $\mathcal{T}$ embeds into a combinatorial model for the connectedness locus $\widetilde{C}(\mathcal{S}_{\mathcal{T}})$, given by identifying non-hyperbolic combinatorial classes.

Chapter 2

Background

Here we recall some of the key results used in this thesis. For more thorough introductions to the field of complex dynamics we refer the reader to the books of Milnor [Mil06] and Lyubich [Lyu].

2.1 Polynomial-like maps, and generalized polynomial-like maps

2.1.1 Polynomial-like maps and Straightening

The notion of polynomial-like maps were introduced in [DH85].

Definition 2.1.1. Let U, V be open topological disks in $\mathbb{C}$ with U compactly contained in V. A polynomial-like map g is a holomorphic degree $d \geq 2$ branched cover $g\colon U \to V$.

The *filled Julia set* or *nonescaping set*, $K(g)$ of a polynomial-like map is the the intersection $\bigcap \overline{g^{-n}(U)}$. Two polynomial-like maps $g\colon U \to V$, $\tilde{g}\colon \tilde{U} \to \tilde{V}$ are said to be *hybrid conjugate* if there is a quasiconformal conjugacy $\varphi\colon U \to \tilde{U}$ such that φ is conformal on $K(g)$.

One source of examples for polynomial-like mappings is the restriction of polynomials to suitable domains. The following straightening theorem of Douady-Hubbard states that up to a hybrid conjugacy *all* polynomial like mappings are such restrictions.

Theorem 2.1.1. *[DH85] Any polynomial-like map $g\colon U \to V$ is hybrid conjugate to the restriction of a polynomial $f\colon \mathbb{C} \to \mathbb{C}$. Furthermore, if the filled Julia set of g is connected then the polynomial f is unique up to an affine conjugacy.*

2.1.2 Generalized Polynomial-Like Mappings and Rigidity

We now introduce a generalization of polynomial like mappings, called *generalized polynomial-like mappings* (also referred to as *complex box mappings*).

Definition 2.1.2. [Cla+22] Let V be finite union of Jordan domains with disjoint closures, $U \subset V$ be an open set so that each component of U is either itself a component of V or compactly contained in V.

A *generalized polynomial-like map* is a map $f\colon U \to V$, which is proper when restricted to any component of U.

In [Cla+22] they further consider conditions of "dynamical naturality" to ensure that certain pathologies do not occur. The conditions are:

1. No permutation condition: For each component $U_j \subset U$ there is some $n \geq 0$ so that $F^n(U) \setminus U \neq \emptyset$.

2. Define $K_{\text{off-crit}}(F)$ to be the subset of $K(F)$ consisting of points whose orbit stays disjoint from a puzzle neighborhood of the critical points of F. The Lebesgue measure of $K_{\text{off-crit}}(F)$ is zero.

3. For every $x \in K(F)$ there is some $\delta = \delta(x)$ so that

$$\limsup_k \operatorname{mod}\left(\operatorname{Comp}_{F^k(x)}(V) \setminus \overline{\operatorname{Comp}_{F^k(x)}(U)}\right) > \delta.$$

A generalized polynomial-like map $f: U \to V$ is said to be non-renormalizable if there is no restriction of f to a polynomial-like map with connected Julia set.

The main result of [Cla+22], generalizing the results of [Avi+09] is the following.

Theorem 2.1.2. *[Cla+22, Theorem 6.1, and the remark afterwards] Suppose that two dynamically natural complex box mappings are combinatorially equivalent. Then they are quasiconformally conjugate. Furthermore, if the Julia sets are nowhere dense then the maps are conformally conjugate.*

2.2 Postcritically Finite Polynomials

2.2.1 Hubbard trees and Poirier's realization theorem

Theorem 2.2.1. *[Poi13, Theorem 5.1] A normalized orientation-reversing angled tree dynamics can be realized as the dynamics associated to a postcritically finite anti-holomorphic polynomial map if and only if it is expanding. In other words, all anti-Hubbard trees — and only them — can be realized. Such a realization is unique up to affine conjugation in the dynamical plane.*

Chapter 3

Schwarz Reflection Maps as Matings of Anti-Holomorphic Parabolic Maps with Kleinian Reflection Groups

3.1 The Dynamics of Schwarz Reflections

3.1.1 Schwarz reflections associated to univalent rational maps

By definition, a domain $\Omega \subsetneq \widehat{\mathbb{C}}$ satisfying $\infty \notin \partial\Omega$ and $\Omega = \operatorname{int}\overline{\Omega}$ is a *quadrature domain* if there exists a continuous function $\sigma : \overline{\Omega} \to \widehat{\mathbb{C}}$ such that σ is anti-meromorphic in Ω and $\sigma(z) = z$ on the boundary $\partial\Omega$. Such a function σ is unique (if it exists), and is called the *Schwarz reflection map* associated with Ω. It is well known that except for a finite number of *singular* points (cusps and double points), the boundary of a quadrature domain consists of finitely many disjoint real-analytic curves [Sak91]. Every non-singular boundary point has a neighborhood where the local reflection in $\partial\Omega$ is well-defined. The (global) Schwarz reflection σ is an antiholomorphic continuation of all such local reflections.

Round disks on the Riemann sphere are the simplest examples of quadrature domains. Their Schwarz reflections are just the usual circle reflections. Further examples can be constructed using univalent polynomials or rational functions. In fact, simply connected quadrature domains admit a simple characterization.

Proposition 3.1.1. *[AS76, Theorem 1][LMM21, Proposition 2.3] A simply connected domain $\Omega \subsetneq \widehat{\mathbb{C}}$ with $\infty \notin \partial\Omega$ and $\operatorname{int}\overline{\Omega} = \Omega$ is a quadrature domain if and only if the Riemann uniformization $R : \mathbb{D} \to \Omega$ extends to a rational map on $\widehat{\mathbb{C}}$. The Schwarz reflection map σ of Ω is given by $R \circ \eta \circ (R|_{\mathbb{D}})^{-1}$.*

$$\begin{array}{ccc} \overline{\mathbb{D}} & \xrightarrow{R} & \overline{\Omega} \\ {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \sigma} \\ \widehat{\mathbb{C}} \setminus \mathbb{D} & \xrightarrow[R]{} & \widehat{\mathbb{C}}. \end{array}$$

In this case, if the degree of the rational map R is $d+1$, then $\sigma : \sigma^{-1}(\Omega) \to \Omega$ is a (branched) covering of degree d, and $\sigma : \sigma^{-1}(\operatorname{int}\Omega^c) \to \operatorname{int}\Omega^c$ is a (branched) covering of degree $d+1$.

We refer the reader to [AS76], [LM16], [Lee+18a, §3], [LMM21, §2] for more background on quadrature domains and Schwarz reflection maps.

For $d \geq 2$, the preimage $\sigma^{-1}(\Omega)$ is never relatively compact in Ω, and the Schwarz reflection does not restrict to an anti-polynomial-like map. It is convenient then to give a weaker notion generalizing this semi-local behavior.

We say that a *polygon* is a Jordan domain whose boundary consists of finitely many closed smooth arcs. The points of intersection of these arcs will be denoted as the corners of the polygon. A *pinched polygon* is a union of domains in $\widehat{\mathbb{C}}$ whose closure is homeomorphic to a closed disk quotiented by a finite geodesic lamination, and whose boundary is given by finitely many closed smooth arcs. The separating points of the closure of a pinched polygon will be called its pinched points.

Definition 3.1.1. Let $V \subset \widehat{\mathbb{C}}$ be a polygon, and let $U \subset V$ be a pinched polygon with $\overline{U} \cap \overline{V}$ consisting of the corners of V.

Suppose that there is a (anti-)holomorphic map $g \colon U \to V$ which is a branched cover from each component of U onto V, extends continuously to the boundary of U, and such that at the corners of V, the map g has local degree 1. We further suppose that corners and pinchings of U are the preimages of the corners of V.

We then call the triple $(g, \overline{U}, \overline{V})$ a *pinched (anti-)polynomial-like map.*

(See Figure 3.1.)

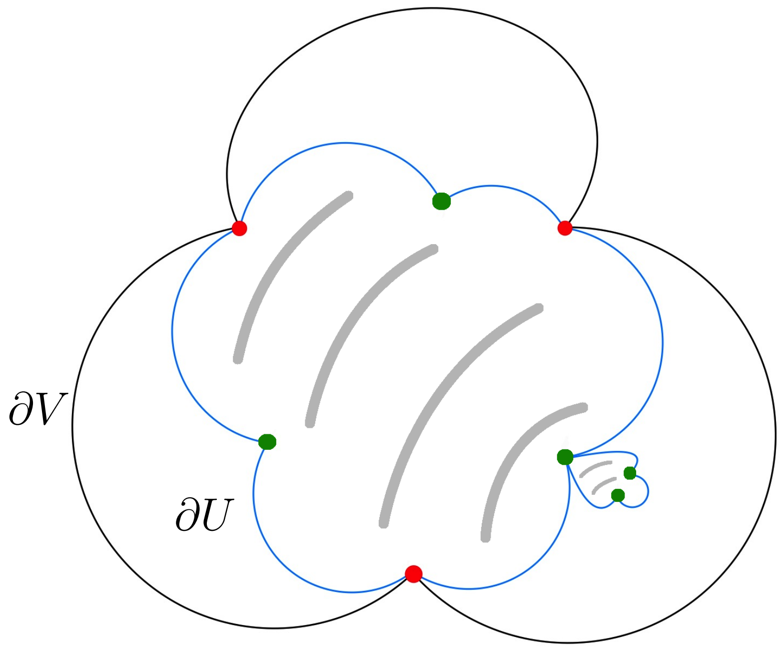

Figure 3.1: Pictured is the domain and codomain of a pinched (anti-)polynomial-like map. Here, V is the interior of the black polygon with three corners (marked in red). The interior of the blue pinched polygon is U. The pinched point and the additional corner points of ∂U are marked in green.

As with polynomial-like maps, we define the *filled Julia set* or *non-escaping set* of a pinched (anti-)polynomial-like map to be $K(g) = \bigcap_{n\geq 0} g^{-n}(\overline{U})$, and denote it by $K(g)$. Analogous to classical polynomial-like maps [DH85], the filled Julia set $K(g)$ of a pinched (anti-)polynomial-like map is connected if and only if it contains all of the critical values of g.

Let us give two important examples of pinched polynomial like maps.

1. For a simply connected quadrature domain Ω with Schwarz reflection σ the restriction $\sigma\colon \overline{\sigma^{-1}(\Omega)} \to \Omega$ is a pinched anti-polynomial-like map.

2. Let $R\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a(n) (anti-)rational map, with a parabolic fixed point and an associated fully invariant parabolic basin of attraction, B. Let $P \subset B$ be an attracting petal for the parabolic fixed point. Then the restriction $R\colon \hat{\mathbb{C}} \setminus R^{-1}(P) \to \hat{\mathbb{C}} \setminus P$ is a pinched (anti-)polynomial-like map.

In general there is no straightening theorem for all pinched polynomial-like maps as there is for standard polynomial-like maps. In a later section however we will prove a straightening theorem under additional assumptions.

Let us now return to the degree $(d+1)$ rational map f that is univalent on $\overline{\mathbb{D}}$. We set $\Omega := f(\mathbb{D})$ and denote the associated Schwarz reflection map by σ.

We define $T(\sigma) := \widehat{\mathbb{C}} \setminus \Omega$ and $S(\sigma)$ to be the singular set of $\partial T(\sigma)$. We further set $T^0(\sigma) := T(\sigma) \setminus S(\sigma)$, and

$$T^\infty(\sigma) := \bigcup_{n \geq 0} \sigma^{-n}(T^0(\sigma)).$$

We will call $T^\infty(\sigma)$ the *tiling set* of σ. For any $n \geq 0$, the connected components of $\sigma^{-n}(T^0(\sigma))$ are called *tiles* of rank n. Two distinct tiles have disjoint interior. The *non-escaping set* of σ is defined as

$$K(\sigma) := \widehat{\mathbb{C}} \setminus T^\infty(\sigma) \subset \Omega \cup S(\sigma).$$

The common boundary of the non-escaping set $K(\sigma)$ and the tiling set $T^\infty(\sigma)$ is called the *limit set* of σ, denoted by $\Lambda(\sigma)$.

Proposition 3.1.2. *The tiling set $T^\infty(\sigma)$ is open, and hence the non-escaping set $K(\sigma)$ is closed.*

Proof. Let us denote the union of the tiles of rank 0 through k by E^k.

If $z \in T^\infty(\sigma)$ belongs to the interior of a tile of rank k, then it clearly belongs to $\operatorname{int} E^k$. On the other hand, if $z \in T^\infty(\sigma)$ belongs to the boundary of a tile of rank k, then z lies in $\operatorname{int} E^{k+1}$. Hence,

$$T^\infty(\sigma) = \bigcup_{k \geq 0} \operatorname{int} E^k.$$

So $T^\infty(\sigma)$ is a union of open sets. The result now follows. $\square$

Proposition 3.1.3. *For Ω a quadrature domain and σ the associated Schwarz reflection, the following are equivalent.*

1. *The critical values of σ which lie in Ω also lie in $K(\sigma)$.*

2. *$T^\infty(\sigma)$ is a simply connected domain.*

3. *$K(\sigma_a)$ is connected.*

Proof. ($1 \implies 2$) Let E^k be the union of the tiles of rank $\leq k$.

Note that since $\boldsymbol{v_w} \in \Delta_a$, we have $y_2 \in \Omega_a$. Hence, $\sigma_a : \sigma_a^{-1}(\operatorname{int} T^0(\sigma_a)) \to \operatorname{int} T^0(\sigma_a)$ is a degree $(d+1)$ branched cover branched only at $f(a)$. It now follows from the Riemann-Hurwitz formula that $\sigma_a^{-1}(\operatorname{int} T^0(\sigma_a))$ is a simply connected domain. Moreover, we have $\partial T^0(\sigma_a) \subset \partial \sigma_a^{-1}(T^0(\sigma_a))$. Hence, $\operatorname{int} E^1$ is a simply connected domain.

If $y_2 \in K(\sigma_a)$, then every tile of rank ≥ 2 is unramified, and we can iterate the arguments of the previous paragraph to conclude that $\operatorname{int} E^k$ is a simply connected domain, for each k. Since $T^\infty(\sigma_a) = \bigcup_{k\geq 0} \operatorname{int} E^k$ (see Proposition 3.1.2), we conclude that $T^\infty(\sigma_a)$ is a simply connected domain.

($2 \implies 3$) The complement of a simply connected domain is a full continuum.

($3 \implies 1$) Suppose that $K(\sigma_a)$ is connected. If $y_2 \in T^\infty(\sigma_a)$, then the tile containing a critical point of σ_a with corresponding critical value y_2 would be ramified, and disconnect $K(\sigma_a)$. Therefore, y_2 must lie in $K(\sigma_a)$. $\square$

3.2 The space $\mathcal{S}_{\mathcal{R}_d}$ of Schwarz reflections

The goal of this section is to introduce a space of Schwarz reflection maps that will give rise to matings of parabolic Bers anti-rational maps and Hecke reflection groups, and describe their dynamics on the tiling set. These Schwarz reflections arise from quadrature domains which are Jordan domains with exactly one singularity on the boundary.

Suppose that there exists a degree $d+1$ rational map f that admits a univalent restriction $f|_{\overline{\mathbb{D}}}$ such that $T^\infty(\sigma)$ (where σ is the Schwarz reflection map of the quadrature domain $f(\mathbb{D})$) is a simply connected domain containing exactly one critical value v_0 of f, and $v_0 \in \operatorname{int} T^0(\sigma)$ with $f^{-1}(v_0)$ a singleton. By the commutative diagram of Subsection 3.1.1, the set of critical points of σ is given by

$$\{f(\eta(c)) : c \in \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}, \text{ and } c \text{ is a critical point of } f\}.$$

One now sees that σ must have a unique critical point in its tiling set $T^\infty(\sigma)$ and this critical point maps to $v_0 \in \operatorname{int} T^0(\sigma)$ with local degree $d+1$. It is this property of σ on its tiling set that will be captured by the anti-Farey map $\mathcal{R}_d$ defined below (cf. [Lee+21, §4.4]).

3.2.1 The anti-Farey map $\mathcal{R}_d$ and the anti-Hecke group Γ_d

Consider the Euclidean circles $\widetilde{C}_1, \cdots, \widetilde{C}_{d+1}$ where $\widetilde{C}_j$ intersects $\{|z| = 1\}$ at right angles at the roots of unity $\exp\left(\frac{2\pi i\cdot(j-1)}{d+1}\right)$, $\exp\left(\frac{2\pi i\cdot j}{d+1}\right)$. We denote the intersection of $\mathbb{D} \cap \widetilde{C}_j$ by C_j. Then $C_1, \cdots, C_{d+1}$ are hyperbolic geodesics in $\mathbb{D}$, and they form a closed ideal polygon (in the topology of $\mathbb{D}$) which we call Π.

Let ρ_j be reflection with respect to the circle $\widetilde{C}_j$, V_j be the bounded connected component of $\widehat{\mathbb{C}} \setminus \widetilde{C}_j$, and $\mathbb{D}_j := V_j \cap \mathbb{D}$ (see Figure 3.2). Note that V_j is the symmetrization of $\mathbb{D}_j$ with respect to the unit circle. The maps $\rho_1, \cdots, \rho_{d+1}$ generate a subgroup $\mathcal{G}_d$ of $\operatorname{Aut}^\pm(\mathbb{D})$. As an abstract group, it is given by the generators and relations

$$\langle \rho_1, \cdots, \rho_{d+1} : \rho_1^2 = \cdots = \rho_{d+1}^2 = \mathrm{id} \rangle.$$

3.2.1.1 The anti-Farey map $\mathcal{R}_d$

We define $M_\omega : z \mapsto \omega z$, and consider the (orbifold) Riemann surfaces $\mathcal{Q} := \mathbb{D}/\langle M_\omega \rangle$ and $\widetilde{\mathcal{Q}} := \widehat{\mathbb{C}}/\langle M_\omega \rangle$, where $\omega := e^{\frac{2\pi i}{d+1}}$. Note that a (closed) fundamental domain for the action of $\langle M_\omega \rangle$ on $\widehat{\mathbb{C}}$ is given by

$$\{z \in \mathbb{C} : \ 0 \le \arg z \le \frac{2\pi}{d+1}\} \cup \{0, \infty\},$$

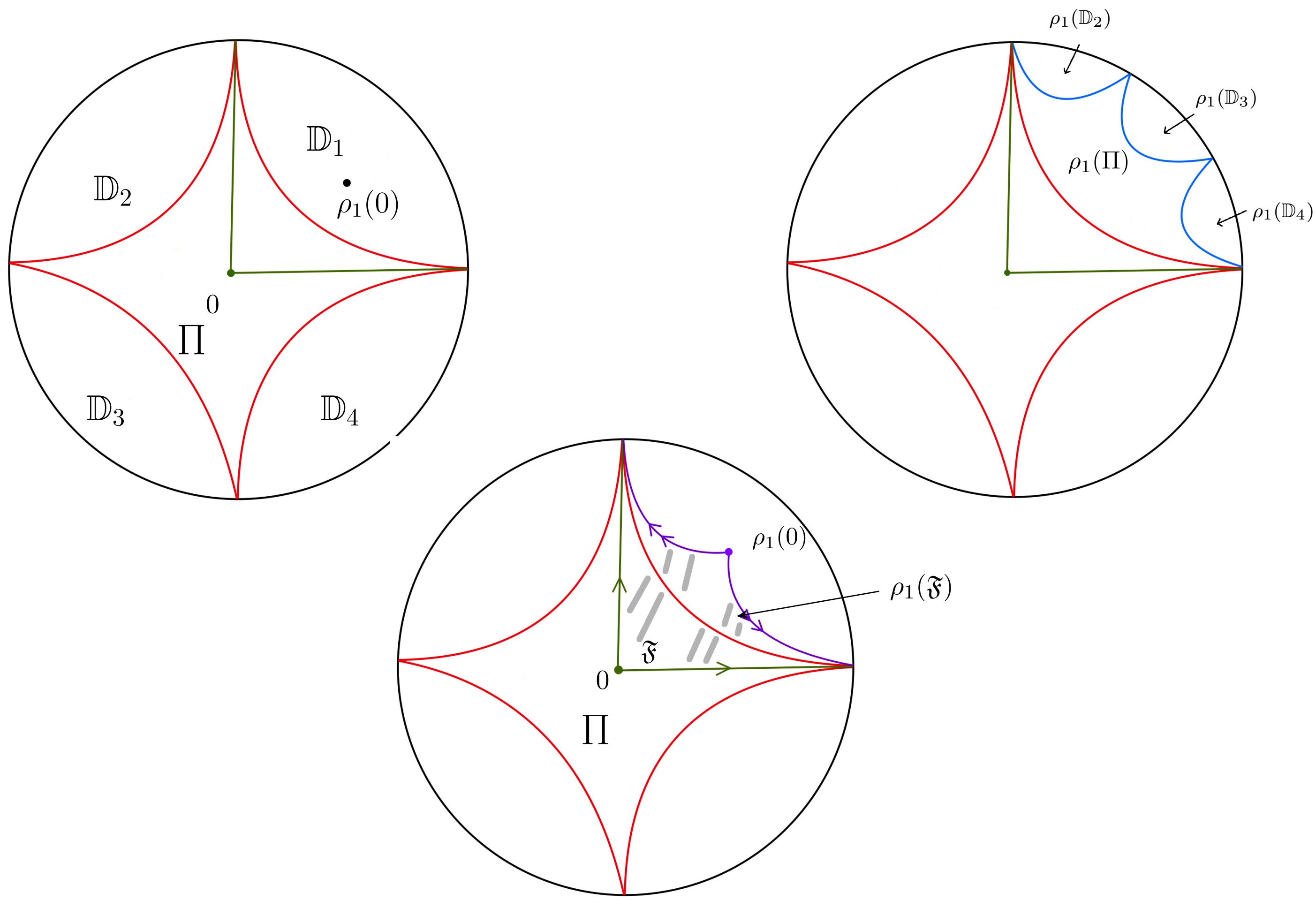

Figure 3.2: Top: Depicted is the ideal polygon Π for $d = 3$. The set $\rho_1(\Pi)$ is also shown. $\rho_1(\Pi)$ is mapped by $\mathcal{R}_3$ as a $4:1$ branched cover onto $\mathcal{Q}_1 = {}^{\Pi}\!\big/_{\langle M_i \rangle} \subset \mathcal{Q}$ (where $M_i : z \mapsto iz$). The unique critical point of $\mathcal{R}_3$ is $\rho_1(0)$. Bottom: $\mathfrak{F}$ is a fundamental domain for the action of the group $\mathbf{\Gamma}_3$, generated by ρ_1 and M_i, on $\mathbb{D}$. A fundamental domain for the action of the associated index two Fuchsian subgroup $\widetilde{\mathbf{\Gamma}}_3$ on $\mathbb{D}$ is given by $\widetilde{\mathfrak{F}} = \mathfrak{F} \cup \rho_1(\mathfrak{F})$. The generators M_i and $\rho_1 \circ M_i \circ \rho_1$ of $\widetilde{\mathbf{\Gamma}}_3$ pair the sides of $\widetilde{\mathfrak{F}}$ as indicated by the arrows. This shows that ${}^{\mathbb{D}}\!\big/_{\widetilde{\mathbf{\Gamma}}_3}$ is a sphere with one puncture and two orbifold points of order four.

and a (closed) fundamental domain for its action on $\mathbb{D}$ is given by

$$\{|z| < 1,\ 0 \leq \arg z \leq \frac{2\pi}{d+1}\} \cup \{0\}.$$

Thus, $\mathcal{Q}$ (respectively, $\widetilde{\mathcal{Q}}$) is biholomorphic to the surface obtained from the above fundamental domain by identifying the radial line segments $\{r : 0 < r < 1\}$ and $\{re^{\frac{2\pi i}{d+1}} : 0 < r < 1\}$ (respectively, the infinite radial rays at angles 0 and $\frac{2\pi i}{d+1}$) by M_ω. This endows $\mathcal{Q}$ and $\widetilde{\mathcal{Q}}$ with preferred choices of complex coordinates. With these coordinates, the identity map is an embedding of the (bordered) surface $\mathbb{D}_1 \cup C_1$ (respectively, $V_1 \cup \widetilde{C}_1$) into $\mathcal{Q}$ (respectively, $\widetilde{\mathcal{Q}}$).

The map ρ_1 induces a map

$$\mathcal{R}_d \colon V_1 \cup \widetilde{C}_1 \xrightarrow{\rho_1} \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}/\langle M_\omega \rangle = \widetilde{\mathcal{Q}}.$$

We note that $\partial\mathcal{Q} := \mathbb{S}^1/\langle M_\omega \rangle$ is topologically a circle, and $\mathcal{R}_d$ restricts to an orientation-reversing degree d covering of $\partial\mathcal{Q} \subset \widetilde{\mathcal{Q}}$ with a unique neutral fixed point (at 1). By [Lyu+20, Lemma 3.7], $\mathcal{R}_d|_{\partial\mathcal{Q}}$ is expansive. Moreover, $\mathcal{R}_d$ has a critical point of multiplicity d at $\rho_1(0)$ with associated critical value 0. We also note that all points in $\mathbb{D}_1 \cup C_1$ eventually escape to $\mathcal{Q}_1 := \Pi/\langle M_\omega \rangle \subset \mathcal{Q}$ under iterates of $\mathcal{R}_d$. We refer to the map $\mathcal{R}_d$ as the degree d anti-Farey map (see [Lod+23, §9] for connections between the map $\mathcal{R}_2$ and an orientation-reversing version of the classical Farey map).

Note that the map $z \mapsto z^{d+1}$ yields a conformal isomorphism ξ between the surface $\widetilde{\mathcal{Q}}$ and the Riemann sphere $\widehat{\mathbb{C}}$. This isomorphism restricts to a homeomorphism between $\partial\mathcal{Q}$ and $\mathbb{S}^1$.

3.2.1.2 The anti-Hecke group $\mathbf{\Gamma}_d$

Consider the subgroup $\mathbf{\Gamma}_d$ of $\mathrm{Aut}^{\pm}(\mathbb{D})$ generated by ρ_1 and M_ω. It is easy to see that $\mathbf{\Gamma}_d$ is a discrete group isomorphic to Γ_d, and a (closed) fundamental domain $\mathfrak{F}$ for the $\mathbf{\Gamma}_d$-action on $\mathbb{D}$ is given by

$$\mathfrak{F} := \{z \in \Pi :\ 0 \leq \arg z \leq \frac{2\pi}{d+1}\} \cup \{0\}.$$

The index two Fuchsian subgroup $\widetilde{\mathbf{\Gamma}}_d$ of $\mathbf{\Gamma}_d$ is generated by M_ω and $\rho_1 \circ M_\omega \circ \rho_1$. A (closed) fundamental domain for the $\widetilde{\mathbf{\Gamma}}_d$-action on $\mathbb{D}$ is given by $\widetilde{\mathfrak{F}} := \mathfrak{F} \cup \rho_1(\mathfrak{F})$, which is the double of $\mathfrak{F}$. It is easy to check that $\mathbb{D}/\widetilde{\mathbf{\Gamma}}_d$ is a sphere with one puncture and two orbifold points of order $d+1$ (see Figure 3.2).

3.2.2 Schwarz reflections with external class $\mathcal{R}_d$

We will now show that restrictions on the uniformizing map f forces the external class of σ to be the anti-Farey map $\mathcal{R}_d$.

Proposition 3.2.1. *Let f be a rational map of degree $d+1$ that is injective on $\overline{\mathbb{D}}$, $\Omega := f(\mathbb{D})$, and σ the Schwarz reflection map associated with Ω. Then the following are equivalent.*

1. *$T^\infty(\sigma)$ is a simply connected domain containing exactly one critical value v_0 of f. Moreover, $v_0 \in \operatorname{int} T^0(\sigma)$ with $f^{-1}(v_0)$ a singleton.*

2. *Ω is a Jordan domain with a unique conformal cusp on its boundary. Moreover, σ has a unique critical point in its tiling set $T^\infty(\sigma)$, and this critical point maps to $v_0 \in \operatorname{int} T^0(\sigma)$ with local degree $d+1$.*

3. *There exists a conformal conjugacy ψ between*

$$\mathcal{R}_d : \mathcal{Q} \setminus \operatorname{int} \mathcal{Q}_1 \longrightarrow \mathcal{Q} \quad \text{and} \quad \sigma : T^\infty(\sigma) \setminus \operatorname{int} T^0(\sigma) \longrightarrow T^\infty(\sigma).$$

   In particular, $T^\infty(\sigma)$ is simply connected.

4. *After possibly conjugating σ by a Möbius map and pre-composing f with an element of* $\operatorname{Aut}(\mathbb{D})$, *the uniformizing map f can be chosen to be a polynomial with a unique critical point on $\mathbb{S}^1$. Moreover, $K(\sigma)$ is connected.*

Proof. <u>(1) $\implies$ (2)</u>: That Ω is a Jordan domain follows from injectivity of f on $\overline{\mathbb{D}}$. By the classification of singular points on boundaries of quadrature domains, there is no double point on $\partial\Omega$; i.e., any singularity of $\partial\Omega$ must be a conformal cusp. We also note that $\operatorname{int} T^0(\sigma) = \widehat{\mathbb{C}} \setminus \overline{\Omega}$ is a Jordan domain.

Since $\operatorname{int} T^0(\sigma) \subsetneq T^\infty(\sigma)$ contains exactly one critical value v_0 of f and $f^{-1}(v_0)$ is a singleton, it follows that v_0 is the unique critical value of σ in $\operatorname{int} T^0(\sigma)$ and $\sigma^{-1}(v_0)$ is a singleton. It follows by Riemann-Hurwitz formula that $\sigma^{-1}(\operatorname{int} T^0(\sigma))$ is a simply connected domain.

Let us suppose, by way of contradiction, that $\partial\Omega$ is non-singular. Then, $T^0(\sigma) = \widehat{\mathbb{C}} \setminus \Omega$ is a closed Jordan domain, and the rank one tile $\sigma^{-1}(T^0(\sigma))$ contains a one-sided annular neighborhood of $\partial\Omega$. Since $\sigma^{-1}(T^0(\sigma))$ is simply connected, it must be equal to $\overline{\Omega}$, and hence $T^0(\sigma) \cup \sigma^{-1}(T^0(\sigma))$ must be the whole Riemann sphere. But this contradicts the fact that $\sigma^{-1}(\Omega) \neq \emptyset$. Hence, $\partial\Omega$ must have at least one conformal cusp.

We claim that $\partial\Omega$ cannot have more than one conformal cusp. By way of contradiction, assume that it has at least two cusps x_1, x_2. Due to connectivity of $\sigma^{-1}(\operatorname{int} T^0(\sigma))$, the union $T^0(\sigma) \cup \sigma^{-1}(T^0(\sigma))$ of the rank zero and rank one tiles must contain a simple closed curve γ in its interior such that x_1 and x_1 lie in different components of $\widehat{\mathbb{C}} \setminus \gamma$. As $x_1, x_2 \in K(\sigma)$, we conclude that $T^\infty(\sigma)$ is not simply connected. This contradicts the hypothesis, and proves our claim.

Thus, we have demonstrated that Ω is a Jordan domain with a unique conformal cusp on its boundary. By the commutative diagram of Subsection 3.1.1, a critical value of σ is also a critical value of f. Hence, σ has a unique critical value in $T^\infty(\sigma)$. Since $\sigma^{-1}(v_0) = f|_{\mathbb{D}}(\eta(f^{-1}(v_0))$ and $f^{-1}(v_0)$ is a singleton, we conclude that σ has a unique critical point in $T^\infty(\sigma)$, and the associated critical value is v_0. Moreover, since f has global degree $d+1$, it follows that the unique critical point of σ in the tiling set maps to $v_0 \in \operatorname{int} T^0(\sigma)$ with local degree $d+1$.

<u>(2) $\implies$ (3)</u>: As $\mathcal{Q}_1$ is simply connected, we can choose a homeomorphism

$$\psi : \mathcal{Q}_1 \to T^0(\sigma)$$

such that it is conformal on the interior (note that both $\mathcal{Q}_1$ and $T^0(\sigma)$ are closed topological disks with one boundary point removed). We can further assume that $\psi(0) = v_0$, and its continuous extension sends the cusp point $1 \in \partial\mathcal{Q}_1$ to the unique cusp on $\partial T^0(\sigma) = \partial\Omega$.

Note that $\sigma : \sigma^{-1}(T^0(\sigma)) \to T^0(\sigma)$ is a $(d+1) : 1$ branched cover branched only at $\sigma^{-1}(v_0)$, and $\mathcal{R}_d : \rho_1(\Pi) \to \mathcal{Q}_1$ is a $(d+1) : 1$ branched cover branched only at $\rho_1(0)$. Moreover, σ fixes $\partial T^0(\sigma)$ pointwise, and $\mathcal{R}_d$ fixes $C_2 \cup \{1\} \cong \partial \mathcal{Q}_1$ pointwise.

This allows one to lift ψ to a conformal isomorphism from $\rho_1(\Pi)$ onto $\sigma^{-1}(T^0(\sigma))$ such that the lifted map sends $\rho_1(0)$ to $\sigma^{-1}(v_0)$, and continuously matches with the initial map ψ on $\mathcal{Q}_1$. Abusing notation, we denote this extended conformal isomorphism by ψ. By construction, ψ is equivariant with respect to the actions of $\mathcal{R}_d$ and σ on $\partial\rho_1(\Pi)$ and $\partial\sigma^{-1}(T^0(\sigma))$, respectively.

Since $T^\infty(\sigma)$ contains no other critical point of σ, every tile of $T^\infty(\sigma)$ of rank greater than one maps diffeomorphically onto $\sigma^{-1}(T^0(\sigma))$ under some iterate of σ. Similarly, each tile of $\mathbb{D}_1$ of rank greater than one maps diffeomorphically onto $\rho_1(\Pi)$ under some iterate of $\mathcal{R}_d$. This fact, along with the equivariance property of ψ mentioned above, enables us to lift ψ to all tiles using the iterates of $\mathcal{R}_d$ and σ. This produces the desired biholomorphism ψ between $\mathcal{Q}$ and $T^\infty(\sigma)$ which conjugates the anti-Farey map $\mathcal{R}_d$ to the Schwarz reflection σ.

(3) $\implies$ (4): Simple connectivity of $T^\infty(\sigma)$ follows from the same property of $\mathcal{Q}$, and this implies connectivity of $K(\sigma)$. By hypothesis, σ has a unique critical point in $\sigma^{-1}(T^0(\sigma)) \subset T^\infty(\sigma)$. We denote this critical point by c_∞. Conjugating σ by a Möbius map, we can assume that this critical point maps with local degree $d+1$ to ∞. We can normalize f (which amounts to pre-composing it with an element of $\mathrm{Aut}(\mathbb{D})$) so that it sends 0 to c_∞. The commutative diagram in Subsection 3.1.1 now implies that f sends ∞ to itself with local degree $d+1$. Consequently, f is a degree $d+1$ polynomial. It remains to prove that f has a unique critical point on $\mathbb{S}^1$. This will follow from the next paragraph, where we argue that $\partial\Omega$ has a unique singular point, which is a conformal cusp.

The biholomorphism ψ induces a homeomorphism between $\partial\mathcal{Q}_1$ (boundary taken in $\widetilde{\mathcal{Q}}$, see Subsection 3.2.1) and $\partial T^0(\sigma)$. Note also that the map $\mathcal{R}_d$ admits local anti-conformal extensions around each point of C_1 (see Subsection 3.2.1), but does not have any such extension in a relative neighborhood of 1 in $\overline{\mathcal{Q}}$ (closure taken in $\widetilde{\mathcal{Q}}$). It follows via the conjugacy ψ that σ admits local anti-conformal extensions around each point of $\partial T^0(\sigma) \setminus \{\psi(1)\}$, but does not have any such extension in a neighborhood of $\psi(1)$. This implies that the Jordan curve $\partial T^0(\sigma) = \partial\Omega$ has a unique singular point at $\psi(1)$, which must be a conformal cusp.

$\underline{(4) \implies (1)}$: Connectedness of $K(\sigma)$ implies that $T^\infty(\sigma)$ is simply connected.

Since f is a polynomial, σ has a d-fold critical point at $f(0)$ with associated critical value $\infty \in \operatorname{int} T^0(\sigma)$. Moreover, $f^{-1}(\infty) = \{\infty\}$. If any tile of $T^\infty(\sigma)$ of rank greater than one contains a critical point of σ, then such a tile would be ramified, and disconnect $K(\sigma)$ (cf. [Lee+18a, Proposition 5.23]). Therefore, $T^\infty(\sigma)$ does not contain any other critical value of σ and hence does not contain any critical value of f other than $v_0 = \infty$. □

Definition 3.2.1. We define $\mathcal{S}_{\mathcal{R}_d}$ to be the space of pairs (Ω, σ), where

1. Ω is a Jordan quadrature domain with associated Schwarz reflection map $\sigma : \overline{\Omega} \to \widehat{\mathbb{C}}$, and
2. there exists a conformal map $\psi : (\mathcal{Q}, 0) \to (T^\infty(\sigma), \infty)$ that conjugates $\mathcal{R}_d : \mathcal{Q} \setminus \operatorname{int} \mathcal{Q}_1 \longrightarrow \mathcal{Q}$ to $\sigma : T^\infty(\sigma) \setminus \operatorname{int} T^0(\sigma) \longrightarrow T^\infty(\sigma)$.

We endow this space with the Carathéodory topology (cf. [McM94, §5.1]).

Remark 3.2.2. The family $\mathcal{S}_{\mathcal{R}_d}$ can be thought of as a Bers slice in the space of Schwarz reflection maps, since all maps in this family have the same external dynamics $\mathcal{R}_d$.

The next corollary follows from Proposition 3.2.1.

Corollary 3.2.3. *Let $(\Omega, \sigma) \in \mathcal{S}_{\mathcal{R}_d}$. Then $\partial\Omega$ has a unique conformal cusp $\boldsymbol{y}$ on its boundary. Moreover, there exists a polynomial f of degree $d+1$ with a unique critical point on $\mathbb{S}^1$ such that f carries $\overline{\mathbb{D}}$ injectively onto $\overline{\Omega}$.*

The following proposition shows that $\mathcal{S}_{\mathcal{R}_d}$ is large. The proof relies on *David surgery* and a full proof can be found in [LMM23]

Let us recall that an anti-rational map is said to be *semi-hyperbolic* if it has no parabolic cycles and all critical points in its Julia set are non-recurrent.

Proposition 3.2.4. *Let p be a degree d semi-hyperbolic anti-polynomial with a connected Julia set. Then, there exist*

- $(\Omega, \sigma) \in \mathcal{S}_{\mathcal{R}_d}$, *and*
- *a global David homeomorphism $\mathfrak{H}$ that is conformal on* $\operatorname{int} \mathcal{K}(p)$,

such that $\mathfrak{H}$ conjugates $p|_{\mathcal{K}(p)}$ to $\sigma|_{K(\sigma)}$.

Definition 3.2.2. We call $(\Omega, \sigma) \in \mathcal{S}_{\mathcal{R}_d}$ *relatively hyperbolic* if the forward σ-orbit of each critical point of σ in $K(\sigma)$ converges to an attracting cycle.

Remark 3.2.5. (1) There are two major difficulties in carrying out the arguments of Proposition 3.2.4 for an arbitrary degree d anti-polynomial with a connected Julia set. Firstly, the Julia set of p may not be locally connected, in which case the proof breaks down. Secondly, even if the Julia set is locally connected, lack of expansion along the postcritical set of p may result in loss of control of the geometry of $\mathcal{B}_\infty(p)$. This may, in turn, imply that the Beltrami coefficient constructed in the proof of Proposition 3.2.4 is not a David coefficient (note that Johnness of the basin of infinity for semi-hyperbolic maps was used crucially in our proof).

(2) We use the term relatively hyperbolic (as opposed to hyperbolic) because the external map of every Schwarz reflection map in $\mathcal{S}_{\mathcal{R}_d}$ has a parabolic fixed point. Thus, there is no expanding conformal metric in a neighborhood of the limit set of a relatively hyperbolic map in $\mathcal{S}_{\mathcal{R}_d}$.

(3) A member of $\mathcal{S}_{\mathcal{R}_d}$ obtained by applying Proposition 3.2.4 on a hyperbolic anti-polynomial p with connected Julia set is relatively hyperbolic.

Note that there is a natural action of $\operatorname{Aut}(\mathbb{C})$ on $\mathcal{S}_{\mathcal{R}_d}$ given by $A\cdot(\Omega, \sigma) := (A(\Omega), A\circ\sigma\circ A^{-1})$. We will define the set of equivalence classes by

$$[\mathcal{S}_{\mathcal{R}_d}] := \mathcal{S}_{\mathcal{R}_d} / \operatorname{Aut}(\mathbb{C}),$$

and denote the equivalence class of (Ω, σ) by $[\Omega, \sigma]$.

The next result shows that $\mathcal{S}_{\mathcal{R}_d}$ also contains the closure of relatively hyperbolic maps.

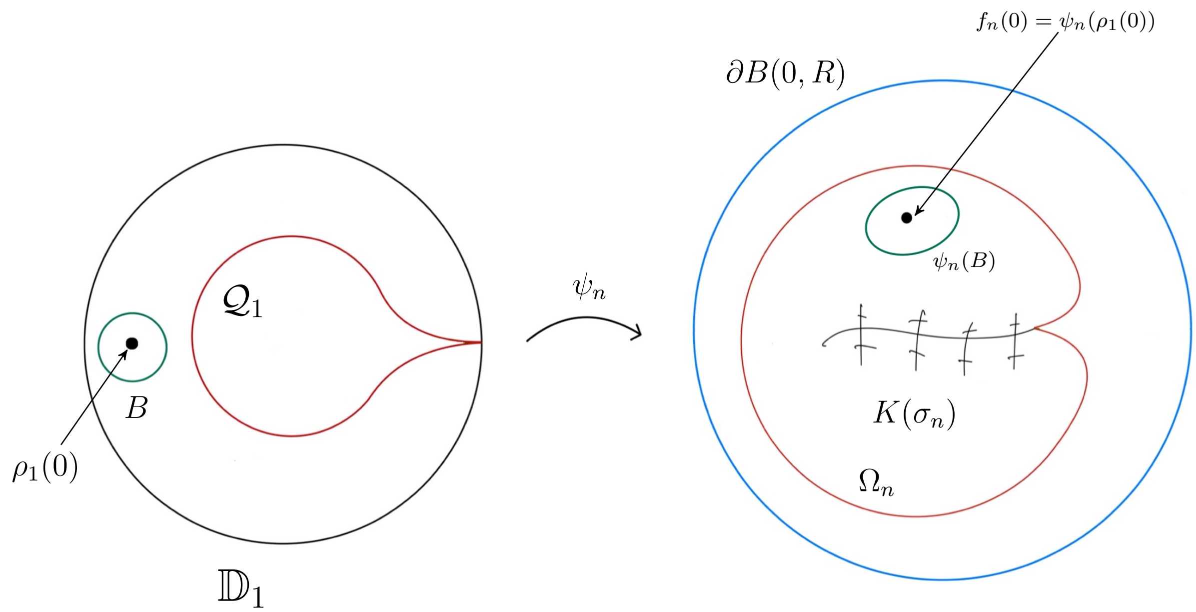

Figure 3.3: The pointed domains $(\Omega_n, f_n(0))$ have a non-trivial Carathéodory limit since all of them contain an open set $\psi_n(B)$ of definite size and are contained in $B(0, R)$.

Proposition 3.2.6. *The moduli space* $[\mathcal{S}_{\mathcal{R}_d}]$ *is compact.*

Proof. Let $\{[\Omega_n, \sigma_n]\}$ be a sequence in $[\mathcal{S}_{\mathcal{R}_d}]$ where $\psi_n : \mathcal{Q} \to T^\infty(\sigma_n)$ is a conformal conjugacy between $\mathcal{R}_d$ and σ_n. We will show that there is a convergent subsequence. We can choose a representative from $[\Omega_n, \sigma_n]$ for which $\psi_n(0) = \infty$, and $\psi_n(z) = 1/z + O(z)$ as $z \to 0$ (this amounts to replacing Ω_n by an affine image of it). By the normality of schlicht maps we may pass to a convergent subsequence, whose limit we denote by ψ_∞.

Let $f_n : \mathbb{D} \to \Omega_n$ be a uniformizing map. We normalize f_n as in Proposition 3.2.1 so that it extends to a degree $d+1$ polynomial on $\widehat{\mathbb{C}}$. The normalization of ψ_n and the Koebe 1/4 theorem imply that there is some $R > 0$ such that $\psi_n(\mathcal{Q}_1) \supset \widehat{\mathbb{C}} \setminus B(0, R)$, and hence $\Omega_n \subset \overline{B(0, R)}$ for all n (see Figure 3.3). This implies that the coefficients of f_n are uniformly bounded, and so after passing to a subsequence there is a limit polynomial f_∞ of degree at most $d+1$ which is univalent on $\mathbb{D}$.

Take B to be a neighborhood of $\rho_1(0)$ which is compactly contained in $\mathbb{D}_1$. We note that for all n sufficiently large that $\psi_\infty(B) \subset \psi_n(\mathbb{D}_1) \subset f_n(\mathbb{D}) = \Omega_n$ (see Figure 3.3). Furthermore

$f_n(0) = \psi_n(\rho_1(0)) \in \psi_\infty(B)$ for all n large enough. It follows that f_∞ has degree at least 1. By the Carathéodory kernel theorem, the pointed disks $(\Omega_n, f_n(0))$ converges to $(f_\infty(\mathbb{D}), f_\infty(0))$ in the Carathéodory topology.

The curve $\gamma := \partial\psi_\infty(\mathcal{Q}_1)$ is a real-algebraic curve, since it is the limit of the real-algebraic curves $\psi_n(\partial\mathcal{Q}_1) = f_n(\partial\mathbb{D})$ of uniformly bounded degree. Thus, γ is locally connected, and hence $\psi_\infty|_{\mathcal{Q}_1}$ extends continuously to $\overline{\mathcal{Q}_1}$. Conformality of ψ_∞ on $\mathcal{Q}$ now implies that $\psi_\infty|_{\partial\mathcal{Q}_1}$ is a homeomorphism. Therefore, γ is a Jordan curve. We also know that $f_\infty(\partial\mathbb{D})$ is a closed curve which is contained in γ, and hence is the same Jordan curve. This shows that $f_\infty(\mathbb{D}) = \widehat{\mathbb{C}} \setminus \overline{\psi_\infty(\mathcal{Q}_1)}$.

Let σ_∞ be the Schwarz reflection map for the quadrature domain $f_\infty(\mathbb{D})$, which we have just shown to be a Jordan domain with a single cusp on its boundary. We know that ψ_∞ conjugates $\mathcal{R}_d$ to σ_∞ where defined, and as $\mathcal{Q} = \bigcup_{n\geq 0} \mathcal{R}_d^{-n}(\mathcal{Q}_1)$, it follows that $T^\infty(\sigma_\infty) = \bigcup_{n\geq 0} \sigma_\infty^{-n}(T^0(\sigma_\infty)) = \psi_\infty(\mathcal{Q})$. Thus, the action of σ_∞ on its tiling set is conformally conjugate (via ψ_∞) to the action of $\mathcal{R}_d$ on $\mathcal{Q}$. □

3.2.3 Relation between $\mathcal{S}_{\mathcal{R}_d}$ and correspondences

Let $(\Omega, \sigma) \in \mathcal{S}_{\mathcal{R}_d}$ and $f : \mathbb{D} \to \Omega$ be the uniformizing degree $d+1$ polynomial given by Corollary 3.2.3. Associated to f is the anti-holomorphic $d : d$ correspondence $\mathfrak{C}^*$ where $(z, w) \in \mathfrak{C}^*$ if

$$\frac{f(w) - f(\iota(z))}{w - \iota(z)} = 0,$$

where $\iota(z) = 1/\bar{z}$. These form the anti-holomorphic analogues to the correspondences considered in [BP94; BF03; BF05; BL20; BL22] considered by Bullett, Penrose, Lomonaco, and other co-authors. We refer the reader to [LMM23, Section 2] for a detailed description of the dynamics of $\mathfrak{C}^*$.

3.3 Straightening members of $\mathcal{S}_{\mathcal{R}_d}$

3.3.1 A space $\mathcal{F}_d$ of parabolic anti-rational maps

We begin with some background on parabolic points in antiholomorphic dynamics. Let z_0 be a parabolic fixed point for an anti-rational map R (i.e., $R(z_0) = z_0$ and $(R^{\circ 2})'(z_0) = 1$) with an invariant Fatou component (a parabolic basin) U such that $z_0 \in \partial U$ and $R^{\circ n}|_U \to z_0$ as $n \to +\infty$. Then according to [HS14, Lemma 2.3], there is an f-invariant open subset $V \subset U$ with $z_0 \in \partial V$ so that for every $z \in U$, there is an $n \in \mathbb{N}$ with $f^{\circ n}(z) \in V$. Moreover, there is a univalent map $\varphi^{\text{att}} : V \to \mathbb{C}$, called the *Fatou coordinate*, with

$$\varphi^{\text{att}}(R(z)) = \overline{\varphi^{\text{att}}(z)} + 1/2, \quad z \in V,$$

and φ^{att} contains a right half-plane. The map φ^{att} is unique up to real translations. Note that the antiholomorphic map R interchanges the two ends of the attracting cylinder $V/R \cong \mathbb{C}/\mathbb{Z}$, and hence fixes a unique horizontal round circle around this cylinder, which we call the *attracting equator*. By construction, φ^{att} sends the equator to the real axis. We can extend φ^{att} analytically to the entire Fatou component U as a semi-conjugacy between R and $\zeta \to \overline{\zeta}+1/2$. For $z \in U$, we call $\text{Im}(\varphi^{\text{att}}(z))$ (which is well-defined) the *Écalle height* of z.

Note that the anti-Blaschke product

$$B_d(z) = \frac{(d+1)\overline{z}^d + (d-1)}{(d-1)\overline{z}^d + (d+1)}$$

has a parabolic fixed point at 1 and $\mathbb{D}$ is an invariant parabolic basin of this fixed point. Due to real-symmetry of the map B_d, the unique critical point 0 of B_d in $\mathbb{D}$ has Écalle height zero.

Definition 3.3.1. We define the family $\mathcal{F}_d$ to be the collection of degree $d \geq 2$ anti-rational maps R with the following properties.

1. ∞ is a parabolic fixed point for R.

2. There is a marked parabolic basin $\mathcal{B}(R)$ of ∞ which is simply connected and completely invariant.

3. $R|_{\mathcal{B}(R)}$ is conformally conjugate to $B_d|_{\mathbb{D}}$.

Note that each $R \in \mathcal{F}_d$ is a Bers-like anti-rational map with filled Julia set $\mathcal{K}(R) = \widehat{\mathbb{C}} \setminus \mathcal{B}(R)$.

In consistence with the terminology for Schwarz reflections, we call $R \in \mathcal{F}_d$ *relatively hyperbolic* if the forward orbit of each critical point of R in $\mathcal{K}(R)$ converges to an attracting cycle (compare Definition 3.2.2).

Remark 3.3.1. By [Lyu+20, Example 4.2, Theorem 4.12], any circle homeomorphism conjugating $\overline{z}^d|_{\mathbb{S}^1}$ to $B_d|_{\mathbb{S}^1}$ continuously extends to a David homeomorphism of $\mathbb{D}$. Using this fact, one can perform a David surgery (as in the proof Proposition 3.2.4) to glue the map $B_d|_{\mathbb{D}}$ outside the filled Julia set of a semi-hyperbolic anti-polynomial (with connected Julia set). This would prove that for any degree d hyperbolic anti-polynomial p with a connected Julia set, there exists a relatively hyperbolic map $R \in \mathcal{F}_d$ such that $R|_{\mathcal{K}(R)}$ is topologically conjugate to $p|_{\mathcal{K}(p)}$ with the conjugacy being conformal on the interior.

An alternative way of constructing relatively hyperbolic maps in $\mathcal{F}_d$ is to appeal to the Cui-Tan theory of characterization of geometrically finite rational maps (cf. [CT18]).

Proposition 3.3.2. *The moduli space* $[\mathcal{F}_d] := {}^{\mathcal{F}_d}\!/\!_{\mathrm{Aut}(\mathbb{C})}$ *is compact.*

Proof. Let R_n be a sequence in $\mathcal{F}_d$ and $\mathcal{B}(R_n)$ be their corresponding marked basins, conjugated by affine transformations appropriately so that the maps $\varphi_n \colon \mathbb{D} \to \mathcal{B}(R_n)$ which conjugate B_d to R_n satisfy $\varphi_n(0) = 0, \varphi_n'(0) = 1$. As these are schlicht functions, we may pass to a subsequence such that φ_n converge to some map φ_∞. We have that the pointed domains $(\mathcal{B}(R_n), 0)$ converge in the Carathéodory topology to $(\varphi_\infty(\mathbb{D}), 0)$, and as $R_n|_{\mathcal{B}(R_n)} = \varphi_n \circ B_d \circ \varphi_n^{-1}$, these anti-rational maps converge to some map $R_\infty \colon \varphi_\infty(\mathbb{D}) \to \widehat{\mathbb{C}}$. Since R_∞ is the locally uniform limit of anti-rational maps of degree d it must extend to an anti-rational map itself, of degree at most d. Furthermore, R_∞ is conformally conjugate to B_d on $\varphi_\infty(\mathbb{D})$ (via φ_∞^{-1}), so that it must have degree at least d, and therefore has degree d.

It is easy to see that R_∞ has a parabolic point at ∞ and that $\varphi_\infty(\mathbb{D})$ is the desired marked parabolic basin of ∞. $\square$

3.3.2 Hybrid conjugacies for Schwarz and anti-rational maps

For a map $R \in \mathcal{F}_d$, let $\mathcal{P} \subset \mathcal{B}(R)$ be a petal at ∞ such that the critical value of R in $\mathcal{B}(R)$ lies in $\mathcal{P}$, the corresponding critical point (of multiplicity $d-1$) lies on $\partial\mathcal{P}$, and $\partial\mathcal{P} \setminus \{\infty\}$ is smooth. This can be arranged so that $V := \widehat{\mathbb{C}} \setminus \overline{\mathcal{P}}$ is a polygon. We now set $U := R^{-1}(V)$, and observe that $(R|_{\overline{U}}, \overline{U}, \overline{V})$ is a pinched anti-polynomial-like map, as in Definition 3.1.1, and this pinched anti-polynomial-like restriction has the same filled Julia set as R. Any two such restrictions are clearly hybrid equivalent.

Convention: We will associate maps $R \in \mathcal{F}_d$ with the above choice of pinched anti-polynomial-like restrictions when discussing hybrid conjugacies.

We now show that elements of $[\mathcal{F}_d]$ and $[\mathcal{S}_{\mathcal{R}_d}]$ are completely determined by their hybrid classes.

Lemma 3.3.3. *1) Let $R_1, R_2 \in \mathcal{F}_d$ be hybrid conjugate. Then R_1 and R_2 are affinely conjugate.*

2) Let $(\Omega_1, \sigma_1), (\Omega_2, \sigma_2) \in \mathcal{S}_{\mathcal{R}_d}$ be hybrid conjugate. Then σ_1 and σ_2 are affinely conjugate.

Proof. 1) Let $\Phi\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a quasiconformal homeomorphism inducing the hybrid conjugacy between R_1, R_2. Also recall that there are conformal maps $\psi_i\colon \mathbb{D} \to \mathcal{B}(R_i)$ which conjugate B_d to R_i, $i \in \{1, 2\}$.

We now define the map

$$H = \begin{cases} \Phi & \text{on } \mathcal{K}(R_1), \\ \psi_2 \circ \psi_1^{-1} & \text{on } \mathcal{B}(R_1). \end{cases}$$

We wish to show that H is continuous. By the arguments of [DH85, §1.5, Lemma 1], it suffices to show that Φ and $\psi_2 \circ \psi_1^{-1}$ agree on the fixed prime ends of $\mathcal{B}(R_1)$. Since $\psi_2 \circ \psi_1^{-1}$ conjugates R_1 to R_2 on their parabolic basins of ∞, it clearly takes fixed prime ends to fixed prime ends while mapping the prime end of $\mathcal{B}(R_1)$ corresponding to ∞ to the prime end of $\mathcal{B}(R_2)$ corresponding to ∞.

On the other hand, since Φ is a global homeomorphism that conjugates pinched anti-polynomial-like restrictions of R_1 and R_2, it follows that Φ also carries fixed prime ends to fixed prime ends and maps the prime end of $\mathcal{B}(R_1)$ corresponding to ∞ to the prime end of $\mathcal{B}(R_2)$ corresponding to ∞. As the fixed prime ends of $\mathcal{B}(R_i)$ are circularly ordered, Φ must agree with $\psi_2 \circ \psi_1^{-1}$ on each of them, and hence H is continuous.

By the Bers-Rickman gluing lemma (see [DH85, §1.5, Lemma 2]), H is a quasiconformal homeomorphism of the sphere. By design it conjugates R_1 to R_2, and is conformal almost everywhere. By Weyl's lemma, it follows that H is in fact conformal and thus an affine map as it fixes ∞.

2) Let $h_1 : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a quasiconformal homeomorphism inducing the hybrid conjugacy between σ_1 and σ_2. Furthermore, by definition of $\mathcal{S}_{\mathcal{R}_d}$, there is a conformal map $h_2 \colon T^\infty(\sigma_1) \to T^\infty(\sigma_2)$ which conjugates the Schwarz reflections, where defined. We now define the map

$$h(z) := \begin{cases} h_1 & \text{on } K(\sigma_1), \\ h_2 & \text{on } T^\infty(\sigma_1). \end{cases}$$

If we prove that h_1 and h_2 agree on the fixed prime ends of $T^\infty(\sigma_1)$, then the arguments of the previous part would apply mutatis mutandis to show that σ_1 and σ_2 are Möbius conjugate. As each σ_i has a unique critical value in its tiling set; namely at ∞, such a Möbius conjugacy must send ∞ to ∞. Hence, σ_1 and σ_2 would be affinely conjugate.

To complete the proof, we now proceed to establish the above statement about prime ends. Since h_2 conjugates σ_1 to σ_2 on their tiling sets, it takes fixed prime ends to fixed prime ends while mapping the prime end of $T^\infty(\sigma_1)$ corresponding to $\boldsymbol{y}_1$ to the prime end of $T^\infty(\sigma_2)$ corresponding to $\boldsymbol{y}_2$.

On the other hand, since h_1 is a global homeomorphism that conjugates pinched anti-polynomial-like restrictions of σ_1 and σ_2, it follows that h_1 also carries fixed prime ends to fixed prime ends and maps the prime end of $T^\infty(\sigma_1)$ corresponding to $\boldsymbol{y}_1$ to the prime end of $T^\infty(\sigma_2)$ corresponding to $\boldsymbol{y}_2$. As the fixed prime ends of $T^\infty(\sigma_i)$ are circularly ordered, h_2

must agree with h_1 on each of them. $\square$

3.3.3 Two straightening results

The goal of this subsection is twofold. The first one is to prove a straightening result for a restricted class of pinched anti-polynomial-like maps (see Definition 3.1.1), and the second one is to establish the fact that the external classes $\mathcal{R}_d$ and B_d are quaiconformally compatible. These results form the technical core of this section.

3.3.3.1 Simple pinched anti-polynomial-like maps

Definition 3.3.2. Let $(F, \overline{U}, \overline{V})$ be a pinched anti-polynomial-like map as defined in Definition 3.1.1. We impose the following conditions on U and V.

(a) $\partial U \cap \partial V = \{\infty\}$, and ∞ is the only corner of $\overline{V}$.

(b) There exists some sufficiently large R such that

$$\partial V \setminus B(0, R) = \{te^{\pm 2\pi i/3} \mid t \geq R\},$$

and $-t \in V$ for $t > R$.

Furthermore, we restrict $F\colon \overline{U} \to \overline{V}$ such that:

1. There is some neighborhood U' of $\overline{U} \setminus F^{-1}(\infty)$ on which F extends to an antiholomorphic map.
2. Each access from $\widehat{\mathbb{C}} \setminus \overline{U}$ to each point of $F^{-1}(\infty)$ has a positive angle.
3. The point ∞ is fixed under F and $F(z) = \overline{z} + \frac{1}{2} + O(1/\overline{z})$ as $z \to \infty$.
4. The critical values of F lie either in V or at ∞.

We then say that the triple $(F, \overline{U}, \overline{V})$ is a *simple pinched anti-polynomial-like map.*

(See Figure 3.5.) We will often refer to a simple pinched anti-polynomial-like map simply by F, implicitly assuming the domain and codomain to be given.

Remark 3.3.4. By Condition (3) we have that near ∞, ∂U is asymptotic to linear rays at angles $\pm 2\pi/3$.

Definition 3.3.3.

1. We define

$$\mathcal{S}_{\mathcal{R}_d}^{\text{simp}} := \{(\Omega, \sigma) \in \mathcal{S}_{\mathcal{R}_d} : \text{the unique cusp of } \partial\Omega \text{ is of type } (3,2)\},$$

   and set $\mathcal{S}_{\mathcal{R}_d}^{\text{high}} := \mathcal{S}_{\mathcal{R}_d} \setminus \mathcal{S}_{\mathcal{R}_d}^{\text{simp}}$.

2. We define

$$\mathcal{F}_d^{\text{simp}} := \{R \in \mathcal{F}_d : \infty \text{ is a simple parabolic fixed point of } R\},$$

   and set $\mathcal{F}_d^{\text{high}} := \mathcal{F}_d \setminus \mathcal{F}_d^{\text{simp}}$.

For a map $R \in \mathcal{F}_d^{\text{simp}}$ the associated pinched anti-polynomial-like map will be simple.

Our main result of this subsection is the following straightening theorem for simple pinched anti-polynomial-like maps, which is of independent interest.

Theorem 3.3.5.

1. Let $(F, \overline{U}, \overline{V})$ be a simple pinched anti-polynomial-like map of degree $d \geq 2$. Then F is hybrid conjugate to a simple pinched anti-polynomial-like restriction of a degree d anti-rational map R with a simple parabolic fixed point.

2. If the filled Julia set of F is connected, then R is unique up to Möbius conjugacy, and has a unique representative in $\left[\mathcal{F}_d^{\text{simp}}\right]$.

Proof. 1) We will use quasiconformal surgery to attach the dynamics of an appropriate degree d antiholomorphic map to the exterior of V. Let $p_d(z) = \overline{z}^d + c_d$, where $c_d = (d-1)d^{\frac{-d}{d-1}}$. We

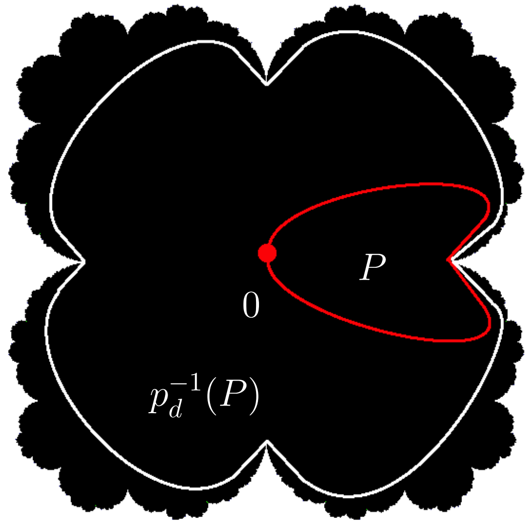

Figure 3.4: The interior of the red curve is the petal P introduced in the proof of Theorem 3.3.5, and the interior of the white curve is its preimage $p_d^{-1}(P)$.

note that p_d has a simple parabolic fixed point at $z_0 = d^{\frac{-1}{d-1}}$ and the orbit of the critical point 0 has Écalle height zero (see the discussion in the beginning of Subsection 3.3.1). Let P be an attracting petal for this fixed point, such that the boundary ∂P near z_0 consists of straight lines which subtend an angle of $4\pi/3$, the boundary ∂P is smooth away from z_0, and $0 \in \partial P$ so that $p_d^{-1}(P)$ is connected (see Figure 3.4). We make a change of variables $z \mapsto -c/(z - z_0)$ which sends z_0 to infinity and denote the image of the petal by $\mathfrak{P}$ and the conjugated map by $\mathfrak{p}$. We choose $c > 0$ so that the asymptotics of $\mathfrak{p}$ at ∞ is given by $z \mapsto \overline{z} + \frac{1}{2} + O(1/\overline{z})$. Denote $\mathfrak{Q} = \mathfrak{p}^{-1}(\mathfrak{P})$ and note that $\partial\mathfrak{Q}$ (like ∂U) is asymptotically linear near ∞ and smooth away from $\mathfrak{p}^{-1}(\infty)$ (see Figure 3.5).

Let $\Phi\colon \mathfrak{P} \to \mathbb{C} \setminus \overline{V}$ be a conformal map whose continuous boundary extension fixes ∞. Since the angle that $\partial\mathfrak{P}$ makes at ∞ is equal to the angle that ∂V makes at ∞, this map is of the form $\Phi(z) = \lambda z + o(z)$, for some $\lambda > 0$, near ∞. By the Carathéodory-Torhorst theorem, Φ extends continuously as a map from $\partial\mathfrak{P}$ to ∂V and in fact smoothly away from ∞. The d components of $\partial\mathfrak{Q} \setminus \mathfrak{p}^{-1}(\infty)$ are circularly ordered by position relative to infinity. There is a corresponding circular ordering of the components of $\partial U \setminus F^{-1}(\infty)$. We then equivariantly

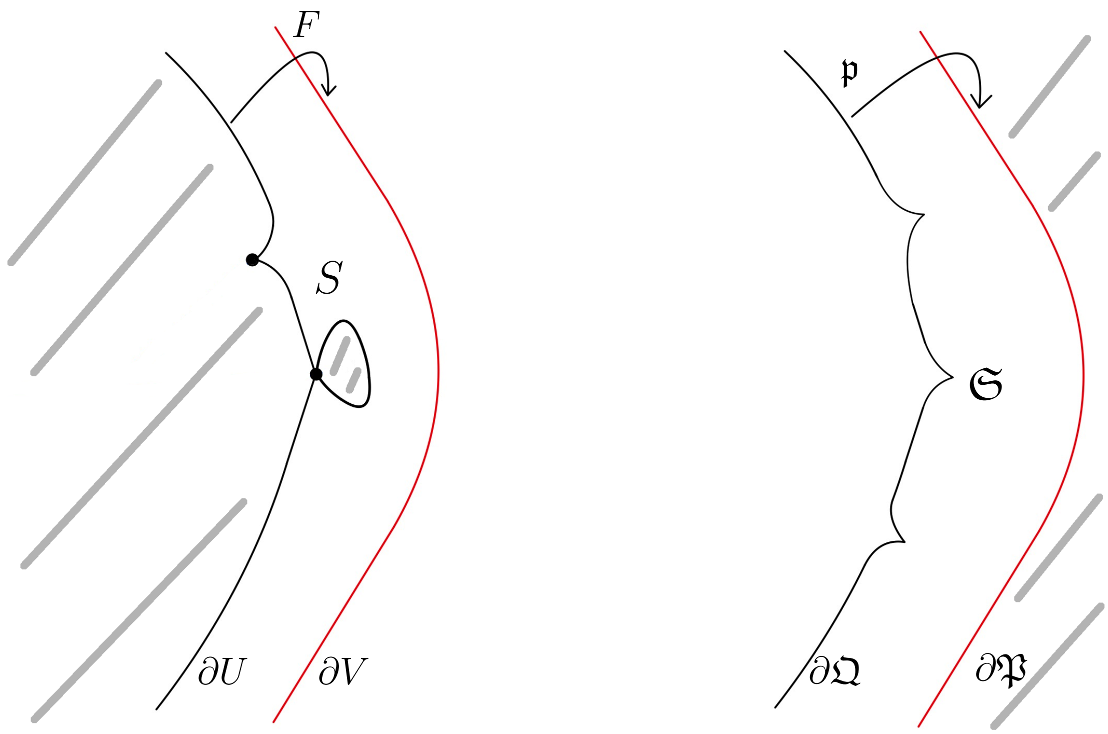

Figure 3.5: Left: Depicted are the domain and range of the simple pinched anti-polynomial-like map F. The shaded region is the pinched polygon U, and the region to the left of the red curve is its F-image V. Right: Depicted are the domain and range of the map $\mathfrak{p}$ obtained by Möbius conjugating the restriction $p_d : p_d^{-1}(P) \to P$ of Figure 3.4. The shaded region is the 'petal' $\mathfrak{P}$, and the region to the right of the black curve is its $\mathfrak{p}$-preimage $\mathfrak{Q}$.

lift $\Phi : \partial\mathfrak{P} \setminus \{\infty\} \to \partial V \setminus \{\infty\}$ to a map $\Phi : \partial\mathfrak{Q} \setminus \mathfrak{p}^{-1}(\infty) \to \partial U \setminus F^{-1}(\infty)$. More precisely, Φ is extended as $F^{-1} \circ \Phi \circ \mathfrak{p}$, where we choose the branch of F^{-1} so that components of $\partial\mathfrak{Q} \setminus \mathfrak{p}^{-1}(\infty)$ map to corresponding components of $\partial U \setminus F^{-1}(\infty)$. By continuity Φ extends to a map from $\partial\mathfrak{Q}$ to ∂U, which will not be injective at the preimages of the pinched points of $\overline{U}$. Since the accesses from $\widehat{\mathbb{C}} \setminus \overline{U}$ to the pinched points have positive angles, the images of local arcs of $\partial\mathfrak{Q}$ are quasiarcs. In particular, Φ is locally a quasi-symmetry. Also note that due to the asymptotics of F and $\mathfrak{p}$ near ∞, this lifted map Φ on the boundary also has the asymptotics $\Phi(z) = \lambda z + o(z)$ as $z \to \infty$.

In fact we will say more. Let $z_1 \in \partial\mathfrak{P}$, $z_2 \in \partial\mathfrak{Q}$ be given points which are sufficiently close to infinity and a distance less than 1 apart. This is possible as $\partial\mathfrak{Q}$ is asymptotic to a pair of rays which are parallel to $\partial\mathfrak{P}$ and a distance of $\sqrt{3}/4$ from it. Now note that by choosing z_1

and z_2 sufficiently close to ∞ we have that $\overline{\mathfrak{p}(z_2)} \in \partial\mathfrak{P} \cap B(z_2, 1)$. Thus $|\overline{\mathfrak{p}(z_2)} - z_1| < 2$. It follows from continuity of Φ that $|\Phi(\overline{\mathfrak{p}(z_2)}) - \Phi(z_1)|$ is bounded, and thus together with the asymptotics on Φ near infnity that

$$|\overline{\Phi(\mathfrak{p}(z_2))} - \Phi(z_1)| \leq |\overline{\Phi(\mathfrak{p}(z_2))} - \Phi(\overline{\mathfrak{p}(z_2)})| + |\Phi(\overline{\mathfrak{p}(z_2)}) - \Phi(z_1)| < M$$

for some sufficiently large M which depends only on how close to ∞ the points z_1 and z_2 were chosen to be, and not on the choice of points. By definition of the conjugacy and asymptotics of F, we have that

$$\Phi(\mathfrak{p}(z_2)) = F(\Phi(z_2)) = \overline{\Phi(z_2)} + 1/2 + O(1/\Phi(z_2)) = \overline{\Phi(z_2)} + \frac{1}{2} + O(1/z).$$

We conclude that $\Phi(z_1)$ and $\Phi(z_2)$ are at a uniformly bounded distance from each other.

Let S be the strip $V \setminus \overline{U}$ and $\mathfrak{S} = \mathfrak{Q} \setminus \overline{\mathfrak{P}}$ be the corresponding strip to be glued in (see Figure 3.5). In order to interpolate the boundary map $\Phi\colon \partial\mathfrak{P} \cup \partial\mathfrak{Q} \to \partial S$ to a quasiconformal map on all of $\mathfrak{S}$ we decompose S into two parts: the unbounded parts asymptotic to half-strips and the bounded pinched polygon.

Let E_1 be a top end of S, such that E_1 is a rotation and translation of the right half-strip bounded by curves $x = 0, y = 0, y = \sqrt{3}/4 + O(1/x)$ as $x \to \infty$. This is possible as ∂V is a linear ray close enough to ∞ and $\partial U = F^{-1}(\partial V)$ is asymptotically linear as noted in Remark 3.3.4. By [War42], there is a uniformizing map $\alpha\colon E_1 \to T := \{(x, y) \mid x > 0, y \in (0, \sqrt{3}/4)\}$ with asymptotics given by $\alpha(z) = e^{-2\pi i/3}z + o(z)$. There is an analogous region $E_1' \subset \mathfrak{S}$ with $\partial E_1' \cap \partial\mathfrak{S} = \Phi^{-1}(\partial E_1 \cap \partial S)$ and an analogous uniformizing map $\alpha' : E_1' \to T$ with the asymptotics $\alpha'(z) = e^{-2\pi i/3}z + o(z)$. Thus, we have an induced map

$$\widetilde{\Phi} := \alpha \circ \Phi \circ \alpha'^{-1} \ : \partial T \to \partial T$$

on the upper and lower boundary rays and by the identity on the vertical line segment $\{it \mid t \in (0, \sqrt{3}/4)\}$. Note that $\widetilde{\Phi}$ is smooth on the upper and lower boundary lines of T. Moreover, by the computation above, it is asymptotic to $w \mapsto \lambda w + o(w)$ as $\partial T \ni w \to \infty$, and the maps on the upper and lower boundaries are a bounded distance from each other.

Therefore, linear interpolation yields a homeomorphism $\widetilde{\Phi} : T \to T$ that is quasiconformal on $\operatorname{int} T$ and that continuously agrees with $\widetilde{\Phi}|_{\partial T}$ defined above (cf. [Lee+21, Lemma 5.3]). This extended map lifts to a quasiconformal map from E_1' to E_1 which agrees with Φ on the boundary. The same argument shows the existence of a quasiconformal interpolating map between regions $E_2' \subset \mathfrak{S}$ and $E_2 \subset S$ which are ends of the lower accesses to ∞.

Now consider the regions $\mathfrak{S} \setminus (E_1' \cup E_2')$ and $S \setminus (E_1 \cup E_2)$ which are a conformal polygon and a conformal pinched polygon respectively. Moreover, the edges of these (pinched) polygons are smooth and they meet at positive angles at the vertices. The map Φ, constructed so far, is a quasisymmetric map between the boundaries of the two regions. Each of these regions may be uniformized by the disk $\mathbb{D}$ and two conformal maps φ_1, φ_2 which send the disk to $\mathfrak{S} \setminus (E_1' \cup E_2')$ and $S \setminus (E_1 \cup E_2)$ respectively. We now define the map $\varphi_2^{-1} \circ \Phi \circ \varphi_1 \colon \partial\mathbb{D} \to \partial\mathbb{D}$ where this composition is well defined, and extending continuously using the circular ordering of the pinched points where it is not. Now note that for every point on $\partial(\mathfrak{S} \setminus (E_1' \cup E_2'))$ there is a local neighborhood for which Φ is a quasi-symmetric homeomorphism onto its image. This property lifts to the boundary map, and by the Ahlfors-Beurling theorem the boundary map extends to a quasiconformal map of $\mathbb{D}$. Going back via the conformal maps φ_1, φ_2, we obtain our desired quasiconformal extension $\Phi : \mathfrak{S} \setminus (E_1' \cup E_2') \longrightarrow S \setminus (E_1 \cup E_2)$.

We now have a globally defined continuous map

$$\widetilde{F} \colon \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$$

$$\widetilde{F}(z) = \begin{cases} F(z), & z \in \overline{U} \\ \Phi \circ \mathfrak{p} \circ \Phi^{-1}(z), & z \in \widehat{\mathbb{C}} \setminus \overline{U}. \end{cases}$$

Note that since ∂U is a piecewise smooth curve with finitely many singular points, it is removable for quasiconformal maps. Hence, $\widetilde{F}$ is a degree d anti-quasiregular map of $\widehat{\mathbb{C}}$. In fact, $\widetilde{F}$ is antiholomorphic off the strip S and the pinching points of ∂U are critical points for $\widetilde{F}$. Let μ_0 denote the standard complex structure on $\widehat{\mathbb{C}} \setminus \overline{V}$. Pulling μ_0 back under iterates of $\widetilde{F}$ we obtain a complex structure on $\mathbb{S}^2 \setminus K(F)$, and we complete this to a complex structure μ

on all of $\mathbb{S}^2$ by putting the standard complex structure on $K(F)$. Furthermore, μ is invariant under the action of $\widetilde{F}$. As at most one iterate of $\widetilde{F}$ lands in the strip S, it follows that the eccentricity of the pulled back complex structure is essentially bounded. By applying the measurable Riemann mapping theorem we obtain a map $\Xi\colon \mathbb{S}^2 \to \widehat{\mathbb{C}}$ which sends μ to the standard complex structure. Note that as μ was already the standard complex structure on $K(F)$ we have the Ξ is in fact conformal on $K(F)$. Now $R := \Xi \circ \widetilde{F} \circ \Xi^{-1}\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is an orientation reversing map of the sphere which preserves the standard complex structure, and is thus an anti-rational map. By construction it is hybrid equivalent to F.

It remains to argue that R has a simple parabolic fixed point at $\Xi(\infty)$. That ∞ is a fixed point for R follows from the fact that the anti-quasiregular map $\widetilde{F}$ fixes ∞. Moreover, by Condition 3 of the definition of simple pinched anti-polynomial-like maps and the construction of $\widetilde{F}$, points in U are repelled away from ∞ under iterations of $\widetilde{F}$, while the forward $\widetilde{F}$-orbits of points in $\widehat{\mathbb{C}} \setminus U$ converge to ∞. This translates to the fact that $\Xi(\infty)$ is a parabolic fixed point of R with a unique attracting and a unique repelling petal. In other words, $\Xi(\infty)$ is a simple parabolic fixed point of R.

2) We now assume that the filled Julia set $K(F)$ is connected. We can assume, possibly after a Möbius change of coordinates, that $\Xi(\infty) = \infty$; i.e., R has a simple parabolic fixed point at ∞. Moreover, by construction of R, the forward R-orbits all points outside of $\Xi(K(F))$ converge to ∞. It follows that $\mathcal{B}(R) := \widehat{\mathbb{C}} \setminus \Xi(K(F))$ is a simply connected, completely invariant parabolic basin of ∞.

Also note that connectedness of $K(F)$ is equivalent to the fact that all critical points of F lie in $K(F)$. Therefore, there is a unique critical point (of multiplicity $d-1$) of R in $\mathcal{B}(R)$. Since the critical point 0 of p_d has Écalle height zero and the critical Écalle height is a conformal invariant, it follows that the unique critical point of R in $\mathcal{B}(R)$ also has Écalle height zero. Thus, $R|_{\mathcal{B}(R)}$ is conformally conjugate to the action on $\mathbb{D}$ of a unicritical parabolic anti-Blaschke product with critical Écalle height zero. Up to Möbius conjugacy, B_d is the unique such anti-Blaschke product. We conclude that $R \in \mathcal{F}_d^{\mathrm{simp}}$.

It remains to prove that if $R_1, R_2 \in \mathcal{F}_d^{\mathrm{simp}}$ are two such straightened maps, then they are affinely conjugate. But this follows from Lemma 3.3.3. The proof of the theorem is now complete. $\square$

3.3.3.2 Straightening the external class $\mathcal{R}_d$

Our next technical lemma asserts that the external classes $\mathcal{R}_d$ and B_d are quasiconformally conjugate in a pinched neighborhood of the circle. As the idea of the proof is similar to that of Theorem 3.3.5, we only outline the key steps.

Lemma 3.3.6. *There exists a homeomorphism $\mathfrak{h} : \overline{\mathcal{Q}} \to \overline{\mathbb{D}}$ that is quasiconformal on $\mathbb{D}$, sends 1 to 1, and conjugates the restriction of the anti-Farey map $\mathcal{R}_d$ on the closure of a (one-sided) neighborhood of $\partial\mathcal{Q} \setminus \mathcal{R}_d^{-1}(1)$ to the restriction of the anti-Blaschke product B_d on the closure of a (one-sided) neighborhood of $\mathbb{S}^1 \setminus B_d^{-1}(1)$.*

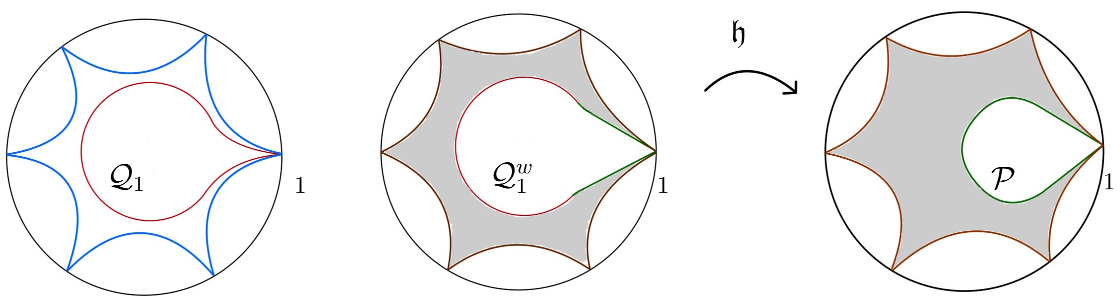

Figure 3.6: We open up the cusp of $\mathcal{Q}_1$ at 1 to obtain a wedge of positive angle. Since this thickened region $\mathcal{Q}_1^w$ and the petal $\mathcal{P}$ of B_d have the same angle at 1, the conformal isomorphism $\mathfrak{h} : \overline{\mathcal{Q}_1^w} \to \overline{\mathcal{P}}$ is asymptotically linear near 1. The shaded regions are fundamental domains for $\mathcal{R}_d$ and B_d.

Sketch of the proof. Let us first thicken $\mathcal{Q}_1$ near 1 to turn the cusp into a wedge of angle θ_0 (for some $\theta_0 \in (0, \pi)$), and call this domain $\mathcal{Q}_1^w$ (see Figure 3.6). Analogously, consider an attracting petal $\mathcal{P} \subset \mathbb{D}$ of B_d at the parabolic point 1 such that $\mathcal{P}$ contains the critical value of B_d in $\mathbb{D}$, the critical point 0 lies on $\partial\mathcal{P}$, and $\partial\mathcal{P}$ subtends an angle θ_0 at 1. Now choose a homeomorphism $\mathfrak{h} : \overline{\mathcal{Q}_1^w} \longrightarrow \overline{\mathcal{P}}$ that is conformal on the interior and sends 1 to 1. Since these regions subtend the same angle at 1, it follows that $\mathfrak{h}$ is asymptotically linear near 1. Note that both $\mathcal{R}_d : \mathcal{R}_d^{-1}(\partial\mathcal{Q}_1^w) \setminus \partial\mathcal{Q}_1^w \longrightarrow \partial\mathcal{Q}_1^w$ and $B_d : B_d^{-1}(\partial\mathcal{P}) \longrightarrow \partial\mathcal{P}$ are degree d orientation-reversing covering maps. We lift the map $\mathfrak{h} : \partial\mathcal{Q}_1^w \longrightarrow \partial\mathcal{P}$ via the above coverings to get a homeomorphism from $\mathcal{R}_d^{-1}(\partial\mathcal{Q}_1^w) \setminus \partial\mathcal{Q}_1^w$ onto $B_d^{-1}(\partial\mathcal{P})$, which we also denote by $\mathfrak{h}$.

The parabolic asymptotics of $\mathcal{R}_d, B_d$ near 1 and the linear asymptotics of $\mathfrak{h}$ near 1 allow one to apply the quasiconformal interpolation arguments of Theorem 3.3.5 to conclude the existence of a quasiconformal homeomorphism $\mathfrak{h}$ between the pinched fundamental annuli $\mathcal{R}_d^{-1}(\overline{\mathcal{Q}_1^w}) \setminus \mathcal{Q}_1^w$ and $B_d^{-1}(\overline{\mathcal{P}}) \setminus \mathcal{P}$ (of $\mathcal{R}_d$ and B_d respectively) that continuously agrees with $\mathfrak{h}$ already defined (these fundamental domains are shade in grey in Figure 3.6). By construction, this map conjugates the actions of $\mathcal{R}_d$ and B_d on the boundaries of their fundamental domains. Finally, pulling $\mathfrak{h}$ back by iterates of $\mathcal{R}_d$ and B_d, one obtains a quasiconformal homeomorphism of $\mathbb{D}$ that conjugates the restriction of $\mathcal{R}_d$ on the closure of a (one-sided) neighborhood of $\partial\mathcal{Q} \setminus \mathcal{R}_d^{-1}(1)$ to the restriction of B_d on the closure of a (one-sided) neighborhood of $\mathbb{S}^1 \setminus B_d^{-1}(1)$. □

Remark 3.3.7. A weaker version of Lemma 3.3.6; namely, the existence of a quasiconformal homeomorphism $\overline{\mathcal{Q}} \to \overline{\mathbb{D}}$ that conjugates $\mathcal{R}_d$ to B_d *only on* $\mathbb{S}^1$, can be deduced from [Lyu+20, Theorem 4.9].

3.3.4 Straightening Schwarz reflections in $\mathcal{S}_{\mathcal{R}_d}$

3.3.4.1 Straightening all maps in $\mathcal{S}_{\mathcal{R}_d}$

Theorem 3.3.8. *Let $(\Omega, \sigma) \in \mathcal{S}_{\mathcal{R}_d}$. Then, there exists a unique $R_\sigma \in [\mathcal{F}_d]$ such that σ is hybrid conjugate to R_σ. Moreover, $R_\sigma \in \mathcal{F}_d^{\text{high}}$ if and only if $(\Omega, \sigma) \in \mathcal{S}_{\mathcal{R}_d}^{\text{high}}$.*

Proof. Let us fix $(\Omega, \sigma) \in \mathcal{S}_{\mathcal{R}_d}$. Recall that there exists a conformal map $\psi : \mathcal{Q} \to T^\infty(\sigma)$ that conjugates $\mathcal{R}_d$ to σ and sends 1 to $\boldsymbol{y}$. Moreover, by Lemma 3.3.6, there exists a quasiconformal homeomorphism $\mathfrak{h} : \mathcal{Q} \to \mathbb{D}$ that conjugates the restriction of $\mathcal{R}_d$ on a (one-sided) neighborhood of $\partial\mathcal{Q} \setminus \mathcal{R}_d^{-1}(1)$ to the restriction of B_d on a (one-sided) neighborhood of $\mathbb{S}^1 \setminus B_d^{-1}(1)$.

Let us now define a map on $\widehat{\mathbb{C}}$ as follows:

$$\widetilde{R_\sigma} := \begin{cases} (\psi \circ \mathfrak{h}^{-1}) \circ B_d \circ (\mathfrak{h} \circ \psi^{-1}) & \text{on } T^\infty(\sigma), \\ \sigma \quad \text{on } K(\sigma). \end{cases}$$

By the conjugation properties of ψ and $\mathfrak{h}$, the map $\widetilde{R_\sigma}$ agrees with σ on the closure of a neighborhood of $K(\sigma) \setminus \sigma^{-1}(\boldsymbol{y})$. Since finitely many points are quasiconformally removable, we conclude that the map $\widetilde{R_\sigma}$ is a global anti-quasiregular map.

Let μ be the Beltrami coefficient on $\widehat{\mathbb{C}}$ given by the pullback of the standard complex structure under the map $\mathfrak{h} \circ \psi^{-1}$ on $T^\infty(\sigma)$ and zero elsewhere. As B_d is an antiholomorphic map, it follows that μ is $\widetilde{R_\sigma}$-invariant. Since $\mathfrak{h} \circ \psi^{-1}$ is quasiconformal, it follows that $||\mu||_\infty < 1$. We conjugate $\widetilde{R_\sigma}$ by a quasiconformal homeomorphism Ξ of $\widehat{\mathbb{C}}$ that solves the Beltrami equation with coefficient μ to obtain an anti-rational map R_σ. By construction, R_σ has a parabolic fixed point at ∞ (after possibly conjugating R_σ by a Möbius map), and this parabolic point has a simply connected, completely invariant Fatou component where the dynamics is conformally conjugate to B_d. Thus, $R_\sigma \in \mathcal{F}_d$. Moreover, $\mathcal{B}(R_\sigma) = \Xi(T^\infty(\sigma))$, and $\mathcal{K}(R_\sigma) = \Xi(K(\sigma))$.

Note that by the normalization of $\mathfrak{h}$, the parabolic fixed point 1 of B_d is glued to the unique cusp of $\partial\Omega$. Hence, Ξ is conformal a.e. on $K(\sigma)$, sends the unique cusp on $\partial\Omega$ to the parabolic fixed point ∞, and conjugates a pinched anti-polynomial-like restriction of σ to a pinched anti-polynomial-like restriction of R_σ.

According to [LMM23, Corollary A.6], $(\Omega, \sigma) \in \mathcal{S}_{\mathcal{R}_d}^{\text{high}}$ if and only if the cusp $\boldsymbol{y}$ has at least one $\sigma^{\circ 2}$-invariant attracting direction in $K(\sigma)$. Since $\Xi(\boldsymbol{y}) = \infty$, it follows that $(\Omega, \sigma) \in \mathcal{S}_{\mathcal{R}_d}^{\text{high}}$ if and only if $R_\sigma^{\circ 2}$ has at least two invariant attracting directions at ∞ (one in $\mathcal{B}(R_\sigma)$ and at least one in $\mathcal{K}(R_\sigma)$). Clearly, this is equivalent to saying that ∞ is a fixed point of $R_\sigma^{\circ 2}$ of multiplicity at least three; i.e., $R_\sigma \in \mathcal{F}_d^{\text{high}}$.

Finally for the uniqueness statement, note that if $R_1, R_2 \in \mathcal{F}_d$ are hybrid conjugate to σ, then they are hybrid conjugate to each other. By Lemma 3.3.3, R_1 and R_2 must be affinely conjugate. $\square$

Remark 3.3.9. Although Theorem 3.3.8 gives a uniform way of straightening all maps in $\mathcal{S}_{\mathcal{R}_d}$, the appearance of the Riemann map of the tiling set in the proof makes this straightening surgery less suitable for parameter space investigations.

3.3.4.2 Straightening Schwarz reflections in $\mathcal{S}_{\mathcal{R}_d}^{\text{simp}}$ via pinched anti-polynomial-like restrictions

We now show that the Straightening Theorem 3.3.5 applies to the Schwarz reflections in $\mathcal{S}_{\mathcal{R}_d}^{\text{simp}}$. The main advantage of this straightening method is that it gives better control on the domains of the hybrid conjugacies.

Lemma 3.3.10. *Let $(\Omega, \sigma) \in \mathcal{S}_{\mathcal{R}_d}^{\text{simp}}$. Then, there exists a Jordan domain $V' \subset \Omega$ with $\overline{V}' \supset K(\sigma)$ and a conformal map $\beta : \overline{V'} \to \widehat{\mathbb{C}}$ such that β conjugates $\sigma \colon \overline{\sigma^{-1}(V')} \to \overline{V'}$ to a simple pinched anti-polynomial-like map $(F, \overline{U}, \overline{V})$ of degree d. Moreover, the filled Julia set of this pinched anti-polynomial-like map is $\beta(K(\sigma))$.*

Proof. Without loss of generality we may assume that the cusp is at 0 and points into the positive real axis.

We begin by opening up the cusp of Ω (i.e., creating a wedge), using the following procedure. For some $\delta > 0$, to be specified later, we define a Jordan domain $V' \subset \Omega$ such that

$$\partial V' \setminus B(0,\delta) = \partial\Omega \setminus B(0,\delta),\ \partial V' \cap B(0,\delta/2) = L^{\pm} := \{te^{\pm 2\pi i/3} : t \in [0, \delta/2)\},$$

and $\partial V'$ is smooth except at 0. Since $(\partial\Omega \setminus \{0\}) \cap K(\sigma) = \emptyset$, we can choose $\delta > 0$ small enough so that $K(\sigma) \subset V' \cup B(0,\delta)$.

By [LMM23, Proposition A.4], σ has a unique invariant direction at 0 given by the positive real axis. By [LMM23, Proposition A.5], this direction is repelling for σ. We apply the change of coordinates β described in [LMM23, Subsection A.4] which conjugates σ (near 0) to $\zeta \mapsto \overline{\zeta} + 1/2 + O(1/\overline{\zeta})$ (near ∞). Moreover, β sends small enough positive reals to large negative reals and the line segments $L^{\pm}$ to the infinite rays at angles $\pm\frac{2\pi}{3}$ meeting at ∞. Since $\beta \circ \sigma \circ \beta^{-1}$ is approximately $\overline{\zeta} + \frac{1}{2}$ for $|\operatorname{Im} \zeta|$ large enough, it follows that points between $\beta(L^{\pm})$ and $\beta(\partial\Omega)$ with sufficiently large imaginary part eventually leave $\beta(\Omega)$. Therefore, we can choose $\delta > 0$ sufficiently small so that points in $B(0,\delta) \setminus \overline{V'}$ eventually leave Ω. It now follows that for such a δ, the non-escaping set $K(\sigma)$ is contained in $\overline{\sigma^{-1}(V')}$. Hence, we have that

$$K(\sigma) = \{z \in \overline{\sigma^{-1}(V')} : \sigma^{\circ n}(z) \in \overline{\sigma^{-1}(V')} \ \forall\ n \geq 0\}. \tag{3.3.1}$$

Note also that $\partial\sigma^{-1}(V') \setminus B(0,\delta) \subset \Omega \setminus B(0,\delta)$, and so $(\partial\sigma^{-1}(V') \setminus B(0,\delta)) \cap \partial V' = \emptyset$. Together with the asymptotics of $\beta \circ \sigma \circ \beta^{-1}$ near ∞, it follows that $\partial\sigma^{-1}(V') \cap \partial V' = \{0\}$.

We now set $V := \beta(V')$, $U := \beta(\sigma^{-1}(V'))$, and $F := \beta \circ \sigma \circ \beta^{-1} : \overline{U} \to \overline{V}$, and claim that F is a simple pinched anti-polynomial-like map. The pinched polygon structure of U follows from the fact that $\overline{\sigma^{-1}(\Omega)}$ is a pinched disk with possible pinched points in $\sigma^{-1}(0)$ (this happens only if the cusp 0 is a critical value of σ). We also note that σ is a proper antiholomorphic map on each component of $\sigma^{-1}(\Omega)$, and hence F is a proper antiholomorphic map on each component of U. The other defining conditions of a simple pinched anti-polynomial-like map are easily checked from the above construction. The fact that the filled Julia set of this pinched anti-polynomial-like map is $\beta(K(\sigma))$ follows from Relation (3.3.1). ☐

Remark 3.3.11. The conformal map $\beta : \overline{V'} \to \beta(\overline{V'})$ can be extended as a quasiconformal homeomorphism of $\widehat{\mathbb{C}}$.

As a slight abuse of notation, we will call $\sigma\colon \overline{\sigma^{-1}(V')} \to \overline{V'}$ a simple pinched anti-polynomial-like restriction of σ.

Theorem 3.3.12. *Let* $(\Omega, \sigma) \in \mathcal{S}_{\mathcal{R}_d}^{\mathrm{simp}}$*. Then,*

1. *σ restricts to a simple pinched anti-polynomial-like map with filled Julia set equal to $K(\sigma)$, and*

2. *this simple pinched anti-polynomial-like map is hybrid conjugate to a unique member* $[R_\sigma] \in \left[\mathcal{F}_d^{\text{simp}}\right]$ *with filled Julia set* $\mathcal{K}(R_\sigma)$.

Proof. This follows from Lemma 3.3.10 and Theorem 3.3.5. □

For $(\Omega, \sigma) \in \mathcal{S}_{\mathcal{R}_d}$, the map R_σ produced by Theorems 3.3.12 and 3.3.8 will be referred to as the *straightening* of σ. Clearly, if $\sigma_1, \sigma_2 \in \mathcal{S}_{\mathcal{R}_d}$ are affinely conjugate, then they have the same straightening.

Definition 3.3.4. We define the *straightening map*

$$\chi : [\mathcal{S}_{\mathcal{R}_d}] \longrightarrow [\mathcal{F}_d], \quad \chi([\Omega, \sigma]) = [R_\sigma],$$

where $[R_\sigma]$ is the straightening of σ; i.e., R_σ is the unique map in $\mathcal{F}_d$, up to affine conjugacy, to which σ is hybrid conjugate.

Abusing notation, we will often write $\chi(\sigma) = R$.

Corollary 3.3.13. *Hybrid conjugacies between* $[\Omega, \sigma] \in \left[\mathcal{S}_{\mathcal{R}_d}^{\text{simp}}\right]$ *and* $\chi([\Omega, \sigma]) \in \left[\mathcal{S}_{\mathcal{R}_d}^{\text{simp}}\right]$ *can be chosen such that*

1. *their dilatations are locally bounded, and*

2. *the domains of definition of these conjugacies depend continuously on parameters.*

Proof. This follows from the construction of hybrid conjugacies given in Theorem 3.3.5 and the facts that the fundamental (pinched) annuli of the simple pinched anti-polynomial-like restrictions of Schwarz reflections constructed in Lemma 3.3.10 move continuously with respect to the parameter and the asymptotics of the maps near the cusps are the same throughout $\mathcal{S}_{\mathcal{R}_d}^{\text{simp}}$ (see [LMM23, Appendix]). □

3.4 Invertibility of the straightening map and proofs of the main Theorems

The main goal of this section is to prove that the straightening map χ is bijective, from which our main theorems will follow. We will demonstrate this by constructing an explicit inverse of χ. The construction of this inverse map is dual to that of χ given in Theorem 3.3.8.

For maps in $\mathcal{F}_d^{\text{simp}}$, we will also give an alternative construction of χ^{-1} that will follow the strategy of the proof of Theorem 3.3.12. This will give us control on the dilatations and the domains of definition of the associated hybrid conjugacies on $\mathcal{F}_d^{\text{simp}}$.

3.4.1 Invertibility of χ

Theorem 3.4.1. *The map* $\chi : [\mathcal{S}_{\mathcal{R}_d}] \longrightarrow [\mathcal{F}_d]$ *is invertible. In particular, the restrictions* $\chi : \left[\mathcal{S}_{\mathcal{R}_d}^{\text{simp}}\right] \to \left[\mathcal{F}_d^{\text{simp}}\right]$ *and* $\chi : \left[\mathcal{S}_{\mathcal{R}_d}^{\text{high}}\right] \to \left[\mathcal{F}_d^{\text{high}}\right]$ *are bijections.*

Proof. Let us fix $R \in \mathcal{F}_d$. Recall that there exists a conformal map $\psi : \mathbb{D} \to \mathcal{B}(R)$ that conjugates B_d to R, and sends 1 to ∞. Also, the quasiconformal homeomorphism $\mathfrak{h} : \mathcal{Q} \to \mathbb{D}$ of Lemma 3.3.6 conjugates the restriction of $\mathcal{R}_d$ on a (one-sided) neighborhood of $\partial\mathcal{Q} \setminus \mathcal{R}_d^{-1}(1)$ to the restriction of B_d on a (one-sided) neighborhood of $\mathbb{S}^1 \setminus B_d^{-1}(1)$.

We now define a map on a subset of $\widehat{\mathbb{C}}$ as follows:

$$\widetilde{\sigma_R} := \begin{cases} (\psi \circ \mathfrak{h}) \circ \mathcal{R}_d \circ (\mathfrak{h}^{-1} \circ \psi^{-1}) & \text{on } \mathcal{B}(R) \setminus \psi(\mathfrak{h}(\operatorname{int} \mathcal{Q}_1)), \\ R \quad \text{on } \mathcal{K}(R). \end{cases}$$

By the conjugation properties of ψ and $\mathfrak{h}$, the map $\widetilde{\sigma_R}$ agrees with R on the closure of a neighborhood of $\mathcal{K}(R) \setminus R^{-1}(\infty)$. Since finitely many points are quasiconformally removable, we conclude that the map $\widetilde{\sigma_R}$ is an anti-quasiregular map on $\widehat{\mathbb{C}} \setminus \overline{\psi(\mathfrak{h}(\mathcal{Q}_1))}$. Moreover, $\widetilde{\sigma_R}$ continuously extends as the identity map to the boundary of its domain of definition, which is a Jordan domain (compare the proof of Proposition 3.2.4).

Let μ be the Beltrami coefficient on $\widehat{\mathbb{C}}$ given by the pullback of the standard complex structure under the map $\mathfrak{h}^{-1}\circ\psi^{-1}$ on $\mathcal{B}(R)$ and zero elsewhere. As $\mathcal{R}_d$ is an antiholomorphic map, it follows that μ is $\widetilde{\sigma_R}$-invariant. Since $\mathfrak{h}^{-1}\circ\psi^{-1}$ is quasiconformal, it follows that $||\mu||_\infty < 1$. We conjugate $\widetilde{\sigma_R}$ by a quasiconformal homeomorphism $\mathfrak{g}$ of $\widehat{\mathbb{C}}$ that solves the Beltrami equation with coefficient μ to obtain an antiholomorphic map σ_R on a Jordan domain that continuously extends as the identity map to the boundary of its domain of definition $\Omega_R = \widehat{\mathbb{C}} \setminus \mathfrak{g}(\psi(\mathfrak{h}(\mathrm{int}\,\mathcal{Q}_1)))$. Thus, Ω_R is a Jordan quadrature domain and σ_R is its Schwarz reflection map.

Arguments used in the last two paragraphs of the proof of Proposition 3.2.4 apply verbatim to the current context to show that the Jordan curve $\partial\Omega_R$ has a unique conformal cusp and the tiling set dynamics of σ_R is conformally conjugate to the action of $\mathcal{R}_d$ on $\mathcal{Q}$. Thus, after possibly a Möbius change of coordinates, we can assume that $(\Omega_R, \Sigma_R) \in \mathcal{S}_{\mathcal{R}_d}$. It also follows from the same arguments that $K(\sigma_R) = \mathfrak{g}(\mathcal{K}(R))$, and $T^\infty(\sigma_R) = \mathfrak{g}(\mathcal{B}(R))$.

Note that by the normalization of $\mathfrak{h}$, the parabolic fixed point 1 of $\mathcal{R}_d$ is glued to the parabolic fixed point ∞ of R. It now follows from the construction that the global quasiconformal map $\mathfrak{g}^{-1}$ (suitably normalized) is conformal a.e. on $K(\sigma_R)$, sends the unique cusp on $\partial\Omega_R$ to ∞, and conjugates a pinched anti-polynomial-like restriction of σ_R to a pinched anti-polynomial-like restriction of R.

By Lemma 3.3.3, the map (Ω_R, σ_R) is the unique element of $\mathcal{S}_{\mathcal{R}_d}$ (up to affine conjugacy) that is hybrid conjugate to R. Hence,

$$\chi^* : \; [\mathcal{F}_d] \;\longrightarrow\; [\mathcal{S}_{\mathcal{R}_d}]\,, \; [R] \mapsto [\Omega_R, \sigma_R]$$

is a well-defined map. Finally, the fact that no two distinct elements of $[\mathcal{S}_{\mathcal{R}_d}]\,, [\mathcal{F}_d]$ have the same hybrid class (again by Lemma 3.3.3) implies that $\chi^*\circ\chi \equiv \mathrm{id}$ on $[\mathcal{S}_{\mathcal{R}_d}]$ and $\chi\circ\chi^* \equiv \mathrm{id}$ on $[\mathcal{F}_d]$. Therefore, χ^* is the desired inverse of χ.

The second statement of the theorem follows from the fact that $\chi([\Omega,\sigma]) \in \left[\mathcal{F}_d^{\mathrm{high}}\right]$ if and only if $[\Omega,\sigma] \in \left[\mathcal{S}_{\mathcal{R}_d}^{\mathrm{high}}\right]$ (see Theorem 3.3.8). $\square$

We will now provide an alternative construction of χ^{-1} on $\left[\mathcal{F}_d^{\mathrm{simp}}\right]$ using the notion of simple pinched anti-polynomial-like maps (in the sense of Definition 3.3.2). This will supply additional control on the corresponding hybrid conjugacies that will be useful in studying topological properties of χ.

Theorem 3.4.2. *Hybrid conjugacies between* $[R] \in \left[\mathcal{F}_d^{\mathrm{simp}}\right]$ *and* $\chi^{-1}([R]) \in \left[\mathcal{S}_{\mathcal{R}_d}^{\mathrm{simp}}\right]$ *can be chosen such that*

1. *their dilatations are locally bounded, and*

2. *the domains of definition of these conjugacies depend continuously on parameters.*

Proof. Let $R \in \mathcal{F}_d^{\mathrm{simp}}$.

By Proposition 3.2.4, there exists $(\Omega_0, \sigma_0) \in \mathcal{S}_{\mathcal{R}_d}$ such that $\sigma_0|_{K(\sigma_0)}$ is topologically conjugate to $\overline{z}^d|_{\overline{\mathbb{D}}}$ with the conjugacy being conformal on the interior. In particular, the cusp of $\partial\Omega_0$ has no attracting direction and hence is of type $(3, 2)$ (by [LMM23, Corollary A.6]). Thus, $(\Omega_0, \sigma_0) \in \mathcal{S}_{\mathcal{R}_d}^{\mathrm{simp}}$. Real-symmetry of $\overline{z}^d$ and $\mathcal{R}_d$ implies that Ω_0 can be chosen to be is real-symmetric (cf. [Lyu+20, Section 11.4]). We can also normalize so that that the cusp of $\partial\Omega_0$ is at the origin.

Recall from Lemma 3.3.10 that there exists a Jordan domain $V' \subset \Omega_0$ with a corner at the origin such that $\overline{V}' \supset K(\sigma_0)$ and

$$\beta : \overline{V'} \to \beta(\overline{V'}),\ z \mapsto c/\sqrt{z}$$

conjugates $\sigma_0 \colon \overline{\sigma_0^{-1}(V')} \to \overline{V'}$ to a degree d simple pinched anti-polynomial-like map whose filled Julia set is $\beta(K(\sigma_0))$ (where $c \in \mathbb{R}_{<0}$ is chosen suitably and the chosen branch of square root sends positive reals to positive reals). In particular, the map β sends the cusp of $\partial\Omega_0$ to ∞, and conjugates σ_0 to a map of the form $\zeta \mapsto \overline{\zeta} + 1/2 + O(1/\overline{\zeta})$ near ∞. We denote this simple pinched anti-polynomial-like map by $(\boldsymbol{\sigma}_0, \overline{U}, \overline{V})$, where $U := \beta(\sigma_0^{-1}(V')), V := \beta(V')$ (see Figure).

Note that the map β extends to a quasiconformal homeomorphism of $\widehat{\mathbb{C}}$. After possibly post-composing β with an affine map, we may assume that $\beta(\infty) = 0$. We set $\mathbf{\Omega}_0 := \beta(\Omega_0)$, and continue to denote the conjugated map $\beta \circ \sigma_0 \circ \beta^{-1}$ on $\mathbf{\Omega}_0$ by $\boldsymbol{\sigma}_0$. Since $\sigma_0^{-1}(\infty)$ is a singleton $\{c_0\}$, it follows that $\boldsymbol{c}_0 := \beta(c_0)$ is a d-fold critical point for $\boldsymbol{\sigma}_0$ with associated critical value 0.

Let us consider a simple pinched anti-polynomial-like restriction $R : \overline{\mathcal{U}} \to \overline{\mathcal{V}}$ of R. By construction, $\widehat{\mathbb{C}} \setminus \overline{\mathcal{V}} \subsetneq \mathcal{B}(R)$ is an attracting petal which subtends an angle of $4\pi/3$ at the parabolic fixed point ∞ such that the petal contains the critical value of R in $\mathcal{B}(R)$ and the corresponding critical point (of multiplicity $d-1$) lies on the petal boundary. Also, $\mathcal{U} := R^{-1}(\mathcal{V})$ (see Figure).

Let $\Psi \colon \widehat{\mathbb{C}} \setminus \overline{V} \longrightarrow \widehat{\mathbb{C}} \setminus \overline{\mathcal{V}}$ be a Riemann map whose homeomorphic boundary extension carries ∞ to ∞ and is asymptotically $z \mapsto \lambda z + o(z)$, for some $\lambda > 0$, near ∞. The arguments of Theorem 3.3.5 apply verbatim to this setting to supply a continuous map $\Psi : \widehat{\mathbb{C}} \setminus U \longrightarrow \widehat{\mathbb{C}} \setminus \mathcal{U}$ that is quasiconformal on the interior of the strip $\overline{V} \setminus U$, conformal on $\widehat{\mathbb{C}} \setminus \overline{V}$ and conjugates $\boldsymbol{\sigma}_0 : \partial U \to \partial V$ to $R : \partial\mathcal{U} \to \partial\mathcal{V}$.

We then define the map

$$F \colon \mathcal{U} \ \cup \ \Psi\left(\overline{\mathbf{\Omega}_0} \setminus U\right) \longrightarrow \widehat{\mathbb{C}}$$

$$F(z) = \begin{cases} R(z), & z \in \mathcal{U} \\ \Psi \circ \boldsymbol{\sigma}_0 \circ \Psi^{-1}(z), & \text{otherwise.} \end{cases}$$

The fact that $\partial\mathcal{U}$ is a piecewise smooth curve with finitely many singular points implies that it is removable for quasiconformal maps and hence, F is anti-quasiregular. Moreover, $\Psi(\boldsymbol{c}_0)$ is a critical point of multiplicity d of F with associated critical value $\Psi(0)$. We also note that under iterates of F, each $z \notin \mathcal{K}(R)$ eventually escapes to $\Psi(\mathbb{C} \setminus \mathbf{\Omega}_0) = \mathbb{C} \setminus \operatorname{int} \operatorname{Dom}(F)$. Finally, the map F fixes $\partial\operatorname{Dom}(F)$ pointwise.

We pull back the standard complex structure on $\widehat{\mathbb{C}} \setminus V'$ under the quasiconformal map $\Psi \circ \beta$ to get a complex structure on $\widehat{\mathbb{C}} \setminus \mathcal{V}$. Pulling this complex structure on $\widehat{\mathbb{C}} \setminus \mathcal{V}$ back by iterates of F and extending by the standard complex structure on $\mathcal{K}(R)$, one obtains an

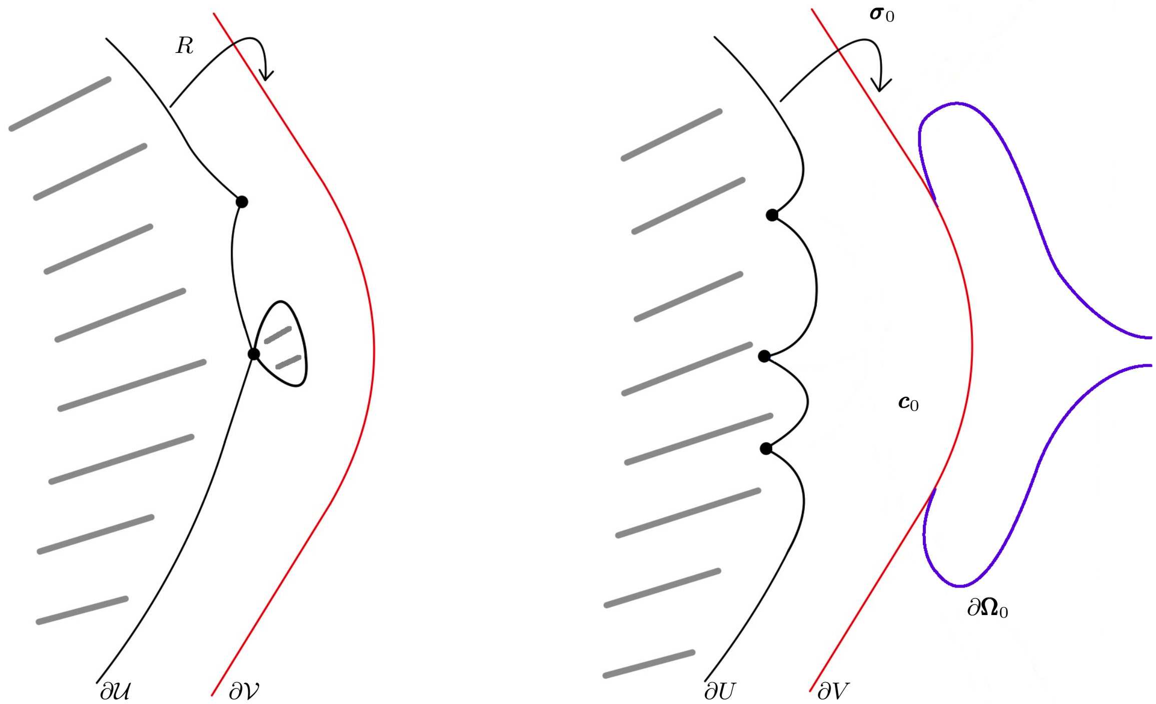

Figure 3.7: Left: The simple pinched anti-polynomial-like map $(R, \overline{\mathcal{U}}, \overline{\mathcal{V}})$ is shown. The shaded region is $\mathcal{U}$, and the region to the left of the red curve is $\mathcal{V}$. Right: The simple pinched anti-polynomial-like map $(\boldsymbol{\sigma}_0, \overline{U}, \overline{V})$ is shown. The shaded region is U, and the region to the left of the red curve is V. The boundary ∂V (in red) consists of a part of $\partial\boldsymbol{\Omega}_0$ and a pair of smooth arcs that meet at ∞ at a positive angle. The purple curves denote the remaining part of $\partial\boldsymbol{\Omega}_0$.

F-invariant Beltrami coefficient μ on $\widehat{\mathbb{C}}$. Since the anti-quasiregular map F is antiholomorphic on $\mathcal{U}$, and the F-orbit of each point meets $\overline{\mathcal{V}} \setminus \mathcal{U}$ at most once, it follows that $||\mu||_\infty < 1$.

Conjugating F by a quasiconformal map H that solves the Beltrami equation with coefficient μ, we obtain an antiholomorphic map $\sigma_R = H \circ F \circ H^{-1}$ defined on the closed Jordan disk $\Omega_R := H(\mathrm{Dom}(F))$. Moreover, σ_R fixes the boundary $\partial\Omega_R$ pointwise. Hence, Ω_R is a simply connected quadrature domain and σ_R is its Schwarz reflection map. After possibly conjugating σ_R by a Möbius map, we can assume that $H(\Psi(0)) = \infty$ and $H(\Psi(\infty)) = H(\infty) = 0$.

We will now justify that $(\Omega_R, \sigma_R) \in \mathcal{S}_{\mathcal{R}_d}^{\mathrm{simp}}$. The mapping properties of F imply that $H(\Psi(0)) = \infty \in \operatorname{int}\Omega_R^c$ is a critical value of σ_R with $\sigma_R^{-1}(\infty) = \{H(\Psi(\boldsymbol{c}_0))\} \in \Omega_R$. In particular, $c := H(\Psi(\boldsymbol{c}_0))$ is a critical point of multiplicity d. It follows that $\sigma_R : \sigma_R^{-1}(\operatorname{int}\Omega_R^c) \to \operatorname{int}\Omega_R^c$ is a degree $d+1$ branched covering. Since Ω_R is a Jordan quadrature domain, it follows from Proposition 3.1.1 that there exists a degree $d+1$ rational map f that carries $\overline{\mathbb{D}}$ injectively onto $\overline{\Omega_R}$. We normalize f so that $f(0) = c$. As $\sigma_R \equiv f \circ \eta \circ (f|_{\overline{\mathbb{D}}})^{-1}$, we conclude that f maps ∞ to itself with local degree $d+1$. Thus, f is a degree $d+1$ polynomial.

Note that as R has $d-1$ critical points in $\mathcal{K}(R)$, the Schwarz reflection σ_R has $d-1$ critical points in $H(\mathcal{K}(R)) \subset \Omega_R$. This implies that f has $d-1$ critical points in $\mathbb{D}^* \setminus \{\infty\}$. As f has d critical points in the plane and none of them can lie in $\mathbb{D}$, it follows that the remaining critical point of f lies on $\mathbb{S}^1$. Thus, f has a unique critical point on $\mathbb{S}^1$, and hence $\partial\Omega_R$ has a unique conformal cusp (and no double point). Further, the fact that $\partial\Omega_0 \setminus \{0\}$ is a non-singular real-analytic arc combined with quasiconformality of β, Ψ and H implies that $\partial\Omega_R \setminus \{0\}$ is a quasi-arc. Hence, the unique conformal cusp of $\partial\Omega_R$ is at 0.

Therefore, $T^0(\sigma_R) = \Omega_R^c \setminus \{0\}$. That each $z \notin \mathcal{K}(R)$ eventually escapes to $\mathbb{C} \setminus \operatorname{int}\operatorname{Dom}(F)$ under F translates to the fact that the non-escaping set (respectively, the tiling set) of σ_R is given by $H(\mathcal{K}(R))$ (respectively, $\widehat{\mathbb{C}} \setminus H(\mathcal{K}(R))$). Thus, the non-escaping set $K(\sigma_R)$ is connected.

In light of Proposition 3.2.1, we conclude that $(\Omega_R, \sigma_R) \in \mathcal{S}_{\mathcal{R}_d}$ (one could alternatively conclude this from the fact that c is the unique critical point of σ_R in its tiling set $T^\infty(\sigma_R)$ and that this critical point maps to $\infty \in \operatorname{int} T^0(\sigma_R)$ with local degree $d+1$). Since $R \in \mathcal{F}_d^{\mathrm{simp}}$, the parabolic fixed point ∞ of R has no attracting direction in $\mathcal{K}(R)$. Under the topological conjugacy H, this translates to the fact that σ_R has no attracting direction in $K(\sigma_R)$. By [LMM23, Corollary A.6], the unique conformal cusp of $\partial\Omega_R$ is of type $(3,2)$. Therefore, $(\Omega_R, \sigma_R) \in \mathcal{S}_{\mathcal{R}_d}^{\mathrm{simp}}$.

Finally, since $\overline{\partial} H = 0$ a.e. on $\mathcal{K}(R)$, we conclude that H^{-1} induces a hybrid conjugacy between a simple pinched anti-polynomial-like restriction of σ_R (with filled Julia set $K(\sigma_R)$)

and a simple pinched anti-polynomial-like restriction of R (with filled Julia set $\mathcal{K}(R)$). In particular, $\chi^{-1}([R]) = [\Omega_R, \sigma_R]$.

Finally, since the fundamental (pinched) annuli of the simple pinched anti-polynomial-like restrictions of anti-rational maps $[R] \in \left[\mathcal{F}_d^{\mathrm{simp}}\right]$ move continuously with respect to the parameter and the asymptotics of the maps near the parabolic point at ∞ are the same throughout $\mathcal{F}_d^{\mathrm{simp}}$, it follows that the quasiconformal dilatations of the hybrid conjugacies between $[R]$ and $[\chi^{-1}(R)]$ constructed above are locally bounded and the domains of definition of these conjugacies depend continuously on parameters as $[R]$ runs over $\mathcal{F}_d^{\mathrm{simp}}$. $\square$

3.4.2 Proofs of the main theorems

We are now ready to prove precise versions of Theorem A and the first part of Theorem B stated in the introduction. The continuity statement of Theorem B will be proved in the next section.

Theorem 3.4.3. *Let $R \in \mathcal{F}_d$. Then, there exists a polynomial map f of degree $d+1$ with a unique critical point on $\mathbb{S}^1$ such that $f|_{\overline{\mathbb{D}}}$ is univalent, and the associated antiholomorphic correspondence $\mathfrak{C}^*$ given in Section 3.2.3 is a mating of the anti-Hecke group $\mathbf{\Gamma}_d$ and R.*

Moreover, this mating operation yields a bijection between $[\mathcal{F}_d]$ and the space of antiholomorphic correspondences arising from $[\mathcal{S}_{\mathcal{R}_d}]$.

Proof. The statement follows from Theorem 3.4.1 and the definition of χ. $\square$

3.5 Continuity properties of the straightening map

Lemma 3.5.1.

1. Let $\{[\Omega_n, \sigma_n]\} \longrightarrow [\Omega_\infty, \sigma_\infty]$ in $\left[\mathcal{S}_{\mathcal{R}_d}^{\mathrm{simp}}\right]$. Then, all accumulation points of $\{\chi(\sigma_n)\}$ in $[\mathcal{F}_d]$ are quasiconformally conjugate to $\chi(\sigma_\infty)$.

2. Let $\{[R_n]\} \longrightarrow [R_\infty]$ in $\left[\mathcal{F}_d^{\text{simp}}\right]$. Then, all accumulation points of $\{\chi^{-1}(R_n)\}$ in $[\mathcal{S}_{\mathcal{R}_d}]$ are quasiconformally conjugate to $\chi^{-1}(R_\infty)$.

Proof. 1) We set $[R_n] := \chi(\sigma_n)$. According to Theorem 3.3.5, there exist quasiconformal maps φ_n that hybrid conjugate simple pinched anti-polynomial-like restrictions of σ_n (with filled Julia set $K(\sigma_n)$) to simple pinched anti-polynomial-like restrictions of R_n (with filled Julia set $\mathcal{K}(R_n)$). Moreover by Corollary 3.3.13, the quasiconformal dilatations of the hybrid conjugacies φ_n are bounded. Thus, we may assume after passing to a subsequence that the global quasiconformal maps φ_n converge uniformly to some quasiconformal homeomorphism φ_∞ of $\widehat{\mathbb{C}}$.

According to Proposition 3.3.2, $\mathcal{F}_d$ is compact. Thus, we can assume possibly after passing to a further subsequence that $[R_n] \to [R_\infty] \in \mathcal{F}_d$. We also recall from Corollary 3.3.13 that the domains of definition of the hybrid conjugacies φ_n depend continuously on parameters. Thus, the domains of the simple pinched anti-polynomial-like restrictions of σ_n constructed in Lemma 3.3.10 converge to that of σ_∞ (with associated non-escaping set $K(\sigma_\infty)$). The equivariance property of φ_n now implies that φ_∞ is a conjugacy between a pinched anti-polynomial-like restriction of σ_∞ to a pinched anti-polynomial-like restriction of R_∞.

On the other hand, there exists a quasiconformal map φ that hybrid conjugates a simple pinched anti-polynomial-like restriction of σ_∞ (with filled Julia set $K(\sigma_\infty)$) to a simple pinched anti-polynomial-like restriction of $\chi(\sigma_\infty)$ (with filled Julia set $\mathcal{K}(\chi(\sigma_\infty))$). Thus, the map R_∞ and $\chi(\sigma_\infty)$ are quasiconformally conjugate on some pinched neighborhoods of their filled Julia sets. As these two maps are also conformally conjugate on their parabolic basin of ∞, it follows by the arguments of Lemma 3.3.3 that R_∞ and $\chi(\sigma_\infty)$ are globally quasiconformally conjugate.

2) Since χ is bijective, we may set $[\Omega_n, \sigma_n] := \chi^{-1}(R_n)$. Theorem 3.4.2 provides with global quasiconformal homeomorphisms ψ_n that hybrid conjugate simple pinched anti-polynomial-like restrictions of R_n (with filled Julia set $\mathcal{K}(R_n)$) to simple pinched anti-polynomial-like restrictions of σ_n (with filled Julia set $K(\sigma_n)$) such that the quasiconformal dilatations of

the hybrid conjugacies ψ_n are bounded. Thus, we may assume after passing to a subsequence that the global quasiconformal maps ψ_n converge uniformly to some quasiconformal homeomorphism ψ_∞ of $\widehat{\mathbb{C}}$.

By Proposition 3.2.6, $\mathcal{S}_{\mathcal{R}_d}$ is compact, and hence we can assume possibly after passing to a further subsequence that $[\Omega_n, \sigma_n] \to [\Omega_\infty, \sigma_\infty] \in \mathcal{S}_{\mathcal{R}_d}$. Theorem 3.4.2 also guarantees that the domains of definition of the hybrid conjugacies ψ_n depend continuously on parameters, and hence the domains of the conventional simple pinched anti-polynomial-like restrictions of R_n converge to that of R_∞ (with associated filled Julia set $\mathcal{K}(R_\infty)$). Hence, ψ_∞ is a conjugacy between a pinched anti-polynomial-like restriction of R_∞ to a pinched anti-polynomial-like restriction of σ_∞.

On the other hand, there exists a quasiconformal map ψ that hybrid conjugates a simple pinched anti-polynomial-like restriction of R_∞ (with filled Julia set $\mathcal{K}(R_\infty)$) to a simple pinched anti-polynomial-like restriction of $\chi^{-1}(R_\infty)$ (with filled Julia set $K(\chi^{-1}(R_\infty))$). Thus, the map σ_∞ and $\chi^{-1}(R_\infty)$ are quasiconformally conjugate on some pinched neighborhoods of their non-escaping sets. As these two maps are also conformally conjugate on their tiling set, it follows by the arguments of Lemma 3.3.3 that σ_∞ and $\chi^{-1}(R_\infty)$ are globally quasiconformally conjugate. □

Proposition 3.5.2. *The sequence* $\{[\Omega_n, \sigma_n]\} \subset \left[\mathcal{S}_{\mathcal{R}_d}^{\text{simp}}\right]$ *has no accumulation point in* $\left[\mathcal{S}_{\mathcal{R}_d}^{\text{high}}\right]$ $\iff$ *the sequence* $\{[\chi(\sigma_n)]\} \subset \left[\mathcal{F}_d^{\text{simp}}\right]$ *has no accumulation point in* $\left[\mathcal{F}_d^{\text{high}}\right]$.

Proof. Suppose that all accumulation points of $\{[\Omega_n, \sigma_n]\} \subset \left[\mathcal{S}_{\mathcal{R}_d}^{\text{simp}}\right]$ lie in $\left[\mathcal{S}_{\mathcal{R}_d}^{\text{simp}}\right]$. Then by part (1) of Lemma 3.5.1, all accumulation points of $\{[\chi(\sigma_n)]\}$ are quasiconformally conjugate to maps in $\left[\mathcal{F}_d^{\text{simp}}\right]$. As the multiplicity of a parabolic fixed point is a topological invariant, it follows that all accumulation points of $\{[\chi(\sigma_n)]\}$ lie in $\left[\mathcal{F}_d^{\text{simp}}\right]$.

Conversely, assume that $\{[\Omega_n, \sigma_n]\} \subset \left[\mathcal{S}_{\mathcal{R}_d}^{\text{simp}}\right]$ and all accumulation points of $\{[\chi(\sigma_n)]\}$ lie in $\left[\mathcal{F}_d^{\text{simp}}\right]$. Then by part (2) of Lemma 3.5.1, all accumulation points of $\{[\Omega_n, \sigma_n]\}$ are quasiconformally conjugate to maps in $\left[\mathcal{S}_{\mathcal{R}_d}^{\text{simp}}\right]$. By [LMM23, Corollary A.6], the condition of having a $(3,2)$-cusp on the quadrature domain boundary is a topological conjugacy invariant for maps in $\mathcal{S}_{\mathcal{R}_d}$. Therefore, all accumulation points of $\{[\Omega_n, \sigma_n]\}$ lie in $\left[\mathcal{S}_{\mathcal{R}_d}^{\text{simp}}\right]$. $\square$

Proposition 3.5.3. *The straightening map χ is continuous at quasiconformally rigid and at relatively hyperbolic parameters in* $\left[\mathcal{S}_{\mathcal{R}_d}^{\text{simp}}\right]$.

Proof. We first note that $(\Omega, \sigma) \in \mathcal{S}_{\mathcal{R}_d}$ is a quasiconformally rigid parameter if and only if $\chi(\sigma)$ is so. Continuity of χ at quasiconformally rigid parameters in $\left[\mathcal{S}_{\mathcal{R}_d}^{\text{simp}}\right]$ now follows from Lemma 3.5.1.

A straightforward adaptation of [Mil12, Theorem 5.1] shows that the relatively hyperbolic components of $\left[\mathcal{S}_{\mathcal{R}_d}^{\text{simp}}\right]$ and $\left[\mathcal{F}_d^{\text{simp}}\right]$ are diffeomorphic to appropriate spaces of fibrewise anti-Blaschke products. The construction of such a diffeomorphism and the fact that hybrid conjugacies are conformal on the interior of the non-escaping sets imply that χ carries each relatively hyperbolic component of $\left[\mathcal{S}_{\mathcal{R}_d}^{\text{simp}}\right]$ to a corresponding component of $\left[\mathcal{F}_d^{\text{simp}}\right]$, and χ factors (through a space of fibrewise anti-Blaschke products) as the composition of two diffeomorphisms. The result follows. $\square$

Chapter 4

One Parameter Families of Maps Associated With Belyi-Shabat Polynomials

4.1 One-parameter families in $\mathcal{F}_d$ and associated Belyi Schwarz reflections

4.1.1 Slices $\mathfrak{L}_{\mathcal{T}}$ of Belyi anti-rational maps

An anti-rational maps is called *Belyi* if it has at most three critical values in $\widehat{\mathbb{C}}$. If $R \in \mathcal{F}_d$ is Belyi with exactly three critical values, then the filled Julia set $\mathcal{K}(R)$ contains two of the three critical values of R. As in the holomorphic case, we can define the *dessin d'enfant* of R to be the bicolored plane tree obtained by taking the R-preimage of a simple arc γ connecting the two critical values of R in $\mathcal{K}(R)$. As the third critical value of R (lying in $\mathcal{B}(R)$) is fully ramified, it follows that the dessin d'enfant of R is a tree.

If $R \in \mathcal{F}_d$ is Belyi with exactly two critical values, then $\mathcal{K}(R)$ contains exactly one critical value of R which must be fully ramified. Thus, this critical value must be different from the parabolic fixed point ∞. In this case, one can still construct a dessin d'enfant of R by choosing γ to be a simple arc that connects ∞ to the unique critical value of R in $\mathcal{K}(R)$.

We will denote this (combinatorial) bicolored plane tree by $\mathcal{T}(R)$.

A natural way of defining dynamically natural sub-families of Belyi anti-rational maps in $\mathcal{F}_d$ is to introduce critical orbit relations.

Definition 4.1.1. We define

$$\mathfrak{L} := \{R \in \mathcal{F}_d : R \text{ is Belyi, and if } R \text{ has three critical values, then the parabolic fixed point } \infty \text{ is a critical value of } R\}.$$

Note that for $R \in \mathfrak{L}$, the parabolic fixed point ∞ gives rise to a marked vertex of valence one (also called an endpoint or a tip) on $\mathcal{T}(R)$. As a convention, we will color this vertex black and denote it by v_b. The adjacent white vertex (of valence at least two) is denoted by v'_w.

Definition 4.1.2. We define

$$\mathfrak{L}_{\mathcal{T}} := \{R \in \mathfrak{L} : (\mathcal{T}(R), v_b) \cong (\mathcal{T}, O)\},$$

where $(\mathcal{T}, O)$ is a bicolored plane tree with a black vertex O of valence one as a root, and the isomorphism is understood to preserve the root and the bicolored plane structure. We also set

$$\mathfrak{L}_{\mathcal{T}}^{\text{simp}} := \mathfrak{L}_{\mathcal{T}} \cap \mathcal{F}_d^{\text{simp}} = \{R \in \mathfrak{L}_{\mathcal{T}} : \infty \text{ is a simple parabolic fixed point of } R\}.$$

We also remark that if $\mathcal{T}$ is a star-tree, then each map in $\mathfrak{L}_{\mathcal{T}}$ has a unique critical value in $\mathcal{K}(R)$.

Lemma 4.1.1. *$\operatorname{int} \mathfrak{L}_{\mathcal{T}} \neq \emptyset$.*

Proof. Perform David surgery to glue $B_d|_{\mathbb{D}}$ outside the filled Julia set of the dynamically Shabat anti-polynomial p (if $\mathcal{T}$ is a star-tree, perform the surgery on $\overline{z}^d$). We can also arrange so that the unique parabolic fixed point of B_d corresponds to the repelling fixed point $\mathrm{emb}(v_b)$ of p (this is possible because $\mathrm{emb}(v_b)$ is the landing point of the external dynamical ray of p at angle 0). This would produce an anti-rational map R with a parabolic fixed point at ∞ (after possibly a Möbius change of coordinates) such that R has a simply connected, completely invariant parabolic basin (of the parabolic fixed point ∞) where the dynamics is conformally conjugate to B_d. Moreover, R has at most three critical values, and when R has exactly three critical values, then the parabolic fixed point ∞ is one of them. Hence, $R \in \mathfrak{L}$. That the dessin d'enfant of R is isomorphic to $\mathcal{T}$ follows from the fact that $p|_{\mathcal{K}(p)}$ is conjugate to $R|_{\mathcal{K}(R)}$ under a global orientation-preserving homeomorphism. This proves the existence of a map $R \in \mathfrak{L}_{\mathcal{T}}$ with a superattracting fixed point. One can now construct an open neighborhood of R in $\mathfrak{L}_{\mathcal{T}}$ by a standard quasiconformal surgery that changes this superattracting fixed point into a linearly attracting one. □

4.1.2 The associated Belyi Schwarz reflections $\mathcal{S}_{\mathcal{T}}$

We will now study the preimage of the family $\mathfrak{L}_{\mathcal{T}}$ under the straightening map χ, and see that the corresponding Schwarz reflections come from univalent restrictions of Shabat polynomials whose dessin d'enfant can be explicitly read from $\mathcal{T}$.

Proposition 4.1.2. *Let $[\Omega, \sigma] \in \chi^{-1}(\mathfrak{L}_{\mathcal{T}})$, and $f : \mathbb{D} \to \Omega$ be a uniformizing polynomial map. Then the following hold.*

1. *f is a Shabat polynomial whose dessin d'enfant $\mathcal{T}^{\mathrm{aug}}$ is obtained by adding a single edge to $\mathcal{T}^{\mathrm{op}}$ at the black vertex v_b. (The other endpoint of this new edge is necessarily white, and we denote it by v_w.)*

   In particular, $\mathcal{T}^{\mathrm{aug}}$ has a black vertex v_b of valence 2 that lies between a terminal white vertex v_w and a white vertex v_w' of valence at least two.

2. *If emb : $\mathcal{T}^{\mathrm{aug}} \to \mathbb{C}$ induces an isomorphism between the combinatorial tree $\mathcal{T}^{\mathrm{aug}}$ and a planar realization of it, then emb$(v_b) \in \mathbb{S}^1$ and emb$(v_w) \in \mathbb{D}$.*

Remark 4.1.3. (1) Recall that the vertex set of the embedded tree $\mathcal{T}_\gamma(f)$ (where γ is an arc connecting the finite critical values of f) and the isomorphism $emb : \mathcal{T}(f) \to \mathcal{T}_\gamma(f)$ appearing in the statement of Proposition 4.1.2 are independent of the choice γ.

(2) We use the notation $\mathcal{T}^{\mathrm{aug}}$ to remind the reader that it is an augmentation of the original tree $\mathcal{T}$.

Proof. 1) Let $\chi(\Omega, \sigma) = R$. Then, $\sigma|_{K(\sigma)}$ is hybird conjugate to $R|_{\mathcal{K}(R)}$.

Let us denote the unique cusp on $\partial\Omega$ by y_1. Note that y_1 is fixed under σ, and corresponds to the fixed point ∞ of R under the hybrid conjugacy. It now follows that σ has exactly $d-1$ critical points (counted with multiplicities) and at most two critical values in $K(\sigma)$. More precisely, if $\mathcal{T}$ is a star-tree, then σ has a unique critical point of multiplicity $d-1$ and hence a single critical value in $K(\sigma)$; otherwise, σ has exactly two critical values $K(\sigma)$ one of which is y_1.

Consider a simple arc $\gamma_\sigma \subset \Omega$ connecting these two critical values of σ (if σ has only one critical value in $K(\sigma)$, then we choose γ_σ to be an arc connecting this critical value to y_1). The existence of a hybrid conjugacy between σ and R implies that $\mathcal{T}_{\gamma_\sigma}(\sigma) := \sigma^{-1}(\gamma_\sigma)$ is a tree with a plane bicolored structure. We denote this combinatorial tree by $\mathcal{T}(\sigma)$, and note that it is is isomorphic to $\mathcal{T}(R) \cong \mathcal{T}$. Moreover, the root v_b of $\mathcal{T}(R)$ defines a root point for $\mathcal{T}(\sigma)$, and this root corresponds to the vertex y_1 of $\mathcal{T}_{\gamma_\sigma}(\sigma)$. Abusing notation, we denote this root point of $\mathcal{T}(\sigma)$ by v_b. One can think of $\mathcal{T}(\sigma)$ as an analogue of dessin d'enfant for the Schwarz reflection map σ.

Recall that f has a unique (simple) critical point on $\mathbb{S}^1$ with associated critical value y_1. As f has no critical point in $\mathbb{D}$, we conclude that f has precisely $d-1$ finite critical points in $\mathbb{D}^*$ (counted with multiplicities). Since $\sigma = f \circ \eta \circ (f|_{\mathbb{D}})^{-1}$, we see that these critical points are given by $\eta((f|_{\mathbb{D}})^{-1}(\mathrm{crit}(\sigma)))$, and they are mapped by f to the two critical values of σ (respectively, to the unique critical value and y_1) in $K(\sigma)$. Therefore, f is a Shabat polynomial.

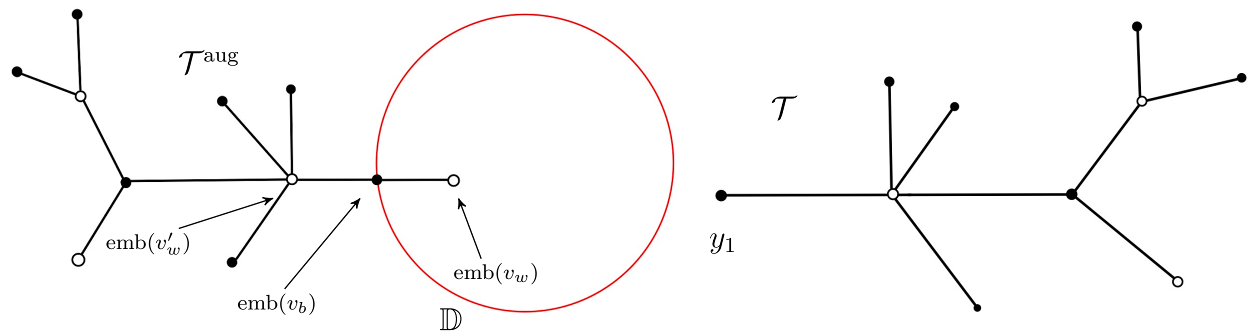

Figure 4.1: Relation among the dessin d'enfant $\mathcal{T}^{\text{aug}} \cong f^{-1}(\gamma_\sigma)$ of f and the dessin d'enfant $\mathcal{T} \cong \sigma^{-1}(\gamma_\sigma)$ of the Schwarz reflection map σ.

It is easy to see from the relation between f and σ that

$$f^{-1}(\gamma_\sigma) = \eta((f|_{\overline{\mathbb{D}}})^{-1}(\sigma^{-1}(\gamma_\sigma))) \bigcup (f|_{\overline{\mathbb{D}}})^{-1}(\gamma_\sigma). \tag{4.1.1}$$

Moreover, the topological closed interval $(f|_{\overline{\mathbb{D}}})^{-1}(\gamma_\sigma)$ is contained in $\overline{\mathbb{D}}$ and intersects $\mathbb{S}^1$ only at $(f|_{\overline{\mathbb{D}}})^{-1}(y_1)$, and $\eta((f|_{\overline{\mathbb{D}}})^{-1}(\sigma^{-1}(\gamma_\sigma)))$ is a topological tree (with d edges) contained in $\overline{\mathbb{D}^*}$ intersecting $\mathbb{S}^1$ only at $(f|_{\overline{\mathbb{D}}})^{-1}(y_1)$ (see Figure 4.1). We conclude that combinatorially, the dessin d'enfant of f is obtained by adding an edge to $\mathcal{T}^{\text{op}}$ at the vertex v_b (note the appearance of the orientation-reversing map η in Relation (4.1.1)). We denote this combinatorial tree by $\mathcal{T}^{\text{aug}}$, and call the newly added vertex v_w (it corresponds to the endpoint of $(f|_{\overline{\mathbb{D}}})^{-1}(\gamma_\sigma)$ different from $(f|_{\overline{\mathbb{D}}})^{-1}(y_1)$).

Consequently, $\mathcal{T}^{\text{aug}}$ has a black vertex v_b of valence 2 with an adjacent white vertex v_w of valence 1. We further note that f is univalent on the closed disk $\overline{\mathbb{D}}$ which has the vertex $(f|_{\overline{\mathbb{D}}})^{-1}(y_1)$ on its boundary, and contains the vertex $(f|_{\overline{\mathbb{D}}})^{-1}(y_2)$ (where y_2 is the unique critical value of σ in $K(\sigma)$ different from y_1) in its interior.

2) We now consider an isomorphism *emb* between the combinatorial bicolored plane tree $\mathcal{T}^{\text{aug}}$ and its planar realization $f^{-1}(\gamma_\sigma)$. By construction, $emb(v_b) = (f|_{\overline{\mathbb{D}}})^{-1}(y_1) \in \mathbb{S}^1$ and $emb(v_w) = (f|_{\overline{\mathbb{D}}})^{-1}(y_2) \in \mathbb{D}$, where y_2 is the unique critical value of σ (in $K(\sigma)$) different from y_1. $\square$

We now wish to give an explicit description of $\chi^{-1}(\mathfrak{L}_{\mathcal{T}})$ as a real two-dimensional family of

Schwarz reflections. Observe that by Proposition 4.1.2 , all $[\Omega, \sigma] \in \chi^{-1}(\mathfrak{L}_{\mathcal{T}})$, the polynomial uniformizations $f : \mathbb{D} \to \Omega$ are equivalent Shabat polynomials. After possibly conjugating σ by an affine map (which amounts to replacing Ω by an affine image of it), we can require that all such σ have the same marked critical values. Then, the corresponding uniformizing polynomials f only differ by pre-composition by an affine map. Instead of fixing the domain of univalence $\mathbb{D}$ and varying the polynomial uniformizations (that differ by pre-composition by affine maps), it will be slightly more convenient to fix a polynomial uniformization and restrict it to various disks of uniavlence (that are affine images of $\mathbb{D}$). This leads to the following space of Schwarz reflections.

Definition 4.1.3. Fix a degree $d+1$ Shabat polynomial $\boldsymbol{f}$ produced by Proposition 4.1.2 together with an isomorphism $\boldsymbol{emb}$ of the combinatorial tree $\mathcal{T}^{\text{aug}}$ (which is the dessin d'enfant of $\boldsymbol{f}$) and a planar realization of it. We also set $\boldsymbol{v_b} := \boldsymbol{emb}(v_b)$, $\boldsymbol{v_w} := \boldsymbol{emb}(v_w)$, $\boldsymbol{f}(\boldsymbol{v_b}) = y_1$, and $\boldsymbol{f}(\boldsymbol{v_w}) = y_2$.

Define

$$S_{\mathcal{T}} := \{a \in \mathbb{C} : \boldsymbol{v_w} \in \Delta_a := B(a, |\boldsymbol{v_b} - a|) \text{ and } \boldsymbol{f}|_{\overline{\Delta_a}} \text{ is univalent}\},$$

and

$$S_{\mathcal{T}}^{\text{simp}} := \{a \in S_{\mathcal{T}} : \text{the unique cusp } y_1 \text{ on } \partial\Omega_a \text{ is simple; i.e., of type } (3,2)\}.$$

We also set $\Omega_a := \boldsymbol{f}(\Delta_a)$.

Remark 4.1.4. (1) Since $\boldsymbol{v_b} \in \mathbb{S}^1$ and $\boldsymbol{v_w} \in \mathbb{D}$, we have that $\Delta_0 = \mathbb{D}$ and thus $0 \in S$ (see Proposition 4.1.2).

(2) For each $a \in S$, the quadrature domain Ω_a contains y_2 and its boundary $\partial\Omega_a$ has a conformal cusp at y_1. In particular,

$$\overline{\Delta_a} \cap \boldsymbol{f}^{-1}(\{y_1, y_2\}) = \{\boldsymbol{v_b}, \boldsymbol{v_w}\}.$$

We denote reflection in the circle $\partial\Delta_a$ by η_a, and the Schwarz reflection map of the quadrature domain Ω_a by $\sigma_a = \boldsymbol{f} \circ \eta_a \circ (\boldsymbol{f}|_{\overline{\Delta_a}})^{-1}$. By definition, the critical points of the

Schwarz reflection map σ_a are given by

$$\{\boldsymbol{f}(\eta_a(\zeta)) : \boldsymbol{f}'(\zeta) = 0,\ \zeta \neq \boldsymbol{v_b}\}.$$

If $\boldsymbol{v_b}$ is the only critical point of $\boldsymbol{f}$ over y_1, then the only critical value of σ_a is y_2. Otherwise, the set of critical values of σ_a is $\{y_1, y_2\}$. In light of this fact, we call the map σ_a a *Belyi* map.

Definition 4.1.4. The family $\mathcal{S}_{\mathcal{T}}$ of Schwarz reflection maps is defined as

$$\mathcal{S}_{\mathcal{T}} := \{\sigma_a : \Omega_a \to \widehat{\mathbb{C}} : a \in S_{\mathcal{T}}\},$$

while its sub-family $\mathcal{S}_{\mathcal{T}}^{\mathrm{simp}}$ is defined as

$$\mathcal{S}_{\mathcal{T}}^{\mathrm{simp}} := \{\sigma_a : \Omega_a \to \widehat{\mathbb{C}} : a \in S_{\mathcal{T}}^{\mathrm{simp}}\}.$$

Remark 4.1.5. The family $\mathcal{S}_{\mathcal{T}}$ can be seen as a generalization of the family of Schwarz reflection maps associated with the cubic Chebyshev polynomial studied in [Lee+21].

Proposition 4.1.6. *χ induces a bijection between $\mathcal{S}_{\mathcal{T}} \cap \mathcal{S}_{\mathcal{R}_d}$ and ${}^{\mathfrak{L}_{\mathcal{T}}}\!/_{\mathrm{Aut}(\mathbb{C})}$.*

Proof. Let us first note that no two maps $\sigma_{a_1}, \sigma_{a_2} \in \mathcal{S}_{\mathcal{T}}^{\mathrm{simp}} \cap \mathcal{S}_{\mathcal{R}_d}$ are Möbius conjugate. This is because any Möbius map conjugating σ_{a_1} to σ_{a_2} would fix the unique critical value ∞ in the tiling sets $T^\infty(\sigma_{a_i})$, the conformal cusp y_1 on the boundaries $\partial\Omega_{a_i}$ and the other critical value $y_2 \in K(\sigma_{a_i})$ of σ_{a_i}.

We proceed to show that $\chi^{-1}(\mathfrak{L}_{\mathcal{T}})$ is contained in $\mathcal{S}_{\mathcal{T}} \cap \mathcal{S}_{\mathcal{R}_d}$. To this end, let $(\Omega, \sigma) \in \chi^{-1}(\mathfrak{L}_{\mathcal{T}})$. By Proposition 4.1.2, there exists a Shabat polynomial f_1 such that the dessin d'enfant of f_1 is isomorphic to $\mathcal{T}^{\mathrm{aug}}$ and $f_1 : \overline{\mathbb{D}} \to \overline{\Omega}$ is a homeomorphism. After possibly replacing Ω by an affine image, we can assume that the unique cusp on $\partial\Omega$ is y_1 and the only other critical value of σ in $K(\sigma)$ is y_2. Then, the proof of Proposition 4.1.2 shows that the vertices v_b, v_w of the combinatorial tree $\mathcal{T}^{\mathrm{aug}}$ correspond to the critical and co-critical values $(f_1|_{\overline{\mathbb{D}}})^{-1}(y_1), (f_1|_{\overline{\mathbb{D}}})^{-1}(y_2)$, respectively.

The classification of Shabat polynomials now implies that there exists an affine map A with $A(\boldsymbol{v_b}) = (f_1|_{\overline{\mathbb{D}}})^{-1}(y_1)$, $A(\boldsymbol{v_w}) = (f_1|_{\overline{\mathbb{D}}})^{-1}(y_2)$, and $\boldsymbol{f} \equiv f_1 \circ A$. Setting $a := A^{-1}(0)$, we

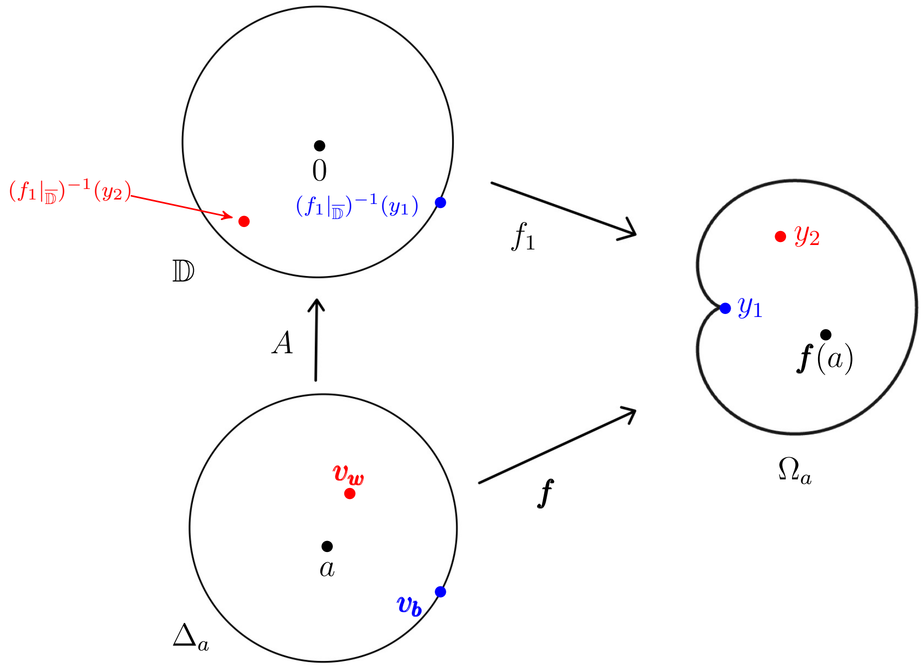

Figure 4.2: Pictured is the affine change of coordinate A appearing in the proof of Proposition 4.1.6. The corresponding points are marked in the same color.

conclude that $A^{-1}(\mathbb{D})$ is a round disk centered at a having $\boldsymbol{v_b}$ on its boundary such that $\boldsymbol{f}$ is univalent on $A^{-1}(\overline{\mathbb{D}}) = \overline{\Delta_a}$ (see Figure 4.2). Thus, $\Omega = f_1(\mathbb{D}) = \boldsymbol{f}(A^{-1}(\mathbb{D})) = \Omega_a$, and hence $\sigma \equiv \sigma_a$. Finally, $\boldsymbol{v_w} = A^{-1}((f_1|_{\overline{\mathbb{D}}})^{-1}(y_2)) \in \Delta_a$. It follows that $a_o \in S_{\mathcal{T}}$, and hence $(\Omega, \sigma) \in \mathcal{S}_{\mathcal{T}}$. Finally, the fact that the χ-preimage of any map in $\mathcal{F}_d^{\mathrm{simp}}$ lies in $\mathcal{S}_{\mathcal{R}_d}^{\mathrm{simp}}$ implies that $(\Omega, \sigma) \in \mathcal{S}_{\mathcal{T}}^{\mathrm{simp}} \cap \mathcal{S}_{\mathcal{R}_d}$.

Conversely, let $(\Omega_a, \sigma_a) \in \mathcal{S}_{\mathcal{T}} \cap \mathcal{S}_{\mathcal{R}_d}$. We need to argue that $R := \chi(\sigma_a) \in \mathfrak{L}_{\mathcal{T}}$. Since σ_a has at most two critical values in $K(\sigma_a)$, it follows that R has at most two critical values in $\mathcal{K}(R)$. Hence, R has at most three critical values. Moreover, R has three critical values if and only if y_1 is a critical value of σ_a. Since the hybrid conjugacy between σ_a and R carries y_1 to ∞, we conclude that if R has three critical values, then ∞ is one of them. Therefore, $R \in \mathfrak{L}$.

We will now describe the dessin d'enfant of R by dualizing some of the arguments of Proposition 4.1.2. Let $\gamma_a \subset \overline{\Omega_a}$ be a simple arc connecting y_1 and y_2. Since $\boldsymbol{f} : \overline{\Delta_a} \to \overline{\Omega_a}$ is a homeomorphism, it follows that $\boldsymbol{f}^{-1}(\gamma_a) \cap \overline{\Delta_a}$ is a simple arc that connects $\boldsymbol{v_b}$ to $\boldsymbol{v_w}$. Note that the embedded tree $\boldsymbol{f}^{-1}(\gamma_a) \setminus \Delta_a$ is isomorphic to $\mathcal{T}^{\mathrm{op}}$ as a combinatorial bicolored plane tree such that the isomorphism sends $(\boldsymbol{f}|_{\Delta_a}^{-1})(y_1)$ to v_b. Thus, the tree

$$\sigma_a^{-1}(\gamma_a) = \boldsymbol{f}(\eta_a(\boldsymbol{f}^{-1}(\gamma_a) \setminus \Delta_a)) \subset \overline{\Omega_a}$$

is isomorphic to $\mathcal{T}$ as a combinatorial bi-colored plane tree. Clearly, the hybrid conjugacy yields an isomorphism between $\sigma_a^{-1}(\gamma_a)$ and a planar realization of the dessin d'enfant $\mathcal{T}(R)$ of R. We conclude that $\mathcal{T}(R) \cong \mathcal{T}$; i.e., $R \in \mathfrak{L}_{\mathcal{T}}$. Finally, the fact that the image of χ is contained in $\mathcal{F}_d^{\mathrm{simp}}$ implies that $R \in \mathfrak{L}_{\mathcal{T}}^{\mathrm{simp}}$. □

Corollary 4.1.7. $\operatorname{int} S_{\mathcal{T}} \neq \emptyset$.

Proof. This follows from Lemma 4.1.1 and Proposition 4.1.6. □

Corollary 4.1.8. *Let $a \in S_{\mathcal{T}}$, $\gamma_0 \subset \overline{\Omega_a}$ be a simple arc connecting y_1 and y_2. Then, the embedded tree*

$$\sigma_a^{-1}(\gamma_a) = \boldsymbol{f}(\eta_a(\boldsymbol{f}^{-1}(\gamma_a) \setminus \Delta_a)) \subset \overline{\Omega_a}$$

is isomorphic to $\mathcal{T}$ as a combinatorial bi-colored plane tree, and the isomorphism identifies the vertex y_1 of $\sigma_a^{-1}(\gamma_a)$ with the root of $\mathcal{T}$.

Corollary 4.1.8 can be restated as follows: σ_a is a Belyi map whose *dessin d'enfant* is given by $\mathcal{T}$.

The relationship between the dessin d'enfant of $\boldsymbol{f}$ and that of σ_a can be used to prove that the family $\mathcal{S}_{\mathcal{T}}$ is quasiconformally closed. Although we will not have need for this statement, it seems worth recording this fact.

Proposition 4.1.9. *Let $a \in S_{\mathcal{T}}$, μ be a σ_a-invariant Beltrami coefficient on $\widehat{\mathbb{C}}$, and Φ be the quasiconformal map solving the Beltrami equation with coefficient μ such that Φ fixes y_1, y_2, and ∞. Then, there exists $a' \in S_{\mathcal{T}}$ such that $\Phi(\Omega_a) = \Omega_{a'}$, and $\Phi \circ \sigma_a \circ \Phi^{-1} = \sigma_{a'}$ on $\Omega_{a'}$.*

Sketch of Proof. The assumption that μ is σ_a-invariant implies that $\check{\sigma} := \Phi \circ \sigma_a \circ \Phi^{-1}$ is anti-meromorphic on $\check{\Omega} := \Phi(\Omega_a)$, and continuously extends to the identity map on $\partial\check{\Omega}$. Thus, $\check{\Omega}$ is a simply connected quadrature domain with Jordan boundary and Schwarz reflection map $\check{\sigma}$. Also, $\check{\sigma} : \check{\sigma}^{-1}(\operatorname{int}\check{\Omega}^c) \to \operatorname{int}\check{\Omega}^c$ is a $d+1:1$ branched cover with a critical point of local degree $d+1$. Arguing as in Proposition 3.2.1, one concludes that there is a degree $d+1$ polynomial g which carries $\overline{\mathbb{D}}$ injectively onto $\overline{\check{\Omega}}$. Thus, $\check{\sigma} \equiv g \circ \eta \circ (g|_{\overline{\mathbb{D}}})^{-1}$ on $\overline{\check{\Omega}}$. The fact that $\check{\sigma}$ has at most two critical points in $K(\check{\sigma})$ (recall that the same is true for σ) implies that g has at most two finite critical values. Thus, g is a Shabat polynomial.

That the dessin d'enfant of σ_a is isomorphic to $\mathcal{T}$ implies that the same is true for $\check{\sigma}$. Arguing as in Proposition 4.1.2, one concludes that the dessin d'enfant of g is isomorphic to $\mathcal{T}^{\text{aug}}$. As g and $\boldsymbol{f}$ have isomorphic dessin d'enfants and the same marked critical values, one can use the arguments of Proposition 4.1.6 to deduce that $\check{\sigma}$ lies in the family $\mathcal{S}_{\mathcal{T}}$. $\square$

4.1.3 Connectedness locus of $\mathcal{S}_{\mathcal{T}}$ and relation to $\mathcal{S}_{\mathcal{R}_d}$

Recall proposition 3.1.3, which states that for a parameter a the filled Julia set $K(\sigma_a)$ is connected if and only if the free critical value of σ_a is non-escaping. leads to the following definition.

Definition 4.1.5 (Connectedness locus and escape locus)**.** The connectedness locus of the family $\mathcal{S}_{\mathcal{T}}$ is defined as

$$\mathcal{C}(\mathcal{S}_{\mathcal{T}}) = \{a \in S_{\mathcal{T}} : y_2 \in K(\sigma_a)\} = \{a \in S_{\mathcal{T}} : K(\sigma_a) \text{ is connected}\}.$$

The complement of the connectedness locus in the parameter space $S_{\mathcal{T}}$ is called the *escape locus*. We similarly define $\mathcal{C}(\mathcal{S}_{\mathcal{T}}^{\text{simp}})$ as the set of parameters $a \in \mathcal{S}_{\mathcal{T}}^{\text{simp}}$ with connected $K(\sigma_a)$.

Definition 4.1.6 (Depth)**.** For any a in the escape locus of $\mathcal{S}_{\mathcal{T}}$, the *smallest* positive integer $n(a)$ such that $\sigma_a^{\circ n(a)}(y_2) \in T^0(\sigma_a)$ is called the *depth* of a.

Lemma 4.1.10. *1) For $a \in \mathcal{C}(\mathcal{S}_{\mathcal{T}})$, the map $\sigma_a : T^\infty(\sigma_a) \setminus \operatorname{int} T^0(\sigma_a) \to T^\infty(\sigma_a)$ is conformally conjugate to $\mathcal{R}_d : \mathbb{D}_1 \cup C_1 \to \mathcal{Q}$.*

2) For $a \in S_{\mathcal{T}} \setminus \mathcal{C}(\mathcal{S}_{\mathcal{T}})$,

$$\sigma_a : \bigcup_{n=1}^{n(a)} \sigma_a^{-n}(T^0(\sigma_a)) \to \bigcup_{n=0}^{n(a)-1} \sigma_a^{-n}(T^0(\sigma_a))$$

is conformally conjugate to

$$\mathcal{R}_d : \bigcup_{n=1}^{n(a)} \mathcal{R}_d^{-n}(\mathcal{Q}_1) \to \bigcup_{n=0}^{n(a)-1} \mathcal{R}_d^{-n}(\mathcal{Q}_1).$$

Proof. Since $\mathcal{Q}_1$ is simply connected, we can choose a homeomorphism

$$\psi_a : \mathcal{Q}_1 \to T^0(\sigma_a)$$

such that it is conformal on the interior. We can further assume that $\psi_a(0) = \infty$, and its continuous extension sends the cusp point $1 \in \partial\mathcal{Q}_1$ to the point $y_1 \in \partial T^0(\sigma_a)$.

Note that $\sigma_a : \sigma_a^{-1}(T^0(\sigma_a)) \to T^0(\sigma_a)$ is a $(d+1) : 1$ branched cover branched only at $\boldsymbol{f}(a)$, and $\mathcal{R}_d : \rho_1(\Pi) \to \mathcal{Q}_1$ is a $(d+1) : 1$ branched cover branched only at $\rho_1(0)$. Moreover, σ_a fixes $\partial T^0(\sigma_a)$ pointwise, and $\mathcal{R}_d$ fixes $C_2 \cup \{1\} \cong \partial\mathcal{Q}_1$ pointwise.

This allows one to lift ψ_a to a conformal isomorphism from $\rho_1(\Pi)$ onto $\sigma_a^{-1}(T^0(\sigma_a))$ such that the lifted map sends $\rho_1(0)$ to $\boldsymbol{f}(a)$, and continuously matches with the initial map ψ_a on $\mathcal{Q}_1$. We denote this extended conformal isomorphism by ψ_a. By construction, ψ_a is equivariant with respect to the actions of $\mathcal{R}_d$ and σ_a on $\rho_1(\Pi)$ and $\partial\sigma_a^{-1}(T^0(\sigma_a))$, respectively.

1) If $a \in \mathcal{C}(\mathcal{S}_\mathcal{T})$, then every tile of $T^\infty(\sigma_a)$ (of rank greater than one) maps diffeomorphically onto $\sigma_a^{-1}(T^0(\sigma_a))$ under some iterate of σ_a, and each tile of $\mathbb{D}_1$ (of rank greater than one) maps diffeomorphically onto $\rho_1(\Pi)$ under some iterate of $\mathcal{R}_d$. This fact, along with the equivariance property of ψ_a mentioned above, enables us to lift ψ_a to all tiles using the iterates of $\mathcal{R}_d$ and σ_a. This produces the desired biholomorphism ψ_a between $\mathcal{Q}$ and $T^\infty(\sigma_a)$ which conjugates $\mathcal{R}_d$ to σ_a.

2) For $a \in S_\mathcal{T} \setminus \mathcal{C}(\mathcal{S}_\mathcal{T})$, the above construction of ψ_a can be carried out onto the tiles of $T^\infty(\sigma_a)$ that map diffeomorphically onto $\sigma_a^{-1}(T^0(\sigma_a))$, which includes all tiles of rank up to $n(a)$. This completes the proof. □

Definition 4.1.7 (Dynamical Rays of σ_a)**.** The pre-image of a $\mathcal{G}_d$-ray at angle $\theta \in [0, \frac{1}{d+1})$ in $\mathcal{Q}$ under the map ψ_a (see Lemma 4.1.10) is called a θ-dynamical ray of σ_a.

Clearly, the image of (the tail of) a dynamical θ-ray under σ_a is (the tail of) a dynamical ray angle $\mathcal{R}_d(\theta)$.

Proposition 4.1.11. $\mathcal{S}_{\mathcal{T}} \cap \mathcal{S}_{\mathcal{R}_d} = \mathcal{C}(\mathcal{S}_{\mathcal{T}})$. *Consequently,*

$$\chi : \mathcal{C}(\mathcal{S}_{\mathcal{T}}^{\text{simp}}) \longrightarrow \mathfrak{L}_{\mathcal{T}}^{\text{simp}} \big/ \text{Aut}(\mathbb{C})$$

is a bijection.

Proof. Clearly, if $(\Omega, \sigma) \in \mathcal{S}_{\mathcal{T}} \cap \mathcal{S}_{\mathcal{R}_d}$, then the tiling set $T^{\infty}(\sigma)$ is biholomorphic to a round disk and hence its complement $K(\sigma)$ is connected.

Conversely, let $\sigma_a \in \mathcal{C}(\mathcal{S}_{\mathcal{T}})$. By Lemma 4.1.10, the tiling set dynamics of σ_a is conformally conjugate to the dynamics of $\mathcal{R}_d$ of $\mathcal{Q}$ via ψ_a. Hence, $(\Omega_a, \sigma_a) \in \mathcal{S}_{\mathcal{T}} \cap \mathcal{S}_{\mathcal{R}_d}$. We conclude that $\mathcal{S}_{\mathcal{T}} \cap \mathcal{S}_{\mathcal{R}_d} = \mathcal{C}(\mathcal{S}_{\mathcal{T}})$.

The second assertion now follows from the above equality and Proposition 4.1.6. □

4.2 Combinatorics of connected filled Julia sets

Before turning to the parameter space for the $\mathcal{S}_{\mathcal{T}}$ we collect some useful results for the combinatorics of maps within the family, and a criteria for realization of PCF maps.

Notation: We denote the vertices of $\mathcal{T}$ as $\{b_j\}$ and $\{w_j\}$ and set b_0 to be the root of $\mathcal{T}$ and w_1 to be the unique white vertex adjacent to b_0. We also denote $v_a = \sigma_a(w_j)$ to be the free critcal value for the parameter a.

Proposition 4.2.1. *For all $a \in \mathcal{C}_{\mathcal{T}}$ for which $K(\sigma_a)$ is locally connected, the tree $\mathcal{T}$ embeds into $K(\sigma_a)$, and the embedding is unique up to a homotopy fixing the vertices.*

Proof. Let γ be a path in $K(\sigma_a)$ with endpoints b_0 and v. Then $\sigma_a^{-1}(\gamma)$ is an embedding of the path. The simple connectivity of $K(\sigma_a)$ guarantees the uniqueness. □

We will show that even when the Julia set is not locally connected that the combinatorial structure of $\mathcal{T}$ is still present in $K(\sigma_a)$.

Proposition 4.2.2. *Periodic dynamic rays land.*

Proof. (See [Mil06, Theorem 18.10]) Let $R_\theta(a)$ be a periodic ray of period p. In the hyperbolic metric on $\widehat{\mathbb{C}} \setminus K(\sigma_a)$ the ray is a union of geodesic segments tending towards infinity. In particular, it must accumulate to at least one point on $K(\sigma_a)$, which we denote as z_0. We now show that z_0 is a periodic point with period dividing p. Let U be a neighborhood of z_0. Let I_{pk} denote a fundamental segment for the ray, with endpoints at a depth pk tile and a depth $p(k+1)$ tile. There is some k large enough that $I_k \subset U$. But now as $U \cap f(U) \supset I_k \cap I_{k-1} \neq \emptyset$, it follows that z_0 is fixed by σ_a^p.

The set of accumulation points of a ray must be connected, and thus either a single point or is uncountable. But as the number of periodic points of σ_a of period dividing p is finite, it follows that z_0 must be the only accumulation point of this ray. □

A version of the following proposition can be found in [Lyu, Theorem 24.5]

Proposition 4.2.3. *Let z_0 be a periodic repelling or parabolic point in the Julia set. Then there is a ray which lands at z_0.*

Proposition 4.2.4. *Suppose multiple rays land at z. Then z is a cut point for the filled Julia set.*

Proof. If θ_1 and θ_2 are distinct angles whose σ_a dynamical rays land at the same point then the union of these rays, together with the landing point and ∞ forms a closed topological loop in $\widehat{\mathbb{C}}$. The cusp of Ω_a lies in one complemenatary component of this loop. At the same time, all rational $\theta \in (\theta_1, \theta_2)$ land on the filled Julia set, which is in the other complementary component of said closed loop. □

Corollary 4.2.5. *For $a \in \mathcal{C}(\mathcal{T})$ no two distinct vertices of $\mathcal{T}$ can lie in the same Fatou component.*

Proof. Let c_1 and c_2 be critical points of σ_a. If it were the case that they lay in the same Fatou component there would exist a path connecting them not passing through $\partial K(\sigma_a)$.

Any path between critical points c_1 and c_2 of σ_a is homotopic within $K(\sigma_a)$, rel the endpoints, to a path in $\mathcal{T}$. Such a path must pass through a critical preimage of 0, which are cut points of $K(\sigma_a)$. It follows that the original path must also contain the said cut point, which is necessarily in ∂K. □

Define the *principal wakes* for a Schwarz reflection $\sigma_a \in \mathcal{S}_{\mathcal{T}}$ to be the planar regions whose boundaries are given by the dynamical rays landing at preimages of the cusp, and such that no such dynamical rays are in the interior. When $\sigma_a \in \mathcal{C}(\mathcal{S}_{\mathcal{T}}$ we may intersect these principal wakes with $K(\sigma_a)$ to arrive at the following analogue of 4.2.1:

Proposition 4.2.6. *Let $a \in \mathcal{C}_{\mathcal{T}}$. Then $K(\sigma_a)$ is the union of finitely many hulls, any two of which intersect in at most one point, namely at a critical pre-image of the cusp of Ω_a. Each such hull contains exactly one pre-image of the free critical value v.*

Let G be the graph whose vertices are preimages of the cusp of Ω_a and the free critical value, and with edges connecting a preimage of the cusp of Ω_a to a preimage of the free critical value if they line in the same hull as above. Then G is isomorphic as a bi-colored planar rooted tree to $\mathcal{T}$.

Definition 4.2.1. We will say that a *Hubbard tree* for a post-critically finite map σ_a is the (unique) tree contained in $K(\sigma_a)$ which contains all preimages of the critical values of σ_a, and whose intersections with Fatou components are given by internal rays.

Poirier's realization theorem, together with theorem C of [LMM23] allows the construction of many post-critically finite maps in $\mathcal{S}$. We give a construction below showing that S contains maps which are semi-conjugate in a natural way to the dynamics of PCF unicritical anti-holomorphic polynomials.

Lemma 4.2.7. *Let w_j be a white vertex of degree $d > 1$, and let p be any PCF, unicritical, anti-holomorphic polynomial of degree d. Then there exists some PCF $\sigma_a \in S$ such that $\sigma_a(\mathcal{P}_j) \supset \mathcal{P}_j$, and with a David hybrid conjugacy from a neighborhood of $J(p)$ to a neighborhood $U \supset \mathcal{P}_j$.*

Proof. Denote the Hubbard tree of p as T_p. Now consider the tree given by adjoining T_p to $\mathcal{T}$ at w_j. We associate a topological map to this tree given by sending each white vertex to the critical value in T_p, sending each black vertex to the root of $\mathcal{T}$, and maintaining the dynamics on T_p. The dynamics on this tree is expanding. This is an admissible Hubbard tree and therefore is realized by some anti-holomorphic polynomial, q. By theorem C of [LMM23] we have that q is David hybrid equivalent to some $\sigma_a \in S$, which has the desired properties.

□

4.3 The parameter space $S_{\mathcal{T}}$.

In this section we show that the parameter space $S_{\mathcal{T}}$ is a topological quadrilateral with boundary given by real analytic arcs, and analyze the dynamics of the associated maps at those boundary components.

4.3.1 Boundedness of $S_{\mathcal{T}}$

Note that $\partial\Delta_a$ contains a fixed base point $\boldsymbol{v_b}$. As $|a| \to \infty$, the radius $|a - \boldsymbol{v_b}|$ of the disk Δ_a also goes to infinity. Hence, as $a \to \infty$ along some θ-ray $\{\boldsymbol{v_b} + Re^{i\theta} : R \in (0, \infty)\}$, the disk $\overline{\Delta_a}$ converges to some closed half-plane in $\widehat{\mathbb{C}}$. Since the polynomial $\boldsymbol{f}$ behaves like $c \cdot z^{d+1}$ near ∞ with $d + 1 \geq 3$, it follows that $\boldsymbol{f}$ cannot be injective on a half-plane near ∞. Hence, for $|a|$ large enough, $\boldsymbol{f}$ cannot be injective on Δ_a.

This gives the following.

Lemma 4.3.1. *The parameter space $S_{\mathcal{T}}$ is a bounded subset of $\mathbb{C}$.*

4.3.2 Dynamics near cusp points

Definition 4.3.1.

1. We define

$$\Gamma^{\mathrm{hoc}} := \{a \in S_{\mathcal{T}} : y_1 = \boldsymbol{f}(\boldsymbol{v_b}) \text{ is a cusp of type } (\nu, 2),\ \nu \geq 5\},$$

and call Γ^{hoc} the *higher order cusp locus.*

2. We define

$$\Gamma^{\mathrm{pb}} := \{a \in \mathbb{C} : \boldsymbol{v_w} \in \partial\Delta_a,\ \boldsymbol{f}|_{\overline{\Delta_a}} \text{ is univalent}\} \ \subset$$

The perpendicular bisector of the straight line segment joining $\boldsymbol{v_w}$ and $\boldsymbol{v_b}$.

For $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, denote by α_θ the curve germ at $\boldsymbol{v_b}$ that maps under $\boldsymbol{f}$ to a straight line segment $\{y_1 - [0, \varepsilon]e^{2i\theta}\}$ (for $\varepsilon > 0$ small).

The following statement can be found in [LMM23, Appendix A]

Lemma 4.3.2. *Let $a \in S_{\mathcal{T}}$. Then the following are equivalent.*

1. *$a \in \Gamma^{\mathrm{hoc}}$; i.e., y_1 is a cusp of type $(\nu, 2)$ of $\partial\Omega_a$ with $\nu > 3$.*

2. *The circle $\partial\Delta_a$ is an osculating circle to the curve germ α_θ at $\boldsymbol{v_b}$, where $\theta = \arg(a - \boldsymbol{v_b})$.*

3. *The second iterate $\sigma_a^{\circ 2}$ has at least one attracting and at least one repelling direction in Ω_a at the cusp y_1. In particular, the unique free critical orbit of σ_a (i.e., the forward orbit of y_2) non-trivially converges to y_1.*

Corollary 4.3.3. *Let $a \in S_{\mathcal{T}}$. Then the following statements hold.*

1. *y_1 is a $(\nu, 2)$-cusp of $\partial\Omega_a$, where $\nu \in \{3, 5, 7\}$.*

2. *The invariant direction $\{y_1 + [0, \varepsilon]e^{2i\arg(a - \boldsymbol{v_b})}\}$ is a repelling (respectively, attracting) direction for σ_a at y_1 if y_1 is a $(3, 2)$ or $(7, 2)$ (respectively, $(5, 2)$) cusp of $\partial\Omega_a$.*

Proof. 1) Suppose that $\nu > 7$. By [LMM23, Proposition A.4], there are at least seven $\sigma_a^{\circ 2}$-invariant direction at y_1, of which at least three are attracting. By the proof of [Lee+21, Proposition 4.5], there must be three infinite critical orbits of $\sigma_a^{\circ 2}$ converging to y_1 in this case. But this is absurd as $\sigma_a^{\circ 2}$ has at most two infinite critical orbits. Thus, $\nu \leq 7$.

2) This follows from [LMM23, Proposition A.5]. □

Lemma 4.3.4. $\Gamma^{\mathrm{hoc}} \cap \overline{\Gamma^{\mathrm{pb}}} = \emptyset$.

Proof. For $a \in \Gamma^{\mathrm{hoc}}$, the cusp point y_1 non-trivially attracts the forward orbit of y_2. On the other hand, for $a \in \overline{\Gamma^{\mathrm{pb}}}$, the critical value y_2 lies on $\partial\Omega_a \setminus \{y_1\}$, and is thus fixed. Hence, $\overline{\Gamma^{\mathrm{pb}}} \cap \Gamma^{\mathrm{hoc}} = \emptyset$. □

4.3.3 Dynamics near double points

A point $p \in \partial\Omega_a$ is said to be a *double point* if for all sufficiently small $\varepsilon > 0$, the intersection $B(p, \varepsilon) \cap \Omega_a$ is a union of two Jordan domains, and p is a non-singular boundary point of each of them. In particular, two distinct non-singular (real-analytic) local branches of $\partial\Omega_a$ intersect tangentially at a double point p. One can further classify such double points according to the order of contact of the two real-analytic branches γ_1 and γ_2 of $\partial\Omega_a$ at p. Let ι_1 and ι_2 be the local Schwarz reflection maps associated with γ_1 and γ_1. It is easily checked that if γ_1 and γ_2 have contact of order k at p, then $\iota_1 \circ \iota_2$ is a parabolic germ of the form $z \mapsto z + a(z-p)^{k+1} + o((z-p)^{k+1})$ with $a \neq 0$. Moreover, $\iota_2 \circ \iota_1$ is the inverse of $\iota_1 \circ \iota_2$, and these two germs are anti-conformally conjugate via ι_1. We also note that k is necessarily odd since otherwise the two branches γ_1 and γ_2 would cross at p.

Definition 4.3.2. We define

$$\Gamma^{\mathrm{dp}} := \{a \in \mathbb{C} : \boldsymbol{v_w} \in \Delta_a,\ f|_{\Delta_a} \text{ is univalent, and } \partial\Omega_a \text{ has a double point}\},$$

and call Γ^{dp} the *double point locus.*

We now study the local dynamics of σ_a near a double point p of $\partial\Omega_a$.

Lemma 4.3.5. *Let p be a double point on $\partial\Omega_a$. Then the following assertions hold.*

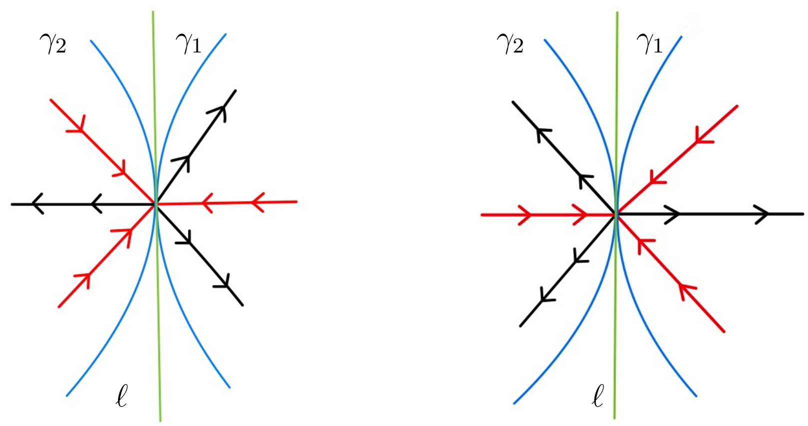

Figure 4.3: The situations considered in the two sub-cases 2a and 2b in the proof of Lemma 4.3.5 are depicted. The arrows indicate attracting/repelling directions for the parabolic germ $\iota_2 \circ \iota_1$.

1. *The two non-singular branches of $\partial\Omega_a$ at p have contact of order one or three.*

2. *If the contact is of order one, then the two associated normal directions at p are repelling directions for $\sigma_a^{\circ 2}$ and there is no attracting direction for $\sigma_a^{\circ 2}$ at p.*

3. *If the contact is of order three, then the two associated normal directions at p are the only attracting directions for $\sigma_a^{\circ 2}$, and the unique free critical orbit of σ_a non-trivially converges to p. In particular, $\partial\Omega_a$ has at most one such double point. Furthermore, there are four repelling directions for $\sigma_a^{\circ 2}$.*

Proof. We denote the common tangent line for γ_1 and γ_2 at p by ℓ.

Case 1 ($k = 1$). Up to second order, the local power series of ι_1, ι_2 depend only on the (signed) curvature of γ_1, γ_2 at p. Hence, it suffices to assume that ι_1, ι_2 are Schwarz reflections with respect to the osculating circles to γ_1, γ_2 at p (cf. [Dav74, Sec.7]). A simple computation now shows that the inward normal to γ_1 (respectively, the outward normal to γ_1) at p is a repelling direction for $\gamma_2 \circ \gamma_1$ (respectively, for $\gamma_1 \circ \gamma_2$). In other words, these normal directions are repelling directions for $\sigma_a^{\circ 2}$.

Case 2 ($k = 3$). In this case, the germ $\iota_2 \circ \iota_1$ has three attracting and three repelling directions. Note that the inward normal to γ_1 at p is invariant under this germ.

Sub-case 2a. If the inward normal to γ_1 at p is an attracting direction for $\iota_2 \circ \iota_1$, then the other two attracting directions of $\iota_2 \circ \iota_1$ are on the opposite side of ℓ (see Figure 4.3 (left)). Thus, these two directions are repelling directions for $\iota_1 \circ \iota_2$ and hence for $\sigma_a^{\circ 2}$. On the other hand, the outward normal to γ_1 at p is attracting for $\iota_1 \circ \iota_2$ (since it is repelling for $\iota_2 \circ \iota_1$). Therefore, the only attracting directions for $\sigma_a^{\circ 2}$ at p are the above two normal vectors, and they are exchanged by σ_a (see [Lee+18a, Proposition 5.15] for the same situation in the circle-and-cardioid family of Schwarz reflections). By Fatou-type arguments, the unique free critical orbit of σ_a must converge to p asymptotic to this 2-cycle of attracting directions (cf. [Lee+18a, Propositions 5.30, 5.32]). The statement about the repelling directions for $\sigma_a^{\circ 2}$ also follows.

Sub-case 2b. If the inward normal to γ_1 at p is a repelling direction for $\iota_2 \circ \iota_1$, then so is the outward normal to γ_1 at p for the inverse germ $\iota_1 \circ \iota_2$. In this case, $\sigma_a^{\circ 2}$ has four attracting directions which are pairwise exchanged by σ_a (see Figure 4.3 (right)). Once again, a Fatou-type argument shows that there are two period two cycles of parabolic basins of σ_a (at p) and each such cycle contains an infinite critical orbit of σ_a. But this contradicts the fact that σ_a has at most one infinite critical orbit. Hence, this sub-case cannot occur.

Case 3 ($k \geq 5$). The same arguments as in the previous case show that there must be at least two period two cycles of parabolic basins of σ_a at p. But this would require at least two infinite critical orbits of σ_a, implying that this case is impossible. □

Definition 4.3.3. We call a double point of $\partial\Omega_a$ *regular* (respectively, *special*) if the two non-singular branches of $\partial\Omega_a$ at p have contact of order one (respectively, three).

For $a \in \Gamma^{\mathrm{dp}}$, the desingularized droplet $T^0(\sigma_a)$ is defined as the set obtained by removing the cusp and the double points from $\widehat{\mathbb{C}} \setminus \Omega_a$. Note that in this case, $T^0(\sigma_a)$ contains a unique unbounded component, denoted by $T^0_u(\sigma_a)$, and finitely many bounded components, whose union is denoted by $T^0_b(\sigma_a)$.

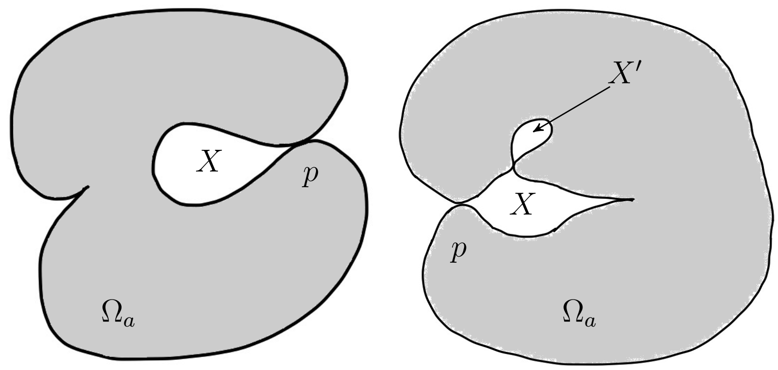

Figure 4.4: Pictured are two a priori possible configurations of the desingularized droplet that are disallowed by Lemma 4.3.6.

Lemma 4.3.6. *Let $a \in \Gamma^{\mathrm{dp}}$, and X be a component of $T_b^0(\sigma_a)$. Then, the set of singular points on ∂X is either exactly two double points of $\partial\Omega_a$, or exactly one double point and the unique cusp of $\partial\Omega_a$.*

Proof. Clearly, ∂X must contain at least one double point p of $\partial\Omega_a$.

We will first argue that p cannot be the only singular point of $\partial\Omega_a$ on ∂X. By way of contradiction, let us assume that this is the case (see Figure 4.4 (left)). Then, $\partial X \setminus \{p\}$ is a non-singular real-analytic arc, and hence there exists a component Y of $\sigma_a^{-1}(X)$ such that $\partial X \subsetneq \partial Y$. Since $a \in \Gamma^{\mathrm{dp}}$, the free critical value of σ_a lies in $\overline{\Omega_a}$, which is disjoint from $\operatorname{int} X$. Hence, $\sigma_a : \operatorname{int} Y \to \operatorname{int} X$ is a covering map. As $\operatorname{int} X$ is a topological disk, $\sigma_a : \operatorname{int} Y \to \operatorname{int} X$ must be a homeomorphism. But this is impossible because each point on $\partial X \setminus \{p\}$ has at least two preimages under σ_a on the boundary of Y.

Therefore, ∂X must contain at least two double points of $\partial\Omega_a$ or a double point and the unique cusp of $\partial\Omega_a$. Also note that since $\partial\Omega_a$ is a real-algebraic curve, it has at most finitely many double points. Hence, if ∂X contains an additional double point of $\partial\Omega_a$, then there must exist some component $X' \neq X$ of $T_b^0(\sigma_a)$ such that the only singularity on $\partial X'$ is a double point of $\partial\Omega_a$ (see Figure 4.4 (right)). But this contradicts the conclusion of the previous paragraph. This completes the proof. □

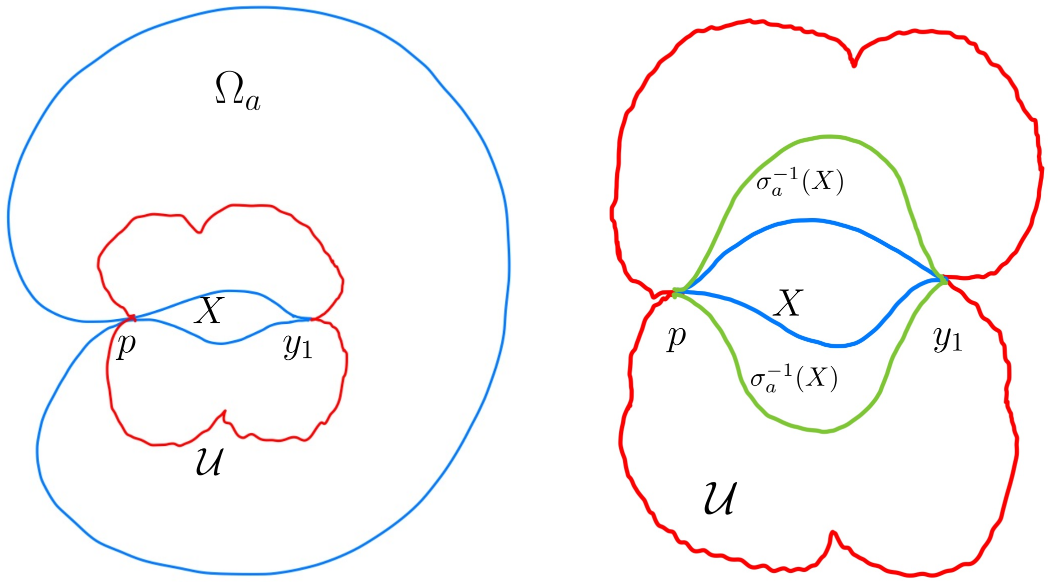

Figure 4.5: The component X of $T_a^0(\sigma_a)$ (introduced in Proposition 4.3.7) and its σ_a-preimages in the tiling component $\mathcal{U}$ are displayed.

We now proceed to study the relation between components of $T_b^0(\sigma_a)$ and the free critical orbit of σ_a.

Proposition 4.3.7. *Let $a \in \Gamma^{\mathrm{dp}}$. Then $T_b^0(\sigma_a)$ is connected and some forward iterate of the critical value y_2 of σ_a lands in $T_b^0(\sigma_a)$.*

Proof. By Proposition 4.3.6 and the fact that $\partial\Omega_a$ has at most finitely many singular points, there must exist a component X of $T_a^0(\sigma_a)$ which has the cusp y_1 and a double point p (of $\partial\Omega_a$) on its boundary, and such that $\partial X \setminus \{p, y_1\}$ is the union of a pair of non-singular real-analytic arcs. In particular, $\partial X \setminus \{p, y_1\}$ has a small neighborhood contained in the tiling set of σ_a. We denote the component of $T^\infty(\sigma_a)$ containing X by $\mathcal{U}$ (see Figure 4.5).

We will first suppose that the orbit $\sigma_a^{\circ n}(y_2)$ does not intersect X and study its dynamical consequence. Under this assumption, the Riemann-Hurwitz formula implies that $\sigma_a^{-1}(\operatorname{int} X) \cap \mathcal{U}$ is the union of two simply connected domains each of which maps homeomorphically onto $\operatorname{int} X$. Moreover, $\partial X \subsetneq \partial(\sigma_a^{-1}(X) \cap \mathcal{U})$, and $\operatorname{int}(X \cup (\sigma_a^{-1}(X) \cap \mathcal{U}))$ is a simply connected domain (see Figure 4.5). Since the free critical orbit never meets X, one can now apply the above argument inductively to conclude that for each $n \geq 1$,

- $\sigma_a^{-n}(\operatorname{int} X) \cap \mathcal{U}$ is the union of two simply connected domains each of which maps homeomorphically onto $\operatorname{int} X$ under $\sigma_a^{\circ n}$, and

- $\operatorname{int} \bigcup_{i=0}^{n} (\sigma_a^{-i}(X) \cap \mathcal{U})$ is a simply connected domain.

Since $\mathcal{U} = \bigcup_{n=1}^{\infty} \left(\operatorname{int} \bigcup_{i=0}^{n} (\sigma_a^{-i}(X) \cap \mathcal{U}) \right)$ is an increasing union of simply connected domains, we conclude that $\mathcal{U}$ is also a simply connected domain. We denote the two connected components of $\mathcal{U} \setminus (X \cup \sigma_a^{-1}(X))$ by $\mathcal{V}_1$ and $\mathcal{V}_2$, and the two connected components of $\mathcal{U} \setminus X$ by $\mathcal{W}_1$ and $\mathcal{W}_2$ such that $\mathcal{W}_j \supset \mathcal{V}_j$, $j \in \{1, 2\}$. Note that the above arguments also demonstrates simple connectivity of the domains $\mathcal{V}_j, \mathcal{W}_j$, $j \in \{1, 2\}$. Moreover, $\sigma_a^{\circ 2} : \mathcal{V}_1 \to \operatorname{int}(X \cup \mathcal{W}_1)$ is a conformal isomorphism. We denote the corresponding inverse branch by $g : \operatorname{int}(X \cup \mathcal{W}_1) \to \mathcal{V}_1 \subsetneq \operatorname{int}(X \cup \mathcal{W}_1)$, and observe that g is a contraction with respect to the hyperbolic metric of $\operatorname{int}(X \cup \mathcal{W}_1)$.

It follows by the previous paragraph and the local dynamics of $\sigma_a^{\circ 2}$ near p (see Lemma 4.3.5) that there are points in $\mathcal{U}$ close to p whose g-orbits converge to p (asymptotically to a repelling direction of $\sigma_a^{\circ 2}$ at p). By [Mil06, Lemma 5.5], all g-orbits must converge to p.

By Lemma 4.3.2 and Corollary 4.3.3, the cusp point y_1 also has a repelling direction in Ω_a. Hence, there are points in Ω_a near the cusp that eventually land in X. Consequently, there are points in $\mathcal{U}$ close to y_1 whose g-orbits converge to y_1. By [Mil06, Lemma 5.5], all g-orbits must converge to y_1. This contradicts the conclusion of the previous paragraph, and proves that the orbit $\sigma_a^{\circ n}(y_2)$ must intersect X.

If $T_b^0(\sigma_a)$ had a component X' other than X, then one can repeat the above argument to conclude that the orbit $\sigma_a^{\circ n}(y_2)$ must also intersect X', which is impossible. Thus, we conclude that $T_b^0(\sigma_a) = X$. □

Corollary 4.3.8. *Let $a \in \Gamma^{\mathrm{dp}}$. Then the unique double point on $\partial\Omega_a$ is a regular double point.*

Proof. By Lemma 4.3.5 and Proposition 4.3.7, the existence of a special double point on $\partial\Omega_a$

would force the free critical orbit of σ_a to non-trivially converge to the special double point as well as escape to $T_b^0(\sigma_a)$, which is absurd. □

Corollary 4.3.9. $\Gamma^{\mathrm{hoc}} \cap \Gamma^{\mathrm{dp}} = \emptyset$.

Proof. By way of contradiction, assume that $a \in \Gamma^{\mathrm{hoc}} \cap \Gamma^{\mathrm{dp}}$. By Proposition 4.3.7, the free critical orbit of σ_a escapes to $T_b^0(\sigma_a)$. But this is impossible since the free critical orbit of σ_a must also converge non-trivially to the cusp point y_1 by Lemma 4.3.2. □

Corollary 4.3.10. *Let $a \in \Gamma^{\mathrm{dp}}$. Then the forward orbit of the free critical value of σ_a is disjoint from $\overline{T_u^0(\sigma_a)} \cup \{y_1\}$.*

Proof. This follows from Propositions 4.3.7. □

4.3.4 Description of the boundary and interior of the parameter space $S_{\mathcal{T}}$

Lemma 4.3.11. $\Gamma^{\mathrm{hoc}} \bigcup \overline{\Gamma^{\mathrm{dp}}} \bigcup \Gamma^{\mathrm{pb}} \subset \partial S_{\mathcal{T}}$.

Proof. Let $a \in \Gamma^{\mathrm{hoc}}$. By definition, $a \in S_{\mathcal{T}}$. By Lemma 4.3.2, the circle $\partial\Delta_a$ is an osculating circle to the curve germ α_θ at $\boldsymbol{v_b}$ that maps under $\boldsymbol{f}$ to a straight line segment $\{y_1 - [0,\varepsilon]e^{2i\theta}\}$, where $\theta = \arg(a - \boldsymbol{v_b})$. Let $\overrightarrow{\ell_a}$ be the infinite ray from $\boldsymbol{v_b}$ to ∞ passing through a, and a' be a parameter obtained by pushing a slightly along $\overrightarrow{\ell_a}$ away from $\boldsymbol{v_b}$. Then, $\Delta_{a'}$ contains points of α_θ that are identified under $\boldsymbol{f}$, and hence $\boldsymbol{f}$ is not injective on $\Delta_{a'}$. Therefore, $a \in \partial S_{\mathcal{T}}$; i.e., $\Gamma^{\mathrm{hoc}} \subset \partial S_{\mathcal{T}}$.

Let $a \in \Gamma^{\mathrm{pb}}$. By Lemma 4.3.4 and the definition of Γ^{pb}, $\boldsymbol{f}$ is univalent on $\overline{\Delta_a}$ and y_1 is a $(3,2)$ cusp. Thus, by Lemma 4.3.2, the circle $\partial\Delta_a$ is not an osculating circle to the curve germ α_θ at $\boldsymbol{v_b}$. Hence, pushing a along $\overrightarrow{\ell_a}$ slightly away from $\boldsymbol{v_b}$ produces parameters a' such that $\boldsymbol{f}$ is injective on $\overline{\Delta_{a'}}$ and $\boldsymbol{v_w} \in \Delta_{a'}$. Hence, $a \in \partial S_{\mathcal{T}}$. This proves that $\Gamma^{\mathrm{pb}} \subset \partial S_{\mathcal{T}}$.

Let $a \in \Gamma^{\mathrm{dp}}$. Once again, pushing a along $\overrightarrow{\ell_a}$ slightly away from $\boldsymbol{v_b}$ produces parameters a' such that $\boldsymbol{f}$ is not injective on $\Delta_{a'}$ (since $\Delta_{a'} \cup \{\boldsymbol{v_b}\} \supset \overline{\Delta_a}$). On the other hand, pushing

a slightly along $\overrightarrow{\ell_a}$ towards $\boldsymbol{v_b}$ produces parameters a' such that $\boldsymbol{f}$ is injective on $\overline{\Delta_{a'}}$ (since $\overline{\Delta_{a'}} \subset \Delta_a \cup \{\boldsymbol{v_b}\}$) and $\boldsymbol{v_w} \in \Delta_{a'}$ (see Figure 4.6). Hence, $a \in \partial S_{\mathcal{T}}$. This proves that $\Gamma^{\mathrm{dp}} \subset \partial S_{\mathcal{T}}$. The result now follows by taking closures. □

Theorem 4.3.12. *We have*

$$\partial S_{\mathcal{T}} = \Gamma^{\mathrm{hoc}} \bigcup \overline{\Gamma^{\mathrm{dp}}} \bigcup \Gamma^{\mathrm{pb}},$$

and

$$\operatorname{int} S_{\mathcal{T}} = \{a \in \mathbb{C} : \boldsymbol{v_w} \in \Delta_a,\ f|_{\overline{\Delta_a}} \textit{ is univalent, and } y_1 \textit{ is a } (3,2) \textit{ cusp}\}.$$

Proof. First we recall that if $\boldsymbol{f}|_{\overline{\Delta_a}}$ is univalent and the cusp $y_1 \in \partial\Omega_a$ is of type $(3,2)$, then a has a neighborhood such that for all parameters a' in this neighborhood, $\boldsymbol{f}|_{\overline{\Delta_{a'}}}$ is univalent. Since parameters in $S_{\mathcal{T}}$ for which y_1 is a higher order cusp belong to the boundary of $S_{\mathcal{T}}$ (by Lemma 4.3.11), the description of the interior of $S_{\mathcal{T}}$ given in the statement of the theorem follows.

Now let $a \in \partial S_{\mathcal{T}}$. We consider two cases.

Case 1: $a \in S_{\mathcal{T}}$**.** By the description of $\operatorname{int} S_{\mathcal{T}}$, we have that y_1 is a cusp of type $(\nu, 2)$ of $\partial\Omega_a$, with $\nu \geq 5$; i.e., $a \in \Gamma^{\mathrm{hoc}}$.

Case 2: $a \notin S_{\mathcal{T}}$**.** Since $\boldsymbol{f}|_{\Delta_a}$ is the local uniform limit of a sequence of injective holomorphic maps and $\boldsymbol{f}$ is non-constant, we have that $\boldsymbol{f}|_{\Delta_a}$ is injective. The assumption that $a \notin S_{\mathcal{T}}$ now implies that either $\boldsymbol{f}$ is not injective on $\partial\Delta_a$ or $\boldsymbol{v_w} \in \partial\Delta_a$ (or both). In the former case, there is a double point on $\partial\Omega_a$. Moreover, as $a \in \overline{S_{\mathcal{T}}}$, we have that $\boldsymbol{v_w} \in \overline{\Omega_a}$. Hence, $a \in \overline{\Gamma^{\mathrm{dp}}}$. In the latter case, either $a \in \overline{\Gamma^{\mathrm{dp}}}$ or $a \in \Gamma^{\mathrm{pb}}$ depending on whether there is a double point on $\partial\Omega_a$ or not.

Combining the two cases, we conclude that

$$\partial S_{\mathcal{T}} \subset \Gamma^{\mathrm{hoc}} \cup \overline{\Gamma^{\mathrm{dp}}} \cup \Gamma^{\mathrm{pb}}.$$

The description of the boundary of $S_{\mathcal{T}}$ follows from the above containment and Lemma 4.3.11.

□

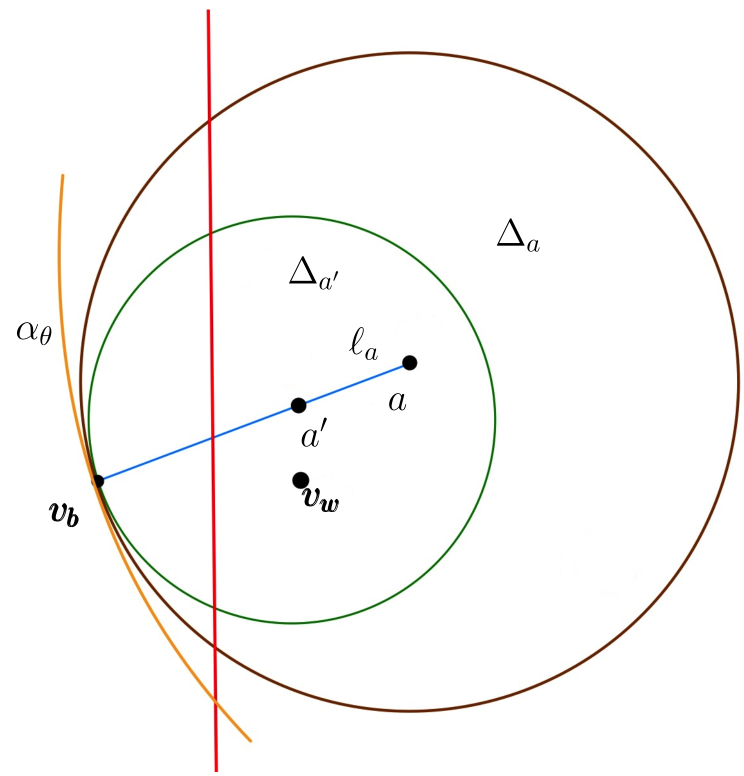

Figure 4.6: The red straight line is the perpendicular bisector of the line segment joining $\boldsymbol{v_b}$ and $\boldsymbol{v_w}$, the half-plane to the right of the red straight line is $\mathbf{P}$, and α_θ is the curve germ at $\boldsymbol{v_b}$ that maps under $\boldsymbol{f}$ to a straight line segment $\{y_1 - [0, \varepsilon]e^{2i\theta}\}$, where $\theta = \arg(a - \boldsymbol{v_b})$. There is a nesting structure of the disks Δ_a when a lies on a fixed straight ray emanating from $\boldsymbol{v_b}$.

4.3.5 Connectedness of $S_{\mathcal{T}}$

Our next goal is to prove that the parameter space $S_{\mathcal{T}}$ is connected. This will be done through a series of lemmas.

Lemma 4.3.13. *Each connected component of* $\operatorname{int} S_{\mathcal{T}}$ *is a Jordan domain. Moreover,* $\overline{S_{\mathcal{T}}} = \overline{\operatorname{int} S_{\mathcal{T}}}$.

Proof. Recall that by Corollary 4.1.7, the interior of $S_{\mathcal{T}}$ is non-empty.

Given any $a \in \mathbb{C}$, let ℓ_a be the straight line segment connecting $\boldsymbol{v_b}$ to a. Now let $a' \in \ell_a$. By definition, $|a - \boldsymbol{v_b}| > |a' - \boldsymbol{v_b}|$ and hence $\overline{\Delta_{a'}} \subset \overline{\Delta_a}$. Thus, if $\boldsymbol{f}$ is univalent on $\overline{\Delta_a}$ then it is also univalent on $\overline{\Delta_{a'}}$. This means that if $a \in S_{\mathcal{T}}$, then the intersection of ℓ_a with the half plane $\mathbf{P} := \{z \in \mathbb{C} : |z - \boldsymbol{v_w}| < |z - \boldsymbol{v_b}|\}$ (which is one of the complementary components of the perpendicular bisector of the line segment joining $\boldsymbol{v_w}$ and $\boldsymbol{v_b}$) also lies in the parameter space (see Figure 4.6). This implies that each connected component of $S_{\mathcal{T}}$ is simply connected.

Now suppose that $a \in \Gamma^{\mathrm{dp}} \cup \Gamma^{\mathrm{hoc}}$, and $a' \in \ell_a \cap \mathbf{P}$, $a' \neq a$. The above argument and the ones used in the proof of Lemma 4.3.11 show that $\boldsymbol{f}$ is injective on $\overline{\Delta_{a'}}$ and $\boldsymbol{v_w} \in \Delta_{a'}$ (see Figure 4.6). We also claim that y_1 is a $(3,2)$ cusp on $\partial\Omega_{a'}$. Indeed, if this were not true, then pushing a' along ℓ_a in the direction of a (i.e., away from $\boldsymbol{v_b}$) would result in $\boldsymbol{f}$ to be non-univalent on the corresponding disk contradicting the assumption that $f|_{\Delta_a}$ is univalent. Therefore, $\ell_a \cap \mathbf{P} \setminus \{a\} \subset \operatorname{int} S_{\mathcal{T}}$ for all $a \in \Gamma^{\mathrm{dp}} \cup \Gamma^{\mathrm{hoc}}$ (by the description of $\operatorname{int} S_{\mathcal{T}}$ given in Theorem 4.3.12). It follows that each component of $\operatorname{int} S_{\mathcal{T}}$ is a Jordan domain.

By Theorem 4.3.12, if $a \in S_{\mathcal{T}} \setminus \operatorname{int} S_{\mathcal{T}}$, then a must lie on Γ^{hoc}. But the above argument demonstrates that in this case, there are points $a' \in \ell_a$ arbitrarily close to a such that $a' \in \operatorname{int} S_{\mathcal{T}}$. This proves that $S_{\mathcal{T}} \subset \overline{\operatorname{int} S_{\mathcal{T}}}$, and hence, $\overline{S_{\mathcal{T}}} = \overline{\operatorname{int} S_{\mathcal{T}}}$. □

Lemma 4.3.14. $\overline{\Gamma^{\mathrm{pb}}} \cap \overline{\Gamma^{\mathrm{dp}}}$ *contains at most two points.*

Proof. Let $a \in \overline{\Gamma^{\mathrm{pb}}} \cap \overline{\Gamma^{\mathrm{dp}}}$. Then the free critical value y_2 of σ_a lies on $\partial\Omega_a$, and thus is fixed. Moreover, by Proposition 4.3.7, y_2 lies on $\partial T_b^0(\sigma_a)$, but is not the (unique) double point or the (unique) cusp point of $\partial\Omega_a$.

Therefore, for each $a \in \overline{\Gamma^{\mathrm{pb}}} \cap \overline{\Gamma^{\mathrm{dp}}}$, the interior of $T_b^0(\sigma_a)$ is a Jordan domain whose boundary contains three distinct distinguished points; namely, the unique double point of $\partial\Omega_a$, the unique cusp of $\partial\Omega_a$, and the unique free critical value y_2 of σ_a. On the other hand, the interior of $T_u^0(\sigma_a)$ is also a Jordan domain which contains the fully branched critical value ∞ of σ_a and has the unique cusp of $\partial\Omega_a$ on its boundary.

Let us now assume that there are two parameters $a_1, a_2 \in \overline{\Gamma^{\mathrm{pb}}} \cap \overline{\Gamma^{\mathrm{dp}}}$ such that for both parameters, the unique double point of $\partial\Omega_{a_i}$, the unique cusp of $\partial\Omega_{a_i}$, and the unique free critical value y_2 of σ_{a_i} lie in the same cyclic order on $\partial T_b^0(\sigma_{a_i})$. We claim that $a_1 = a_2$. Since these three distinguished points on the boundary of $T_b^0(\sigma_a)$ can lie in exactly two different cyclic orders, the proof will be complete once the claim is established.

Let $\mathfrak{g}_b : \operatorname{int} T_b^0(\sigma_{a_1}) \to \operatorname{int} T_b^0(\sigma_{a_2})$ be the conformal isomorphism whose homeomorphic boundary extension (also denoted by $\mathfrak{g}_b$) carries the cusp, double point and the free critical value associated with a_1 to those associated with a_2. Furthermore, let $\mathfrak{g}_u : \operatorname{int} T_u^0(\sigma_{a_1}) \to \operatorname{int} T_u^0(\sigma_{a_2})$ be the conformal isomorphism which sends the fully branched critical value ∞ of σ_{a_1} to the fully branched critical value ∞ of σ_{a_1}, and whose homeomorphic boundary extension (also denoted by $\mathfrak{g}_b$) takes the unique double point of $\partial\Omega_{a_1}$ to the unique double point of $\partial\Omega_{a_2}$. Since y_2 is fixed under σ_{a_i} $(i \in \{1,2\})$, Lemmas 4.3.2 and 4.3.5 imply that the double points on $\partial\Omega_{a_1}$, $\partial\Omega_{a_2}$ are regular and the cusps on them are of type $(3,2)$. Hence, one can apply the arguments of [Lee+18b, Lemmas 8.10, 8.11] or [LMM21, Lemmas 5.3, 5.4] to conclude that there exists a global K-quasiconformal map $\mathfrak{G}_0$ of the Riemann sphere that continuously matches with $\mathfrak{g}_u$ and $\mathfrak{g}_b$ on their domains of definition.

Since the free critical orbits of $\sigma_{a_1}, \sigma_{a_2}$ lie in the respective tiling sets, one can apply classical arguments of Fatou to see that $\operatorname{int} K(\sigma_{a_1}) = \operatorname{int} K(\sigma_{a_2}) = \emptyset$ (see [Lee+18a, Propositions 5.30, 5.32]). Hence, $\partial T^\infty(\sigma_{a_i}) = K(\sigma_{a_i})$; i.e., $\overline{T^\infty(\sigma_{a_i})} = \widehat{\mathbb{C}}$. Moreover, the same fact also allows one to show that these non-escaping sets have zero area (cf. [Lee+18a, Corollary 6.3]).

As the Schwarz reflection maps act as identity on the boundaries of the desingularized droplets, a standard pullback argument as in [Lee+18b, Proposition 8.13] or [LMM21, Theorem 5.1] can be employed to construct a sequence of K-quasiconformal maps $\{\mathfrak{G}_n\}$ such that

1. $\sigma_{a_2} \circ \mathfrak{G}_n = \mathfrak{G}_{n-1} \circ \sigma_{a_1}$ on $\widehat{\mathbb{C}} \setminus \operatorname{int} T^0(\sigma_{a_1})$,
2. $\mathfrak{G}_n$ is conformal on $\displaystyle\bigcup_{i=0}^{n} \sigma_{a_1}^{-i}(T^0(\sigma_{a_1}))$, and
3. $\mathfrak{G}_n = \mathfrak{G}_{n-1}$ on $\displaystyle\bigcup_{i=0}^{n-1} \sigma_{a_1}^{-i}(T^0(\sigma_{a_1}))$.

By compactness of the family of K-quasiconformal homeomorphisms and Conditions (1), (2), there exists a quasiconformal homeomorphism $\mathfrak{G}_\infty$ of $\widehat{\mathbb{C}}$ that conjugates σ_{a_1} to σ_{a_2} on the tiling set. By continuity and density of the tiling sets of $\sigma_{a_1}, \sigma_{a_2}$ in $\widehat{\mathbb{C}}$, the conjugacy relation holds on the entire domain of definition of σ_{a_1}. Also, Condition 3 implies that $\mathfrak{G}_\infty$ is conformal on the tiling set of σ_{a_1}. Since the non-escaping set of σ_{a_1} has zero area, it follows by Weyl's lemma that $\mathfrak{G}_\infty$ is a Möbius map of $\widehat{\mathbb{C}}$. Finally, since the conjugacy $\mathfrak{G}_\infty$ fixes ∞, y_2 and y_1, it must be the identity map. Hence, $a_1 = a_2$. □

Lemma 4.3.15. $\Gamma^{\mathrm{hoc}} \subset \partial S_{\mathcal{T}}$ *is a closed real-analytic arc.*

Proof. We will first show that $\Gamma^{\mathrm{hoc}} \neq \emptyset$. To do this, we apply the arguments of A on the dynamically Shabat anti-polynomial p (as usual, if $\mathcal{T}$ is a star-tree, we perform the surgery on $\overline{z}^d$) to replace the dynamics on the basin of infinity with $\mathcal{R}_d|_{\mathcal{Q}}$ and the dynamics on the bounded fixed critical Fatou component with the unicritical parabolic anti-Blaschke product $B_d|_{\mathbb{D}}$ such that the unique parabolic fixed point of $\mathcal{R}_d$ as well as the unique parabolic fixed point of B_d correspond to the repelling fixed point $\mathrm{emb}(v_b)$ of p. This produces a parameter $a_0 \in S_{\mathcal{T}}$ such that $\sigma_{a_0}^{\circ 2}$ has a unique attracting direction in $K(\sigma_{a_0})$ at the cusp y_1 of $\partial\Omega_{a_0}$. Hence, by Lemma 4.3.2 and Corollary 4.3.3, y_1 is a $(5, 2)$ cusp of $\partial\Omega_{a_0}$; and hence in particular, $a_0 \in \Gamma^{\mathrm{hoc}}$. Note that $\mathcal{J}(p)$ is removable for $W^{1,1}$ functions (by [JS00, Theorem 4] and the fact that $\mathcal{B}_\infty(p)$ is a John domain). Thus, according to [Lyu+20, Theorem 2.7], the limit set $\partial K(\sigma_{a_0})$ is conformally removable.

The discussion in the beginning of Subsection 3.3.1, applied to a forward-invariant attracting petal of σ_{a_0} at the cusp point y_1, furnishes a Fatou coordinate (unique up to real translations) on such a petal that conjugates σ_{a_0} to the glide reflection $\zeta \mapsto \overline{\zeta} + \frac{1}{2}$ on a right half-plane. By construction of σ_{a_0}, such an attracting petal at y_1 contains the tail of the $\sigma_{a_0}^{\circ 2}$-orbit of the free critical value y_2 (of σ_{a_0}). We refer to the imaginary part of $\sigma_{a_0}^{\circ 2n}(y_2)$ (for n large enough) in this coordinate as the critical *Écalle height* of σ_{a_0}. It is readily seen that the critical Écalle height is a conformal conjugacy invariant of the map σ_{a_0}. As in Subsection 3.3.1, the real-symmetry of B_d tells us that the critical Écalle height of the map σ_{a_0} is 0.

One can now apply a quasiconformal deformation argument as in the proof of [Lee+21, Proposition 6.6 (part 2)] (cf. [MNS17, Lemma 3.1]) to obtain an open real-analytic arc

$\Gamma' \subset S_{\mathcal{T}}$ containing a_0 such that for each $t \in \mathbb{R}$, there exists a unique parameter $a(t) \in \Gamma'$ with the following properties:

1. the cusp y_1 has a unique attracting direction under $\sigma_{a(t)}$, and hence y_1 is a $(5, 2)$ cusp of $\partial\Omega_{a(t)}$,

2. the forward orbit of the free critical value y_2 of $\sigma_{a(t)}$ converges to y_1, and

3. the critical Écalle height of $\sigma_{a(t)}$ is t.

Clearly, Γ' is contained in Γ^{hoc}. Since conformal removability is preserved under quasiconformal maps, the limit set of each map produced above is conformally removable.

We claim that there are no other parameters in Γ^{hoc} such that y_1 is a $(5, 2)$-cusp on the corresponding quadrature domain boundary. Indeed, let $a' \in \Gamma^{\text{hoc}}$ be such that y_1 is a $(5, 2)$-cusp of $\partial\Omega_{a'}$. Then the unique free critical orbit of $\sigma_{a'}$ converges to y_1 and hence the map has a finite critical Écalle height t_0. We claim that $a' = a(t_0)$, where $a(t_0)$ is the unique parameter on Γ' with critical Écalle height t_0. Since both $\sigma_{a'}, \sigma_{a(t_0)}$ have a unique free critical orbit, one can adapt the arguments of [Lee+18a, Propositions 5.30, 5.32] to show that both $\operatorname{int} K(\sigma_{a'})$, $\operatorname{int} K(\sigma_{a(t_0)})$ equal the basin of attraction of the cusp y_1. Moreover, the proofs of [Lee+18b, Lemma 8.5, Proposition 8.6] (or more generally, that of [Kiw01, Proposition 6.19]) apply to the maps $\sigma_{a'}, \sigma_{a(t_0)}$ and imply that both $\partial T^\infty(\sigma_{a'}), \partial T^\infty(\sigma_{a(t_0)})$ are quotients of $\partial\mathcal{Q} \cong \mathbb{S}^1$ under the closed $\mathcal{R}_d$-invariant equivalence relation generated by the angles of the dynamical rays landing at the preimages of y_1. Note that the angles of these rays only depend on the plane tree $\mathcal{T}$, and hence the corresponding equivalence relation is the same for the two maps $\sigma_{a'}, \sigma_{a(t_0)}$. It now follows that the non-escaping set dynamics of $\sigma_{a'}$ and $\sigma_{a(t_0)}$ are topologically conjugate where the conjugacy is conformal on the interior (conformality is a consequence of the fact that the maps have the same critical Écalle height). Moreover by Lemma 4.1.10, their tiling set dynamics are also conformally conjugate. These two conjugacies match up to yield a global orientation-preserving topological conjugacy between the two Schwarz reflection maps such that the conjugacy is conformal off the limit set. Conformal removability of the limit set of $\sigma_{a(t_0)}$ now implies that $\sigma_{a'}$ and $\sigma_{a(t_0)}$ are Möbius conjugate and hence equal. (Alternatively, one can employ the pullback arguments of [Lee+18b, Proposition 9.4] to prove rigidity of parameters with $(5,2)$-cusps.)

Since Γ^{hoc} is contained in a real-algebraic curve (defined by the higher order cusp condition), each end of the arc Γ' has a unique limit point in Γ^{hoc}. We claim that the two ends of Γ' have distinct endpoints. If this were not true, since $\Gamma^{\mathrm{hoc}} \cap \overline{\Gamma^{\mathrm{pb}}} = \emptyset$ (by Lemma 4.3.4), the closure of Γ' (in $\mathbb{C}$) would be a topological circle contained in the open half plane $\mathbf{P} = \{z \in \mathbb{C} : |z - \boldsymbol{v_w}| < |z - \boldsymbol{v_b}|\}$. As $\Gamma^{\mathrm{hoc}} \subset \partial S_{\mathcal{T}}$, this topological circle must entirely lie on the boundary of $S_{\mathcal{T}}$. On the other hand, there must exist sub-arcs I', I'' of this topological circle such that the union of the line segments connecting points of I' to $\boldsymbol{v_b}$ contain I'' in its interior. The 'projection argument' of Lemma 4.3.13 now implies that I'' is contained in the interior of $S_{\mathcal{T}}$, a contradiction.

Let a' be a limit point of Γ'. We will now argue that $a' \in S_{\mathcal{T}}$, and that y_1 is a $(7,2)$-cusp of $\partial\Omega_{a'}$. Clearly, $\boldsymbol{f}$ is univalent on $\Delta_{a'}$, $\boldsymbol{v_w} \in \overline{\Delta_{a'}}$, and y_1 is a cusp of type $(\nu, 2)$ with $\nu > 3$ on $\partial\Omega_{a'}$, where $\Omega_{a'} := \boldsymbol{f}(\Delta_{a'})$. Thus, by the arguments of Lemma 4.3.2, the forward orbit of y_2 under the associated Schwarz reflection map $\sigma_{a'}$ converges non-trivially to the cusp y_1. It follows that $y_2 = \boldsymbol{f}(\boldsymbol{v_w}) \notin \partial\Omega_{a'}$; i.e., $\boldsymbol{v_w} \in \Delta_{a'}$. If there were a double point on $\partial\Omega_{a'}$, then Proposition 4.3.7 would prevent the critical value y_2 of $\sigma_{a'}$ to converge non-trivially to y_1. Therefore, $\boldsymbol{f}$ is injective on $\overline{\Delta_{a'}}$, and hence $a' \in S_{\mathcal{T}}$. Thanks to Corollary 4.3.3, it now suffices to show that y_1 is not a $(5,2)$-cusp on $\partial\Omega_{a'}$. We have already established that the parameters in $S_{\mathcal{T}}$ for which y_1 is a $(5,2)$ cusp on the corresponding quadrature domain boundaries comprise Γ'. Thus, we need to prove that Γ' does not accumulate on itself. But this follows directly from the fact that $\ell_a \cap \mathbf{P} \setminus \{a\} \subset \operatorname{int} S_{\mathcal{T}}$ for all $a \in \Gamma^{\mathrm{hoc}}$ (see the proof of Lemma 4.3.13). Hence, y_1 is a $(7,2)$-cusp of $\partial\Omega_{a'}$.

As in the case of $(5,2)$-cusps, one can now apply the arguments of [Lee+18b, Proposition 8.15] to prove a rigidity statement for parameters with $(7,2)$-cusps and conclude that there are no other parameters in Γ^{hoc} such that y_1 is a $(7,2)$-cusp of $\partial\Omega_a$.

By Corollary 4.3.3, these are all the parameters in Γ^{hoc}. Therefore, $\Gamma^{\mathrm{hoc}} = \overline{\Gamma'}$ is a closed real-analytic arc. □

Corollary 4.3.16. *For all $a \in \Gamma^{\mathrm{hoc}}$, the free critical orbit $\{\sigma_a^{\circ n}(y_2)\}$ converges to the cusp y_1.*

Remark 4.3.17. The existence of two distinct parameters a for which y_1 is a $(7,2)$-cusp of $\partial\Omega_a$ can be interpreted as follows. For such a parameter, the interior of $K(\sigma_a)$ is the basin of attraction of y_1, and every component of $\mathrm{int}\, K(\sigma_a)$ is mapped eventually to the 2-cycle of Fatou components touching at y_1 (which correspond to the two attracting directions of $\sigma_a^{\circ 2}$ at y_1). Moreover, the first return map $\sigma_a^{\circ 2}$ to each of these two components is a unicritical holomorphic Blaschke product of degree equal to the valence of v'_w in $\mathcal{T}$. Since such Blaschke products are rigid and since the external class of σ_a is frozen, it follows that all parameters with a $(7,2)$-cusp at y_1 have conformally conjugate dynamics on the union of the tiling set and the interior of the non-escaping set. However, the internal and external conjugacies between two such maps would agree if and only if the circular order of the two periodic Fatou components (which are marked as one of them contains $\boldsymbol{v'_w}$) and the droplet at y_1 are the same. This implies that there are at most two maps σ_a for which y_1 is a $(7,2)$-cusp of $\partial\Omega_a$, and the proof of Lemma 4.3.15 confirms that both possibilities are realized.

Theorem 4.3.18.

1. $\mathrm{int}\, S_{\mathcal{T}}$ *is a bounded Jordan domain.*

2. $\overline{S_{\mathcal{T}}}$ *is a topological quadrilateral whose sides are given by* Γ^{hoc}, $\overline{\Gamma^{\mathrm{pb}}}$, *and the two connected components of* $\overline{\Gamma^{\mathrm{dp}}}$.

Proof. 1) By Lemma 4.3.13, it suffices to show that $\mathrm{int}\, S_{\mathcal{T}}$ is connected.

Let us first note that by the projection argument of Lemma 4.3.13, the boundary of each component of $\mathrm{int}\, S_{\mathcal{T}}$ contains a non-degenerate interval in $\overline{\Gamma^{\mathrm{pb}}}$. As $\Gamma^{\mathrm{hoc}} \cap \overline{\Gamma^{\mathrm{pb}}} = \emptyset$, we conclude that the boundary of each component of $\mathrm{int}\, S_{\mathcal{T}}$ must contain at least two points of $\overline{\Gamma^{\mathrm{pb}}} \cap \overline{\Gamma^{\mathrm{dp}}}$. It follows that if $\mathrm{int}\, S_{\mathcal{T}}$ had two distinct components, then $\overline{\Gamma^{\mathrm{pb}}} \cap \overline{\Gamma^{\mathrm{dp}}}$ would contain at least three distinct points, which would contradict Lemma 4.3.14. Hence, $\operatorname{int} S_{\mathcal{T}}$ is connected.

2) By Lemma 4.3.13, $\overline{S_{\mathcal{T}}} = \overline{\operatorname{int} S_{\mathcal{T}}}$. By the proof of the first part of the theorem, $\partial S_{\mathcal{T}}$ is a Jordan curve in $\mathbb{C}$ consisting of a closed real-analytic arc Γ^{hoc}, a closed interval $\overline{\Gamma^{\mathrm{pb}}}$, and a pair of closed real-analytic arcs $\overline{\Gamma^{\mathrm{dp}}}$ that connect the two endpoints of Γ^{hoc} to the two endpoints of $\overline{\Gamma^{\mathrm{pb}}}$. This yields the desired topological quadrilateral structure of $\overline{S_{\mathcal{T}}}$. □

4.4 Hyperbolic components in $S_{\mathcal{T}}$

We say that a parameter a is *hyperbolic* if σ_a has an attracting periodic cycle[1]. Let U_a be the union of the connected Fatou components which contain the attracting periodic cycle. Necessarily there is a critical point c contained in U_a [Mil06]. The following is then a consequence of Crollary 4.2.

Proposition 4.4.1. *There are no critical or co-critical points in U_a other than c_a.*

Proof. Since U_a is contained in the interiror of Ω_a, it follows that the cusp does not belong to U_a and therefore neither does any preimage. If $\tilde{c} \in \sigma^{-1}$ must necessarily be contained in the same component of U_a as c, since they are both mapped to v. At the same time no connected component of $\sigma_a^{-1}(\Omega_a)$ contains two different vertices of $\mathcal{T}$. □

Definition 4.4.1. We define the *primary* hyperbolic component $\mathcal{H}$ to be the hyperbolic component of $\mathcal{C}(S_{\mathcal{T}})$ such that critical point in the periodic Fatou component is w_1, the critical point adjacent in $\mathcal{T}$ to the pararabolic point.

Proposition 4.4.2. $\Gamma^{\mathrm{hoc}} \subset \overline{\mathcal{H}} \subset \mathcal{C}(\mathcal{S}_{\mathcal{T}})$. *Moreover, any sequence in $S_{\mathcal{T}}$ accumulating on the interior of* Γ^{hoc} *is eventually contained in $\mathcal{H}$.*

Proof. Follows from the proof of Lemma 4.3.15. □

[1] Strictly speaking, we should call such parameters *relatively* hyperbolic, as there is always a fixed point with parabolic dynamics. However, as this behavior is persistent throughout our parameter space, we call these maps hyperbolic for the sake of brevity.

4.4.1 Uniformizing hyperbolic parameters

For a hyperbolic map σ_a the free critical value has a bounded orbit, and thus the Julia set is connected. The periodic Fatou component is then topologically equivalent to $\mathbb{D}$. The Riemann mapping theorem produces a map $\varphi\colon \mathbb{D} \to U$, and the first return map $\sigma_a^p|_U$ is a proper (anti-)holomorphic map, and hence $\varphi^{-1} \circ f^p \circ \varphi\colon \mathbb{D} \to \mathbb{D}$ is a Blaschke product.

The holomorphic and anti-holomorphic unicritical Blaschke products as

$$B^+_{a,\lambda,d}(z) = \lambda \left(\frac{z-a}{\bar{a}z-1}\right)^d \text{ and } B^-_{a,\lambda,d}(z) = \overline{\lambda \left(\frac{z-a}{\bar{a}z-1}\right)^d},$$

where $|\lambda| = 1$ and $a \in \mathbb{D}$. Such maps, if normalized to fix 1, are unique up to $d-1$ choices in the holomorphic case, and $d+1$ maps in the anti-holomorphic setting. This argument, as seen in [NS03][Lemma 3.2] gives rise to the following proposition.

Proposition 4.4.3. *Let $\sigma_a \in S$ be hyperbolic, let U be a periodic Fatou component, and suppose that the critical point of σ_a contained in the periodic Fatou cycle has degree d. Then there are $a \in \mathbb{D}, \lambda \in S^1$ such that the first return map $\sigma_a^p|_U$ is conformally conjugate either to $B^+_{a,\lambda,d}$ if p is even or $B^-_{a,\lambda,d}$ if p is odd.*

Normalizing in such a way that 1 is a fixed point of the conjugated map, there are $d-1$ choices of (a, λ) in the holomorphic case, and $d+1$ choices of (a, λ) in the anti-holomorphic case.

Remark 4.4.4. As noted in [NS03], not every choice of (a, λ) will give rise to a fixed point in the interior of $\mathbb{D}$.

We will show a parameter version of this statement. Let $\mathcal{B}^\pm_d$ be the spaces of maps $B^\pm_{a,\lambda,d}$ such that there exists an attracting fixed point in $\mathbb{D}$.

For an anti-holomorphic map f with an attracting fixed point (say at 0), [NS03] showed that there exists a local conformal coordinate $\varphi\colon U \to \mathbb{D}$ which linearizes f, in the sense that $\varphi \circ f = \bar{\partial} f(\zeta) \cdot \overline{\varphi}$.

For a hyperbolic map $\sigma_a \in S$, let w_j be the critical point in the periodic Fatou coordinate, and let ζ be the periodic point that lies in the same Fatou coordinate as w_j. The first return map σ_a^p has the local linearizing coordinate above, which can be extended to a closed neighborhood $\overline{U} \ni \zeta$ in such a way that $w_j \in \partial\overline{U}$. The linearizing coordinate is unique if we choose $\varphi(w_j) = 1$.

Definition 4.4.2. The critical value map for σ_a is defined to be $\sigma_a \mapsto \varphi(\sigma_a^p(w_j))$.

Theorem 4.4.5. *(see also [NS03] theorems 5.6, 5.9). Every even period hyperbolic component whose critical point in the periodic Fatou coordinate has degree d is homeomorphic to $\mathcal{B}_d^+$, and the homeomorphism respects the multiplier of the attracting fixed point. Likewise, every odd period hyperbolic component whose critical point in the periodic Fatou coordinate has degree d is homeomorphic to $\mathcal{B}_d^-$, and the homeomorphism respects the critical value map.*

4.4.2 Boundaries of hyperbolic components

The unicriticality of our hyperbolic components allows many statments about the boundaries of hyperbolic components in the Multicorn family to carry over to our setting as well. The following proposition however is specific to our setting.

We say that hyperbolic components of $\mathcal{C}(\mathcal{S}_\mathcal{T})$ are adjacent if their closures are not disjoint.

Proposition 4.4.6. *Let H_1 and H_2 be two adjacent hyperbolic components. The ass in $\mathcal{T}$.*

Proof. Suppose to the contrary that there were two adjacent hyperbolic components whose periodic Fatou components contained critical points with different addresses in $\mathcal{T}$, and denote these addresses by w and w'. Argue that when moving parameters from σ_a to $\sigma_{a'}$ that v must pass through a point which eventually lands at the cusp. But such Misiurewicz parameters cannot be on the boundary of hyperbolic components. □

The above proposition allows us to consider bifurcations of hyperbolic parameters analgously to the standard unicritical case.

We will say that a parameter is *parabolic* if σ_a has a periodic cycle other than root whose multiplier is a root of unity.

Proposition 4.4.7 (Netural parameters on boundary of hyperbolic components)**.** *If σ_a has a neutral periodic point of period k, then every neighborhood of a in S contains parameters with attracting periodic points of period k, so the parameter a is on the boundary of a hyperbolic component of period k of $\mathcal{C}(\mathcal{S}_{\mathcal{T}})$.*

Proof. See [MNS17, Theorem 2.1] for a proof in the multicorn family. Since the proof given there only uses local dynamical properties of anti-holomorphic maps near neutral periodic points, it applies to the family $\mathcal{S}_{\mathcal{T}}$ as well. □

Another result which carries over from the Multicorn setting ([MNS17, Theorem 1.1]) is the following.

Proposition 4.4.8. *If σ_a has a 2k-periodic cycle with multiplier $e^{2\pi p/q}$ with $\gcd(p,q)=1$, then the parameter a sits on the boundary of a hyperbolic component of period $2kq$ (and is the root thereof) of $\mathcal{C}(\mathcal{S}_{\mathcal{T}})$.*

Proposition 4.4.9. *1) The boundary of a hyperbolic component of odd period k of $\mathcal{C}(\mathcal{S}_{\mathcal{T}})$ which is not the primary hyperbolic component is contained in the interior of $S_{\mathcal{T}}$, and consists entirely of parameters having a parabolic orbit of exact period k. In suitable local conformal coordinates, the $2k$-th iterate of such a map has the form $z \mapsto z + z^{q+1} + \cdots$ with $q \in \{1,2\}$. 2) Every parameter on the boundary of the primary hyperbolic component $\mathcal{H}$ is either contained in Γ^{hoc} or has a parabolic fixed point (with local power series as above).*

Proof. 1) The boundary of any hyperbolic component H will either intersect Γ^{hoc} or be contained in the interior of $\mathcal{C}(\mathcal{S}_{\mathcal{T}})$. By 4.4.2 it follows that for a non-primary hyperbolic component that its boundary is contained in the interior of $\mathcal{C}(\mathcal{S}_{\mathcal{T}})$. Now applying [MNS17, Lemma 2.5] combined with the fact that the Schwarz reflection maps under consideration have unique free critical values also show that for every parameter on the boundary of H, the

k-cycle to which the critical orbit converges must be parabolic with the desired local Taylor series expansion. 2) The proof is analogous to part 1), accounting for the fact that $\Gamma^{\mathrm{hoc}} \subset \overline{\mathcal{H}}$ by 4.4.2. □

Definition 4.4.3. Let a be a parameter with a parabolic periodic point of odd period. If for the above proposition $q = 1$ we say that it is a *simple* parabolic parameter. If $q = 2$, we say that it is a *parabolic cusp.*

Proposition 4.4.10. *Let $\tilde{a}$ be a simple parabolic parameter of odd period. Then $\tilde{a}$ is on a parabolic arc in the following sense: there exists a real-analytic arc of simple parabolic parameters $a(h)$ (for $h \in \mathbb{R}$) with quasiconformally equivalent but conformally distinct dynamics of which $\tilde{a}$ is an interior point, and the Écalle height of the free critical value of $\sigma_{a(h)}$ is h. This arc is called a parabolic arc.*

Proposition 4.4.11. *Every parabolic arc of odd period $k > 1$ intersects the boundary of a hyperbolic component of period $2k$ along an arc consisting of the set of parameters where the parabolic fixed point index is at least 1. In particular, every parabolic arc has, at both ends, an interval of positive length at which bifurcation from a hyperbolic component of odd period $k$to a hyperbolic component of period $2k$ occurs.*

For a critical point of degree d, there are $d + 1$ distinct combinatorial ways for parabolic arcs to occur. This allows us to conclude the following (see [Lee+18b]

Proposition 4.4.12. *1) The boundary of every non-primary hyperbolic component of odd period k of $\mathcal{C}(\mathcal{S}_{\mathcal{T}})$ is a topological polygon with $d + 1$ sides having parabolic cusps as vertices and parabolic arcs as sides. Here d is the degree of the critical point contained in the periodic Fatou coordinate. 2) The boundary of the primary hyperbolic component consists of d parabolic arcs, $d - 1$ parabolic cusps, and Γ^{hoc}.*

4.5 Tessellation of the escape locus of $\mathcal{S}_{\mathcal{T}}$

Theorem 4.5.1 (Uniformization of The Escape Locus). *The map*

$$\mathbf{\Psi}: S_{\mathcal{T}} \setminus \mathcal{C}(\mathcal{S}_{\mathcal{T}}) \to \mathbb{D}_1,$$

$$a \mapsto \psi_a(y_2)$$

is a homeomorphism.

Proof. The proof is analogous to that of [Lee+18b, Theorem 1.3]. We only indicate the key differences.

Note that for all $a \in S_{\mathcal{T}}$, the critical value y_2 of σ_a lies in Ω_a; i.e., $y_2 \notin T^0(\sigma_a)$. It now follows from the definition of ψ_a that $\psi_a(y_2) \in \mathbb{D}_1$ for each $a \in S_{\mathcal{T}} \setminus \mathcal{C}(\mathcal{S}_{\mathcal{T}})$.

The map $\mathbf{\Psi}$ is easily seen to be continuous. We will show that $\mathbf{\Psi}$ is proper, and locally invertible. This will imply that $\mathbf{\Psi}$ is a covering map from $S_{\mathcal{T}} \setminus \mathcal{C}(\mathcal{S}_{\mathcal{T}})$ onto the simply connected domain $\mathbb{D}_1$, and hence a homeomorphism from each connected component of $S_{\mathcal{T}} \setminus \mathcal{C}(\mathcal{S}_{\mathcal{T}})$ onto $\mathbb{D}_1$. However, $a_0 = \boldsymbol{v_w}$ is the only possible parameter in $S_{\mathcal{T}} \setminus \mathcal{C}(\mathcal{S}_{\mathcal{T}})$ satisfying $\mathbf{\Psi}(a_0) = \rho_1(0)$. So, $S_{\mathcal{T}} \setminus \mathcal{C}(\mathcal{S}_{\mathcal{T}})$ must be connected; i.e., $\mathbf{\Psi}$ is a homeomorphism.

Local invertibility follows from a quasiconformal deformation/surgery argument as in [Lee+18b, Theorem 1.3].

We need to consider several cases to show that $\mathbf{\Psi}$ is proper. Let us first assume that $\{a_k\}_k$ is a sequence in $S_{\mathcal{T}} \setminus \mathcal{C}(\mathcal{S}_{\mathcal{T}})$ such that $a_k \to \overline{\Gamma^{\mathrm{pb}}}$. It follows from the definition of Γ^{pb} that the spherical distance between the co-critical point $\boldsymbol{v_w}$ of $\boldsymbol{f}$ and the circle $\partial\Delta_a$ tends to zero as $k \to \infty$. Hence, $d_{\mathrm{sph}}(y_2, \partial\Omega_{a_k})$ tends to 0 as $k \to \infty$. Therefore, $\mathbf{\Psi}(a_k) = \psi_{a_k}(y_2)$ accumulates on $C_1 \subset \partial\mathbb{D}_1$.

Now suppose that $\{a_k\}_k \subset S_{\mathcal{T}} \setminus \mathcal{C}(\mathcal{S}_{\mathcal{T}})$ is a sequence with $\{a_k\}_k \to a \in \Gamma^{\mathrm{dp}}$. It then follows from Proposition 4.3.7 that the free critical value y_2 of σ_a lands in the bounded component $T_b^0(\sigma_a)$ of the corresponding desingularized droplet $T^0(\sigma_a)$ under $\sigma_a^{\circ n}$, for some $n \equiv n(a) \geq 1$. Note that for k sufficiently large, σ_{a_k} is a small perturbation of σ_a. We set

$$U_k := \mathrm{int}\,(T^0(\sigma_{a_k}) \cup \sigma_{a_k}^{-1}(T^0(\sigma_{a_k}))).$$

Then, for k large enough, the critical value y_2 of σ_{a_k} lands in $T^0(\sigma_{a_k})$ under $\sigma_{a_k}^{\circ n'}$, where $n' \in \{n, n+1\}$. Moreover, the hyperbolic geodesic in U_k connecting $\sigma_{a_k}^{\circ n'}(y_2)$ and ∞ passes through an extremely narrow channel formed by the splitting of the double point on $\partial T^0(\sigma_a)$ such that the thickness of this channel decreases as $k \to \infty$, and gets pinched in the limit. Since the Euclidean distance between this part of the geodesic and the boundary of U_k tends to zero as $k \to \infty$, the hyperbolic distance between $\sigma_{a_k}^{\circ n'}(y_2)$ and ∞ (in U_k) tends to ∞ as $k \to \infty$. Furthermore, as ψ_{a_k} is a conformal isomorphism from U_k onto $\mathcal{Q}_1 \cup \mathcal{R}_d^{-1}(\mathcal{Q}_1)$, we have that the hyperbolic distance between $\psi_{a_k}(\sigma_{a_k}^{\circ n'}(y_2))$ and 0 (in $\mathcal{Q}_1 \cup \mathcal{R}_d^{-1}(\mathcal{Q}_1)$) tends to ∞ as k increases. Consequently, $\{\psi_{a_k}(\sigma_{a_k}^{\circ n'}(y_2))\}_k$ escapes to the boundary of $\mathcal{Q}_1 \cup \mathcal{R}_d^{-1}(\mathcal{Q}_1)$ as $k \to \infty$. But the sequence $\{\psi_{a_k}(\sigma_{a_k}^{\circ n'}(y_2))\}_k$ is contained in $\mathcal{Q}_1$, and hence, $\{\psi_{a_k}(\sigma_{a_k}^{\circ n'}(y_2))\}_k$ must converge to $1 \in \partial\mathcal{Q}_1$. In fact, the dynamical properties of σ_{a_k} and the geometry of $T^0(\sigma_{a_k})$ now imply that for k sufficiently large, each $\psi_{a_k}(\sigma_{a_k}^{\circ j}(y_2))$ $(0 \leq j \leq n')$ is close to $1 \in \partial\mathbb{D}_1$, and hence $\mathbf{\Psi}(a_k) = \psi_{a_k}(y_2)$ converges to $1 \in \partial\mathbb{D}_1$ as $k \to \infty$.

Finally let $\{a_k\}_k \subset S_{\mathcal{T}} \setminus \mathcal{C}(\mathcal{S}_{\mathcal{T}})$ be a sequence accumulating on $\mathcal{C}(\mathcal{S}_{\mathcal{T}})$. Suppose that $\{\mathbf{\Psi}(a_k)\}_k$ converges to some $u \in \mathbb{D}_1$. Then, $\{\psi_{a_k}(y_2)\}_k$ is contained in a compact subset $\mathcal{X}$ of $\mathbb{D}_1$. After passing to a subsequence, we can assume that $\mathcal{X}$ is contained in a single tile of $\mathbb{D}_1$. But this implies that each a_k has a common depth n_0 (see Definition 4.1.6), and $\psi_{a_k}(\sigma_{a_k}^{\circ n_0}(y_2))$ is contained in the compact set $\mathcal{R}_d^{\circ n_0}(\mathcal{X}) \subset \mathcal{Q}_1$ for each k. Note that the map $\sigma_{a'}$, the fundamental tile $T^0(\sigma_{a'})$ as well as (the continuous extension of) the conformal isomorphism $\psi_{a'}^{-1} : \mathcal{Q}_1 \to T^0(\sigma_{a'})$ change continuously with the parameter as a' runs over $S_{\mathcal{T}}$. Therefore, for every accumulation point a of $\{a_k\}_k$, the point $\sigma_a^{\circ n_0}(y_2)$ belongs to the compact set $\psi_a^{-1}(\mathcal{R}_d^{\circ n_0}(\mathcal{X}))$. In particular, the critical value y_2 of σ_a lies in the tiling set $T^\infty(\sigma_a)$. This contradicts the assumption that $\{a_k\}_k$ accumulates on $\mathcal{C}(\mathcal{S}_{\mathcal{T}})$, and proves that $\{\mathbf{\Psi}(a_k)\}_k$ must accumulate on the boundary of $\mathbb{D}_1$. □

Definition 4.5.1 (Parameter Rays of $S_{\mathcal{T}}$)**.** The pre-image of a $\mathcal{G}_d$-ray at angle $\theta \in [0, \frac{1}{d+1})$ in $\mathbb{D}_1$ under the map $\mathbf{\Psi}$ is called a θ-parameter ray of $\mathcal{S}_{\mathcal{T}}$.

Proposition 4.5.2. *Parameters rays at pre-periodic angles land at critically pre-periodic* *parameters. Furthermore, the corresponding dynamical rays land at the free critical value for this landing parameter. (cf. [Lee+21] lemma 8.14)*

Proof. Let $\mathfrak{R}^\theta$ be a (strictly) pre-periodic parameter ray, let a_0 be an accumulation point of this parameter ray, and let $\mathcal{R}_a^\theta$ be the dynamical ray of the corresponding angle for a Schwarz reflection σ_a. Also recall, that by definition, there exist a close to a_0 for which $\mathcal{R}_a^\theta$ contains the free critical value of $\sigma_{a'}$.

Note that $\mathcal{R}_{a_0}^\theta$ is strictly pre-periodic and thus lands at a pre-periodic point, which we denote by z_0. The point z_0 has a forward orbit which is either a repelling or a parabolic periodic point.

First, we rule out the possibility that z_0 eventually hits a parabolic periodic point, with the possible exception of the Schwarz reflection cusp. To see this, suppose the opposite for the sake of contradiction. Note that z_0 is not contained in the closure of the free critical orbit, and so we may find two repelling periodic points near to z_0 and an arc contained in the filled Julia set $K(\sigma_a)$ which connects these two points and separates z_0 from the free critical value. Such curves persist under peturbations, contradicting the fact that $\mathcal{R}_{a'}^\theta$ lands at the free critical value for nearby parameters a'.

This means that the orbit of z_0 is eventually a repelling periodic point. If z_0 is neither pre-critical nor equal to the free critical value, then the ray $\mathcal{R}_a^\theta$ will still land at a periodic point for all a near enough to a_0, giving rise to a contradiction.

This means that z_0 is pre-critical, and hence σ_{a_0} is critically pre-periodic and hence locally connected. Then the same cut-line argument as above implies that if z_0 is not equal to the free critical value that we may separate the two for all nearby parameters, giving rise to a contradiction. Hence z_0 is the free critical value.

Lastly we show that a_0 is the unique accumulation point of $\mathfrak{R}^\theta$. Any accumulation point must be critically pre-periodic parameter with preperiod defined by θ. Specifically, there exist n, p which are minimal such that $\sigma_{a_0}^n(v) = \sigma_{a_0}^{n+p}(v)$. At most finitely many parameters satisfy this property for the given n and p, since there are at most finitely many Hubbard trees which do. As the accumulation set of a ray must be connected, and in particular the accumulation set is either a single point or infinitely many, it follows that a_0 is the only accumulation point. $\square$

4.6 Puzzles, renormalizability, and combinatorial rigidity

The purpose of this section is to show twofold: We show that the connectedness locus $\mathcal{C}_\mathcal{T}$ contains many combinatorial copies of multibrots and multicorns, which correspond to the the connectedness loci of families of (anti-)polynomial-like mappings. We also show that the parameters which do not lie inside of these little multibrot and multicorns are combinatorially rigid.

We say that a parameter $\sigma_a \in \mathcal{S}_\mathcal{T}$ is (pinched-)renormalizable if there is some domain V containing the critical value and some component U of $\sigma_a^{-p}(V)$ with $v \in U \subset V$, such that $\sigma^p \colon U \to V$ is a unicritical (pinched) polynomial-like map of degree $d \geq 2$, with connected non-escaping set.

Theorem 4.6.1. *Suppose that a map $\sigma_a \in \mathcal{C}(\mathcal{S}_\mathcal{T})$ is non-renormalizable. Then it has locally connected Julia set.*

Theorem 4.6.2. *A non-renormalizable map is combinatorially rigid.*

4.6.0.1 Period 1 renormalizations

Before describing the puzzle structure associated with our Schwarz reflection maps, let us first mention a case with particularly simple renormalization combinatorics.

For a fixed map $\sigma_a \in \mathcal{C}(\mathcal{S}_\mathcal{T})$, denote by $\Omega_1, \Omega_2, \cdots, \Omega_k$ the preimages of Ω under σ_a. As a convention we will always choose Ω_1 to be the component which contains the root on its boundary.

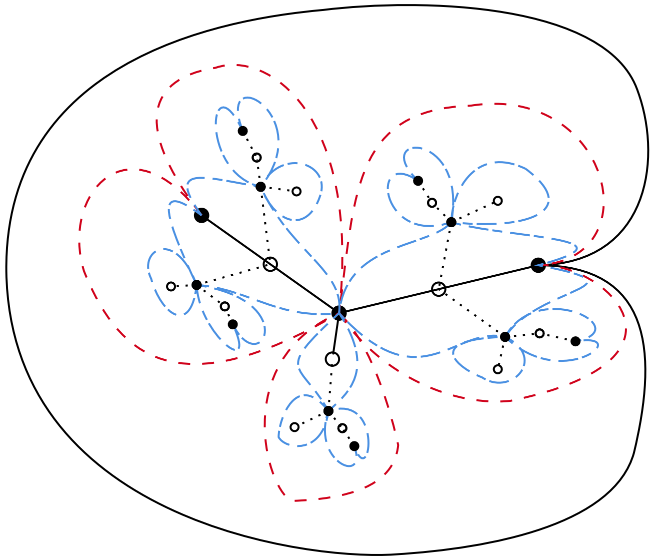

Figure 4.7: The depth 0, 1, and 2 puzzle pieces of a Schwarz reflection for a given combinatorics

Now suppose that σ_a is such that the critical orbit $\sigma^n(v)$ lies within a single component Ω_j, which contains a critical point. If $j \neq 1$ then $\sigma_a|_{\Omega_j} \colon \Omega_j \to \Omega$ is an anti-polynomial like map, with connected non-escaping set. For the case $j = 1$ there are two possible cases. Either $a \in \Gamma^{\mathrm{hoc}}$, or there is a restriction of σ_a to a subset of $\overline{\Omega_1}$ which is a simple pinched-anti-polynomial-like mapping.

We will refer to the special case in which we have a period 1 renormalization and such that $\sigma_a^n(v) \in \Omega$ as the map σ_a being *pinched* renormalizable.

Theorem 4.6.3. *For a renormalizable parameter a which is not pinched renormalizable there is a compact set $\mathcal{M}_a \subset \mathcal{C}(\mathcal{S}_{\mathcal{T}})$ which contains a and is homeomorphic to the closure of the hyperbolic components of a multibrot set or a multicorn set.*

Theorem 4.6.4. *The pinched renormalizable parameters form a set combinatorially equivalent to a parabolic multicorn.*

4.6.1 Puzzle pieces

Here we describe a "puzzle piece" structure associated to Schwarz reflections in our family.

Denote by $\Omega(a)$ the quadrature domain associated with $\sigma_a \in \mathcal{S}_{\mathcal{T}}$, and let $\Omega_j(a)$ denote the preimages of Ω under σ. We denote the preimage with boundary intersecting the cusp as Ω_1. We let $c_j \in \Omega_j$ denote the unique preimage of v, the free critical value.

Throughout this section we will assume that the free critical orbit, $\mathrm{orb}(v)$ never lands on the cusp of Ω. In such cases the associated map will be postcritically finite, and the results of this section can be obtained from standard arguments.

Definition 4.6.1. For a map $\sigma \in \mathcal{C}(\mathcal{S}_{\mathcal{T}})$, the *puzzle pieces of depth* $k \geq 0$ are the components of $\sigma^{-k}(\Omega)$.

These are a collection of disjoint topological such that the boundaries of any two intersect in at most one point. For a Jordan disk V, let $\mathrm{Dom}R_V = \{z \in V \mid \sigma^k(z) \in V \text{ for some } k > 0\}$. We define $R_V \colon \mathrm{Dom}R_V \to V$ to be the first return map under σ. Note that for a puzzle piece V, components of $\mathrm{Dom}R_V$ are themselves puzzle pieces.

The situation we have above does not immediately fall into the above setting, as puzzle pieces all contain non-compactly contained deeper puzzle pieces. We deal with this as follows: First, suppose that there is some $k \geq 1$ such that $\sigma^k(c_j) \in \Omega_i \neq \Omega_1$; otherwise we are in a renormalizable situation by considering the map restriction $\sigma \colon \Omega_1 \to \Omega$. We will deal with this situation later.

Proposition 4.6.5. *If σ is combinatorially recurrent there is a domain V which is the union of puzzle pieces such that the first return map to V is a dynamically natural complex box mapping, containing the postcritical set of σ.*

Proof. Let $V_j = \sigma^{-(k+1)}(\Omega_i)|c_j$ be the depth $k+2$ critical puzzle pieces. As $\Omega_i \subset \Omega$ is a compact containment, it follows that the puzzle pieces V_j are all compactly contained in puzzle pieces of one lower depth. Let D be any component of $\mathrm{Dom}R_{V_j}$, and let $D \subset D' \subset V_j$, where D' is the puzzle piece containing D of one lower depth. If $R_{V_j}|_D \equiv \sigma^t$, then as $\sigma^t(D) = V_j \subset \sigma^t(D')$, it follows that D is compactly contained in D', and hence in all puzzle pieces of less depth. In particular, $D \subset V_j$ is a compact containment.

□

4.6.2 Combinatorics of Puzzles

For a puzzle piece V_j^k of depth k, let $c_j^k \in V_j^k$ be the point so that $\sigma^k(c_j^k) = v$. Let $b = b^0$ denote the conformal cusp of Ω, and denote by b_j^k the elements of $\sigma^{-k}(b)$.

Using arcs contained in V_j^k, connect each c_j^k to any points in $\sigma^{-k}(b) \cap \partial V_j^k$. This clearly gives rise to a bi-colored planar embedded graph. We will let $\mathcal{T}^k$ denote the abstract graph isomorphic to this one (retaining the planar embedding data), and refer to it as the *depth k dessin.* Note that the depth 1 dessin is $\mathcal{T}$, the original dessin generating our family.

Alternatively, we may say that $\mathcal{T}^k$ is the graph isotopic to $\bigcup \overline{V_j^k}$ where the isotopy is taken rel $\sigma^{-k}(b)$.

We have two maps from $\mathcal{T}^k \to \mathcal{T}^{k-1}$. One is a branched covering map induced by σ; we map $c_j^k \to c_\ell^{k-1}$ if $\sigma(V_j^k) = V_\ell^{k-1}$ and map b_j^k to $\sigma(b_j^k)$.

The other map is the forgetful one, induced by the inclusion of puzzle into ones of less depth.

Proposition 4.6.6. *The graphs $\mathcal{T}^k$ are trees.*

Proof. We will prove this inductively. It is clearly true for $\mathcal{T}^1$, which is the generating dessin for the family. Now suppose that $\mathcal{T}^{k-1}$ is a tree. If there is a cycle in $\mathcal{T}^k$, then it must lie inside of a single puzzle piece of depth $k-1$, which we will denote by P. The interior component of this cycle gives rise to a topological disk $D \subset P \setminus \bigcup \overline{V_j^k}$, where the union is taken over all depth k puzzle pieces contained in P. The boundary of D is given by a cycle of depth k puzzle pieces, and so the boundary of $\sigma(D)$ is given by a cycle of depth $k-1$ puzzle pieces, a contradiction. □

Definition 4.6.2. We define the *depth k kneading sequence* $\{\vartheta_n^k\}$ to be the address of the critical orbit $\sigma^n(v)$ in the depth k puzzles.

Proposition 4.6.7. *The depth k dessin and the depth k kneading sequence uniquely determine the depth $k+1$ dessin. That is, if $\vartheta_j^k(a) = \vartheta_j^k(a')$, then $\mathcal{T}^{k+1}(a) = \mathcal{T}^{k+1}(a')$.*

A puzzle piece is called *critical* if it contains a critical point c_j. Suppose that there is some critical puzzle piece $V \ni c$, and a component U of the first return map R_V to V which also contains c. If $\sigma^p \colon U \to V$ is the first return map (necessarily a unicritical branched cover), and $\sigma^{np}(c) \in U$ for all n, then we say that our map is *renormalizable with period p.*

Note that for all depths k, the kneading sequence ϑ_j^k is periodic with period p for a renormalizable map.

4.6.3 Renormalization combinatorics have associated Multibrots or Multicorns

We will use this section to prove

Proposition 4.6.8. *For a given renormalization combinatorics of period p, there exists a unique critically periodic parameter with the same combinatorics.*

Proof. We will use the depth k dessins to produce a Hubbard tree for this map. More specifically, let V be a depth k critical puzzle piece, let $U \ni c_j$ be the component of the first return map containing c_j, and suppose that $R_V|_U \equiv \sigma^p$ with $\sigma^{np}(c_j) \in U$ for all $n \geq 1$.

As described in the previous subsection there are maps $\sigma \colon \mathcal{T}^{k+\ell} \to \mathcal{T}^{k+\ell-1}$ and $\iota \colon \mathcal{T}^k \to \mathcal{T}^{k+p}$. Then $\iota \circ \sigma^k \colon \mathcal{T}^{k+p} \to \mathcal{T}^{k+p}$, which can then be realized as the Hubbard tree of an anti-holomorphic polynomial. □

4.6.3.1 Non-pinched renormalizations

Let $U_j(a)$ be the component of $\sigma_a^{-1}(\Omega)$ containing the critical point $\mathbf{w}_j(a)$ of degree $d_j \geq 2$. For the remainder of this sub-subsection we assume that w_j is not adjacent to the cusp (i.e. $j \neq 1$), so that $U_j(a)$ is compactly contained in Ω, and $\sigma_a|_{U_j(a)} \colon U_j(a) \to \Omega$ is a family of anti-holomorphic, unicritical, polynomial-like maps of degree d_j. The *little filled Julia set* of this polynomial-like map is defined to be the set of points which do not escape from U_j, and will be denoted by $K_j(a)$. Define the set $\mathcal{C}_j \subset \mathcal{C}(\mathcal{S}_{\mathcal{T}})$ as the set of parameters for which $K_j(a)$ is connected.

Following [IM16, Section 5], there exists quasiconformal map $\rho_a \colon U_j(a) \to \mathbb{C}$ which is a hybrid conjugacy between the polynomial-like map $\sigma_a|_{U_j(a)}$ and a unicritical anti-holomorphic polynomial $z \mapsto \overline{z}^{d_j} + c$. The parameter c here is not unique; it is only determined up to a $d_j + 1$ order symmetry. To fix the polynomial to which we straighten, we will canonically mark the fixed points of the polynomial-like map.

Proposition 4.6.9. *Let $\sigma_a \in \mathcal{C}_j$. Then there exists a unique point $\beta(a) \in U_j(a) \cap K(\sigma_a)$ which disconnects $\mathbf{w}_j$ from the parabolic root $\mathbf{b}$*

Proof sketch. Let a_j be such that $\mathbf{w}_j(a_j) = \mathbf{v}$ (a parameter which can be argued to exist by [Poi13]) the dessin $\mathcal{T}$ naturally embeds into $K(\sigma_{a_j})$, using internal angles for arcs intersecting Fatou coordinates. The interval between $\mathbf{w}_j$ and the adjacent black vertex closer to the cusp is mapped onto a strictly larger interval, and therefore contains a fixed point. Furthermore, since there are no critical points in the interior of this interval it is mapped univalently and thus the fixed point is unique. Any other fixed points in $U_j(a_j)$ will therefore not disconnect $\mathbf{w}_j$ from $\mathbf{b}$. Call this fixed point $\beta(a_j)$. As it disconnects the filled Julia set $K(\sigma_{a_j})$ it will have multiple dynamical rays land at it. These rays persist throughout $\mathcal{C}_j$. □

Remark 4.6.10. The rays which land at $\beta(a)$ will be a period 2 cycle. The corresponding parameter rays land at the root of the little multicorn.

In light of the above proposition, fix the hybrid conjugacy so that $\rho_a(\beta(a))$ is sent to the landing point of the 0 ray. We define a straightening map $\chi_j \colon \mathcal{C}_j \to \mathcal{M}^*_{d_j}$, by applying a dynamical straightening to each parameter in $\mathcal{C}_j$.

The main result of this subsection is the following:

Theorem 4.6.11. *Let $\mathcal{C}_j$ be as above.*

1. *The straightening map* χ_j *is injective.*

2. *The image of the straightneing map* $\chi_j(\mathcal{C}_j)$ *contains the closure of hyperbolic parameters in* $\mathcal{M}^*_{d_j}$.

We will refer to sets $\mathcal{C}_j$ as *period 1 little combinatorial multicorns.* The above theorem justifies this name.

We first prove a number of preperatory propositions.

Proposition 4.6.12. *Let* $\sigma_a \in \mathcal{C}(\mathcal{S}_\mathcal{T})$*. Almost every point* $z \in J_a$ *converges to* $\omega(\mathbf{v}(a))$*.*

Proof. By the main result of [LMM23] every such Schwarz reflection is quasiconformally conjugate to a (anti-holomorphic) rational map. For rational maps the postcritical set forms a measure-theoretic attractor for the Julia set, and as quasiconformal maps preserve the Lebesgue measure class this property is preserved for σ_a. Thus, almost every point in J_a converges to either $\omega(\mathbf{v}(a))$ or the cusp $\mathbf{b}$. If $a \notin \Gamma^{\text{hoc}}$ then J_a is contained in a repelling petal of $\mathbf{b}$. Therefore the only points which converge to $\mathbf{b}$ are its preimages, which form a countable, and thus measure-zero, set.

On the other hand if $a \in \Gamma^{\text{hoc}}$ then $\omega(\mathbf{v}) = \{\mathbf{b}\}$ and we are done. □

Remark 4.6.13. Alternatively, if $a \in \Gamma^{\text{hoc}}$ then one can show that the Julia set J_a has zero Lebesgue measure.

Definition 4.6.3. For a postcritically finite (possibly anti-holomorphic) unicritical polynomial p, there exists a unique, up to homotopy forward invariant tree contained in the filled Julia set of p which contains the critical point and its forward orbit, as well as the landing point of the 0 ray and all its preimages. Define the *augmented* Hubbard tree of p to be the associated combinatorial tree and the induced action on it.

Proposition 4.6.14. *The image of the straightening map* $\chi_j(\mathcal{C}_j)$ *contains all postcritically finite parameters of* $\mathcal{M}^*_{d_j}$.

Proof. Let $f \in \mathcal{M}^*_{d_j}$ be a postcritically finite map. We consider an augmented Hubbard tree for it, denoted by T_f, which contains the landing point for the dynamical ray at angle 0, which we denote β, as well as all preimages of β. We produce a tuned tree as follows: Delete from $\mathcal{T}$ the point w_j, giving d_j components which are circularly ordered. Attach the component of $\mathcal{T} \setminus w_j$ which contains the root of $\mathcal{T}$ to T_f at β, and attach the remaining components of $\mathcal{T} \setminus \{w_j\}$ to the preimages of β, maintaining the circular ordering. Denote this tuned tree as $\widehat{T}_f$. The embedding $T_f \hookrightarrow \widehat{T}_f$ induces dynamics on a forward-invariant part of the new tree. We induce dynamics on the rest of it by mapping all black points from $\mathcal{T}$ to the root, and mapping all remaining white points of $\mathcal{T}$ to the critical value contained in the embedding of T_f.

By [Poi13, Theorem 5.1], to find a postcritically finite, anti-holomorphic polynomial, $p_f \colon \mathbb{C} \to \mathbb{C}$ which realizes this Hubbard tree $\widehat{T}_f$, it suffices to check that Note that p_f has two finite critical values, one of which is fixed. Furthermore for an interval I containing the two critical values, $p_f^{-1}(I)$ is isomorphic as a planar embedded tree to $\mathcal{T}$. By [LMM23, Theorem C], p_f is hybrid equivalent to a Schwarz reflection, and by the critical orbit relation just described, it is in fact hybrid equivalent to some $\sigma_{a(f)} \in \mathcal{C}(\mathcal{S}_{\mathcal{T}})$.

Lastly, it remains to be shown that $\chi_j(\sigma_{a(f)}) = f$. Note first that $\sigma_{a(f)}$ has Hubbard tree $\widehat{T}_f$. Need to argue that the embedded image of T_f is contained in U_j, and is the only forward-invariant part of the tree for which that is true. Then after straightening and by uniqueness we are done. □

Proposition 4.6.15. *The image of the straightening map $\chi(\mathcal{C}_j)$ is closed under quasiconformal conjugations.*

Proof. Lift line fields on filled Julia set under straightening map to filled Julia set of σ_a. Then we use measurable Riemann mapping theorem to find another $\sigma_{a'}$ which straightens to desired map. □

Proof of Theorem 4.6.11. Injectivity: We make use of the "pullback argument" first developed by W. Thurston [Lyu]. Suppose that $\chi(\sigma_a) = \chi(\sigma_{\widetilde{a}})$. By composing the dynamical straightening maps we obtain a quasiconformal map $\rho_{\widetilde{a}}^{-1} \circ \rho_a \colon U_j \to \widetilde{U}_j$ such that $\bar{\partial}\psi_0 \equiv 0$ almost everywhere on $K_j(a)$, which conjugates the polynomial-like restrictions. We obtain neighborhoods $V, \widetilde{V}$ which are mapped to eachother by $\rho_{\widetilde{a}}^{-1} \circ \rho_a$. Now define $\varphi_0 \colon \Omega(a) \to \Omega(\widetilde{a})$ to be such that

$$\varphi_0(z) = \begin{cases} \rho_{\widetilde{a}}^{-1}(z) \text{ for } z \in V \\ \psi_{\widetilde{a}}^{-1} \circ \psi_a(z) \text{ for } z \in \Omega(a) \setminus U_j(a) \end{cases} .$$

This map is quasiconformal, conjugates σ_a to $\sigma_{\widetilde{a}}$ on a neighborhood of $K_j(a)$, and satisfies $\bar{\partial}\varphi_0 \equiv 0$ on $K_j(a)$.

We now proceed to lift this conjugacy to preimages of U_j. For any i, lift φ_0 to the map $\varphi_{1,i} \colon \sigma_a^{-1}(V) \cap U_i \to \sigma_{\widetilde{a}}^{-1}(\widetilde{U}_j) \cap \widetilde{U}_i$, so that $\psi_0 \circ \sigma_a = \sigma_{\widetilde{a}} \circ \psi_{1,i}$. There will be $\deg(w_i)$ choices for this lift, which in general will be greater than 1. However, there will only be one preimage of $\beta(a)$ (corresponding $\beta(\widetilde{a})$ in U_i ($\widetilde{U}_i$) which disconnects the preimage of $\mathbf{v}$ from $\mathbf{b}$. The lift $\varphi_{1,i}$ is chosen to map the corresponding marked preimages, and is therefore uniquely determined. Define $\varphi_1 \colon \Omega(a) \to \Omega(\widetilde{a})$ to be $\varphi_{1,i}$ on $\sigma^{-1}(U_j) \cap U_i$, and the uniformizing map $\psi_{\widetilde{a}}^{-1} \circ \psi_a$ on $\Omega \setminus \sigma_a^{-1}(U_j)$. Note that ψ_1 has the same maximum dilatation as ψ_0. Furthermore, $\overline{\partial}\psi_1 \equiv 0$ almost everywhere on $\sigma_a^{-1}(K_j(a))$.

We now inductively repeat this process to obtain a sequence of uniformly quasiconformal maps $\varphi_n \colon \Omega(a) \to \Omega(\widetilde{a})$ which conjugate $\sigma_a|_{\sigma_a^{-n}(V)}$ to $\sigma_{\widetilde{a}}|_{\sigma_{\widetilde{a}}^{-n}(\widetilde{V})}$, and with $\bar{\partial}\varphi_n \equiv 0$ almost everywhere on pullbacks of the little filled Julia set. The φ_n converge to some global quasiconformal conjugacy $\varphi \colon \Omega(a) \to \Omega(\widetilde{a})$, which is conformal almost everywhere on the pullbacks of $K_j(a)$. Note that K_j contains every periodic Fatou component, as well as $\omega(\mathbf{v})$. By proposition 4.6.12 almost every point in $K(\sigma_a)$ lies in a preimage of K_j. In particular, φ is conformal almost everywhere on $K(\sigma_a)$.

It was shown in [LMM23] that the hybrid class of any parameter in $\mathcal{C}(\mathcal{S}_{\mathcal{T}})$ is a single point, and therefore $a = \widetilde{a}$.

Almost surjectivity: It is easy to extend proposition 4.6.14 to show that the image of χ_j contains all hyperbolic parameters by using the uniformization of hyperbolic parameters from section 4.4. Now suppose that $p_{c_n} \in \mathcal{M}^*_{d_j}$ are hyperbolic parameters in the associated multicorn set with $c_n \to c$, and let $a_n \in \mathcal{C}_j$ be the preimages under χ_j. After passing to a subsequence, we assume that $a_n \to a \in \mathcal{C}_j$. Denote the associated dynamical straightening maps as ρ_n, extending them as necessary so that they all share a common domain, and let ρ be the straightening map for σ_a. Note that the domains $U_j(a)$ and $\Omega(a)$ all vary continuously, and so the modulus of the annulus $\Omega(a) \setminus U_j(a)$ is bounded below. This tells us that the associated straightening maps ρ_n can be chosen to have uniformly bounded dilatation, so in particular, after passing to a subsequnce, there is a limit $\rho_n \to \widetilde{\rho}$. It follows that $p_c = \widetilde{\rho} \circ \sigma_a \circ \widetilde{\rho}^{-1} = \widetilde{\rho} \circ \rho^{-1} \circ \chi_j(a) \circ \rho \circ \widetilde{\rho}^{-1}$. But now as the image of the straightening is quasiconformally closed, it follows that $p_c \in \chi_j(\mathcal{C}_j)$. □

4.6.3.2 Pinched period 1 renormalizations

We now consider the case in which $j = 1$, i.e. the case in which the critical point adjacent to the cusp has a period 1 renormalization. It is no longer true that $U_1 \subset \Omega$ is compactly contained, and so the classical straightening theorem will not work.

Instead, Theroem 3.3.5 allows us to straighten such maps to *parabolic* anti-rational maps, which are a unicritical. The same proof as above works with minor modifications.

4.7 Model of the connectedness locus $\mathcal{C}(\mathcal{S}_\mathcal{T})$

4.7.1 Laminations and the pinched-disk model

We begin this section with some definitions.

Definition 4.7.1 (Pre-periodic Laminations and Combinatorial Classes)**.** 1. For $a \in \mathcal{C}(\mathcal{S}_\mathcal{T})$ the *pre-periodic lamination* of σ_a is defined as the equivalence relation on $\mathrm{Per}(\rho) \subset \partial Q =$

$[0, \frac{1}{d+1}]/\{0 \sim \frac{1}{d+1}\}$ such that $\theta, \theta' \in \mathrm{Per}(\rho)$ are related if and only if the corresponding dynamical rays land at the same point of ∂K_a.

2. Two parameters a and a' in $\mathcal{C}(\mathcal{S}_{\mathcal{T}})$ are said to be *combinatorially equivalent* if they have the same pre-periodic lamination.

3. The combinatorial class $\mathrm{Comb}(a)$ of $a \in \mathcal{C}(\mathcal{S}_{\mathcal{T}})$ is defined as the set of all paraemters in $\mathcal{C}(\mathcal{S}_{\mathcal{T}})$ which are combinatorially equivalent to a.

4. A combinatorial class $\mathrm{Comb}(a)$ is called *periodically repelling* if for every $a' \in \mathrm{Comb}(a)$, each periodic orbit of σ_a is repelling.

Proposition 4.7.1. *Two parameters a and a' in $\mathcal{C}(\mathcal{S}_{\mathcal{T}})$ are combinatorially equivalent if and only if $\chi(a), \chi(a') \in F_{\mathcal{T}}$ are as well.*

Proof. This follows from the hybrid equivalence of the straightening. □

Proposition 4.7.2. *Every combinatorial class Comb(a) of $\mathcal{C}(\mathcal{S}_{\mathcal{T}})$ is one of the following types.*

- *Comb(a) consists of an even period hyperbolic component that does not bifurcate from an odd period hyperbolic component, its root point, and the irrationally neutral parameters on its boundary.*

- *Comb(a) consists of an even period hyperbolic component that bifurcates from an odd period hyperbolic component, the unique parabolic cusp and the irrationally neutral parameters on its boundary.*

- *Comb(a) consists of an odd period hyperbolic component and the parabolic arcs on its boundary.*

- *Comb(a) is periodically repelling.*

We define an equivalence relation $\sim$ on $\mathbb{S}^2$ as follows:

- identify all points in the closure of each periodically repelling combinatorial class of $\mathcal{C}(\mathcal{S}_{\mathcal{T}})$,
- identify all points in the closure of the non-bifurcating sub-arc of each parabolic arc of $\mathcal{C}(\mathcal{S}_{\mathcal{T}})$, and
- identify all points in Γ^{hoc}.

All equivalence classes are connected and non-separating, and therefore by Moore's theorem the image of $\mathbb{S}^2$ under this relation is again $\mathbb{S}^2$.

Definition 4.7.2. The abstract connectedness locus of $\mathcal{S}$ is defined as the image of $\mathcal{C}(\mathcal{S}_{\mathcal{T}})$ under the above equivalence relation, and will be denoted as $\widetilde{\mathcal{C}}_{\mathcal{T}}$.

Proposition 4.7.3. *There is an embedding of $\mathcal{T} \setminus \{root\}$ into $\widetilde{\mathcal{C}}_{\mathcal{T}}$, which maps the white vertices of $\mathcal{T}$ which are of valence at least two to the combinatorial classes associated with the centers of period one hyperbolic components, and maps the remaining white vertices and the black vertices to the combinatorial classes associated with Misiurewicz points.*

Proof. This tree will exactly be the image of those parameters σ_a for which the free critical value v_a lies on the spine of $K(\sigma_a)$. The combinatorial classes of these parameters are given by the rays which land on the spine. □

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