The Stability of the Spacetime Penrose Inequality in Spherical Symmetry

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Abstract of the Dissertation

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We formulate and prove the stability statement associated with the spacetime Penrose inequality for n-dimensional spherically symmetric, asymptotically flat initial data satisfying the dominant energy condition. We assume that the ADM mass is close to the half area radius of the outermost apparent horizon and, following the generalized Jang equation approach, show that the initial data must arise from an isometric embedding into a static spacetime close to to the exterior region of a Schwarzschild spacetime in the following sense. Namely, the time-slice is close to the Schwarzschild time-slice in the volume preserving intrinsic flat distance, the static potentials are close in L^2_{loc}, and the initial data extrinsic curvature is close to the second fundamental form of the embedding in L^2.

I dedicate this work to my friends and compatriots at Stony Brook, who have given me boundless knowledge and confidence over the years. Because of you I have had a wonderful, wonderful time; because of you I am ready and happy to move on to what's next.

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Chapter 1

Introduction

The Penrose inequality, given in 3 dimensions by

$$m \geq \sqrt{\frac{|A|}{16\pi}}, \tag{1.0.1}$$

posits that the mass of a model of the universe, represented on the left-hand side of the inequality by m, should be bounded below by a fixed function of the area of the boundary black holes within the model, a quantity denoted on the right by $|A|$. We will specify these quantities much more rigorously in the next section, focusing for now on discussing the motivation behind this statement.

The Penrose inequality is an important conjecture within the field of general relativity. In addition to being mathematically useful in its own right, Roger Penrose initially developed the inequality, and a heuristic argument in favor, to study the cosmic censorship conjecture. To wit: we may view spacetime as a Lorentzian manifold governed by the Einstein field equations. We have both mathematical and observational evidence that this manifold contains points at which time and space are not well-defined. Such points are known as singularities and pose a significant problem to the scientific study of physics. If there are points at which there is no sensible notion of time and space, how can we study and describe the relationship between those points and the rest of spacetime? How do we make sense of the past and future when these points might introduce behavior we cannot mathematically predict?

The weak cosmic censorship conjecture proposes a solution to this problem by "censoring" these singularities by hiding them behind black holes. A black hole is a special region of space time characterized by the fact that events which take place within this region of spacetime may not have any effect on events that lie outside of the black hole. The boundary of the black hole is known as an event horizon. Once any matter or energy, including light, passes through an event horizon and into a black hole, it may never escape. If the Penrose inequality were not true, and there exists a universe which has mass less than the area of its black holes, it is most likely that such a universe would violate the cosmic censorship conjecture, containing unguarded singularities that swallow mass. Therefore, studying the Penrose inequality is an important avenue of investigation in understanding the nature of the universe.

The Penrose inequality has another facet known as the rigidity statement, which studies what happens if a model of the universe observes equality in (1.0.1). The conjecture in that case is that, if we have this equality, then the universe must exactly be a time-slice of

Schwarzschild spacetime. Schwarzschild spacetime is the simplest model of a universe that exhibits a black hole, and was the first solution found to the Einstein field equations which has such a singularity. If the rigidity statement of the Penrose inequality is true, then the Schwarzschild spacetime is the unique model spacetime with a black hole.

1.1 Formalization

Formalizing the Penrose inequality into a statement which is mathematically rigorous is a little involved, as it can be difficult to translate our physical observations and intuition into mathematical statements. We model spacetime as a Lorentzian manifold $(\mathcal{M}, h)$ with signature $(-, +, +, +)$, where the signature represents the signs associated to the eigenvalues of the metric h at any point $p \in \mathcal{M}$. Note that this is in contrast to a positive definite, or Riemannian, metric. For the purposes of this discussion, we will restrict ourselves to a 4-dimensional manifold $\mathcal{M}$, with 3 spacial dimensions and 1 time dimension, although later we will prove results for $\mathcal{M}$ of general dimension. We use the metric to encode which directions in $\mathcal{M}$ are spatial and which are time-like: for $v \in T_p\mathcal{M}$, we say that v is...

- space-like if $g_p(v, v) > 0$,
- null if $g_p(v, v) = 0$, and
- time-like if $g_p(v, v) < 0$.

We impose on $\mathcal{M}$ the condition that it is time-orientable, meaning that that the preceding decomposition of vector spaces varies smoothly across $\mathcal{M}$, and we can fix a time-orientation so that time-like vectors are either past or future directed. We say that a submanifold M of $\mathcal{M}$ is space-like if $g_p(v, v) > 0$ for all $v \in T_p\mathcal{M}|_M$; ie, if all of the vectors tangent to M are space-like. We will often refer to such manifolds as time-slices, and we consider these submanifolds as possible models of the physical universe.

Now that we have a general way of representing the physcial universe in the form of a time-slice, we may discuss how a point in space moves through time to a new point. For two time-slices M_1 and M_2 and points $p \in M_1$, $q \in M_2$, any curve γ that satisfies:

$$\gamma : [0, 1] \to \mathcal{M} \text{ so that } \gamma(0) = p,\ \gamma(1) = q, \text{ and } g_p(\dot{\gamma}, \dot{\gamma}) < 0 \tag{1.1.1}$$

describes a path that an observer at point p can travel along through time to arrive at point q. If $\dot{\gamma}$ is future-pointing, we say that q is in the future cone of p. The boundary of the future cone of p is known as the light cone and is composed of points reachable from p only by traveling at the speed of light. If an observer at p on M_1 would have to travel faster than the speed of light to reach q on M_2, then q lies outside of the future cone of p and no such time-like path γ can exist between the two points. Thus, the signature of the Lorentzian metric encodes the physical restriction we observe that nothing can travel faster than the speed of light. For more background on the principles of general relativity, the interested reader can refer to the following sources: [Wal84; Mis+73; OO14; GN14; Ste12].

We turn next to another important consideration we must take into account when building our mathematical model of the universe, which is that the matter in the universe influences how the universe is shaped. Physicists have, for example, observed how light bends around massive objects, indicating that space near those objects is actually curved. In order to capture the relationship between the matter and energy in a universe and its geometry, we appeal to the Einstein field equations, which Einstein published in 1912 and which describe a relationship between the stress-energy tensor T of $\mathcal{M}$ – which captures the flow of energy and momentum in spacetime – and the metric h, Ricci curvature Ric_h, and scalar curvature R_h of the spacetime $\mathcal{M}$ – which establish the mathematical geometry. The equations are given by

$$Ric_h - \frac{1}{2}R_h h - \Lambda h = \kappa T \tag{1.1.2}$$

where κ is the Einstein gravitational constant and Λ is the cosmological constant which represents the vacuum energy of space.

The field equations detail a system of PDEs which the metric h and stress-energy tensor T on a Lorentzian manifold must satisfy in order for the manifold $(\mathcal{M}, h)$ to be considered a valid model of spacetime with energy distribution T. However, there is currently no mathematically coherent way of identifying a set of tensors on $\mathcal{M}$ which are reasonable possibilities for the stress-energy tensor that takes into consideration the physical properties of the universe that we observe. With this in mind, there is currently no mathematically rigorous way of using the Einstein equations to restrict which Lorentzian manifolds might represent spacetime.

In lieu of a direct approach, researchers have developed various strategies for studying the information that the Einstein equations might give us about models of spacetime. One such approach is to use an initial value formulation in which a Riemannian manifold (M, g) and symmetric 2-tensor k constitute initial data for the Einstein PDE, the natural question being whether we can evolve this data to obtain a reasonable representation of spacetime. We represent the initial data by the tuple (M, g, k), where the symmetric 2-tensor k determines the geometry of how M sits in a hypothetical spacetime $\mathcal{M}$. Mathematically, k is the second fundamental form of the embedding of (M, g) into $(\mathcal{M}, h)$. We are therefore interested in using the initial data to interpret quantities associated to spacetime, which has an intuitive justification in that we can observe the physical properties of our own time-slice, but have little knowledge of the global nature of spacetime. A special case of initial data arises when we set $k = 0$; this case is known as the time-symmetric case. That the second fundamental form k is null implies that the geometry of the Riemannian manifold (M, g) is independent of the spacetime $(\mathcal{M}, g)$, and this case reduces to the study of Riemannian manifolds. The case in which $k \neq 0$ is more complex and is the case that we will primarily deal with here.

Now that we have reduced our study of spacetime to the study of an initial data set (M, g, k), we next consider how to best capture our quantities of interest: the mass and the size of the black holes.

We know that the energy density at a point is determined by the stress-energy tensor on $\mathcal{M}$. However, expanding this definition to a region larger than a point poses some difficulties as there is no observer-invariant way of accounting for the energy of gravity in the stress-energy tensor. Instead, we can appeal to one of a variety of quasi-local measures of mass, which calculate the mass within a pre-defined region of space. Such definitions include the Hawking mass and the Geroch mass. For a more in-depth exposition, see [Wan15].

We concentrate on the Hawking mass, which, heuristically, interprets the mass in a given region of space by measuring how much light emitted from the boundary of the region bends relative to the boundary. The more mass within the region, the more gravitational energy that region exerts on light, the more the light bends. Mathematically, the Hawking mass m_H of a compact 3-dimensional region $U \subset M$ is

$$m_H(U) = \sqrt{\frac{|\partial U|}{16\pi}} \left(1 - \frac{1}{16\pi} \int_{\partial U} H^2 d\sigma\right) \tag{1.1.3}$$

where $|\partial U|$ is the area of the boundary of U and H is the mean curvature of U in M, a quantity determined by the extrinsic curvature of the boundary ∂U relative to M.

Given the Hawking mass, we might hypothesize that we can measure the total mass of space by just taking the limit of the Hawking mass of infinitely expanding regions, such as a compact exhaustion of M. However, it turns out that, if we don't make an additional restriction on our initial data, that limit will not be well-defined. The problem is that, for general initial data, we can choose different coordinates near infinity that force the limit to converge to different values. We want the computation to be coordinate independent, meaning that we don't want the way we look at the space to change how big the space is.

Fortunately, there is a physically intuitive way of solving this problem: we impose conditions on what the initial data looks like near infinity. These conditions are called asymptotics and they determine how the metric behaves – and therefore what the initial data must look like – near infinity. If the metric decays to the Euclidean metric, we say that the initial data is asymptotically flat; if the metric decays to the hyperbolic metric, we say that the initial data is asymptotically hyperbolic. We will focus on asymptotically flat initial data, which is the assumption backed by the vast majority of physical evidence measured from our own universe. However, asymptotically hyperbolic initial data sets are mathematically interesting as well, see, for example: [Bra+21; ALM20; Ner17; All+16; CCS15; Nar10; XZ08; Sak21; SS17; CKS16; DS21; DGS12; DGS13; Sak10].

Once we restrict our metric to decay to the Euclidean metric within a specified rate, and thus restrict what our time-slice looks like, we can show that taking the limit of the Hawking mass as described above is well-defined. For a discussion of the nuances of this proof, see [Nat21]. We may then define the Arnowitt-Deser-Misner (ADM) mass of the asypmtotically flat time-slice M as

$$m_{ADM}(M) = \lim_{r \to \infty} m_H(S_r) \tag{1.1.4}$$

where S_r is a sphere near infinity, ie, a sphere with massive radius, $r >> 0$.

We have obtained a mathematical formulation of the left-hand side of the Penrose inequality. However, we still need to figure out how to formalize the right-hand side by computing the size of the black holes that the cosmic censorship conjecture assigns to each singularity. The most intuitive thing to do would be to compute the area of the event horizon as follows: the event horizon $\mathcal{H}$ intersects our initial data set M in some hypersurface $M \cap \mathcal{H}$, and we should then measure the area of this hypersurface. Unfortunately, there is no straightforward way of identifying the event horizon of $\mathcal{M}$ given only the initial data M.

Instead, we can make an intermediate definition motivated by our intuitive understanding of what happens beyond an event horizon. A trapped surface is a surface $S \subset M$ for which a shell of light emitted from S is pulled inward – in other words, a trapped surface is one from which nothing can move out into the rest of the initial data; instead, anything on S, including light, can only move in toward the singularity. We can mathematically formalize what it means for a surface to be trapped as follows. Recall that the mean curvature of S in $\mathcal{M}$, denoted by $\mathcal{H}$, is the rate of change of the area of an embedded surface subject to variation [Li98]. We can model a shell of light emitted from a surface S as a future directed light-like vector field ℓ on S, and we say that the surface is trapped if the rate of change of the area of the shell of light is negative, i.e., $h(\mathcal{H}, \ell) < 0$.

While this is a valid formalization of the notion of a trapped surface, it still depends on more knowledge of the Lorentzian manifold $\mathcal{M}$ than we have access to. We want to be able to formulate our definition using only the information provided by the initial data (M, g, k); in particular, k represents the second fundamental form of the embedding of M into $\mathcal{M}$. Fixing a time orientation relative to M, we can find a unit length space-like vector field n_1 which is normal to S and a unit length time-like, future-pointed vector field n_2 which is normal to S and n_1. We can then decompose the spacetime mean curvature vector $\mathcal{H}$ as

$$\mathcal{H} = Hn_1 - (\mathrm{tr}_S k)n_2 \tag{1.1.5}$$

where H is the mean curvature of S in M. With this decomposition, we can see that $\mathcal{H}$ is future pointing if the coefficient of the time-like component is positive, meaning that we must have $\mathrm{tr}_S k < 0$. From earlier, we know that if S is trapped, then

$$0 > g(\mathcal{H}, \mathcal{H}) = H^2 - (\mathrm{tr}_S k)^2 = (H + \mathrm{tr}_S k)(H - \mathrm{tr}_S k). \tag{1.1.6}$$

We can label each component of this product as $\Theta_+ = H + \mathrm{tr}_S k$ and $\Theta_- = H - \mathrm{tr}_S k$ and will, in future, refer to these components as the future null expansion and the past null expansion, respectively. By (1.1.6), we can see that for S to be trapped we must have that $|H| < |\mathrm{tr}_S k|$. Combining this with the observation that $\mathrm{tr}_S k < 0$, we get that $\Theta_+ < 0$ for a future trapped surface. While this derivation has relied on knowledge of the Lorentzian metric, we can use it to define a time-slice future trapped surface as a surface S for which $\Theta_+ < 0$, a condition which relies only on the information given by the initial data set. Correspondingly, a past trapped surface is one for which $\Theta_- < 0$.

Now that we can identify the set all the future trapped surfaces $\{S_\alpha\}$ in an initial data set, we can denote the boundary of all the future trapped surfaces as $A = \partial\left(\bigcup_\alpha S_\alpha\right)$. We call this boundary the future apparent horizon and suggestively label it A because it approximates the event horizon using only the information given by the initial data set. Indeed, the event horizon cannot lie beyond the future apparent horizon A, as anything outside of A can expand into the asymptotic end of the initial data. Further, we may prove that a surface A_0 is a connected component of the future apparent horizon if and only if $H_{A_0} - \mathrm{tr}_{A_0} k = 0$.

Because anything bounded inside the apparent horizon A of an initial data set (M, g, k) cannot influence the evolution of the initial data set, we can, without loss of generality, assume that $A = \partial M$. We sometimes refer to M in this case as an asypmtotic end. Further, the Hawking Black Hole Topology Theorem gives us that, for a 3-dimensional initial data set, the connected components of the apparent horizon boundary ∂M must be topological 2-spheres. This established, if we let ρ_i be the radius of the connected component A_i of the boundary and assume that $g|_{\partial M} = g_{\mathbb{E}^2}$, we can simplify the right hand side of the Penrose inequality as

$$\sqrt{\frac{|\partial M|}{16\pi}} = \frac{1}{2}\sqrt{\frac{\sum_i \rho_i^2 \omega_2}{\omega_2}} = \frac{1}{2}\sum_i \rho_i \tag{1.1.7}$$

where $\omega_2 = 4\pi$ is the area of the unit 2-sphere in Euclidean space. We thus often call the right-hand side of the Penrose inequality the "half area radius" of the apparent horizon boundary of M. In 2006, Galloway and Schoen proved an extension of Hawking's Topology Theorem for higher dimensional initial data sets, specifically, that the apparent horizon must be of positive Yamabe type, admitting metrics of positive scalar curvature [GS06]. While we prove our results for general dimension, we do not need to invoke this result as, in the spherically symmetric case, the right-hand side is exactly $\frac{1}{2}\rho_0$, where $\partial M = A_0$ is the only boundary component and is exactly a Euclidean sphere.

In order to relate the ADM mass to the half area radius, we follow Penrose's original heuristic argument as outlined by Mars [Mar09]. First, the cosmic censorship conjecture implies that we can determine the whole future of the spacetime. Under this assumption, we expect the spacetime to converge to to a vacuum spacetime with stationary black hole: ie, we expect the spacetime to converge to the Kerr spacetime. The area of the event horizon $|A|$ in Kerr spacetime is independent of the time-slice, and can be computed as

$$|A| = 8\pi m(m + \sqrt{m^2 - L^2/m^2}) \tag{1.1.8}$$

where m is the mass and L is the total angular momentum. We can estimate this area from above then by dropping the L^2/m^2 term, from which we obtain the Penrose inequality in Kerr spacetime. We relate the inequality for the initial data set in the limiting Kerr spacetime by considering how the ADM mass and the event horizon evolve. In particular, we expect the mass to only decrease as matter escapes off to infinity, and Hawking's Black Hole Area Theorem tells us that the area of the black hole can only increase. It follows that the Penrose inequality must hold for the time-slice we started with as well.

However, there is an additional class of assumptions hiding in this heuristic argument, a class known as the positive energy conditions. Such conditions, generally, guarantee that matter in the universe carries positive mass. Thus, as matter escapes to infinity, the total mass of the universe decreases. Hawking's Black Hole Area Theorem also relies on a particular positive energy property. The positive energy property that we focus on when stating and proving the Penrose inequality is known as the dominant energy condition. Physically, the dominant energy condition requires that energy cannot flow faster than light, and restricts pockets of negative energy from developing. Mathematically, we may write the dominant energy condition as follows. From the Einstein equations (1.1.2), we can derive equations that describe the local matter, denoted by μ, and the density of momentum, denoted by J:

$$16\pi\mu = R_g - (\mathrm{tr}_g k)^2 - |k|_g^2 \quad \text{and} \quad 8\pi J = \mathrm{div}_g(k - (\mathrm{tr}_g k)g). \tag{1.1.9}$$

The dominant energy condition is then stated as

$$\mu \geq |J|_g. \tag{1.1.10}$$

We may now state our formalized, generalized Penrose inequality. Let (M, g, k) be an initial data set for the Einstein equations: (M, g) is an n-dimensional Riemannian manifold, $n \geq 3$, and k is a symmetric 2-tensor on M. We assume that the boundary of M is the outermost apparent horizon. Further, we assume that (M, g, k) is asymptotically flat in the sense that there is an asymptotic end that is diffeomorphic to the complement of a ball $\mathbb{R}^n \setminus B_0(R)$ and there exists a uniform constant C so that in the coordinates x provided by this asymptotic diffeomorphism we have the following fall off conditions:

$$\begin{aligned} \left|\partial^{\beta_1}(g_{ij} - \delta_{ij})\right| &\leq \frac{C}{|x|^{n-2+|\beta_1|}}, & \left|\partial^{\beta_2} k_{ij}\right| &\leq \frac{C}{|x|^{n-1+|\beta_2|}}, \\ |R_g| &\leq \frac{C}{|x|^{n+1}}, & |\mathrm{tr}_g k| &\leq \frac{C}{|x|^n} \end{aligned} \tag{1.1.11}$$

for multi-indices $\beta_1 \leq 2$, $\beta_2 \leq 1$. Additionally, we assume the dominant energy condition as in (1.1.10). The Penrose conjecture then asserts that

$$m_{ADM}(M) \geq \frac{1}{2}\left(\frac{|\partial M|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}, \tag{1.1.12}$$

and, if equality holds, then (M, g) is isometric to a time-slice of Schwarzschild spacetime with half area radius $m_0 = \frac{1}{2}\left(\frac{|\partial M|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}$, which can be written in radial coordinates as

$$(\mathrm{Sch}(m_0), g_S) = \left([\rho_0, \infty] \times \mathbb{S}^{n-1}, \frac{1}{\phi_S^2} d\rho^2 + \rho^2 d\Omega^2\right) \tag{1.1.13}$$

where $\rho_0 = (2m_0)^{1/(n-2)}$, $\phi_S = \sqrt{1 - \frac{2m_0}{\rho^{n-2}}}$, and $d\Omega^2$ is the standard metric on the $(n-1)$-sphere.

1.2 History

1.2.1 Inequality and Equality

Penrose first formulated this inequality in 1973 [Pen73], but the first mathematically rigorous results did not appear until much later. In 1977, Jang and Wald explored applying the inverse mean curvature flow, an idea developed by Geroch, to relate the area of the boundary of a time-slice of spacetime to its mass; however, they were not able to account for the singularities that appeared along the flow. In 2001, Huisken and Ilmanen solved these problems by formulating the flow as an elliptic equation, the solution of which is a function whose level sets describe the leaves of the flow and is constant on singularities. Using these ideas, Huisken and Ilmanen proved the Penrose inequality for 3-dimensional time-symmetric time-slices with one boundary component [HI01]. Shortly thereafter, Bray proved the result for 3-dimensional time-symmetric time-slices with general boundary by conformally deforming the metric, without changing the boundary and without increasing the mass, to a metric which is known to satisfy the inequality [Bra01]. Both arguments established the rigidity case as well, and Bray and Lee later generalized the conformal flow argument to dimensions less than 8 [BL09].

While in the time-symmetric (or Riemannian) case mathematicians were able to prove the Penrose inequality with no additional assumptions on the geometry of the manifold, the full (or spacetime) Penrose inequality has only been proven in the case of spherical symmetry. In 1996, Hayward proved the inequality for a spherically symmetric time-slice but did not handle the case of equality [Hay96], while Bray and Khuri published a proof of the inequality and the rigidity statement for a spherically symmetric time-slice in 2010 which relied on ideas adapted by Jang [BK10]. Although their proof is only explicitly stated in dimension 3, it can be generalized to all dimensions due to spherical symmetry.

The study of the Penrose inequality in many ways goes hand-in-hand with another important inequality in general relativity: the Positive Mass Theorem. In fact, the Positive Mass Theorem is a special case of the Penrose inequality – the case in which the spacetime contains no singularities. This theorem states that a universe with no singularities should have mass greater than or equal to zero, and that if the mass equals zero, a time-symmetric time-slice must be Euclidean, and the spacetime must be the simplest possible model of spacetime, known as Minkowski spacetime.

Schoen and Yau proved the time-symmetric Positive Mass Theorem in 1979, then extended their result two years later to the general case using ideas from Jang. Bray's proof of the Riemannian Penrose inequality relied on a key result from Schoen and Yau regarding the perturbation of metrics; the Jang equation idea that Schoen and Yau used to prove the Positive Mass Theorem is what Bray and Khuri later extended in their proof of the spherically symmetric spacetime Penrose inequality.

1.2.2 Stability

Both the Riemannian and spacetime Penrose inequalities have stability conjectures corresponding to the case of equality. Such conjectures ask whether the manifold or the associated static spacetime is close to Schwarzschild when the ADM mass is close to the half area radius of the black hole.

We will appeal to the intrinsic flat distance to judge whether two manifolds are close together. The intrinsic flat distance was established by Sormani and Wenger in [SW11] as a generalization of the flat distance in a manner analogous to how the Gromov-Hausdorff distance generalizes the Hausdorff distance; instead of computing the flat distance between integral currents in Euclidean space, the intrinsic flat distance is taken as an infimum over all distance-preserving embeddings into all complete metric spaces. The weak nature of the intrinsic flat distance makes it ideal for problems like the Penrose inequality, which rely on assumptions regarding scalar curvature as opposed to stronger conditions regarding Ricci curvature.

Bryden, Khuri, and Sormani used the Lakzian-Sormani technique [LS12] to show the intrinsic flat distance stability of the spacetime Positive Mass Theorem in the case of spherical symmetry [BKS20]. However, the Penrose inequality, for which the boundary of the initial data remains nontrivial, poses complications when it comes to establishing stability. In 2011, Lee and Sormani proved a type of stability result for the Riemannian Penrose inequality in spherical symmetry using the Lipschitz distance, which in turn implied intrinsic flat closeness [LS11]. In their paper, Lee and Sormani highlight that a pure stability result is impossible in the Riemannian case because any reasonable assumptions on the boundary do not prohibit the manifold from developing a cylindrical end of arbitrary length. Instead, they show that if the ADM mass is close to the half area radius, then the manifold is close to a so-called appended Schwarzschild space constructed by attaching a cylinder to the boundary of the Schwarzschild time-slice with the same boundary area. Moreover, in [MS14], Mantoulidis and Schoen published a counterexample to the stability of the Riemannian Penrose inequality by gluing an almost-cylinder with a poorly behaved minimal boundary onto a portion of the exterior region of Schwarzschild space. While their resulting manifold is not spherically symmetric, it provides further evidence that the behavior near the boundary cannot quite be controlled by assumptions regarding only scalar curvature and mass.

Sakovich and Sormani have proven a stability result for the spacetime Positive Mass Theorem in the asymptotically hyperbolic setting [SS17]. In [Sak21], Sakovich solves the Jang equation for asymptotically hyperboloidal initial data in order give a non-spinor proof of the hyperbolic Positive Mass Theorem. Allen has studied stability of the 3-dimensional Riemannian Positive Mass Theorem and Penrose inequality in a number of cases using inverse mean curvature flow, see [All17a; All17b; All18]. Dong, along with Song, has also studied the stability of the 3-dimensional Riemannian Positive Mass Theorem and Penrose inequality using the pointed measured Gromov-Hausdorff topology and novel methods for selecting the sets on which to show convergence [Don24a; DS23; Don24b].

We prove a stability result for the spacetime Penrose inequality in spherical symmetry with asymptotically flat initial data using the generalized Jang equation approach. While we are able to adapt the Jang equation proof of the Penrose inequality presented by Bray and Khuri [BK10; BK11] to the stability case, we find that an additional assumption is needed regarding the mean curvature of spheres in order to properly control the geometry outside of the asymptotic end. This assumption is closely tied to the outermost apparent horizon condition, and while technical in nature, is indispensable for the argument. Additionally, we must appropriately address the difficulties presented by lack of control near the boundary, which, similar to the case presented by Lee and Sormani, could develop into a cylinder of positive length. Finally, unlike [BKS20], we cannot use the Lakzian-Sormani technique to show intrinsic flat closeness. The reason is that our initial data might have countably many regions in which we cannot guarantee metric closeness and therefore we need a technique that is more robust to measure the boundary of the regions on which we are evaluating intrinsic flat closeness. Instead, we apply the new boundary version of the Volume Above Distance Below Theorem [AP20].

Chapter 2

Setup and Background

We use a Jang equation approach to relate the ADM mass and the half area radius, and we will use intrinsic flat convergence to achieve stability. Note that we establish in this chapter many of the conventions and notation that we use in subsequent chapters, and will refer back to certain sections and equations as necessary.

2.1 Inequality and Equality: The Jang Equation

In order to study the Positive Mass Theorem in 1977, Jang introduced a quasilinear elliptic differential equation that later became known as the Jang equation. He derived this equation as a necessary condition for an initial data set to be a time-slice of Minkowski space $\mathcal{M}_{Min} = (\mathbb{R}^4, -dt^2 + g_{\mathbb{E}^3})$; if an initial data set (M, g, k) is Minkowski, then the embedding $M \hookrightarrow \mathcal{M}_{Min}$ must satisfy

$$H - \text{tr}K = 0 \tag{2.1.1}$$

where H is the mean curvature of the embedding and K is an extension of the symmetric 3-tensor k to a symmetric 4-tensor on $\mathcal{M}_{Min}$. Thus, the solutions to this equation are are of interest as candidates for embeddings that might satisfy nice properties. In particular, in the original paper, Jang showed that the embedded surface obtained from a solution to (2.1.1) inherits the positive energy property from the initial data [Jan77].

More specifically, given an initial data sot (M, g, k), a solution f to the Jang equation (2.1.1) gives an embedding

$$f : M \to M \times \mathbb{R} \tag{2.1.2}$$

where the metric on $M \times \mathbb{R}$ is given by $g + dt^2$. Denoting the image of M under the embedding f by Σ, we can show that $m_{ADM}(\Sigma) = m_{ADM}(M)$, so proving the Positive Mass Theorem for Σ is equivalent to proving the Positive Mass Theorem for M. Further, as shown by Jang, the metric induced on Σ, $\bar{g} = g + df^2$, will have a positive energy property, ie, that $\bar{R} > 0$ where $\bar{R}$ is the scalar curvature of $\bar{g}$. The positivity of the scalar curvature can then be used to prove the Positive Mass Theorem. It was in this way that, in 1981, that Schoen and Yau leveraged the Jang equation to reduce the spacetime Positive Mass Theoremto the

Riemannian case, which they had proven in 1979. Notably, they took the trivial extension of K, where the coordinates corresponding to the time direction were set to zero.

Part of the tractability of solving the Jang equation in the case of the Positive Mass Theorem is that the metric on the constructed space $M \times \mathbb{R}$ is simply the product metric $g + dt^2$. This idea was suggested by the fact that the metric on Minkowski space is a product metric, and Minkowski space is the equality case of the Positive Mass Theorem. On the other hand, the equality case of the Penrose inequality should be Schwarzschild space, the metric on which is a warped product. Bray and Khuri observed in 2010 [BK10] that the trivial extension of K is no longer the natural choice under the warped product metric. This observation led them to develop the generalized Jang equation, which they solved in the case of spherical symmetry and used the solutions to prove the Penrose inequality for spherically symmetric, 3-dimensional initial data.

As our work follows their program closely, we will here establish the formalities of the setup, quoting necessary results and, in some cases, re-proving these results for general dimension. For initial data (M, g, k), we look for a surface Σ embedded in $(M \times \mathbb{R}, g + \phi^2 dt^2)$ as an image of M under a map f which satisfies the generalized Jang equation

$$H_\Sigma - \mathrm{tr}_\Sigma K = 0 \tag{2.1.3}$$

where ϕ is a suitable warping factor, K is an extension of k to $n+1$ dimensions given by

$$K_{ij} = \begin{cases} k_{ij} & \text{for } i, j \leq n \\ 0 & \text{for } i = n+1, j \neq n+1 \\ \langle N, \phi \nabla_g \phi \rangle_{g+\phi^2 dt^2} & \text{for } i = j = n+1, \end{cases} \tag{2.1.4}$$

and N is the normal vector to Σ inside $M \times \mathbb{R}$. We denote the second fundamental form of the Jang surface $(\Sigma, \bar{g})$ by h. The choice of extension for k to K in (2.1.4) is informed by the desire to obtain a positivity property for the scalar curvature of $\bar{g}$ when we assume the dominant energy condition, which we will see later. Once we have the embedding $F(x) = (x, f(x))$ to specify Σ inside of $M \times \mathbb{R}$, we can define

$$G : (M, g) \to (\Sigma \times \mathbb{R}, \bar{g} - \phi^2 dt^2) \quad \text{as} \quad G(x) = (x, f(x)) \tag{2.1.5}$$

so that the induced metric on $G(M)$ is $g = \bar{g} - \phi^2 df^2$. We denote the second fundamental form of $G(M)$ by π and rely on context to avoid confusion with the more common usage of π. If $\pi - k = 0$, then k is the second fundamental form of the embedding of M into our constructed spacetime. If $\bar{g} = g_S$ is the Schwarzschild metric and $\phi = \phi_S$, then the map G gives (M, g) as the image of an isometric embedding of Σ into Schwarzschild spacetime.

While so far we have avoided invoking spherical symmetry, we must do so now in order to proceed to solving the Jang equation. In spherical symmetry, we can write the metric g globally as

$$g = g_{11}(r) dr^2 + \rho^2(r) d\Omega^2 \tag{2.1.6}$$

for radial functions g_{11} and ρ where $d\Omega^2$ is the round metric on $\mathbb{S}^{n-1}$. If η is the outer unit normal vector to the sphere of radius r, we can write k as

$$k_{ij} = \eta_i \eta_j k_n(r) + (g_{ij} - \eta_i \eta_j) k_t(r) \tag{2.1.7}$$

for radial functions k_n and k_t, where k_n represents the component of k normal to spheres and k_t the tangent component.

In order to guarantee our boundary is an apparent horizon, we assume that the null expansions satisfy

$$\Theta_{\pm} := 2\left(\frac{\rho_{,r}}{\rho\sqrt{g_{11}}} \pm k_n\right)(r) > 0, \tag{2.1.8}$$

where $\rho_{,r}$ denotes the derivative of ρ with respect to r. In his 1979 paper [Jan77], Jang proved the equivalence of the existence and uniqueness of a new function v to the existence and uniqueness of the graph determined by f, where, in the generalized Jang case, v is given in r coordinates by

$$v = \frac{\phi\sqrt{g^{11}}f_{,r}}{\sqrt{1+\phi^2 g^{11} f_{,r}^2}} \tag{2.1.9}$$

and arises from analyzing the spherically symmetric structure of the Jang equation (2.1.3), see [BK10]. We refer to f and v both as solutions to the Jang equation.

The first major result regarding the generalized Jang equation is that the scalar scalar curvature of the surface Σ has a very nice expression. Indeed, we have that the scalar curvature of the Jang surface, denoted by $\bar{R}$ is given by

$$\bar{R} = 16\pi(\mu - J(w)) + |h - K|_{\Sigma}|_{\bar{g}}^2 + 2|q|_{\bar{g}}^2 - 2\phi^{-1}\text{div}_{\bar{g}}(\phi q), \tag{2.1.10}$$

where h is the second fundamental form, $K|_{\Sigma}$ is the restriction to Σ of the extended tensor K, q is a 1-form, and w is a vector with $|w|_g \leq 1$ given by

$$w = \frac{f^i \partial_{x^i}}{\sqrt{\phi^{-2} + |\nabla_g f|^2}}, \text{ and } q_i = \frac{g^{ij} f_{,i}}{\sqrt{\phi^{-2} + |\nabla_g f|^2}}(h_{ij} - (K|_{\Sigma})_{ij}). \tag{2.1.11}$$

The reader familiar with the expression for the scalar curvature for the trivial extension of the Jang equation should note the differences appear primarily in the divergence term, which is multiplied by the inverse of the warping factor. We can see from this expression for $\bar{R}$ that once we assume the dominant energy condition (1.1.10), our the generalized Jang scalar curvature is positive modulo the divergence term. The crux of the stability result we prove here rests on studying the behavior of this divergence term. Although we do not get pointwise control, L^2 control is enough to prove the intrinsic flat convergence.

Another difference between (2.1.10) and the expression for the trivial Jang extension appears in the second term:

$$|h - K|_{\Sigma}|_{\bar{g}}^2. \tag{2.1.12}$$

Note that h is the second fundamental form of Σ as the image of M in $(M \times \mathbb{R}, g + \phi^2 dt^2)$, not of M inside $(\Sigma \times \mathbb{R}, \bar{g} - \phi^2 dt^2)$ which we have denoted by π. Moreover, $K|_{\Sigma}$ is the extension of k, not k itself. In order to show that $\pi - k$ is somehow close to zero, we need that

$$h - K|_{\Sigma} = \pi - k. \tag{2.1.13}$$

This is the content of the next lemma, which otherwise appears in Appendix B of [BK10]. We leave out the details of computations appearing in Appendix A of [BK10].

Lemma 2.1.1. *Let h be the second fundamental form of the Jang surface $(\Sigma, \bar{g})$ inside of $(M\times\mathbb{R}, g+\phi^2 dt^2)$ and let π be the second fundamental form of $G(M)$ inside of $(\Sigma\times\mathbb{R}, \bar{g}-\phi^2 dt^2)$. Recall that K is the extension of the initial data k given by (2.1.4). Then we have (2.1.13).*

Proof. Let

$$X_i = \partial_{x^i} + f_{,i}\partial_{x^{n+1}}, \quad i = 1, 2, 3 \tag{2.1.14}$$

be tangent vectors to Σ. Then by equation (42) of [BK10],

$$h_{ij} = \frac{\nabla_{ij} f + (\log\phi)_{,i} f_{,j} + (\log\phi)_{,j} f_{,i} + g^{\ell p}\phi\phi_{,\ell} f_{,p} f_{,i} f_{,j}}{\sqrt{\phi^{-2} + |\nabla_g f|^2}} \tag{2.1.15}$$

where ∇_{ij} represents covariant differentiation with respect to g. Meanwhile, if $\bar{\nabla}$ represents covariant differentiation with respect to $\bar{g}$, then

$$\pi_{ij} := \frac{\bar{\nabla}_{ij} f + (\log\phi)_{,i} f_{,j} + (\log\phi)_{,j} f_{,i} - \bar{g}^{\ell p}\phi\phi_{,\ell} f_{,p} f_{,i} f_{,j}}{\sqrt{\phi^{-2} + |\nabla_{\bar{g}} f|^2}}. \tag{2.1.16}$$

Calculations making use of the equality $g = \bar{g} - \phi^2 df^2$ show that

$$\begin{aligned} &\nabla_{ij} f + (\log\phi)_{,i} f_{,j} + (\log\phi)_{,j} f_{,i} + g^{\ell p}\phi\phi_{,\ell} f_{,p} f_{,i} f_{,j} \\ &\qquad = \frac{\phi^{-2}}{\phi^{-2} - |\nabla_{\bar{g}} f|^2}\left(\bar{\nabla}_{ij} f + (\log\phi)_{,i} f_{,j} + (\log\phi)_{,j} f_{,i}\right) \end{aligned} \tag{2.1.17}$$

and

$$\frac{1}{\sqrt{\phi^{-2} + |\nabla_g f|^2}} = \phi^2\sqrt{\phi^{-2} - |\nabla_{\bar{g}} f|^2}. \tag{2.1.18}$$

It follows that

$$\begin{aligned} h_{ij} &= \frac{\sqrt{\phi^{-2} - |\nabla_{\bar{g}} f|^2}}{\phi^{-2} - |\nabla_{\bar{g}} f|^2}\left(\bar{\nabla}_{ij} f + (\log\phi)_{,i} f_{,j} + (\log\phi)_{,j} f_{,i}\right) \\ &= \pi_{ij} + \frac{\bar{g}^{\ell p}\phi\phi_{,\ell} f_{,p} f_{,i} f_{,j}}{\sqrt{\phi^{-2} + |\nabla_{\bar{g}} f|^2}} \\ &= \pi_{ij} + \frac{\langle\phi\nabla_g\phi, \nabla_g f\rangle_g}{\sqrt{\phi^{-2} + |\nabla_g f|^2}} f_{,i} f_{,j}. \end{aligned} \tag{2.1.19}$$

On the other hand, we have

$$(K|_\Sigma)_{ij} = K(X_i, X_j) = k_{ij} + \frac{\langle\phi\nabla_g\phi, \nabla_g f\rangle_g}{\sqrt{\phi^{-2} + |\nabla_g f|^2}} f_{,i} f_{,j}. \tag{2.1.20}$$

It is immediate then that (2.1.13) holds. $\square$

Note that (2.1.10) and Lemma 2.1.1 are both independent of the choice of warping factor ϕ and the assumption that the initial data is spherically symmetric. However, the existence and uniqueness of the Jang equations relies heavily on the choice of ϕ. Inspired by studying the warping factor of Schwarzschild spacetime, we may couple (2.1.3) to the mean curvature of codimension 1 spheres in Σ by choosing ϕ as follows. Let

$$s = \int_0^r \sqrt{g_{11} + \phi^2 f_{,r}^2} \tag{2.1.21}$$

be the radial arc length parameter in the $\bar{g}$ metric. In these coordinates,

$$\bar{g} = ds^2 + \rho^2(s) d\Omega^2 \tag{2.1.22}$$

and the mean curvature of a sphere of radius s is

$$\bar{H}_{S_s} = \frac{n-1}{\rho} \rho_{,s}. \tag{2.1.23}$$

We then set

$$\phi = \rho_{,s} = \frac{\bar{H}_{S_s}}{n-1} \left(\frac{|S_s|}{\omega_{n-1}} \right)^{1/(n-1)}, \tag{2.1.24}$$

coupling the Jang equation to the inverse mean curvature flow, which is solvable in spherical symmetry. The solvability of the flow produces a solvable Jang system. In a sense, this technique is a combination of the Huisken and Ilmanen approach with the original approach of Schoen and Yau to the spacetime Positive Mass Theorem [SY81].

Under this framework, then, [BK10], gives a unique solution as follows. If we assume that the initial data are smooth, satisfy the outermost apparent horizon condition (2.1.8), and the asymptotics (1.1.11), then there exists a unique solution to (2.1.3) given by $v \in C^\infty((0,\infty)) \cap C^1([0,\infty))$ so that

$$0 \le |v| \le 1 \tag{2.1.25}$$

which satisfies asymptotics to ensure that the Jang surface Σ is also asymptotically flat:

$$|v(r)| + r|v_{,r}(r)| \le Cr^{1-n} \text{ as } r \to \infty \tag{2.1.26}$$

for a constant C depending only on $|g|_{C^1((0,\infty))}$ and $|k|_{C^0((0,\infty))}$. We choose boundary data $\alpha = \pm 1$ so that the Jang solution v will blow up to an apparent horizon and the Jang solution will have the same mass information as the initial data. Now we are ready to state and prove the spacetime Penrose inequality in spherical symmetry.

Theorem 2.1.1 (Spacetime Penrose Inequality in Spherical Symmetry)**.** *Suppose n-dimensional spherically symmetric initial data (M, g, k) satisfies* (2.1.8)*, and* (1.1.11)*. Suppose further that the initial data satisfies the dominant energy condition* (1.1.10)*. Then*

$$m_{ADM} \ge m_0 \tag{2.1.27}$$

and if $m_{ADM} = m_0$ then (M, g) arises as an isometric embedding into Schwarzschild spacetime with second fundamental form k.

Proof. We can solve the Jang equation for (M, g, k) to get the Jang surface $(\Sigma, \bar{g}, h)$. Moreover, the asymptotics given by (1.1.11) and the choice of horizon boundary for the Jang solution means that $m_{ADM} = m_{ADM}(\Sigma)$ and $m_0 = m_0(\Sigma)$ so that it suffices to prove the inequality for $(\Sigma, \bar{g})$.

For the first part of the proof, we work in arc length radial coordinates denoted by (s, θ) on $(\Sigma, \bar{g})$. In spherical symmetry, the Hawking mass (or Misner-Sharp mass) of the $(n-1)$-sphere S_s generalizes as

$$m(s) := \frac{1}{2}\rho^{n-2}(1-\rho_{,s}^2) \tag{2.1.28}$$

$$= \frac{1}{2}\left(\frac{|S_s|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}\left[1 - \frac{1}{(n-1)^2(\omega_{n-1}^2|S_s|^{n-3})^{\frac{1}{n-2}}}\int_{S_s} H_{S_s}^2\right] \tag{2.1.29}$$

where H_{S_s} in $\bar{g}$ as computed in (2.1.23). We also note that in spherical symmetry the scalar curvature of $\bar{g}$ is

$$\bar{R} = \frac{n-1}{\rho^2}\left((n-2)(1-\rho_{,s}^2) - 2\rho\rho_{,ss}\right). \tag{2.1.30}$$

Differentiate the expression for $m(s)$ in (2.1.28) with respect to s and use the Fundamental Theorem of Calculus to get that

$$m(\infty) - m(0) = \int_0^\infty \frac{1}{2(n-1)}\rho^{n-1}\rho_{,s}\bar{R}ds \tag{2.1.31}$$

where we have simplified the expression using (2.1.30). We leverage spherical symmetry to write the integral over Σ with volume form $d\omega_{\bar{g}}$ and then substitute the formula for $\bar{R}$ given by (2.1.10) to get

$$m_{ADM} - m_0 = \frac{1}{2\omega_{n-1}(n-1)}\int_\Sigma \rho_{,s}\bar{R}d\omega_{\bar{g}} \tag{2.1.32}$$

$$= \frac{1}{2\omega_{n-1}(n-1)}\int_\Sigma \phi\left(16\pi(\mu - J(w)) + |h-K|_\Sigma|_{\bar{g}}^2 + 2|q|_{\bar{g}}^2\right)d\omega_{\bar{g}} \tag{2.1.33}$$

$$- \frac{1}{\omega_{n-1}(n-1)}\int_\Sigma \mathrm{div}_{\bar{g}}(\phi q)d\omega_{\bar{g}}. \tag{2.1.34}$$

By adding the null expansions in (2.1.8), we can see that

$$0 < H_{S_r} = 2\frac{\rho_{,r}}{\rho\sqrt{g_{11}}}, \tag{2.1.35}$$

which guarantees that the mean curvature of spheres is positive, and, by (2.1.24), we get that ϕ is positive as well. This combined with the dominant energy condition (1.1.10) gives that (2.1.33) is strictly positive. We can apply Stokes Theorem to (2.1.34) and arrive at

$$m_{ADM} - m_0 \geq -\frac{1}{\omega_{n-1}(n-1)}\int_{S_\infty \cup S_0} \phi\bar{g}(q, n_{\bar{g}})d\sigma_{\bar{g}} \tag{2.1.36}$$

where $n_{\bar{g}}$ is the outer unit normal and $d\sigma_{\bar{g}}$ is the appropriate surface form.

The computation in Appendix C of [BK10] shows that the part of q normal to spheres can be expressed in r coordinates as

$$q(\partial_r) = -2\sqrt{g_{11}}\frac{v}{1-v^2}\left(\frac{\rho_{,r}}{\rho\sqrt{g_{11}}}v - k(\eta,\eta)\right), \tag{2.1.37}$$

and that in the diagonal metric, $q_i = 0$ for $i > 1$. For a sphere of radius r,

$$\begin{aligned} -\frac{1}{\omega_{n-1}(n-1)}\int_{S_r}\phi\bar{g}(q, n_{\bar{g}})d\sigma_{\bar{g}} &= \frac{\rho^{n-1}\phi q_1}{(n-1)\sqrt{g_{11}+\phi^2 f_{,r}^2}} \\ &= \pm\frac{2\rho_{,r}v}{\sqrt{g_{11}}}\left(\frac{\rho_{,r}}{\rho\sqrt{g_{11}}}v - k(\eta,\eta)\right) \end{aligned} \tag{2.1.38}$$

depending on whether the boundary is a past or future horizon. By the preservation of the apparent horizon under the choice of v,

$$\frac{\rho_{,r}}{\rho\sqrt{g_{11}}}v - k(\eta,\eta) = 0 \tag{2.1.39}$$

on the inner boundary so that divergence term vanishes there. Moreover, the asymptotics for v given by (2.1.26) and the fall off conditions in (1.1.11) guarantee that the divergence term vanishes at infinity. It follows that

$$m_{ADM} - m_0 \geq 0 \tag{2.1.40}$$

which proves the inequality.

In the case of equality, the evaluation of the boundary term still holds so that from (2.1.33) we have

$$0 = \frac{1}{2\omega_{n-1}(n-1)}\int_{\Sigma}\phi\left(16\pi(\mu - J(w)) + |h - K|_{\Sigma}|_{\bar{g}}^2 + 2|q|_{\bar{g}}^2\right)d\omega_{\bar{g}} \tag{2.1.41}$$

which immediately implies that

$$\mu - |J|_g \equiv 0, \quad h - K|_{\Sigma} \equiv 0, \text{ and } \quad q \equiv 0. \tag{2.1.42}$$

It follows that $\bar{R} = 0$, and from the time symmetric Penrose inequality we deduce that $\bar{g} \cong g_S$ so that, in particular, $\phi = \phi_S$. Moreover,

$$g = \bar{g} - \phi^2 df^2 = g_S - \phi^2 df^2 \tag{2.1.43}$$

so that the graph map $G : M \to \mathrm{Sch}(M_0) \times \mathbb{R}$ is an isometric embedding of the initial data into Schwarschild spacetime.

Finally, by Lemma 2.1.1, $h - K|_{\Sigma} \equiv 0$ implies that $\pi - k \equiv 0$, and k is indeed the second fundamental form of the embedding into Schwarzschild spacetime. □

2.2 Stability: Intrinsic Flat Distance

Now that we have established the Penrose inequality and equality case in spherical symmetry, we explore what it means for a Jang solution to be "close" to Schwarzschild space. A fundamental question of geometric analysis is: How can we describe the differences between two sets in a metric space? A first-pass approach is to define this distance as the smallest tubular radius needed to enclose both sets with a tubular neighborhood around each set – this is known as the Hausdorff distance. To formalize this, we may define the tubular neighborhood of radius r of a set X in a metric space N with distance d_N as

$$T_r(X) = \{p \in N \mid \text{ there exists } x \in X, d_N(x,p) < r\} \tag{2.2.1}$$

and then write the Hausdorff distance as

$$d_H(X,Y) := \inf\{r > 0 \mid X \subset T_r(Y) \text{ and } Y \subset T_r(X)\}. \tag{2.2.2}$$

One thing to note about the Hausdorff distance is that it conveys information about the distance between two sets relative to their embedding in the metric space. As we can see in Figure 2.1, the Hausdorff distance between X and Y can be quite large, even if X and Y appear to be almost identical.

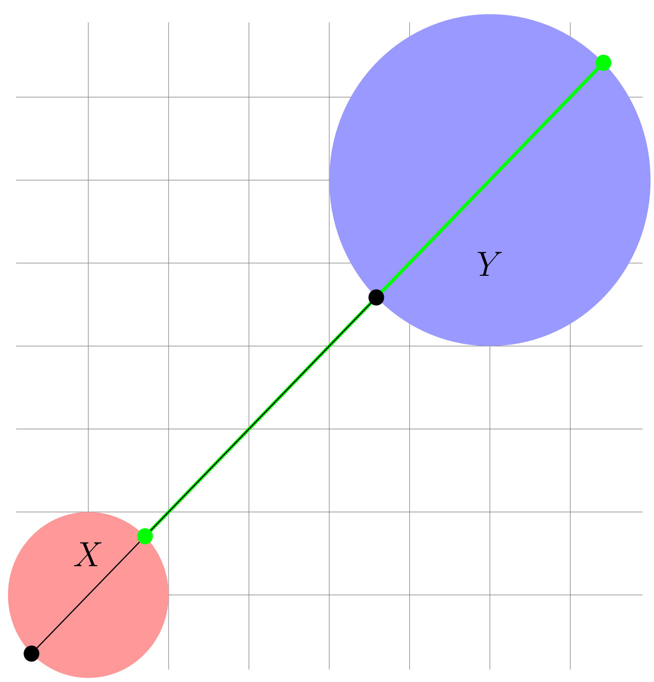

Figure 2.1. *A representation of the Hausdorff distance between two disks X and Y in $\mathbb{E}^2$. The black line, representing a tubular radius centered on Y which encloses X, has length $r \approx 6$. The green line, representing a tubular radius centered on X which encloses Y, has length $r \approx 8$. Thus, we may estimate that the Hausdorff distance between these two sets is $d_H(X,Y) = \inf\{r > 0 \mid X \subset T_r(Y)$ and $Y \subset T_r(X)\} \approx 8$.*

Say we want to compare X and Y more directly, without information about the space in which they are embedded getting in the way. We still want X and Y to live in some common metric space so we may have a consistent way of measuring how their properties compare. However, we can minimize the impact that this parent space has on our distance measurement by taking the infimum of the distance between two sets across all possible distance-preserving embeddings of X and Y into all possible metric spaces. When we take the distance between our sets to be the Hausdorff distance, this infimum achieves a new notion of distance known as the Gromov-Hausdorff distance [Gro06; Bur01]:

$$d_{GH}(X,Y) = \inf_{f,g,(N,d)} d_H(f(X), g(Y)) \tag{2.2.3}$$

where $f : X \to N$ is an isometric embedding of X into N and likewise for $g : Y \to N$. Using the Gromov-Hausdorff distance, we are able to reposition the sets X and Y relative to each other as we see fit so long as we preserve the geometry, as in Figure 2.2.

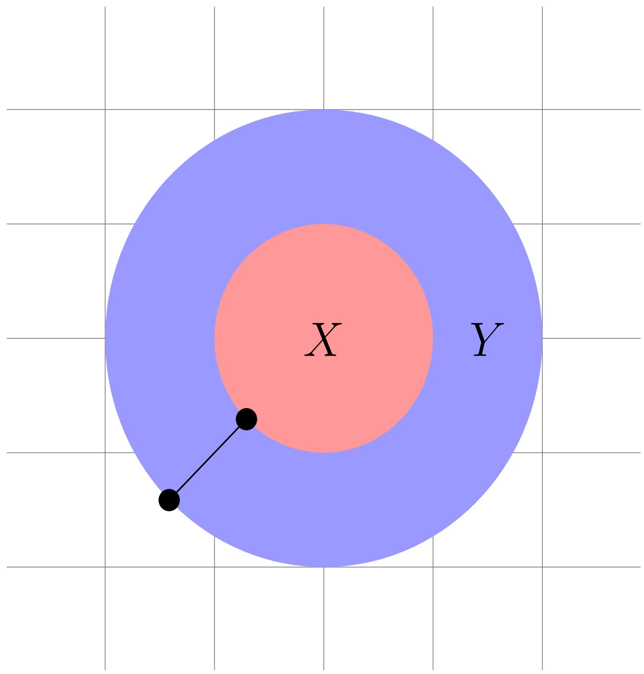

Figure 2.2. *A representation of the Gromov-Hausdorff distance between the same disks X and Y in $\mathbb{E}^2$. Under the Gromov-Hausdorff distance, we are free to position X and Y relative to each other as we please in order to minimize the Hausdorff distance as much as possible. Here, we can see that any tubular neighborhood of Y also encloses X, and we require only a tubular neighborhood with $r = 1$ around X to enclose Y.*

While the Gromov-Hausdorff distance is useful for computing the distance between sets which are “nice” in some sense, we run into issues when our sets have anomalies. We are often interested in studying the distance between some model space and a set which arises as the limit of an evolution of manifolds. However, many limiting sets have strange properties, developing spikes and holes that will inflate the Gromov-Hausdorff distance between that set and a model space. For example, in Figure 2.3, the Gromov-Hausdorff distance between X and Y will remain large no matter how we reposition these sets in relationship to each other because of the line protruding from Y, despite the fact that what we might consider the “majority” of X and Y can overlap perfectly.

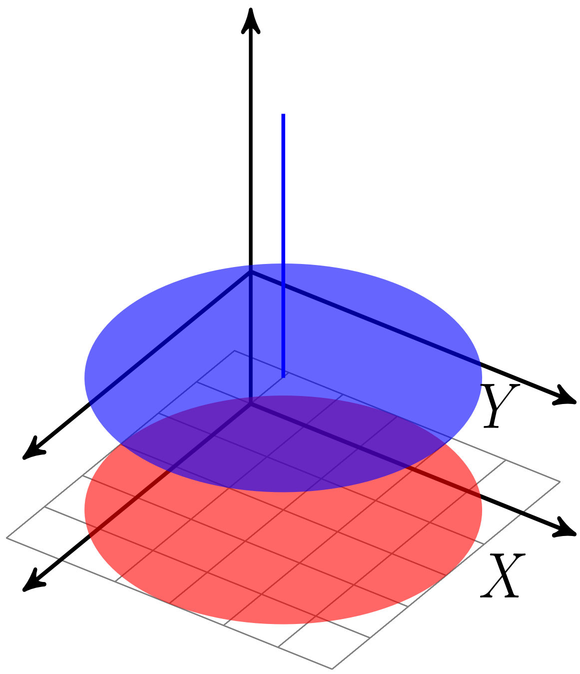

Figure 2.3. *Two sets in $\mathbb{E}^3$. X, which lies on the xy-plane, is a simple disk. On the other hand, Y, which lies in the $z = 1$ plane, is a disk of the same radius as X with a line attached at its center. The addition of the line means that these two sets will have large Gromov-Hausdorff distance, despite the fact that all of their 2-dimensional space is identical.*

To deal with this case, we want to construct a way of measuring the distance between these two sets that would allow us to forget about the spike on Y – more generally, we want a weak form of distance that allows us to ignore lower dimensional subsets. In order to do so, we need to introduce a few technical concepts. The first thing we need to address is how to determine the dimension of a set, and in order to do this, we need a way of measuring a set without first knowing its dimension.

We define the η-dimensional Hausdorff measure of measurable sets $E \subset \mathbb{E}^n$ as

$$m_\eta(E) := \lim_{\delta \to 0} \inf \left\{ \sum_i (\mathrm{diam} U_i)^\eta \mid E \subset \bigcup_i U_i, \mathrm{diam} U_i \leq \delta \text{ for all } k \right\}. \tag{2.2.4}$$

Note that we take the collection $\{U_i\}$ to be a countable cover of E but make no assumptions on the dimensions of these sets. All we measure with regard to the sets U_i is their Euclidean diameter, which can be interpreted as a 1-dimensional measure of these sets. We turn that 1-dimensional measure into an η-dimensional measure when we take the η power of $\mathrm{diam} U_i$. Note that as η gets larger, $(\mathrm{diam} U_i)^\eta$ gets smaller for small diameters. This is instructive when we consider the values of $m_\eta(E)$ for different values of η and the same set E; indeed, if the dimension of E is less than the dimension of the measure η, $m_\eta(E) = 0$ as the diameters will be shrunk too much by the power η; on the other hand, if the dimension of E is greater than the dimension η, we have $m_\eta(E) = \infty$ and the diameters are not shrunk enough to allow finite convergence. We formalize this observation in the definition of the Hausdorff dimension of E as

$$\dim_H(E) := \inf\{\eta \geq 0 \mid m_\eta(E) = 0\}. \tag{2.2.5}$$

Hausdorff dimension is particularly useful for describing irregular sets, such as fractals, which may have fractional Hausdorff dimension. For details and examples, see [SS05].

We are interested in the Hausdorff dimension of a set as a way of detecting the dimension of a set without having to assume it ahead of time. With this tool in hand we may move on to our second technical definition. Federer and Fleming defined the notion of the integral current on Euclidean space, and then Ambrosio and Kirchheim extended the notion to metric spaces [Fed96; AK00]. We are interested in a particular class of integral currents that may be described in the following way. Let $\mathcal{A} = \{A_i, \psi_i, z_i\}$ be a collection of Borel sets with maximum Hausdorff dimension $d \in \mathbb{Z}$, Lipschitz maps $\psi_i : A_i \to \mathbb{E}^D$ for D large enough with pairwise disjoint images, and $z_i \in \mathbb{Z}$. We then construct the d-dimensional integral current $T_{\mathcal{A}}$ as

$$T_{\mathcal{A}}(\omega) = \sum_i z_i \int_{A_i} \psi_i^* \omega \tag{2.2.6}$$

for $\omega \in \Lambda^d \mathbb{E}^D$.

Under this formulation, we may now define two 2-dimensional integral currents to represent our example spaces X (the disk) and Y (the disk with the spike):

$$T_X(\omega) = \int_X \psi_X^* \omega \quad \text{and} \quad T_Y(\omega) = \int_Y \psi_Y^* \omega \tag{2.2.7}$$

for the maps $\psi_X : X \to \mathbb{E}^3$ and $\psi_Y : Y \to \mathbb{E}^3$ as in Fig 2.3. We see that, as integral currents, $T_X(\omega) = T_Y(\omega)$ for any $\omega \in \Lambda^2 \mathbb{E}^3$ because these sets are the same except on a set of measure zero.

We may follow the theory of Federer and Fleming [Fed96] to develop the representation of manifolds as integral currents into a notion of convergence for manifolds. We say that a sequence d-dimensional manifolds $\{M_i, \psi_i\}$, $\psi_i : M_i \to \mathbb{E}^D$, converges weakly to an d-dimensional integral current T if for any d-form on $\mathbb{E}^D$ we have

$$\int_{M_i} \psi_i^* \omega \to T(\omega). \tag{2.2.8}$$

Thus, we obtain a notion of convergence for manifolds that is robust to the limiting set not necessarily being a manifold. To learn more about this convergence and its application to the classic Plateau problem, see [DHS10].

While the weak distance compares the integral currents generated by manifolds as linear functionals, we may define another notion of distance for integral currents, known as the flat distance. The flat distance, originally defined by Federer and Fleming by building upon the work of Whitney, measures the difference between two integral currents by analyzing the relationship between the manifolds that generate them. Let M_1 and M_2 be two d-dimensional submanifolds of $\mathbb{E}^{d+1}$. We may relate these submanifolds to each other by choosing a d-dimensional submanifold A so that $A \cup M_1 \cup M_2$ is closed with empty boundary; we choose

A with the correct orientation so that Stoke's Theorem can be applied. We then set B to be the interior of this union: the compact, d-dimensional region for which $\partial B = A \cup M_1 \cup M_2$. We now define the Euclidean flat distance between the integral currents T_1 and T_2 generated by M_1 and M_2 as

$$d_F^{\mathbb{E}^d}(T_1, T_2) = \inf_{A,B}\{m_d(A) + m_{d+1}(B) \mid T_1 - T_2 = A + \partial B\}, \tag{2.2.9}$$

where m_d is the d-dimensional Lebesgue measure. The flat distance is quite nice to work with, as evaluating $m_d(A) + m_{d+1}(B)$ for test sets A and B provides an upper estimate.

In 2000, Ambrosio and Kirchheim generalized the notions of integral currents and the weak convergence of integral currents to metric spaces, which required them to generalize the notion of differential forms to spaces without smooth structures [AK00]. Seven years later, Wenger generalized the definition of flat distance to integral currents over metric spaces. Moreover, Wenger showed that under the conditions of the Federer-Fleming compactness theorem, flat and weak convergence agree in general metric spaces [Wen06]. The coincidence of these two distances allows us to define a new distance which combines the nice properties of the weak distance with the tractability of the flat distance and is obtained by generalizing the flat distance in the same spirit as the Gromov-Hausdorff generalization of the Hausdorff distance.

Given two compact, oriented, d-dimensional Riemannian manifolds, we may choose isometric embeddings $\psi_i : M_i \to Z$ to a common metric space Z, and define the intrinsic flat distance between them as

$$d_{\mathcal{F}}(M_1, M_2) = \inf_{\psi_i, Z}\{d_F^Z(\psi_{1\#}[M_1], \psi_{2\#}[M_2])\} \tag{2.2.10}$$

where $\psi_{i\#}[M_i]$ denotes the pushforward by ψ_i of the integral current $[M_i]$ to Z. This notion of distance was established by Sormani and Wenger in 2011 [SW11] and is the one we choose to evaluate how similar our Jang surface is to a time-slice of Schwarzschild spacetime.

Perhaps after studying the definition, one might wonder in what way estimating the intrinsic flat distance is tractable. Indeed, it is often impossible to compute the exact intrinsic flat distance between two manifolds, as it necessitates analysis over all possible isometric embeddings of the two manifolds into all possible metric spaces. However, finding upper estimates for the intrinsic flat distance has proven to be fertile ground. Such upper estimates are developed by constructing a suitably nice metric space with suitably nice embeddings, the flat distance in which can be estimated by quantities determined by the original manifolds.

In the spirit of the example shown in Figure 2.3, these estimates also must account for the fact that these two manifolds might have relatively small sets on which the manifolds are very different, and do so by distinguishing between a "good" region on which the manifolds are similar, and a "bad" region on which the manifolds are different. The intrinsic flat distance is then estimated by the difference on the good region, which is small because this is where the manifolds are similar, and the difference on the bad region, which is controlled by the size of the bad region. One significant such result is the Lakzian-Sormani estimate [LS12]; this is the estimate that, in 2020, Bryden, Khuri, and Sormani used to prove stability for the Positive Mass Theorem [BKS20]. In 2020, Allen, Perales, and Sormani proved another such estimate which is called Volume Above Distance Below [APS20]. Also in 2020, Allen and Perales adapted the argument in [APS20] to the case of Riemannian manifolds with continuous metrics and boundary [AP20]. We quote this last result here, as it is the result we will leverage in order to get our estimates on intrinsic flat distance.

Theorem 2.2.1 (Volume Above, Distance Below with Boundary)**.** *Let U be an oriented and compact manifold, $U_1 = (U, g_1)$ and $U_0 = (U, g_0)$ be continuous Riemannian manifolds with $Diam(U_1) \leq D$, $Vol(U_1) \leq V$, $Vol(\partial U_1) \leq A$ and $F_1 : U_1 \to U_0$ a biLipschitz and distance non-increasing map with a C^1 inverse. Let $W_1 \subset U_1$ be a measurable set with*

$$Vol(U_1 \setminus W_1) \leq V_1 \tag{2.2.11}$$

and assume there exists an $\alpha_1 > 0$ so that for all $x, y \in W_1$,

$$d_1(x, y) \leq d_0(F_1(x), F_1(y)) + 2\alpha_1 \tag{2.2.12}$$

and that $h_1 \geq \sqrt{2\alpha_1 D + \alpha_1^2}$. Then

$$d_{\mathcal{F}}(U_0, U_1) \leq 2V_1 + h_1 V + h_1 A. \tag{2.2.13}$$

The theorem is proven by constructing an explicit metric space Z in which U_0, U_1 embed in a distance preserving manner. Although we will find that the diffeomorphism $F_1 : \Sigma \to \mathrm{Sch}(m_0)$ which restricts to a map $F_1 : U^A_{a(\epsilon_1)} \to \widetilde{U}^A_{a(\epsilon_1)}$ is not necessarily distance decreasing, spherical symmetry allows us to construct an intermediary metric space for which there exists a distance non-increasing diffeomorphism. To show this, we will reference the following lemma from [APS20].

Lemma 2.2.1. *Let $U_1 = (U, g_1)$ and $U_0 = (U, g_0)$ be Riemannian manifolds and $F : U_1 \to U_0$ be a C^1 diffeomorphism. Then*

$$g_0(dF(v), dF(v)) \leq g_1(v, v) \text{ for all } v \in TU_1 \tag{2.2.14}$$

if and only if

$$d_0(F(p), F(q)) \leq d_1(p, q) \text{ for all } p, q \in U_1. \tag{2.2.15}$$

We note that the proof of this lemma may be extended to continuous metrics. For a more thorough overview of the intrinsic flat distance, we refer the reader to Section 2.3 of [APS20].

We add an extra term to the intrinsic flat distance to obtain the volume preserving intrinsic flat distance. This term controls for the convergence of the global volumes:

$$d_{\mathcal{VF}}(\Omega, \widetilde{U}^A_{a(\epsilon_1)}) = d_{\mathcal{F}}(\Omega, \widetilde{U}^A_{a(\epsilon_1)}) + |\mathrm{Vol}_g(\Omega) - \mathrm{Vol}_{g'}(\widetilde{U}^A_{a(\epsilon_1)})|. \tag{2.2.16}$$

Portegies and Jauregue-Lee have shown that volume preserving intrinsic flat convergence implies that the volumes of balls centered on a convergent sequence converge to the volume of the ball in the limit space centered at the limit point [JL19; Por15]. This distance is technically stronger than the unmodified intrinsic flat convergence, but in our case the work needed to upgrade intrinsic flat convergence to volume preserving intrinsic flat convergence is minimal.

Chapter 3

Stability Result

We may now proceed to stating and proving a stability result for the Penrose inequality in spherical symmetry.

3.1 Problem Setup

We follow the setup established in Chapter 2, wherein we take an n-dimensional spherically symmetric initial data set for the Einstein equations (M, g, k), where (M, g) is a Riemannian manifold and k is a symmetric 2-tensor. Recall that in spherical symmetry, we may write g and k as

$$g = g_{11}(r)dr^2 + \rho^2(r)d\Omega^2 \quad \text{and} \quad k_{ij} = \eta_i\eta_j k_n(r) + (g_{ij} - \eta_i\eta_j)k_t(r) \tag{3.1.1}$$

for radial functions g_{11}, ρ, k_n, and k_t, and $d\Omega^2$ the round metric on $\mathbb{S}^{n-1}$. To prove the Penrose inequality, we assumed the apparent horizon condition given by (2.1.8). However, we need a slightly stronger assumption to prove stability; we must additionally assume that the null expansions are uniformly bounded:

$$0 < \Theta_\pm(r) \leq C < \infty. \tag{3.1.2}$$

We may derive from (3.1.2) that

$$0 < H_{S_r} = 2\frac{\rho_{,r}}{\rho\sqrt{g_{11}}} \leq C, \tag{3.1.3}$$

where H_{S_r} is the mean curvature to spheres in g. The asymptotics (1.1.11) guarantee that the null expansions decay at infinity, so the uniform boundedness condition prevents singularites in the radial length from appearing in the interior.

Recall additionally that the half area radius is given by

$$m_0 = \frac{1}{2}\left(\frac{|\partial M|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} \tag{3.1.4}$$

where $|\partial M|$ denotes the area of the boundary, and that, in arc length coordinates, we may write the ADM mass as the limit of the Hawking mass as $s \to \infty$ of spheres of radius s, denoted by S_s:

$$m_{ADM} = \lim_{s\to\infty} m(s) = \lim_{s\to\infty} \frac{1}{2}\left(\frac{|S_s|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} \left[1 - \frac{1}{(n-1)^2(\omega_{n-1}^2|S_s|^{n-3})^{\frac{1}{n-2}}}\int_{S_s} H_{S_s}^2\right] \quad (3.1.5)$$

where ω_{n-1} is the area of the $(n-1)$-sphere and H_{S_s} is the mean curvature of the sphere of radius s in M.

To show the stability of the Penrose inequality given by (1.1.12), we draw analogy to the way in which the equality case is established in [BK10]. If $\bar{g} = g_S$ is the Schwarzschild metric, $\phi = \phi_S$, and $\pi - k = 0$, then the map G found from the Jang equation gives (M, g) as the image of an isometric embedding of Σ into Schwarzschild spacetime with second fundamental form k. To prove stability, we want to show that (M, g) is close to $(\mathrm{Sch}(m_0), g_S)$, ϕ is close to ϕ_S, and k is close to π. Once these are established, we may conclude that (M, g, k) is close to being realized as an embedded time-slice of Schwarzschild spacetime where we denote the Schwarzschild space with half area radius m_0 and metric g_S by $(\mathrm{Sch}(m_0), g_S)$.

In order to show that (M, g) is close to $(\mathrm{Sch}(m_0), g_S)$ in the intrinsic flat sense, we must develop a consistent way of referring to important subsets of these spaces. We can write the Jang metric in radial coordinates

$$\bar{g} = \frac{1}{\rho_{,s}^2} d\rho^2 + \rho^2 d\Omega^2. \quad (3.1.6)$$

That ρ is strictly increasing implies that, in these coordinates, there is a bijection between the set of radial values and the set of areas, so distinguishing a spherically symmetric subset by its radial values is the same as distinguishing it by the areas of the spheres it contains: the unique sphere of radius ρ has area $\rho^{n-1}\omega_{n-1}$. In this way, we see that these coordinates have geometric meaning and represent a valid way of establishing our sets of interest.

We wish to isolate the minimal radial value. Again, as ρ is strictly increasing, this will be the radial value corresponding to the area of the innermost boundary. We chose the Jang data to preserve the half area radius m_0, so we may compute that, in spherical symmetry, the minimal radial value is

$$\rho_0 = (2m_0)^{1/(n-2)}. \quad (3.1.7)$$

It follows that a point $x \in \Sigma$ can be written as $x = (\rho, \theta)$ for $\rho \in [\rho_0, \infty)$ and $\theta \in \mathbb{S}^{n-1}$. We compare to Schwarzschild space with the same half area radius m_0, given in radial coordinates by

$$(\mathrm{Sch}(m_0), g_S) = \left([\rho_0, \infty] \times \mathbb{S}^{n-1}, \frac{1}{\phi_S^2} d\rho^2 + \rho^2 d\Omega^2\right) \quad (3.1.8)$$

where $\phi_S = \sqrt{1 - \frac{2m_0}{\rho^{n-2}}}$, $d\Omega^2$ is the standard metric on the $(n-1)$-sphere, and $\rho_0^{n-2} = 2m_0$. There is thus a natural diffeomorphism $F_1 : \Sigma \to \mathrm{Sch}(m_0)$ so that $(\rho, \theta) \mapsto (\rho, \theta)$.

We distinguish spheres and annuli in $(\Sigma, \bar{g})$ and $(\mathrm{Sch}(m_0), g_S)$ with the convention that, if S is a set in Σ, $\widetilde{S}$ is a set in $\mathrm{Sch}(m_0)$:

$$\begin{aligned} S_A &= \{(\rho,\theta) \in \Sigma \,:\, \rho = A\}, & U_a^A &= \{(\rho,\theta) \in \Sigma \,:\, a \le \rho \le A\}, \\ \widetilde{S}_A &= \{(\rho,\theta) \in \mathrm{Sch}(m_0) \,:\, \rho = A\}, & \widetilde{U}_a^A &= \{(\rho,\theta) \in \mathrm{Sch}(m_0) \,:\, a \le \rho \le A\}, \end{aligned} \quad (3.1.9)$$

and let

$$T_D(S_A) = \{x \in \Sigma \mid d_{\bar{g}}(x,y) \leq D \text{ for } y \in S_A\} \tag{3.1.10}$$

be the tubular neighborhood of radius D around S_A.

We choose $A >> \rho_0$. Given the anchor surface S_A, we select the annulus on which we prove intrinsic flat convergence. To do so, we must be careful of how radially long our annulus is. We must also take care that the inner boundary of our annulus is not too close to the boundary of the Jang surface. We first control for the radial length by choosing the distance from which we measure down on the anchor surface S_A. Let

$$\widetilde{D} = \min\left\{d_{g_S}(x,y) \mid x \in \widetilde{S}_A, y \in \partial\text{Sch}(m_0) = \widetilde{S}_{\rho_0}\right\} \tag{3.1.11}$$

be the distance between the anchor surface in $\text{Sch}(m_0)$ and $\partial\text{Sch}(m_0)$. We set

$$a(\epsilon_1) := \min\{\rho \mid (\rho,\theta) \in T_{\widetilde{D}}(S_A)\} + \epsilon_1 \tag{3.1.12}$$

for a fixed $\epsilon_1 > 0$. The constant ϵ_1 controls for the case that the radial distance between S_A and $\partial\Sigma$ is $\widetilde{D}$ or less. Specifically, we are preventing $T_{\widetilde{D}}(S_A) \cap U^A_{\rho_0} = U^A_{\rho_0}$: by adding ϵ_1 we guarantee that the inclusion $U^A_{a(\epsilon_1)} \subset U^A_{\rho_0}$ is strict. See Figure 3.1 for an illustration of $U^A_{a(\epsilon_1)}$ in Σ.

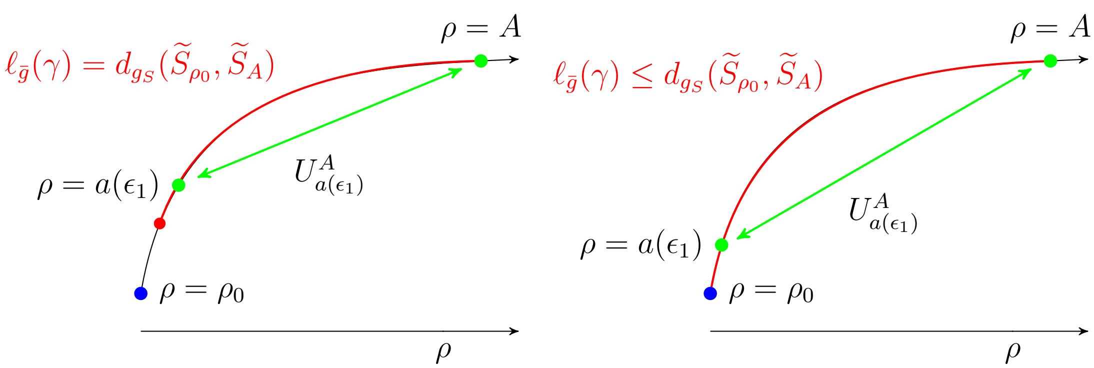

Figure 3.1. *Here we have sketched radial cross-sections of Σ illustrating the two different cases that might occur when choosing the set $U^A_{a(\epsilon_1)}$. The first, pictured left, is that the radial distance between S_{ρ_0} and S_A is greater than $d_{g_S}(\widetilde{S}_{\rho_0}, \widetilde{S}_A)$. In this case, the radial geodesic γ that drops down toward S_{ρ_0} and has length $\ell_{\bar{g}}(\gamma) = d_{g_S}(\widetilde{S}_{\rho_0}, \widetilde{S}_A)$ will terminate at some radial value bigger than ρ_0, depicted by the red dot. The set $U^A_{a(\epsilon_1)}$ is then the part of Σ in between the green dots and described by the green arrow. In the second case, pictured right, the radial distance between $S_{\rho_)}$ and S_A is less than or equal to $d_{g_S}(\widetilde{S}_{\rho_0}, \widetilde{S}_A)$ so that $\ell_{\bar{g}}(\gamma) \leq d_{g_S}(\widetilde{S}_{\rho_0}, \widetilde{S}_A)$ and γ terminates at ρ_0. It is this second case in which the control by ϵ_1 is crucial for ensuring the set $U^A_{a(\epsilon_1)}$ does not extend to the boundary. In both cases, the radial depth of $U^A_{a(\epsilon_1)}$ is strictly less than $d_{g_S}(\widetilde{S}_{\rho_0}, \widetilde{S}_A)$ in a manner controlled by ϵ_1.*

We define a restricted Jang space by

$$\Sigma_{\epsilon_1} := \{(\rho,\theta) \in \Sigma \mid a(\epsilon_1) \leq \rho\}\,. \tag{3.1.13}$$

The construction of the subsets $U^A_{a(\epsilon_1)}$ and Σ_{ϵ_1} succeed in keeping our study of convergence away from the boundary where we do not have specific control over the geometry. We have chosen $U^A_{a(\epsilon_1)}$ specifically to control the radial length, which prevents us from falling down a cylindrical region near the boundary, see Figure 3.1. However, to justify this restriction, we need to include as part of our argument that $a(\epsilon_1)$ approaches ρ_0 as we take ϵ_1 and δ to zero. We are now ready to state our theorem.

Theorem 3.1.1. *Suppose (M, g, k) is a spherically symmetric initial data set satisfying the dominant energy condition* (1.1.10)*, asymptotic flatness given by* (1.1.11)*, and the uniform bounded outermost expansion condition* (3.1.2)*. Further, suppose that $m_{ADM} = (1 + \delta)m_0$. Then, there exists a Riemannian manifold $(\Sigma, \bar{g})$ diffeomorphic to (M, g) with graphical isometric embeddings*

$$\begin{gathered} G : (M,g) \to (\Sigma \times \mathbb{R}, \bar{g} - \phi^2 dt^2), \quad G(x) = (x, f(x)), \text{ and} \\ g = G^*(\bar{g} - \phi^2 dt^2) = \bar{g} - \phi^2 df^2 \end{gathered} \tag{3.1.14}$$

for ϕ as in (2.1.24) *such that the static spacetime $(\Sigma \times \mathbb{R}, \bar{g} - \phi^2 dt^2)$ converges to Schwarzschild spacetime*

$$\left(Sch(m_0) \times \mathbb{R}, g_S - \phi^2 dt^2\right) \tag{3.1.15}$$

in the sense that the restricted base manifold Σ_{ϵ_1} is close to $Sch(m_0)$ for ϵ_1 and δ very small. In other words, for each arbitrarily small $\epsilon > 0$ there exists ϵ_1 and δ small enough such that

$$d_{\mathcal{VF}}(U^A_{a(\epsilon_1)}, \widetilde{U}^A_{a(\epsilon_1)}) < \epsilon \tag{3.1.16}$$

where $d_{\mathcal{VF}}(,)$ is the volume preserving intrinsic flat distance. Moreover, we have the following convergences of the warping factor and the second fundamental form:

$$\|\phi - \phi_S\|_{L^2_{loc}(U^A_{a(\epsilon_1)}, \bar{g})} \to 0 \text{ and } \left\|\sqrt{\phi}(k - \pi)\right\|_{L^2(\Sigma, \bar{g})} \to 0. \tag{3.1.17}$$

Remark 3.1.1. We stress that our data (M, g, k) and subsequent Jang surface depends on δ, so that the uniformity of our assumptions is determined with respect to δ. It is possible to prove the theorem for a sequence of inital data (M_j, g_j, k_j); moreover, the theorem holds if the $m^j_0 = m_0(M_j)$ is allowed to vary as long as we assume that $m^j_{ADM} = (1 + \delta_j)m^j_0 \to m_0$ for a sequence of δ_j converging to zero. In particular, this forces $m^j_0 \to m_0$. However, these considerations require more complicated notation and technical details without any fundamental differences of proof, so we omit them from this paper.

3.2 Preliminaries

We now have all the background we need to set up the proof of Theorem 3.1.1. Recall that $U^A_{\rho_0} = \{(\rho, \theta) \in \Sigma \mid \rho_0 \le \rho \le A\}$ and $U^A_{a(\epsilon_1)} = \{(\rho, \theta) \in \Sigma \mid a(\epsilon_1) \le \rho \le A\}$ where

$$a(\epsilon_1) := \min\{\rho \mid (\rho, \theta) \in T_{\widetilde{D}}(S_A)\} + \epsilon_1 \tag{3.2.1}$$

for

$$\widetilde{D} = \min\left\{d_{gS}(x,y) \mid x \in \widetilde{S}_A, y \in \partial\mathrm{Sch}(m_0) = \widetilde{S}_{\rho_0}\right\}. \tag{3.2.2}$$

Recall also that we may write our Jang metric as

$$\bar{g} = \frac{1}{\rho_{,s}^2} d\rho^2 + \rho^2 d\Omega^2. \tag{3.2.3}$$

We begin with some preliminary results to control the warping factor ϕ and the divergence term in the scalar curvature equation (2.1.10). The control over the divergence term is essential for the remainder of our argument. As a consequence, we prove the local convergence of ϕ to ϕ_S. Our first result gives a local upper bound for ϕ, for which we need the bounded expansion condition.

Lemma 3.2.1. *If we assume the initial data (M, g, k) are smooth, asymptotically flat, spherically symmetric, and satisfy the apparent horizon condition, we can solve the Jang equation for $\phi = \rho_{,s}$ where s is the arc length parameter on $(\Sigma, \bar{g})$. Additionally, if we assume the bounded expansion condition* (3.1.2)*, we have that*

$$\phi \leq AC \tag{3.2.4}$$

on $U^A_{\rho_0}$.

Proof. We can re-write ϕ in r coordinates by noting that

$$\frac{dr}{ds} = \frac{1}{\sqrt{g_{11} + \phi^2 f_{,r}^2}} = \frac{\sqrt{1 - v^2}}{\sqrt{g_{11}}} \tag{3.2.5}$$

so that, because $\rho_{,r}$ is positive,

$$\phi = \frac{\sqrt{1 - v^2}}{\sqrt{g_{11}}}\rho_{,r} \leq \frac{\rho_{,r}}{\sqrt{g_{11}}}. \tag{3.2.6}$$

As described previously, we can deduce from the the bounded expansion condition (3.1.2) that

$$H^g_{S_r} = \frac{2\rho_{,r}}{\rho\sqrt{g_{11}}} \leq C \tag{3.2.7}$$

and thus at any given $r \in \Sigma$, $\frac{\rho_{,r}}{\sqrt{g_{11}}} \leq \rho C$ so that on $U^A_{\rho_0}$,

$$\phi \leq \frac{\rho_{,r}}{\sqrt{g_{11}}} \leq AC. \tag{3.2.8}$$

□

Now we seek to control the divergence term in the expression for scalar curvature. Following (2.1.31), we have in radial coordinates that

$$\begin{aligned} m(\rho) - m_0 &= \frac{1}{2\omega_{n-1}(n-1)} \int_{U^\rho_{\rho_0}} \rho_{,s} \bar{R} d\omega_{\bar{g}} \\ &= \frac{1}{2\omega_{n-1}(n-1)} \int_{U^\rho_{\rho_0}} \phi \left(16\pi(\mu - J(w)) + |h - K|_\Sigma|^2_{\bar{g}} + 2|q|^2_{\bar{g}}\right) d\omega_{\bar{g}} \\ &\quad - \frac{1}{\omega_{n-1}(n-1)} \int_{S_\rho} \phi \bar{g}(q, n_{\bar{g}}) d\sigma_{\bar{g}} \\ &= \frac{1}{2\omega_{n-1}(n-1)} \int_{U^\rho_{\rho_0}} \phi \left(16\pi(\mu - J(w)) + |h - K|_\Sigma|^2_{\bar{g}} + 2|q|^2_{\bar{g}}\right) d\omega_{\bar{g}} \\ &\quad - \frac{\rho^{n-1}}{n-1} \phi^2 q_1 \end{aligned} \tag{3.2.9}$$

where

$$q_1 := q(\partial_\rho). \tag{3.2.10}$$

We may therefore write

$$m(\rho) = P(\rho) + Q(\rho) + m_0 \tag{3.2.11}$$

where

$$P(\rho) = \frac{1}{2\omega_{n-1}(n-1)} \int_{U^\rho_{\rho_0}} 16\pi(\mu - J(w)) + |h - K|_\Sigma|^2_{\bar{g}} + 2|q|^2_{\bar{g}} d\omega_{\bar{g}} \tag{3.2.12}$$

and

$$Q(\rho) = -\frac{\rho^{n-1}}{n-1} \phi^2 q_1. \tag{3.2.13}$$

We can see by this definition that P is a positive, increasing function of ρ. The computations in Theorem 2.1.1 tell us that

$$\delta m_0 = m_{ADM} - m_0 = \lim_{\rho \to \infty} P(\rho) \tag{3.2.14}$$

so that $0 \leq P(\rho) \leq \delta m_0$. Our next lemma gives control over Q.

Lemma 3.2.2. *If the initial data (M, g, k) are smooth, asymptotically flat as in* (1.1.11)*, and spherically symmetric and satisfy the dominant energy condition* (1.1.10) *and bounded expansion condition* (3.1.2)*, we have*

$$\|Q\|^2_{L^2(U^A_{\rho_0}, \bar{g})} \leq \delta \omega_{n-1} m_0 A^{2n-1} C \tag{3.2.15}$$

and, in particular, $\|Q\|_{L^2(U^A_{\rho_0}, \bar{g})} \to 0$ as $\delta \to 0$.

Proof. By the analysis in Theorem 2.1.1,

$$\begin{aligned}\delta m_0 &= m_{ADM} - m_0 \\ &\geq \frac{1}{2\omega_{n-1}(n-1)} \int_\Sigma \phi \left(16\pi(\mu - J(w)) + |h - K|_\Sigma|_{\bar{g}}^2 + 2|q|_{\bar{g}}^2\right) d\omega_{\bar{g}} \\ &\geq \frac{1}{\omega_{n-1}(n-1)} \int_{U_{\rho_0}^A} \phi|q|_{\bar{g}}^2 d\omega_{\bar{g}} \\ &= \frac{1}{\omega_{n-1}(n-1)} \int_{U_{\rho_0}^A} \phi^3 q_1^2 d\omega_{\bar{g}}.\end{aligned} \tag{3.2.16}$$

From this inequality, we get that

$$\begin{aligned}\delta\omega_{n-1} m_0(n-1) &\geq \int_{U_{\rho_0}^A} \left(\frac{\rho^{n-1}}{n-1}\phi^2 q_1\right)^2 \frac{(n-1)^2}{\rho^{2(n-1)}\phi} d\omega_{\bar{g}} \\ &\geq \frac{(n-1)^2}{A^{2n-1}C} \int_{U_{\rho_0}^A} |Q|^2 d\omega_{\bar{g}}\end{aligned} \tag{3.2.17}$$

where in the last line we have used (3.2.4). It follows that

$$\|Q\|_{L^2(U_{\rho_0}^A,\bar{g})}^2 \leq \frac{\delta\omega_{n-1} m_0 A^{2n-1} C}{n-1} \tag{3.2.18}$$

and we have the result. □

The motivation for controlling the divergence term is made more clear if we re-write the metric $\bar{g}$ in the form of a Schwarzschild metric. Indeed, if for some function $m(\rho)$ we set

$$\bar{g} = \frac{1}{\phi^2} d\rho^2 + \rho^2 d\Omega^2 = \left(1 - \frac{2m(\rho)}{\rho^{n-2}}\right)^{-1} d\rho^2 + \rho^2 d\Omega^2 \tag{3.2.19}$$

we solve to get that $m(\rho) = \frac{1}{2}\rho^{n-2}(1 - \rho_{,s}^2)$, which is exactly the Hawking mass at S_ρ as in (2.1.28). From this, and the decomposition of the Hawking mass as in (3.2.11), we see that

$$\bar{g} = \left(1 - \frac{2(m_0 + P(\rho) + Q(\rho))}{\rho^{n-2}}\right)^{-1} d\rho^2 + \rho^2 d\Omega^2 \tag{3.2.20}$$

and the difference between $\bar{g}$ and g_S is then controlled by the terms $P(\rho)$ and $Q(\rho)$, which determine the deviation of the Hawking mass from the half area radius m_0. We have seen that $0 \leq P(\rho) \leq \delta m_0$; moreover, P is increasing. Therefore, the behavior of the metric is controlled by Q, and the L^2 control of Q by δ is thus essential to proving intrinsic flat convergence.

3.3 Convergence of the warping factor and its consequences

Now we turn to the study of the warping factor ϕ and show convergence to ϕ_S as a consequence of Lemma 3.2.2. To prove this result we will need to restrict away from the boundary. Recall that

$$\phi = \sqrt{1 - \frac{2m(\rho)}{\rho^{n-2}}} \quad \text{and} \quad \phi_S = \sqrt{1 - \frac{2m_0}{\rho^{n-2}}}. \tag{3.3.1}$$

Lemma 3.3.1. *If the initial data (M, g, k) are smooth, asymptotically flat, and spherically symmetric and satisfy the dominant energy condition* (1.1.10) *and bounded expansion condition* (3.1.2)*, then for any $\epsilon_1 > 0$ and $\epsilon > 0$ there is $\delta(\epsilon_1, \epsilon) > 0$ small enough so that $\|\phi - \phi_S\|_{L^2(U^A_{a(\epsilon_1)}, \bar{g})} < \epsilon$.*

Proof. We compute

$$\int_{U^A_{a(\epsilon_1)}} |\phi - \phi_S|^2 \, d\omega_{\bar{g}} = \int_{U^A_{a(\epsilon_1)}} \frac{4}{\rho^{2(n-2)}(\phi + \phi_S)^2} |m_0 - m(\rho)|^2 \, d\omega_{\bar{g}} \tag{3.3.2}$$

by rationalizing $(\phi - \phi_S)^2$. Now, because $\rho \geq a(\epsilon_1)$, $\phi \geq 0$, and $\phi_S \geq \phi_S(a(\epsilon_1))$ on $U^A_{a(\epsilon_1)}$, we may estimate that on $U^A_{a(\epsilon_1)}$,

$$\begin{aligned} \frac{4}{\rho^{2(n-2)}(\phi + \phi_S)^2} &\leq \frac{4}{(a(\epsilon_1)^{n-2}\phi_S(a(\epsilon_1)))^2} \\ &= \frac{4}{(a(\epsilon_1))^{n-2}(a(\epsilon_1)^{n-2} - 2m_0)}. \end{aligned} \tag{3.3.3}$$

Using this and the decomposition (3.2.11), we have that

$$\begin{aligned} \int_{U^A_{a(\epsilon_1)}} |\phi - \phi_S|^2 \, d\omega_{\bar{g}} &\leq \frac{4}{(a(\epsilon_1))^{n-2}(a(\epsilon_1)^{n-2} - 2m_0)} \int_{U^A_{a(\epsilon_1)}} |Q + P|^2 d\omega_{\bar{g}} \\ &\leq \frac{8}{(a(\epsilon_1))^{n-2}(a(\epsilon_1)^{n-2} - 2m_0)} \left(\int_{U^A_{a(\epsilon_1)}} |Q|^2 d\omega_{\bar{g}} + \int_{U^A_{a(\epsilon_1)}} |P|^2 d\omega_{\bar{g}} \right). \end{aligned} \tag{3.3.4}$$

We know that $0 \leq P(\rho) \leq \delta$ for any $\rho \in \Sigma$ and we can apply the estimate (3.2.15) from Lemma 3.2.2 to see that

$$\begin{aligned} \int_{U^A_{a(\epsilon_1)}} |\phi - \phi_S|^2 \, d\omega_{\bar{g}} &\leq \frac{8}{(a(\epsilon_1))^{n-2}(a(\epsilon_1)^{n-2} - 2m_0)} \left(\delta \omega_{n-1} m_0 A^{2n-1} C + \int_{U^A_{a(\epsilon_1)}} \delta^2 d\omega_{\bar{g}} \right) \\ &\leq \frac{8}{(a(\epsilon_1))^{n-2}(a(\epsilon_1)^{n-2} - 2m_0)} \left(\delta \omega_{n-1} m_0 A^{2n-1} C + \delta^2 \omega_{n-1} A^{n-1} \widetilde{D} \right) \end{aligned} \tag{3.3.5}$$

where we have estimated $\mathrm{Vol}(U^A_{a(\epsilon_1)})$ by $\omega_{n-1} A^{n-1} \widetilde{D}$ using the coarea formula. We find therefore that for δ small enough with respect to ϵ_1 we have $\|\phi - \phi_S\|_{L^2(U^A_{a(\epsilon_1)}, \bar{g})} < \epsilon$. □

The next result describes the behavior of $a(\epsilon_1)$ as we take $\delta \to 0$. Specifically, we find that $a(\epsilon_1) \to \rho_0 + \epsilon_1$, meaning that $T_{\widetilde{D}}(S_A) \cap U^A_{\rho_0} \subset \Sigma$ will have inner boundary close to the Schwarzschild radius for small enough δ and ϵ_1.

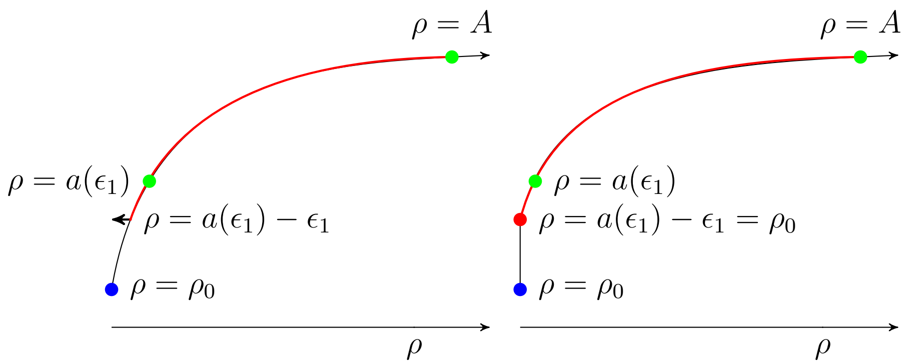

Figure 3.2. *Here we show the nontrivial case of $a(\epsilon_1)$ convergence, which occurs in the first case described in Figure 3.1. The black arrow emanating from $\rho = a(\epsilon_1) - \epsilon_1$ in the left diagram demonstrates the convergence of this value to ρ_0. The arrow is drawn horizontally to make it clear that, while this convergence occurs by Lemma 3.3.2, the set $U_{\rho_0}^{a(\epsilon_1)-\epsilon_1}$ will not necessarily collapse. In the case that it does not collapse, this set converges to a cylindrical end, as shown in the diagram on the right.*

Lemma 3.3.2. *If the initial data (M, g, k) are smooth, asymptotically flat, and spherically symmetric and satisfy the dominant energy condition* (1.1.10) *and bounded expansion condition* (3.1.2)*, then as $\delta \to 0$ we have that $a(\epsilon_1) \to \rho_0 + \epsilon_1$.*

Proof. Recall that we have defined

$$a(\epsilon_1) := \min\{\rho \mid (\rho, \theta) \in T_{\widetilde{D}} \cap U_{a(\epsilon_1)}^A\} + \epsilon_1 \tag{3.3.6}$$

where $\widetilde{D}$ is the Schwarzschild distance between $\widetilde{S}_A$ and $\partial\text{Sch}(m_0)$. In the case that the radial depth of $U_{\rho_0}^A$ is less than or equal to $\widetilde{D}$, we have that $a(\epsilon_1) = \rho_0 + \epsilon_1$. Otherwise, $a(\epsilon_1) > \rho_0 + \epsilon_1$. In that case, set

$$\rho_\delta = a(\epsilon_1) - \epsilon_1 = \min\{\rho \mid (\rho, \theta) \in T_{\widetilde{D}} \cap U_{\rho_0}^A\}. \tag{3.3.7}$$

We assume, for contradiction, that convergence does not occur, so that $\rho_\delta - \rho_0 \geq c$ as $\delta \to 0$ for some small $c > 0$. By definition of $U_{\rho_\delta}^A = T_{\widetilde{D}} \cap U_{\rho_0}^A$, we have that the radial distance between S_A and $S_{\rho_\delta} \subset \Sigma$ is

$$\int_{\rho_\delta}^A \frac{1}{\phi} d\rho = \widetilde{D} \tag{3.3.8}$$

where $S_A \cup S_{\rho_\delta} = \partial U_{a(\epsilon_1)}^A$. We use this to estimate the difference between $\widetilde{D}$ and the radial distance between $\widetilde{S}_A$ and $\widetilde{S}_{\rho_\delta} \subset \mathrm{Sch}(m_0)$ as follows:

$$\begin{aligned} \left|\widetilde{D} - \int_{\rho_\delta}^{A} \frac{1}{\phi_S} d\rho\right| &\leq \int_{\rho_\delta}^{A} \left|\frac{1}{\phi} - \frac{1}{\phi_S}\right| d\rho \\ &= \int_{\rho_\delta}^{A} \frac{1}{\phi\phi_S} \left|\phi_S - \phi\right| d\rho \\ &\leq \frac{1}{\omega_{n-1}} \int_{\rho_\delta}^{A} \int_{S_\rho} \frac{1}{\phi_S \rho^{n-1}} |\phi_S - \phi| \frac{\rho^{n-1}}{\phi} d\rho d\sigma_{\bar{g}} \\ &\leq \frac{1}{\omega_{n-1}\phi_S(\rho_\delta)\rho_\delta^{n-1}} \int_{U^A_{\rho_\delta}} \left|\phi_S - \phi\right| d\omega_{\bar{g}} \end{aligned} \tag{3.3.9}$$

where in the last line we have used that $d\omega_{\bar{g}} = \frac{1}{\phi}\rho^{n-1} d\rho d\sigma_{\bar{g}}$ and estimated from above on $U^A_{\rho_\delta}$. By Hölder's inequality and the coarea formula,

$$\begin{aligned} \left|\widetilde{D} - \int_{\rho_\delta}^{A} \frac{1}{\phi_S} d\rho\right|^2 &\leq \left(\frac{1}{\omega_{n-1}\phi_S(\rho_\delta)\rho_\delta^{n-1}}\right)^2 \mathrm{Vol}_{\bar{g}}(U^A_{\rho_\delta}) \int_{U^A_{\rho_\delta}} \left|\phi_S - \phi\right|^2 d\omega_{\bar{g}} \\ &\leq \left(\frac{1}{\omega_{n-1}\phi_S(\rho_\delta)\rho_\delta^{n-1}}\right)^2 \left(A^{n-1}\widetilde{D}\right) \int_{U^A_{\rho_\delta}} \left|\phi_S - \phi\right|^2 d\omega_{\bar{g}}. \end{aligned} \tag{3.3.10}$$

Our assumption that ρ_δ stays at least c away from ρ_0 means that

$$\phi_S(\rho_\delta) = \sqrt{1 - \frac{\rho_0^{n-2}}{\rho_\delta^{n-2}}} \geq \sqrt{\frac{(\rho_0 + c)^{n-2} - \rho_0^{n-2}}{(\rho_0 + c)^{n-2}}} > 0 \tag{3.3.11}$$

and so by Lemma 3.3.1 the right hand side becomes arbitrarily small as $\delta \to 0$. It follows that

$$\int_{\rho_\delta}^{A} \frac{1}{\phi_S} d\rho \to \widetilde{D} = \int_{\rho_0}^{A} \frac{1}{\phi_S} d\rho \tag{3.3.12}$$

as $\delta \to 0$. However, $\int_{\rho_\delta}^{A} \frac{1}{\phi_S} d\rho$ increases as ρ_δ decreases with maximum $\widetilde{D}$ which is only achieved when $\rho_\delta = \rho_0$. We then have a contradiction that $\rho_\delta - \rho_0 \geq c > 0$ for all δ. □

In the next proposition, we show that $\left\|\widetilde{U}^{a(\epsilon_1)}_{\rho_0}\right\|_{\mathcal{F}}$ is arbitrarily small when ϵ_1 and δ are small enough. In other words, the "remainder" of the Schwarzschild space is small.

Proposition 3.3.1. *Suppose the initial data* (M, g, k) *are smooth, asymptotically flat, and spherically symmetric and satisfy the dominant energy condition* (1.1.10) *and bounded expansion condition* (3.1.2). *Then for any* $\epsilon > 0$, *there exist* $\epsilon_1(\epsilon)$, $\delta(\epsilon_1)$ *small enough so that* $\left\|\widetilde{U}^{a(\epsilon_1)}_{\rho_0}\right\|_{\mathcal{F}} < \epsilon$.

Proof. We have that

$$\begin{aligned}
\left\|\widetilde{U}_{\rho_0}^{a(\epsilon_1)}\right\|_{\mathcal{F}} &\leq \mathrm{Vol}_{g_S}\left(\widetilde{U}_{\rho_0}^{a(\epsilon_1)}\right) \\
&= \int_{\widetilde{U}_{\rho_0}^{a(\epsilon_1)}} d\omega_{g_S} \\
&\leq \omega_{n-1}(a(\epsilon_1))^{n-1}\int_{\rho_0}^{a(\epsilon_1)} \frac{1}{\phi_S} d\rho
\end{aligned} \tag{3.3.13}$$

where the last line follows by the coarea formula. By Lemma 3.3.2, for δ small enough we have $a(\epsilon_1) \leq \rho_0 + 2\epsilon_1$ so that

$$\begin{aligned}
\left\|\widetilde{U}_{\rho_0}^{a(\epsilon_1)}\right\|_{\mathcal{F}} &\leq \omega_{n-1}(\rho_0+2\epsilon_1)^{n-1}\int_{\rho_0}^{\rho_0+2\epsilon_1} \sqrt{\frac{\rho^{n-2}}{\rho^{n-2}-\rho_0^{n-2}}} d\rho \\
&\leq \omega_{n-1}(\rho_0+2\epsilon_1)^{n-1}\int_{\rho_0}^{\rho_0+2\epsilon_1} \sqrt{\frac{\rho}{\rho-\rho_0}}\sqrt{\frac{\rho^{n-3}}{\sum_{i=0}^{n-3}\rho^i\rho_0^{n-3-i}}} d\rho \\
&\leq \omega_{n-1}(\rho_0+2\epsilon_1)^{n-1}\sqrt{\frac{(\rho_0+2\epsilon_1)^{n-3}}{(n-3)\rho_0^{n-3}}}\int_{\rho_0}^{\rho_0+2\epsilon_1} \sqrt{\frac{\rho}{\rho-\rho_0}} d\rho
\end{aligned} \tag{3.3.14}$$

where in the last line we have applied the estimate on $[\rho_0, \rho_0 + 2\epsilon_1]$ that

$$\sqrt{\frac{\rho^{n-3}}{\sum_{i=0}^{n-3}\rho^i\rho_0^{n-3-i}}} \leq \sqrt{\frac{(\rho_0+2\epsilon_1)^{n-3}}{\sum_{i=0}^{n-3}\rho_0^i\rho_0^{n-3-i}}} \leq \sqrt{\frac{(\rho_0+2\epsilon_1)^{n-3}}{(n-3)\rho_0^{n-3}}}. \tag{3.3.15}$$

The function $\sqrt{\frac{\rho}{\rho-\rho_0}}$ is integrable on $[\rho, \rho+2\epsilon_1]$, from which it follows that for ϵ_1 and δ small enough we may obtain

$$\left\|\widetilde{U}_{\rho_0}^{a(\epsilon_1)}\right\|_{\mathcal{F}} < \epsilon. \tag{3.3.16}$$

□

3.4 Application of VADB with Boundary

In this section, we prove the next main ingredient for the theorem, ie, that

$$d_{\mathcal{F}}(U^A_{a(\epsilon_1)}, \widetilde{U}^A_{a(\epsilon_1)}) \tag{3.4.1}$$

can be made as small as we like. However, we cannot do this directly: recall that, as in (3.2.20), we can write each metric as

$$\bar{g} = \left(1 - \frac{2(m_0+P(\rho)+Q(\rho))}{\rho^{n-2}}\right)^{-1} d\rho^2 + \rho^2 d\Omega^2 \quad g_S = \left(1-\frac{2m_0}{\rho^{n-2}}\right)^{-1} d\rho^2 + \rho^2 d\Omega^2. \tag{3.4.2}$$

By inspecting the metrics, we can see that the diffeomorphism $F_1 : U^A_{a(\epsilon_1)} \to \widetilde{U}^A_{a(\epsilon_1)}$ is not distance decreasing. Indeed, we can compute that $g_S \geq \bar{g}$ when $P \geq -Q$; as we have no control over the sign of Q this is certainly a possibility. However, we may define an intermediary metric space which will admit a distance decreasing map from both sets.

Define the space $U_0 = (U, g_0)$ with U diffeomorphic to $U^A_{a(\epsilon_1)}$ and $\widetilde{U}^A_{a(\epsilon_1)}$. Define g_0 as follows. Let $V = \{(\rho,\theta) \in U \mid P(\rho) + Q(\rho) \leq 0\}$. This is a closed set composed of countably many disjoint annular regions: denote these connected components by $\{V_j\}$. For each V_j, let $\rho_j = \min\{\rho \mid (\rho,\theta) \in V_j\}$. Let $V' = \bigcup_j (V_j - S_{\rho_j})$ so that we have deleted the innermost circle from each component of V. Let g_0 be a metric on U defined as follows:

$$g_0 := \begin{cases} \bar{g} & \text{when } x \in V' \\ g_S & \text{when } x \in U \setminus V'. \end{cases} \tag{3.4.3}$$

Note that by definition, $\bar{g} = g_S$ on $\partial V'$ so that the metric g_0 is continuous and, for every annulus on which $g_0 = \bar{g}$ and $g_0 = g_S$, the metric is smooth up to the outer boundary. Moreover, $\bar{g} \leq g_S$ on V' and $g_S \leq \bar{g}$ on $U \setminus V'$ so by Lemma 2.2.1, the diffeomorphisms $J_1 : U^A_{a(\epsilon_1)} \to U_0$, $J_2 : \widetilde{U}^A_{a(\epsilon_1)} \to U_0$ for which $(\rho,\theta) \mapsto (\rho,\theta)$ are distance decreasing.

In this section, we prove the following proposition.

Proposition 3.4.1. *Suppose the initial data (M, g, k) are smooth, asymptotically flat, and spherically symmetric and satisfy the dominant energy condition* (1.1.10) *and bounded expansion condition* (3.1.2)*. Then, for any $\epsilon, \epsilon_1 > 0$ there exists $\delta(\epsilon,\epsilon_1) > 0$ small enough so that*

$$\begin{aligned} d_{\mathcal{F}}(U_0, U^A_{a(\epsilon_1)}) &< \epsilon/3, \\ d_{\mathcal{F}}(U_0, \widetilde{U}^A_{a(\epsilon_1)}) &< \epsilon/3 \end{aligned} \tag{3.4.4}$$

when $m_{ADM} = (1+\delta)m_0$.

First, we prove bounds on diameters, volumes, and volumes of the boundary for each $U^A_{a(\epsilon_1)}$, $\widetilde{U}^A_{a(\epsilon_1)}$. Then, we choose the set W on which we can get the estimate (2.2.12) and prove the distance estimate. Finally, we prove the proposition by applying Theorem 2.2.1.

3.4.1 Volumes, Areas, and Diameters

We now estimate the volumes, boundary areas, and diameters of the sets $U^A_{a(\epsilon_1)}$ and $\widetilde{U}^A_{a(\epsilon_1)}$.

Lemma 3.4.1. *Suppose the initial data (M, g, k) are smooth, asymptotically flat, and spherically symmetric and satisfy the dominant energy condition* (1.1.10) *and bounded expansion condition* (3.1.2)*. Then, the volumes of the diffeomorphic subregions $U^A_{a(\epsilon_1)}$ and $\widetilde{U}^A_{a(\epsilon_1)}$ may be estimated by*

$$Vol_{\bar{g}}(U^A_{a(\epsilon_1)}),\ Vol_{g_S}(\widetilde{U}^A_{a(\epsilon_1)}) \leq \omega_{n-1} A^{n-1} \widetilde{D}. \tag{3.4.5}$$

Proof. Both $U^A_{a(\epsilon_1)}$ and $\widetilde{U}^A_{a(\epsilon_1)}$ have a radial depth of at most $\widetilde{D}$ and a largest sphere of radius A. By the coarea formula, we have

$$\begin{aligned} \mathrm{Vol}_{\bar{g}}(U^A_{a(\epsilon_1)}) &= \int_{U^A_{a(\epsilon_1)}} d\omega_{\bar{g}} \\ &\leq \mathrm{Area}_{\bar{g}}(S_A) \int_{\rho_0+\epsilon_1}^{A} \frac{1}{\phi} d\rho \\ &\leq \omega_{n-1} A^{n-1} \widetilde{D}. \end{aligned} \tag{3.4.6}$$

The same estimate holds for $\widetilde{U}^A_{a(\epsilon_1)}$. □

Lemma 3.4.2. *Suppose the initial data* (M, g, k) *are smooth, asymptotically flat, and spherically symmetric and satisfy the dominant energy condition* (1.1.10) *and bounded expansion condition* (3.1.2)*. Then, the boundary areas of the diffeomorphic subregions* $U^A_{a(\epsilon_1)}$ *and* $\widetilde{U}^A_{a(\epsilon_1)}$ *may be estimated by*

$$Vol_{\bar{g}}(\partial U^A_{a(\epsilon_1)}),\, Vol_{g_S}(\partial \widetilde{U}^A_{a(\epsilon_1)}) = \omega_{n-1}(A^{n-1} + a(\epsilon_1)^{n-1}). \tag{3.4.7}$$

Proof. This follows by spherical symmetry and the radial values on the boundary. □

Lemma 3.4.3. *Suppose the initial data* (M, g, k) *are smooth, asymptotically flat, and spherically symmetric and satisfy the dominant energy condition* (1.1.10) *and bounded expansion condition* (3.1.2)*. Let* $D_0 \geq A, \widetilde{D}$*. Then*

$$\max\{diam_{\bar{g}}(U^A_{a(\epsilon_1)}), diam_{g_S}(\widetilde{U}^A_{a(\epsilon_1)})\} \leq 4\pi D_0. \tag{3.4.8}$$

Proof. The depth of $U^A_{a(\epsilon_1)}$ and $\widetilde{U}^A_{a(\epsilon_1)}$ is at most $2\widetilde{D}$, and the largest symmetry sphere satisfies

$$\mathrm{diam}_{g_S}(S_A) = \mathrm{diam}_{g_S}(\widetilde{S}_A) = \pi A. \tag{3.4.9}$$

By the triangle inequality the diameters of $U^A_{a(\epsilon_1)}$ and $\widetilde{U}^A_{a(\epsilon_1)}$ are no larger than $4\widetilde{D} + \pi A \leq 4\pi D_0$. □

3.4.2 Metric Estimate

Although we will eventually compare the metrics g_S to g_0 and $\bar{g}$ to g_0, because g_0 is an amalgam of g_S and $\bar{g}$ we may first compare these metrics. In this section, we define the the "good" region W on which $\bar{g}$ will be close to g_S and "bad" region $B := U - W$ on which the metric is not close to g_S. Once again, we can write each metric as

$$\bar{g} = \left(1 - \frac{2(m_0 + P(\rho) + Q(\rho))}{\rho^{n-2}}\right)^{-1} d\rho^2 + \rho^2 d\Omega^2 \quad g_S = \left(1 - \frac{2m_0}{\rho^{n-2}}\right)^{-1} d\rho^2 + \rho^2 d\Omega^2 \tag{3.4.10}$$

which suggests that choosing the set on which $\bar{g}$ and g_S are close is equivalent to choosing the set on which Q is forced to be small in some precise way.

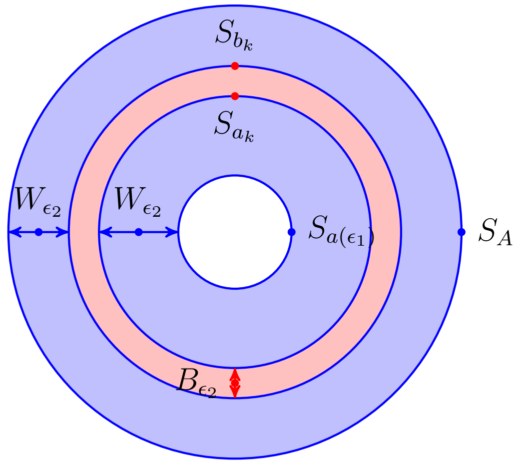

Figure 3.3. *We show a sketch of the regions of interest inside of Σ. The blue regions labeled W_{ϵ_2} represent the part of $U^A_{a(\epsilon_1)}$ for which $|Q| \leq \epsilon_2$ and the red region is the complement of this set in $U^A_{a(\epsilon_1)}$. Note that the spherical symmetry forces the components of W_{ϵ_2} and B_{ϵ_2} to be annular regions as displayed here.*

We introduce a new parameter $\epsilon_2 > 0$ and let

$$\begin{aligned} B_{\epsilon_2} &:= \{(\rho, \theta) \in U^A_{a(\epsilon_1)} \mid |Q(\rho)| > \epsilon_2\} \text{ and} \\ W_{\epsilon_2} &:= U^A_{a(\epsilon_1)} - B_{\epsilon_2}. \end{aligned} \tag{3.4.11}$$

By definition, $B_{\epsilon_2} - \partial U^A_{a(\epsilon_1)}$ is open and composed of perhaps infinitely many disjoint annular components; however, as $U^A_{a(\epsilon_1)}$ is second countable, there may be at most countably many of these components. Let B_k be a component of B_{ϵ_2} so that $\partial B_k = S_{a_k} \cup S_{b_k}$ with $a_k < b_k$ when $a_k \neq a(\epsilon_1)$ and $b_k \neq A$; otherwise, we might have $a_k = b_k$. We can thus describe ∂B_{ϵ_2} by the radial values of the inner and outer boundaries of each region – ie, the set $\{(a_k, b_k)\}$. See Figure 3.3. We denote the diffeomorphic counterparts of these sets in $(\mathrm{Sch}(m_0), g_S)$ by $\widetilde{B}_{\epsilon_2}$ and $\widetilde{W}_{\epsilon_2}$. When we identify these regions via the diffeomorphism, we refer to them as B and W.

Lemma 3.4.4. *Suppose the initial data (M, g, k) are smooth, asymptotically flat, and spherically symmetric and satisfy the dominant energy condition* (1.1.10) *and bounded expansion condition* (3.1.2). *Let ϵ_1, $\epsilon_3 > 0$. Then, for $\epsilon_2(\epsilon_1, \epsilon_3)$ and $\delta(\epsilon_1, \epsilon_2, \epsilon_3)$ small enough, we have*

$$\bar{g} \leq (1 + \epsilon_3)^2 g_S \text{ and } g_S \leq (1 + \epsilon_3)^2 \bar{g} \tag{3.4.12}$$

on W.

Proof. The proof is done by direct estimate. First, we find a function $c_1(\rho) \geq 1$ that we can use to get the comparison $\bar{g} \leq c_1(\rho) g_S$. Second, we show that for a given ϵ_1 and ϵ_3 there exist δ and ϵ_2 small enough so that $c_1(\rho) \leq (1 + \epsilon_3)^2$ on W. We repeat the procedure to find $c_2(\rho)$ so that $g_S \leq c_2(\rho)\bar{g}$ and $c_2(\rho) \leq (1 + \epsilon_3)^2$ on W for δ and ϵ_2 small enough.

In ρ coordinates we have

$$\begin{aligned}\bar{g} &= \left(1 - \frac{2m(\rho)}{\rho^{n-2}}\right)^{-1} d\rho^2 + \rho^2 d\Omega^2 \\ &= \left(1 - \frac{2(m_0 + P(\rho) + Q(\rho))}{\rho^{n-2}}\right)^{-1} d\rho^2 + \rho^2 d\Omega^2.\end{aligned} \tag{3.4.13}$$

We have that $P \leq \delta m_0$ on all of Σ. Restricting to W, we get $Q \leq \epsilon_2$ as well. We therefore have the estimate

$$1 - \frac{2(m_0 + P(\rho) + Q(\rho))}{\rho^{n-2}} \geq 1 - \frac{2(m_0 + \delta m_0 + \epsilon_2)}{\rho^{n-2}}. \tag{3.4.14}$$

When

$$\rho^{n-2} - 2(m_0 + \delta m_0 + \epsilon_2) > 0 \tag{3.4.15}$$

we may invert both sides to get that the radial metric coefficient is estimated by

$$\left(1 - \frac{2(m_0 + P(\rho) + Q(\rho))}{\rho^{n-2}}\right)^{-1} \leq \left(1 - \frac{2(m_0 + \delta m_0 + \epsilon_2)}{\rho^{n-2}}\right)^{-1}. \tag{3.4.16}$$

Let $c_1(\rho) = \left(1 - \frac{2(m_0+\delta m_0+\epsilon_2)}{\rho^{n-2}}\right)^{-1}\left(1 - \frac{2m_0}{\rho^{n-2}}\right)$. We compute that

$$1 \leq c_1(\rho) = \frac{\rho^{n-2} - 2m_0}{\rho^{n-2} - 2(m_0 + \delta m_0 + \epsilon_2)} \tag{3.4.17}$$

when (3.4.15) holds. It follows that

$$c_1(\rho) g_S = \left(1 - \frac{2(m_0 + \delta m_0 + \epsilon_2)}{\rho^{n-2}}\right)^{-1} d\rho^2 + c_1(\rho)\rho^2 d\Omega^2 \geq \bar{g}. \tag{3.4.18}$$

We finish the proof of the first metric estimate by showing there exists $\epsilon_2, \delta m_0$ small enough so that $c_1(\rho) \leq (1+\epsilon_3)^2$ by finding an upper bound for $c_1(\rho)$ and showing this bound can be controlled as needed. Recall that $\rho_0^{n-2} = 2m_0$ and set

$$\xi(\epsilon_1) := (\rho_0 + \epsilon_1)^{n-2} - 2m_0 = \sum_{k=0}^{n-3} \binom{n-2}{k} \epsilon_1^{n-2-k} \rho_0^k. \tag{3.4.19}$$

Note that $c_1(\rho)$ is decreasing in ρ so that the right hand side of (3.4.17) will be bounded above on W by $c_1(\rho_0 + \epsilon_1)$. We have therefore that

$$c_1(\rho) \leq c_1(\rho_0 + \epsilon_1) = \frac{\xi(\epsilon_1)}{\xi(\epsilon_1) - 2\delta m_0 - 2\epsilon_2}. \tag{3.4.20}$$

A brief computation shows that $\frac{\xi(\epsilon_1)}{\xi(\epsilon_1) - 2\delta m_0 - 2\epsilon_2} \leq (1+\epsilon_3)^2$ when

$$\delta m_0 + \epsilon_2 \leq \frac{\xi(\epsilon_1)(2\epsilon_3 + \epsilon_3^2)}{2(1+\epsilon_3)^2}. \tag{3.4.21}$$

Note that if (3.4.21) holds then (3.4.15) holds, so all inversions and calculations are valid given (3.4.21). It follows that

$$\bar{g} \leq c_1(\rho) g_S \leq (1+\epsilon_3)^2 g_s \tag{3.4.22}$$

on W when δm_0 and ϵ_2 satisfy (3.4.21).

We follow the same idea to show that for $\epsilon_2, \delta m_0$ small enough we have

$$g_S \leq c_2(\rho)\bar{g} \leq (1+\epsilon_3)^2 \bar{g}. \tag{3.4.23}$$

First, we study the Schwarzschild metric. We have by definition of the set W that $Q(\rho)+\epsilon_2 \geq 0$ on W. Therefore, we have

$$0 \leq 2(P(\rho)+Q(\rho)+\epsilon_2) \leq 2\delta m_0 + 4\epsilon_2 \tag{3.4.24}$$

so that when

$$2\delta m_0 + 4\epsilon_2 < \xi(\epsilon_1) \tag{3.4.25}$$

we have

$$\rho^{n-2} - 2m_0 - 2(P(\rho)+Q(\rho)+\epsilon_2) \geq \xi(\epsilon_1) - 2(P(\rho)+Q(\rho)+\epsilon_2) > 0 \tag{3.4.26}$$

and

$$1 - \frac{2(m_0+P(\rho)+Q(\rho)+\epsilon_2)}{\rho^{n-2}} > 0. \tag{3.4.27}$$

We then have

$$\begin{aligned} g_S &= \left(1-\frac{2m_0}{\rho^{n-2}}\right)^{-1} d\rho^2 + \rho^2 d\Omega^2 \\ &\leq \left(1 - \frac{2(m_0+P(\rho)+Q(\rho)+\epsilon_2)}{\rho^{n-2}}\right)^{-1} d\rho^2 + \rho^2 d\Omega^2 \\ &= \left(1-\frac{2(m(\rho)+\epsilon_2)}{\rho^{n-2}}\right)^{-1} d\rho^2 + \rho^2 d\Omega^2 \end{aligned} \tag{3.4.28}$$

where the second inequality follows because $0 \leq P(\rho)+Q(\rho)+\epsilon_2$. Therefore, we choose

$$c_2(\rho) = \left(1-\frac{2(m(\rho)+\epsilon_2)}{\rho^{n-2}}\right)^{-1}\left(1-\frac{2m(\rho)}{\rho^{n-2}}\right) \tag{3.4.29}$$

and can check that $c_2(\rho) \geq 1$ when

$$\rho^{n-2} - 2(m(\rho)+\epsilon_2) > 0 \tag{3.4.30}$$

which we have already guaranteed by (3.4.25). Now, we have

$$c_2(\rho)\bar{g} = \left(1-\frac{2(m(\rho)+\epsilon_2)}{\rho}\right)^{-1} d\rho^2 + c_2(\rho)\rho^2 d\Omega^2 \geq g_S \tag{3.4.31}$$

by the last line of (3.4.28) and that $c_2(\rho) \geq 1$.

To complete the proof, we check the conditions under which $c_2(\rho) \leq (1+\epsilon_3)^2$. First, we will find an upper bound for $c_2(\rho)$ in terms of $\xi(\epsilon_1)$, ϵ_2, and δm_0. Observe that

$$\begin{aligned} c_2(\rho) &= \frac{\rho^{n-2} - 2m(\rho)}{\rho^{n-2} - 2m(\rho) - 2\epsilon_2} \\ &= \frac{1}{1 - \frac{2\epsilon_2}{\rho^{n-2} - 2m(\rho)}} \end{aligned} \tag{3.4.32}$$

from which we can see that we may find an upper bound for $c_2(\rho)$ by finding a lower bound for $\rho^{n-2} - 2m(\rho)$. Indeed, we have that $\rho^{n-2} - 2m(\rho) \geq \xi(\epsilon_1) - 2(\delta m_0 + \epsilon_2)$ so that

$$\begin{aligned} c_2(\rho) &\leq \frac{1}{1 - \frac{2\epsilon_2}{\xi(\epsilon_1) - 2(\delta m_0 + \epsilon_2)}} \\ &\leq \frac{1}{1 - \frac{2\epsilon_2 + 2\delta m_0}{\xi(\epsilon_1) - 2(\delta m_0 + \epsilon_2)}} \end{aligned} \tag{3.4.33}$$

where in the last line we have once more increased the upper bound by subtracting $2\delta m_0/(\xi(\epsilon_1) - 2(\delta m_0 + \epsilon_2))$ from the denominator, which is valid when

$$4(\delta m_0 + \epsilon_2) < \xi(\epsilon_1). \tag{3.4.34}$$

Now we check when

$$c_2(\rho) \leq \frac{1}{1 - \frac{2(\epsilon_2 + \delta m_0)}{\xi(\epsilon_1) - 2(\delta m_0 + \epsilon_2)}} < (1+\epsilon_3)^2. \tag{3.4.35}$$

First, we get that this holds when

$$\frac{2(\epsilon_2 + \delta m_0)}{\xi(\epsilon_1) - 2(\delta m_0 + \epsilon_2)} < \frac{\epsilon_3^2 + 2\epsilon_3}{(1+\epsilon_3)^2}. \tag{3.4.36}$$

Then we solve for $2(\epsilon_2 + \delta m_0)$ to get that we have the inequality when

$$\delta m_0 + \epsilon_2 < \frac{\xi(\epsilon_1)\,(\epsilon_3^2 + 2\epsilon_3)}{2((1+\epsilon_3)^2 + \epsilon_3^2 + 2\epsilon_3)}. \tag{3.4.37}$$

Note that (3.4.21), (3.4.34), and (3.4.37) hold when we have

$$\delta m_0 + \epsilon_2 < \frac{\xi(\epsilon_1)\,(\epsilon_3^2 + 2\epsilon_3)}{4((1+\epsilon_3)^2 + \epsilon_3^2 + 2\epsilon_3)} \tag{3.4.38}$$

Thus, we have both metric estimates (3.4.12) on W when ϵ_2 and δ satisfy (3.4.38). □

As a consequence, we get the desired comparison to the g_0 metric.

Corollary 3.4.1. *Suppose the initial data (M, g, k) are smooth, asymptotically flat, and spherically symmetric and satisfy the dominant energy condition* (1.1.10) *and bounded expansion condition* (3.1.2)*. Let ϵ_1, $\epsilon_3 > 0$. Then, for $\epsilon_2(\epsilon_1, \epsilon_3)$ and $\delta(\epsilon_1, \epsilon_2, \epsilon_3)$ small enough, we have*

$$\bar{g} \leq (1+\epsilon_3)^2 g_0 \text{ and } g_S \leq (1+\epsilon_3)^2 g_0 \tag{3.4.39}$$

on W.

Proof. At any point $x \in W$ we have that $g_0 = \bar{g}$ or $g_0 = g_S$. In either case, both estimates hold by Lemma 3.4.4 when δ and ϵ_2 are chosen thusly with respect to ϵ_1 and ϵ_3. □

3.4.3 Volume of the Bad Region

Now that we have validated the choices of W_{ϵ_2} and B_{ϵ_2} by expressing the metric estimate on W_{ϵ_2}, we need to estimate from above the volume of the sets B_{ϵ_2} and $\widetilde{B}_{\epsilon_2}$. This estimate relies on the bound on the L^2 norm of Q given by (3.2.15).

Lemma 3.4.5. *Suppose the initial data (M, g, k) are smooth, asymptotically flat, and spherically symmetric and satisfy the dominant energy condition* (1.1.10) *and bounded expansion condition* (3.1.2)*. Then, the volumes outside the diffeomorphic subregions, denoted by B_{ϵ_2} and $\widetilde{B}_{\epsilon_2}$, may be estimated by*

$$Vol_{\bar{g}}(B_{\epsilon_2}) \leq \frac{\delta \omega_{n-1} m_0 A^{2n-1} C}{\epsilon_2^2}, \tag{3.4.40}$$

and

$$Vol_{g_S}(\widetilde{B}_{\epsilon_2}) \leq \frac{\delta \omega_{n-1} m_0 A^n C^2 \sqrt{(\rho_0+\epsilon_1)^{n-2}}}{\sqrt{\xi(\epsilon_1)}\epsilon_2^2}. \tag{3.4.41}$$

In particular, for small enough ϵ_1

$$Vol_{\bar{g}}(B_{\epsilon_2}),\, Vol_{g_S}(\widetilde{B}_{\epsilon_2}) \leq \frac{\delta \omega_{n-1} m_0 A^n C^2 \sqrt{(\rho_0+\epsilon_1)^{n-2}}}{\sqrt{\xi(\epsilon_1)}\epsilon_2^2}. \tag{3.4.42}$$

Proof. From (3.2.15) and the definition of B_{ϵ_2},

$$\begin{aligned} \delta \omega_{n-1} m_0 A^{2n-1} C &\geq \int_{U^A_{a(\epsilon_1)}} |Q|^2 d\omega_{\bar{g}} \\ &\geq \int_{B_{\epsilon_2}} \epsilon_2^2 d\omega_{\bar{g}} \end{aligned} \tag{3.4.43}$$

from which (3.4.40) follows by dividing by ϵ_2^2. To get (3.4.41), we calculate that

$$\begin{aligned} \mathrm{Vol}_{\bar{g}}(B_{\epsilon_2}) &= \int_{B_{\epsilon_2}} d\omega_{\bar{g}} \\ &= \int_{\widetilde{B}_{\epsilon_2}} \frac{\phi_S}{\phi} d\omega_{g_S} \\ &\geq \frac{\phi_S(a(\epsilon_1))}{AC} \mathrm{Vol}_{g_S}(\widetilde{B}_{\epsilon_2}) \end{aligned} \tag{3.4.44}$$

where in the last line we have used the upper bound on ϕ given by (3.2.4) in Lemma 3.2.1. By (3.4.40), we then have that

$$\mathrm{Vol}_{g_S}(\widetilde{B}_{\epsilon_2}) \leq \frac{AC}{\phi_S(a(\epsilon_1))}\frac{\delta\omega_{n-1}m_0A^{2n-1}C}{\epsilon_2^2}. \tag{3.4.45}$$

Note that $a(\epsilon_1) \geq \rho_0+\epsilon_1$ so that $\phi_S(a(\epsilon_1)) \geq \phi_S(\rho_0+\epsilon_1) = \sqrt{\frac{\xi(\epsilon_1)}{(\rho_0+\epsilon_1)^{n-2}}}$ where $\xi(\epsilon_1)$ is defined in (3.4.19). The result follows. □

3.4.4 Distance Estimate

Now we consider our intermediary metric space (U_0, g_0) to compute the estimate (2.2.12) for g_S and $\bar{g}$. We wish to obtain some estimates

$$\begin{aligned} d_S(x,y) - d_0(x,y) &\leq 2\alpha \\ d_{\bar{g}}(x,y) - d_0(x,y) &\leq 2\alpha. \end{aligned} \tag{3.4.46}$$

In this section, we compute α. First, we need a lemma estimating radial distances in the bad region in the $\bar{g}$ and g_S metrics.

Lemma 3.4.6. *Suppose the initial data* (M, g, k) *are smooth, asymptotically flat, and spherically symmetric and satisfy the dominant energy condition* (1.1.10) *and bounded expansion condition* (3.1.2)*. Let* (a_k, b_k) *denote the radii which correspond to the inner and outer boundaries of the components of* B*. Then*

$$\sum_k \int_{a_k}^{b_k} \frac{1}{\phi}d\rho, \sum_k \int_{a_k}^{b_k} \frac{1}{\phi_S}d\rho \leq \frac{\delta m_0 A^n C^2}{\sqrt{(\rho_0+\epsilon_1)^n \xi(\epsilon_1)}\epsilon_2^2}. \tag{3.4.47}$$

Proof. By (3.4.42) we have

$$\sum_k \int_{a_k}^{b_k}\int_{S_\rho} d\omega_{\bar{g}} \leq \frac{\delta\omega_{n-1}m_0A^nC^2\sqrt{(\rho_0+\epsilon_1)^{n-2}}}{\sqrt{\xi(\epsilon_1)}\epsilon_2^2}. \tag{3.4.48}$$

We can minimize ρ by $\rho_0+\epsilon_1$ to get that

$$(\rho_0+\epsilon_1)^{n-1}\omega_{n-1}\sum_k \int_{a_k}^{b_k} \frac{1}{\phi}d\rho \leq \frac{\delta\omega_{n-1}m_0A^nC^2\sqrt{(\rho_0+\epsilon_1)^{n-2}}}{\sqrt{\xi(\epsilon_1)}\epsilon_2^2}. \tag{3.4.49}$$

from which the estimate follows. The estimate for the Schwarzschild distance follows in the same way. □

Before we compute the α parameter, we need another result concerning the nature of geodesics in spherically symmetric annuli which may be foliated by spheres of positive mean curvature. We say that a curve

$$c : [t_1, t_2] \to M, \quad c(t) = (r(t), \theta(t)) \tag{3.4.50}$$

on such a manifold has a radial relative maximum on an interval $[a,b] \subsetneq [t_1,t_2]$ if $r(t)$ is constant on $[a,b]$ and there exists δ small enough such that $r(t)$ is strictly increasing on $[a-\delta,a)$ and strictly decreasing on $(b,b+\delta]$. We prove that a geodesic on such a manifold may not have a radial relative maximum. Note that the relative radial maximum may only occur on the outer boundary of the annulus and not the inner boundary. This is why we chose the metric g_0 to be smooth up to the outer boundary of each annulus and continuous up to the inner boundary.

Lemma 3.4.7. *Let V be a spherically symmetric annulus with boundary with Riemannian metric $\hat{g}$ which is smooth up to the outer boundary, smooth in the interior, and continuous up to the inner boundary. Let V be foliated by spheres of positive mean curvature. Then for points $x,y \in V$, the distance minimizing geodesic which realizes $d_{\hat{g}}(x,y) = \ell_{\hat{g}}(\gamma)$ may not achieve a radial relative maximum.*

Proof. We write our spherically symmetric metric in arc length coordinates as

$$\hat{g} = ds^2 + \rho^2(s) g_S. \tag{3.4.51}$$

Suppose $\gamma(t)$ is a distance minimizing geodesic between two points x and y. Suppose further that $\gamma(t)$ has a radial relative maximum on $[a,b]$ at a radius s_1 which might be on the outermost boundary of the annulus. Let δ_1, δ_2 be such that $\gamma(a-\delta_1) = (s_2,\theta_1)$ and $\gamma(b+\delta_2) = (s_2,\theta_2)$. Further, choose δ_1 and δ_2 small enough enough so that $\gamma : (a-\delta_1, b+\delta_2) \to M$ lies above S_{s_2}. Such a choice of δ_j is always possible because $[a,b]$ is a radial relative maximum. Let $s \circ \gamma : [a,b] \to \mathbb{R}_+$ be the distance between γ and S_{s_2}.

We may compute the mean curvature as

$$0 < H = \frac{(n-1)\rho_{,s}}{\rho} \tag{3.4.52}$$

and the second fundamental form is

$$2A = 2\partial_s(\rho^2(s) g_S) = \rho\rho_{,s} g_S \tag{3.4.53}$$

which is positive because positive mean curvature implies $\rho_{,s}$ is positive. Further, we write

$$A = \nabla^2_S s \tag{3.4.54}$$

because, as a distance function, $|\nabla s| = 1$. We express the Hessian of s as

$$\mathrm{Hess}(s) = \nabla^2 s(\partial_s, \partial_s) + A \tag{3.4.55}$$

where

$$\nabla^2 s(\partial_s, \partial_s) = \langle \nabla_{\partial_s} s, \partial_s \rangle = 0. \tag{3.4.56}$$

Note that $\mathrm{Hess}(s)(v,v)$ may be zero if and only if v is strictly radial. Recall that $\dot{\gamma}(t)$ is tangent to S_{s_1} on $[a,b]$. As $\dot{\gamma}(t)$ varies smoothly in a smooth metric, we may choose δ small enough so that $\dot{\gamma}(t)$ may not have a strictly radial component on $(a-\delta, a+\delta)$. It follows that

$$\mathrm{Hess}(s)(\dot{\gamma}(t), \dot{\gamma}(t)) > 0 \tag{3.4.57}$$

on that section, which contradicts that $s \circ \gamma$ is concave down or flat on that interval; ie $[a,b]$ cannot be a relative radial maximum for γ. □

A consequence of this same argument is that a geodesic may not be radially constant except on the innermost boundary. In other words, there are no radial shortcuts except at the inner boundary. With this result in hand, we estimate the difference in distances directly. This is the result in which we need the specific definition of g_0, which is smooth up to the outer boundary of each annulus and continuous on the inner boundary.

Lemma 3.4.8. *Suppose the initial data* (M, g, k) *are smooth, asymptotically flat, and spherically symmetric and satisfy the dominant energy condition* (1.1.10) *and bounded expansion condition* (3.1.2)*. Then we can compute that the difference in distances with respect to the* $g_S, \bar{g}$*, and* g_0 *metrics for* $x, y \in W$ *satisfies*

$$\begin{aligned} d_S(x,y) - d_0(x,y) &\leq 2\alpha \\ d_{\bar{g}}(x,y) - d_0(x,y) &\leq 2\alpha \end{aligned} \tag{3.4.58}$$

where

$$\alpha = 2\epsilon_3 \pi D_0 + \frac{\delta m_0 A^n C^2}{\sqrt{(\rho_0+\epsilon_1)^n \xi(\epsilon_1)\epsilon_2^2}}. \tag{3.4.59}$$

Proof. For x and $y \in W$, let γ^0 by a geodesic connecting them in the g_0 metric, and γ^S a geodesic in the Schwarzschild metric g_S. Let ℓ_{g_0} and ℓ_{g_S} denote the lengths in each metric.

To begin, we have

$$d_{g_S}(x,y) = \ell_{g_S}(\gamma^S) \leq \ell_{g_S}(\gamma^0) \tag{3.4.60}$$

because γ^S and γ^0 both connect points the points x and y in W. Now we break up γ^0 into the disjoint sets $\gamma^0 \cap W$ and $\gamma^0 \cap B$ and estimate $\gamma^0 \cap W$ in terms of g_0 using the metric comparison Corollary 3.4.1:

$$\begin{aligned} d_{g_S}(x,y) &\leq \ell_{g_S}(\gamma^0 \cap W) + \ell_{g_S}(\gamma^0 \cap B) \\ &\leq (1+\epsilon_3)\ell_{g_0}(\gamma^0 \cap W) + \ell_{g_S}(\gamma^0 \cap B). \end{aligned} \tag{3.4.61}$$

If we express the distance between x and y in g_0 as

$$d_{g_0}(x,y) = \ell_{g_0}(\gamma^0 \cap W) + \ell_{g_0}(\gamma^0 \cap B) \tag{3.4.62}$$

we get that

$$d_{g_S}(x,y) - d_{g_0}(x,y) \leq \epsilon_3 \ell_{g_0}(\gamma^0 \cap W) + \ell_{g_S}(\gamma^0 \cap B) - \ell_{g_0}(\gamma^0 \cap B). \tag{3.4.63}$$

Repeating the procedure with $\bar{g}$ and g_0 gives

$$d_{\bar{g}}(x,y) - d_{g_0}(x,y) \leq \epsilon_3 \ell_{g_0}(\gamma^0 \cap W) + \ell_{\bar{g}}(\gamma^0 \cap B) - \ell_{g_0}(\gamma^0 \cap B). \tag{3.4.64}$$

If we can estimate the right hand side of (3.4.63) and (3.4.64) by the same constant independent of x and y we will have α.

By Lemma 3.4.3, $\ell_{g_0}(\gamma^0 \cap W) \leq 4\pi D_0$. It remains only to estimate the differences

$$\ell_{g_S}(\gamma^0 \cap B) - \ell_{g_0}(\gamma^0 \cap B) \quad \text{and} \quad \ell_{\bar{g}}(\gamma^0 \cap B) - \ell_{g_0}(\gamma^0 \cap B). \tag{3.4.65}$$

By Lemma 3.4.7, for each annular region in which $g_0 = \bar{g}$ or $g_0 = g_S$, γ_0 may not achieve a relative maximum either on the interior of the region or on the outermost boundary. It follows that we can split γ^0 into at two disjoint components: one which is radially decreasing and one which is radially increasing.

Let $x = (\rho_1, \theta_1)$, $y = (\rho_2, \theta_2)$ and, without loss of generality, let $\gamma_1^0(t) = (\rho_1 - t, \theta_1(t))$ be radially decreasing for $t \in [0, t_3]$ and $\gamma_2^0(t) = (\rho_{min} + t, \theta_2(t))$ be radially increasing for $t \in [t_3, \rho_2 - \rho_{min}]$. We may compute

$$\begin{aligned} \ell_{g_S}(\gamma^0 \cap B) - \ell_{g_0}(\gamma^0 \cap B) = &\int_{\gamma_1^0 \cap B} \sqrt{\frac{1}{\phi_S^2} + \rho^2 \beta_{n-1}} - \sqrt{\frac{1}{\phi_0^2} + \rho^2 \beta_{n-1}} dt \\ &+ \int_{\gamma_2^0 \cap B} \sqrt{\frac{1}{\phi_S^2} + \rho^2 \beta_{n-1}} - \sqrt{\frac{1}{\phi_0^2} + \rho^2 \beta_{n-1}} dt \end{aligned} \tag{3.4.66}$$

where β_{n-1} is the distance on the $(n-1)$-sphere and we abuse notation slightly so $\gamma_1^0 \cap B$ is the set of t values for which γ_1^0 lies in B. Then,

$$\begin{aligned} \ell_{g_S}(\gamma_1^0 \cap B) - \ell_{g_0}(\gamma_1^0 \cap B) &= \int_{\gamma_1^0 \cap B} \frac{\frac{1}{\phi_S^2} - \frac{1}{\phi_0^2}}{\sqrt{\frac{1}{\phi_S^2} + \beta_{n-1}} + \sqrt{\frac{1}{\phi_0^2} + \beta_{n-1}}} dt \\ &\leq \int_{\gamma_1^0 \cap B} \frac{\frac{1}{\phi_S^2} - \frac{1}{\phi_0^2}}{\sqrt{\frac{1}{\phi_S^2} + \frac{1}{\phi_0^2}}} dt \\ &\leq \int_{\gamma_1^0 \cap B} \frac{\frac{1}{\phi_S^2} - \frac{1}{\phi_0^2}}{\sqrt{\frac{1}{\phi_S^2} - \frac{1}{\phi_0^2}}} dt \\ &\leq \int_{\gamma_1^0 \cap B} \sqrt{\frac{1}{\phi_S^2} - \frac{1}{\phi_0^2}} dt \\ &\leq \int_{\gamma_1^0 \cap B} \frac{1}{\phi_S} \\ &\leq \sum_k \int_{a_k}^{b_k} \frac{1}{\phi_S} d\rho \end{aligned} \tag{3.4.67}$$

where the last estimate holds for any possible choice of x and y because, by virtue of being monotonic in ρ, γ_1^0 may be in each component of B at most once. The same estimates hold for γ_2^0, so by Proposition 3.4.6 we have now that

$$\ell_{g_S}(\gamma^0 \cap B) - \ell_{g_0}(\gamma^0 \cap B) \leq 2 \frac{\delta m_0 A^n C^2}{\sqrt{(\rho_0 + \epsilon_1)^n \xi(\epsilon_1) \epsilon_2^2}}. \tag{3.4.68}$$

The same procedure for $\bar{g}$ gives that

$$\ell_{\bar{g}}(\gamma^0 \cap B) - \ell_{g_0}(\gamma^0 \cap B) \leq 2 \frac{\delta m_0 A^n C^2}{\sqrt{(\rho_0 + \epsilon_1)^n \xi(\epsilon_1) \epsilon_2^2}} \tag{3.4.69}$$

and we have

$$\begin{aligned} d_S(x,y) - d_0(x,y) &\le 2\left(2\epsilon_3\pi D_0 + \frac{\delta m_0 A^n C^2}{\sqrt{(\rho_0+\epsilon_1)^n \xi(\epsilon_1)\epsilon_2^2}}\right) \\ d_{\bar{g}}(x,y) - d_0(x,y) &\le 2\left(2\epsilon_3\pi D_0 + \frac{\delta m_0 A^n C^2}{\sqrt{(\rho_0+\epsilon_1)^n \xi(\epsilon_1)\epsilon_2^2}}\right) \end{aligned} \tag{3.4.70}$$

for all $x, y \in W$, so we may choose

$$\alpha = 2\epsilon_3\pi D_0 + \frac{\delta m_0 A^n C^2}{\sqrt{(\rho_0+\epsilon_1)^n \xi(\epsilon_1)\epsilon_2^2}}. \tag{3.4.71}$$

□

3.4.5 Proof of Proposition

We may now apply Theorem 2.2.1 to prove Proposition 3.4.1.

Proof of Proposition 2. Let $\epsilon > 0$. First, consider $d_{\mathcal{F}}(U_0, U^A_{a(\epsilon_1)})$. The map $F : U^A_{a(\epsilon_1)} \to U_0$ which identifies $(\rho,\theta) \in U^A_{a(\epsilon_1)}$ with $(\rho,\theta) \in U_0$ is a smooth diffeomorphism and so is biLipschitz with smooth inverse. Also, by definition of U_0, this map is distance nonincreasing. If we take α as in Lemma 3.4.8, we have $d_{\bar{g}}(x,y) \le d_0(x,y) + 2\alpha$ for all $x, y \in W$.

It follows by Theorem 2.2.1 and Lemmas 3.4.1, 3.4.2, 3.4.3, 3.4.5, and 3.4.8 that

$$d_{\mathcal{F}}(U_0, U^A_{a(\epsilon_1)}) \le 2\left(\frac{\delta\omega_{n-1} m_0 A^n C^2 \sqrt{(\rho_0+\epsilon_1)^{n-2}}}{\sqrt{\xi(\epsilon_1)}\epsilon_2^2}\right) + h(\omega_{n-1} A\widetilde{D}) + h(\omega_{n-1}(A^{n-1} + a(\epsilon_1)^{n-1})) \tag{3.4.72}$$

where

$$h = \sqrt{8\alpha\pi D_0 + \alpha^2}. \tag{3.4.73}$$

and $\alpha = 2\epsilon_3\pi D_0 + \frac{\delta m_0 A^n C^2}{\sqrt{(\rho_0+\epsilon_1)^n \xi(\epsilon_1)\epsilon_2^2}}$. For each $\epsilon_1, \epsilon > 0$ there exists ϵ_3, ϵ_2, and δ small enough so that each term on the right hand side is less than $\epsilon/9$ and we have

$$d_{\mathcal{F}}(U_0, U^A_{a(\epsilon_1)}) < \epsilon/3.$$

The proof for $\widetilde{U}^A_{a(\epsilon_1)}$ follows in the same way. □

Remark 3.4.1. In the proof of Proposition 2, we first have ϵ_1, ϵ small, then choose ϵ_3 sufficiently small so that $2\epsilon_3\pi D_0$ is small. We then must choose ϵ_2 and δ small enough to satisfy Corollary 3.4.1. This, as well as the fact that $\xi(\epsilon_1)$ has largest term $\rho_0^{n-3}\epsilon_1$, means we must choose δ much smaller than $\rho_0^{n-1}\epsilon_1\epsilon_2^2$ so that $\frac{\delta\omega_{n-1} m_0 A^n C^2 \sqrt{(\rho_0+\epsilon_1)^{n-2}}}{\sqrt{\xi(\epsilon_1)}\epsilon_2^2}$ and $\frac{\delta m_0 A^n C^2}{\sqrt{(\rho_0+\epsilon_1)^n \xi(\epsilon_1)\epsilon_2^2}}$ are small.

3.5 Proof of Theorem

We prove first the estimate on the difference in volumes, then the convergence of the second fundamental forms in separate arguments.

Proposition 3.5.1. *Suppose the initial data* (M, g, k) *are smooth, asymptotically flat, and spherically symmetric and satisfy the dominant energy condition* (1.1.10) *and bounded expansion condition* (3.1.2)*. Then, the difference of total volumes may be estimated by*

$$\begin{aligned}|\,Vol_{\bar{g}}(U^A_{a(\epsilon_1)}) - Vol_{g_S}(\widetilde{U}^A_{a(\epsilon_1)})| \leq \, & \frac{2\delta\omega_{n-1}m_0A^nC^2\sqrt{(\rho_0+\epsilon_1)^{n-2}}}{\sqrt{\xi(\epsilon_1)}\epsilon_2^2} \\ & + ((1+\epsilon_3)^n - 1)\widetilde{D}A^{n-1}\omega_{n-1}.\end{aligned} \tag{3.5.1}$$

In particular for any $\epsilon > 0$*, there exists* $\epsilon_3, \epsilon_2, \epsilon_1$*, and* δ *small enough that*

$$|\,Vol_{\bar{g}}(U^A_{a(\epsilon_1)}) - Vol_{g_S}(\widetilde{U}^A_{a(\epsilon_1)})| < \epsilon/4. \tag{3.5.2}$$

Proof. We estimate each term on the right hand side of

$$\begin{aligned}|\mathrm{Vol}_{\bar{g}_j}(U^A_{a(\epsilon_1)}) - \mathrm{Vol}_{g_S}(\widetilde{U}^A_{a(\epsilon_1)})| \leq & |\mathrm{Vol}_{\bar{g}}(U^A_{a(\epsilon_1)}) - \mathrm{Vol}_{\bar{g}}(W_{\epsilon_2})| + |\mathrm{Vol}_{\bar{g}}(W_{\epsilon_2}) - \mathrm{Vol}_{g_S}(\widetilde{W}_{\epsilon_2})| \\ & + |\mathrm{Vol}_{g_S}(\widetilde{W}_{\epsilon_2}) - \mathrm{Vol}_{g_S}(\widetilde{U}^A_{a(\epsilon_1)})|.\end{aligned} \tag{3.5.3}$$

By Lemma 3.4.5, we have

$$|\mathrm{Vol}_{\bar{g}}(U^A_{a(\epsilon_1)}) - \mathrm{Vol}_{\bar{g}}(W_{\epsilon_2})|, |\mathrm{Vol}_{g_S}(\widetilde{W}_{\epsilon_2}) - \mathrm{Vol}_{g_S}(\widetilde{U}^A_{a(\epsilon_1)})| \leq \frac{\delta\omega_{n-1}m_0A^nC^2\sqrt{(\rho_0+\epsilon_1)^{n-2}}}{\sqrt{\xi(\epsilon_1)}\epsilon_2^2}. \tag{3.5.4}$$

By Lemmas 3.4.4 and 3.4.1 we have

$$\begin{aligned}|\mathrm{Vol}_{\bar{g}}(W_{\epsilon_2}) - \mathrm{Vol}_{g_S}(\widetilde{W}_{\epsilon_2})| & \leq ((1+\epsilon_3)^n - 1)\mathrm{Vol}_{g_S}(\widetilde{W}_{\epsilon_2}) \\ & \leq ((1+\epsilon_3)^n - 1)\widetilde{D}A^{n-1}\omega_{n-1}.\end{aligned} \tag{3.5.5}$$

□

Lemma 3.5.1. *If the initial data* (M, g, k) *are smooth, asymptotically flat, and spherically symmetric and satisfy the dominant energy condition* (1.1.10) *and bounded expansion condition* (3.1.2)*, then for any* $\epsilon > 0$ *there is* $\delta > 0$ *small enough so that* $\left\|\sqrt{\phi}(k-\pi)\right\|_{L^2(\Sigma,\bar{g})} < \epsilon$*.*

Proof. From Theorem 2.1.1, we have

$$\begin{aligned}\delta m_0 = m_{ADM} - m_0 & = \frac{1}{2\omega_{n-1}(n-1)}\int_\Sigma \phi\left(16\pi(\mu - J(w)) + |h - K|_\Sigma|^2_{\bar{g}} + 2|q|^2_{\bar{g}}\right) d\omega_{\bar{g}} \\ & \geq \frac{1}{2\omega_{n-1}(n-1)}\int_\Sigma \phi|h - K|_\Sigma|^2_{\bar{g}} d\omega_{\bar{g}}\end{aligned} \tag{3.5.6}$$

so that the L^2 norm of $\sqrt{\phi}(h - K|_\Sigma) = \sqrt{\phi}(k-\pi)$ is controlled by δ on all of Σ, from which the second fundamental form convergence follows. □

Now we may prove the theorem.

Proof of Theorem 1. By Lemma 3.3.1 we have the L^2_{loc} convergence of ϕ to ϕ_S and by Lemma 3.5.1 we have the L^2 convergence of the second fundamental forms. It remains to show that

$$d_{\mathcal{VF}}(U^A_{a(\epsilon_1)}, \widetilde{U}^A_{a(\epsilon_1)}) < \epsilon \tag{3.5.7}$$

for small enough δ.

By the triangle inequality, we may write

$$\begin{aligned} d_{\mathcal{VF}}(U^A_{a(\epsilon_1)}, \widetilde{U}^A_{a(\epsilon_1)}) &= d_{\mathcal{F}}(U^A_{a(\epsilon_1)}, \widetilde{U}^A_{a(\epsilon_1)}) + |\mathrm{Vol}_{\bar{g}}(U^A_{a(\epsilon_1)}) - \mathrm{Vol}_{g_S}(\widetilde{U}^A_{a(\epsilon_1)})| \\ &\le d_{\mathcal{F}}(U^A_{a(\epsilon_1)}, U_0) + d_{\mathcal{F}}(\widetilde{U}^A_{a(\epsilon_1)}, U_0) \\ &\quad + |\mathrm{Vol}_{\bar{g}}(U^A_{a(\epsilon_1)}) - \mathrm{Vol}_{g_S}(\widetilde{U}^A_{a(\epsilon_1)})|. \end{aligned} \tag{3.5.8}$$

By Propositions 3.3.1, 3.4.1, and 3.5.1 we have for small enough ϵ_3, ϵ_2, and δ that

$$\begin{aligned} d_{\mathcal{F}}(U^A_{a(\epsilon_1)}, U_0) &< \epsilon/3 \\ d_{\mathcal{F}}(\widetilde{U}^A_{a(\epsilon_1)}, U_0) &< \epsilon/3 \\ |\mathrm{Vol}_{\bar{g}}(U^A_{a(\epsilon_1)}) - \mathrm{Vol}_{g_S}(\widetilde{U}^A_{a(\epsilon_1)})| &< \epsilon/3. \end{aligned} \tag{3.5.9}$$

The bound follows. ∎

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