Stability for the Positive Mass Theorem and the Penrose inequality

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Abstract of the Dissertation

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Let (M^3, g) be a complete, smooth, asymptotically flat 3-manifold which has nonnegative scalar curvature, ADM mass $m(g)$, and an outermost minimal boundary with area A. The Riemannian Penrose inequality states that $m(g) \geq \sqrt{\frac{A}{16\pi}}$, and the equality holds if and only if (M^3, g) is isometric to the Schwarzschild 3-manifold of mass $m(g)$. In particular, this implies the Positive Mass Theorem, which states that $m(g) \geq 0$, and $m(g) = 0$ if and only if (M^3, g) is isometric to the Euclidean 3-space $(\mathbb{R}^3, g_{\mathrm{Eucl}})$. In this thesis, we study the stability problems of these two geometric inequalities. We show that when $m(g)$ is almost zero, (M^3, g) is close to the Euclidean 3-space in the pointed measured Gromov-Hausdorff topology modulo negligible spikes; when $m(g)$ is almost equal to $\sqrt{\frac{A}{16\pi}}$, (M^3, g) is close to the Schwarzschild 3-manifold in the pointed measured Gromov-Hausdorff topology modulo negligible spikes and boundary area perturbations.

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Chapter 1

Introduction

A smooth orientable connected complete Riemannian 3-manifold (M^3, g) is called asymptotically flat if there exists a compact subset $K \subset M$ such that $M \setminus K = \sqcup_{k=1}^{N} M_{\text{end}}^k$ consists of finite pairwise disjoint ends, and for each $1 \le k \le N$, there exist constants $C > 0, \sigma > \frac{1}{2}$, and a C^∞-diffeomorphism $\Phi_k : M_{\text{end}}^k \to \mathbb{R}^3 \setminus B(1)$ such that under this identification,

$$|\partial^l (g_{ij} - \delta_{ij})(x)| \le C|x|^{-\sigma - |l|},$$

for all multi-indices $|l| = 0, 1, 2$ and any $x \in \mathbb{R}^3 \setminus B(1)$, where $B(1)$ is a unit ball in $\mathbb{R}^3$. Furthermore, we always assume the scalar curvature R_g is integrable over (M^3, g).

In General Relativity, following Arnowitt-Deser-Misner [ADM61], the ADM mass of each end M_{end}^k, $1 \le k \le N$, is then defined by

$$m_k(g) := \lim_{r \to \infty} \frac{1}{16\pi} \int_{S_r} \sum_i (g_{ij,i} - g_{ii,j}) \nu^j dA$$

where ν is the unit outer normal to the coordinate sphere S_r of radius $|x| = r$ in the given end, and dA is its area element. It has been shown in [Bar86] that $m_k(g)$ is a geometric invariant of the given end and is independent on the choice of coordinates. We can also define the mass of (M^3, g) as the maximum mass of its ends.

The Riemannian case of the Positive Mass Theorem, firstly proved by Schoen-Yau [SY79], states that an asymptotically flat 3-manifold of non-negative scalar curvature has

$$m(g) \geq 0.$$

Furthermore, the mass is zero if and only if the manifold is isometric to the Euclidean 3-space $\mathbb{R}^3$. This theorem has since been proven by numerous new methods: these include [Wit81] using harmonic spinors; [HI01] using the inverse mean curvature flow; [Li18] using Ricci flow with surgery; [Bra+22] using harmonic functions; [AMO21] using Green's function; and others. There are also generalizations in higher dimensions including [Wit81; SY17; LUY21], and spacetime variations including [SY81; Wit81; Bar86; Eic13; Eic+15] etc.

Motivated by physics, one is also interested in the case when an asymptotically flat manifold has minimal boundary. Given an asymptotically flat 3-manifold (M^3, g) with one end and possibly non-empty boundary ∂M, we say that ∂M is outermost minimal if it is compact and consists of minimal surfaces, and M contains no other compact minimal surfaces. In 1973, Penrose [Pen73] conjectured that for an asymptotically flat 3-manifold with outermost minimal boundary and nonnegative scalar curvature,

$$m(g) \geq \sqrt{\frac{\text{Area}(\partial M)}{16\pi}}.$$

Furthermore, the equality holds if and only if (M, g) is isometric to the Schwarzschild 3-manifold.

The Schwarzschild 3-manifold M^3_{sc} of positive mass $m > 0$ is the manifold $\mathbb{R}^3 \setminus B(\frac{m}{2})$, where $B(\frac{m}{2})$ is the Euclidean ball with radius $\frac{m}{2}$ around 0, equipped with the metric $g_{sc} = \left(1 + \frac{m}{2|x|}\right)^4 \delta$. Notice that M_{sc} is an asymptotically flat manifold with mass m, and ∂M_{sc} is an outermost minimal boundary with $\text{Area}(\partial M_{sc}) = 16\pi m^2$.

Huisken-Ilmanen [HI01] firstly proved Penrose's inequality by using the inverse mean curvature flow when the boundary is assumed to be connected, and later Bray [Bra01] proved the general case by using a different flow of conformal metrics together with the result of the positive mass theorem. There are also many additional extensions including [BL09; KWY17; Ago+22] and others.

Recently there has been growing interest in establishing stability results for geometric inequalities. The stability problem related to the Positive Mass Theorem and the Penrose inequality to then ask is whether the almost equality in these theorems would imply that the manifold is close to the model case when the equality holds in some topology. Physically, this problem helps us understand the interplay between the mass and the space geometry more deeply. Mathematically, this problem falls within the larger effort to understand the geometry for metrics with lower scalar curvature bounds [Gro19].

When studying such stability problems, a main difficulty we need to deal with is determining the appropriate topology. To measure how close two Riemannian manifolds are, the preferred topology is the $C^{k,\alpha}$ topology for some $k \in \mathbb{N}$ and $\alpha \in (0,1)$. Recall that a sequence of pointed complete Riemannian manifolds is said to converge in the pointed $C^{k,\alpha}$ topology, $(M_i, g_i, p_i) \to (M, g, p)$, if for envery $D > 0$ we can find a domain $\Omega \subset M$ with $B(p, D) \subset \Omega$ and embeddings $\Phi_i : \Omega \to M_i$ for all large enough i such that $\Phi_i(p) = p_i$, $B(p_i, D) \subset \Phi_i(\Omega)$, and $\Phi_i^* g_i \to g$ on Ω in the $C^{k,\alpha}$ topology. Usually, one can hope to establish control in the $C^{k,\alpha}$ topology when strong geometric conditions are imposed. Some notable results include: Cheeger [Che67] proved that the class of Riemannian manifolds with uniformly bounded sectional curvature and uniformly positive volume lower bound is precompact in the pointed $C^{1,\alpha}$ topology; Anderson [And90] proved that the class of Riemannian manifolds with uniformly bounded Ricci curvature and uniformly positive injectivity radius is precompact in the pointed $C^{1,\alpha}$ topology; Anderson-Cheeger [AC92] proved that the class of Riemannian manifolds with uniformly Ricci curvature lower bound and uniformly positive injectivity radius is precompact in the pointed C^{α} topology.

In general, one cannot expect to achieve $C^{k,\alpha}$ precompactness. We need to extend the class of Riemannian manifolds to a class of 'weak' Riemannian manifolds and study 'weak' convergence in a broader set of spaces. To preserve the Riemannian metric structure, a natural generalization is metric spaces. Recall that a metric space (X, d) consists of a set X and a metric function d on $X \times X$ satisfying that for any $x, y, z \in X$, $d(x, y) = d(y, x) \geq 0$,

$d(x,y)=0$ iff $x=y$, and the triangle inequality $d(x,z)\le d(x,y)+d(y,z)$. For any Riemannian manifold (M,g), the canonical metric structure is given by $d(x,y):=\inf\{\mathrm{Length}_g(\gamma)=\int_\gamma|\gamma'|_g:\gamma$ is a piecewise smooth curve between x and $y\}$. So we can consider the class of Riemannian manifolds as a subset of the class of metric spaces, and study the topology for the class of metric spaces.

At the beginning of the 20th century, Hausdorff introduced the so-called Hausdorff distance between subsets of a metric space. Given a metric space (X,d), for any two subsets A,B, the Hausdorff distance is defined by

$$d_H(A,B):=\inf\{\varepsilon:A\subset B_\varepsilon(B),B\subset B_\varepsilon(A)\},$$

where $B_\varepsilon(A):=\{x\in X:d(x,y)<\varepsilon$ for some $y\in A\}$ is the ε neighborhood of A. Around 1980, Gromov extended Hausdorff's definition to introduce a so-called Gromov-Hausdorff (GH) distance in the class of metric spaces, not limited to subsets of a fixed metric space. Given two abstract pointed metric spaces (X,d_X,x) and (Y,d_Y,y), we say d is an admissible metric on the disjoint union $X\sqcup Y$ if $d|_X=d_X$ and $d|_Y=d_Y$. Then the pointed GH distance is defined as

$$d_{GH}((X,d_X,x),(Y,d_Y,y)):=\inf\{d_H(X,Y)+d(x,y):\text{ admissible metric } d \text{ on } X\sqcup Y\}.$$

Notice that for compact metric spaces, $d_{GH}(X,Y)=0$ iff X and Y are isometric. And the collection of all isometric classes of compact metric spaces is complete. A very important property for GH distance is the Gromov's precompactness theorem. Let $\mathcal{M}(D)$ be the collection of isometric classes of metric spaces whose diameters are bounded by $D<\infty$. For any $X\in\mathcal{M}(D)$, we can define the capacity and covering functions as $\mathrm{Cap}_X(\varepsilon)=$ maximum number of disjoint ε-balls in X, $\mathrm{Cov}_X(\varepsilon)=$ minimum number of ε-balls it takes to cover X.

Theorem 1.0.1 (Gromov's precompactness theorem)**.** *For a class $\mathcal{C}\subset(\mathcal{M}(D),d_{GH})$, $\mathcal{C}$ is precompact iff there is a function $N_1(\varepsilon):(0,\alpha)\to(0,\infty)$ such that $\mathrm{Cap}_X(\varepsilon)\le N_1(\varepsilon)$ for all $X\in\mathcal{C}$ iff there is a function $N_2(\varepsilon):(0,\alpha)\to(0,\infty)$ such that $\mathrm{Cov}_X(\varepsilon)\le N_2(\varepsilon)$ for all $X\in\mathcal{C}$.*

The Gromov-Hausdorff distance proves highly useful when sectional curvature or Ricci curvature has a uniform lower bound. This is due to a significant corollary of Gromov's precompactness theorem and the Bishop-Gromov comparison theorem, which states that the collection of pointed complete Riemannian n-manifolds with a uniform Ricci lower bound is precompact in the pointed GH topology. Based on this fundamental fact, the geometry of metrics with sectional curvature or Ricci curvature bounded from below has been extensively studied in the past several years. The study of metrics with sectional curvature bounded from below is now well-known as Alexandrov geometry. Refer to [BGP92; Per91] and others for the explosion of work on the intrinsic theory of Alexandrov spaces using the GH topology. The examination of metrics with Ricci curvature bounded from below, along with the analysis of the limit space in GH convergence, was initially comprehensively explored in the series of works by Cheeger and Colding [Col96b; Col96a; Col97; CC96; CC97; CC00a; CC00b]. Since then, the study of a synthetic approach to Ricci curvature has been very active, including the so-called RCD space. In the case of Ricci lower bound and non-collapsing, many analytic tools have also been developed for GH convergence; refer to [Gig18] and others for reference.

In cases where the scalar curvature has a uniform lower bound, as in the stability problems we mentioned previously, one also hopes to achieve GH convergence and develop similar analytic tools. Unfortunately, there are many examples showing that GH topology is not suitable in these cases. As shown in [HI01; LS14], one can construct metrics with a uniform lower bound on the scalar curvature; however, these metrics are not precompact in the GH topology. Even if the GH limit exists, the resulting metrics may appear quite peculiar when compared to our intuitive expectations. As mentioned in [HI01], the main issue here could be that adding arbitrarily many long, thin cylindrical spikes does not violate positive scalar curvature. Moreover, neither the number nor the volume of these spikes can be bounded. Motivated by these examples, Huisken-Ilmanen formulated the following conjecture regarding the stability of the Positive Mass Theorem:

Conjecture 1.0.2 ([HI01])**.** *Suppose M_i is a sequence of asymptotically flat* 3*-manifolds with*

ADM mass tending to zero. Then there is a set $Z_i \subset M_i$ such that $|\partial Z_i| \to 0$ and $M_i \setminus Z_i$ converges to $\mathbb{R}^3$ in the Gromov-Hausdorff topology.

Later, researchers have explored various newer topologies for metrics with a scalar curvature lower bound explored by researchers, notably the intrinsic flat topology in [SW11] (refer to [Sor16; Sor23] for a survey on those questions) and the d_p convergence developed in [LNN20].

We will now introduce another new topology, a modification of the GH topology, as proposed in Huisken-Ilmanen's conjecture. For simplicity, we only consider manifolds, but one can easily generalize it to general metric measured spaces.

Definition 1.0.1. For a sequence of pointed Riemannian n-manifolds (M_i, g_i, p_i) and (M, g, p), we say that (M_i, g_i, p_i) converges to (M, g, p) in the pointed measured Gromov-Hausdorff topology modulo negligible spikes, if there exist open subsets $Z_i \subset M_i$ such that $\mathcal{H}^{n-1}(\partial Z_i) \to 0, p_i \in M_i \setminus Z_i$ and

$$(M_i \setminus Z_i, \hat{d}_{g_i}, p_i) \to (M, d_g, p)$$

in the pointed measured Gromov-Hausdorff topology for the induced length metric.

Recall that for any subset $A \subset (M^n, g)$, and any $x, y \in A$, the induced length metric on A of the metric g is defined as $\hat{d}_{g,A}(x, y) := \inf_\gamma \{\text{Length}_g(\gamma)\}$, where the infimum is taken among all rectifiable curves $\gamma \subset A$ connecting x and y.

Given this definition, we can state our first main theorem jointly with Antoine Song, which proves Huisken-Ilmanen's conjecture:

Theorem 1.0.3 ([DS23])**.** *The Euclidean 3-space is stable for the Positive Mass Theorem in the pointed GH topology modulo negligible spikes.*

More precisely, let (M_i^3, g_i) be a sequence of asymptotically flat 3-manifolds with nonnegative scalar curvature and suppose that the ADM mass $m(g_i)$ is positive and converges to 0. Then for all i, for each end in M_i, there is a domain Z_i in M_i with smooth boundary such *that the area of the boundary ∂Z_i converges to 0, $M_i \setminus Z_i$ contains the given end, and*

$$(M_i \setminus Z_i, \hat{d}_{g_i}, p_i) \to (\mathbb{R}^3, d_{\mathrm{Eucl}}, 0)$$

in the pointed measured Gromov-Hausdorff topology, where $p_i \in M_i \setminus Z_i$ is any choice of base point and $\hat{d}_{g_i}$ is the length metric on $M_i \setminus Z_i$ induced by g_i.

Moreover, the area of ∂Z_i is almost bounded quadratically by the mass in the following sense. For any positive continuous function $\xi : (0, \infty) \to (0, \infty)$ with

$$\lim_{x \to 0^+} \xi(x) = 0,$$

for all large i, we have

$$\mathrm{Area}(\partial Z_i) \leq \frac{m(g_i)^2}{\xi(m(g_i))}.$$

Similarly, we can also reformulate the stability problem for the Penrose inequality. Before giving the appropriate conjecture, let's review an example constructed by Mantoulidis-Schoen. In [MS15], for some non-standard metric on 2-sphere (S^2, g), they constructed a sequence of asymptotically flat 3-manifolds (M_i^3, g_i) with nonnegative scalar curvature such that ∂M_i^3 is isometric to (S^2, g) and minimal, and ADM mass $m(g_i) \to \sqrt{\frac{\mathrm{Area}_g(S^2)}{16\pi}}$. Also, outside a neighborhood of ∂M_i, which could be an arbitrarily long neck, g_i is isometric to a Schwarzschild metric with mass $m(g_i)$. That is, we need to cut out the long necks in these examples such that the remaining part has almost the same boundary area and is stable for the Penrose inequality. Our second main theorem states that up to such boundary area perturbations, the Schwarzschild 3-manifold is stable.

Theorem 1.0.4 ([Don24])**.** *The Schwarzschild 3-manifold is stable for the Penrose inequality in the pointed GH topology modulo negligible spikes and boundary area perturbations.*

More precisely, let $A_0 \geq 0$ be a fixed constant and (M_i^3, g_i) be a sequence of asymptotically flat 3-manifolds, each of which has nonnegative scalar curvature and a compact connected outermost minimal boundary with area A_0. Suppose that the ADM mass $m(g_i)$ converges to

$\sqrt{\frac{A_0}{16\pi}}$, *then for all* i, *there is a smooth submanifold* $N_i \subset M_i$ *such that* $\mathrm{Area}_{g_i}(\partial N_i) \to A_0$, *a spike domain* $Z_i \subset N_i$ *with smooth boundary such that* $\mathrm{Area}_{g_i}(\partial Z_i) \to 0$, *and for* $\check{N}_i := N_i \setminus Z_i$,

$$(\check{N}_i, \hat{d}_{g_i, \check{N}_i}, p_i) \to (M^3_{sc}, g_{sc}, x_o)$$

in the pointed measured Gromov-Hausdorff topology, and

$$(\partial^* \check{N}_i, \hat{d}_{g_i, \partial^* \check{N}_i}) \to (\partial M^3_{sc}, \hat{d}_{g_{sc}, \partial M^3_{sc}})$$

in the measured Gromov-Hausdorff topology,where (M^3_{sc}, g_{sc}, x_o) *is the standard Schwarzschild* 3*-manifold with boundary area* $\mathrm{Area}(\partial M^3_{sc}) = A_0$, *mass* $m(g_{sc}) = \sqrt{\frac{A_0}{16\pi}}$ *and* $x_o \in \partial M^3_{sc}$, $\partial^* \check{N}_i = \partial N_i \setminus Z_i$, *and* $\hat{d}_{g_i, \cdot}$ *are the length metrics on the corresponding spaces induced by* g_i *respectively, and* p_i *is any base point on* $\partial^* \check{N}_i$.

Now, let's provide a brief overview of the proof ideas for these two main theorems. Recently, Kazaras-Khuri-Lee [KKL21] were able to solve the stability problem for the Positive Mass Theorem under Ricci curvature lower bounds and a uniform asymptotic flatness assumption. The author [Don22] also made progress on the original version of the stability problem, without additional curvature assumptions but still under a uniform asymptotic flatness assumption. Similarly to [KKL21; Don22], our proof builds on the recent new proof of the Positive Mass Theorem by Bray-Kazaras-Khuri-Stern [Bra+22]. There, the authors employ level sets of harmonic maps. This method was initially explored by [Ste22], and there have been many other applications and generalizations in recent studies, including [Bra+23; AMO21; Ago+22] and others.

The result of [Bra+22] gives a quantitative mass inequality, which bounds the mass $m(g)$ of an asymptotically flat manifold (M, g) with nonnegative scalar curvature as follows: let M_{ext} be an exterior region of (M, g), i.e. M_{ext} is an asymptotically flat manifold with an outermost minimal boundary, then

$$m(g) \geq \frac{1}{16\pi} \int_{M_{ext}} \left(\frac{|\nabla^2 u|^2}{|\nabla u|} + R_g |\nabla u| \right) \mathrm{dvol}_g \tag{1.0.1}$$

where $u : M_{ext} \to \mathbb{R}$ is any harmonic function "asymptotic to one of the asymptotically flat coordinate functions on M_{ext}", see [Bra+22].

Let $\mathbf{u} := (u^1, u^2, u^3) : M_{ext} \to \mathbb{R}^3$, where u^j $(j = 1, 2, 3)$ is the harmonic function asymptotic to the asymptotically flat coordinate function x^j on M_{ext} (we fix a coordinate chart at infinity). Assuming that the mass $m(g)$ is positive and close to 0, we consider as in [Don22] the subset Ω of M_{ext} where the differential of $\mathbf{u}$ is ϵ_0-close to the identity map, for some small positive ϵ_0. More concretely define $F : (M_{ext}, g) \to \mathbb{R}$ by

$$F(x) := \sum_{j,k=1}^{3} \left(g(\nabla u^j, \nabla u^k) - \delta_{jk}\right)^2 .$$

Then set $\Omega := F^{-1}[0, \epsilon_0]$. On such a set, $\mathbf{u}$ is close to being a Riemannian isometry onto its image in $\mathbb{R}^3$.

In the first step, we show that one level set S of F has small area, bounded by $m(g)^2$ times a constant depending on ϵ_0. Inequality (1.0.1) bounds the L^2 norm of the gradient of F. The crucial point is then to observe that the classical capacity-volume inequality of Poincaré-Faber-Szegö generalizes to our setting, and enables us to find a region between two level sets of F with small volume. Then one can apply the coarea formula to find the small area level set S.

Consider the connected component A of $M_{ext} \setminus S$ containing the end. An issue is that even though the area of ∂A is small, the component A is in general not close to Euclidean 3-space for the induced length structure on A. The second step of the proof consists of modifying the subset A to another subset A' containing the end, so that $\partial A'$ still has small area but moreover A' is close to Euclidean 3-space in the pointed Gromov-Hausdorff topology with respect to its own length metric. This is shown by dividing the full space into small cube-like regions and by a repeated use of the coarea formula.

Regarding the stability for the Penrose inequality, we will first prove a similar stability result for the mass-capacity inequality, then up to a boundary area perturbation, Theorem 1.0.4 will be a corollary by an argument in [Bra01, Section 7].

Recall that for an asymptotically flat 3-manifold (M^3, g) with an outermost minimal boundary Σ and an end ∞_1, the capacity of Σ in (M^3, g) is defined by

$$\mathcal{C}(\Sigma, g) := \inf\{\frac{1}{2\pi}\int_M |\nabla\varphi|^2 \mathrm{dvol}_g : \varphi \in C^\infty(M), \varphi = \frac{1}{2} \text{ on } \Sigma, \lim_{x\to\infty_1} \varphi(x) = 1\}.$$

Then, as a corollary of the Positive Mass theorem, it was shown by [Bra01, Theorem 9] that $m(g) \geq 2\mathcal{C}(\Sigma, g)$, and equality holds if and only if (M^3, g) is the Schwarzschild manifold. For more details, please refer to Section 4.1.

Theorem 1.0.5 ([Don24])**.** *Let $m_0 > 0$ be a fixed constant and (M_i^3, g_i) be a sequence of asymptotically flat 3-manifolds, each of which has nonnegative scalar curvature and a compact connected outermost minimal boundary. Suppose that $m(g_i) - 2\mathcal{C}(\partial M_i, g_i) \to 0$ and $m(g_i) \to m_0$, then the same conclusion of Theorem 1.0.4 holds.*

Assume the conditions stated in Theorem 1.0.5, i.e., (M^3, g) is an asymptotically flat 3-manifold with nonnegative scalar curvature and a connected outermost minimal boundary Σ such that $m(g) - 2\mathcal{C}(\Sigma, g) \ll 1$. By employing a doubling technique and allowing for a perturbation as in [Bra01], we can assume that $\bar{M} = M \cup_\Sigma M$ is a smooth asymptotically flat 3-manifold with nonnegative scalar curvature and two ends ∞_1, ∞_2. Then the infimum in the definition of capacity is achieved by the Green's function f defined on $\bar{M}$, which satisfies

$$\Delta_g f = 0,$$
$$\lim_{x\to\infty_1} f(x) = 1,$$
$$\lim_{x\to\infty_2} f(x) = 0.$$

From symmetry of $\bar{M}$, f equals $\frac{1}{2}$ on Σ. We now consider the conformal metric $h = f^4 g$ on $\bar{M}$. Then ∞_2 can be compactified such that $\bar{M}^* := \bar{M} \cup \{\infty_2\}$ is a smooth manifold and $(\bar{M}^*, h)$ is an asymptotically flat 3-manifold with nonnegative scalar curvature. Also it can be shown that the ADM mass $m(h) = m(g) - 2\mathcal{C}(\Sigma, g)$ as in [Bra01].

Based on our assumptions, $m(h) \ll 1$, the case discussed in [DS23]. So modulo negligible spikes, $(\bar{M}^*, h)$ is close to the Euclidean 3-space $\mathbb{R}^3$ in the pointed measured Gromov-Hausdorff topology. For the original metric $g = f^{-4}h$, we will show that f is uniformly close to the conformal factor in the Schwarzschild metric in the following two steps.

We first prove a new integral inequality involving scalar curvature and the hessian of f (c.f. Proposition 4.2.1):

$$\frac{8\pi(m(g)^2 - (2\mathcal{C}(\Sigma, g))^2)}{m(g)^2} \geq$$
$$\int_M \left(\frac{|\nabla^2 f - (1-f)^{-1} f^{-1}(2f-1)|\nabla f|^2 (g - 3\nu \otimes \nu)|^2}{|\nabla f|} + R_g |\nabla f| \right) \mathrm{dvol}_g,$$

where $\nu = \frac{\nabla f}{|\nabla f|}$ and the integration is taken over regular set of f. This formula is very similar to the mass inequality proved by [Bra+22, Theorem 1.2]. One can follow the proof of [Bra+22, Theorem 1.2] and employ the technique of integration over level sets of f to control the integral in above inequality. But by this argument, we can only get an integral inequality with a coarse upper bound. To obtain the final effective estimate, we study a weighted volume, and use the nonnegative scalar curvature conditions together with ODE comparison to prove a weighted volume comparison (c.f. Lemma 4.2.3), which finally implies the desired estimate. As a corollary, we also get a mass-area-capacity inequality (c.f. Proposition 4.2.5).

Then by using the integral inequality of f, together with the techniques used in [DS23], we are able to find a region $\mathcal{E} \subset \bar{M}^*$ with a small area boundary. In this region $\mathcal{E}$, the behavior of $|\nabla f|$ closely resembles the conformal factor in the Schwarzschild metric, and particularly $|\nabla f|$ is uniformly bounded. By Arzelà-Ascoli theorem, along the convergence of $(\mathcal{E}, h)$, we can take a limit of such f and get a limit function f_∞ defined on $\mathbb{R}^3$. However, it is not immediately clear whether f_∞ is precisely the conformal factor in the Schwarzschild metric. To address this, we note that a favorable property of the pointed measured Gromov-Hausdorff convergence modulo negligible spikes, is that f also converges to f_∞ in the $W^{1,2}$-sense (c.f. Lemma 4.4.1). Therefore, the elliptic equations satisfied by f are preserved in this convergence, and f_∞ also satisfies an elliptic equation on $\mathbb{R}^3$. The fact that the area of the boundary $\partial\mathcal{E}$ converges to 0 has been used essentially here. The elliptic equation satisfied by f_∞ together with some growth conditions would imply the rigidity of f_∞ (c.f. Section 4.4).

Finally, using metric geometry tools and similar arguments in [DS23, Section 4], we can utilize these properties of f and h to prove Theorem 1.0.5.

From these main theorems together with their proofs, it seems that the GH topology modulo negligible spikes could be an appropriate tology when studying metrics with a uniform scalar curvature lower bound. So a natural follow-up question is:

Question 1.0.6. *Do we have precompactness for pointed complete Riemannian manifolds with a uniform scalar curvature lower bound in the pointed GH topology modulo negligible spikes?*

Also, we can ask that

Question 1.0.7. *If we have a sequence of metrics with a uniform scalar curvature lower bound converging in the pointed GH topology modulo negligible spikes, can we develop analysis on the limit space? Can we define some weak scalar curvature on the limit space?*

Chapter 2

Preliminaries

2.0.1 Notations

We will use C, C' to denote a universal positive constant (which may be different from line to line); $\Psi(t), \Psi(t|a, b, \cdots)$ denote small constants depending on $a, b, \cdots$ and satisfying

$$\lim_{t\to 0} \Psi(t) = 0,\ \lim_{t\to 0} \Psi(t|a, b\cdots) = 0,$$

for each fixed $a, b, \cdots$.

We denote the Euclidean metric by g_{Eucl} or δ, and the induced geometric quantities with subindex Eucl or δ.

For a general Riemannian manifold (M, g) and any $\pi \in M$, the geodesic ball with center p and radius r is denoted by $B_g(p, r)$ or $B(p, r)$ if the underlying metric is clear. Given a Riemannian metric, for a surface Σ and a domain Ω, Area(Σ) is the area of Σ, Vol(Ω) is the volume of Ω with respect to the metric.

Finally we introduce some notations about length metric that will be used later. Given a subset U in a Riemannian manifold (M, g), let $(U, \hat{d}_{g,U})$ be the induced length metric on U of the metric g, that is, for any $x_1, x_2 \in U$,

$$\hat{d}_{g,U}(x_1, x_2) := \inf\{L_g(\gamma) : \gamma \text{ is a rectifiable curve connecting } x_1, x_2 \text{ and } \gamma \subset U\},$$

where $L_g(\gamma) = \int_0^1 |\gamma'|_g$ is the length of γ with respect to metric g. For any $D > 0$ and $p \in U$, we use $\hat{B}_{g,U}(p, D)$ to denote the geodesic ball inside $(U, \hat{d}_{g,U})$, that is

$$\hat{B}_{g,U}(p, D) := \{x \in U : \hat{d}_{g,U}(p, x) \leq D\}.$$

2.0.2 Basics of Riemannian geometry

We review some basics of Riemannian geometry. Most of the content in this subsection is derived from [Pet06].

A Riemannian n-manifold (M^n, g) consists of a C^∞-manifold M (Hausdorff and second countable) and a Euclidean inner product g_p or $g|_p$ on each of the tangent spaces $T_pM \cong \mathbb{R}^n$ of M. In addition we assume that $p \mapsto g_p$ varies smoothly. The tensor g is referred to as the Riemannian metric or simply the metric. Usually, we also use $\langle \cdot, \cdot \rangle_g$ to denote the metric.

In local coordinate charts $\{x^i\}_{i=1}^n$ with $\partial_i = \frac{\partial}{\partial x^i}$, we can write

$$g = g_{ij} dx^i \cdot dx^j,$$

where $g_{ij} = g(\partial_i, \partial_j)$. The canonical flat metric on $\mathbb{R}^n$ in the identity chart is

$$g_{\text{Eucl}} = \delta_{ij} dx^i dx^j = \sum_{i=1}^n dx^i dx^i.$$

There is a canonical Riemannian volume n-form dvol_g, which is defined locally by

$$\text{dvol}_g = \sqrt{\det(g_{ij})} dx^1 \wedge dx^2 \cdots \wedge dx^n.$$

Given two vector fields $X = X^i \partial_i, Y = Y^j \partial_j$, the Levi-Civita connection ∇ is defined locally as

$$\nabla_Y X = Y^j (\partial_j X^i) \frac{\partial}{\partial x^i} + Y^j X^i \Gamma_{ij}^k \frac{\partial}{\partial x^k},$$

where

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right).$$

Then the curvature tensor is defined as

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \ \forall X, Y, Z \in TM.$$

In local coordinate charts, we can write $R(\partial_i, \partial_j)\partial_k = R^l_{ijk}\partial_l$, so

$$R^l_{ijk} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^s_{jk}\Gamma^l_{is} - \Gamma^s_{ik}\Gamma^l_{js}.$$

At any $p \in M$, for any $v, w \in T_pM$, the sectional curvature of (v, w) is defined as

$$\sec_{\mathrm{g}}(v,w) = \frac{g(R(w,v)v,w)}{g(v,v)g(w,w) - g(v,w)^2}.$$

The Ricci curvature is defined as

$$\mathrm{Ric}_g(v,w) = \sum_{i,j=1}^{n} g^{ij} g(R(v,\partial_i)\partial_j, w).$$

We say that $\mathrm{Ric} \geq \lambda$ if $\mathrm{Ric}(v,v) \geq \lambda g(v,v)$ for all $v \in T_pM$.

The scalar curvature is defined as

$$R_g = \sum_{i,j=1}^{n} g^{ij} \mathrm{Ric}_g(\partial_i, \partial_j).$$

For a smooth function $f : M \to \mathbb{R}$, we can define its gradient ∇f by $\langle \nabla f, X \rangle = X(f), \forall X \in TM$; its hessian $\nabla^2 f = \mathrm{Hess} f$ by

$$\nabla^2 f(X,Y) = \nabla_Y (\nabla f)(X) = \nabla_Y \nabla_X f - \nabla_{\nabla_Y X} f, \ \forall X, Y \in TM,$$

and its Laplacian by

$$\Delta f = \mathrm{tr}_g \nabla^2 f = \sum_{i,j=1}^{n} g^{ij} \nabla^2 f(\partial_i, \partial_j).$$

Locally, we have

$$\Delta f = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{\det g} \frac{\partial f}{\partial x^j} \right),$$

where g^{ij} is the inverse matrix of g_{ij}.

If $n \geq 3$, for the conformal metric $\tilde{g} = e^{2\varphi} g$, we have the relation between the scalar curvature

$$R_{\tilde{g}} = e^{-2\varphi}\left(R_g - \frac{4(n-1)}{(n-2)} e^{-\frac{(n-2)\varphi}{2}} \Delta_g (e^{\frac{(n-2)\varphi}{2}})\right),$$

and the Laplacian on functions:

$$\Delta_{\tilde{g}} f = e^{-2\varphi}\left(\Delta_g f + (n-2)\left\langle \nabla\varphi, \nabla f\right\rangle_g\right).$$

2.0.3 Geometry of Schwarzschild metric

For a positve number $m > 0$, the Schwarzschild 3-manifold (M^3_{sc}, g_{sc}) (with mass m) is given by the following warped product metric on $\mathbb{S}^2 \times [0, \infty)$:

$$g_{sc} = ds^2 + u_m(s)^2 g_{\mathbb{S}^2}, \quad s \in [0, \infty), \tag{2.0.1}$$

where $g_{\mathbb{S}^2}$ is the spherical metric with $\text{Area}(\mathbb{S}^2, g_{\mathbb{S}^2}) = 4\pi$, and u_m is a positive increasing function satisfying

$$u_m(0) = 2m,\ u_m'(0) = 0,\ u_m'(s) = \left(1 - \frac{2m}{u_m(s)}\right)^{\frac{1}{2}},\ u_m''(s) = \frac{m}{u_m(s)^2}. \tag{2.0.2}$$

Then the scalar curvature of g_{sc} is identically zero, and the boundary $\Sigma_{sc} := \partial M^3_{sc}$ is the only minimal surface inside M^3_{sc}.

Under Cartesian coordinate, M^3_{sc} is diffeomorphic to $\mathbb{R}^3 \setminus B(\frac{m}{2})$, where $B(\frac{m}{2})$ is the Euclidean ball with radius $\frac{m}{2}$ around the center, and we have

$$g_{sc,ij}(x) = \left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij},\ \forall |x| \geq \frac{m}{2}. \tag{2.0.3}$$

This metric can also be extended to give a Schwarzschild metric g_{sc} defined on $\mathbb{R}^3 \setminus \{0\}$.

Define $\rho_m(x) := \text{dist}(x, \Sigma_{sc})$. Then

$$\rho_m(x) = \int_{\frac{m}{2}}^{|x|} \left(1 + \frac{m}{2t}\right)^2 dt = |x| - \frac{m^2}{4|x|} + m \log \frac{2|x|}{m},$$

which implies that

$$\rho_m(x) - m \log \frac{2\rho_m(x)}{m} \leq |x| \leq \rho_m(x). \tag{2.0.4}$$

Using these two representations (2.0.1) and (2.0.3) of g_{sc} to compute the area of geodesic spheres, we have

$$u_m(\rho_m(x))^2 \cdot 4\pi = \left(1 + \frac{m}{2|x|}\right)^4 \cdot 4\pi |x|^2,$$

i.e.

$$u_m(\rho_m(x)) = \left(1 + \frac{m}{2|x|}\right)^2 \cdot |x|. \tag{2.0.5}$$

In particular, together with (2.0.4),

$$\lim_{r \to \infty} \frac{u_m(r)}{r} = 1. \tag{2.0.6}$$

Now we introduce another harmonic function f_m and rewrite above identities using f_m instead of $|x|$. Define

$$f_m(x) := \left(1 + \frac{m}{2|x|}\right)^{-1}.$$

Standard computations imply that

$$\Delta_{g_{sc}} f_m = 0, \quad f_m = \frac{1}{2} \text{ on } \Sigma_{sc}, \quad \lim_{|x| \to \infty} f_m(x) = 1.$$

Then

$$|x| = \frac{m}{2} \cdot \frac{f_m(x)}{1 - f_m(x)},$$

and

$$\rho_m(x) = \rho_m(f_m(x)) = \frac{m}{2}\left(\frac{1}{1 - f_m(x)} - \frac{1}{f_m(x)}\right) + m \log \frac{f_m(x)}{1 - f_m(x)}. \tag{2.0.7}$$

Moreover,

$$u_m(\rho_m(x)) = \frac{m}{2} \cdot \frac{1}{f_m(x)(1 - f_m(x))}. \tag{2.0.8}$$

2.0.4 Asymptotically flat 3-manifolds

A smooth orientable connected complete Riemannian 3-manifold (M^3, g) is called asymptotically flat if there exists a compact subset $K \subset M$ such that $M \setminus K = \sqcup_{k=1}^{N} M_{\text{end}}^k$ consists of finite pairwise disjoint ends, and for each $1 \leq k \leq N$, there exist $C > 0, \sigma > \frac{1}{2}$, and a C^∞-diffeomorphism $\Phi_k : M_{\text{end}}^k \to \mathbb{R}^3 \setminus B(1)$ such that under this identification,

$$|\partial^l (g_{ij} - \delta_{ij})(x)| \leq C|x|^{-\sigma-|l|},$$

for all multi-indices $|l| = 0, 1, 2$ and any $x \in \mathbb{R}^3 \setminus B(1)$. Furthermore, we always assume the scalar curvature R_g is integrable over (M^3, g). The ADM mass from general relativity of each end M_{end}^k, $1 \leq k \leq N$, is then well-defined (see [ADM61; Bar86]) and given by

$$m_k(g) := \lim_{r\to\infty} \frac{1}{16\pi} \int_{S_r} \sum_i (g_{ij,i} - g_{ii,j})\nu^j dA$$

where ν is the unit outer normal to the coordinate sphere S_r of radius $|x| = r$ in the given end, and dA is its area element.

Definition 2.0.1. A surface $\Sigma \subset (M^3, g)$ is called a horizon if it is a minimal surface. It is called an outermost horizon if it is a horizon and it is not enclosed by another minimal surface in (M^3, g).

Let (M^3, g) be an asymptotically flat 3-manifold. By Lemma 4.1 in [HI01], we know that inside M^3, there is a trapped compact region T whose topological boundary consists of smooth embedded minimal 2-spheres. An "exterior region" M_{ext}^3 is defined as the metric completion of any connected component of $M \setminus T$ containing one end. Then M_{ext}^3 is connected, asymptotically flat, has a compact minimal boundary ∂M_{ext}^3 (∂M_{ext}^3 may be empty), and contains no other compact minimal surfaces, that is, M_{ext}^3 is an asymptotically flat 3-manifold with outermost horizon boundary.

We will be able to perturb an asymptotically flat metric to a metric with nicer behavior at infinity in each end because of the following definition and proposition.

Definition 2.0.2. We say that (M^3, g) is harmonically flat at infinity if $(M^3 \setminus K, g)$ is isometric to a finite disjoint union of regions with zero scalar curvature which are conformal to $(\mathbb{R}^3 \setminus B, \delta)$ for some compact set K in M^3 and some ball B in $\mathbb{R}^3$ centered around the origin.

By definition, if (M^3, g) is harmonically flat, then on each end, $g_{ij}(x) = V(x)\delta_{ij}$ for some bounded positive δ-harmonic function $V(x)$, which satisfies that $\Delta_\delta V(x) = 0$ and (c.f. [Bra01, Equation (10)])

$$V(x) = a + \frac{b}{|x|} + O\left(\frac{1}{|x|^2}\right). \tag{2.0.9}$$

In this case, its ADM mass on this end is given by $2ab$.

Proposition 2.0.1 ([SY81])**.** *Let (M^3, g) be a complete, asymptotically flat 3-manifold with $R_g \geq 0$ and ADM mass $m_k(g)$ in the k-th end. For any $\epsilon > 0$, there exists a metric $\hat{g}$ such that $e^{-\epsilon} g \leq \hat{g} \leq e^{\epsilon} g$, $R_{\hat{g}} \geq 0$, $(M^3, \hat{g})$ is harmonically flat at infinity, and $|m_k(\hat{g}) - m_k(g)| \leq \epsilon$, where $m_k(\hat{g})$ is the ADM mass of $\hat{g}$ in the k-th end.*

2.0.5 pm-GH convergence modulo negligible spikes

In this subsection, we recall some definitions for the pointed measured Gromov-Hausdorff topology.

Assume $(X, d_X, x), (Y, d_Y, y)$ are two pointed metric spaces. The pointed Gromov-Hausdorff (or pGH-) distance is defined in the following way. A pointed map $f : (X, d_X, x) \to (Y, d_Y, y)$ is called an ε-pointed Gromov-Hausdorff approximation (or ε-pGH approximation) if it satisfies the following conditions:

(1) $f(x) = y$;

(2) $B(y, \frac{1}{\varepsilon}) \subset B_\varepsilon(f(B(x, \frac{1}{\varepsilon})))$;

(3) $|d_X(x_1, x_2) - d_Y(f(x_1), f(x_2))| < \varepsilon$ for all $x_1, x_2 \in B(x, \frac{1}{\varepsilon})$.

The pGH-distance is defined by

$$d_{pGH}((X,d_X,x),(Y,d_Y,y)) := \inf\{\varepsilon>0 : \exists\ \varepsilon\text{-pGH approximation } f:(X,d_X,x)\to(Y,d_Y,y)\}.$$

We say that a sequence of pointed metric spaces (X_i, d_i, p_i) converges to a pointed metric space (X, d, p) in the pointed Gromov-Hausdorff topology, if the following holds

$$d_{pGH}((X_i,d_i,p_i),(X,d,p))\to 0.$$

If (X_i, d_i) are length metric spaces, i.e. for any two points $x, y \in X_i$,

$$d_i(x,y)=\inf\{L_{d_i}(\gamma):\gamma \text{ is a rectifiable curve connecting } x,y\},$$

where $L_{d_i}(\gamma)$ is the length of γ induced by the metric d_i, then equivalently,

$$d_{pGH}((X_i,d_i,p_i),(X,d,p))\to 0$$

if and only if for all $D > 0$,

$$d_{pGH}((B(p_i,D),d_i),(B(p,D),d))\to 0,$$

where $B(p_i, D)$ are the geodesic balls of metric d_i.

A pointed metric measure space is a structure (X, d_X, μ, x) where (X, d_X) is a complete separable metric space, μ a Radon measure on X and $x \in \mathrm{supp}(\mu)$.

We say that a sequence of pointed metric measure length spaces (X_i, d_i, μ_i, p_i) converges to a pointed metric measure length space (X, d, μ, p) in the pointed measured Gromov-Hausdorff (or pm-GH) topology, if for any $\varepsilon > 0, D > 0$, there exists $N(\varepsilon, D) \in \mathbb{Z}_+$ such that for all $i \geq N(\varepsilon, D)$, there exists a Borel ε-pGH approximation

$$f_i^{D,\varepsilon}:(B(p_i,D),d_i,p_i)\to(B(p,D+\varepsilon),d,p)$$

satisfying

$$(f_i^{D,\varepsilon})_\sharp(\mu_i|_{B(p_i,D)}) \text{ weakly converges to } \mu|_{B(p,D)} \text{ as } i\to\infty, \text{ for } a.e. D>0.$$

In the case when X_i is an n-dimensional manifold, without extra explanations, we will always consider (X_i, d_i, p_i) as a pointed metric measure space equipped with the n-dimensional Hausdorff measure $\mathcal{H}^n_{d_i}$ induced by d_i.

Finally, we introduce a notation about the topology that we used in this paper. For simplicity, we only consider manifolds, and one can easily generalize it to general metric measured spaces.

Definition 2.0.3. For a sequence of pointed Riemannian n-manifolds (M_i, g_i, p_i) and (M, g, p), we say that (M_i, g_i, p_i) converges to (M, g, p) in the pointed measured Gromov-Hausdorff topology modulo negligible spikes, if there exist open subsets $Z_i \subset M_i$ such that $\mathcal{H}^{n-1}(\partial Z_i) \to 0$, $p_i \in M_i \setminus Z_i$ and

$$(M_i \setminus Z_i, \hat{d}_{g_i}, p_i) \to (M, d_g, p)$$

in the pointed measured Gromov-Hausdorff topology for the induced length metric.

Chapter 3

Stability for the Positive Mass Theorem

3.1 Regular region with small area boundary

In this section, for any given asymptotically flat 3-manifold (M^3, g) with nonnegative scalar curvature, we will find out an unbounded domain in M_{ext} containing the end such that its boundary has small area depending on $m(g)$. We always assume $0 < m(g) \ll 1$.

Consider the harmonic map $\mathbf{u} = (u^1, u^2, u^3) : M_{ext} \to \mathbb{R}^3$ associated to one end as in previous section, where for each $j \in \{1, 2, 3\}$, u^j is a harmonic function with Neumann boundary condition if $\partial M_{ext} \neq \emptyset$ and asymptotical to a coordinate function around the end.

Define the C^∞-function $F : (M_{ext}, g) \to \mathbb{R}$ by

$$F(x) := \sum_{j,k=1}^{3} \left(g(\nabla u^j, \nabla u^k) - \delta_{jk}\right)^2.$$

Fix a small number $0 < \epsilon \ll 1$. We use the following notations:

$$\forall t \in [0, 6\epsilon],\ S_t := F^{-1}(t),$$

$$\forall 0 \leq a < b \leq 6\epsilon,\ \Omega_{a,b} := F^{-1}([a, b]).$$

Notice that for any $t \in (0, 6\epsilon]$, S_t is compact, $S_t \cap \partial M_{ext} = \emptyset$ since ϵ is small, and the complement of $\Omega_{0,t}$ is compact in M_{ext}. By Sard's theorem, we will always consider $t \in (0, 6\epsilon]$ outside of the measure zero set of critical values of F, and we will call such t "generic". For generic choices of $0 < a < b < 6\epsilon$, we have $\partial\Omega_{a,b} = S_a \cup S_b$.

We will always consider the restricted map $\mathbf{u} : \Omega_{0,6\epsilon} \to \mathbb{R}^3$. We choose $\epsilon \ll 1$ such that for any $x \in \Omega_{0,6\epsilon}$, the Jacobian of $\mathbf{u}$ satisfies (with abuse of notations):

$$|\mathrm{Jac}\mathbf{u}(x) - \mathrm{Id}| \leq \epsilon', \tag{3.1.1}$$

for

$$\epsilon' := 100\sqrt{\epsilon} \ll 1,$$

so that in particular $\mathbf{u}$ is a local diffeomorphism. What we mean by (3.1.1) is that there exist orthonormal bases of T_xM and $\mathbb{R}^3$ such that with respect to these bases, $\mathrm{Jac}\mathbf{u}(x)$ is ϵ'-close to the identity map.

Lemma 3.1.1.

$$\int_{\Omega_{0,6\epsilon}} |\nabla F|^2 \leq Cm(g).$$

Proof. By definitions, we have

$$|\nabla u^j|(x) \leq \sqrt{1 + \sqrt{6\epsilon}},\ \forall x \in \Omega_{0,6\epsilon},\ \forall j \in \{1, 2, 3\}. \tag{3.1.2}$$

We readily obtain that for all $x \in \Omega_{0,6\epsilon}$,

$$\begin{aligned} |\nabla F|(x) &\leq \sum_{j,k=1}^{3} 2|g(\nabla u^j, \nabla u^k) - \delta_{jk}| \cdot (|\nabla u^j||\nabla^2 u^k| + |\nabla^2 u^j||\nabla u^k|) \\ &\leq C \sum_{j=1}^{3} |\nabla^2 u^j|. \end{aligned} \tag{3.1.3}$$

So by inequalities (3.1.2) and (**??**), we have

$$\begin{aligned}\int_{\Omega_{0,6\epsilon}} |\nabla F|^2 &\leq C\sum_{j=1}^{3}\int_{\Omega_{0,6\epsilon}} |\nabla^2 u^j|^2 \\ &\leq C\sum_{j=1}^{3}\int_{\Omega_{0,6\epsilon}} \frac{|\nabla^2 u^j|^2}{|\nabla u^j|} \\ &\leq Cm(g).\end{aligned} \tag{3.1.4}$$

□

In the context of General Relativity, the notion of capacity of a set has been studied in [Bra01; BM08; Jau20; Mia22] (see also references therein). We will use capacity in a different way. Recall that the classical Poincaré-Faber-Szegö inequality relates the capacity of a set in Euclidean space to its volume, see [PS51; Jau12]. In the following lemma, which is a key step in this section, we prove a Poincaré-Faber-Szegö type inequality. This will be used to find a smooth level set of F with small area.

Lemma 3.1.2. *If* $\inf_{s\in(0,5\epsilon)} \mathrm{Area}(S_s) > 0$*, where the infimum is taken over all generic regular values, then there exists* $s_0 \in (0, 5\epsilon)$ *such that*

$$\mathrm{Vol}(\Omega_{s_0,s_0+\epsilon} \cap \{|\nabla F| \neq 0\}) \leq C\left(\frac{m(g)}{\epsilon^2}\right)^3.$$

Proof. Since $\inf_{s\in(0,5\epsilon)} \mathrm{Area}(S_s) > 0$, we can find a generic $s_0 \in (0, 5\epsilon)$ such that S_{s_0} is a smooth surface satisfying

$$\mathrm{Area}(S_{s_0}) \leq 2 \inf_{s\in(0,5\epsilon)} \mathrm{Area}(S_s).$$

For the reader's convenience, we follow the presentation of the note [Jau12] when we can.

Case 1): $s_0 \in [3\epsilon, 5\epsilon)$.

For any regular value $s \in (\epsilon, s_0)$, define the function $\chi : \mathbb{R}^3 \to \mathbb{R}$ by

$$\chi(x) := \mathcal{H}^0(\mathbf{u}^{-1}(x) \cap \Omega_{s,s_0}).$$

By the isoperimetric inequality [EG15, Theorem 5.10 (i)],

$$\|\chi\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \leq C_{\mathrm{isop}}\|D\chi\|(\mathbb{R}^3).$$

Since $\mathbf{u}$ is a local diffeomorphism and $|\mathrm{Jac}\mathbf{u} - \mathrm{Id}| \leq \epsilon'$ by (3.1.1), this means that

$$\mathrm{Vol}(\Omega_{s,s_0})^{\frac{2}{3}} \leq 2C_{\mathrm{isop}}(\mathrm{Area}(S_s) + \mathrm{Area}(S_{s_0})). \tag{3.1.5}$$

By the definition of s_0, we have

$$\mathrm{Vol}(\Omega_{s,s_0})^{\frac{2}{3}} \leq 6C_{\mathrm{isop}}\mathrm{Area}(S_s). \tag{3.1.6}$$

By the coarea formula,

$$\int_{\Omega_{\epsilon,s_0}} |\nabla F|^2 \mathrm{dvol}_g = \int_0^{s_0-\epsilon} \int_{S_{s_0-t}} |\nabla F| dA_g dt.$$

For $t \in (0, s_0 - \epsilon)$, define

$$V(t) := \mathrm{Vol}(\Omega_{s_0-t,s_0}),\ S(t) := \mathrm{Area}(S_{s_0-t}).$$

By Sard's theorem and the assumption that $\inf_{s\in(0,5\epsilon)} \mathrm{Area}(S_s) > 0$, we know that except possibly for a measure zero set in $(0, s_0 - \epsilon)$, $S_{s_0-t} \neq \emptyset$ and $|\nabla F| \neq 0$ over S_{s_0-t}. Then

$$W(t) := \int_0^t \int_{S_{s_0-s}} \frac{1}{|\nabla F|} dA_g ds$$

is a strictly increasing continuous function, where the integral is taken over regular values of F, and

$$W'(t) = \int_{S_{s_0-t}} \frac{1}{|\nabla F|} dA_g > 0$$

is well-defined for a.e. $t \in (0, s_0 - \epsilon)$. Notice that for $t \in (0, s_0 - \epsilon)$,

$$0 < W(t) = \mathrm{Vol}(\Omega_{s_0-t,s_0} \cap \{|\nabla F| \neq 0\}) \leq V(t).$$

Since for a.e. $t \in (0, s_0 - \epsilon)$,

$$S(t)^2 \leq \int_{S_{s_0-t}} |\nabla F| dA_g \cdot \int_{S_{s_0-t}} \frac{1}{|\nabla F|} dA_g,$$

we have

$$\int_0^{s_0-\epsilon} \frac{S(t)^2}{W'(t)} dt \leq \int_{\Omega_{\epsilon,s_0}} |\nabla F|^2 \mathrm{dvol}_g.$$

By the isoperimetric inequality (3.1.6) obtained above,

$$W(t)^{\frac{2}{3}} \leq V(t)^{\frac{2}{3}} \leq 6C_{\mathrm{isop}}S(t),$$

so we have

$$\int_0^{s_0-\epsilon} \frac{W(t)^{\frac{4}{3}}}{W'(t)}dt \leq 36C_{\mathrm{isop}}^2 \int_{\Omega_{\epsilon,s_0}} |\nabla F|^2 \mathrm{dvol}_g. \tag{3.1.7}$$

Denote by ω_n the volume of the unit Euclidean n-ball. For any $t \in (0, s_0 - \epsilon)$, define $R(t)$ by

$$\omega_3 R(t)^3 = W(t).$$

Note that $R(0) = 0$, $R(s_0 - \epsilon) < \infty$, and the derivative $R'(t)$ is well-defined and positive almost everywhere.

Define the function $\tilde{F} : B_{\mathrm{Eucl}}(0, R(s_0 - \epsilon)) \to \mathbb{R}$ by

$$\tilde{F} := t \text{ on } \partial B_{\mathrm{Eucl}}(0, R(t)).$$

Then for a.e. $t \in (0, s_0 - \epsilon)$,

$$|\nabla \tilde{F}| = \frac{1}{R'(t)} \text{ on } \partial B_{\mathrm{Eucl}}(0, R(t)).$$

By (3.1.7), for some uniform constant $C > 0$,

$$\begin{aligned} C\int_{\Omega_{\epsilon,s_0}} |\nabla F|^2 \mathrm{dvol}_g &\geq \int_0^{s_0-\epsilon} 3\omega_3 \frac{R(t)^2}{R'(t)}dt \\ &= \int_0^{s_0-\epsilon} \int_{\partial B_{\mathrm{Eucl}}(0,R(t))} |\nabla \tilde{F}| dA dt \\ &= \int_{B_{\mathrm{Eucl}}(0,R(s_0-\epsilon))} |\nabla \tilde{F}|^2 dV. \end{aligned} \tag{3.1.8}$$

The above inequality gives an upper bound of the capacity of the Euclidean ball $B_{\mathrm{Eucl}}(0, R(s_0-\epsilon))$.

Let us recall the definition and some properties of the capacity (see [EG15, Definition 4.10, Theorem 4.15]). Set

$$K := \{f : \mathbb{R}^n \to \mathbb{R} : f \geq 0, f \in L^{2^*}(\mathbb{R}^n), |\nabla f| \in L^2(\mathbb{R}^n; \mathbb{R}^n)\}.$$

For any open subset $A \subset \mathbb{R}^n$, the capacity of A is defined as

$$\mathrm{Cap}(A) := \inf\{\int_{\mathbb{R}^n} |\nabla f|^2 dV : f \in K, A \subset \{f \geq 1\}^0\}.$$

Then for any $x \in \mathbb{R}^n$,

$$\mathrm{Cap}(B_{\mathrm{Eucl}}(x,r)) = r^{n-2}\mathrm{Cap}(B_{\mathrm{Eucl}}(1)) > 0. \tag{3.1.9}$$

In our case, modifying $\tilde{F}$, we define a test function $\hat{\varphi} : \mathbb{R}^3 \to \mathbb{R}$ by

$$\hat{\varphi} := \frac{s_0 - \epsilon - \tilde{F}}{s_0 - \frac{5}{2}\epsilon} \text{ on } B_{\mathrm{Eucl}}(0, R(s_0 - \epsilon)),$$

$$\hat{\varphi} := 0 \text{ on } \mathbb{R}^3 \setminus B_{\mathrm{Eucl}}(0, R(s_0 - \epsilon)).$$

With this definition, we have $\hat{\varphi} \in K$ and

$$B_{\mathrm{Eucl}}(0, R(\epsilon)) \subset \{\hat{\varphi} > 1\} = \{\hat{\varphi} \geq 1\}^0.$$

This implies that

$$\mathrm{Cap}(B_{\mathrm{Eucl}}(0, R(\epsilon))) \leq \int_{\mathbb{R}^3} |\nabla\hat{\varphi}|^2 dV = \frac{1}{(s_0 - \frac{5}{2}\epsilon)^2} \int_{B_{\mathrm{Eucl}}(0,R(s_0-\epsilon))} |\nabla\tilde{F}|^2 dV.$$

Together with (3.1.8), we deduce

$$\mathrm{Cap}(B_{\mathrm{Eucl}}(0, R(\epsilon))) \leq \frac{C}{(s_0 - \frac{5}{2}\epsilon)^2} \int_{\Omega_{\epsilon,s_0}} |\nabla F|^2 \mathrm{dvol}_g.$$

From (3.1.9), $s_0 \geq 3\epsilon$ and Lemma 3.1.1,

$$R(\epsilon) \leq \frac{Cm(g)}{\epsilon^2}.$$

Since $\mathrm{Vol}(\Omega_{s_0-\epsilon,s_0} \cap \{|\nabla F| \neq 0\}) = W(\epsilon) = \omega_3 R(\epsilon)^3$, we conclude:

$$\mathrm{Vol}(\Omega_{s_0-\epsilon,s_0} \cap \{|\nabla F| \neq 0\}) \leq C\left(\frac{m(g)}{\epsilon^2}\right)^3.$$

This is the desired conclusion, up to renaming $s_0 - \epsilon$ and s_0.

Case 2): $s_0 \in (0, 3\epsilon)$.

That case is completely similar to the first case. For any regular value $s \in (s_0, 5\epsilon)$, we have

$$\mathrm{Vol}(\Omega_{s_0,s})^{\frac{2}{3}} \leq 6C_{\mathrm{isop}}\mathrm{Area}(S_s).$$

For $t \in (0, 5\epsilon - s_0)$, define

$$W(t) := \mathrm{Vol}(\Omega_{s_0,s_0+t} \cap \{|\nabla F| \neq 0\}),\ \ S(t) := \mathrm{Area}(S_{s_0+t}),$$

and

$$\omega_3 R(t)^3 = W(t),$$

with $R(0) = 0$ and $R(5\epsilon - s_0) < \infty$.

Define $\tilde{F} : B_{\mathrm{Eucl}}(0, R(5\epsilon - s_0)) \to \mathbb{R}$ by

$$\tilde{F} := t \text{ on } \partial B_{\mathrm{Eucl}}(0, R(t)).$$

Then as in Case 1, we get

$$\int_{B_{\mathrm{Eucl}}(0,R(5\epsilon - s_0))} |\nabla \tilde{F}|^2 dV \leq C \int_{\Omega_{s_0,5\epsilon}} |\nabla F|^2 \mathrm{dvol}_g.$$

Modifying $\tilde{F}$, we define a test function $\hat{\varphi} : \mathbb{R}^3 \to \mathbb{R}$ by

$$\hat{\varphi} := \frac{5\epsilon - s_0 - \tilde{F}}{\frac{7}{2}\epsilon - s_0} \text{ on } B_{\mathrm{Eucl}}(0, R(5\epsilon - s_0)),$$

$$\hat{\varphi} := 0 \text{ on } \mathbb{R}^3 \setminus B_{\mathrm{Eucl}}(0, R(5\epsilon - s_0)).$$

So $\hat{\varphi} \in K$ and $B_{\mathrm{Eucl}}(0, R(\epsilon)) \subset \{\hat{\varphi} > 1\} = \{\hat{\varphi} \geq 1\}^0$. This implies that

$$\mathrm{Cap}(B_{\mathrm{Eucl}}(0, R(\epsilon))) \leq \frac{1}{(\frac{7}{2}\epsilon - s_0)^2} \int_{B_{\mathrm{Eucl}}(0,R(5\epsilon - s_0))} |\nabla \tilde{F}|^2 dV.$$

Thus we have

$$R(\epsilon) \leq \frac{Cm(g)}{\epsilon^2},$$

and since $\mathrm{Vol}(\Omega_{s_0,s_0+\epsilon} \cap \{|\nabla F| \neq 0\}) = W(\epsilon) = \omega_3 R(\epsilon)^3$, we conclude:

$$\mathrm{Vol}(\Omega_{s_0,s_0+\epsilon} \cap \{|\nabla F| \neq 0\}) \leq C \cdot \left(\frac{m(g)}{\epsilon^2}\right)^3.$$

□

Lemma 3.1.3. *There exists a generic $\tau_0 \in (0, 6\epsilon)$ such that S_{τ_0} is a smooth surface satisfying*

$$\text{Area}(S_{\tau_0}) \leq C\left(\frac{m(g)}{\epsilon^2}\right)^2.$$

Proof. If $\inf_{s\in(0,5\epsilon)} \text{Area}(S_s) = 0$, then we can take a generic $\tau_0 \in (0, 5\epsilon)$ such that S_{τ_0} is a smooth surface and satisfies

$$\text{Area}(S_{\tau_0}) \leq \frac{m(g)^2}{\epsilon^4}.$$

If $\inf_{s\in(0,5\epsilon)} \text{Area}(S_s) > 0$, then we can choose $s_0 \in (0, 5\epsilon)$ as in Lemma 3.1.2. By the coarea formula,

$$\begin{aligned}
\int_{s_0}^{s_0+\epsilon} \text{Area}(S_t)dt &= \int_{\Omega_{s_0,s_0+\epsilon}} |\nabla F| \\
&\leq \left(\int_{\Omega_{s_0,s_0+\epsilon}} |\nabla F|^2\right)^{\frac{1}{2}} \cdot (\text{Vol}(\Omega_{s_0,s_0+\epsilon} \cap \{|\nabla F| \neq 0\}))^{\frac{1}{2}} \qquad (3.1.10) \\
&\leq C\sqrt{m(g)}\left(\frac{m(g)}{\epsilon^2}\right)^{\frac{3}{2}},
\end{aligned}$$

where in the last inequality we used Lemma 3.1.1 and Lemma 3.1.2. So there exists a generic $\tau_0 \in (s_0, s_0 + \epsilon) \subset (0, 6\epsilon)$ such that S_{τ_0} is smooth and

$$\text{Area}(S_{\tau_0}) \leq \frac{Cm(g)^2}{\epsilon^4}.$$

□

For τ_0 in above lemma, the domain Ω_{0,τ_0} has smooth boundary S_{τ_0}, whose area is very small depending on $m(g)$, and contains the end of M_{ext}. In general, $\mathbf{u} : \Omega_{0,\tau_0} \to \mathbb{R}^3$ is only a local but not global diffeomorphism. For this reason, we need to restrict $\mathbf{u}$ to a smaller region.

By definition of asymptotic flatness and the construction of $\mathbf{u}$, we clearly have the following lemma (see for instance [Don22, Lemma 3.3, 3.4]).

Lemma 3.1.4. $\mathbf{u}$ *is one-to-one around the end at infinity. That is, for some big number $L > 0$ (not uniform in general), there exists an unbounded domain $U \subset M_{ext}$ containing the end such that $\mathbf{u} : U \to \mathbb{R}^3 \setminus B_{\text{Eucl}}(0, L)$ is injective and onto.*

Proposition 3.1.5. *Assume (M^3, g) is an asymptotically flat 3-manifold with nonnegative scalar curvature. For any end of (M, g), let M_{ext} be the exterior region associated to this end. Then there exists a connected region $E \subset M_{ext}$ with smooth boundary, such that the restricted harmonic map*

$$\mathbf{u} : E \to Y \subset \mathbb{R}^3$$

is a diffeomorphism onto its image $Y := \mathbf{u}(E)$, E contains the end of M_{ext}, Y contains the end of $\mathbb{R}^3$, and

$$\text{Area}(\partial E) \leq \frac{Cm(g)^2}{\epsilon^4}.$$

Proof. Take τ_0 in Lemma 3.1.3. Then $\mathbf{u} : \Omega_{0,\tau_0} \to \mathbb{R}^3$ is a local diffeomorphism and $\partial\Omega_{0,\tau_0} = S_{\tau_0}$ has small area. Let $Y_1 \subset \mathbf{u}(\Omega_{0,\tau_0})$ be the connected component containing the end of $\mathbb{R}^3$, and

$$Y_2 := \{y \in Y_1 : \mathcal{H}^0(\mathbf{u}^{-1}\{y\} \cap \Omega_{0,\tau_0}) = 1\}.$$

By Lemma 3.1.4, $Y_2 \neq \emptyset$ and contains the end of $\mathbb{R}^3$. Notice that Y_2 is open in Y_1 and $\partial Y_2 \subset \mathbf{u}(S_{\tau_0})$.

Since $\mathbf{u}(S_{\tau_0})$ is a smooth immersed surface in $\mathbb{R}^3$, we can choose a slightly smaller region Y_3 such that $Y_3 \cup \partial Y_3 \subset Y_2$, ∂Y_3 is a smooth embedded surface and $\text{Area}(\partial Y_3) \leq 2\text{Area}(\partial Y_2)$.

Define Y to be the connected component of Y_3 containing the end of $\mathbb{R}^3$, and $E := \mathbf{u}^{-1}(Y)$. By construction, $\mathbf{u} : E \to Y$ is a diffeomorphism. Moreover by (3.1.1),

$$\text{Area}(\partial E) = \int_{\partial Y} |\text{Jac}(\mathbf{u}\big|_{\partial E})^{-1}| \leq C(1 + \epsilon')\text{Area}(\partial Y) \leq C\text{Area}(S_{\tau_0}).$$

□

3.2 Proof of Theorem 1.0.3

In general a regular region E such as the one given by Proposition 3.1.5 is not sufficient to get Gromov-Hausdorff convergence. In this section, we will construct a more refined subregion over which we have pointed measured Gromov-Hausdorff convergence for the induced length metric.

Fix a continuous function $\xi : (0, \infty) \to (0, \infty)$ with $\xi(0) = 0$ and

$$\lim_{x \to 0^+} \xi(x) = 0.$$

For our purposes we can assume without loss of generality that

$$\lim_{x \to 0^+} \frac{x}{\xi(x)} = 0.$$

Choose continuous functions $\xi_0, \xi_1 : (0, \infty) \to (0, \infty)$ with $\xi_0(0) = \xi_1(0) = 0$, and

$$\lim_{x \to 0^+} \frac{\xi(x)}{\xi_0^{100}(x)} = \lim_{x \to 0^+} \frac{\xi_0(x)}{\xi_1^{100}(x)} = 0.$$

The reader can think of these functions ξ, ξ_0, ξ_1, as converging to 0 very slowly as $x \to 0^+$. Set

$$\delta_0 := \xi_0(m(g)), \quad \delta_1 := \xi_1(m(g)).$$

Then

$$\delta_1^{100} \gg \delta_0 \gg \xi(m(g))^{\frac{1}{100}} \gg m(g)^{\frac{1}{100}}$$

when $0 < m(g) \ll 1$. In the following, we will always assume $0 < m(g) \ll 1$.

Let E be the regular region given by Proposition 3.1.5 and its image $Y := \mathbf{u}(E) \subset \mathbb{R}^3$. Then $\mathbf{u} : E \to Y$ is a diffeomorphism and

$$\text{Area}(\partial Y) \leq \frac{Cm(g)^2}{\epsilon^4}. \tag{3.2.1}$$

For any subset $U \subset E$, let $(U, \hat{d}_{g,U})$ be the induced length metric on U of the metric g, that is, for any $x_1, x_2 \in U$,

$$\hat{d}_{g,U}(x_1, x_2) := \inf\{L_g(\gamma) : \gamma \text{ is a rectifiable curve connecting } x_1, x_2 \text{ and } \gamma \subset U\},$$

where $L_g(\gamma) = \int_0^1 |\gamma'|_g$ is the length of γ with respect to metric g.

Similarly, for any $V \subset Y \subset \mathbb{R}^3$, let $(V, \hat{d}_{\mathrm{Eucl},V})$ be the induced length metric on V of the standard Euclidean metric g_{Eucl}, that is, for any $y_1, y_2 \in V$,

$$\hat{d}_{\mathrm{Eucl},V}(y_1, y_2) := \inf\{L_{\mathrm{Eucl}}(\gamma) : \gamma \text{ is a rectifiable curve connecting } y_1, y_2 \text{ and } \gamma \subset V\},$$

where $L_{\mathrm{Eucl}}(\gamma)$ is the length of γ with respect to the Euclidean metric g_{Eucl}.

Write

$$\Sigma := \partial Y \subset \mathbb{R}^3$$

and let $\mathcal{W}$ be the compact domain bounded by Σ in $\mathbb{R}^3$, so $\partial \mathcal{W} = \Sigma$.

The main part of this section is devoted to the proofs of Proposition 3.3.7 and the lemmas leading to it. In Proposition 3.3.7, we construct a subregion inside $Y \subset \mathbb{R}^3$ with small area boundary, over which the induced length metric of g_{Eucl} is close to the restriction of the Euclidean metric.

By the isoperimetric inequality and (3.2.1),

$$\mathrm{Vol}(\mathcal{W}) \leq C \mathrm{Area}(\Sigma)^{\frac{3}{2}} \leq \frac{C m(g)^3}{\epsilon^6}.$$

Take

$$\epsilon = \delta_0$$

(in particular in this section ϵ depends on $m(g)$). Recall that

$$\epsilon' := 100\sqrt{\epsilon} \ll 1.$$

Then

$$\mathrm{Area}(\Sigma) \leq \frac{C m(g)^2}{\delta_0^4} \ll 1, \ \mathrm{Vol}(\mathcal{W}) \leq \frac{C m(g)^3}{\delta_0^6} \ll 1.$$

For any triple $\mathbf{k} = (k_1, k_2, k_3) \in \mathbb{Z}^3$, consider the cube $\mathbf{C}_{\mathbf{k}}(\delta_1)$ defined by

$$\mathbf{C}_{\mathbf{k}}(\delta_1) := (k_1\delta_1, (k_1+1)\delta_1) \times (k_2\delta_1, (k_2+1)\delta_1) \times (k_3\delta_1, (k_3+1)\delta_1) \subset \mathbb{R}^3.$$

Let $B_{\mathbf{k}}(r)$ be the Euclidean ball with center the same as $\mathbf{C}_{\mathbf{k}}(\delta_1)$ and radius r. By applying the coarea formula, we can find $r \in (3\delta_1, 3\delta_1 + \delta_0)$ such that $\mathcal{W} \cap \partial B_{\mathbf{k}}(r)$ consists of smooth surfaces and

$$\operatorname{Area}(\mathcal{W}\cap\partial B_{\mathbf{k}}(r))\leq\frac{\operatorname{Vol}(\mathcal{W}\cap B_{\mathbf{k}}(4\delta_1))}{\delta_0}. \tag{3.2.2}$$

Since $\operatorname{Vol}(B_{\mathbf{k}}(4\delta_1))\geq 16\omega_3\delta_1^3$, where ω_3 is the Euclidean volume of unit ball in $\mathbb{R}^3$, we know

$$\operatorname{Vol}(\mathcal{W}\cap B_{\mathbf{k}}(4\delta_1))\leq\operatorname{Vol}(\mathcal{W})\leq\frac{Cm(g)^3}{\delta_0^6}\ll\delta_1^3<\frac{1}{10}\operatorname{Vol}(B_{\mathbf{k}}(4\delta_1)).$$

By the relative isoperimetric inequality [EG15, Theorem 5.11 (ii)],

$$\begin{aligned}\operatorname{Vol}(\mathcal{W}\cap B_{\mathbf{k}}(4\delta_1))&\leq C\operatorname{Area}(\Sigma\cap B_{\mathbf{k}}(4\delta_1))^{\frac{3}{2}}\\&\leq\frac{Cm(g)}{\delta_0^2}\operatorname{Area}(\Sigma\cap B_{\mathbf{k}}(4\delta_1)).\end{aligned}$$

So by (3.2.2),

$$\begin{aligned}\operatorname{Area}(\mathcal{W}\cap\partial B_{\mathbf{k}}(r))&\leq\frac{Cm(g)}{\delta_0^3}\operatorname{Area}(\Sigma\cap B_{\mathbf{k}}(4\delta_1))\\&\leq\operatorname{Area}(\Sigma\cap B_{\mathbf{k}}(4\delta_1)).\end{aligned}$$

Then we can smooth the surface $(\Sigma\cap B_{\mathbf{k}}(r))\cup(\mathcal{W}\cap\partial B_{\mathbf{k}}(r))$ to get a closed embedded surface $\Sigma_{\mathbf{k}}\subset B_{\mathbf{k}}(4\delta_1)$, which coincides with Σ inside $\mathbf{C}_{\mathbf{k}}(\delta_1)$, and which satisfies

$$\begin{aligned}\operatorname{Area}(\Sigma_{\mathbf{k}})&\leq 2(\operatorname{Area}(\Sigma\cap B_{\mathbf{k}}(4\delta_1))+\operatorname{Area}(\mathcal{W}\cap\partial B_{\mathbf{k}}(r)))\\&\leq 4\operatorname{Area}(\Sigma\cap B_{\mathbf{k}}(4\delta_1)).\end{aligned} \tag{3.2.3}$$

For $t\in\mathbb{R}$, define the plane

$$A_{\mathbf{k},\delta_1}(t):=\{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_3=(k_3+t)\delta_1\}.$$

By definition $\mathbf{C}_{\mathbf{k}}(\delta_1)\subset\bigcup_{t\in[0,1]}A_{\mathbf{k},\delta_1}(t)$.

By the coarea formula, there exists $t_{\mathbf{k}}\in(\frac{1}{2},\frac{1}{2}+\delta_0)$ such that $A_{\mathbf{k},\delta_1}(t_{\mathbf{k}})\cap\Sigma_{\mathbf{k}}$ consists of smooth curves and

$$\begin{aligned}\operatorname{Length}(A_{\mathbf{k},\delta_1}(t_{\mathbf{k}})\cap\Sigma_{\mathbf{k}})&\leq\frac{\operatorname{Area}(\Sigma_{\mathbf{k}})}{\delta_0\delta_1}\\&\leq\frac{Cm(g)^2}{\delta_0^5\delta_1}\\&\leq m(g).\end{aligned} \tag{3.2.4}$$

Define $D'_{\mathbf{k}}$ as the connected component of $(\mathbf{C}_{\mathbf{k}}(\delta_1) \cap A_{\mathbf{k},\delta_1}(t_{\mathbf{k}})) \setminus \Sigma$ with largest area and $D''_{\mathbf{k}} := (\mathbf{C}_{\mathbf{k}}(\delta_1) \cap A_{\mathbf{k},\delta_1}(t_{\mathbf{k}})) \setminus D'_{\mathbf{k}}$. By the relative isoperimetric inequality and (3.2.4), we know that

$$\begin{aligned}\text{Area}(D''_{\mathbf{k}}) &\leq C\text{Length}(A_{\mathbf{k},\delta_1}(t_{\mathbf{k}}) \cap \Sigma_{\mathbf{k}})^2 \\ &\leq Cm(g)\frac{\text{Area}(\Sigma_{\mathbf{k}})}{\delta_0\delta_1} \\ &\leq \text{Area}(\Sigma_{\mathbf{k}}).\end{aligned} \tag{3.2.5}$$

Let $\pi_{\mathbf{k}} : \mathbb{R}^3 \to A_{\mathbf{k},\delta_1}(t_{\mathbf{k}})$ be the orthogonal projection. Define

$$\mathbf{C}_{\mathbf{k}}(\delta_1)' := D'_{\mathbf{k}} \cup \left(\mathbf{C}_{\mathbf{k}}(\delta_1) \cap \pi_{\mathbf{k}}^{-1}(D'_{\mathbf{k}} \setminus \pi_{\mathbf{k}}(\Sigma_{\mathbf{k}}))\right).$$

Lemma 3.2.1. *$\mathbf{C}_{\mathbf{k}}(\delta_1)'$ is path connected.*

Proof. By definition, for any point $x \in \mathbf{C}_{\mathbf{k}}(\delta_1) \cap \pi_{\mathbf{k}}^{-1}(D'_{\mathbf{k}} \setminus \pi_{\mathbf{k}}(\Sigma_{\mathbf{k}}))$, the line segment $L_x \subset \mathbf{C}_{\mathbf{k}}(\delta_1)$ through x and orthogonal to $A_{\mathbf{k},\delta_1}(t_{\mathbf{k}})$ satisfies $L_x \cap D'_{\mathbf{k}} \neq \emptyset$. Since $D'_{\mathbf{k}}$ is path connected, $\mathbf{C}_{\mathbf{k}}(\delta_1)'$ is also path connected. □

Lemma 3.2.2. $\text{Vol}(\mathbf{C}_{\mathbf{k}}(\delta_1) \setminus \mathbf{C}_{\mathbf{k}}(\delta_1)') \leq 8\delta_1\text{Area}(\Sigma \cap B_{\mathbf{k}}(4\delta_1)) \leq m(g)\delta_1^3$.

Proof. Since

$$\text{Area}(\pi_{\mathbf{k}}(\Sigma_{\mathbf{k}})) \leq \text{Area}(\Sigma_{\mathbf{k}}),$$

and since by (3.2.5),

$$\text{Area}(D''_{\mathbf{k}}) \leq \text{Area}(\Sigma_{\mathbf{k}}),$$

we have

$$\begin{aligned}\text{Vol}(\mathbf{C}_{\mathbf{k}}(\delta_1)) \setminus \text{Vol}(\mathbf{C}_{\mathbf{k}}(\delta_1)') &\leq 2\delta_1\text{Area}(\Sigma_{\mathbf{k}}) \\ &\leq 8\delta_1\text{Area}(\Sigma \cap B_{\mathbf{k}}(4\delta_1)).\end{aligned} \tag{3.2.6}$$

We can use (3.2.1) to conclude the proof. □

Clearly by construction,

$$\mathbf{C}_{\mathbf{k}}(\delta_1)' \subset Y.$$

Define

$$Y' := \cup_{\mathbf{k}\in\mathbb{Z}^3}\mathbf{C}_{\mathbf{k}}(\delta_1)' \subset Y.$$

Notice that when $|\mathbf{k}|$ is big enough, one can certainly ensure that $\mathbf{C}_{\mathbf{k}}(\delta_1)' = \mathbf{C}_{\mathbf{k}}(\delta_1)$, so that $Y \setminus Y'$ is a bounded set. Choosing Y slightly bigger in Proposition 3.1.5, we can assume that $\partial Y \cap Y' = \emptyset$.

For any subset $V \subset Y$, let V_t be the t-neighborhood of V inside $(Y, \hat{d}_{\mathrm{Eucl},Y})$ in terms of the length metric $\hat{d}_{\mathrm{Eucl},Y}$, i.e.

$$V_t := \{y \in Y : \exists z \in V \text{ such that } \hat{d}_{\mathrm{Eucl},Y}(y,z) \leq t\}.$$

So $(Y')_t$ is the t-neighborhood of Y' inside $(Y, \hat{d}_{\mathrm{Eucl},Y})$.

In the following lemma, by modifying some $(Y')_t$, we construct a domain with smooth boundary such that its boundary area is small and it is very close to Y' in the Gromov-Hausdorff topology with respect to a length metric.

Lemma 3.2.3. *There exists Y'' with smooth boundary such that $Y' \subset Y'' \subset (Y')_{6\delta_0}$,*

$$\mathrm{Area}(\partial Y'') \leq \frac{m(g)^2}{\delta_0^5},$$

and Y'' is contained in the $6\delta_0$-neighborhood of Y' inside Y'', with respect to its length metric $\hat{d}_{\mathrm{Eucl},Y''}$.

Proof. Smoothing the Lipschitz function $\hat{d}_{\mathrm{Eucl},Y}(Y',\cdot)$, we can get a smooth function $\phi : Y \to \mathbb{R}$ such that $|\phi - \hat{d}_{\mathrm{Eucl},Y}(Y',\cdot)| \leq \delta_0$ and $|\nabla\phi| \leq 2$ (see for instance [GW79, Proposition 2.1]). Applying coarea formula to ϕ, we have

$$\int_{3\delta_0}^{4\delta_0} \mathrm{Area}(\phi^{-1}(t) \cap Y)dt = \int_{\{3\delta_0<\phi<4\delta_0\}\cap Y} |\nabla\phi| \mathrm{dvol} \leq 2\mathrm{Vol}(Y \setminus Y').$$

By Lemma 3.2.2, for each $\mathbf{k} \in \mathbb{Z}^3$,

$$0 \leq \mathrm{Vol}(\mathbf{C}_{\mathbf{k}}(\delta_1)) - \mathrm{Vol}(\mathbf{C}'_{\mathbf{k}}(\delta_1)) \leq 8\delta_1 \mathrm{Area}(\Sigma \cap B_{\mathbf{k}}(4\delta_1)).$$

Since the number of overlaps of $\{B_{\mathbf{k}}(4\delta_1)\}_{\mathbf{k}\in\mathbb{Z}^3}$ is uniformly bounded,

$$\begin{aligned} 0 \le \mathrm{Vol}(Y) - \mathrm{Vol}(Y') &\le 8\delta_1 \sum_{\mathbf{k}\in\mathbb{Z}^3} \mathrm{Area}(\Sigma \cap B_{\mathbf{k}}(4\delta_1)) \\ &\le C\delta_1 \mathrm{Area}(\Sigma) \\ &\le \frac{C\delta_1 m(g)^2}{\delta_0^4}. \end{aligned} \tag{3.2.7}$$

So we can find a generic regular value $t \in (3\delta_0, 4\delta_0)$ of ϕ such that $\phi^{-1}(t)$ is smooth and

$$\mathrm{Area}(\phi^{-1}(t) \cap Y) \le \frac{C\delta_1 m(g)^2}{\delta_0^5} \le \frac{m(g)^2}{8\delta_0^5}.$$

Smoothing $(\phi^{-1}(t) \cap Y) \cup (\partial Y \cap \{\phi < t\})$ inside Y, we can get a smooth surface S_1 with $S_1 \subset (Y')_{5\delta_0} \setminus Y'$ and

$$\mathrm{Area}(S_1) \le 2(\mathrm{Area}(\phi^{-1}(t) \cap Y) + \mathrm{Area}(\partial Y)) \le \frac{m(g)^2}{4\delta_0^5} + \frac{Cm(g)^2}{\delta_0^4} \le \frac{m(g)^2}{2\delta_0^5}.$$

Denote by Y_1 the connected component such that

$$Y' \subset Y_1 \subset (Y')_{5\delta_0} \subset Y \quad \text{and} \quad \partial Y_1 \subset S_1. \tag{3.2.8}$$

At this point, Y_1 is close to Y' in the Hausdorff topology with respect to $\hat{d}_{\mathrm{Eucl},Y}$, but possibly not with respect to its own length metric $\hat{d}_{\mathrm{Eucl},Y_1}$. To remedy this, choose a finite subset $\{x_j\}$ consisting of δ_0-dense discrete points of $(Y_1 \setminus Y', \hat{d}_{\mathrm{Eucl},Y_1})$ and denote by $\gamma_j \subset Y$ a smooth curve connecting x_j to Y' with minimal length with respect to the length metric $\hat{d}_{\mathrm{Eucl},Y}$. Then by (3.3.11), γ_j has length at most $5\delta_0$, and so $\gamma_j \subset (Y')_{5\delta_0}$. By thickening each γ_j, we can get thin solid tubes T_j inside δ_0-neighborhood of γ_j with arbitrarily small boundary area. Let $Y_2 := Y_1 \cup (\cup_j T_j)$. By smoothing the corners of Y_2, we have a connected domain Y'' with smooth boundary such that

$$Y' \subset Y'' \subset Y_2 \subset Y'_{6\delta_0}$$

and

$$\mathrm{Area}(\partial Y'') \le 2\mathrm{Area}(S_1) \le \frac{m(g)^2}{\delta_0^5}.$$

For any $y \in Y''\backslash Y'$, by our construction, there exists some j such that either $\hat{d}_{\text{Eucl},Y_1}(y, x_j) \leq \delta_0$ or $y \in T_j$. In each case, there exists a smooth curve $\sigma_{y,j} \subset Y''$ connecting y to a point in γ_j and $\text{Length}(\sigma_{y,j}) \leq \delta_0$. Since $\text{Length}(\gamma_j) \leq 5\delta_0$, $\sigma_{y,j} \cup \gamma_j$ is a piecewise smooth curve inside Y'' connecting y to Y' with length smaller than $6\delta_0$. So inside the length space $(Y'', \hat{d}_{\text{Eucl},Y''})$, Y'' is in the $6\delta_0$-neighborhood of Y' as desired.

□

Let Y'' be as in Lemma 3.2.3. Recall that $\hat{d}_{\text{Eucl},Y''}$ is defined as the length metric on Y'' induced by g_{Eucl}. Since $Y' \subset Y'' \subset Y$, we have $d_{\text{Eucl}} \leq \hat{d}_{\text{Eucl},Y} \leq \hat{d}_{\text{Eucl},Y''}$.

Lemma 3.2.4. $\text{diam}_{\hat{d}_{\text{Eucl},Y''}}(\mathbf{C}_{\mathbf{k}}(\delta_1)') \leq 5\delta_1$.

Proof. For any two points $x_1, x_2 \in \mathbf{C}_{\mathbf{k}}(\delta_1)'$, let L_{x_1}, L_{x_2} be the line segments inside $\mathbf{C}_{\mathbf{k}}(\delta_1)$ through x_1, x_2 and orthogonal to $A_{\mathbf{k},\delta_1}(t_{\mathbf{k}})$ respectively. Let $x_1' = L_{x_1} \cap D_{\mathbf{k}}', x_2' = L_{x_2} \cap D_{\mathbf{k}}'$. Then by modifying the line segment between x_1', x_2' if necessary, and by (3.2.4), we can find a curve γ between x_1', x_2' inside $D_{\mathbf{k}}'$ such that

$$\begin{aligned} \text{Length}_{\text{Eucl}}(\gamma) &\leq d_{\text{Eucl}}(x_1', x_2') + \text{Length}(A_{\mathbf{k},\delta_1}(t_{\mathbf{k}}) \cap \Sigma_{\mathbf{k}}) \\ &\leq d_{\text{Eucl}}(x_1', x_2') + m(g). \end{aligned}$$

Consider the curve $\tilde{\gamma}$ consisting of three parts: the line segment $[x_1 x_1']$ between x_1, x_1', γ, and the line segment $[x_2' x_2]$ between x_2', x_2. We have $\tilde{\gamma} \subset \mathbf{C}_{\mathbf{k}}(\delta_1)' \subset Y'$, so

$$\hat{d}_{\text{Eucl},Y''}(x_1, x_2) \leq L_{\text{Eucl}}(\tilde{\gamma}) \leq 4\delta_1 + m(g) \leq 5\delta_1.$$

□

Lemma 3.2.5. *For any base point $q \in Y'$ and any $D > 0$,*

$$d_{pGH}((Y' \cap B_{\text{Eucl}}(q, D), \hat{d}_{\text{Eucl},Y''}, q), (Y' \cap B_{\text{Eucl}}(q, D), d_{\text{Eucl}}, q)) \leq \Psi(m(g)).$$

Proof. Let $x_0, y_0 \in Y' \cap B_{\text{Eucl}}(q, D)$ be two points and $x_0 \in \mathbf{C}_{\mathbf{k}}(\delta_1)', y_0 \in \mathbf{C}_{\mathbf{l}}(\delta_1)'$ for some $\mathbf{k}, \mathbf{l} \in \mathbb{Z}^3$. Since $d_{\text{Eucl}} \leq \hat{d}_{\text{Eucl},Y''}$, it's enough to show

$$\hat{d}_{\text{Eucl},Y''}(x_0, y_0) \leq d_{\text{Eucl}}(x_0, y_0) + \Psi(m(g)). \qquad (3.2.9)$$

Let $T_{\mathbf{k},\mathbf{l}}$ be the translation which maps $\mathbf{C}_{\mathbf{k}}(\delta_1)$ to $\mathbf{C}_{\mathbf{l}}(\delta_1)$. Then by Lemma 3.2.2,

$$\begin{aligned}\mathrm{Vol}(T_{\mathbf{k},\mathbf{l}}(\mathbf{C}_{\mathbf{k}}(\delta_1)') \cap \mathbf{C}_{\mathbf{l}}(\delta_1)') \geq & \mathrm{Vol}(\mathbf{C}_{\mathbf{k}}(\delta_1)) - (\mathrm{Vol}(\mathbf{C}_{\mathbf{k}}(\delta_1) \setminus \mathbf{C}_{\mathbf{k}}(\delta_1)')) \\ & - (\mathrm{Vol}(\mathbf{C}_{\mathbf{l}}(\delta_1) \setminus \mathbf{C}_{\mathbf{l}}(\delta_1)')) \\ \geq & (1 - 2m(g))\delta_1^3.\end{aligned}$$

If $\mathbf{k} = \mathbf{l}$, then by Lemma 3.2.4, we know that

$$\hat{d}_{\mathrm{Eucl},Y''}(x_0, y_0) \leq 4\delta_1 \leq d_{\mathrm{Eucl}}(x_0, y_0) + 4\delta_1.$$

Suppose that $\mathbf{k} \neq \mathbf{l}$. For any $x \in \mathbb{R}^3$, the straight line between x and $T_{\mathbf{k},\mathbf{l}}(x)$ meets the set $T_{\mathbf{k},\mathbf{l}}(\mathbf{C}_{\mathbf{k}}(\delta_1)') \cap \mathbf{C}_{\mathbf{l}}(\delta_1)'$ in a subset of total length at most say $10\delta_1$. We claim that there is at least one point $x_0' \in \mathbf{C}_{\mathbf{k}}(\delta_1)'$ such that $T_{\mathbf{k},\mathbf{l}}(x_0') \in \mathbf{C}_{\mathbf{l}}(\delta_1)'$ and the line segment $[x_0' T_{\mathbf{k},\mathbf{l}}(x_0')]$ between these two points has no intersection with $\partial Y''$. Otherwise, by the coarea formula and Lemma 3.2.3, we would get

$$\mathrm{Vol}(T_{\mathbf{k},\mathbf{l}}(\mathbf{C}_{\mathbf{k}}(\delta_1)') \cap \mathbf{C}_{\mathbf{l}}(\delta_1)') \leq 10\delta_1 \mathrm{Area}(\partial Y'') \leq 10\delta_1 m(g),$$

which together with the above estimate on the left hand side would give

$$(1 - 2m(g))\delta_1^3 \leq 10\delta_1 m(g),$$

a contradiction when $0 < m(g) \ll 1$.

Since from the paragraph above, $[x_0' T_{\mathbf{k},\mathbf{l}}(x_0')] \subset Y''$, we estimate

$$\begin{aligned}\hat{d}_{\mathrm{Eucl},Y''}(x_0, y_0) &\leq \hat{d}_{\mathrm{Eucl},Y''}(x_0, x_0') + \mathrm{Length}_{\mathrm{Eucl}}([x_0' T_{\mathbf{k},\mathbf{l}}(x_0')]) + \hat{d}_{\mathrm{Eucl},Y''}(T_{\mathbf{k},\mathbf{l}}(x_0'), y_0) \\ &\leq 4\delta_1 + d_{\mathrm{Eucl}}(x_0', T_{\mathbf{k},\mathbf{l}}(x_0')) + 4\delta_1 \\ &\leq d_{\mathrm{Eucl}}(x_0, y_0) + 16\delta_1.\end{aligned}$$

□

Proposition 3.2.6. *For any base point $q \in Y''$ and any $D > 0$,*

$$d_{pGH}((Y'' \cap B_{\mathrm{Eucl}}(q, D), \hat{d}_{\mathrm{Eucl},Y''}, q), (Y'' \cap B_{\mathrm{Eucl}}(q, D), d_{\mathrm{Eucl}}, q)) \leq \Psi(m(g)).$$

Proof. By Lemma 3.2.3, Y'' lies in the $6\delta_0$-neighborhood of Y' inside $(Y'', \hat{d}_{\mathrm{Eucl},Y''})$. This clearly implies for any $q \in Y'$:

$$d_{pGH}((Y' \cap B_{\mathrm{Eucl}}(q,D), \hat{d}_{\mathrm{Eucl},Y''}, q), (Y'' \cap B_{\mathrm{Eucl}}(q,D), \hat{d}_{\mathrm{Eucl},Y''}, q)) \leq \Psi(m(g)).$$

Similarly, since $d_{\mathrm{Eucl}} \leq \hat{d}_{\mathrm{Eucl},Y''}$, Y'' lies in the $6\delta_0$-neighborhood of Y' in terms of d_{Eucl} and for any $q \in Y'$:

$$d_{pGH}((Y' \cap B_{\mathrm{Eucl}}(q,D), d_{\mathrm{Eucl}}, q), (Y'' \cap B_{\mathrm{Eucl}}(q,D), d_{\mathrm{Eucl}}, q)) \leq \Psi(m(g)).$$

Together with Lemma 3.2.5 and the triangle inequality, we have the conclusion for in fact any base point $q \in Y''$ (using again that Y'' lies in the $6\delta_0$-neighborhood of Y' inside $(Y'', \hat{d}_{\mathrm{Eucl},Y''})$).

□

Next we can construct a subregion in $E \subset M_{ext}$ by pulling back the subregion constructed above through the diffeomorphism $\mathbf{u}$. Set

$$E'' := \mathbf{u}^{-1}(Y'').$$

For any $p \in E''$ and $D > 0$, denote by $\hat{B}_{g,E''}(p,D)$ the geodesic ball in $(E'', \hat{d}_{g,E''})$, that is,

$$\hat{B}_{g,E''}(p,D) := \{x \in E'' : \hat{d}_{g,E''}(p,x) \leq D\}.$$

Similarly, denote by $\hat{B}_{\mathrm{Eucl},Y''}(q,D)$ the geodesic ball in $(Y'', \hat{d}_{\mathrm{Eucl},Y''})$.

Lemma 3.2.7. *For any base point $q \in Y''$ and any $D > 0$,*

$$d_{pGH}((Y'' \cap B_{\mathrm{Eucl}}(q,D), \hat{d}_{\mathrm{Eucl},Y''}, q), (\hat{B}_{\mathrm{Eucl},Y''}(q,D), \hat{d}_{\mathrm{Eucl},Y''}, q)) \leq \Psi(m(g)).$$

Proof. From Lemma 3.2.3 and (3.2.9) in the proof of Lemma 3.2.5, for any $q, x \in Y''$,

$$d_{\mathrm{Eucl}}(q,x) \leq \hat{d}_{\mathrm{Eucl},Y''}(q,x) \leq d_{\mathrm{Eucl}}(q,x) + \Psi(m(g)), \tag{3.2.10}$$

so

$$\hat{B}_{\mathrm{Eucl},Y''}(q,D) \subset Y'' \cap B_{\mathrm{Eucl}}(q,D) \subset \hat{B}_{\mathrm{Eucl},Y''}(q, D + \Psi(m(g))).$$

□

Lemma 3.2.8. *For any base point $p \in E''$ and any $D > 0$,*

$$d_{pGH}((\hat{B}_{g,E''}(p,D), \hat{d}_{g,E''}, p), (\hat{B}_{\mathrm{Eucl},Y''}(\mathbf{u}(p), D), \hat{d}_{\mathrm{Eucl},Y''}, \mathbf{u}(p))) \leq \Psi(m(g)|D).$$

Proof. It is enough to show that $\mathbf{u}$ gives the desired GH approximation. Under the diffeomorphism $\mathbf{u}$, if we still denote the metric $(\mathbf{u}^{-1})^* g$ by g, then

$$(E'', \hat{d}_{g,E''}, p) = (Y'', \hat{d}_{g,Y''}, \mathbf{u}(p)).$$

Since $|\mathrm{Jac}\mathbf{u} - \mathrm{Id}| \leq \epsilon'$, we have

$$\hat{d}_{g,Y''}(x_1, x_2) \leq (1+\epsilon')\hat{d}_{\mathrm{Eucl},Y''}(x_1,x_2) \leq (1+\epsilon')^2 \hat{d}_{g,Y''}(x_1,x_2).$$

Note that we have taken $\epsilon = \delta_0$. So for any fixed $D > 0$, if $x_1, x_2 \in \hat{B}_{\mathrm{Eucl},Y''}(\mathbf{u}(p), D)$,

$$|\hat{d}_{g,Y''}(x_1,x_2) - \hat{d}_{\mathrm{Eucl},Y''}(x_1,x_2)| \leq \Psi(m(g)|D).$$

□

From Proposition 3.3.7, Lemma 3.2.7 and Lemma 3.2.8, we immediately have the following.

Lemma 3.2.9. *For any $p \in E''$ and $D > 0$,*

$$d_{pGH}((\hat{B}_{g,E''}(p,D), \hat{d}_{g,E''}, p), (Y'' \cap B_{\mathrm{Eucl}}(\mathbf{u}(p), D), d_{\mathrm{Eucl}}, \mathbf{u}(p)) \leq \Psi(m(g)|D).$$

To compare those metric spaces to the Euclidean 3-space $(\mathbb{R}^3, g_{\mathrm{Eucl}})$, we need the following lemma, which is a corollary of the fact that $\mathrm{Area}(\partial Y'') \leq \Psi(m(g))$.

Lemma 3.2.10. *For any $q \in Y''$ and $D > 0$,*

$$d_{pGH}((Y'' \cap B_{\mathrm{Eucl}}(q,D), d_{\mathrm{Eucl}}, q), (B_{\mathrm{Eucl}}(0,D), d_{\mathrm{Eucl}}, 0)) \leq \Psi(m(g)).$$

Proof. Under a translation diffeomorphism, we can assume $q = 0$. By (3.2.10), it suffices to show that $B_{\mathrm{Eucl}}(q,D)$ lies in a $\Psi(m(g))$-neighborhood of Y''. If that were not the case, there would be a $\mu > 0$, independent of $m(g)$, such that for all small enough $0 < m(g) \ll 1$, there exists a $x \in B_{\mathrm{Eucl}}(q, D)$ with $B_{\mathrm{Eucl}}(x, \mu) \cap Y'' = \emptyset$. But from the isoperimetric inequality, we should have

$$\mathrm{Vol}(\mathbb{R}^3 \setminus Y'') \leq C\mathrm{Area}(\partial Y'')^{\frac{3}{2}} \leq \Psi(m(g)),$$

which would imply that

$$\omega_3 \mu^3 = \mathrm{Vol}(B_{\mathrm{Eucl}}(x, \mu)) \leq \Psi(m(g)),$$

a contradiction. □

Summarizing above arguments, we have proved the following. Recall that ξ is a fixed function defined at the begining of this section.

Proposition 3.2.11. *Assume (M^3, g) is an asymptotically flat 3-manifold with nonnegative scalar curvature and mass $0 < m(g) \ll 1$. For any end of (M, g), there exists a connected region $\mathcal{E} \subset M$ containing this end, with smooth boundary, such that*

$$\mathrm{Area}(\partial \mathcal{E}) \leq \frac{m(g)^2}{\xi(m(g))},$$

and there is a harmonic diffeomorphism $\mathbf{u} : \mathcal{E} \to \mathcal{Y}$ with $\mathcal{Y} := \mathbf{u}(\mathcal{E}) \subset \mathbb{R}^3$ such that the Jacobian satisfies

$$|\mathrm{Jac}\mathbf{u} - \mathrm{Id}| \leq \Psi(m(g)).$$

Moreover, for any base point $p \in \mathcal{E}$, any $D > 0$,

$$d_{pGH}((\hat{B}_{g,\mathcal{E}}(p, D), \hat{d}_{g,\mathcal{E}}, p), (B_{\mathrm{Eucl}}(0, D), d_{\mathrm{Eucl}}, 0)) \leq \Psi(m(g)|D),$$

and $\Phi_{\mathbf{u}(p)} \circ \mathbf{u}$ gives a $\Psi(m(g)|D)$-pGH approximation, where $\Phi_{\mathbf{u}(p)}$ is the translation diffeomorphism of $\mathbb{R}^3$ mapping $\mathbf{u}(p)$ to 0.

Proof. With the same notations as above, we take $\mathcal{E} := E''$ and $\mathcal{Y} := Y''$. Notice that by Lemma 3.2.3, by the fact that $|\mathrm{Jac}\mathbf{u} - \mathrm{Id}| \leq \epsilon'$ (see (3.1.1)), and by our choice of δ_0 and ξ_0, when $0 < m(g) \ll 1$,

$$\mathrm{Area}(\partial \mathcal{E}) \leq 2\frac{m(g)^2}{\delta_0^5} = 2\frac{m(g)^2}{\xi_0^5(m(g))} \leq \frac{m(g)^2}{\xi(m(g))}.$$

The rest of the statement follows from Lemma 3.2.9 and Lemma 3.3.9. □

Proof of Theorem 1.0.3. Assume (M_i^3, g_i) is a sequence of asymptotically flat 3-manifolds with nonnegative scalar curvature and positive mass $m(g_i) \to 0$. Assume ξ is any fixed continuous function as in the statement of Theorem 1.0.3. For any end of M_i, for all large i, Proposition 3.2.11 gives the existence of a region $\mathcal{E}_i$ containing this end, which satisfies

$$\mathrm{Area}_{g_i}(\partial\mathcal{E}_i) \leq \frac{m(g_i)^2}{\xi(m(g_i))},$$

and a harmonic diffeomorphism $\mathbf{u}_i : \mathcal{E}_i \to \mathcal{Y}_i \subset \mathbb{R}^3$ with $\mathcal{Y}_i = \mathbf{u}_i(\mathcal{E}_i)$.

By Proposition 3.2.11, for any base point $p_i \in \mathcal{E}_i$, any $D > 0$, up to a translation diffeomorphism of $\mathbb{R}^3$, we can assume $\mathbf{u}_i(p_i) = 0$, and then $\mathbf{u}_i$ is an $\Psi(m(g_i)|D)$-pGH approximation, and as $i \to \infty$,

$$d_{pGH}((\hat{B}_{g_i,\mathcal{E}_i}(p_i, D), \hat{d}_{g_i,\mathcal{E}_i}, p_i), (B_{\mathrm{Eucl}}(0, D), d_{\mathrm{Eucl}}, 0)) \leq \Psi(m(g_i)|D) \to 0.$$

In other words,

$$(\mathcal{E}_i, \hat{d}_{g_i,\mathcal{E}_i}, p_i) \to (\mathbb{R}^3, d_{\mathrm{Eucl}}, 0)$$

in the pointed Gromov-Hausdorff topology.

We claim that $(\mathcal{E}_i, \hat{d}_{g_i,\mathcal{E}_i}, p_i) \to (\mathbb{R}^3, d_{\mathrm{Eucl}}, 0)$ also in the pointed measured Gromov-Hausdorff topology. Since the Hausdorff measure induced by $\hat{d}_{g_i,\mathcal{E}_i}$ is the same as dvol_{g_i}, it suffices to show that for a.e. $D > 0$,

$$(\mathbf{u}_i)_\sharp(\mathrm{dvol}_{g_i}|_{\hat{B}_{g_i,\mathcal{E}_i}(p_i,D)}) \text{ weakly converges to } \mathrm{dvol}_{\mathrm{Eucl}}|_{B(0,D)} \text{ as } i \to \infty. \tag{3.2.11}$$

By construction and the isoperimetric inequality,

$$\mathrm{Vol}(\mathbb{R}^3 \setminus \mathcal{Y}_i) \leq \Psi(m(g_i)),$$

and so $(\mathcal{Y}_i \cap B_{\mathrm{Eucl}}(0, D), \mathrm{dvol}_{\mathrm{Eucl}})$ converges weakly to $(B_{\mathrm{Eucl}}(0, D), \mathrm{dvol}_{\mathrm{Eucl}})$. Since we have (by abuse of notations):

$$|\mathrm{Jac}\mathbf{u}_i - \mathrm{Id}| \leq \Psi(m(g_i)),$$

it is now simple to check (3.2.11) using Lemma 3.2.7 and Lemma 3.2.8.

We finish the proof by defining $Z_i := M_i \setminus \mathcal{E}_i$.

□

3.3 GH convergence modulo negligible spikes in general dimensions

We observe that the metric geometric arguments presented in the previous section are valid in higher dimensions. In this section, we will prove the following theorem using the same arguments, which the author hopes will be useful in future studies.

Theorem 3.3.1. *Assume that $n \geq 2$, and $\mathcal{W}_i \subset \mathbb{R}^n$ is a sequence of smooth bounded domains whose boundary $\Sigma_i := \partial \mathcal{W}_i$ consists of smooth closed hypersurfaces such that $\mathcal{H}^{n-1}(\Sigma_i) \to 0$, then for $Y_i := \mathbb{R}^n \setminus \mathcal{W}_i$, there exists a sequence of smooth closed subsets $Y_i'' \subset Y_i$ such that $\mathcal{H}^{n-1}(\partial Y_i'') \to 0$ and for any base point $p_i \in Y_i''$,*

$$(Y_i'', p_i, \hat{d}_{\mathrm{Eucl}, Y_i''}) \xrightarrow{pGH} (\mathbb{R}^n, 0, \hat{d}_{\mathrm{Eucl}}).$$

Moreover, we have a quantitative version: there exists $\varepsilon(n) \ll 1$ such that if $\mathcal{H}^{n-1}(\Sigma) \leq \varepsilon(n)$, then for the perturbation $Y'' \subset Y$, $\mathcal{H}^{n-1}(\partial Y'') \leq (\mathcal{H}^{n-1}(\Sigma))^{1-10^{-4}n^{-1}}$, and

$$d_{pGH}((Y'', \hat{d}_{Y''}, p), (\mathbb{R}^n, \hat{d}_{\mathrm{Eucl}}, 0)) \leq (\mathcal{H}^{n-1}(\Sigma))^{2^{-n}}.$$

Proof. Notice that the case when $n = 2$ is obvious by definition, and the case when $n = 3$ has been proved in previous section. So we assume that $n \geq 4$. In the following, we will prove the theorem by induction and assume that the conclusion holds for all dimensions less than or equal to $n - 1$. For simplicity, we will omit the subindex and all metric tensors are taken as the Euclidean metric in the following of this section.

We ignore the index i for now. We will find out $\varepsilon(n)$ inductively. Firstly take $\varepsilon(n) \ll 1$ such that $4\varepsilon(n)^{1-(10n)^{-1}} \leq \varepsilon(n-1)$. Let $\varepsilon := \mathcal{H}^{n-1}(\Sigma) \leq \varepsilon(n) \ll 1$. By the isoperimetric inequality,

$$\mathcal{H}^n(\mathcal{W}) \leq C(n)\mathcal{H}^{n-1}(\Sigma)^{\frac{n}{n-1}} \leq C(n)\varepsilon^{\frac{n}{n-1}} \leq \varepsilon.$$

Take $\delta_0 = \varepsilon^{10^{-2}n^{-1}}, \delta_1 = \varepsilon^{10^{-4}n^{-1}}$. For any $\mathbf{k} = (k_1, k_2, \cdots, k_n) \in \mathbb{Z}^n$, consider the cube $\mathbf{C}_\mathbf{k}(\delta_1)$ defined by

$$\mathbf{C}_\mathbf{k}(\delta_1) := (k_1\delta_1, (k_1+1)\delta_1) \times \cdots \times (k_n\delta_1, (k_n+1)\delta_1) \subset \mathbb{R}^n.$$

Let $B_\mathbf{k}(r)$ be the ball with center the same as $\mathbf{C}_\mathbf{k}(\delta_1)$ and radius r. By applying the co-area formula, we can find $r \in (3\delta_1, 3\delta_1 + \delta_0)$ such that $\mathcal{W} \cap \partial B_\mathbf{k}(r)$ consists of smooth surfaces and

$$\mathcal{H}^{n-1}(\mathcal{W} \cap \partial B_\mathbf{k}(r)) \leq \frac{\mathcal{H}^n(\mathcal{W} \cap B_\mathbf{k}(4\delta_1))}{\delta_0}. \tag{3.3.1}$$

Since $\mathcal{H}^n(B_\mathbf{k}(4\delta_1)) \geq 16\omega_n\delta_1^n$, where ω_n is the Euclidean volume of unit ball in $\mathbb{R}^n$, we know

$$\mathcal{H}^n(\mathcal{W} \cap B_\mathbf{k}(4\delta_1)) \leq \mathcal{H}^n(\mathcal{W}) \leq \varepsilon \ll \delta_1^n < \frac{1}{10}\mathcal{H}^n(B_\mathbf{k}(4\delta_1)). \tag{3.3.2}$$

By the relative isoperimetric inequality [EG15, Theorem 5.11 (ii)],

$$\begin{aligned}\mathcal{H}^n(\mathcal{W} \cap B_\mathbf{k}(4\delta_1)) &\leq C(n)\mathcal{H}^{n-1}(\Sigma \cap B_\mathbf{k}(4\delta_1))^{\frac{n}{n-1}} \\ &\leq C(n)\varepsilon^{\frac{1}{n-1}}\mathcal{H}^{n-1}(\Sigma \cap B_\mathbf{k}(4\delta_1)).\end{aligned}$$

So by (3.3.1),

$$\begin{aligned}\mathcal{H}^{n-1}(\mathcal{W} \cap \partial B_\mathbf{k}(r)) &\leq \frac{C(n)\varepsilon^{\frac{1}{n-1}}}{\delta_0}\mathcal{H}^{n-1}(\Sigma \cap B_\mathbf{k}(4\delta_1)) \\ &= C(n)\varepsilon^{(1-10^{-2})n^{-1}}\mathcal{H}^{n-1}(\Sigma \cap B_\mathbf{k}(4\delta_1)) \\ &\leq \mathcal{H}^{n-1}(\Sigma \cap B_\mathbf{k}(4\delta_1)).\end{aligned}$$

Then we can smooth the surface $(\Sigma \cap B_\mathbf{k}(r)) \cup (\mathcal{W} \cap \partial B_\mathbf{k}(r))$ to get a closed embedded hypersurface $\Sigma_\mathbf{k} \subset B_\mathbf{k}(4\delta_1)$, which coincides with Σ inside $\mathbf{C}_\mathbf{k}(\delta_1)$, and which satisfies

$$\begin{aligned}\mathcal{H}^{n-1}(\Sigma_\mathbf{k}) &\leq 2(\mathcal{H}^{n-1}(\Sigma \cap B_\mathbf{k}(4\delta_1)) + \mathcal{H}^{n-1}(\mathcal{W} \cap \partial B_\mathbf{k}(r))) \\ &\leq 4\mathcal{H}^{n-1}(\Sigma \cap B_\mathbf{k}(4\delta_1)).\end{aligned} \tag{3.3.3}$$

For $t \in \mathbb{R}$, define the plane

$$A_{\mathbf{k},\delta_1}(t) := \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_n = (k_n + t)\delta_1\}.$$

By definition $\mathbf{C}_{\mathbf{k}}(\delta_1) \subset \bigcup_{t\in[0,1]} A_{\mathbf{k},\delta_1}(t)$.

By the coarea formula, there exists $t_{\mathbf{k}} \in (\frac{1}{2}, \frac{1}{2}+\delta_0)$ such that $A_{\mathbf{k},\delta_1}(t_{\mathbf{k}}) \cap \Sigma$ consists of $(n-2)$-submanifolds and

$$\begin{aligned}\mathcal{H}^{n-2}(A_{\mathbf{k},\delta_1}(t_{\mathbf{k}}) \cap \Sigma) &\le \frac{\mathcal{H}^{n-1}(\Sigma)}{\delta_0\delta_1} \\ &\le 4\varepsilon^{1-(10n)^{-1}} \\ &\le \varepsilon(n-1).\end{aligned} \tag{3.3.4}$$

By induction assumption, there is a perturbation $\tilde{\Sigma}^{n-2}(t_{\mathbf{k}}) \subset A_{\mathbf{k},\delta_1}(t_{\mathbf{k}})$ of $\Sigma^{n-2}(t_{\mathbf{k}}) := A_{\mathbf{k},\delta_1}(t_{\mathbf{k}}) \cap \Sigma$ such that the domain bounded by $\Sigma^{n-2}(t_{\mathbf{k}})$ is inside the domain bounded by $\tilde{\Sigma}^{n-2}(t_{\mathbf{k}})$,

$$\mathcal{H}^{n-2}(\tilde{\Sigma}^{n-2}(t_{\mathbf{k}})) \le 4\mathcal{H}^{n-2}(A_{\mathbf{k},\delta_1}(t_{\mathbf{k}}) \cap \Sigma), \tag{3.3.5}$$

and for any two points $x, y \in \tilde{Y}(t_{\mathbf{k}}) := A_{\mathbf{k},\delta_1}(t_{\mathbf{k}}) \setminus \tilde{Z}(t_{\mathbf{k}})$, where $\tilde{Z}(t_{\mathbf{k}})$ is the domain bounded by $\tilde{\Sigma}^{n-2}(t_{\mathbf{k}})$ inside $A_{\mathbf{k},\delta_1}(t_{\mathbf{k}})$, we have

$$|\hat{d}_{\tilde{Y}(t_{\mathbf{k}})}(x,y) - \hat{d}_{\mathbb{R}^{n-1}}(x,y)| \le (4\varepsilon^{1-(10n)^{-1}})^{2^{-(n-1)}} \le \frac{1}{8}\varepsilon^{2^{-n}}, \tag{3.3.6}$$

if we choose $\varepsilon(n)$ small enough depending on n.

Define $D'_{\mathbf{k}}$ as the connected component of $\mathbf{C}_{\mathbf{k}}(\delta_1) \cap \tilde{Y}(t_{\mathbf{k}})$ with largest $\mathcal{H}^{n-1}$-measure and $D''_{\mathbf{k}} := (\mathbf{C}_{\mathbf{k}}(\delta_1) \cap A_{\mathbf{k},\delta_1}(t_{\mathbf{k}})) \setminus D'_{\mathbf{k}}$. By the relative isoperimetric inequality and (3.3.4), we know that

$$\begin{aligned}\mathcal{H}^{n-1}(D''_{\mathbf{k}}) &\le C(n)\mathcal{H}^{n-2}(A_{\mathbf{k},\delta_1}(t_{\mathbf{k}}) \cap \Sigma_{\mathbf{k}})^{\frac{n-1}{n-2}} \\ &\le \varepsilon^{(2n)^{-1}}\mathcal{H}^{n-2}(A_{\mathbf{k},\delta_1}(t_{\mathbf{k}}) \cap \Sigma_{\mathbf{k}}) \\ &\le \mathcal{H}^{n-1}(\Sigma_{\mathbf{k}}).\end{aligned} \tag{3.3.7}$$

We can take a ε-net of $D'_{\mathbf{k}}$ and define $\tilde{D}'_{\mathbf{k}}$ as the union of $D'_{\mathbf{k}}$ and all almost $\hat{d}_{\tilde{Y}(t_{\mathbf{k}})}$-geodesics connecting points in the ε-net. In this way, for any $x, y \in \tilde{D}'_{\mathbf{k}}$, we have a smooth curve $\gamma \subset \tilde{D}'_{\mathbf{k}}$ connecting x and y such that

$$|\text{Length}_{\text{Eucl}}(\gamma) - d_{\text{Eucl}}(x,y)| \le \varepsilon + \frac{1}{8}\varepsilon^{2^{-n}} \le \frac{1}{4}\varepsilon^{2^{-n}}. \tag{3.3.8}$$

For simplicity, we still denote $\tilde{D}'_{\mathbf{k}}$ by $D'_{\mathbf{k}}$. Let $\pi_{\mathbf{k}} : \mathbb{R}^n \to A_{\mathbf{k},\delta_1}(t_{\mathbf{k}})$ be the orthogonal projection. Notice that by comparing $\mathcal{H}^{n-1}$-measures, $D'_{\mathbf{k}} \setminus \pi_{\mathbf{k}}(\Sigma_{\mathbf{k}}) \neq \emptyset$.

Define

$$\mathbf{C}_{\mathbf{k}}(\delta_1)' := D'_{\mathbf{k}} \cup \left(\mathbf{C}_{\mathbf{k}}(\delta_1) \cap \pi_{\mathbf{k}}^{-1}(D'_{\mathbf{k}} \setminus \pi_{\mathbf{k}}(\Sigma_{\mathbf{k}}))\right).$$

Lemma 3.3.2. $\mathbf{C}_{\mathbf{k}}(\delta_1)'$ *is path connected.*

Proof. By definition, for any point $x \in \mathbf{C}_{\mathbf{k}}(\delta_1) \cap \pi_{\mathbf{k}}^{-1}(D'_{\mathbf{k}} \setminus \pi_{\mathbf{k}}(\Sigma_{\mathbf{k}}))$, the line segment $L_x \subset \mathbf{C}_{\mathbf{k}}(\delta_1)$ through x and orthogonal to $A_{\mathbf{k},\delta_1}(t_{\mathbf{k}})$ satisfies $L_x \cap D'_{\mathbf{k}} \neq \emptyset$. Since $D'_{\mathbf{k}}$ is path connected, $\mathbf{C}_{\mathbf{k}}(\delta_1)'$ is also path connected. □

Lemma 3.3.3. $\mathcal{H}^n(\mathbf{C}_{\mathbf{k}}(\delta_1) \setminus \mathbf{C}_{\mathbf{k}}(\delta_1)') \leq 8\delta_1 \mathcal{H}^{n-1}(\Sigma \cap B_{\mathbf{k}}(4\delta_1)) \leq 8\varepsilon^{1+10^{-4}n^{-1}}$.

Proof. Since

$$\mathcal{H}^{n-1}(\pi_{\mathbf{k}}(\Sigma_{\mathbf{k}})) \leq \mathcal{H}^{n-1}(\Sigma_{\mathbf{k}}),$$

and since by (3.3.7),

$$\mathcal{H}^{n-1}(D''_{\mathbf{k}}) \leq \mathcal{H}^{n-1}(\Sigma_{\mathbf{k}}),$$

we have

$$\begin{aligned} \mathcal{H}^n(\mathbf{C}_{\mathbf{k}}(\delta_1)) \setminus \mathrm{Vol}(\mathbf{C}_{\mathbf{k}}(\delta_1)') &\leq 2\delta_1 \mathcal{H}^{n-1}(\Sigma_{\mathbf{k}}) \\ &\leq 8\delta_1 \mathcal{H}^{n-1}(\Sigma \cap B_{\mathbf{k}}(4\delta_1)). \end{aligned} \tag{3.3.9}$$

□

Clearly by construction,

$$\mathbf{C}_{\mathbf{k}}(\delta_1)' \subset Y.$$

Define

$$Y' := \cup_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{C}_{\mathbf{k}}(\delta_1)' \subset Y.$$

Notice that when $|\mathbf{k}|$ is big enough, one can certainly ensure that $\mathbf{C}_{\mathbf{k}}(\delta_1)' = \mathbf{C}_{\mathbf{k}}(\delta_1)$, so that $Y \setminus Y'$ is a bounded set. Choosing Y slightly bigger, we can assume that $\partial Y \cap Y' = \emptyset$.

For any subset $V \subset Y$, let V_t be the t-neighborhood of V inside $(Y, \hat{d}_{\mathrm{Eucl},Y})$ in terms of the length metric $\hat{d}_{\mathrm{Eucl},Y}$, i.e.

$$V_t := \{y \in Y : \exists z \in V \text{ such that } \hat{d}_{\mathrm{Eucl},Y}(y,z) \leq t\}.$$

So $(Y')_t$ is the t-neighborhood of Y' inside $(Y, \hat{d}_{\mathrm{Eucl},Y})$.

In the following lemma, by modifying some $(Y')_t$, we construct a domain with smooth boundary such that its boundary area is small and it is very close to Y' in the Gromov-Hausdorff topology with respect to a length metric.

Lemma 3.3.4. *There exists Y'' with smooth boundary such that $Y' \subset Y'' \subset (Y')_{6\delta_0}$,*

$$\mathcal{H}^{n-1}(\partial Y'') \leq \delta_0^{-1}\mathcal{H}^{n-1}(\partial Y) \leq \varepsilon^{1-10^{-2}n^{-1}},$$

and Y'' is contained in the $6\delta_0$-neighborhood of Y' inside Y'', with respect to its length metric $\hat{d}_{\mathrm{Eucl},Y''}$.

Proof. Smoothing the Lipschitz function $\hat{d}_{\mathrm{Eucl},Y}(Y',\cdot)$, we can get a smooth function $\phi : Y \to \mathbb{R}$ such that $|\phi - \hat{d}_{\mathrm{Eucl},Y}(Y',\cdot)| \leq \delta_0$ and $|\nabla\phi| \leq 2$ (see for instance [GW79, Proposition 2.1]). Applying coarea formula to ϕ, we have

$$\int_{3\delta_0}^{4\delta_0} \mathcal{H}^{n-1}(\phi^{-1}(t) \cap Y)dt = \int_{\{3\delta_0<\phi<4\delta_0\}\cap Y} |\nabla\phi| \mathrm{dvol} \leq 2\mathcal{H}^n(Y \setminus Y').$$

By Lemma 3.3.3, for each $\mathbf{k} \in \mathbb{Z}^3$,

$$0 \leq \mathcal{H}^n(\mathbf{C}_\mathbf{k}(\delta_1)) - \mathcal{H}^n(\mathbf{C}'_\mathbf{k}(\delta_1)) \leq 8\delta_1\mathcal{H}^{n-1}(\Sigma \cap B_\mathbf{k}(4\delta_1)).$$

Since the number of overlaps of $\{B_\mathbf{k}(4\delta_1)\}_{\mathbf{k}\in\mathbb{Z}^n}$ is uniformly bounded,

$$\begin{aligned} 0 \leq \mathcal{H}^n(Y) - \mathcal{H}^n(Y') &\leq 8\delta_1 \sum_{\mathbf{k}\in\mathbb{Z}^n} \mathcal{H}^{n-1}(\Sigma \cap B_\mathbf{k}(4\delta_1)) \\ &\leq C(n)\delta_1\mathcal{H}^{n-1}(\Sigma) \\ &\leq C(n)\varepsilon^{1+10^{-4}n^{-1}}. \end{aligned} \tag{3.3.10}$$

So we can find a generic regular value $t \in (3\delta_0, 4\delta_0)$ of ϕ such that $\phi^{-1}(t)$ is smooth and

$$\mathcal{H}^{n-1}(\phi^{-1}(t) \cap Y) \le \frac{C(n)\varepsilon^{1+10^{-4}n^{-1}}}{\delta_0} \le C(n)\varepsilon^{1-10^{-2}n^{-1}+10^{-4}n^{-1}}.$$

Smoothing $(\phi^{-1}(t) \cap Y) \cup (\partial Y \cap \{\phi < t\})$ inside Y, we can get a smooth surface S_1 with $S_1 \subset (Y')_{5\delta_0} \setminus Y'$ and

$$\mathcal{H}^{n-1}(S_1) \le 2(\mathcal{H}^{n-1}(\phi^{-1}(t) \cap Y) + \mathcal{H}^{n-1}(\partial Y)) \le C(n)\varepsilon^{1-10^{-2}n^{-1}+10^{-4}n^{-1}}.$$

Denote by Y_1 the connected component such that

$$Y' \subset Y_1 \subset (Y')_{5\delta_0} \subset Y \quad \text{and} \quad \partial Y_1 \subset S_1. \tag{3.3.11}$$

At this point, Y_1 is close to Y' in the Hausdorff topology with respect to $\hat{d}_{\mathrm{Eucl},Y}$, but possibly not with respect to its own length metric $\hat{d}_{\mathrm{Eucl},Y_1}$. To remedy this, choose a finite subset $\{x_j\}$ consisting of δ_0-dense discrete points of $(Y_1 \setminus Y', \hat{d}_{\mathrm{Eucl},Y_1})$ and denote by $\gamma_j \subset Y$ a smooth curve connecting x_j to Y' with minimal length with respect to the length metric $\hat{d}_{\mathrm{Eucl},Y}$. Then by (3.3.11), γ_j has length at most $5\delta_0$, and so $\gamma_j \subset (Y')_{5\delta_0}$. By thickening each γ_j, we can get thin solid tubes T_j inside δ_0-neighborhood of γ_j with arbitrarily small boundary area. Let $Y_2 := Y_1 \cup (\cup_j T_j)$. By smoothing the corners of Y_2, we have a connected domain Y'' with smooth boundary such that

$$Y' \subset Y'' \subset Y_2 \subset Y'_{6\delta_0}$$

and

$$\mathcal{H}^{n-1}(\partial Y'') \le 2\mathcal{H}^{n-1}(S_1) \le C(n)\varepsilon^{1-10^{-2}n^{-1}+10^{-4}n^{-1}} \le \delta_0^{-1}\mathcal{H}^{n-1}(\partial Y).$$

For any $y \in Y'' \setminus Y'$, by our construction, there exists some j such that either $\hat{d}_{\mathrm{Eucl},Y_1}(y, x_j) \le \delta_0$ or $y \in T_j$. In each case, there exists a smooth curve $\sigma_{y,j} \subset Y''$ connecting y to a point in γ_j and $\mathrm{Length}(\sigma_{y,j}) \le \delta_0$. Since $\mathrm{Length}(\gamma_j) \le 5\delta_0$, $\sigma_{y,j} \cup \gamma_j$ is a piecewise smooth curve inside Y'' connecting y to Y' with length smaller than $6\delta_0$. So inside the length space $(Y'', \hat{d}_{\mathrm{Eucl},Y''})$, Y'' is in the $6\delta_0$-neighborhood of Y' as desired.

□

Let Y'' be as in Lemma 3.3.4. Recall that $\hat{d}_{\mathrm{Eucl},Y''}$ is defined as the length metric on Y'' induced by g_{Eucl}. Since $Y' \subset Y'' \subset Y$, we have $d_{\mathrm{Eucl}} \leq \hat{d}_{\mathrm{Eucl},Y} \leq \hat{d}_{\mathrm{Eucl},Y''}$.

Lemma 3.3.5. $\mathrm{diam}_{\hat{d}_{\mathrm{Eucl},Y''}}(\mathbf{C_k}(\delta_1)') \leq (n+2)\delta_1 + \frac{1}{4}\varepsilon^{2^{-n}}$.

Proof. For any two points $x_1, x_2 \in \mathbf{C_k}(\delta_1)'$, let L_{x_1}, L_{x_2} be the line segments inside $\mathbf{C_k}(\delta_1)$ through x_1, x_2 and orthogonal to $A_{\mathbf{k},\delta_1}(t_{\mathbf{k}})$ respectively. Let $x_1' = L_{x_1} \cap D_{\mathbf{k}}', x_2' = L_{x_2} \cap D_{\mathbf{k}}'$. Then by (3.3.8) we can find a curve γ between x_1', x_2' inside $D_{\mathbf{k}}'$ such that

$$\mathrm{Length}_{\mathrm{Eucl}}(\gamma) \leq d_{\mathrm{Eucl}}(x_1', x_2') + \frac{1}{4}\varepsilon^{2^{-n}}.$$

Consider the curve $\tilde{\gamma}$ consisting of three parts: the line segment $[x_1 x_1']$ between x_1, x_1', γ, and the line segment $[x_2' x_2]$ between x_2', x_2. We have $\tilde{\gamma} \subset \mathbf{C_k}(\delta_1)' \subset Y'$, so

$$\hat{d}_{\mathrm{Eucl},Y''}(x_1, x_2) \leq L_{\mathrm{Eucl}}(\tilde{\gamma}) \leq (n+2)\delta_1 + \frac{1}{4}\varepsilon^{2^{-n}}.$$

□

Lemma 3.3.6. *For any base point $q \in Y'$ and any $D > 0$,*

$$d_{pGH}((Y' \cap B_{\mathrm{Eucl}}(q, D), \hat{d}_{\mathrm{Eucl},Y''}, q), (Y' \cap B_{\mathrm{Eucl}}(q, D), d_{\mathrm{Eucl}}, q)) \leq \frac{3}{4}\varepsilon^{2^{-n}}.$$

Proof. Let $x_0, y_0 \in Y' \cap B_{\mathrm{Eucl}}(q, D)$ be two points and $x_0 \in \mathbf{C_k}(\delta_1)', y_0 \in \mathbf{C_l}(\delta_1)'$ for some $\mathbf{k}, \mathbf{l} \in \mathbb{Z}^n$. Since $d_{\mathrm{Eucl}} \leq \hat{d}_{\mathrm{Eucl},Y''}$, it's enough to show

$$\hat{d}_{\mathrm{Eucl},Y''}(x_0, y_0) \leq d_{\mathrm{Eucl}}(x_0, y_0) + \frac{3}{4}\varepsilon^{2^{-n}}. \tag{3.3.12}$$

Let $T_{\mathbf{k},\mathbf{l}}$ be the translation which maps $\mathbf{C_k}(\delta_1)$ to $\mathbf{C_l}(\delta_1)$. Then by Lemma 3.3.3,

$$\begin{aligned}\mathcal{H}^n(T_{\mathbf{k},\mathbf{l}}(\mathbf{C_k}(\delta_1)') \cap \mathbf{C_l}(\delta_1)') \geq & \mathcal{H}^n(\mathbf{C_k}(\delta_1)) - (\mathcal{H}^n(\mathbf{C_k}(\delta_1) \setminus \mathbf{C_k}(\delta_1)')) \\ & - (\mathcal{H}^n(\mathbf{C_l}(\delta_1) \setminus \mathbf{C_l}(\delta_1)')) \\ \geq & \delta_1^3 - 16\varepsilon^{1+10^{-4}n^{-1}}.\end{aligned}$$

If $\mathbf{k} = \mathbf{l}$, then by Lemma 3.3.5, we know that by choosing $\varepsilon(n)$ small enough,

$$\hat{d}_{\mathrm{Eucl},Y''}(x_0, y_0) \leq (n+2)\varepsilon^{10^{-4}n^{-1}} + \frac{1}{4}\varepsilon^{2^{-n}} \leq d_{\mathrm{Eucl}}(x_0, y_0) + \frac{3}{4}\varepsilon^{2^{-n}}.$$

Suppose that $\mathbf{k} \neq \mathbf{l}$. For any $x \in \mathbb{R}^3$, the straight line between x and $T_{\mathbf{k},\mathbf{l}}(x)$ meets the set $T_{\mathbf{k},\mathbf{l}}(\mathbf{C}_{\mathbf{k}}(\delta_1)') \cap \mathbf{C}_{\mathbf{l}}(\delta_1)'$ in a subset of total length at most say $10\delta_1$. We claim that there is at least one point $x_0' \in \mathbf{C}_{\mathbf{k}}(\delta_1)'$ such that $T_{\mathbf{k},\mathbf{l}}(x_0') \in \mathbf{C}_{\mathbf{l}}(\delta_1)'$ and the line segment $[x_0' T_{\mathbf{k},\mathbf{l}}(x_0')]$ between these two points has no intersection with $\partial Y''$. Otherwise, by the coarea formula and Lemma 3.3.4, we would get

$$\mathcal{H}^n(T_{\mathbf{k},\mathbf{l}}(\mathbf{C}_{\mathbf{k}}(\delta_1)') \cap \mathbf{C}_{\mathbf{l}}(\delta_1)') \leq 10\delta_1 \mathcal{H}^{n-1}(\partial Y'') \leq 10\delta_1 \varepsilon^{1-10^{-2}n^{-1}},$$

which together with the above estimate on the left hand side would give

$$\delta_1^3 - 16\varepsilon^{1+10^{-4}n^{-1}} \leq 10\delta_1 \varepsilon^{1-10^{-2}n^{-1}},$$

a contradiction when $\varepsilon(n) \ll 1$.

Since from the paragraph above, $[x_0' T_{\mathbf{k},\mathbf{l}}(x_0')] \subset Y''$, we estimate

$$\begin{aligned}
\hat{d}_{\mathrm{Eucl},Y''}(x_0, y_0) &\leq \hat{d}_{\mathrm{Eucl},Y''}(x_0, x_0') + \mathrm{Length}_{\mathrm{Eucl}}([x_0' T_{\mathbf{k},\mathbf{l}}(x_0')]) + \hat{d}_{\mathrm{Eucl},Y''}(T_{\mathbf{k},\mathbf{l}}(x_0'), y_0) \\
&\leq d_{\mathrm{Eucl}}(x_0, y_0) + 2(n+2)\delta_1 + \frac{1}{2}\varepsilon^{2^{-n}} \\
&\leq d_{\mathrm{Eucl}}(x_0, y_0) + \frac{3}{4}\varepsilon^{2^{-n}}.
\end{aligned}$$

□

Proposition 3.3.7. *For any base point $q \in Y''$ and any $D > 0$,*

$$d_{pGH}((Y'' \cap B_{\mathrm{Eucl}}(q, D), \hat{d}_{\mathrm{Eucl},Y''}, q), (Y'' \cap B_{\mathrm{Eucl}}(q, D), d_{\mathrm{Eucl}}, q)) \leq \varepsilon^{2^{-n}}.$$

Proof. By Lemma 3.3.4, Y'' lies in the $6\delta_0$-neighborhood of Y' inside $(Y'', \hat{d}_{\mathrm{Eucl},Y''})$. This clearly implies for any $q \in Y'$:

$$d_{pGH}((Y' \cap B_{\mathrm{Eucl}}(q, D), \hat{d}_{\mathrm{Eucl},Y''}, q), (Y'' \cap B_{\mathrm{Eucl}}(q, D), \hat{d}_{\mathrm{Eucl},Y''}, q)) \leq 10\delta_0.$$

Similarly, since $d_{\mathrm{Eucl}} \leq \hat{d}_{\mathrm{Eucl},Y''}$, Y'' lies in the $6\delta_0$-neighborhood of Y' in terms of d_{Eucl} and for any $q \in Y'$:

$$d_{pGH}((Y' \cap B_{\mathrm{Eucl}}(q, D), d_{\mathrm{Eucl}}, q), (Y'' \cap B_{\mathrm{Eucl}}(q, D), d_{\mathrm{Eucl}}, q)) \leq 10\delta_0.$$

Together with Lemma 3.3.6 and the triangle inequality, we have the conclusion for in fact any base point $q \in Y''$ (using again that Y'' lies in the $6\delta_0$-neighborhood of Y' inside $(Y'', \hat{d}_{\mathrm{Eucl},Y''})$).

□

For any $p \in Y''$ and $D > 0$, denote by $\hat{B}_{Y''}(p, D)$ the geodesic ball in $(Y'', \hat{d}_{\mathrm{Eucl},Y''})$, that is,

$$\hat{B}_{Y''}(p, D) := \{x \in Y'' : \hat{d}_{\mathrm{Eucl},Y''}(p, x) \leq D\}.$$

Lemma 3.3.8. *For any base point $q \in Y''$ and any $D > 0$,*

$$d_{pGH}((Y'' \cap B_{\mathrm{Eucl}}(q, D), \hat{d}_{\mathrm{Eucl},Y''}, q), (\hat{B}_{\mathrm{Eucl},Y''}(q, D), \hat{d}_{\mathrm{Eucl},Y''}, q)) \leq \varepsilon^{2^{-n}}.$$

Proof. From Lemma 3.3.4 and (3.3.12) in the proof of Lemma 3.3.6, for any $q, x \in Y''$,

$$d_{\mathrm{Eucl}}(q, x) \leq \hat{d}_{\mathrm{Eucl},Y''}(q, x) \leq d_{\mathrm{Eucl}}(q, x) + \frac{3}{4}\varepsilon^{2^{-n}}, \tag{3.3.13}$$

so

$$\hat{B}_{\mathrm{Eucl},Y''}(q, D) \subset Y'' \cap B_{\mathrm{Eucl}}(q, D) \subset \hat{B}_{\mathrm{Eucl},Y''}(q, D + \frac{3}{4}\varepsilon^{2^{-n}}).$$

□

To compare those metric spaces to the Euclidean 3-space $(\mathbb{R}^3, g_{\mathrm{Eucl}})$, we need the following lemma, which is a corollary of the fact that $\mathcal{H}^{n-1}(\partial Y'') \leq \delta_1^{-1}\varepsilon$.

Lemma 3.3.9. *For any $q \in Y''$ and $D > 0$,*

$$d_{pGH}((Y'' \cap B_{\mathrm{Eucl}}(q, D), d_{\mathrm{Eucl}}, q), (B_{\mathrm{Eucl}}(0, D), d_{\mathrm{Eucl}}, 0)) \leq \varepsilon^{2^{-n}}.$$

Proof. Under a translation diffeomorphism, we can assume $q = 0$. By (3.3.13), it suffices to show that $B_{\mathrm{Eucl}}(q, D)$ lies in a $\frac{1}{4}\varepsilon^{2^{-n}}$-neighborhood of Y''. If that were not the case, there would be a $\mu > \frac{1}{4}\varepsilon^{2^{-n}}$, and an $x \in B_{\mathrm{Eucl}}(q, D)$ with $B_{\mathrm{Eucl}}(x, \mu) \cap Y'' = \emptyset$. But from the isoperimetric inequality, we have

$$\mathcal{H}^n(\mathbb{R}^3 \setminus Y'') \leq C(n)\mathcal{H}^{n-1}(\partial Y'')^{\frac{n}{n-1}} \leq C(n)\varepsilon^{(1-10^{-4}n^{-1})\cdot\frac{n}{n-1}},$$

which would imply that

$$\omega_n \cdot \frac{\varepsilon^{\frac{n}{2n}}}{4^n} \leq \omega_n \mu^n = \mathcal{H}^n(B_{\mathrm{Eucl}}(x,\mu)) \leq C(n)\varepsilon^{\frac{n}{2(n-1)}},$$

a contradiction when $\varepsilon(n) \ll 1$. □

This concludes the proof of the theorem. □

Chapter 4

Stability for the Penrose inequality

4.1 Capacity of horizon and Green's function

In this section, we introduce some properties of capacity and Green's function, which are known in the literature and will be used later in this paper. Most of this section follows from [Bra01].

Let's firstly introduce the capacity of a surface in the special case when it is the horizon of an asymptotically flat 3-manifold which is harmonically flat at infinity.

Definition 4.1.1. Given a complete, asymptotically flat 3-manifold (M^3, g) with a connected outermost horizon boundary Σ, nonnegative scalar curvature and one harmonically flat end ∞_1, the capacity of Σ in (M^3, g) is defined by

$$\mathcal{C}(\Sigma, g) := \inf\{\frac{1}{2\pi}\int_{M^3} |\nabla\varphi|^2 \mathrm{dvol}_g : \varphi \in C^\infty(M),\ \varphi = \frac{1}{2} \text{ on } \Sigma,\ \lim_{x\to\infty_1}\varphi(x) = 1\}.$$

From standard theory (c.f. [Bar86]), the infimum in the definition of $\mathcal{C}(\Sigma, g)$ is achieved by the Green's function $\varphi \in C^\infty(M^3)$ which satisfies

$$\begin{aligned} \Delta_g\varphi &= 0, \\ \varphi &= \frac{1}{2} \text{ on } \Sigma, \qquad (4.1.1) \\ \lim_{x\to\infty_1}\varphi(x) &= 1. \end{aligned}$$

By maximum principle, $\varphi(x) \in [\frac{1}{2}, 1)$ for any $x \in M^3$. Define the level sets of φ to be

$$\Sigma_t^{\varphi} := \{x \in M^3 : \varphi(x) = t\}.$$

Then by Sard's theorem, Σ_t^{φ} is a smooth surface for almost all $t \in (\frac{1}{2}, 1)$. By the co-area formula,

$$\mathcal{C}(\Sigma, g) = \frac{1}{2\pi} \int_{\frac{1}{2}}^{1} \int_{\Sigma_t^{\varphi}} |\nabla \varphi|.$$

For any regular value $t \in (\frac{1}{2}, 1)$, integrating $\Delta \varphi = 0$ over $\{\frac{1}{2} \leq \varphi \leq t\}$, and using Stokes' theorem, we have

$$\int_{\Sigma} |\nabla \varphi| = \int_{\Sigma_t^{\varphi}} |\nabla \varphi|. \tag{4.1.2}$$

So

$$\mathcal{C}(\Sigma, g) = \frac{1}{4\pi} \int_{\Sigma} |\nabla \varphi|. \tag{4.1.3}$$

Since (M^3, g) is harmonically flat at infinity, we have the following expansion at infinity of the Green's function (c.f. [Bar86]):

$$\varphi(x) = 1 - \frac{\mathcal{C}(\Sigma, g)}{|x|} + O\left(\frac{1}{|x|^2}\right) \text{ as } x \to \infty_1. \tag{4.1.4}$$

Now we introduce another definition which is closely related to the capacity of a horizon surface.

Definition 4.1.2. Given a complete, asymptotically flat 3-manifold $(\bar{M}^3, \bar{g})$ with multiple harmonically flat ends and one chosen end ∞_1, define

$$\mathcal{C}(\bar{g}) := \inf\{\frac{1}{2\pi} \int_{\bar{M}^3} |\nabla \phi|^2 \mathrm{dvol}_{\bar{g}} : \phi \in \mathrm{Lip}(\bar{M}),\ \lim_{x \to \infty_1} \phi(x) = 1,\ \lim_{x \to \{\infty_k\}_{k \geq 2}} \phi(x) = 0\}.$$

Similarly the infimum in the definition of $\mathcal{C}(\bar{g})$ is achieved by the Green's function ϕ which satisfies

$$\begin{aligned} &\Delta_{\bar{g}} \phi = 0, \\ &\lim_{x \to \infty_1} \phi(x) = 1, \\ &\lim_{x \to \infty_k} \phi(x) = 0 \text{ for all } k \geq 2. \end{aligned} \tag{4.1.5}$$

For a complete asymptotically flat 3-manifold (M^3, g) with a connected outermost horizon boundary Σ, nonnegative scalar curvature and one end ∞_1, we can take another copy of (M^3, g) and glue them together along the boundary Σ to get a new metric space $(\bar{M}, \bar{g})$. In general, $(\bar{M}, \bar{g})$ is only a Lipschitz manifold with two asymptotically flat ends $\{\infty_1, \infty_2\}$. From the proof of [Bra01, Theorem 9], for any $\delta > 0$ small enough, we can smooth out $(\bar{M}, \bar{g})$ and construct a smooth complete 3-manifold $(\tilde{M}_\delta, \tilde{g}_\delta)$ with nonnegative scalar curvature and two asymptotically flat ends which, in the limit as $\delta \to 0$, approaches $(\bar{M}, \bar{g})$ uniformly. For reader's convenience, we recall the details of [Bra01] in the following.

Let $(M_1^3, g), (M_2^3, g)$ be the two copies of (M^3, g). A first step is to construct a smooth manifold (c.f. [Bra01, Equation (92)])

$$(\tilde{M}_\delta, \bar{g}_\delta) := (M_1^3, g) \sqcup (\Sigma \times (0, 2\delta), G) \sqcup (M_2^3, g),$$

where $\Sigma \times \{0\}$ and $\Sigma \times \{2\delta\}$ are identified with $\Sigma \subset (M^3, g)$, G is a warped product metric and symmetric about $t = \delta$, and $\Sigma \times \{\delta\} \subset (\Sigma \times (0, 2\delta), G)$ is totally geodesic. In general, the scalar curvature of G only satisfies $R_G \geq R_0$ for some constant $R_0 \leq 0$ independent of δ, and may not be nonnegative.

Then a second step is to take a conformal deformation of $\bar{g}_\delta$ to get a new metric with nonnegative scalar curvature. Define a smooth function $\mathcal{R}_\delta$, which equals R_0 in $\Sigma \times [0, 2\delta]$, equals 0 for x more than a distance δ from $\Sigma \times [0, 2\delta]$, takes values in $[R_0, 0]$ everywhere and symmetric about $\Sigma \times \{\delta\}$. In particular, $R_{\bar{g}_\delta}(x) \geq \mathcal{R}_\delta(x)$ for any $x \in \tilde{M}_\delta$. Define $u_\delta(x)$ such that (c.f. [Bra01, Equation (101)])

$$\begin{aligned} (-8\Delta_{\bar{g}_\delta} + \mathcal{R}_\delta(x))u_\delta(x) &= 0, \\ \lim_{x \to \{\infty_1, \infty_2\}} u_\delta(x) &= 1. \end{aligned} \tag{4.1.6}$$

Then u_δ is a smooth function and satisfies that (c.f. [Bra01, Equation (102)])

$$1 \leq u_\delta(x) \leq 1 + \epsilon(\delta)$$

where ϵ goes to 0 as $\delta \to 0$. Define

$$\tilde{g}_\delta := u_\delta^4 \cdot \bar{g}_\delta.$$

The scalar curvature of $\tilde{g}_\delta$ satisfies

$$\begin{aligned} R_{\tilde{g}_\delta} &= u_\delta^{-5} \left(R_{\bar{g}_\delta} u_\delta - 8\Delta_{\bar{g}_\delta} u_\delta \right) \\ &= u_\delta^{-4} \left(R_{\bar{g}_\delta} - \mathcal{R}_\delta \right) \\ &\geq 0. \end{aligned}$$

By definition, $\lim_{\delta \to 0} m(\tilde{g}_\delta) = m(\bar{g})$ and $\lim_{\delta \to 0} \mathcal{C}(\tilde{g}_\delta) = \mathcal{C}(\bar{g})$.

To see the relation between $\mathcal{C}(\bar{g})$ and $\mathcal{C}(\Sigma, g)$, we define the reflection map

$$\Phi : M_1^3 \cup_\Sigma M_2^3 \to M_1^3 \cup_\Sigma M_2^3$$

such that for any $x \in M_1^3$, $\Phi(x) \in M_2^3$ is the same point under the identification $M_1^3 = M_2^3 = M^3$, $\Phi^2 = \mathrm{Id}$ and $\Phi|_\Sigma = \mathrm{Id}$. If ϕ satisfies (4.1.5), then $1 - \phi \circ \Phi$ also satisfies (4.1.5) and by the uniqueness we have $\phi(x) = 1 - \phi \circ \Phi(x)$, which implies that $\phi|_\Sigma = \frac{1}{2}$. So $\phi|_{M_1^3}$ also satisfies (4.1.1), which implies that

$$\mathcal{C}(\bar{g}) = 2\mathcal{C}(\Sigma, g). \tag{4.1.7}$$

Similarly, we can define the reflection map $\Phi_\delta : \tilde{M}_\delta \to \tilde{M}_\delta$ and from the equation (4.1.6) and the fact that $\bar{g}_\delta = \bar{g}_\delta \circ \Phi_\delta, \mathcal{R}_\delta = \mathcal{R}_\delta \circ \Phi_\delta$, we know u_δ is also symmetric about $\Sigma \times \{\delta\}$ and particularly $\langle \nabla u_\delta, \vec{n} \rangle_{\bar{g}_\delta} = 0$ on $\Sigma \times \{\delta\}$, where $\vec{n}$ is the normal vector of $\Sigma \times \{\delta\} \subset (\tilde{M}_\delta, \bar{g}_\delta)$. Thus, $\tilde{g}_\delta$ is symmetric about $\Sigma \times \{\delta\}$ and the mean curvature of $\Sigma \times \{\delta\} \subset (\tilde{M}_\delta, \tilde{g}_\delta)$ is

$$H_{(\Sigma \times \{\delta\}, \tilde{g}_\delta)} = u_\delta^{-2} H_{(\Sigma \times \{\delta\}, \bar{g}_\delta)} - 2 \left\langle \nabla u_\delta^{-2}, \vec{n} \right\rangle_{\bar{g}_\delta} = 0.$$

Let $(M_\delta, \tilde{g}_\delta)$ be one half of $(\tilde{M}_\delta, \tilde{g}_\delta)$ with minimal boundary $\Sigma_\delta := \Sigma \times \{\delta\}$ and one asymptotically flat end. Then $(M_\delta, \tilde{g}_\delta, \Sigma_\delta)$ converges to (M^3, g, Σ) uniformly as $\delta \to 0$.

Without loss of generality, by applying Proposition 2.0.1, we can assume that $(\tilde{M}_\delta, \tilde{g}_\delta)$ is also harmonically flat at infinity. In summary, we have the following proposition.

Proposition 4.1.1. *Given a complete asymptotically flat one-ended* 3*-manifold* (M^3, g) *with a connected outermost horizon boundary* Σ *and nonnegative scalar curvature, there is a sequence of smooth complete* 3*-manifolds* $(\tilde{M}^3_\delta, \tilde{g}_\delta)$*, which have nonnegative scalar curvature and two harmonically flat ends, and are symmetric about a minimal surface* $\Sigma_\delta \subset (\tilde{M}^3_\delta, \tilde{g}_\delta)$*, such that* $(\tilde{M}_\delta, \tilde{g}_\delta) \to (\bar{M}, \bar{g})$ *and* $(M_\delta, \tilde{g}_\delta) \to (M, g)$ *uniformly as* $\delta \to 0$*, where* $(\bar{M}, \bar{g})$ *is the doubling of* (M, g) *along the boundary* Σ*, and* $(M_\delta, \tilde{g}_\delta)$ *is one half of* $(\tilde{M}_\delta, \tilde{g}_\delta)$ *with minimal boundary* Σ_δ*.*

To conclude this section, we give a remark about the relations between the mass, capacity and boundary area of the outermost horizon. Let's briefly recall Bray's proof of the Penrose inequality in [Bra01]. Given a complete smooth 3-manifold (M^3, g_0) with a harmonically flat end, nonnegative scalar curvature, an outermost minimizing horizon Σ_0 of total area A_0 and total mass m_0. Then for all $t \geq 0$, we can construct a continuous family of conformal metrics g_t on M^3 which are asymptotically flat with nonnegative scalar curvature and total mass $m(t)$. Let $\Sigma(t)$ be the outermost minimal enclosure of Σ_0 in (M^3, g_t). Then $\Sigma(t)$ is a smooth outermost horizon in (M^3, g_t) with area $A(t)$ being a constant function about t. It was shown that $m(t)$ is decreasing. And as $t \to \infty$, (M^3, g_t) approaches a Schwarzschild manifold $(\mathbb{R}^3 \setminus \{0\}, g_{sc})$ with total mass $\lim_{t\to\infty} m(t) = \sqrt{\frac{A_0}{16\pi}}$. In particular, $m_0 \geq \sqrt{\frac{A_0}{16\pi}}$, which proves the Penrose inequality.

If we assume that $m_0 - \sqrt{\frac{A_0}{16\pi}} \leq \varepsilon \ll 1$, then the total variation of $m(t)$ is bounded by ε. From [Bra01, Section 7], we know that for a.e. t, $m'(t) = 2(2\mathcal{C}(\Sigma(t), g_t) - m(t))$ (notice that our definition of capacity differs by a scale). Then we can choose a small perturbation, say $t_\varepsilon \in (0, \sqrt{\varepsilon})$ so that $m(t_\varepsilon) - 2\mathcal{C}(\Sigma(t_\varepsilon), g_{t_\varepsilon}) \leq \frac{1}{2}\sqrt{\varepsilon} \ll 1$. And as $\varepsilon \to 0$, g_{t_ε} approaches uniformly to g_0. In particular, $\mathrm{Area}_{g_{t_\varepsilon}}(\Sigma_{t_\varepsilon}) = \mathrm{Area}_{g_0}(\Sigma_0)$ and $|m(g_{t_\varepsilon}) - m(g_0)| \leq \Psi(\varepsilon)$.

4.2 Integral estimate and weighted volume comparison

In this and the following section, we assume that $(\tilde{M}^3, g)$ is a complete, asymptotically flat 3-manifold with two harmonically flat ends $\{\infty_1, \infty_2\}$ and nonnegative scalar curvature obtained as in Proposition 4.1.1. In particular, the topology of $\tilde{M}^3$ is $\mathbb{R}^3 \setminus \{0\}$, and there is a minimal surface $\Sigma \subset (\tilde{M}^3, g)$ such that g is symmetric about Σ. Let (M^3, g) be the half of $(\tilde{M}^3, g)$ which contains the end ∞_1 and has minimal boundary Σ. So Σ is diffeomorphic to a 2-sphere, M^3 is diffeomorphic to $\mathbb{R}^3 \setminus B(1)$, and $(\tilde{M}^3, g) = (M^3, g) \cup_\Sigma (M^{3,\prime}, g)$, where we use M' to denote a copy of M containing the other end ∞_2.

Let $f(x)$ be the solution to (4.1.5) on $(\tilde{M}^3, g)$, that is

$$\begin{aligned} \Delta_g f &= 0, \\ \lim_{x \to \infty_1} f(x) &= 1, \\ \lim_{x \to \infty_2} f(x) &= 0. \end{aligned} \tag{4.2.1}$$

Then f is a smooth function satisfying $0 < f < 1$ and the following expansion at infinity (c.f. [Bar86; Bra01])

$$\begin{aligned} f(x) &= 1 - \frac{c_1}{|x|} + O\left(\frac{1}{|x|^2}\right) \text{ as } x \to \infty_1, \\ f(x) &= \frac{c_2}{|x|} + O\left(\frac{1}{|x|^2}\right) \text{ as } x \to \infty_2, \end{aligned} \tag{4.2.2}$$

where c_k are positive constants for $k = 1, 2$. Moreover, for some $\tau \in (0, 1)$,

$$\begin{aligned} \partial_j f(x) &= \frac{c_1}{|x|^2} \cdot \frac{x^j}{|x|} + O\left(\frac{1}{|x|^{2+\tau}}\right), \\ \partial_j \partial_k f(x) &= \frac{c_1 \delta_{jk}}{|x|^3} - \frac{3c_1}{|x|^3} \cdot \frac{x^j x^k}{|x|^2} + O\left(\frac{1}{|x|^{3+\tau}}\right). \end{aligned} \tag{4.2.3}$$

By the symmetry of $\tilde{g}$ about Σ, we know that on (M^3, g), f satisfies (4.1.1), that is

$$\begin{aligned} \Delta_g f &= 0, \\ f &= \frac{1}{2} \text{ on } \Sigma, \\ \lim_{x \to \infty_1} f(x) &= 1. \end{aligned} \tag{4.2.4}$$

So by (4.1.3) and (4.1.4),

$$c_1 = \mathcal{C}(\Sigma, g) = \frac{1}{4\pi}\int_\Sigma |\nabla f|. \tag{4.2.5}$$

Similarly, on (M', g), $1-f$ also satisfies (4.1.1), so

$$c_2 = \mathcal{C}(\Sigma, g) = c_1. \tag{4.2.6}$$

Since f is a proper smooth map, by Sard's theorem, the regular values of f is an open dense subset of $[\frac{1}{2}, 1)$. For any regular value $t \in (\frac{1}{2}, 1)$, we define

$$M_t := \{\frac{1}{2} \leq f \leq t\}, \quad \Sigma_t := \{f = t\}.$$

Notice that by maximum principle, a regular level set Σ_t is connected and separates Σ from ∞_1. In particular, Σ_t is a 2-sphere.

Our main goal in this section is to prove the following quantitative integral estimate. In the proof, we will use the technique of integration over level sets of f, as well as a comparison lemma for weighted volumes (c.f. Lemma 4.2.3).

Proposition 4.2.1. *We have the following integration inequality for the Green's function f on (M^3, g):*

$$8\pi \cdot \frac{m(g)^2 - (2\mathcal{C}(\Sigma, g))^2}{m(g)^2} \geq$$
$$\int_{M^3} \left(\frac{\left|\nabla^2 f - f^{-1}(1-f)^{-1}(2f-1)|\nabla f|^2 (g - 3\nu \otimes \nu)\right|^2}{|\nabla f|} + R_g |\nabla f| \right) \mathrm{dvol}_g,$$

where $\nu = \frac{\nabla f}{|\nabla f|}$, and the integral is taken over the regular set of f.

Proof. We first smooth $|\nabla f|$ by defining for any $\epsilon > 0$,

$$\phi_\epsilon := \sqrt{|\nabla f|^2 + \epsilon}.$$

If Σ_t is a regular level set of f, then the Gauss-Codazzi equation implies that

$$R_g - 2\mathrm{Ric}\,(\nu, \nu) = R_{\Sigma_t} + |II|^2 - H^2,$$

where $\nu = \frac{\nabla f}{|\nabla f|}$ and $II = \frac{\nabla^2_{\Sigma_t} f}{|\nabla f|}, H = \mathrm{tr}_{\Sigma_t} II$ are the second fundamental form and mean curvature of Σ_t in (M^3, g) respectively. So

$$\begin{aligned}(R_g - R_{\Sigma_t})|\nabla f|^2 &= 2\mathrm{Ric}(\nabla f, \nabla f) + |\nabla^2 f|^2 - 2|\nabla|\nabla f||^2 \\ &\quad - (\Delta f)^2 + 2\Delta f \cdot \nabla^2 f(\nu,\nu) \\ &= 2\mathrm{Ric}(\nabla f, \nabla f) + |\nabla^2 f|^2 - 2|\nabla|\nabla f||^2,\end{aligned} \tag{4.2.7}$$

where we used the equation $\Delta f = 0$. Together with the Bochner formula

$$\Delta|\nabla f|^2 = 2|\nabla^2 f|^2 + 2\langle \nabla \Delta f, \nabla f\rangle + 2\mathrm{Ric}(\nabla f, \nabla f),$$

we have

$$\Delta|\nabla f|^2 = |\nabla^2 f|^2 + 2|\nabla|\nabla f||^2 + (R_g - R_{\Sigma_t})|\nabla f|^2.$$

So at any point on a regular level set Σ_t, we can compute

$$\begin{aligned}\Delta\phi_\epsilon &= \frac{1}{2}\phi_\epsilon^{-1}\Delta|\nabla f|^2 - \frac{1}{4}\phi_\epsilon^{-3}|\nabla|\nabla f|^2|^2 \\ &= \frac{1}{2}\phi_\epsilon^{-1}\left(|\nabla^2 f|^2 + (R_g - R_{\Sigma_t})|\nabla f|^2\right) + \frac{\epsilon}{\phi_\epsilon^3}|\nabla|\nabla f||^2 \\ &\geq \frac{1}{2}\phi_\epsilon^{-1}\left(|\nabla^2 f|^2 + (R_g - R_{\Sigma_t})|\nabla f|^2\right).\end{aligned}$$

Notice that the mean curvature of Σ_t is

$$H_{\Sigma_t} = \frac{\Delta f - \nabla^2 f(\nu,\nu)}{|\nabla f|} = -\frac{\nabla^2 f(\nu,\nu)}{|\nabla f|}.$$

Taking integration on M_t and using integration by parts and co-area formula, we have

$$\int_{\Sigma_t}\langle\nabla\phi_\epsilon, \nu\rangle \geq \frac{1}{2}\int_{M_t}\phi_\epsilon^{-1}\left(|\nabla^2 f|^2 + R_g|\nabla f|^2\right) - \frac{1}{2}\int_{\frac{1}{2}}^{t}\int_{\Sigma_s} R_{\Sigma_s}\frac{|\nabla f|}{\phi_\epsilon}ds, \tag{4.2.8}$$

where we have used the fact that $\int_\Sigma \langle\nabla\phi_\epsilon, \nu\rangle = -\int_\Sigma \frac{H_\Sigma|\nabla f|^2}{\phi_\epsilon} = 0$ since $H_\Sigma = 0$.

Define the tensor

$$\mathcal{T} := \nabla^2 f - f^{-1}(1-f)^{-1}(2f-1)|\nabla f|^2(g - 3\nu\otimes\nu).$$

Then

$$\frac{1}{2}\phi_\epsilon^{-1}|\nabla^2 f|^2 = \frac{1}{2}\phi_\epsilon^{-1}|\mathcal{T}|^2 - 3\phi_\epsilon^{-1}f^{-2}(1-f)^{-2}(2f-1)^2|\nabla f|^4 \\ - 3\phi_\epsilon^{-1}f^{-1}(1-f)^{-1}(2f-1)\nabla^2 f(\nabla f, \nabla f). \tag{4.2.9}$$

Notice that

$$\begin{aligned}
&\phi_\epsilon^{-1}f^{-1}(1-f)^{-1}(2f-1)\nabla^2 f(\nabla f, \nabla f) \\
&\quad = \frac{1}{2}\phi_\epsilon^{-1}f^{-1}(1-f)^{-1}(2f-1)\left\langle \nabla|\nabla f|^2, \nabla f\right\rangle \\
&\quad = \frac{1}{2}\text{div}\left(|\nabla f|^2 \cdot \phi_\epsilon^{-1}f^{-1}(1-f)^{-1}(2f-1)\nabla f\right) \\
&\quad\quad - \frac{1}{2}|\nabla f|^2\text{div}\left(\phi_\epsilon^{-1}f^{-1}(1-f)^{-1}(2f-1)\nabla f\right) \\
&\quad = \frac{1}{2}\text{div}\left(|\nabla f|^2 \cdot \phi_\epsilon^{-1}f^{-1}(1-f)^{-1}(2f-1)\nabla f\right) \\
&\quad\quad + \frac{1}{2}\frac{|\nabla f|^2}{\phi_\epsilon^2}\phi_\epsilon^{-1}f^{-1}(1-f)^{-1}(2f-1)\nabla^2 f(\nabla f, \nabla f) \\
&\quad\quad - \frac{1}{2}\phi_\epsilon^{-1}f^{-2}(1-f)^{-2}(2f-1)^2|\nabla f|^4 - \phi_\epsilon^{-1}f^{-1}(1-f)^{-1}|\nabla f|^4 \\
&\quad = \frac{1}{2}\text{div}\left(|\nabla f|^2 \cdot \phi_\epsilon^{-1}f^{-1}(1-f)^{-1}(2f-1)\nabla f\right) \\
&\quad\quad + \frac{1}{2}\phi_\epsilon^{-1}f^{-1}(1-f)^{-1}(2f-1)\nabla^2 f(\nabla f, \nabla f) \\
&\quad\quad - \frac{1}{2}\frac{\epsilon}{\phi_\epsilon^2}\phi_\epsilon^{-1}f^{-1}(1-f)^{-1}(2f-1)\nabla^2 f(\nabla f, \nabla f) \\
&\quad\quad - \frac{1}{2}\phi_\epsilon^{-1}f^{-2}(1-f)^{-2}(2f-1)^2|\nabla f|^4 - \phi_\epsilon^{-1}f^{-1}(1-f)^{-1}|\nabla f|^4,
\end{aligned}$$

so

$$\begin{aligned}
&\phi_\epsilon^{-1}f^{-1}(1-f)^{-1}(2f-1)\nabla^2 f(\nabla f, \nabla f) \\
&\quad = \text{div}\left(|\nabla f|^2 \cdot \phi_\epsilon^{-1}f^{-1}(1-f)^{-1}(2f-1)\nabla f\right) \\
&\quad\quad - \frac{\epsilon}{\phi_\epsilon^2}\phi_\epsilon^{-1}f^{-1}(1-f)^{-1}(2f-1)\nabla^2 f(\nabla f, \nabla f) \\
&\quad\quad - \phi_\epsilon^{-1}f^{-2}(1-f)^{-2}(2f-1)^2|\nabla f|^4 - 2\phi_\epsilon^{-1}f^{-1}(1-f)^{-1}|\nabla f|^4.
\end{aligned} \tag{4.2.10}$$

Substituting (4.2.10) into (4.2.9), we have

$$\begin{aligned}\frac{1}{2}\phi_\epsilon^{-1}|\nabla^2 f|^2 = &\frac{1}{2}\phi_\epsilon^{-1}|\mathcal{T}|^2 - 3\mathrm{div}\left(|\nabla f|^2\phi_\epsilon^{-1}f^{-1}(1-f)^{-1}(2f-1)\nabla f\right)\\ &+6\phi_\epsilon^{-1}f^{-1}(1-f)^{-1}|\nabla f|^4\\ &+\frac{3\epsilon}{\phi_\epsilon^2}\phi_\epsilon^{-1}f^{-1}(1-f)^{-1}(2f-1)\nabla^2 f(\nabla f,\nabla f).\end{aligned} \tag{4.2.11}$$

So (4.2.8) is equivalent to

$$\begin{aligned}\int_{\Sigma_t}\langle\nabla\phi_\epsilon,\nu\rangle \geq &\frac{1}{2}\int_{M_t}\phi_\epsilon^{-1}\left(|\mathcal{T}|^2+R_g|\nabla f|^2\right) - \frac{1}{2}\int_{\frac{1}{2}}^t\int_{\Sigma_s}R_{\Sigma_s}ds\\ &-3\int_{\Sigma_t}\phi_\epsilon^{-1}f^{-1}(1-f)^{-1}(2f-1)|\nabla f|^3 + 6\int_{M_t}\phi_\epsilon^{-1}f^{-1}(1-f)^{-1}|\nabla f|^4\\ &+\int_{M_t}\frac{3\epsilon}{\phi_\epsilon^2}\phi_\epsilon^{-1}f^{-1}(1-f)^{-1}(2f-1)\nabla^2 f(\nabla f,\nabla f)\\ &+\frac{1}{2}\int_{\frac{1}{2}}^t\int_{\Sigma_s}R_{\Sigma_s}\cdot\frac{\epsilon}{\phi_\epsilon^2+\phi_\epsilon|\nabla f|}ds.\end{aligned} \tag{4.2.12}$$

For any regular value t of f, define

$$h_\epsilon(t) := \int_{\Sigma_t}\phi_\epsilon|\nabla f|.$$

Notice that by co-area formula, integration by parts and $\Delta f = 0$,

$$\begin{aligned}\int_{\frac{1}{2}}^t\int_{\Sigma_s}\langle\nabla\phi_\epsilon,\nu\rangle\,ds &= \int_{M_t}\langle\nabla\phi_\epsilon,\nabla f\rangle\\ &= \int_{\Sigma_t}\phi_\epsilon|\nabla f| - \int_{\Sigma}\phi_\epsilon|\nabla f|.\end{aligned}$$

So $h_\epsilon(t)$ can be defined for any $t\in[\frac{1}{2},1)$ and it is a Lipschitz function with

$$h'_\epsilon(t) = \int_{\Sigma_t}\langle\nabla\phi_\epsilon,\nu\rangle \text{ for a.e. } t\in(\frac{1}{2},1).$$

Together with Gauss-Bonnet formula, we can rewrite (4.2.12) as

$$\begin{aligned}
h_\epsilon'(t) \geq& \frac{1}{2}\int_{M_t} \phi_\epsilon^{-1}\left(|\mathcal{T}|^2 + R_g|\nabla f|^2\right) \\
&- 4\pi(t-\frac{1}{2}) - \frac{3(2t-1)}{t(1-t)}\int_{\Sigma_t}\phi_\epsilon|\nabla f| + 6\int_{M_t} f^{-1}(1-f)^{-1}\phi_\epsilon|\nabla f|^2 \\
&+ 3\epsilon\int_{\Sigma_t}\phi_\epsilon^{-1}f^{-1}(1-f)^{-1}(2f-1) - 6\epsilon\int_{M_t}\phi_\epsilon^{-1}f^{-1}(1-f)^{-1}|\nabla f|^2 \\
&+ 3\epsilon\int_{M_t}\phi_\epsilon^{-3}f^{-1}(1-f)^{-1}(2f-1)\nabla^2 f(\nabla f,\nabla f) \\
&+ \frac{\epsilon}{2}\int_{\frac{1}{2}}^t\int_{\Sigma_s}\frac{R_{\Sigma_s}}{\phi_\epsilon^2+\phi_\epsilon|\nabla f|}ds \\
\geq& \frac{1}{2}\int_{M_t}\phi_\epsilon^{-1}\left(|\mathcal{T}|^2 + R_g|\nabla f|^2\right) \\
&- 4\pi(t-\frac{1}{2}) - \frac{3(2t-1)}{t(1-t)}h_\epsilon(t) + 6\int_{\frac{1}{2}}^t\frac{h_\epsilon(s)}{s(1-s)}ds \\
&- 6\epsilon\int_{M_t}\phi_\epsilon^{-1}f^{-1}(1-f)^{-1}|\nabla f|^2 \\
&+ 3\epsilon\int_{M_t}\phi_\epsilon^{-3}f^{-1}(1-f)^{-1}(2f-1)\nabla^2 f(\nabla f,\nabla f) \\
&+ \frac{\epsilon}{2}\int_{\frac{1}{2}}^t\int_{\Sigma_s}\frac{R_{\Sigma_s}}{\phi_\epsilon^2+\phi_\epsilon|\nabla f|}ds.
\end{aligned} \tag{4.2.13}$$

Using (4.2.3), $|\nabla f| = \frac{c_1}{|x|^2} + O(\frac{1}{|x|^3})$, $|\nabla^2 f| = O(\frac{1}{|x|^3})$, and (4.1.2), we notice that

$$\begin{aligned}
\epsilon\int_{M_t}\phi_\epsilon^{-1}f^{-1}(1-f)^{-1}|\nabla f|^2 &\leq 2\sqrt{\epsilon}\int_{M_t}(1-f)^{-1}|\nabla f|^2 \\
&= 2\sqrt{\epsilon}\int_{\frac{1}{2}}^1\frac{1}{s(1-s)}\int_{\Sigma_s}|\nabla f|ds \\
&= 16\pi\mathcal{C}(\Sigma,g)\sqrt{\epsilon}\int_{\frac{1}{2}}^t\frac{1}{1-s}ds \\
&\leq C(g)\sqrt{\epsilon}\cdot\log\frac{1}{1-t},
\end{aligned}$$

$$\begin{aligned}
\epsilon\int_{M_t}\phi_\epsilon^{-3}f^{-1}(1-f)^{-1}(2f-1)|\nabla^2 f(\nabla f,\nabla f)| &\leq C\sqrt{\epsilon}\int_{M_t}(1-f)^{-1}|\nabla^2 f| \\
&\leq C(g)\sqrt{\epsilon}\cdot\frac{1}{1-t},
\end{aligned}$$

and by (4.2.7), $|R_{\Sigma_t}| = O((1-t)^2)$, so

$$\epsilon\int_{\frac{1}{2}}^t\int_{\Sigma_s}\frac{|R_{\Sigma_s}|}{\phi_\epsilon^2+\phi_\epsilon|\nabla f|}ds \leq C(g)\sqrt{\epsilon}\cdot\frac{1}{1-t}.$$

Thus for a.e. $t \in (\frac{1}{2}, 1)$,

$$\begin{aligned} h_\epsilon'(t) \geq \frac{1}{2} \int_{M_t} \phi_\epsilon^{-1} \left(|\mathcal{T}|^2 + R_g |\nabla f|^2 \right) \\ - 4\pi (t - \frac{1}{2}) - \frac{3(2t-1)}{t(1-t)} h_\epsilon(t) + 6 \int_{\frac{1}{2}}^t \frac{h_\epsilon(s)}{s(1-s)} ds - \frac{C(g)\sqrt{\epsilon}}{1-t}. \end{aligned} \tag{4.2.14}$$

For any regular value $t \in (\frac{1}{2}, 1)$ of f, as in the proof of [Bra+22, Theorem 1.2], we can divide the integrals into two disjoint parts such that one is the integral over preimage of an open set containing the critical values of f, then letting $\epsilon \to 0$ and using Sard's theorem, we can have

$$\begin{aligned} \int_{\Sigma_t} \langle \nabla |\nabla f|, \nu \rangle \geq \frac{1}{2} \int_{M_t} \frac{1}{|\nabla f|} \left(|\mathcal{T}|^2 + R_g |\nabla f|^2 \right) \\ - 4\pi (t - \frac{1}{2}) - \frac{3(2t-1)}{t(1-t)} \int_{\Sigma_t} |\nabla f|^2 + 6 \int_{\frac{1}{2}}^t \frac{1}{s(1-s)} \int_{\Sigma_s} |\nabla f|^2 ds. \end{aligned}$$

Choose a sequence of regular values $t_i \to 1$. Notice that by (4.2.3),

$$\lim_{t_i \to 1} \int_{\Sigma_{t_i}} \langle \nabla |\nabla f|, \nu \rangle = \lim_{t_i \to 1} \frac{|\nabla^2 f(\nu, \nu)|}{|\nabla f|} = \lim_{|x| \to \infty} \frac{\frac{2c_1}{|x|^3}}{\frac{c_1}{|x|^2}} = 0,$$

and

$$\lim_{t_i \to 1} \frac{3(2t_i - 1)}{t_i(1-t_i)} \int_{\Sigma_{t_i}} |\nabla f|^2 = 12\pi c_1 \lim_{t_i \to 1} \frac{|\nabla f|}{1 - t_i} = 0.$$

So we have the following global integral inequality:

Lemma 4.2.2.

$$2\pi \geq \frac{1}{2} \int_M \frac{1}{|\nabla f|} \left(|\mathcal{T}|^2 + R_g |\nabla f|^2 \right) + 6 \int_M \frac{|\nabla f|^3}{f(1-f)}. \tag{4.2.15}$$

It remains to estimate $2\pi - 6 \int_M \frac{|\nabla f|^3}{f(1-f)}$.

For any $t \in [\frac{1}{2}, 1]$, define

$$\mathcal{H}_\epsilon(t) := \int_{\frac{1}{2}}^t s(1-s) \left(1 - \frac{h_\epsilon(s)}{4\pi s^2 (1-s)^2} \right) ds.$$

Then $\mathcal{H}_\epsilon \in W^{2,\infty}$, and

$$\begin{aligned} \mathcal{H}_\epsilon'(t) &= t(1-t) - \frac{h_\epsilon(t)}{4\pi t(1-t)}, \\ \mathcal{H}_\epsilon''(t) &= 1 - 2t - \frac{h_\epsilon'(t)}{4\pi t(1-t)} - \frac{(2t-1) h_\epsilon(t)}{4\pi t^2 (1-t)^2}. \end{aligned}$$

Notice that

$$\mathcal{H}_\epsilon(\frac{1}{2}) = 0, \quad \mathcal{H}'_\epsilon(\frac{1}{2}) = \frac{1}{4} - \frac{1}{\pi}\int_\Sigma |\nabla f|\sqrt{|\nabla f|^2+\epsilon} =: a_\epsilon.$$

So (4.2.14) implies that

$$\begin{aligned}\frac{C'\sqrt{\epsilon}}{(1-t)^2} &\geq \mathcal{H}''_\epsilon(t) + \frac{2(2t-1)}{t(1-t)}\mathcal{H}'_\epsilon(t) - \frac{6}{t(1-t)}\mathcal{H}_\epsilon \\ &\quad + \frac{1}{8\pi t(1-t)}\int_{M_t}\phi_\epsilon^{-1}\left(|\mathcal{T}|^2 + R_g|\nabla f|^2\right) \\ &\geq \mathcal{H}''_\epsilon(t) + \frac{2(2t-1)}{t(1-t)}\mathcal{H}'_\epsilon(t) - \frac{6}{t(1-t)}\mathcal{H}_\epsilon,\end{aligned} \tag{4.2.16}$$

where we have used that $R_g \geq 0$ in the last inequality.

Consider the solution $\mathcal{U}$ of the equation

$$\mathcal{U}''(t) + \frac{2(2t-1)}{t(1-t)}\mathcal{U}'(t) - \frac{6}{t(1-t)}\mathcal{U}(t) = \frac{C'\sqrt{\epsilon}}{(1-t)^2}, \tag{4.2.17}$$

satisfying initial conditions

$$\mathcal{U}(\frac{1}{2}) = \mathcal{H}_\epsilon(\frac{1}{2}),\ \mathcal{U}'(\frac{1}{2}) = \mathcal{H}'_\epsilon(\frac{1}{2}).$$

A general solution to (4.2.17) is given by

$$\mathcal{U}(t) = -\frac{C'\sqrt{\epsilon}}{18}\left(6t^3\log\frac{1-t}{t} + 6t^2 + 3t - 1\right) + \alpha_1\left(3t^2 - 3t + 1\right) + \alpha_2(1-t)^3.$$

Since

$$\mathcal{U}'(t) = -\frac{C'\sqrt{\epsilon}}{18}\left(18t^2\log\frac{1-t}{t} - \frac{6t^3}{1-t} - 6t^2 + 12t + 3\right) + 3\alpha_1(2t-1) - 3\alpha_2(1-t)^2,$$

we know

$$\begin{aligned}\mathcal{U}(\frac{1}{2}) &= -\frac{C'\sqrt{\epsilon}}{9} + \frac{1}{4}\alpha_1 + \frac{1}{8}\alpha_2 = 0 \\ \mathcal{U}'(\frac{1}{2}) &= -\frac{C'\sqrt{\epsilon}}{3} - \frac{3}{4}\alpha_2 = a_\epsilon,\end{aligned}$$

i.e.

$$\alpha_1 = \frac{2C'\sqrt{\epsilon}}{3} + \frac{2a_\epsilon}{3}, \quad \alpha_2 = -\frac{4C'\sqrt{\epsilon}}{9} - \frac{4a_\epsilon}{3}.$$

Then from ODE comparison, for any $t \in (\frac{1}{2}, 1)$, we have

$$\mathcal{H}_\epsilon(t) \leq \mathcal{U}(t),$$

i.e.

$$\begin{aligned}\mathcal{H}_\epsilon(t) \leq &\left(\frac{2C'\sqrt{\epsilon}}{3} + \frac{2a_\epsilon}{3}\right)\left(3t^2 - 3t + 1\right) - \left(\frac{4C'\sqrt{\epsilon}}{9} + \frac{4a_\epsilon}{3}\right)(1-t)^3 \\ &- \frac{C'\sqrt{\epsilon}}{18}\left(6t^3 \log\frac{1-t}{t} + 6t^2 + 3t - 1\right).\end{aligned} \tag{4.2.18}$$

For any fixed $t \in (\frac{1}{2}, 1)$, since by definition,

$$\mathcal{H}_\epsilon(t) = \frac{1}{2}t^2 - \frac{1}{3}t^3 - \frac{1}{12} - \frac{1}{4\pi}\int_{M_t} \frac{1}{f(1-f)}|\nabla f|^2\sqrt{|\nabla f|^2 + \epsilon},$$

which is a strictly increasing function as $\epsilon \to 0+$. Since (4.2.15) shows that $\mathcal{H}_\epsilon(t)$ is always bounded, by monotone convergence theorem, we have

$$\lim_{\epsilon\to 0}\mathcal{H}_\epsilon(t) = \frac{1}{2}t^2 - \frac{1}{3}t^3 - \frac{1}{12} - \frac{1}{4\pi}\int_{M_t}\frac{|\nabla f|^3}{f(1-f)}.$$

Together with (4.2.18), we have proved the following weighted volume comparison.

Lemma 4.2.3. *For any* $t \in (\frac{1}{2}, 1)$,

$$\frac{1}{2}t^2 - \frac{1}{3}t^3 - \frac{1}{12} - \frac{1}{4\pi}\int_{M_t}\frac{|\nabla f|^3}{f(1-f)} \leq \frac{2a_0}{3}(3t^2 - 3t + 1) - \frac{4a_0}{3}(1-t)^3, \tag{4.2.19}$$

where $a_0 = \frac{1}{4} - \frac{1}{\pi}\int_\Sigma |\nabla f|^2$.

Taking $t \to 1$, we have

$$\frac{1}{12} - \frac{1}{4\pi}\int_M \frac{|\nabla f|^3}{f(1-f)} \leq \frac{2a_0}{3},$$

which implies that

$$2\pi - 6\int_M \frac{|\nabla f|^3}{f(1-f)} \leq 16\left(\frac{\pi}{4} - \int_\Sigma |\nabla f|^2\right).$$

Together with (4.2.15), we have the following lemma.

Lemma 4.2.4.

$$\int_M \frac{1}{|\nabla f|}\left(|\mathcal{T}|^2 + R_g|\nabla f|^2\right) \leq 32\left(\frac{\pi}{4} - \int_\Sigma |\nabla f|^2\right). \tag{4.2.20}$$

By the Hölder inequality and the Penrose inequality,

$$\begin{aligned}\left(\int_\Sigma |\nabla f|\right)^2 &\leq \int_\Sigma |\nabla f|^2 \cdot \mathrm{Area}(\Sigma)\\ &\leq 16\pi m(g)^2 \int_\Sigma |\nabla f|^2.\end{aligned}$$

Since $\mathcal{C}(\Sigma, g) = \frac{1}{4\pi}\int_\Sigma |\nabla f|$, we have

$$\int_\Sigma |\nabla f|^2 \geq \frac{\pi \mathcal{C}(\Sigma, g)^2}{m(g)^2},$$

i.e.

$$32\left(\frac{\pi}{4} - \int_\Sigma |\nabla f|^2\right) \leq 8\pi\left(1 - \left(\frac{2\mathcal{C}(\Sigma, g)}{m(g)}\right)^2\right).$$

This completes the proof.

□

Notice that from (4.2.20), we also have

$$\int_\Sigma |\nabla f|^2 \leq \frac{\pi}{4}. \tag{4.2.21}$$

Together with Hölder inequality, we have the following lower bound on horizon area:

$$\mathrm{Area}(\Sigma) \geq 64\pi \mathcal{C}(\Sigma, g)^2. \tag{4.2.22}$$

Using Penrose inequality, we have the following mass-area-capacity inequality:

Proposition 4.2.5.

$$16\pi(2\mathcal{C}(\Sigma, g))^2 \leq \mathrm{Area}(\Sigma) \leq 16\pi m(g)^2. \tag{4.2.23}$$

Applying Proposition 4.2.1 to $1 - f$ on (M', g), we also have the following integration inequality:

$$\begin{aligned}&\frac{8\pi((m(g)^2 - (2\mathcal{C}(\Sigma, g))^2)}{m(g)^2} \geq\\ &\int_{M'}\left(\frac{|-\nabla^2 f - (1-f)^{-1}f^{-1}(1-2f)|\nabla f|^2(g - 3\nu\otimes\nu)|^2}{|\nabla f|} + R_g|\nabla f|\right)\mathrm{dvol}_g.\end{aligned}$$

So the following integration inequality holds on $(\tilde{M}, g)$:

$$\begin{aligned} &\frac{8\pi((m(g)^2 - (2\mathcal{C}(\Sigma,g))^2)}{m(g)^2} \geq \\ &\int_{\tilde{M}} \left(\frac{|\nabla^2 f - (1-f)^{-1} f^{-1}(2f-1)|\nabla f|^2 (g - 3\nu\otimes\nu)|^2}{|\nabla f|} + R_g |\nabla f| \right) \mathrm{dvol}_g. \end{aligned} \tag{4.2.24}$$

We can introduce an additional function $\rho(x)$ to simplify above estimate. Using (2.0.7), for $m := m(g)$, we define $\rho : (M, g) \to [0, \infty)$ by

$$\rho(x) := \rho_m(f(x)). \tag{4.2.25}$$

Then by (2.0.8),

$$u_m(\rho(x)) = \frac{m}{2} \cdot \frac{1}{f(1-f)},$$

and by (2.0.2),

$$u_m'(\rho(x)) = 2f(x) - 1.$$

So

$$\nabla\rho = \frac{m}{2}(1-f)^{-2} f^{-2} \nabla f,$$

and

$$\mathcal{T} = \frac{m}{2u_m^2(\rho(x))} \left(\nabla^2 \rho - \frac{u_m'(\rho(x))}{u_m(\rho(x))} |\nabla\rho|^2 (g - \nu\otimes\nu) \right).$$

Equivalently, we can rewrite the integration inequality in Proposition 4.2.1 by

$$\begin{aligned} &\frac{8\pi((m(g)^2 - (2\mathcal{C}(\Sigma,g))^2)}{m(g)^2} \geq \\ &\int_{M^3} \left(\frac{m}{2} \cdot \frac{|\nabla^2\rho - \frac{u_m'}{u_m}|\nabla\rho|^2 (g-\nu\otimes\nu)|^2}{u_m^2 |\nabla\rho|} + \frac{m}{2u_m^2} R_g |\nabla\rho| \right) \mathrm{dvol}_g. \end{aligned} \tag{4.2.26}$$

We also notice that by (4.2.2), as $x \to \infty_1$,

$$|\nabla\rho| \to \frac{m}{2\mathcal{C}(\Sigma, g)}. \tag{4.2.27}$$

At any regular point $x \in M$,

$$
\begin{aligned}
|\nabla|\nabla\rho|^2| &\le 2|\nabla\rho|\cdot|\nabla^2\rho(\nu,\cdot)| \\
&= 2|\nabla\rho|\cdot|(\nabla^2\rho - \frac{u_m'}{u_m}|\nabla\rho|^2(g-\nu\otimes\nu))(\nu,\cdot)|.
\end{aligned} \tag{4.2.28}
$$

If $R_g \ge 0$ and $m(g) \ge m_0 > 0$, using (4.2.26), we have

$$
\int_M \frac{|\nabla|\nabla\rho|^2|^2}{u_m^2|\nabla\rho|^3} \le C\cdot(m(g)^2 - (2\mathcal{C}(\Sigma,g))^2),
$$

or equivalently,

$$
\int_M \frac{|\nabla\left((1-f)^{-4}f^{-4}|\nabla f|^2\right)|^2}{(1-f)^{-8}f^{-8}|\nabla f|^3} \le C\cdot(m(g)^2 - (2\mathcal{C}(\Sigma,g))^2). \tag{4.2.29}
$$

Applying same arguments to $1-f$ on (M',g), we have

$$
\int_{M'} \frac{|\nabla\left((1-f)^{-4}f^{-4}|\nabla f|^2\right)|^2}{(1-f)^{-8}f^{-8}|\nabla f|^3} \le C\cdot(m(g)^2 - (2\mathcal{C}(\Sigma,g))^2). \tag{4.2.30}
$$

Taking sum of (4.2.29) and (4.2.30), we have proved the following proposition.

Proposition 4.2.6. *Let f be a solution to (4.1.5) on $(\tilde{M},g)$, where $(\tilde{M},g)$ is a two-ended asymptotically flat 3-manifolds obtained as in Proposition 4.1.1, and assume $m(g) \ge m_0 > 0$, then there exists a uniform constant C depending only on m_0 such that the following integration inequality holds:*

$$
\int_{\tilde{M}} \frac{|\nabla\left((1-f)^{-4}f^{-4}|\nabla f|_g^2\right)|_g^2}{(1-f)^{-8}f^{-8}|\nabla f|_g^3}\mathrm{dvol}_g \le C\cdot(m(g)^2 - (2\mathcal{C}(\Sigma,g))^2), \tag{4.2.31}
$$

and

$$
\lim_{x\to\infty_1}(1-f)^{-2}f^{-2}|\nabla f|_g = \lim_{x\to\infty_2}(1-f)^{-2}f^{-2}|\nabla f|_g = \frac{1}{\mathcal{C}(\Sigma,g)}. \tag{4.2.32}
$$

4.3 Harmonic coordinate for conformal metric

Let f be the harmonic function defined by (4.2.1) on $(\tilde{M}^3,g)$. We consider the conformal metric $h := f^4g$. Then on the end ∞_2, since g is harmonically flat, we have $h_{ij}(x) = f^4(x)V^4(x)\delta_{ij}$, where $V(x)$ is a positive bounded δ-harmonic function. So on the end ∞_2, h is conformal to a punctured ball with the conformal factor $(fV)^4(x)$, where $(fV)(x)$ is a bounded δ-harmonic function in the punctured ball. Hence, by the removable singularity theorem, fV can be extended to the whole ball, which together with the expansion (4.2.2) and (2.0.9) implies that h can be extended smoothly over the one point compactification $\tilde{M}^* := \tilde{M} \cup \{\infty_2\}$. Moreover, by standard computations, $(\tilde{M}^*, h)$ also has nonnegative scalar curvature and has a single harmonically flat end ∞_1 with ADM mass $m(h) = m(g) - 2\mathcal{C}(\Sigma, g) \geq 0$ (c.f. [Bra01, Equation (84)]).

In the following, we assume $0 \leq m(h) \ll 1$, $|m(g) - m_0| \ll 1$ for a fixed $m_0 > 0$ and follow the arguments in [DS23] to construct a harmonic coordinate map, which is an almost isometry into $\mathbb{R}^3$.

For g-harmonic function f, since $|\nabla f|_g^2 = f^4 |\nabla f|_h^2$ and $\mathrm{dvol}_g = f^{-6}\mathrm{dvol}_h$, (4.2.31) is equivalent to

$$\int_{\tilde{M}^*} \frac{|\nabla \left((1-f)^{-4}|\nabla f|_h^2\right)|_h^2}{(1-f)^{-8}|\nabla f|_h^3} \mathrm{dvol}_h \leq C \cdot m(h). \tag{4.3.1}$$

Notice that we also have

$$\int_{\tilde{M}^*} f^{-2}|\nabla f|_h^2 \mathrm{dvol}_h = 4\pi \mathcal{C}(\Sigma, g). \tag{4.3.2}$$

Let $\{x^j\}_{j=1}^3$ denote the asymptotically flat coordinate system of the end ∞_1. We firstly solve the harmonic coordinate functions u^j, for each $j \in \{1, 2, 3\}$, such that

$$\begin{aligned} \Delta_h u^j &= 0, \\ |u^j(x) - x^j| &= o(|x|^{1-\sigma}) \text{ as } x \to \infty_1, \end{aligned} \tag{4.3.3}$$

where $\sigma > \frac{1}{2}$ is the order of the asymptotic flatness. Denote by $\mathbf{u}$ the resulting harmonic map

$$\mathbf{u} := (u^1, u^2, u^3) : (\tilde{M}^*, h) \to \mathbb{R}^3.$$

For any fixed small $0 < \epsilon \ll 1$, by [DS23], we know that there exists a connected region $\mathcal{E}_1 \subset (\tilde{M}^*, h)$ containing ∞_1, with smooth boundary, such that

$$\mathrm{Area}(\partial \mathcal{E}_1) \leq m(h)^{2-\epsilon},$$

and $\mathbf{u}:\mathcal{E}_1\to\mathcal{Y}_1:=\mathbf{u}(\mathcal{E}_1)\subset\mathbb{R}^3$ is a diffeomorphism with the Jacobian satisfying

$$|\mathrm{Jac}\,\mathbf{u}-\mathrm{Id}|\leq\Psi(m(h)),$$

and under the identification by $\mathbf{u}$, the metric tensor satisfies

$$\sum_{j,k=1}^{3}(h_{jk}-\delta_{jk})^2\leq m(h)^{2\epsilon}.$$

Now we modify $\mathcal{E}_1$ to get a subset such that f is locally uniformly Lipschitz. For this purpose, we denote by $a_g:=\frac{m(g)}{2\mathcal{C}(\Sigma,g)}\geq 1$, and define $P:\tilde{M}^*\to[0,\infty)$ by

$$P(x):=\left(\frac{m(g)^2}{4}(1-f)^{-4}|\nabla f|_h^2-a_g^2\right)^2.$$

By (4.2.32), $\lim_{x\to\infty_1}P(x)=\lim_{x\to\infty_2}P(x)=0$. Notice that $|a_g-1|\ll 1$.

If x is a regular point of f such that $P(x)\leq m(h)^{2\epsilon}$ and $f(x)\leq 1-m(h)^\epsilon$, then

$$\begin{aligned}|\nabla P|_h(x)&\leq 2\sqrt{6}m(h)^\epsilon\cdot\frac{m(g)^2}{4}|\nabla\left((1-f)^{-4}|\nabla f|_h^2\right)|_h\\&=2\sqrt{6}m(h)^\epsilon\cdot\frac{m(g)^2}{4}\frac{|\nabla\left((1-f)^{-4}|\nabla f|_h^2\right)|_h}{(1-f)^{-4}|\nabla f|_h^{\frac{3}{2}}}\cdot(1-f)^{-3}|\nabla f|_h^{\frac{3}{2}}\cdot(1-f)^{-1}\\&\leq C\cdot\frac{|\nabla\left((1-f)^{-4}|\nabla f|_h^2\right)|_h}{(1-f)^{-4}f|\nabla f|_h^{\frac{3}{2}}},\end{aligned}\tag{4.3.4}$$

which together with (4.3.1) implies that

$$\int_{\{P\leq m(h)^{2\epsilon}\}\cap\{f\leq 1-m(h)^\epsilon\}}|\nabla P|_h^2\leq Cm(h).\tag{4.3.5}$$

By the co-area formula, we have

$$\begin{aligned}&\int_0^{m(h)^{2\epsilon}}\mathrm{Area}_h(\{P=s\}\cap\{f\leq 1-m(h)^\epsilon\})ds\\&\qquad=\int_{\{P\leq m(h)^{2\epsilon}\}\cap\{f\leq 1-m(h)^\epsilon\}}|\nabla P|_h\mathrm{dvol}_h\\&\qquad\leq Cm(h)^{\frac{1}{2}}\cdot\left(\mathrm{Vol}_h(\{P\leq m(h)^{2\epsilon}\}\cap\{f\leq 1-m(h)^\epsilon\})\right)^{\frac{1}{2}}.\end{aligned}$$

Since

$$\begin{aligned}\mathrm{Vol}_h(\{P\leq m(h)^{2\epsilon}\}\cap\{f\leq 1-m(h)^\epsilon\})&\leq C\int_{\{f\leq 1-m(h)^\epsilon\}}(1-f)^{-4}|\nabla f|_h^2\mathrm{dvol}_h\\&\leq Cm(h)^{-4\epsilon}\int_{\tilde{M}^*}f^{-2}|\nabla f|_h^2\mathrm{dvol}_h\\&\leq Cm(h)^{-4\epsilon},\end{aligned}$$

where we used (4.3.2) in the last inequality, for a generic $s_\epsilon \in (0, m(h)^{2\epsilon})$, we have

$$\mathrm{Area}_h(\{P = s_\epsilon\} \cap \{f \le 1 - m(h)^\epsilon\}) \le Cm(h)^{\frac{1}{2}-4\epsilon}.$$

Define the region $\mathcal{E}_2$ to be the one of the connected components of $\mathcal{E}_1 \cap \{P \le s_\epsilon\} \cap \{f \le 1 - m(h)^\epsilon\}$ with biggest volume. Then

$$\mathrm{Area}(\partial\mathcal{E}_2 \cap \{f \le 1 - m(h)^\epsilon\}) \le Cm(h)^{\frac{1}{2}-4\epsilon}.$$

By co-area formula and a perturbation argument, we can assume that $\{f = 1 - m(h)^\epsilon\}$ is a smooth surface and $\partial\mathcal{E}_2 \cap \{f = 1 - m(h)^\epsilon\}$ consists of smooth curves whose total length is smaller than $m(h)^{\frac{1}{4}}$. Also by a perturbation we can assume that $\partial\mathcal{E}_2 \cap \{f < 1 - m(h)^\epsilon\}$ is a smooth surface.

Since $\mathrm{Area}(\Sigma) \ge 16\pi(2\mathcal{C}(\Sigma, g))^2 \ge A_0$ for a uniform $A_0 > 0$, $\Sigma \cap \mathcal{E}_2 \ne \emptyset$. We can take a base point $p \in \Sigma \cap \mathcal{E}_2$. For all $0 < m(h) \ll 1$, we claim that

$$\hat{B}_{h,\mathcal{E}_2}(p, 10m(h)^{-\frac{\epsilon}{2}}) \subset \{f < 1 - m(h)^\epsilon\}.$$

Otherwise, if there exists $x \in \hat{B}_{h,\mathcal{E}_2}(p, 10m(h)^{-\frac{\epsilon}{2}})$ such that $f(x) = 1 - m(h)^\epsilon$, then there exists a curve $\gamma \subset \mathcal{E}_2$ such that $\gamma(0) = p$, $\gamma(1) = x$, and $\mathrm{Length}_h(\gamma) \le 10m(h)^{-\frac{\epsilon}{2}}$. But by definition of $\mathcal{E}_2$, we have

$$\begin{aligned}(1 - f(x))^{-1} &\le (1 - f(p))^{-1} + \int_\gamma (1-f)^{-2}|\nabla f|_h \\ &\le 2 + C \cdot m(h)^{-\frac{\epsilon}{2}},\end{aligned}$$

which is a contradiction.

For all $0 < m(h) \ll 1$, we can apply the same arguments of [DS23, Section 4] to $\hat{B}_{h,\mathcal{E}_2}(p, 10m(h)^{-\frac{\epsilon}{2}})$, and as a result, we can find a smooth domain $\mathcal{E}$ such that $\mathrm{Area}(\partial\mathcal{E}) \le Cm(h)^{\frac{1}{2}-4\epsilon}$, the induced length metric of h on $\mathcal{E}$ is almost the same as Euclidean metric, and $\hat{B}_{h,\mathcal{E}}(p, m(h)^{-\frac{\epsilon}{2}}) \subset \mathcal{E}_2$. In summary, we have the following proposition.

Proposition 4.3.1. *There exists a connected region $\mathcal{E} \subset (\tilde{M}^*, h)$ containing ∞_1, with smooth boundary, such that*

$$\mathrm{Area}(\partial \mathcal{E}) \leq \Psi(m(h)),$$

and $\mathbf{u} : \mathcal{E} \to \mathcal{Y} := \mathbf{u}(\mathcal{E}) \subset \mathbb{R}^3$ is a diffeomorphism with the Jacobian satisfying

$$|\mathrm{Jac}\, \mathbf{u} - \mathrm{Id}| \leq \Psi(m(h)),$$

and under the identification by $\mathbf{u}$, the metric tensor satisfies

$$\sum_{j,k=1}^{3} (h_{jk} - \delta_{jk})^2 \leq \Psi(m(h)).$$

For any base point $p \in \mathcal{E} \cap \Sigma$ and any fixed $D > 0$,

$$d_{pGH}\Big((\hat{B}_{h,\mathcal{E}}(p, D), \hat{d}_{h,\mathcal{E}}, p), (B_{\mathrm{Eucl}}(0, D), d_{\mathrm{Eucl}}, 0) \Big) \leq \Psi(m(h)|D),$$

and $\Phi_{\mathbf{u}(p)} \circ \mathbf{u}$ gives a $\Psi(m(h)|D)$-pGH approximation, where $\Phi_{\mathbf{u}(p)}$ is the translation diffeomorphism of $\mathbb{R}^3$ mapping $\mathbf{u}(p)$ to 0.

Moreover, $\forall x \in \hat{B}_{h,\mathcal{E}}(p, D)$, for all $0 < m(h) \ll 1$, $f(x) \leq 1 - \frac{1}{C \cdot D}$ and

$$a_g^2 - \Psi(m(h)|D) \leq \frac{m(g)^2}{4}(1 - f)^{-4} |\nabla f|_h^2 \leq a_g^2 + \Psi(m(h)|D). \tag{4.3.6}$$

4.4 $W^{1,2}$-convergence of elliptic equations

Assume that $(\tilde{M}_i^3, g_i)$ is a sequence of complete asymptotically flat 3-manifolds obtained as in Proposition 4.1.1 with $m(g_i) - 2\mathcal{C}(\Sigma_i, g_i) = \varepsilon_i \to 0$, where $\Sigma_i \subset (\tilde{M}_i^3, g_i)$ is the minimal surface such that $(\tilde{M}_i^3, g_i)$ is symmetric about Σ_i, and $m(g_i) \to m_0 > 0$. Notice that from (4.2.23), $|\mathrm{Area}(\Sigma_i) - 16\pi m_0^2| \to 0$, and particularly $\mathrm{Area}(\Sigma_i) \geq A_0 > 0$ for some uniform $A_0 > 0$.

Let f_i be the harmonic functions defined by (4.2.1) on $(\tilde{M}_i^3, g_i)$, (M_i, g_i) be the half of $(\tilde{M}_i^3, g_i)$ such that f_i satisfies (4.2.4), and $h_i := f_i^4 g_i$ the conformal metrics. Using the same notations as in previous section, let $(\tilde{M}_i^*, h_i)$ be the one point compactification $\tilde{M}_i \cup \{\infty_2\}$, then $(\tilde{M}_i^*, h_i)$ is a sequence of one-ended asymptotically flat 3-manifolds with nonnegative scalar curvature and ADM mass $m(h_i) = \varepsilon_i \to 0$.

Let $\mathcal{E}_i$ be regions given by Proposition 4.3.1. Since $\mathrm{Area}(\partial\mathcal{E}_i) \to 0$ and Σ_i is a minimal surface with $\mathrm{Area}(\Sigma_i) \geq A_0 > 0$, $\mathcal{E}_i \cap \Sigma_i \neq \emptyset$. Taking any base point $p_i \in \mathcal{E}_i \cap \Sigma_i = \mathcal{E}_i \cap \{f_i = \frac{1}{2}\}$, by [DS23, Theorem 1.3],

$$(\mathcal{E}_i, \hat{d}_{h_i,\mathcal{E}_i}, p_i) \to (\mathbb{R}^3, d_{\mathrm{Eucl}}, x_o)$$

in the pointed measured Gromov-Hausdorff topology, where $x_0 = (\frac{m_0}{2}, 0, 0) \in \mathbb{R}^3$, the harmonic maps $\mathbf{u}_i$ with $\mathbf{u}_i(p_i) = x_o$ are $\Psi(\varepsilon_i)$-pGH approximation, and for any $D > 0$, $(\mathbf{u}_i)_\sharp(\mathrm{dvol}_{h_i}|_{\hat{B}_{h_i,\mathcal{E}_i}(p_i,D)})$ weakly converges to $\mathrm{dvol}_{\mathrm{Eucl}}|_{B(x_o,D)}$ as $i \to \infty$.

Now we consider functions $\xi_i(x) := \frac{f_i^2}{(1-f_i)^2}$ defined on $(\tilde{M}_i^*, h_i)$. Since $\Delta_{g_i} f_i = 0$, we have

$$\Delta_{h_i} f_i = 2 f_i^{-1} |\nabla f_i|_{h_i}^2,$$

and

$$\Delta_{h_i} (1 - f_i)^{-1} = 2(1 - f_i)^{-3} f_i^{-1} |\nabla f_i|_{h_i}^2.$$

So

$$\Delta_{h_i} \xi_i = 6(1 - f_i)^{-4} |\nabla f_i|_{h_i}^2. \tag{4.4.1}$$

For any fixed $D > 0$ and any $x \in \hat{B}_{h_i,\mathcal{E}_i}(p_i, D)$, by (4.3.6) we have

$$|\nabla (1 - f_i)^{-1}|_{h_i} = (1 - f_i)^{-2} |\nabla f_i|_{h_i} \leq C,$$

so

$$(1 - f_i)^{-1}(x) \leq 2 + C \cdot \hat{d}_{h_i,\mathcal{E}_i}(p_i, x).$$

Also

$$|\nabla \xi_i|_{h_i}(x) = \frac{2 f_i |\nabla f_i|_{h_i}(x)}{(1 - f_i)^3} \leq \frac{C}{1 - f_i(x)}, \tag{4.4.2}$$

so

$$0 \le \xi_i(x) \le C(1 + \hat{d}_{h_i,\mathcal{E}_i}(p_i, x))^2. \tag{4.4.3}$$

By Arzelà-Ascoli theorem, up to a subsequence, $f_i \to f_\infty$ and $\xi_i \to \xi_\infty$ locally uniformly for some Lipschitz functions f_∞ and ξ_∞ on $\mathbb{R}^3$, and $\xi_\infty = \frac{f_\infty^2}{(1-f_\infty)^2}$.

Lemma 4.4.1. *f_i and ξ_i converges to f_∞ and ξ_∞ in the weakly $W^{1,2}$-sense respectively. That is, for any uniformly converging sequence of compactly supported Lipschitz functions $\psi_i \to \psi$ and $\nabla\psi_i \to \nabla\psi$ in L^2, we have*

$$\lim_{i\to\infty} \int_{\mathcal{E}_i} \xi_i \psi_i \mathrm{dvol}_{h_i} = \int_{\mathbb{R}^3} \xi_\infty \psi \mathrm{dvol}_{\mathrm{Eucl}},$$
$$\lim_{i\to\infty} \int_{\mathcal{E}_i} \langle \nabla\xi_i, \nabla\psi_i \rangle_{h_i} \mathrm{dvol}_{h_i} = \int_{\mathbb{R}^3} \langle \nabla\xi_\infty, \nabla\psi \rangle_\delta \mathrm{dvol}_{\mathrm{Eucl}}.$$

Proof. It's enough to prove for ξ_i. Under the diffeomorphism $\mathbf{u}_i$, we can identify $\mathcal{E}_i$ as a subset in $\mathbb{R}^3$. Suppose that ψ_i, ψ have support in $U \subset B(0, D)$ for some $D > 0$. Then since $\xi_i \to \xi_\infty$ uniformly, by Proposition 4.3.1 we know

$$\begin{aligned}
&\lim_{i\to\infty} \left| \int_{U\cap\mathcal{E}_i} \xi_i \psi_i \mathrm{dvol}_{h_i} - \int_U \xi_\infty \psi \mathrm{dvol}_{\mathrm{Eucl}} \right| \\
&\quad \le \lim_{i\to\infty} \int_{U\cap\mathcal{E}_i} |\xi_i\psi_i - \xi_\infty\psi| \mathrm{dvol}_{h_i} + \lim_{i\to\infty} \left| \int_{U\cap\mathcal{E}_i} \xi_\infty \psi \mathrm{dvol}_{h_i} - \int_U \xi_\infty \psi \mathrm{dvol}_{\mathrm{Eucl}} \right| \\
&\quad \le C \lim_{i\to\infty} \int_{U\cap\mathcal{E}_i} |\xi_i\psi_i - \xi_\infty\psi| \mathrm{dvol}_{\mathrm{Eucl}} \\
&\quad = 0.
\end{aligned}$$

Similarly, since $|\nabla\xi_i|_{h_i}$ and $|\nabla\psi_i|_{h_i}$ are uniformly bounded and h_i converges uniformly to δ,

$$\begin{aligned}
&\lim_{i\to\infty} \left| \int_{U\cap\mathcal{E}_i} \langle \nabla\xi_i, \nabla\psi_i \rangle_{h_i} \mathrm{dvol}_{h_i} - \int_U \langle \nabla\xi_\infty, \nabla\psi \rangle_\delta \mathrm{dvol}_{\mathrm{Eucl}} \right| \\
&\quad \le C \lim_{i\to\infty} \int_{U\cap\mathcal{E}_i} |\langle \nabla\xi_i, \nabla\psi_i \rangle_{h_i} - \langle \nabla\xi_\infty, \nabla\psi \rangle_\delta| \mathrm{dvol}_{\mathrm{Eucl}} \\
&\quad = C \lim_{i\to\infty} \int_{U\cap\mathcal{E}_i} |h_i^{jk} \partial_j\xi_i \partial_k\psi_i - \delta^{jk} \partial_j\xi_\infty \partial_k\psi| \mathrm{dvol}_{\mathrm{Eucl}} \\
&\quad \le C \lim_{i\to\infty} \sum_{j=1}^{3} \int_{U\cap\mathcal{E}_i} (|\langle \partial_j\xi_i - \partial_j\xi_\infty \rangle| \cdot |\partial_j\psi| + |\partial_j\psi_i - \partial_j\psi|) \\
&\quad = 0,
\end{aligned}$$

where in the first inequality we used that $\mathrm{Vol}(U \setminus \mathcal{E}_i) \to 0$, and in the last step, we used the assumption that $\partial_j \psi_i \to \partial_j \psi$ in L^2 and the fact that uniformly Lipschitz sequence ξ_i has a locally $W^{1,2}$-convergent subsequence, which can be seen by a smooth mollifier argument. $\square$

Proposition 4.4.2. *ξ_∞ satisfies the following equation weakly on $\mathbb{R}^3$:*

$$\Delta_\delta \xi_\infty = \frac{24}{m_0^2}. \tag{4.4.4}$$

Proof. For any smooth function ψ on $\mathbb{R}^3$ with compact support in $U \subset B(x_o, D) \subset \mathbb{R}^3$, it is enough to show that

$$-\int_U \langle \nabla \xi_\infty, \nabla \psi \rangle_\delta \, \mathrm{dvol}_{\mathrm{Eucl}} = \frac{24}{m_0^2} \int_U \psi \mathrm{dvol}_{\mathrm{Eucl}}.$$

Define $\psi_i := \psi \circ \mathbf{u}_i$ on $\mathcal{E}_i$. Then ψ_i are uniformly Lipschitz and $\psi_i \to \psi$ uniformly. By (4.4.1) and integration by parts, for $U_i := \mathbf{u}_i^{-1} U \subset \mathcal{E}_i$, we have

$$-\int_{U_i} \langle \nabla \xi_i, \nabla \psi_i \rangle_{h_i} \mathrm{dvol}_{h_i} = \int_{U_i} 6\psi_i (1 - f_i)^{-4} |\nabla f_i|^2_{h_i} \mathrm{dvol}_{h_i} - \int_{\partial U_i} \psi_i \langle \nabla \xi_i, \vec{n} \rangle \, \mathrm{dA}_{h_i}.$$

Notice that $|\psi_i|, |\nabla \xi_i|_{h_i} \le C(D)$ on U_i, so

$$\left| \int_{\partial U_i} \psi \langle \nabla \xi_i, \vec{n} \rangle \, \mathrm{dA}_{h_i} \right| \le C(D) \cdot \mathrm{Area}(\partial \mathcal{E}_i) \to 0.$$

By (4.3.6) and Proposition 4.3.1,

$$\lim_{i \to \infty} \int_{U_i} 6\psi_i (1 - f_i)^{-4} |\nabla f_i|^2_{h_i} \mathrm{dvol}_{h_i} = \lim_{i \to \infty} \int_{\mathbf{u}_i(U_i)} \frac{24}{m_0^2} \psi \mathrm{dvol}_{h_i} = \frac{24}{m_0^2} \int_U \psi \mathrm{dvol}_{\mathrm{Eucl}}.$$

It remains to show $\lim_{i \to \infty} \int_{U_i} \langle \nabla \xi_i, \nabla \psi_i \rangle_{h_i} \mathrm{dvol}_{h_i} = \int_U \langle \nabla \xi_\infty, \nabla \psi \rangle_\delta \, \mathrm{dvol}_{\mathrm{Eucl}}$, which comes from Lemma 4.4.1.

$\square$

So we have the following rigidity:

Lemma 4.4.3.

$$\xi_\infty(x) = \frac{4}{m_0^2} |x|^2.$$

In particular,

$$f_\infty(x) = \left(1 + \frac{m_0}{2|x|}\right)^{-1}. \tag{4.4.5}$$

Proof. By standard theory of elliptic equations, ξ_∞ is a smooth function. So the smooth function $\eta(x) := \xi_\infty(x) - \frac{4}{m_0^2}|x|^2$ is δ-harmonic on $\mathbb{R}^3$, i.e. $\Delta_\delta \eta = 0$. By taking limit of (4.4.3), η has quadratic growth, so η is a polynomial of degree at most 2. Since $\xi_\infty \geq 0$ and $\xi_\infty(x_o) = \xi_\infty((\frac{m_0}{2}, 0, 0)) = 1$, we can assume ξ_∞ has the following form:

$$\xi_\infty(x) = c_{11}\left(x_1 - \frac{m_0}{2}\right)^2 + c_{22}x_2^2 + c_{33}x_3^2 + \frac{4}{m_0^2}|x|^2,$$

where c_{11}, c_{22}, c_{33} are constants to be determined such that $c_{11} + \frac{4}{m_0^2} \geq 0, c_{22} + \frac{4}{m_0^2} \geq 0, c_{33} + \frac{4}{m_0^2} \geq 0$ and $c_{11} + c_{22} + c_{33} = 0$. Taking limit of (4.4.2), $|\nabla \xi_\infty|_\delta^2 \leq \frac{16}{m_0^2}\xi_\infty$, which implies that for any $x_1 \in \mathbb{R}$,

$$\left(c_{11}(x_1 - \frac{m_0}{2}) + \frac{4}{m_0^2}x_1\right)^2 \leq \frac{4}{m_0^2}\left(c_{11}(x_1 - \frac{m_0}{2})^2 + \frac{4}{m_0^2}x_1^2\right)$$

from which we can get that $c_{11} = 0$. So we also have that for any $x_2, x_3 \in \mathbb{R}$,

$$\left(c_{22} + \frac{4}{m_0^2}\right)^2 x_2^2 \leq \frac{4}{m_0^2}(c_{22} + \frac{4}{m_0^2})x_2^2$$
$$\left(c_{33} + \frac{4}{m_0^2}\right)^2 x_3^2 \leq \frac{4}{m_0^2}(c_{33} + \frac{4}{m_0^2})x_3^2,$$

from which we can get that $c_{22}, c_{33} \leq 0$, and thus $c_{22} = c_{33} = 0$ by $c_{22} + c_{33} = 0$. $\square$

In fact, we can show that the convergence is also strongly $W^{1,2}$ by the following lemmas.

Lemma 4.4.4. *For any $D > 0$ and $U \subset B(0, D)$,*

$$\lim_{i\to\infty} \int_{U\cap\mathcal{E}_i} |\nabla \xi_i|_{h_i}^2 = \int_U |\nabla \xi_\infty|_\delta^2,$$
$$\lim_{i\to\infty} \int_{U\cap\mathcal{E}_i} |\nabla f_i|_{h_i}^2 = \int_U |\nabla f_\infty|_\delta^2.$$

Proof. By Lemma 4.4.3, we know $|\nabla \xi_\infty|_\delta^2 = \frac{16}{m_0^2}\xi_\infty$ and $|\nabla f_\infty|_\delta^2 = \frac{4}{m_0^2}(1 - f_\infty)^4$. We only prove it for f_i and it is similar for ξ_i.

It's clear that $|\nabla f_\infty|_\delta \le \lim_{i\to\infty} |\nabla f_i|_{h_i}$, so it is enough to show $\lim_{i\to\infty}\int_{U\cap\mathcal{E}_i} |\nabla f_i|^2_{h_i} \le \int_U |\nabla f_\infty|^2_\delta$. This comes from (4.3.6) as following:

$$\begin{aligned}\int_U |\nabla f_\infty|^2_\delta &= \int_U \frac{4}{m_0^2}(1-f_\infty)^4\\ &= \lim_{i\to\infty}\int_{U\cap\mathcal{E}_i}\frac{4}{m_0^2}(1-f_i)^4\\ &\ge \lim_{i\to\infty}(1-\varepsilon_i)\int_{U\cap\mathcal{E}_i}|\nabla f_i|^2_{h_i}\\ &= \lim_{i\to\infty}\int_{U\cap\mathcal{E}_i}|\nabla f_i|^2_{h_i}.\end{aligned}$$

□

Lemma 4.4.5. *For any $D>0$ and $U\subset B(0,D)$,*

$$\lim_{i\to\infty}\int_{U\cap\mathcal{E}_i}|\nabla\xi_i\circ\mathbf{u}_i^{-1}-\nabla\xi_\infty|^2_\delta = 0,$$
$$\lim_{i\to\infty}\int_{U\cap\mathcal{E}_i}|\nabla f_i\circ\mathbf{u}_i^{-1}-\nabla f_\infty|^2_\delta = 0.$$

Proof. It follows from Lemma 4.4.1 and Lemma 4.4.4. □

4.5 Proof of Theorems 1.0.4 and 1.0.5

We first prove the stability for the mass-capacity inequality.

Proof of Theorem 1.0.5. Assume that (M_i^3, g_i) is a sequence of asymptotically flat 3-manifolds with nonnegative scalar curvature and compact connected outermost horizon boundaries Σ_i. Suppose that $m(g_i) - 2\mathcal{C}(\Sigma_i, g_i) = \varepsilon_i \to 0$ and $m(g_i)\to m_0 > 0$.

By Proposition 4.1.1, without loss of generality, we can assume that there exists a doubling $(\tilde{M}_i, g_i)$ of each (M_i, g_i) such that $(\tilde{M}_i, g_i)$ is symmetric about minimal surface Σ_i, has nonnegative scalar curvature and two harmonically flat ends. Let f_i be harmonic functions defined by (4.1.5) and $h_i = f_i^4 g_i$. Using the same notations in previous arguments of this section, we have proved that there exist smooth regions $\mathcal{E}_i\subset\tilde{M}_i^*$ such that $\mathrm{Area}_{h_i}(\partial\mathcal{E}_i)\to 0$, and for base point $p_i \in \mathcal{E}_i \cap \Sigma_i$,

$$(\mathcal{E}_i, \hat{d}_{h_i,\mathcal{E}_i}, p_i) \xrightarrow{\text{pm}-\text{GH}} (\mathbb{R}^3, d_{\text{Eucl}}, x_o),$$

where $x_o = (\frac{m_0}{2}, 0, 0)$ and

$$f_i \to f_\infty(x) = \left(1 + \frac{m_0}{2|x|}\right)^{-1}$$

locally uniformly.

In the following, we will only consider the manifold M_i, which is equivalent to $\{\frac{1}{2} \leq f_i < 1\}$. Since $h_i = f_i^4 g_i$, we know g_i and h_i are two uniformly equivalent metrics on M_i. When talking about uniform upper or lower bound on volume, area, or distance etc., there is no difference by using either metric, so we will omit the subindex for simplicity. Also we always identify $\mathcal{E}_i$ as a subset in $\mathbb{R}^3$ by using the diffeomorphism $\mathbf{u}_i$.

Fix $0 < \epsilon \ll 1$. By co-area formula,

$$\int_{\frac{1}{2}+2\epsilon}^{\frac{1}{2}+4\epsilon} \text{Length}(\partial\mathcal{E}_i \cap \{f_\infty = t\})dt \leq \int_{\partial\mathcal{E}_i \cap \{\frac{1}{2} \leq f_\infty \leq \frac{3}{4}\}} |\nabla f_\infty| \leq C \cdot \text{Area}(\partial\mathcal{E}_i).$$

So there exists a generic regular value $t_0 \in (\frac{1}{2} + 2\epsilon, \frac{1}{2} + 4\epsilon)$ such that $\partial\mathcal{E}_i \cap \{f_\infty = t_0\}$ consists of smooth curves, and the total length satisfies

$$\text{Length}(\partial\mathcal{E}_i \cap \{f_\infty = t_0\}) \leq \frac{C \cdot \text{Area}(\partial\mathcal{E}_i)}{\epsilon} = \Psi(\varepsilon_i|\epsilon).$$

We can assume $t_0 = \frac{1}{2} + 3\epsilon$ for simplicity. By the uniform convergence $f_i \to f_\infty$, for all large enough i, we have

$$\mathcal{E}_i \cap \{\frac{1}{2} + 3\epsilon \leq f_\infty \leq 1 - \frac{1}{\epsilon}\} \subset M_i.$$

Define $\mathcal{E}_i(\epsilon)$ as the noncompact component of

$$\mathcal{E}_i \cap \{f_\infty \geq \frac{1}{2} + 3\epsilon\} \cap M_i.$$

We make some modifications on $\mathcal{E}_i(\epsilon)$ for later usage as in the proof of [DS23, Lemma 4.3]. Let $\{D_k\}_{k=0}^N \subset \mathcal{E}_i$ be the components of $\{f_\infty = \frac{1}{2} + 3\epsilon\} \setminus \partial\mathcal{E}_i$, and assume D_0 has the largest area. For any $k \geq 1$, then $\text{diam} D_k \leq \Psi(\varepsilon_i|\epsilon)$. Choose $x_k \in D_k$ for each $k \geq 1$. Since

$d_{\mathrm{Eucl}}(D_0, D_k) \le \Psi(\varepsilon_i|\epsilon)$, we know there exists $y_k \in D_0$ such that $\hat{d}_{h_i,\mathcal{E}_i}(x_k, y_k) \le \Psi(\varepsilon_i|\epsilon)$. For each $k \ge 1$, let γ_k be a geodesic between x_k, y_k for the metric $\hat{d}_{h_i,\mathcal{E}_i}$. By thickening each γ_k, we can get thin solid tubes T_k inside $\Psi(\varepsilon_i|\epsilon)$-neighborhood of γ_k with arbitrarily small boundary area. Let $\mathcal{E}_i(\epsilon)' := \mathcal{E}_i(\epsilon) \cup (\cup_k T_k)$. We then get a smooth connected subset by smoothing corners of $\mathcal{E}_i(\epsilon)'$.

For simplicity, we still denote the modified $\mathcal{E}_i(\epsilon)'$ by $\mathcal{E}_i(\epsilon)$. Denote by $\partial^*\mathcal{E}_i(\epsilon)$ the closure of the connected component of $\partial\mathcal{E}_i(\epsilon) \setminus \partial\mathcal{E}_i$. Notice that $\partial^*\mathcal{E}_i(\epsilon)$ lies inside $\Psi(\varepsilon_i|\epsilon)$-neighborhood of $\{f_\infty = \frac{1}{2} + 3\epsilon\} \setminus \partial\mathcal{E}_i$, and the total length of the boundary of $\partial^*\mathcal{E}_i(\epsilon)$ is smaller than $\Psi(\varepsilon_i|\epsilon)$.

Denote by $M_{sc}(\epsilon) := \{f_\infty \ge \frac{1}{2} + 3\epsilon\}$, and $x_o(\epsilon) \in \{f_\infty = \frac{1}{2} + 3\epsilon\}$ a base point.

Proposition 4.5.1.

$$(\mathcal{E}_i(\epsilon), \hat{d}_{g_i,\mathcal{E}_i(\epsilon)}, q_i) \to (M_{sc}(\epsilon), d_{g_{sc}}, x_o(\epsilon)) \tag{4.5.1}$$

in the pointed measured Gromov-Hausdorff topology, and

$$(\partial^*\mathcal{E}_i(\epsilon), \hat{d}_{g_i,\partial^*\mathcal{E}_i(\epsilon)}) \to (\partial M_{sc}(\epsilon), \hat{d}_{g_{sc},\partial M_{sc}(\epsilon)}) \tag{4.5.2}$$

in the measured Gromov-Hausdorff topology for the induced length metrics.

Proof. We firstly show (4.5.2). By the construction of $\mathcal{E}_i(\epsilon)$, we know $\partial^*\mathcal{E}_i(\epsilon)$ is inside $\Psi(\varepsilon_i|\epsilon)$-neighborhood of $\partial^*\mathcal{E}_i(\epsilon) \cap \partial M_{sc}(\epsilon)$, so

$$(\partial^*\mathcal{E}_i(\epsilon), \hat{d}_{g_{sc},\partial^*\mathcal{E}_i(\epsilon)}) \to (\partial M_{sc}(\epsilon), \hat{d}_{g_{sc},\partial M_{sc}})$$

in the Gromov-Hausdorff topology. It is sufficient to show that for any $x, y \in \partial^*\mathcal{E}_i(\epsilon)$,

$$\lim_{i\to\infty} \hat{d}_{g_i,\partial^*\mathcal{E}_i(\epsilon)}(x, y) = \hat{d}_{g_{sc},\partial M_{sc}(\epsilon)}(x, y).$$

Let $\gamma \subset \partial M_{sc}(\epsilon)$ be a geodesic between $x, y \in \partial^*\mathcal{E}_i(\epsilon)$ for $\hat{d}_{g_{sc}}$. Since $\partial M_{sc}(\epsilon) \cap \partial\mathcal{E}_i$ consists of smooth curves with total length converging to 0, we can always perturb γ to get $\tilde{\gamma} \subset \partial M_{sc}(\epsilon) \cap \mathcal{E}_i$ such that $|\mathrm{Length}_{g_{sc}}(\gamma) - \mathrm{Length}_{g_{sc}}(\tilde{\gamma})| \to 0$. Then

$$\begin{aligned}\hat{d}_{g_i,\partial^*\mathcal{E}_i(\epsilon)}(x, y) &\le \int_0^1 |\tilde{\gamma}'|_{g_i} \\ &\le (1 + \Psi(\varepsilon_i)) \int_0^1 |\tilde{\gamma}'|_{g_{sc}},\end{aligned}$$

where we used $f_i \to f_\infty$ uniformly. Taking $i \to \infty$, we have

$$\lim_{i\to\infty} \hat{d}_{g_i,\partial^*\mathcal{E}_i(\epsilon)}(x,y) \le \hat{d}_{g_{sc},\partial M_{sc}(\epsilon)}(x,y).$$

Similarly, it's easy to check

$$\hat{d}_{g_{sc},\partial M_{sc}(\epsilon)}(x,y) \le \lim_{i\to\infty} \hat{d}_{g_i,\partial^*\mathcal{E}_i(\epsilon)}(x,y).$$

This completes the Gromov-Hausdorff convergence in (4.5.2).

Notice that for any $x, y \in \partial M_{sc}(\epsilon)$, there is a uniform constant $C > 0$ such that

$$d_{\mathrm{Eucl}}(x,y) \le \hat{d}_{g_{sc},\partial M_{sc}(\epsilon)}(x,y) \le C d_{\mathrm{Eucl}}(x,y),$$

which implies that, for any $x, y \in \partial^*\mathcal{E}_i(\epsilon)$, for all large enough i,

$$\frac{1}{2}\hat{d}_{h_i,\mathcal{E}_i}(x,y) \le \hat{d}_{g_i,\partial^*\mathcal{E}_i(\epsilon)}(x,y) \le C\hat{d}_{h_i,\mathcal{E}_i}(x,y) + \Psi(\varepsilon_i|\epsilon). \tag{4.5.3}$$

Now we prove (4.5.1). It is enough to show that for any fixed $D > 0$ and any $x, y \in \mathcal{E}_i(\epsilon) \cap B(q_i, D)$,

$$|\hat{d}_{g_i,\mathcal{E}_i(\epsilon)}(x,y) - d_{g_{sc}}(x,y)| \to 0 \text{ as } i \to \infty.$$

It's easy to check that $d_{g_{sc}}(x,y) \le \lim_{i\to\infty} \hat{d}_{g_i,\mathcal{E}_i(\epsilon)}(x,y)$. In the following, we will check the other inequality.

For any fixed $\delta_0 > 0$. We firstly assume that $d_{g_{sc}}(x,y) \ge \delta_0$ and $d_{g_{sc}}(x, \partial M_{sc}(\epsilon)) \ge \delta_0$, $d_{g_{sc}}(y, \partial M_{sc}(\epsilon)) \ge \delta_0$. If γ is a g_{sc}-geodesic between x, y, then $d_{g_{sc}}(\gamma, \partial M_{sc}(\epsilon)) \ge \delta_0$. For any $0 < \delta \ll \delta_0$, we can use piecewise line segments $\{\gamma_j\}_{j=1}^{N(\delta,D)}$ to approximate γ such that $\Sigma_{j=1}^{N}\mathrm{Length}_{g_{sc}}(\gamma_j) \le \mathrm{Length}_{g_{sc}}(\gamma) + \Psi(\delta)$, and $\mathrm{Length}_{\mathrm{Eucl}}(\gamma_j) \ge \delta$. For each γ_j with $\gamma_j(0) = x_j, \gamma_j(1) = y_j$, from the proof of [DS23, Lemma 4.5], we can find peturbed points x_j', y_j' such that the straight line segment $\tilde{\gamma}_j$ between x_j', y_j' lies in $\mathcal{E}_i(\epsilon)$ and $d_{\mathrm{Eucl}}(x_j, x_j') +$

$d_{\mathrm{Eucl}}(y_j, y_j') \leq \Psi(\varepsilon_i)$. Then

$$\begin{aligned}
\hat{d}_{g_i,\mathcal{E}_i(\epsilon)}(x,y) &\leq \sum_j \hat{d}_{g_i,\mathcal{E}_i(\epsilon)}(x_j', y_j') + C\sum_j (d_{\mathrm{Eucl}}(x_j, x_j') + d_{\mathrm{Eucl}}(y_j, y_j')) \\
&\leq \sum_j \int |\tilde{\gamma}_j'|_{g_i} + \Psi(\varepsilon_i|\delta, D) \\
&\leq (1 + \Psi(\varepsilon_i|D)) \sum_j \mathrm{Length}_{g_{sc}}(\tilde{\gamma}_j) + \Psi(\varepsilon_i|\delta, D),
\end{aligned}$$

where we have used that $f_i \to f_\infty$ uniformly. Taking $i \to \infty$, we have

$$\lim_{i\to\infty} \hat{d}_{g_i,\mathcal{E}_i(\epsilon)}(x,y) \leq d_{g_{sc}}(x,y) + \Psi(\delta).$$

Taking $\delta \to 0$ gives

$$\lim_{i\to\infty} \hat{d}_{g_i,\mathcal{E}_i(\epsilon)}(x,y) \leq d_{g_{sc}}(x,y). \tag{4.5.4}$$

If $d_{g_{sc}}(x,y) \leq \delta_0$ and $d_{g_{sc}}(x, \partial M_{sc}(\epsilon)) \leq \delta_0$, then $d_{\mathrm{Eucl}}(x, \partial M_{sc}(\epsilon)) \leq \delta_0$ and $d_{\mathrm{Eucl}}(x,y) \leq \delta_0$. We can take an almost $\hat{d}_{h_i,\mathcal{E}_i}$-geodesic $\gamma \subset \mathcal{E}_i$ between x, y, so $\mathrm{Length}_{h_i}(\gamma) \leq 2\delta_0$. If $\gamma \subset \mathcal{E}_i(\epsilon)$, we have $\hat{d}_{g_i,\mathcal{E}_i(\epsilon)}(x,y) \leq C\delta_0$; otherwise, let x' be the first intersection point of γ and $\partial^*\mathcal{E}_i(\epsilon)$ and y' be the last intersection point. By (4.5.3), we have

$$\hat{d}_{g_i,\mathcal{E}_i(\epsilon)}(x',y') \leq \hat{d}_{g_i,\partial^*\mathcal{E}_i(\epsilon)}(x',y') \leq C\hat{d}_{h_i,\mathcal{E}_i}(x',y') + \Psi(\varepsilon_i|\epsilon) \leq C\delta_0 + \Psi(\varepsilon_i|\epsilon).$$

So

$$\hat{d}_{g_i,\mathcal{E}_i(\epsilon)}(x,y) \leq \hat{d}_{g_i,\mathcal{E}_i(\epsilon)}(x,x') + \hat{d}_{g_i,\mathcal{E}_i(\epsilon)}(x',y') + \hat{d}_{g_i,\mathcal{E}_i(\epsilon)}(y',y) \leq C\delta_0 + \Psi(\varepsilon_i|\epsilon).$$

This shows that the pointed Gromov-Hausdorff distance

$$\begin{aligned}
d_{pGH}((\mathcal{E}_i(\epsilon), \hat{d}_{g_i,\mathcal{E}_i(\epsilon)}, q_i), (\mathcal{E}_i(\epsilon) \cap \{x : d_{g_{sc}}(x, \partial M_{sc}(\epsilon)) \geq \delta_0\}, \hat{d}_{g_i,\mathcal{E}_i(\epsilon)}, q_i)) \\
\leq C\delta_0 + \Psi(\varepsilon_i|\epsilon).
\end{aligned}$$

Together with (4.5.4), we have that

$$d_{pGH}((\mathcal{E}_i(\epsilon), \hat{d}_{g_i,\mathcal{E}_i(\epsilon)}, q_i), (M_{sc}(\epsilon), d_{g_{sc}}, x_o(\epsilon))) \leq C\delta_0 + \Psi(\varepsilon_i|\epsilon).$$

First taking $i \to \infty$ and then taking $\delta_0 \to 0$ gives the conclusion on the pointed Gromov-Hausdorff convergence in (4.5.1).

Since the Hausdorff measure induced by $\hat{d}_{g_i,\mathcal{E}_i(\epsilon)}$ and $\hat{d}_{g_i,\partial^*\mathcal{E}_i(\epsilon)}$ are the same as the volume element dvol_{g_i} and the area element dA_{g_i} respectively, together with isoperimetric inequality, it's standard to check that these measures also converge weakly (c.f. [DS23, Page 22]). In particular, $\mathrm{Area}_{g_i}(\partial^*\mathcal{E}_i(\epsilon)) \to \mathrm{Area}_{g_{sc}}(\partial M_{sc}(\epsilon))$.

□

Choosing a sequence $\epsilon_i \to 0$, and using a diagonal argument, we can take a subsequence such that

$$(\mathcal{E}_i(\epsilon_i), \hat{d}_{g_i,\mathcal{E}_i(\epsilon_i)}, q_i) \to (M_{sc}, d_{g_{sc}}, x_o)$$

in the pointed measured Gromov-Hausdorff topology, and

$$(\partial^*\mathcal{E}_i(\epsilon_i), \hat{d}_{g_i,\partial^*\mathcal{E}_i(\epsilon_i)}) \to (\partial M_{sc}, \hat{d}_{g_{sc},\partial M_{sc}})$$

in the measured Gromov-Hausdorff topology.

Finally, we can take $Z_i = M_i \setminus \mathcal{E}_i$ and $N_i \subset M_i$ to be a smooth submanifold such that $\mathbf{u}_i(N_i \setminus Z_i) = \mathcal{E}_i(\epsilon_i)$. This completes the proof.

□

Then we prove the stability for the Penrose inequality as a corollary of the stability for the mass-capacity inequality.

Proof of Theorem 1.0.4. Let $A_0 \geq 0$ be a fixed constant and (M_i^3, g_i) be a sequence of asymptotically flat 3-manifolds with nonnegative scalar curvature, whose boundaries are compact connected outermost minimal surfaces with area A_0. Suppose that the ADM mass $m(g_i)$ converges to $\sqrt{\frac{A_0}{16\pi}}$.

From the remark in the end of Section 4.1, we can find a smooth subset $M_i' \subset M_i$ and a metric g_i' such that (M_i', g_i') is also an asymptotically flat 3-manifold with nonnegative scalar curvature and a connected outermost horizon Σ_i' with $\text{Area}_{g_i'}(\Sigma_i') = A_0$, the uniform distance between g_i' and g_i on M_i' converges to 0, and $m(g_i') - 2\mathcal{C}(\Sigma_i', g_i') \to 0$.

Then we can apply the stability for mass-capacity inequality to (M_i', g_i'). So the conclusion that up to boundary area perturbations, the Schwarzschild 3-manifold is stable for the Penrose inequality follows.

□

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