Advanced Linear Algebra MAT 315

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Linear maps

Let V and W be vector spaces over a field $\mathbb{F}$.

3.2 Definition A map $T : V \to W$ is said to be **linear** if:

$T(u+v) = Tu + Tv$ for all $u, v \in V$ (T is **additive**);

$T(\lambda v) = \lambda(Tv)$ for all $\lambda \in \mathbb{F}$ and all $v \in V$ (T is **homogeneous**).

Linear maps or linear transformations? Tv or $T(v)$?

3.3 Notation $\mathcal{L}(V, W) = \{\text{all the linear maps } V \to W\}$

Other notations: $\mathrm{Hom}_{\mathbb{F}}(V, W)$ or $\mathrm{Hom}(V, W)$.

Examples of linear maps

Zero $0 \in \mathcal{L}(V, W) : x \mapsto 0$

Identity $I \in \mathcal{L}(V, V) : x \mapsto x$ Other notations: id, or id_V, or 1.

Inclusion $\mathrm{in} \in \mathcal{L}(V, W) : x \mapsto x$ if $V \subset W$

Examples of linear maps

Differentiation $D : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R}) : Dp = p'$

Integration $T : \mathcal{P}(\mathbb{R}) \to \mathbb{R} : Tp = \int_0^1 p(x)\, dx$

Multiplication by x^3 $T : \mathcal{P}(\mathbb{F}) \to \mathcal{P}(\mathbb{F}) : (Tp)(x) = x^3 p(x)$

Backward shift $T \in \mathcal{L}(\mathbb{F}^\infty, \mathbb{F}^\infty) : T(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$

A linear map takes 0 to 0

3.11 **Theorem** Let $T : V \to W$ be a linear map. Then $T(0) = 0$.

Proof $T(0) = T(0 + 0) = T(0) + T(0)$.

So, $T(0) = T(0) + T(0)$.

Add $-T(0)$ to both sides.

$0 = T(0)$. ∎

Linear operations in $\mathcal{L}(V,W)$

3.6 **Definition** Let $S, T : V \to W$ be maps and $\lambda \in \mathbb{F}$.

The **sum** $S + T$ and the **product** λT are maps $V \to W$ defined by

$$(S+T)(v) = Sv + Tv \quad \text{and} \quad (\lambda T)(v) = \lambda(Tv) \quad \text{for all } v \in V.$$

Theorem If S, T are linear maps, then $S + T$ and λT are linear maps.

Proof. **Exercise!** It's easy! ∎

3.7 **Theorem** With the operations of addition and scalar multiplication, **$\mathcal{L}(V,W)$ is a vector space.**

Proof. **Exercise!** It's easy! ∎

Composition

Definition Let $T : U \to V$ and $S : V \to W$ be maps.

The **composition** $S \circ T$ is a map $U \to W$ defined by formula

$$(S \circ T)(u) = S(T(u)) \text{ for all } u \in U. \qquad U \xrightarrow{T} V \xrightarrow{S} W, \quad U \xrightarrow{S\circ T} W$$

Composition is also called a **product**. (Say, in Axler's textbook.)

Often $S \circ T$ is denoted by ST, like a product.

Theorem If S and T are linear maps, then $S \circ T$ is a linear map.

Proof. **Exercise!** It's easy! ∎

3.9 **Algebraic properties of composition**

associativity $(T_1T_2)T_3 = T_1(T_2T_3)$

identity $T\,\mathrm{id}_V = T = \mathrm{id}_W\,T$

distributivity $(S_1 + S_2)T = S_1T + S_2T$ and $(T_1 + T_2)S = T_1S + T_2S$.

Language of categories

Categories

A category provides a framework with a convenient language to speak about objects of unspecified nature, but related to each other in a very specific way.

A **category** consists of:

objects and
morphisms: for any two objects X, Y morphisms $X \to Y$, and
compositions of morphisms: $X \xrightarrow{f} Y \xrightarrow{g} Z$, $X \xrightarrow{g\circ f} Z$

The composition is **associative**: $h \circ (g \circ f) = (h \circ g) \circ f$

$$\begin{array}{ccc} B & \xleftarrow{f} & A \\ {\scriptstyle g}\downarrow & \swarrow{\scriptstyle g\circ f} & \downarrow{\scriptstyle h\circ(g\circ f)} \\ C & \xrightarrow[h]{} & D \end{array} \quad = \quad \begin{array}{ccc} B & \xleftarrow{f} & A \\ {\scriptstyle g}\downarrow & \searrow{\scriptstyle h\circ g} & \downarrow{\scriptstyle (h\circ g)\circ f} \\ C & \xrightarrow[h]{} & D \end{array}$$

With any object X, the **identity morphism** $\mathrm{id}_X : X \to X$ is associated:

for $A \xrightarrow{f} X \xrightarrow{\mathrm{id}_X} X$, $A \xrightarrow{f} X$ we have $\mathrm{id}_X \circ f = f$

and for $X \xrightarrow{\mathrm{id}_X} X \xrightarrow{g} B$, $X \xrightarrow{g} B$ we have $g \circ \mathrm{id}_X = g$.

Examples of categories

Example 1. The category of sets
Objects are sets, morphisms are maps, compositions are compositions of maps.

Example 2. The category of vector spaces over a field $\mathbb{F}$
Objects are vector spaces over $\mathbb{F}$, morphisms are linear maps, compositions are compositions of linear maps.

Example 3. The category of linear maps Let $\mathbb{F}$ be a field.
Objects are linear maps $V \to W$, where V and W are vector spaces over $\mathbb{F}$.
A morphism $(V \xrightarrow{T} W) \to (X \xrightarrow{S} Y)$ is a pair $(V \xrightarrow{L} X, W \xrightarrow{M} Y)$ of linear maps such that $M \circ T = S \circ L$.

It is presented by a diagram: $\begin{array}{ccc} V & \xrightarrow[L]{} & X \\ \downarrow{\scriptstyle T} & & {\scriptstyle S}\downarrow \\ W & \xrightarrow{M} & Y \end{array}$ which is **commutative**: $M \circ T = S \circ L$.

Composition: $$\left(\begin{array}{ccc} A & \xleftarrow[N]{} & X \\ \downarrow{\scriptstyle U} & & {\scriptstyle S}\downarrow \\ B & \xleftarrow{R} & Y \end{array}\right) \circ \left(\begin{array}{ccc} X & \xleftarrow[L]{} & V \\ \downarrow{\scriptstyle S} & & {\scriptstyle T}\downarrow \\ Y & \xleftarrow{M} & W \end{array}\right) = \left(\begin{array}{ccc} A & \xleftarrow[N\circ L]{} & V \\ \downarrow{\scriptstyle U} & & {\scriptstyle T}\downarrow \\ B & \xleftarrow{R\circ M} & W \end{array}\right)$$

Operators

3.67 Definition
A linear map from a vector space to itself is called an **operator**.

Notation $\mathcal{L}(V) = \{\text{all linear maps } V \to V\} = \mathcal{L}(V, V)$.

Category of operators in vectors spaces over a field $\mathbb{F}$

objects are operators $T : V \to V$,

a **morphism** $(V \xrightarrow{T} V) \to (W \xrightarrow{S} W)$

is a linear map $V \xrightarrow{L} W$ such that $S \circ L = L \circ T$.

or, rather, a commutative diagram $\begin{array}{ccc} V & \xrightarrow{L} & W \\ \downarrow{\scriptstyle T} & & {\scriptstyle S}\downarrow \\ V & \xrightarrow{L} & W \end{array}$,

a **composition** of morphisms is the composition of the linear maps.

Axler: "The deepest and most important parts of linear algebra ... deal with operators."

Which categories will be used in this course?

Inverses and invertibles

In any category:

Definition
Morphisms $T : V \to W$ and $S : W \to V$ are said to be **inverse** to each other if $S \circ T = \mathrm{id}_V$ and $T \circ S = \mathrm{id}_W$.
A morphism $T : V \to W$ is called **invertible** if there exists a morphism inverse to T.

3.54 Uniqueness of Inverse If a morphism is invertible then its inverse is unique.

Proof Let S_1 and S_2 be inverse to $T : V \to W$. Then
$$S_1 = S_1 \,\mathrm{id}_W = S_1(TS_2) = (S_1T)S_2 = \mathrm{id}_V \, S_2 = S_2$$ ■

3.55 Notation If T is invertible, then its inverse is denoted by T^{-1}.

For a morphism $T : V \to W$, the inverse morphism T^{-1} is defined by two properties:
$$TT^{-1} = \mathrm{id}_W \quad \text{and} \quad T^{-1}T = \mathrm{id}_V .$$

Isomorphism

In any category:

3.58 **Definition** An invertible morphism is called an **isomorphism**.
Objects V and W are called **isomorphic** if $\exists$ an isomorphism $V \to W$.

Properties of isomorphisms

- An identity morphism is an isomorphism.
- The composition of isomorphisms is an isomorphism.
- The map inverse to an isomorphism is an isomorphism.

Relation of being isomorphic is equivalence.
It is reflexive, symmetric and transitive.

A category does not recognize any difference between its isomorphic objects,
although the objects may be not identically the same.

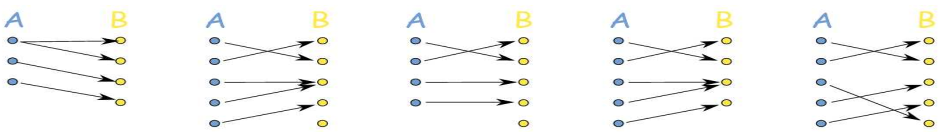
surjectivity, injectivity and bijectivity

Back to the category of sets and maps

3.20 **Definition** A map $T : V \to W$ is called **surjective** if $T(V) = W$.

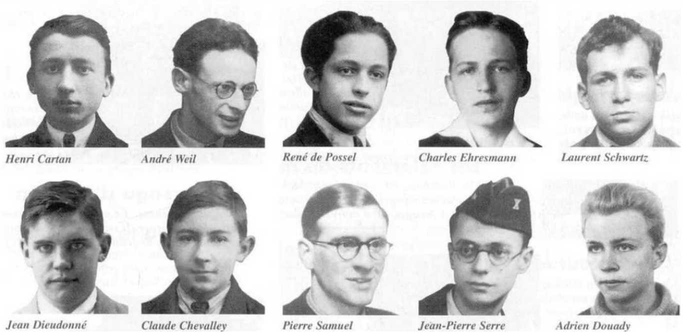
3.15 **Definition** A map $T : V \to W$ is called **injective** if $Tu = Tv \implies u = v$.

Definition
A map $T : V \to W$ is called **bijective** if T is both injective and surjective.



not a map	a map	injection, but not surjection	surjection, but not injection	bijection
		1-to-1	"onto"	invertible

liberté, égalité et fraternité



Nicolas Bourbaki

Invertible = bijection

Which sets are isomorphic in the category of sets and maps?

3.56 **Theorem. Invertibility is equivalent to bijectivity.**

You should know this. If not, see the textbook, page 81.

Null space

3.12 **Definition (reminder)** For $T \in \mathcal{L}(V, W)$, the null space of T is
$$\operatorname{null} T = T^{-1}\{0\} = \{v \in V \mid Tv = 0\}.$$

Another name: **kernel**. Notation: $\operatorname{Ker} T$.

3.13 **Examples**

- For $T : V \to W : v \mapsto 0$, $\operatorname{null} T = V$
- For differentiation $D : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$, $\operatorname{null} D = \{\text{constants}\}$
- For multiplication by x^3 $T : \mathcal{P}(\mathbb{F}) \to \mathcal{P}(\mathbb{F}) : Tp = x^3 p(x)$, $\operatorname{null} T = 0$
- For backward shift $T \in \mathcal{L}(\mathbb{F}^\infty, F^\infty) : T(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$
  $\operatorname{null} T = \{(a, 0, 0, \dots) \mid a \in \mathbb{F}\}$

Null space is a subspace

3.14 **Theorem.** For $T \in \mathcal{L}(V, W)$, $\operatorname{null} T$ is a subspace of V.

Proof. As we know (by 3.11) $T(0) = 0$. Hence $0 \in \operatorname{null} T$.

$u, v \in \operatorname{null} T \implies T(u + v) = T(u) + T(v) = 0 + 0 = 0 \implies u + v \in \operatorname{null} T$.

$u \in \operatorname{null} T, \lambda \in \mathbb{F} \implies T(\lambda u) = \lambda T u = \lambda 0 = 0 \implies \lambda u \in \operatorname{null} T$. ∎

Injectivity and the null space

3.15 **Definition (reminder)**
A map $T : V \to W$ is called **injective** if $Tu = Tv \implies u = v$.

A map $T : V \to W$ is injective $\iff$ $u \neq v \implies Tu \neq Tv$.

3.16 T is injective $\iff$ $\operatorname{null} T = \{0\}$.

Proof

$\implies$ Recall $0 \in \operatorname{null} T$. If $\operatorname{null} T \neq \{0\}$, then $\exists v \in \operatorname{null} T$, $v \neq 0$.
So, $Tv = T0 = 0$ and T is not injective. ■

$\impliedby$ Let $u, v \in V$, $Tu = Tv$. Then $0 = Tu - Tv = T(u - v)$.
Hence $u - v \in \operatorname{null} T = \{0\} \implies u = v$. ■