Review solutions to select problems

Saturday, May 9, 2020 2:55 PM

2.b. $\int_{1/3}^{1} \cos(\underbrace{3\pi\cdot(x-1)}_{u})\,dx$ ← bounds for x

$= \int_{-2\pi}^{0} \cos(u)\cdot\frac{1}{3\pi}\,du$

$= \frac{1}{3\pi}\cdot\int_{-2\pi}^{0}\cos(u)\cdot du$

$= \frac{1}{3\pi}\left[\sin u\right]\Big|_{-2\pi}^{0}$

$= \frac{1}{3\pi}\left[\cancel{\sin 0}^{\,0} - \cancel{\sin(-2\pi)}^{\,0}\right]$

$= 0$

try u-sub with
$u = 3\pi\cdot(x-1) = 3\pi x - 3\pi$
$du = 3\pi\cdot dx \rightsquigarrow dx = \frac{1}{3\pi}\,du$

bounds: $3\pi(1-1) = 0 = u_{top}$
$3\pi(\frac{1}{3}-1) = -2\pi = u_{bot}$

2.d. $\int_{-3}^{0} (x+5)\sqrt{9-x^2}\cdot dx$ ← distribute

$= \underbrace{\int_{-3}^{0} x\cdot\sqrt{9-x^2}\,dx}_{①} + \underbrace{\int_{-3}^{0} 5\cdot\sqrt{9-x^2}\,dx}_{②}$

① $\int_{-3}^{0} x\cdot\sqrt{9-x^2}\,dx$

$= \int_{0}^{9}\sqrt{u}\cdot\left(-\frac{1}{2}\right)du$

$= -\frac{1}{2}\cdot\int_0^9 u^{1/2}\,du$

$= -\frac{1}{2}\cdot\left[\frac{u^{3/2}}{3/2}\right]\Big|_0^9$

$= -\frac{1}{2}\cdot\frac{2}{3}\cdot\left[u^{3/2}\right]\Big|_0^9$

$= -\frac{1}{3}\cdot\left[9^{3/2} - 0^{3/2}\right] = -\frac{1}{3}\cdot\left[(3^2)^{3/2}\right] = -\frac{1}{3}\cdot 27 = -9$

let $u = 9 - x^2$
$du = -2x\cdot dx$
$\hookrightarrow x\cdot dx = -\frac{1}{2}du$

bounds $\left[\begin{array}{l} 9-(-3)^2 = 9-9 = 0 \\ 9-(0)^2 = 9 \end{array}\right.$

② $\int_{-3}^{0} 5\cdot\sqrt{9-x^2}\,dx$

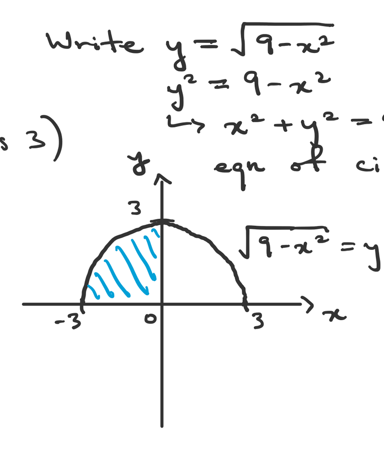
$= 5\cdot\int_{-3}^{0}\sqrt{9-x^2}\,dx$

$= 5\cdot\frac{1}{4}$(Area of circle of radius 3)

$= 5\cdot\frac{1}{4}\cdot\pi\cdot 3^2$

$= \frac{45}{4}\cdot\pi.$

Write $y = \sqrt{9-x^2}$
$y^2 = 9 - x^2$
$\hookrightarrow x^2 + y^2 = 9$
eqn of circle.



2.c. Write $\left(\frac{3-x}{x}\right)^2 = \left(\frac{3}{x} - 1\right)^2$ then distribute.

3.a. $\int x^2\cdot\cos 3x\cdot dx$

$= u\cdot v - \int v\cdot du$

Integration by Parts:
$\int u\cdot dv = u\cdot v - \int v\cdot du$

Let $u = x^2$, $dv = \cos 3x\cdot dx$
$du = 2x\cdot dx$ $\quad v = \int\cos(3x)\cdot dx$
$= \frac{1}{3}\sin(3x)$

$= x^2\cdot\frac{1}{3}\sin(3x) - \int\frac{1}{3}\sin(3x)\cdot 2x\cdot dx$

$= x^2\cdot\frac{1}{3}\cdot\sin(3x) - \frac{2}{3}\cdot\int x\cdot\sin(3x)\,dx$

apply int. by parts one more time.
$u = x$, $dv = \sin(3x)\,dx$

4.c. $\int\frac{\tan^3 x}{\sqrt{\sec x}}\,dx$

$= \int\frac{\tan x}{\sqrt{\sec x}}\cdot(\sec^2 x - 1)\cdot dx$ ← distribute

$= \underbrace{\int\frac{\tan x\cdot\sec^2 x}{\sqrt{\sec x}}\,dx}_{①} - \underbrace{\int\frac{\tan x}{\sqrt{\sec x}}\,dx}_{②}$

$\sin^2 x + \cos^2 x = 1$
$\hookrightarrow \tan^2 x + 1 = \sec^2 x$

① $= \int\tan x\cdot\sec x\cdot\underbrace{\frac{\sec x}{\sqrt{\sec x}}}_{\sqrt{\sec x}}\cdot dx$

apply u-sub
$u = \sec x$
$du = \tan x\cdot\sec x\cdot dx$

$= \int\sqrt{u}\cdot du$

$= \frac{u^{3/2}}{3/2} = \frac{2}{3}\cdot(\sec x)^{3/2} + C.$

② $= -\int\frac{\tan x}{\sqrt{\sec x}}\,dx$

$= -\int\frac{\sin x}{\cos x}\cdot\sqrt{\cos x}\cdot dx$

$\cos x = (\sqrt{\cos x})^2$

$= -\int\frac{\sin x}{\sqrt{\cos x}}\,dx$

$= \int\frac{1}{\sqrt{u}}\cdot du$

$= \int u^{-1/2}\,du$

$= \frac{u^{1/2}}{1/2} = 2\cdot\sqrt{u} = 2\cdot\sqrt{\cos x} + C$

Recall $\tan x = \frac{\sin x}{\cos x}$
$\sec x = \frac{1}{\cos x}$

u-sub with $u = \cos x$
$\boxed{du = -\sin x\,dx}$

3.c. $\int\sin(\ln(2x))\,dx$

$= u\cdot v - \int v\cdot du$

Integration by Parts:
$u = \sin(\ln(2x))$ $\quad dv = dx$
$du = \cos(\ln(2x))\cdot\frac{1}{2x}\cdot 2\cdot dx$ $\quad v = x.$

$= \sin(\ln(2x))\cdot x - \int\cancel{x}\cdot\cos(\ln(2x))\cdot\frac{1}{\cancel{2x}}\cdot\cancel{2}\cdot dx$

$= \sin(\ln(2x))\cdot x - \int\cos(\ln(2x))\cdot dx$

Integrate by Parts:
$u = \cos(\ln(2x))$ $\quad dv = dx$
$du = -\sin(\ln(2x))\cdot\frac{1}{2x}\cdot 2\cdot dx$ $\quad v = x.$

$= \sin(\ln(2x))\cdot x - \left[u\cdot v - \int v\cdot du\right]$

$= \sin(\ln(2x))\cdot x - \left[x\cdot\cos(\ln(2x)) + \int\cancel{x}\cdot(+\sin(\ln(2x)))\cdot\frac{1}{\cancel{2x}}\cdot\cancel{2}\,dx\right]$

distribute

$= \sin(\ln(2x))\cdot x - x\cdot\cos(\ln(2x)) - \int\sin(\ln(2x))\cdot dx = \int\sin(\ln(2x))\,dx$

started with

same

So $2\cdot\int\sin(\ln(2x))\,dx = \sin(\ln(2x))\cdot x - x\cdot\cos(\ln(2x))$

$\Rightarrow \int\sin(\ln(2x))\,dx = \frac{1}{2}\cdot\sin(\ln(2x))\cdot x - \frac{x}{2}\cdot\cos(\ln(2x))$

5.a. use $\begin{cases} x = 4\cdot\sin\theta \\ dx = 4\cdot\cos\theta\cdot d\theta \end{cases}$

5.b. use $\begin{cases} x = 5\cdot\sec\theta \\ dx = 5\cdot\tan\theta\cdot\sec\theta\cdot d\theta \end{cases}$

Then $\sqrt{x^2 - 25}$
$= \sqrt{25\sec^2\theta - 25}$
$= 5\cdot\sqrt{\sec^2\theta - 1}$
$= 5\cdot\sqrt{\tan^2\theta}$
$= 5\cdot\tan\theta$

5.c. use $\begin{cases} x = \tan\theta \\ dx = \sec^2\theta\cdot d\theta. \end{cases}$

6.a. $\int\frac{2x^4}{x^2-2x}\,dx$ ← numerator has higher degree than the denominator.

First apply long divison:

$$\begin{array}{r l} & 2x^2 + 4x + 8 \\ x^2 - 2x \,\big) & 2x^4 + 0\cdot x^3 + 0x^2 + 0x + 0 \\ & -(2x^4 - 4\cdot x^3) \\ & 0x^4 + 4x^3 + 0x^2 \\ & -(4x^3 - 8x^2) \\ & 0x^3 + 8x^2 + 0x \\ & -(8x^2 - 16x) \\ & 0x^2 + 16x \end{array}$$

remainder

smaller degree than $x^2 - 2x$.

This means $\frac{2x^4}{x^2-2x} = 2x^2 + 4x + 8 + \frac{16x}{x^2-2x}$

Partial Frac. Decomp: rewrite $\frac{16x}{x^2-2x} = \boxed{\frac{16x}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}}$

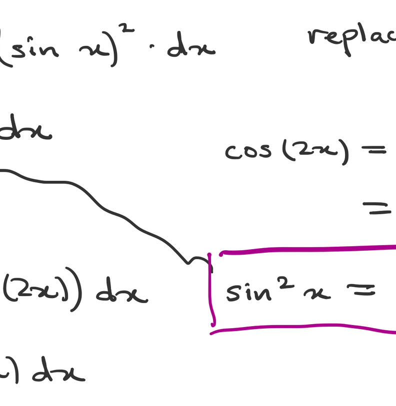
Clear denominators: $16x = A\cdot(x-2) + B\cdot x$
$= (A+B)x - 2A$

$\Rightarrow \begin{array}{l} 16 = A + B \\ 0 = -2A \end{array} \Rightarrow \boxed{A = 0} \quad \boxed{B = 16}$

So $\frac{16x}{x^2-2x} = \frac{16}{x-2}.$

$\int\frac{2x^4}{x^2-2x}\cdot dx = \int\left(2x^2 + 4x + 8 + \frac{16}{x-2}\right)dx$
$= \frac{2}{3}\cdot x^3 + 2\cdot x^2 + 8x + 16\cdot\ln|x-2| + C.$

9.b. Region R: $y = \sin x$
$y = 5\sin x$
$x = 0$
$x = \pi.$



Washer method:

Volume $= \pi\cdot\int_a^b R^2 - r^2\,dx$
$= \pi\cdot\int_0^\pi (5\sin x)^2 - (\sin x)^2\cdot dx$
$= \pi\cdot\int_0^\pi 24\cdot\sin^2 x\cdot dx$

replace with $1 - \sin^2 x$
$\cos(2x) = \cos^2 x - \sin^2 x$
$= 1 - 2\cdot\sin^2 x$

$\boxed{\sin^2 x = \frac{1}{2}\cdot(1 - \cos(2x))}$

$= 24\cdot\pi\cdot\int_0^\pi\frac{1}{2}\cdot(1 - \cos(2x))\,dx$
$= 12\pi\cdot\int_0^\pi 1 - \cos(2x)\,dx$
$= 12\pi\cdot\left[x - \frac{1}{2}\cdot\sin(2x)\right]\Big|_0^\pi$
$= 12\pi\cdot\left[\left(\pi - \cancel{\frac{1}{2}\cdot\sin(2\pi)}\right) - \left(0 - \cancel{\frac{1}{2}\cdot\sin(0)}^{\,0}\right)\right]$
$= 12\cdot\pi^2$

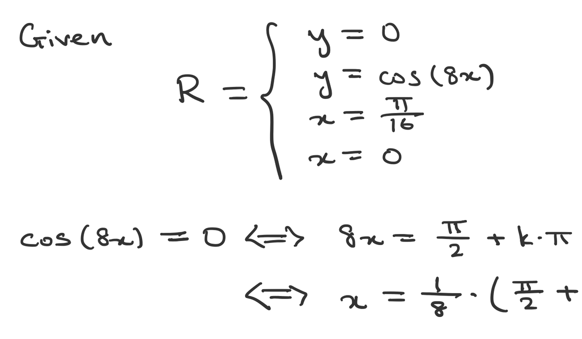
10. Find length of $y = -\frac{1}{2}x + 2$ from $x = 1$ to $x = 4$.

Arclength $= \int_a^b\sqrt{1 + [f'(x)]^2}\cdot dx$

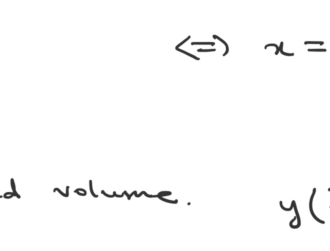
apply this with $f(x) = -\frac{1}{2}x + 2$
$f'(x) = -\frac{1}{2}.$

$= \int_1^4\sqrt{1 + (-\frac{1}{2})^2}\,dx$
$= \int_1^4\sqrt{\frac{5}{4}}\cdot dx$
$= \sqrt{\frac{5}{4}}\cdot[x]\Big|_1^4 = \sqrt{\frac{5}{4}}\cdot 3$

How do we see this geometrically?



$f(x) = -\frac{1}{2}x + 2$ $\quad f(1) = -\frac{1}{2} + 2 = \boxed{\frac{3}{2}}$



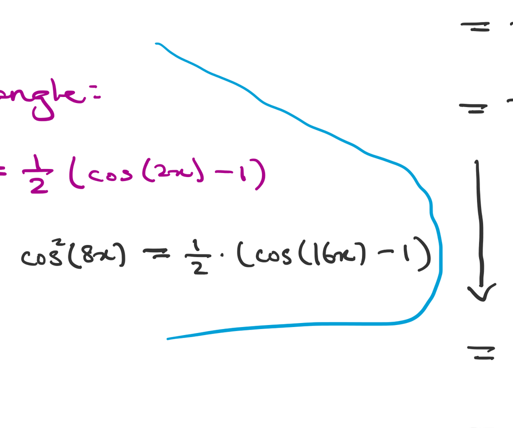
$c^2 = 3^2 + \left(\frac{3}{2}\right)^2 = 9\cdot\left(1 + \frac{1}{4}\right) = 9\cdot\frac{5}{4}$

$c = \sqrt{9\cdot\frac{5}{4}} = 3\cdot\sqrt{\frac{5}{4}}.$

Given $R = \begin{cases} y = 0 \\ y = \cos(8x) \\ x = \frac{\pi}{16} \\ x = 0 \end{cases}$ Rotate around axis $y = -2$.

$\cos(8x) = 0 \iff 8x = \frac{\pi}{2} + k\cdot\pi$ for any integer k.
$\iff x = \frac{1}{8}\cdot\left(\frac{\pi}{2} + k\cdot\pi\right)$
$\iff x = \frac{\pi}{16} + \frac{k\cdot\pi}{8}$ for any integer k.

Find volume. $y\left(\frac{\pi}{16}\right) = \cos\left(8\cdot\frac{\pi}{16}\right) = \cos\left(\frac{\pi}{2}\right) = 0$



$R = \cos(8x) - (-2) = \cos(8x) + 2.$

$r = 0 - (-2) = 2$

Volume $= \pi\cdot\int_0^{\pi/16}(R^2 - r^2)\,dx = \pi\cdot\int_0^{\pi/16}\left[(\cos(8x) + 2)^2 - 2^2\right]dx$
$= \pi\cdot\int_0^{\pi/16}\left[\cos^2(8x) + 4\cdot\cos(8x) + \cancel{4} - \cancel{4}\right]dx$
$= \pi\cdot\int_0^{\pi/16}\underbrace{\frac{1}{2}(\cos(16x) - 1)}_{\frac{1}{2}\cos(16x) - \frac{1}{2}} + 4\cdot\cos(8x)\,dx$

Double-angle:
$\cos^2 x = \frac{1}{2}(\cos(2x) - 1)$
We want $\cos^2(8x) = \frac{1}{2}\cdot(\cos(16x) - 1)$

$= \pi\cdot\left[\frac{1}{2}\cdot\frac{1}{16}\cdot\sin(16x) - \frac{1}{2}x + 4\cdot\frac{1}{8}\cdot\sin(8x)\right]\Big|_0^{\pi/16}$
$= \pi\cdot\left[\frac{1}{32}\cdot\cancel{\sin(\pi)}^{\,0} - \frac{1}{2}\cdot\left(\frac{\pi}{16}\right) + \frac{1}{2}\cdot\cancel{\sin\left(\frac{\pi}{2}\right)}^{\,1}\right]$
$= -\frac{\pi^2}{32} + \frac{1}{2}\pi.$