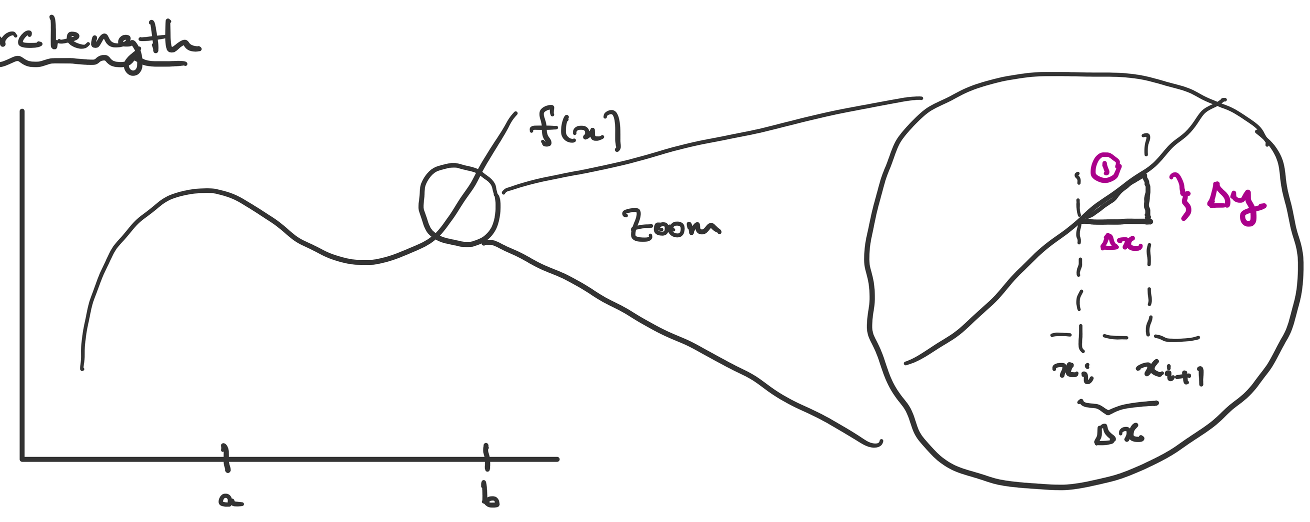
5/5/2020 R02

Tuesday, May 5, 2020 4:01 PM

* Usual office hours this week + next week.
* I will also be in the MLC this Friday 4:30pm.
* Review session on Saturday at 3pm - check website for review sheet.
* Please fill out course evals.

Arclength



length of ① [using Pythagorean thm]

slope of ①

$= f'(x_i) = \frac{\text{rise}}{\text{run}}$

$= \frac{\Delta y}{\Delta x}$

$\Rightarrow \Delta y = \Delta x \cdot f'(x_i)$

$(\Delta x)^2 + (\Delta y)^2 = (\text{length of } ①)^2$

So length of ①

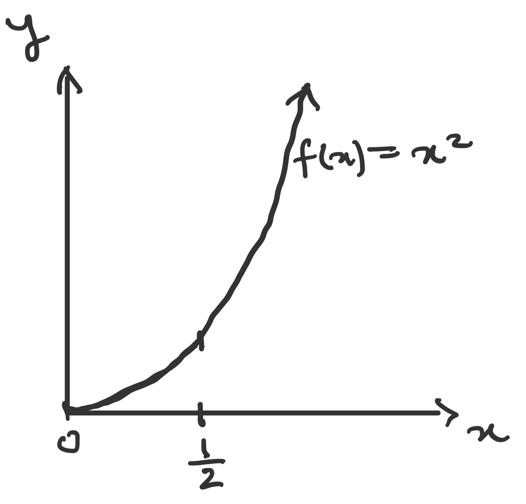
$= \sqrt{(\Delta x)^2 + (\Delta x)^2 \cdot [f'(x_i)]^2}$

$= \sqrt{1 + [f(x_i)]^2} \cdot \Delta x$

So formula for arclength:

$$\int_a^b \sqrt{1 + [f'(x)]^2} \cdot dx$$

Ex $f(x) = x^2$. Find arclength from $x = 0$ to $x = \frac{1}{2}$.



First, $f'(x) = 2x$

$$\text{Arclength} = \int_0^{\frac{1}{2}} \sqrt{1 + [2x]^2} \cdot dx$$

$$= \int_0^{1/2} \sqrt{1 + 4x^2} \cdot dx$$

$$= \int_0^{\pi/4} \sqrt{1 + \tan^2 z} \cdot \frac{1}{2} \sec^2 z \cdot dz$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{2} \cdot \sec^3 z \cdot dz$$

$$(*) \quad = \frac{1}{2} \cdot \int_0^{\pi/4} \sec^3 z \cdot dz$$

$$= \frac{1}{2} \cdot \int_0^{\pi/4} \sec z \cdot (1 + \tan^2 z)\, dz$$

$$= \frac{1}{2}\left[\int_0^{\pi/4} \sec z \cdot dz + \int_0^{\frac{\pi}{4}} \sec z \cdot \tan^2 z \cdot dz\right]$$

Need trig sub:

$x = \frac{1}{2} \tan z$

$dx = \frac{1}{2} \sec^2 z \cdot dz$

[why? $1 + \tan^2 x = \sec^2 x$]

Bounds: $0 = \frac{1}{2} \tan z$

$\Rightarrow z = 0$.

$\frac{1}{2} = \frac{1}{2} \cdot \tan z$

$\Rightarrow z = \frac{\pi}{4}$

$\ln|\sec z + \tan z|$

Use integration by parts

$u = \tan z \qquad dv = \tan z \cdot \sec z \cdot dz$

$du = \sec^2 z \cdot dz \qquad v = \sec z$

$\tan z \cdot \sec z - \int \sec z \cdot \sec^2 z \cdot dz$

$= \tan z \cdot \sec z - \int \sec^3 z\, dz$

In summary

We have

$$\int \sec^3 z \cdot dz = \ln|\sec z + \tan z| + \tan z \cdot \sec z - \int \sec^3 z\, dz$$

$$\int \sec^3 z \cdot dz = (\ln|\sec z + \tan z| + \tan z \cdot \sec z) \cdot \frac{1}{2}$$

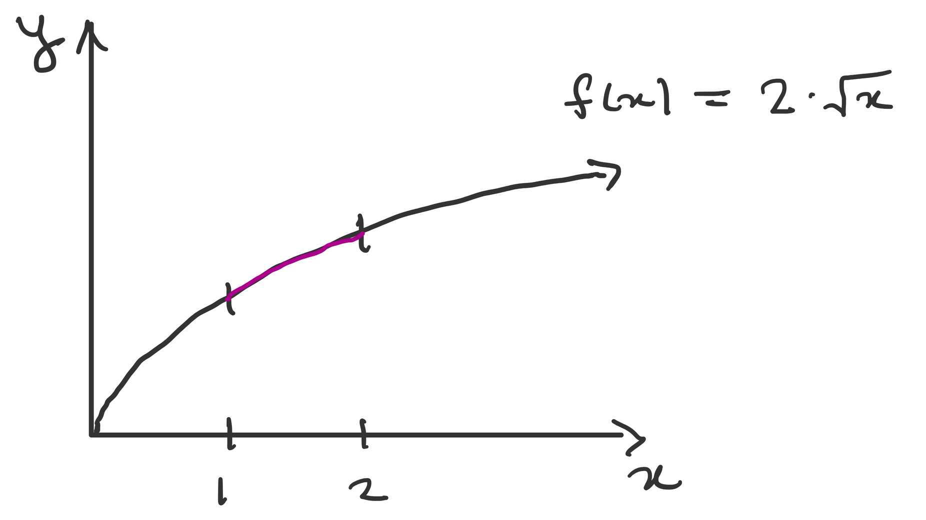
$\sec\frac{\pi}{4} = \frac{1}{\cos(\frac{\pi}{4})} = \sqrt{2}$

So $(*) = \frac{1}{2} \cdot \frac{1}{2} \cdot (\ln|\sec z + \tan z| + \tan z \cdot \sec z)\Big|_0^{\frac{\pi}{4}}$

$= \frac{1}{4} \cdot \left[(\ln|\sqrt{2} + 1| + 1 \cdot \sqrt{2}) - (\underbrace{\cancel{\ln(1+0)}}_{0} + \underbrace{\cancel{1 \cdot 0}}_{0})\right]$

$= \frac{1}{4} \cdot (\ln(\sqrt{2} + 1) + \sqrt{2})$

Ex. $f(x) = 2 \cdot \sqrt{x}$, Arclength from $x = 1$ to $x = 2$:



$f'(x) = 2 \cdot \frac{1}{2} \cdot x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}}$

$$\text{Arclength} = \int_1^2 \sqrt{1 + [f'(x)]^2}\, dx$$

$$= \int_1^2 \sqrt{1 + \frac{1}{x}} \cdot dx$$

$$= \int_1^2 \frac{\sqrt{x+1}}{\sqrt{x}}\, dx$$

$$= \int_1^{\sqrt{2}} \frac{\sqrt{u^2+1}}{\cancel{u}} \cdot 2 \cdot \cancel{u} \cdot du$$

$$= \int_1^{\sqrt{2}} 2 \cdot \sqrt{u^2 + 1}\, du$$

....

$x = u^2$

u-sub with $u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}} \cdot dx$

$\Rightarrow dx = 2 \cdot \sqrt{x} \cdot du$

$= 2 \cdot u \cdot du$

Need trig sub:

$u = \tan z$

$du = \sec^2 z \cdot dz$