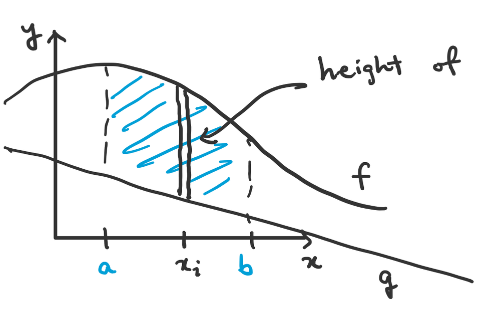
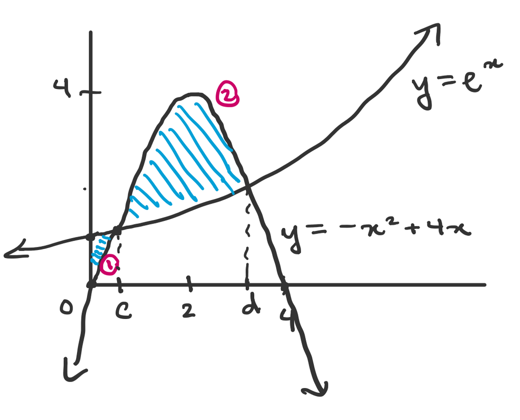
Area between curves



height of rectangle $= f(x_i) - g(x_i)$

Area between f and g $= \int_a^b [f(x) - g(x)]\,dx$

Ex. Find area between



$y = x(4-x) = -x^2+4x$

$y = e^x$

① Top function is $y=e^x$
bottom function is $y = -x^2+4x$

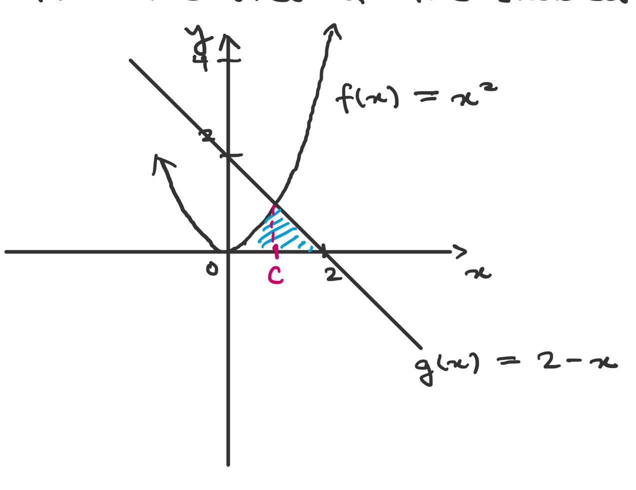
② Top function is $y = -x^2+4x$
bottom function is $y=e^x$

Say that $y=e^x$ and $y=-x^2+4x$ intersect at $x=c$ and $x=d$.

Total Area = Area of ① + Area of ②

$= \int_0^c e^x - (-x^2+4x)\,dx + \int_c^d (-x^2+4x) - e^x\,dx$

Ex Find the area of the shaded region



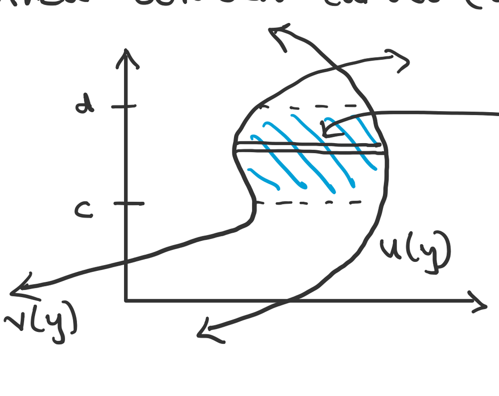
Find x-value such that

$x^2 = 2-x \iff x^2+x-2=0$
$\iff (x+2)(x-1) = 0$
$\iff x=-2$ or $x=1$.

So $c=1$.

$$\text{Area} = \int_0^1 (x^2 - 0)\,dx + \int_1^2 ((2-x)-0)\,dx$$
$$= \left[\frac{x^3}{3}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2$$
$$= \frac{1}{3} + [4-2] - \left[2-\frac{1}{2}\right]$$
$$= \frac{1}{3} + 2 - \frac{3}{2} = 2 + \frac{2}{6} - \frac{9}{6} = \frac{5}{6}.$$

Area between curves (with horizontal rectangles)



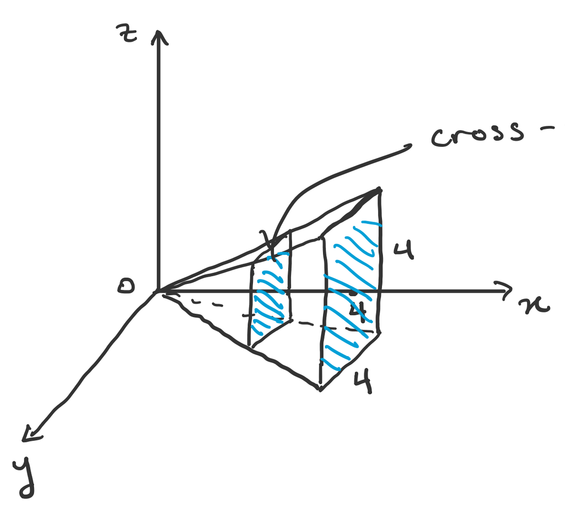
height of rectangle is given by $u(y) - v(y)$

Area is computed by taking "right" function minus "left" function and then integrating

$$\text{Area} = \int_c^d u(y) - v(y)\,dy$$

Ex find area between $y=x^3$ and $y=x^2+x$.
↑ should have two regions.

Volume of solids:

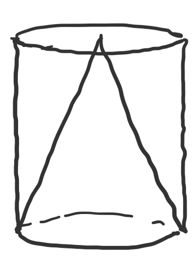


cross-section has area x^2 at x

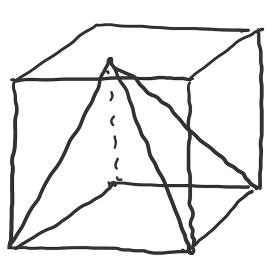
e.g. at $x=4$, cross section has area $4^2 = 16$

$$\text{Volume} = \int_0^4 x^2\,dx$$
$$= \left[\frac{x^3}{3}\right]_0^4$$
$$= \frac{4^3}{3}$$
$$= \frac{1}{3}\cdot 4^3. = \frac{1}{3}\,\text{Vol}_{\text{cube } 4\times4\times4}$$

Given a cone or pyramid, we can find the volume:



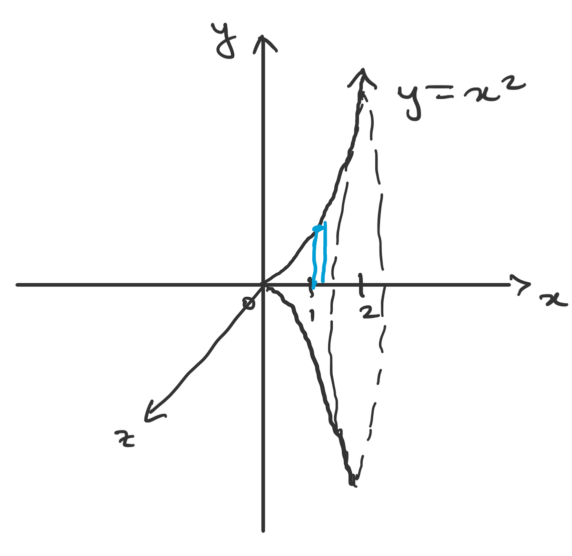
$\text{Vol}_{\text{Cone}} = \frac{1}{3}\cdot\text{Vol}_{\text{cylinder}}$



$\text{Vol}_{\text{Pyramid}} = \frac{1}{3}\,\text{Vol}_{\text{cube/rect. prism}}$.

Solids of revolution:

Take $y=x^2$ and rotate around x-axis. Look at x in $[0,2]$.



Find volume of this region:

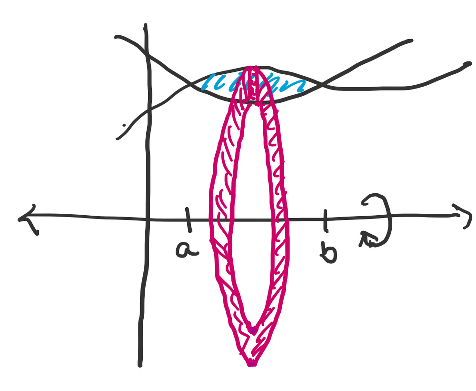
Radius of each disk is $r = x^2$ at x.

Area of each disk $= \pi\cdot r^2 = \pi\cdot x^4$

$$\text{Volume} = \int_0^2 \pi\cdot x^4\cdot dx$$

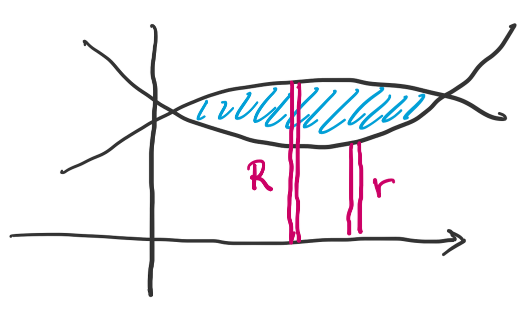
In general: Given radius $r = r(x)$, the solid of revolution on $[a,b]$ has volume $\text{Vol} = \int_a^b \pi\cdot r^2\cdot dx$

Variation:



rotate region around x-axis

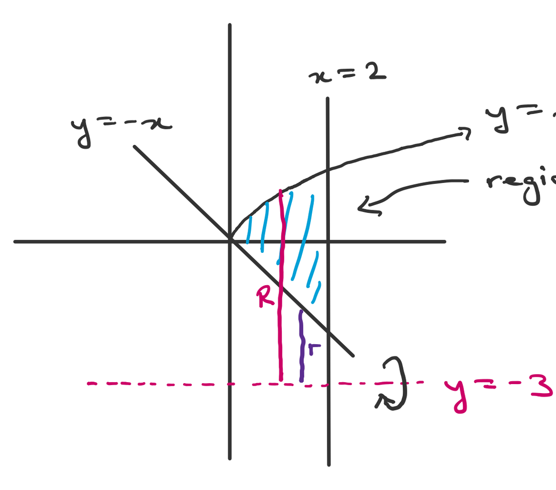
Here, $R = R(x)$ and $r = r(x)$



$$\text{Vol} = \int_a^b (\pi\cdot R^2 - \pi\cdot r^2)\,dx$$
$$= \pi\cdot\int_a^b (R^2 - r^2)\,dx$$

Ex. Find the volume of the solid obtained by rotating the region bounded by $y=\sqrt{x}$, $y=-x$, and $x=2$ around the line $y=-3$

Sol. Here is a sketch of the region:



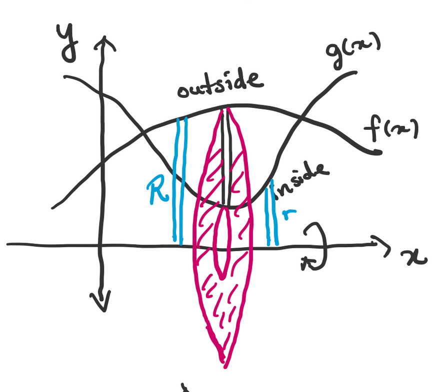
region is bounded between $x=0$ and $x=2$

$R = \sqrt{x} - (-3) = \sqrt{x}+3$

$r = -x-(-3) = 3-x$

So
$$\text{Vol} = \pi\cdot\int_a^b R^2 - r^2\,dx$$
$$= \pi\cdot\int_0^2 (\sqrt{x}+3)^2 - (3-x)^2\,dx$$
$$= \pi\cdot\int_0^2 x + 6\cdot\sqrt{x} + 9 - (9 - 6x + x^2)\,dx$$
$$= \pi\cdot\int_0^2 (7x + 6\cdot x^{1/2} - x^2)\,dx$$
$$= \pi\cdot\left[\frac{7x^2}{2} + \frac{6\cdot x^{3/2}}{3/2} - \frac{x^3}{3}\right]\Big|_0^2$$
$$= \pi\cdot\left[\frac{7\cdot 2^2}{2} + 4\cdot 2^{3/2} - \frac{8}{3}\right]$$
$$= \pi\cdot\left[14 + 4\cdot 2^{3/2} - \frac{8}{3}\right].$$

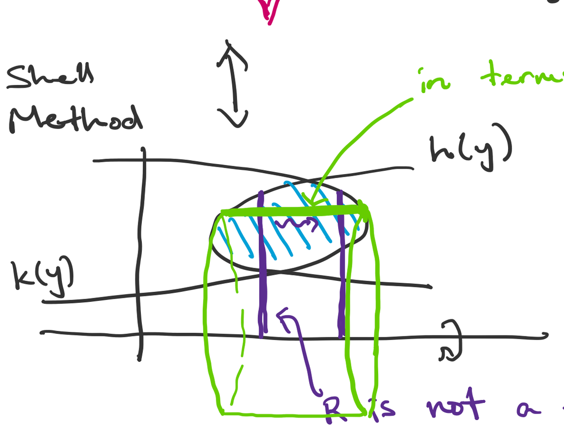
Washer-method: Solids of revolution.



Area of cross-section $= \pi\cdot f(x)^2 - \pi\cdot g(x)^2$
$= \pi\cdot R^2 - \pi\cdot r^2$

g and f pass the vertical line test.

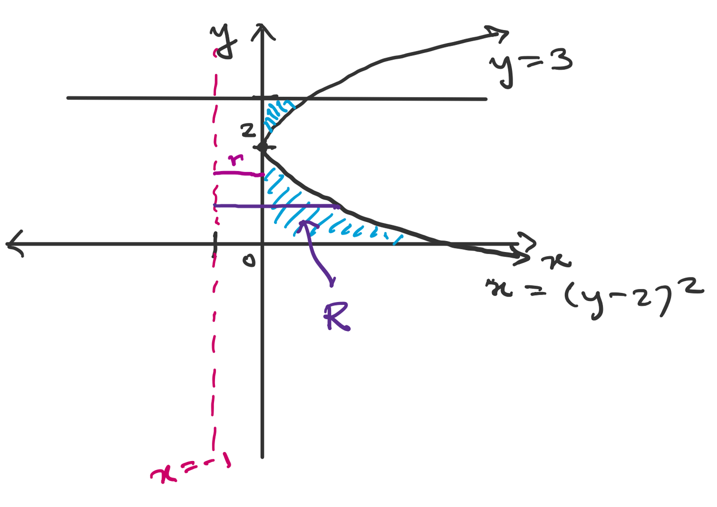
Shell Method ↕



in terms of y. $k(y) - h(y)$

h, k pass the horizontal line test.

Ex. Find the volume obtained by rotating the region bounded by $x=(y-2)^2$, the x-axis, $y=3$, and y-axis around the line $x=-1$.



We can use the washer method.

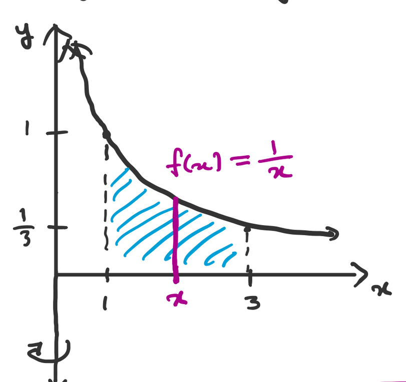
$$\text{Vol} = \int_a^b \pi\cdot R^2 - \pi\cdot r^2\,dy$$

R, r are functions of y

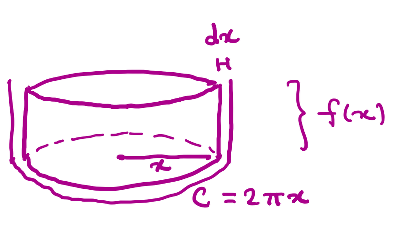
$R = (y-2)^2 - (-1)$ and $r = 0-(-1) = 1$

$$\text{Vol} = \pi\cdot\int_0^3 R^2 - r^2\cdot dy$$
$$= \pi\cdot\int_0^3 [(y-2)^2+1]^2 - 1\,dy$$

Ex. Region given by $f(x) = \frac{1}{x}$, x-axis, $x=1$, and $x=3$.



$$\text{Vol} = \int_1^3 2\pi x\cdot f(x)\,dx = \int_1^3 2\pi\cdot x\cdot\frac{1}{x}\,dx$$
$$= 2\pi\cdot\int_1^3 dx$$
$$= 2\pi\cdot 2 = 4\pi.$$



Final: Wed. May 13th 11:15 am – 1:45 pm

I'll send out Zoom link

Next week – arclength