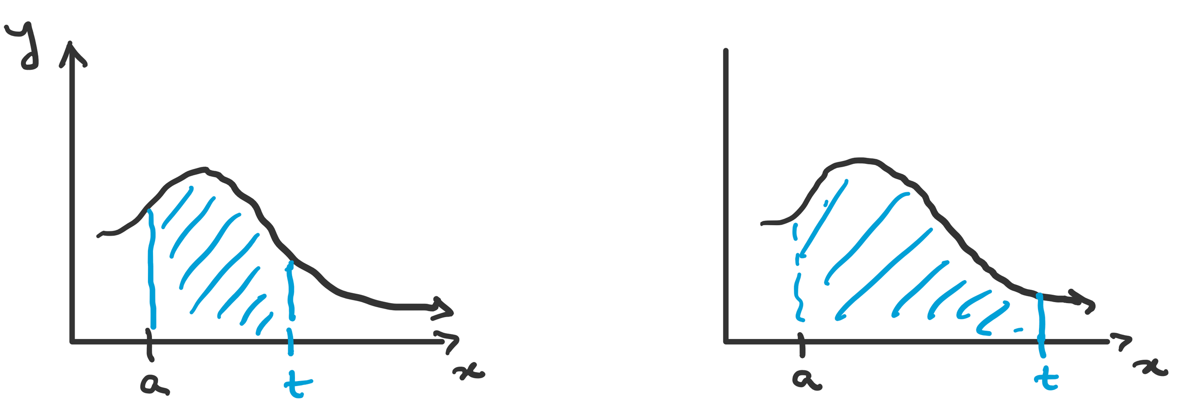
4/21/2020 R03

Tuesday, April 21, 2020 5:30 PM

Improper integrals: * bounds may involve $\pm\infty$
* integrand could have vertical asymptotes (or other discontinuities)

<u>Def.</u> 1) f continuous on $[a, \infty)$. Then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$



2) f continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

3) f continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx$$

only makes sense if both terms on RHS converge (integral is finite)

Otherwise, $\int_{-\infty}^\infty f(x)\,dx$ diverges.

<u>Ex.</u> Is the area between $f(x) = \frac{1}{x}$ and x-axis on $[1, \infty)$ finite?

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$$
$$= \lim_{t\to\infty} \Big[\ln|x|\Big]\Big|_1^t$$
$$= \lim_{t\to\infty} \big(\ln t - \cancel{\ln 1}^{\,0}\big)$$
$$= \infty$$

What happens for $\int_1^\infty \frac{1}{x^k}\,dx$ for $k > 1$?

$$\int_1^\infty \frac{1}{x^k}\,dx = \lim_{t\to\infty} \int_1^t x^{-k}\,dx$$
$$= \lim_{t\to\infty} \left[\frac{x^{-k+1}}{-k+1}\right]\Big|_1^t$$
$$= \lim_{t\to\infty} \frac{t^{-k+1}}{-k+1} - \frac{1}{-k+1}$$
$$= \lim_{t\to\infty} \underbrace{\frac{1}{(-k+1)\cdot t^{k-1}}}_{\to 0} - \frac{1}{(-k+1)}$$

$k-1 > 0$ since $k > 1$.

$$= -\frac{1}{(-k+1)} = \frac{1}{k-1} \quad \text{finite}$$

<u>Ex.</u> What happens to $\int_0^1 \frac{1}{x^k}\,dx$? Converge / diverge?

undefined at $x = 0$.

<u>Def</u> 1) f continuous on $[a, b)$ ← e.g. f could have an asymptote at b.

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

2) f continuous on $(a, b]$

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

3) f continuous on $[a, b]$ except at $x = c$ for some c between a and b.

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

<u>Ex</u> $\int_0^3 x \cdot \ln x \cdot dx$

not defined at $x = 0$.

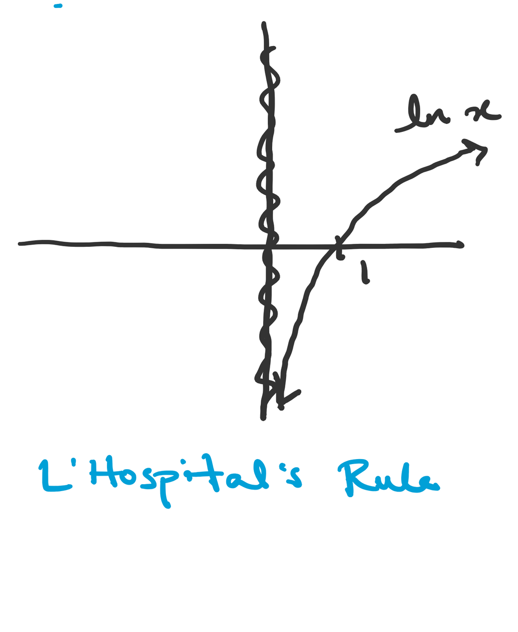
$$= \lim_{t\to0^+} \int_t^3 x \cdot \ln x \cdot dx$$

Try integration by parts
$u = \ln x \qquad du = \frac{1}{x}\,dx$
$dv = x \cdot dx \qquad v = \int x \cdot dx = \frac{x^2}{2}$.

$$= \lim_{t\to0^+} \left(u \cdot v - \int v \cdot du\right)\Big|_t^3$$
$$= \lim_{t\to0^+} \left(\frac{x^2}{2} \cdot \ln x - \int \underbrace{\frac{x^2}{2} \cdot \frac{1}{x}}_{\frac{x}{2}}\,dx\right)\Big|_t^3$$
$$= \lim_{t\to0^+} \left(\frac{x^2}{2} \cdot \ln x - \frac{x^2}{4}\right)\Big|_t^3$$
$$= \lim_{t\to0^+} \underbrace{\left(\frac{9}{2} \cdot \ln 3 - \frac{9}{4}\right)}_{\text{constant}} - \Big(\underbrace{\frac{t^2}{2} \cdot \ln t}_{?} - \cancel{\frac{t^2}{4}}^{\,0}\Big)$$

Suffices to compute

$$\lim_{t\to0^+} \underbrace{\frac{t^2}{2}}_{\to 0} \cdot \underbrace{\ln t}_{\to -\infty}$$



$$= \lim_{t\to0^+} \frac{\ln t \;\to -\infty}{\frac{2}{t^2} \;\to \infty} \qquad \frac{2}{t^2} = 2 \cdot t^{-2}$$

✓ use L'Hospital's Rule

$$= \lim_{t\to0^+} \frac{\frac{1}{t}}{-4 \cdot t^{-3}}$$
$$= \lim_{t\to0^+} \frac{t^2}{-4} = 0$$

$$= \frac{9}{2} \cdot \ln 3 - \frac{9}{4}$$

<u>Ex</u> $\int_{-\infty}^\infty x \cdot \cos x \cdot dx$ ← should diverge

$$\| $$

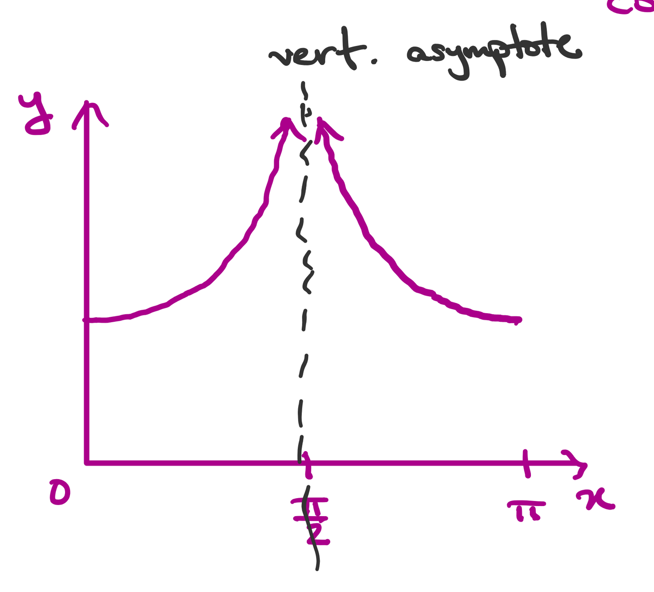
$$\int_{-\infty}^0 x \cdot \cos x\,dx + \int_0^\infty x \cdot \cos x \cdot dx$$

<u>Ex</u>

$$\left.\begin{aligned} \int_0^\pi \sec^2 x \cdot dx &= (\tan x)\Big|_0^\pi \\ &= \tan \pi - \tan 0 \\ &= 0 - 0 = 0 \end{aligned}\right\} \text{This is wrong!}$$

Why is this wrong? $\sec^2 x = \frac{1}{\cos^2 x}$

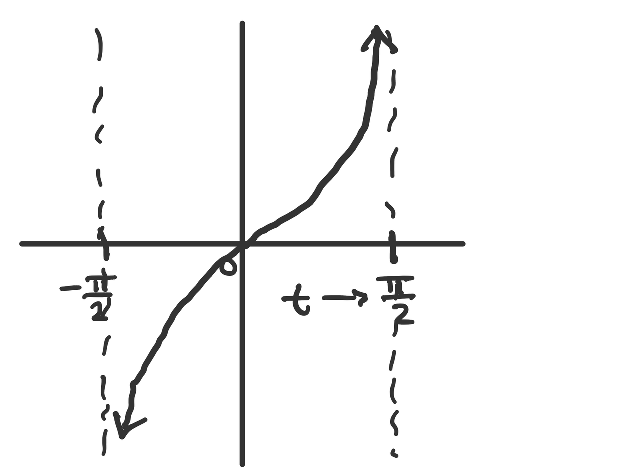
$\cos\left(\frac{\pi}{2}\right) = 0$ has an asymptote at $x = \frac{\pi}{2}$.



Correct solution:

$$\int_0^\pi \sec^2 x \cdot dx = \underbrace{\int_0^{\frac{\pi}{2}} \sec^2 x \cdot dx}_{①} + \underbrace{\int_{\frac{\pi}{2}}^\pi \sec^2 x \cdot dx}_{②}$$

① $$\int_0^{\frac{\pi}{2}} \sec^2 x \cdot dx = \lim_{t\to\frac{\pi}{2}^-} \int_0^t \sec^2 x \cdot dx$$
$$= \lim_{t\to\frac{\pi}{2}^-} (\tan x)\Big|_0^t$$
$$= \lim_{t\to\frac{\pi}{2}^-} \tan t$$
$$= \infty$$



Similarly, ② also diverges to ∞