Questions from/about Midterm 2?

Partial Fraction Decomposition

Helps with integrals of the form

$$\int \frac{p(x)}{q(x)}\,dx \qquad p, q \text{ are polynomials in } x$$

Ex $\int \boxed{\frac{3x}{x^2-x-2}}\,dx$

↑ rewritten

Notice $\frac{3x}{x^2-x-2} = \boxed{\frac{3x}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}}$.

Clear denominators: multiply both sides by $(x-2)(x+1)$

$$3x = A\cdot(x+1) + B\cdot(x-2)$$

Try $x=-1$: $\quad -3 = A\cdot 0 + B\cdot(-3) \qquad B=1.$

Try $x=2$: $\quad 6 = A\cdot 3 + B\cdot 0 \qquad A=2.$

So $\frac{3x}{(x-2)(x+1)} = \frac{2}{x-2} + \frac{1}{x+1}$

$$\int \frac{3x}{x^2-x-2}\,dx = \int \frac{2}{x-2}\,dx + \int \frac{1}{x+1}\,dx$$

$$= 2\cdot \ln|x-2| + \ln|x+1| + C.$$

Ex $\int \frac{x^2+3x+5}{x+1}\,dx$

Long division:

$$\begin{array}{r} x + 2 \\ x+1 \,\big)\, x^2 + 3x + 5 \\ -(x^2 + x) \downarrow \\ \hline 0 + 2x + 5 \\ -(2x+2) \\ \hline 0 + 3 \end{array}$$

$0 + 3$ = remainder

So $\frac{x^2+3x+5}{x+1} = \underbrace{x+2}_{\text{polynomial}} + \boxed{\frac{3}{x+1}}$

↑ ↑ ↑ we know how to integrate this

Ex $\int \frac{x-2}{(2x-1)^2\cdot(x-1)}\,dx$

Two ways to set up.

Repeated factors * $\boxed{\frac{x-2}{(2x-1)^2\cdot(x-1)} = \frac{A}{(2x-1)} + \frac{B}{(2x-1)^2} + \frac{C}{x-1}}$

Higher deg terms in denom. * $\frac{x-2}{(4x^2-4x+1)(x-1)} = \frac{\boxed{Ax+B}}{(4x^2-4x+1)} + \frac{C}{x-1}$ (linear)

numerators are of degree 1 less

Clear denominators: multiply both sides by $(2x-1)^2\cdot(x-1)$

$$(x-2) = A(2x-1)\cdot(x-1) + B\cdot(x-1) + C\cdot(2x-1)^2$$

Try $x=1$: $\quad -1 = A\cdot(1)\cdot 0 + B\cdot 0 + C\cdot(1)^2$

$\Rightarrow \boxed{C=-1.}$

Try $x=\frac{1}{2}$: $\quad -\frac{3}{2} = A\cdot 0 + B\cdot\left(-\frac{1}{2}\right) + C\cdot 0^2$

$\Rightarrow -\frac{1}{2}B = -\frac{3}{2}$

$\Rightarrow \boxed{B = (-2)\cdot\left(-\frac{3}{2}\right) = 3}$

Try $x=0$: $\quad -2 = A\cdot(-1)\cdot(-1) + B\cdot(-1) + C\cdot(-1)^2$

$= A - B + C.$

↳ plug in $B=3$, $C=-1$ $\Rightarrow -2 = A-3-1$

$\Rightarrow \boxed{A=2}$

We have $\frac{x-2}{(2x-1)^2(x-1)} = \frac{2}{2x-1} + \frac{3}{(2x-1)^2} + \frac{(-1)}{(x-1)}$

↑ ↑ ↗ integrate using u-sub.

See list of strategies for setting up PFD (pg. 303–304, Calc Vol. 2)

Ex $\int \frac{2x-3}{x^3+x}\,dx$

$$\frac{2x-3}{x^3+x} = \frac{2x-3}{x(x^2+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x}$$

Clear denominators:

$$2x-3 = x(Ax+B) + (x^2+1)\cdot C$$

Try $x=0$: (disappears) $\quad -3 = 0 + (1)\cdot C \Rightarrow C=-3.$

Try $x=1$: $\quad -1 = 1\cdot(A+B) + 2\cdot C$

$= A+B-6.$

$A+B=5$

Try $x=-1$: $\quad -5 = (-1)\cdot(-A+B) + 2\cdot C$

$= A-B-6$

solve

$A-B=1.$

$2A=6 \Rightarrow A=3 \Rightarrow B=2.$

So $\frac{2x-3}{x^3+x} = \frac{3x+2}{x^2+1} + \frac{(-3)}{x}$

$$\int \frac{2x-3}{x^3+x}\,dx = \int \frac{3x+2}{x^2+1}\,dx + \boxed{\int \frac{-3}{x}\,dx} = -3\cdot\ln|x|$$

$$\int\left(\frac{3x}{x^2+1} + \frac{2}{x^2+1}\right)dx$$

↑ use u-sub. with $u=x^2+1$, $du = 2x\cdot dx$

↖ $\int \frac{2}{x^2+1}\,dx = 2\cdot\arctan(x)$

Ex $\int \frac{3x^2}{x^3-1}\,dx$

Notice that x^3-1 has a root at $x=1$.
So $x-1$ divides x^3-1.

Use long division:

$$\begin{array}{r} \boxed{x^2} + x + 1 \\ x-1 \,\big)\, x^3 + 0x^2 + 0x - 1 \\ -(x^3 - x^2) \\ \hline 0 + x^2 + 0x \\ -(x^2 - x) \\ \hline 0 + x - 1 \\ -(x-1) \\ \hline 0 \end{array}$$

placeholders

So $x^3-1 = (x-1)\cdot(x^2+x+1)$

↑ cannot be factored further.

Apply PFD:

$$\boxed{\frac{3x^2}{x^3-1} = \frac{Ax+B}{x^2+x+1} + \frac{C}{x-1}}$$

Solve by matching coefficients

↳ I think this is what Prof. Kahn did in class.

$$3x^2 = (Ax+B)\cdot(x-1) + C\cdot(x^2+x+1)$$

$$= Ax^2 + Bx - Ax - B + Cx^2 + Cx + C.$$

$3x^2 + 0x + 0 = (A+C)\cdot x^2 + (B-A+C)\,x + C-B$ match colors

$$\begin{cases} 3 = A+C \\ 0 = B-A+C \\ 0 = C-B \end{cases} \Rightarrow \quad B=C.$$

$$\begin{cases} 3 = A+B \\ 0 = B-A+B = 2B-A \end{cases}$$

add the two eqns. $\Rightarrow \quad 3 = 3B$

$\Rightarrow \boxed{B=1} \quad \boxed{C=1}$

$A+B=3 \Rightarrow \boxed{A=2.}$

So $\int \frac{3x^2}{x^3-1}\,dx = \int \frac{2x+1}{x^2+x+1}\,dx + \boxed{\int \frac{1}{x-1}\,dx} = \ln|x-1|$

Try u-sub with $u = x^2+x+1$
$du = (2x+1)\cdot dx$

We get $\int \frac{1}{u}\cdot du = \ln|u| = \ln|x^2+x+1| + C.$